Chapter-3

Monday, May 3, 2021 11:04 AM

3.3.

$\operatorname{dom} g = (f(a), f(b))$.

and $g(f(x)) = x$, $a < x < b$.

Since f is convex and increasing, $f'' \geq 0$ and $f' > 0$.

The function g can be realized as a composition,

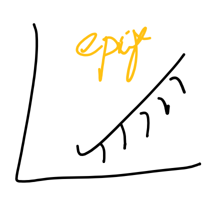
$$g \circ f = g(f(x)), \quad \operatorname{dom} g = \{ f(x) \mid x \in \operatorname{dom} f \}.$$

Since $f(x)$ is convex, g may be convex or concave depending on $\operatorname{dom} g$. Since $f(x)$ is increasing, $g(f(x))$ must be nondecreasing. Thus, g is convex.

3.6.

Epigraph is a halfspace when function is affine linear.

$\operatorname{epi} f = \{ x \mid a^T x = b \}$



Epigraph is a convex cone when $\underbrace{f(x) \text{ is convex.}}_{}$

$\operatorname{epi} f = \{ x \mid f(x) \leq t \}$. $f(x)$ is positively homogenous

Epigraph is a polyhedron when $f(x)$ represents a finite number of inequalities, $\to$ is piecewise-affine

$\operatorname{epi} f = \{ x \mid f_1(x) \leq b, f_2(x) \geq t \}$.

3.8.

Consider f to be convex and $x, y \in \operatorname{dom} f$. Since $\operatorname{dom} f$ is convex, we conclude that $\forall\ 0 < t \leq 1$, $x + t(y-x) \in \operatorname{dom} f$,

$$f(x+t(y-x)) \leq (1-t) f(x) + t f(y).$$

Div. by t,

$$\frac{1}{t} f(x+t(y-x)) \leq \frac{1}{t} f(x) - f(x) + f(y),$$

$$\Rightarrow f(y) \geq \frac{f(x+t(y-x)) - f(x)}{t} + f(x).$$

as $t \to 0$,

$$f(y) \geq f(x) + f'(x)(y-x).$$

Taking deriv. w.r.t. x,

$$0 \geq \cancel{f'(x)} + f''(x)(y-x) - \cancel{f'(x)},$$

$$\Rightarrow f''(x) \leq 0.$$

Now, assume that $\nabla^2 f(x) \succeq 0\ \forall x \in \operatorname{dom} f$,

Let $x \neq y$ and $0 \leq \theta \leq 1$, $z = \theta x + (1-\theta) y$,

$$f''(x) \geq 0, \quad f''(y) \geq 0,$$

and,
$$f''(z) = f''(\theta x + (1-\theta) y),$$
$$\geq \theta \underbrace{f''(x)}_{\geq 0} + (1-\theta) \underbrace{f''(y)}_{\geq 0},$$
$$\Rightarrow f''(z) \geq 0.$$

Thus, $\nabla^2 f(x) \succeq 0$ implies that f is convex.

3.11. Since $f : \mathbb{R}^n \to \mathbb{R}$ is convex,

f satisfies the 1st order condition,

$$f(y) \geq f(x) + f'(x)(y-x),$$

$$\Rightarrow \frac{f(y) - f(x)}{(y-x)} \geq f'(x),$$

and, $f'(y) \geq f'(x)$,

$$\Rightarrow f'(y) - f'(x) \geq 0,$$

$$\Rightarrow \frac{(f'(y) - f'(x))^T (y-x)}{(y-x)} \geq 0$$

$$\Rightarrow (f'(y) - f'(x))^T (y-x) \geq 0.$$

Thus, ∇f is monotone.

