The Math Behind Escher

1º) Transform coordinate plane to complex plane

Use either of these:

↳ Rectangular Form: $z = x + iy$
↳ Polar ": $z = r.\cos(\theta) + r.\sin(\theta)\,i$
↳ Exponential : $z = r.e^{i\theta}$

Note: the rectangular form is the one that
works best in Python, the others have rounding err.

2º) Apply Log to the complex plane → L(z)

called L(w) in paper

This is the term 'Lnr' in Python
(natural log of the radius = Lnr)

Depending on the complex representation,
some tricks apply: ↓

↳ On the exponential repres. of complex:

$$z = |r| e^{i\theta}$$

$\ln(z) = \ln(|r|) + i\theta$ ⇒ this is the one used by the shader code

But Python has rounding errors with this one!

3:) Escher transform

From the paper: $h(w) = w^{\alpha}$
where 'w' is the complex plane

$$\boxed{\alpha = \frac{2\pi i + \ln(256)}{2\pi i} = 1 - \frac{\ln(256)}{2\pi i}}$$

$$\left(\frac{(2\pi i + \ln(256))(-i)}{2\pi i \quad (-i)} \longrightarrow \frac{2\pi \cdot 1}{2\pi \cdot 1} - \frac{\ln(256)\, i}{2\pi \cdot 1} = 1 - \frac{\ln(256)}{2\pi}\right.$$

$(-i)(i) = 1$

↓

4:) Combine all steps → Complex + Log + transform + Exp

$h(w) = w^{\alpha}$ / trick: $e^{L(x)} = x$

$L(a^b) = b\,L(a)$

$h(w) = e^{L(w^{\alpha})}$

$$\boxed{h(w) = e^{\alpha L(w)}}$$

$L(w)$ is the log of the complex plane we had before

↳ 3 operations → 1°) $L(w)$

2°) $\alpha L(w)$

3°) $e^{(\cdot)}$

Note that if we do $e^{L(w)} = w$ → what we had started with

↳ so the term α is the rotation

Inverse

↳ Need to do : $1/\alpha$

with:

$$\alpha = \frac{2\pi i + \ln(256)}{2\pi i} = 1 - \frac{\ln(256)}{2\pi i}$$

$$\left(\frac{(2\pi i + \ln(256))(-i)}{2\pi i \;(-i)}\right) \longrightarrow \frac{2\pi \cdot 1}{2\pi \cdot 1} - \frac{\ln(256)\, i}{2\pi \cdot 1} = 1 - \frac{\ln(256)}{2\pi}$$

$(-i)(i) = 1$

$$\Rightarrow \frac{1}{\alpha} = \frac{2\pi i}{2\pi i + \ln(256)} = 1 - \frac{2\pi \cdot 1}{\ln(256) \cdot i}$$

Forward: $z = w^{\alpha}$ → from straigth to Escher

$z = e^{\alpha L(w)}$ (notation is weird)

Backward: from Escher to straight → from z to w

$$z = e^{\alpha L(w)}$$

$$L(z) = \alpha L(w)$$

$$\frac{1}{\alpha} L(z) = L(w)$$

$$e^{\frac{1}{\alpha} \cdot L(z)} = w$$

we get z → the transfm image

1°) Apply $L(z)$

2°) $\frac{1}{\alpha} L(z)$

3°) $e^{(\;)}$