Chapter-2

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The Eigenvalue Problem.

Let F be the field of real or complex numbers, and A an $n \times n$ matrix with entries in F,

$$Av = \underbrace{\lambda}_{\text{eigenvalue}} \underbrace{v}_{\text{eigenvector}}.$$

Eigenvalues of A are the zeroes of characteristic polynomial,

$$P_A(z) = \det(A - zI).$$

A (linear) subspace S of F^n is a subset of F^n that is closed under linear combinations,

$$\forall x, y \in S, \forall a, b \in F : (ax + by) \in S.$$

The set $\{y_1, \ldots, y_p\}$ of elements of S such that every element can be written as a linear combination of $y_1, \ldots, y_p$ is called a spanning set of S.

S is simply the span of $n \times p$ matrix $Y = [y_1 \cdots y_p]$ and that Y spans S.

$$S = \operatorname{span}(Y) = \{Yx : x \in F^p\} = YF^p.$$

The set of all p-dimensional subspaces of F^n are denoted by $\operatorname{Grass}(p, n)$ and admit a Grassmann manifold.

The kernel $\ker(B)$ of a matrix B is the subspace formed by the vectors x such that $Bx = 0$.

For every symmetric matrix A, there is an orthonormal matrix V and a diagonal matrix Λ such that $A = \Lambda V \Lambda^T$.

Given two $n \times n$ matrices A and B, we say that (λ, v) in an eigenpair of the pencil (A, B) if

$$Av = \lambda Bv.$$

Finding eigenpairs of a matrix pencil is known as the generalized eigenvalue problem.

A subspace Y is called a (generalized) invariant subspace of the symmetric / positive-definite pencil (A, B) if $B^{-1}Ay \in Y$ for $y \in Y$.
Note that the generalized problem reduces to the standard problem for $B = I$.

Research Problems.

1) Singular Value Problem.

Matrices U, Σ and V form a singular value decomposition (SVD) of A if,

$$A = U \Sigma V^T$$

with $U \in \mathbb{R}^{m \times m}$, $U^T U = I_m$, $V \in \mathbb{R}^{n \times n}$, $V^T V = I_n$, $\Sigma \in \mathbb{R}^{m \times n}$ with diagonal entries $\sigma_1 \geq \cdots \sigma_n \geq 0$ as the singular values of A.

An SVD expresses A as a sum of rank-1 matrices,

$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^T.$$

The singular value problem is closely related to the eigenvalue problem, $A^T A = V \Sigma^2 V^T$, which indicates that the squares of singular values of A are the eigenvalues of $A^T A$.

A suitable way to solve this problem is by computing simultaneously a few dominant singular triplets by maximizing,

$$\max f(U, V) = \operatorname{tr}(U^T A V N).$$

$$\text{s.t. } U^T U = I_p$$
$$V^T V = I_p$$
$$N = \operatorname{diag}(\mu_1, \ldots, \mu_p).$$

If (U, V) is a solution of the problem then u_i of U and v_i of V are the i^{th} dominant left and right singular vectors of A.

2) Matrix Approximation.

We aim to solve,

$$\min_{X \in M} \|A - X\|^2_{\underbrace{F}_{\text{Frobenius Norm}}}.$$

For example,

Find $C \in \mathbb{R}^{n \times n}$

to $\min \|C - C_0\|^2$ s.t. $\operatorname{rank}(C) = p$,
$C = C^T$,
$C \succeq 0$.

We can reformulate this by setting $C = Y^T Y$, $Y \in \mathbb{R}_*^{n \times p}$.

$$f : \mathbb{R}_*^{n \times p} \to \mathbb{R} : Y \to \|YY^T - C_0\|^2.$$

This leads to a quotient manifold problem where a set $\{YQ : Q^T Q = I\}$ is identified as one point of quotient manifold.

x ———— x ———— x