Consider the converse by taking $(\psi(x) - \psi(y))^T (x-y) \geq 0$,

We need to show that $\psi(x)$ is the gradient of convex function s.t. $x \in f$,

However, this is not true as the C^1 condition for $f = (f_1, f_2, \dots, f_n)$ does not hold,

$$\Rightarrow \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad 1 \leq i \leq j \leq n.$$

3.15.
(a)
$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},$$

$$\lim_{\alpha \to 0} u_\alpha(x) = \lim_{\alpha \to 0} \frac{x^\alpha \log x}{1} = \log x$$

(b)

Firstly, $u_\alpha(1) = \frac{1^\alpha - 1}{\alpha} = 0$,

$$u_\alpha'(x) = x^{\alpha-1} > 0 \quad \forall\ 0 < \alpha < 1.$$

and, $u_\alpha''(x) = (\alpha - 1) x^{\alpha-2} < 0 \quad \forall\ 0 < \alpha < 2.$

$\Rightarrow u_\alpha''(x)$ is concave, increasing and has $u_\alpha(1) = 0$.

3.17

$$f(x) = \left(\sum_{i=1}^n x_i^p \right)^{1/p}. \quad \Bigg| \quad f(x) = \left(\sum_{i=1}^n x_i^{1/2} \right)^2. \quad f(x) = \left(\sum_{i=1}^n 1/x_i \right)^{-1}.$$

$$\nabla_{x_i} f(x) = \frac{1}{p} \left(\sum_{i=1}^n x_i^p \right)^{1/p - 1} p\, x_i^{(p-1)}$$

$$\nabla^2_{x_i} f(x) = (1-p)\, x_i^{2(p-1)} \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-1-1} + (p-1)\, x_i^{(p-2)} \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-1}.$$

$$\nabla^2_{x_i x_j} f(x) = (1-p)\, x_i^{(p-1)} x_j^{(p-1)} \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-1}$$

Thus,
$$\nabla^2 f(x) = (1-p)\, x_i^{2(p-1)} \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-1-1} + (p-1)\, x_i^{(p-2)} \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-1}$$
$$+ (1-p)\, x_i^{(p-1)} x_j^{(p-1)} \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-1}.$$

$$(1-p)\, x_i^{(p-1)} \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-1} \left[x_i^{(p-1)} \left(\sum_{i=1}^n x_i^p \right)^{-1} - x_i^{-1} + x_j^{(p-1)} \right].$$

$a^2 b^2 \geq (ab)^2$

We need to show that $v^T \nabla^2 f(x) v \leq 0$,

$$-v^2 (p-1)\, x_i^{(p-1)} \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-1} \left[\frac{x_i^{(p-1)} - x_i^{-1} + x_j^{(p-1)} \left(\sum_{i=1}^n x_i^p \right)}{\left(\sum_{i=1}^n x_i^p \right)} \right].$$

$$= -v^2 (p-1)\, x_i^{(p-1)} \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-2} \left[x_i^{(p-1)} - x_i^{-1} + x_j^{(p-1)} \left(\sum_{i=1}^n x_i^p \right) \right].$$

$$= \underbrace{-(p-1)\, x_i^{(p-1)} \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-2}}_{\leq 0} \left[v^2 x_i^2\, x_i^{(p-3)} - v^2 x_i^{-1} + v^2 x_j^2\, x_j^{(p-3)} \left(\sum_{i=1}^n x_i^p \right) \right].$$

Using Cauchy-Schwartz inequality for $v^2 x^2$.

3.32.
(a)
$$f(\theta x + (1-\theta) y)\, g(\theta x + (1-\theta) y),$$
$$\leq (\theta f(x) + (1-\theta) f(y)) (\theta g(x) + (1-\theta) g(y)).$$
$$= \theta f(x) g(x) + (1-\theta) f(y) g(y) + \underbrace{\theta(1-\theta)(f(y) - f(x))(g(x) - g(y))}_{\leq 0}.$$

$$\Rightarrow f(\theta x + (1-\theta) y)\, g(\theta x + (1-\theta) y) \leq \theta f(x) g(x) + (1-\theta) f(y) g(y)$$

(b) Same as above but with reversed ineq

(c) Note that $\frac{1}{g}$ is convex, positive and increasing. Thus, $\frac{f}{g}$ is convex as per (a).

x ——————— x ——————— x