The completion of Escher's "Print Gallery": a dynamical system approach*

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Abstract

This small note presents a visual introduction to the sequence of transformations which have permitted the completion of Escher's "Print Gallery" by the team of Lenstra. The logarithm transformation and its inverse are presented as the limits of their unfoldings. These unfoldings, one of which is the Ecalle-Roussarie compensator, are well known functions in differential equations and dynamical systems.

1 Introduction

The present note has been motivated by the will to make visible the mathematics underlying the completion of Escher's "Print Gallery" by Bart de Smit and Hendrik W. Lenstra [1].

Once Escher's print is completed, what is obtained is a transformation from an original print, in which the original image is reproduced inside the image itself with a contraction factor of $C = \frac{1}{256}$, to a print with infinite spiralling in itself (Figure 1). The publishing of [1] has led to the birth of a large number of programs transforming an image over the complex plane invariant under some linear contraction $z \mapsto Cz$, with $|C| < 1$ into an infinite spiralling image, which we will call its "escherization". The final image is again an image invariant under some contraction $z \mapsto Bz$, with $|B| < 1$. If we restrict the original image to a rectangular frame around the center of the picture, then the image of the frame under the transformation is a spiral. Considering an image as a map $F : D \to \mathscr{C}$, where $\mathscr{C}$ is the set of colors, the different transformations are simply compositions $F \circ G^{-1}$, with $G : D \to D'$. Since conformal transformations G are used, they can of course be written as holomorphic functions of z. The domains considered throughout the paper are either $\mathbb{C}$ or $\mathbb{C}^*$. Indeed, when an image is invariant under a linear contraction $z \mapsto Cz$ with $|C| < 1$, it is impossible to give a color to the origin compatible with the invariance.

We use the notation L_C and P_β for the linear and power maps

$$\begin{cases} L_C(z) = Cz, \\ P_\beta(z) = z^\beta. \end{cases}$$

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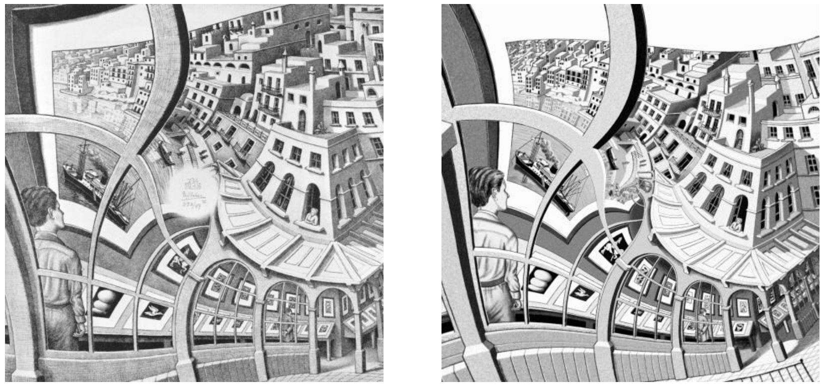


(a) Escher's "Print Gallery" (b) Its completion

Figure 1: Escher's "Print Gallery" and its completion.

We let

$$\begin{cases} \mathscr{I} & = \{F : \mathbb{C} \to \mathscr{C}\}, \\ \mathscr{I}^* & = \{F : \mathbb{C}^* \to \mathscr{C}\}, \\ \mathscr{I}^*_C & = \{F : \mathbb{C}^* \to \mathscr{C} \mid F \circ L_C = F\}. \end{cases}$$

The construction can be summarized through the following steps. We define

$$\begin{cases} \beta = \frac{2\pi i}{2\pi i - \ln C}, \\ B = C^\beta. \end{cases}$$

Considering an initial image $F_0 : \mathbb{C}^* \to \mathscr{C}$, invariant under C, we associate to it

$$\begin{tikzcd} \mathscr{I}^* \arrow[r, "F_0 \mapsto F_1 = F_0 \circ \exp"] \arrow[d, "F_0 \mapsto F_3 = F_0 \circ P_{\frac{1}{\beta}}"'] & \mathscr{I} \arrow[d, "F_1 \mapsto F_1 = F_2 \circ L_{\frac{1}{\beta}}"'] \\ \mathscr{I}^* & \mathscr{I} \arrow[l, "F_2 \mapsto F_3 = F_2 \circ \ln"'] \end{tikzcd}$$

If the initial image F_0 is invariant under L_C, then the resulting image F_3 is invariant under L_B. The construction is adapted to the particular C:

$$\begin{tikzcd} \mathscr{I}^*_C \arrow[r, ".\circ \exp"] \arrow[d, ".\circ P_{\frac{1}{\beta}}"'] & \mathscr{I} \arrow[d, ".\circ L_{\frac{1}{\beta}}"'] \\ \mathscr{I}^*_B & \mathscr{I} \arrow[l, ".\circ \ln"'] \end{tikzcd}$$

where $.\circ G$ denotes the operator $F \mapsto F \circ G$.

