Exercises- 23

Tuesday, March 9, 2021 9:43 AM

1. Cauchy distribution = $\mathcal{T}(x|0,1,1)$.

$$\mathcal{T}(x|\mu,\sigma^2,\nu) = \frac{1}{Z}\left[1+\frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right]^{-\frac{(\nu+1)}{2}}.$$

Thus, $\mathcal{T}(x|0,1,1) = \frac{1}{Z}\left[1+x^2\right]^{-1} = \left(\frac{1}{\pi}\right)\frac{1}{1+x^2}$.

$\hookrightarrow Z$

and cdf of the distribution,

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi}\left(\frac{1}{1+t^2}\right) dt = \frac{\arctan x + \pi/2}{\pi}.$$

and $F^{-1}(p) = \tan\left[\pi\left(p-\frac{1}{2}\right)\right]$.

Thus, we can sample from $\mathrm{Unif}\left[-\frac{1}{2},\frac{1}{2}\right]$ and transform it by $\tan p\pi$.

2. The Cauchy distribution,

$$\mathcal{T}(x|\mu,\sigma^2,1) = \frac{1}{Z}\cdot\frac{1}{1+\left(\frac{x-\mu}{\sigma}\right)^2}$$

where $Z = \pi\sigma$.

The Gamma distribution,

$$Ga(x|a,b) = \frac{b^a}{\Gamma(a)}x^{a-1}e^{-xb}.$$

The optimal coefficient,

$$M = \max_x\left\{\frac{Ga(x|a,b)}{\mathcal{T}(x|\mu,\sigma^2,1)}\right\}.$$

$$\frac{b^a}{\Gamma(a)}x^{a-1}e^{-xb}\; Z\cdot\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

Similarly, the optimal value for μ and σ^2 should be,

$$\underset{\mu,\sigma^2}{\operatorname{argmin}}\{M(\mu,\sigma^2|a,b)\}.$$

3. $z_t = f_t(z_{t-1}) + \mathcal{N}(0,Q_{t-1})$.

$y_t = H_t z_t + \mathcal{N}(0,R_t)$.

Derive $p(z_t|z_{t-1},y_t)$ and $p(y_t|z_{t-1})$.

$$p(z_t|z_{t-1}) = \mathcal{N}(z_t|f_t(z_{t-1}),Q_{t-1}).$$

$$p(y_t|z_t) = \mathcal{N}(y_t|H_t z_t, R_t).$$

Now, $p(z_t|z_{t-1},y_t) = \frac{p(z_t,z_{t-1},y_t)}{p(z_{t-1},y_t)}$.

$$\propto p(z_t|z_{t-1})\,p(y_t|z_t)$$

$$\propto \exp\left\{-\frac{1}{2}(z_t-f_t(z_{t-1}))^T Q_{t-1}^{-1}(z_t-f_t(z_{t-1}))\right\}\cdot \exp\left\{-\frac{1}{2}(y_t-H_t z_t)^T R_t^{-1}(y_t-H_t z_t)\right\}.$$

$$\propto \exp\Big\{-\frac{1}{2}z_t^T Q_{t-1}^{-1}z_t - \frac{1}{2}f_t(z_{t-1})^T Q_{t-1}^{-1}f_t(z_{t-1}) + z_t^T Q_{t-1}^{-1}f_t(z_{t-1}) - \frac{1}{2}y_t^T R_t^{-1}y_t - \frac{1}{2}z_t^T H_t^T R_t^{-1}(H_t z_t) + y_t^T R_t^{-1} z_t H_t^T\Big\}.$$

$$\Sigma = Q_t^{-1} + (H_t^T R_t^{-1} H_t)^{-1}$$

$$\mu = \Sigma(Q_t^{-1}f(z_{t-1}) + H_t^T R_t^{-1} y_t).$$

On the other hand,

$$p(y_t|z_{t-1}) = \int p(y_t,z_t|z_{t-1})\,dz_t.$$

$$= \int p(y_t|z_t)\cdot p(z_t|z_{t-1})\,dz_t,$$

$$\propto \exp\left\{-\frac{1}{2}y_t^T R_t^{-1}y_t\right\}\exp\left\{\frac{1}{2}(f^T Q_t^{-1} + y_t^T R_t^{-1}H_t)\Sigma(Q_t^{-1}f + H_t^T R_t^{-1}y_t)\right\}.$$

Thus, we get,

$$\Sigma' = (R_t^{-1} - R_t^{-1}H_t\Sigma H_t^T R_t^{-1})^{-1}$$

$$\mu' = \Sigma' R_t^{-1}H_t\Sigma Q_t^{-1}f(z_{t-1}).$$

x ———————— x ———————— x