We want to "visualize the process". It is easy with the right vertical arrow of the diagram $F \mapsto F \circ L_{\frac{1}{\beta}}$, since it is a composition of a rotation with a homothety. The two horizontal arrows are more difficult to visualize for someone who has no mathematical background. Let us work with a simple example, namely Figure 2.

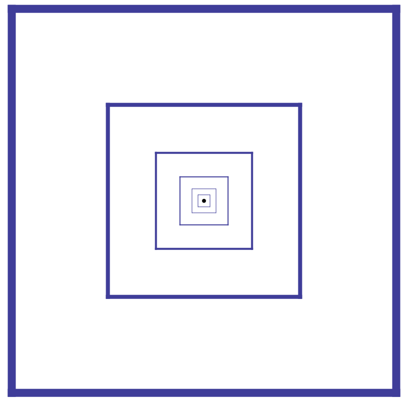


Figure 2: The initial image.

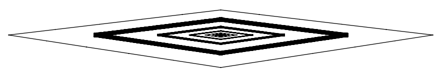


Figure 3: The initial image in 3-dimensional space.

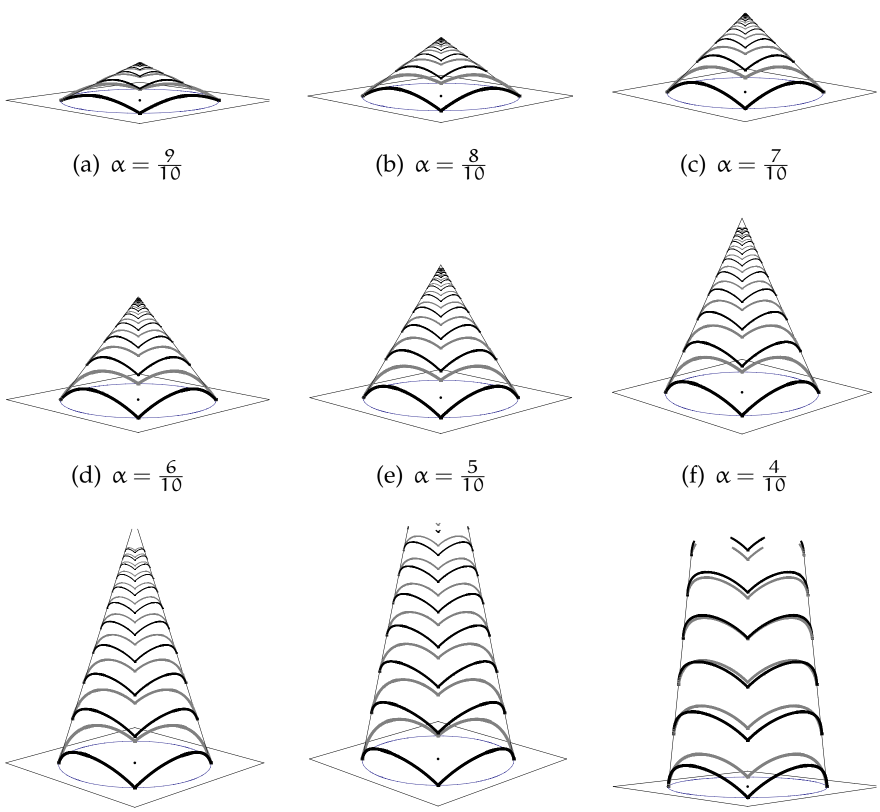
1. **Upper arrow.** It transforms an image invariant under L_C, i.e. with a multiplicative period, into an image with two additive periods. In order to understand how this happens, we will introduce a film between the two images. We imagine ourselves in 3-dimensional space with the image F_0 on the horizontal plane (Figure 3), which we can think of as a table cloth. We lift the (elastic) table cloth by its center, thus transforming it into a cone (Figure 4).

   (Figure 7). During the whole process we keep the unit circle fixed in the horizontal plane. Lifting up the vertex of the cone to infinity, we transform the punctured plane into a cylinder. On the cones we still have invariance under a multiplicative period but, when the cone tends to the cylinder, this multiplicative period tends to 1. In the limit when the vertex of the cone is at infinity we get a translation by a first (vertical) period T_1. The process yields a uniform image on the cylinder. So, when we unroll the cylinder, we get a second independent period T_2 of norm 2π.

2. **Right vertical arrow.** We cut the cylinder along a vertical line and slide it along this line by one vertical period. We also scale it so that its horizontal section remains of constant length.

3. **Lower arrow.** We inverse the process of the upper arrow. The final image appears in Figure 6.

Now, how to put this visualization in equation? We want to use conformal transformations all the way. The natural conformal transformation of a plane to a cone is $z \mapsto z^\alpha$, and if the cone is to become infinitely thin, then α must decrease from 1 to 0. Simultaneously,

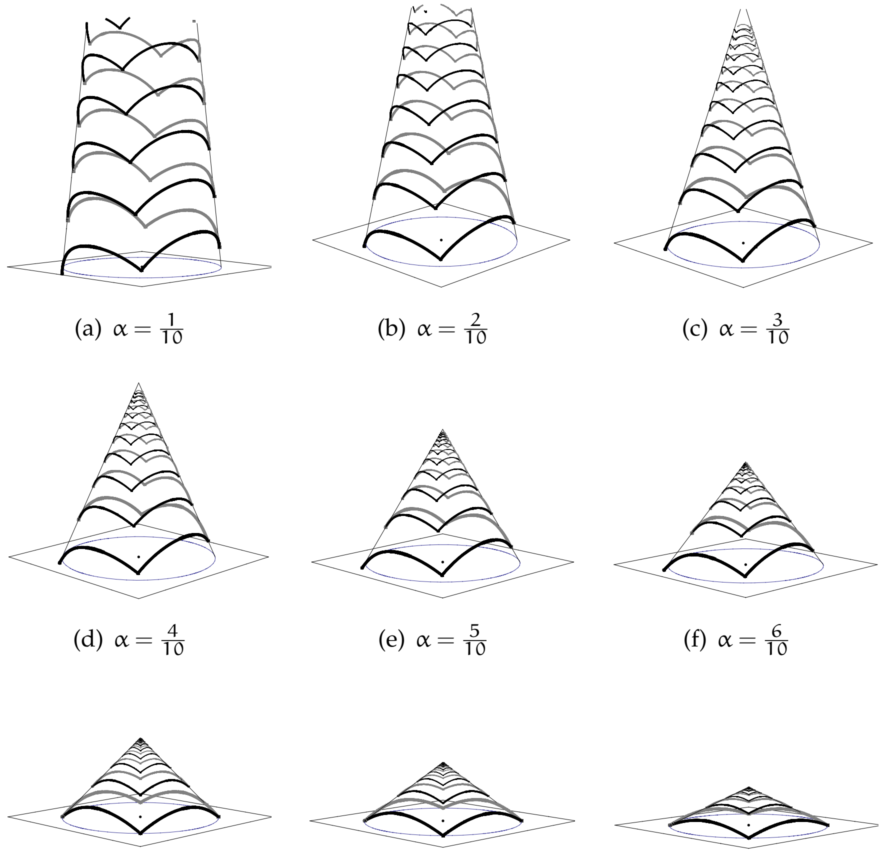


(a) $\alpha = \frac{9}{10}$ (b) $\alpha = \frac{8}{10}$ (c) $\alpha = \frac{7}{10}$

(d) $\alpha = \frac{6}{10}$ (e) $\alpha = \frac{5}{10}$ (f) $\alpha = \frac{4}{10}$

(g) $\alpha = \frac{3}{10}$ (h) $\alpha = \frac{2}{10}$ (i) $\alpha = \frac{1}{10}$

Figure 4: Lifting the image by the center.



(a) $\alpha = \frac{1}{10}$ (b) $\alpha = \frac{2}{10}$ (c) $\alpha = \frac{3}{10}$

(d) $\alpha = \frac{4}{10}$ (e) $\alpha = \frac{5}{10}$ (f) $\alpha = \frac{6}{10}$

(g) $\alpha = \frac{7}{10}$ (h) $\alpha = \frac{8}{10}$ (i) $\alpha = \frac{9}{10}$

Figure 5: Back from the cylinder to the plane.

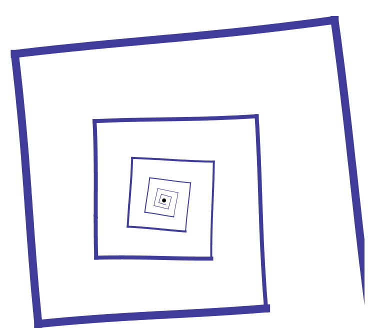
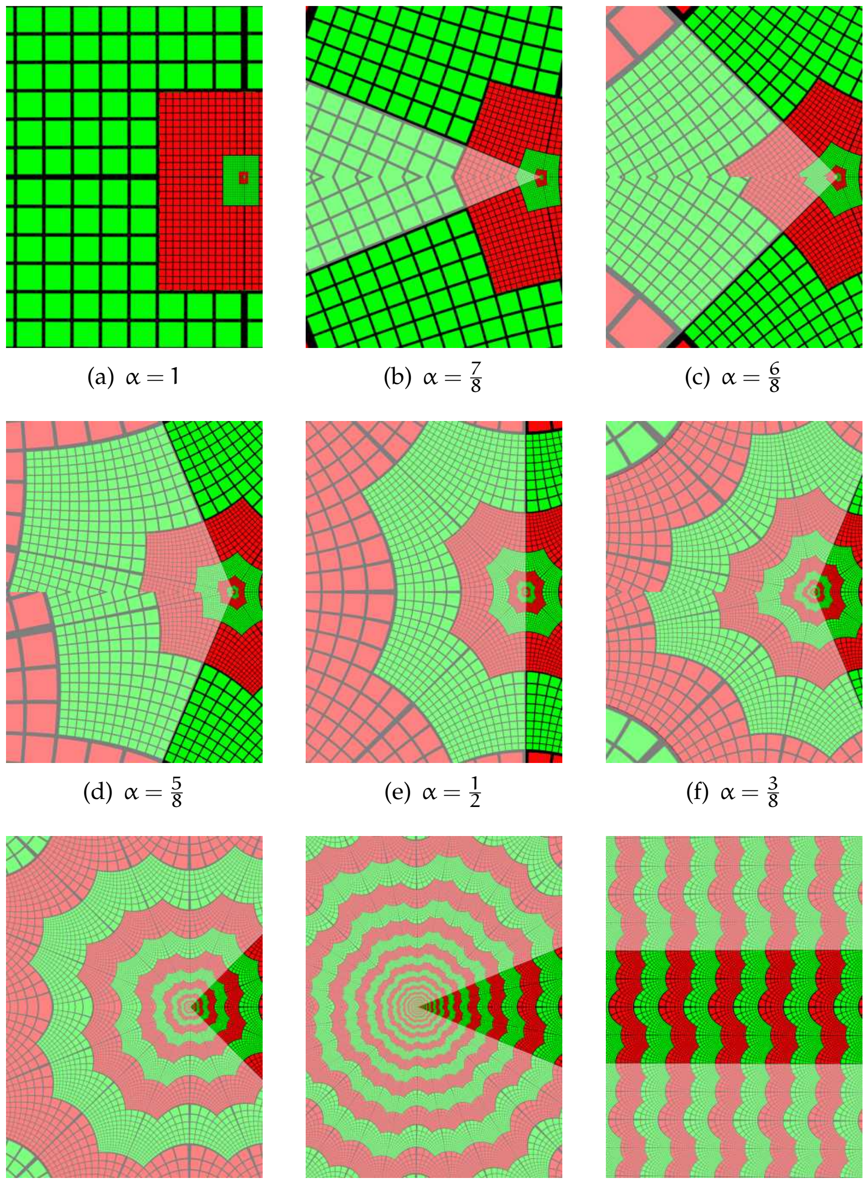


Figure 6: The "escherized" final image.

we must send the origin to ∞ and rescale, so that what we see at finite distance, the image of the circle of radius 1, remains a curve of length 1. A way to do that is to use the function $z \mapsto \frac{z^\alpha - 1}{\alpha}$ with α decreasing from 1 to 0. This just one part of the Ecalle-Leontovich-Roussarie compensator

$$\omega(z,\alpha) = \begin{cases} \frac{z^\alpha - 1}{\alpha}, & \alpha \neq 0, \\ \ln z, & \alpha = 0, \end{cases}$$

which is analytic except at $z = 0$. This function is well known in dynamical systems as



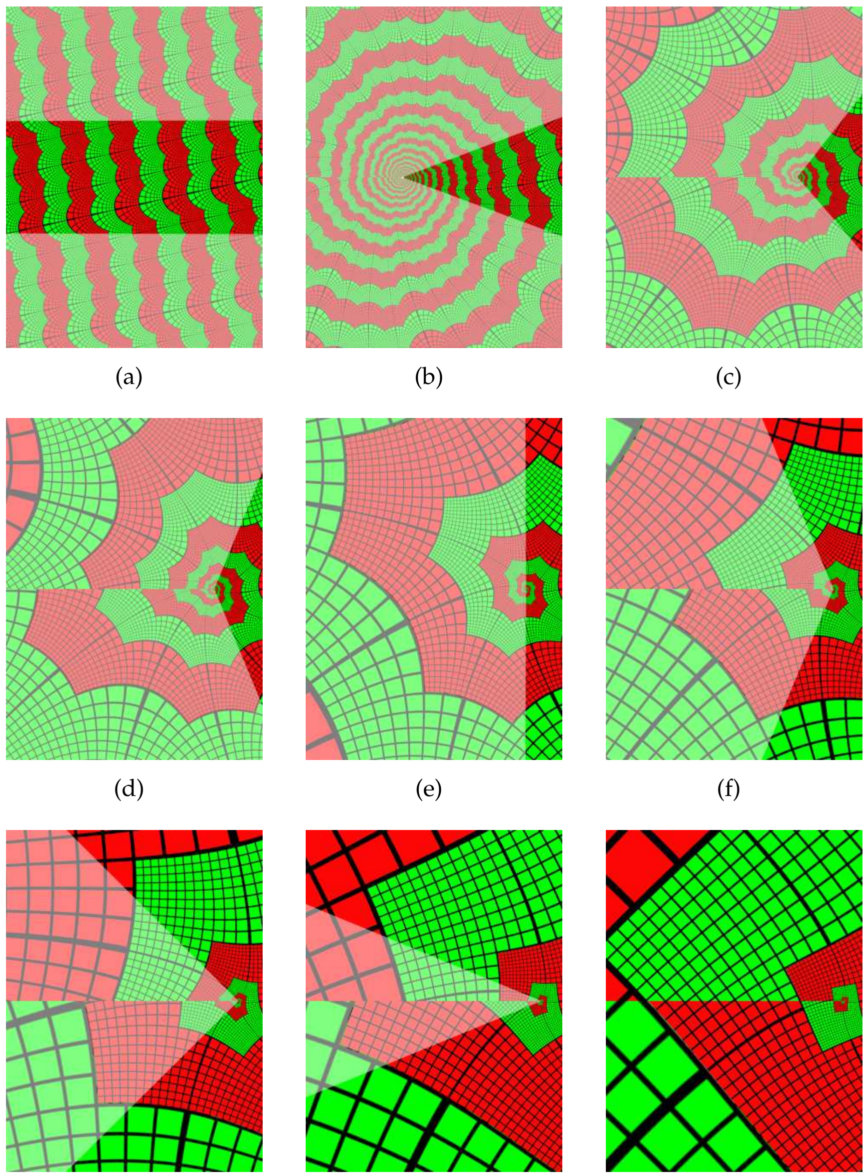
(a) $\alpha = 1$ (b) $\alpha = \frac{7}{8}$ (c) $\alpha = \frac{6}{8}$

(d) $\alpha = \frac{5}{8}$ (e) $\alpha = \frac{1}{2}$ (f) $\alpha = \frac{3}{8}$

(g) $\alpha = \frac{1}{4}$ (h) $\alpha = \frac{1}{8}$ (i) $\alpha = 0.0001$

Figure 7: Deformation to the logarithm function.

the natural unfolding of the logarithm function. It is not unlikely that it be known under different names in other fields of mathematics. The construction of the escherization of F_0 can now be visualized in the Figures 7 and 8. Note that angles are preserved throughout the process.



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 8: Coming back from the logarithm function.

References

[1] B. de Smit and H.W. Lenstra, Artful mathematics: the heritage of M. C. Escher, *Notices of the AMS* **50, no 4 (2003), 446-451.**