Shader Math Explained

Here I explain the math used on this shader:
https://www.shadertoy.com/view/Mdf3zM
Here the author explains a little bit more:
https://reindernijhoff.net/2014/05/escher-droste-effect-webgl-fragment-shader/

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This is what he said on his email:

I think that I have done something like this:

h(w) = w^((2πi + log scale)/(2πi)) = w^(1 - i * log scale / 2π) = w^(1 + i * sn)

where: log scale = L(256)

and

w^(1 + i * sn) = exp(ln(w) * (1 + i * sn)) = exp((ln|w| + i * arg(w)) * (1 + i * sn)) =

exp((lnr + i * th) * (1 + i * sn)) = exp ((lnr - th * sn) + i * (lnr * sn + th)) =

exp (lnr - th*sn) * exp (i *(lnr * sn + th))

so when I want to go back to Cartesian coordinates I get a coordinate that corresponds with a vector of length exp(lnr - th*sn) rotated (sn * lnr + th) around the origin.

(NOTE: I asked him about this: `(0.4/256.)*deformationScale` but he didn't reply on that, he might not remember)

This is my explanation of what he did:

First (unconfirmed) I believe he didn't use the complex plane, this allows him to do some extra tricks with his math.

He started from Escher's formula:
h(w) = w^((2πi + log scale)/(2πi))

Then he applied the factorization we know to arrive to this one:
w^(1 - i * log scale / 2π)

Then he expressed "sn" as a negative quantity on his code
float sn = -log(deformationScale)*(1./(2.*3.1415926));

This allows him to write:
w^(1 + i * sn)

Now you can use the trick: e^ln(x)= x   (i.e. apply exponential and log)

So basically: W^a = exp[L(W^a)]

Then by property of logs: L(W^a) = L(aW) = L(Wa)

exp(ln(w) * (1 + i * sn))

Explanation of what he did here:

exp((ln|w| + i * arg(w)) * (1 + i * sn))

so, this is tricky… imagine “w” represents the complex plane (which is usually represented as “Z” in math books). So what you have is really:

w = r.e^i.th //note that “th” = arg(w)

Then if we take the natural log:
Ln(w) = Ln(r.e^i.th) = Ln(|r|) + i.th   // where r = sqrt(x^2+y^2) → but this is actually a real number

So he went ahead and wrote this, which it’s not OK, as “w” should now be “r”
exp((ln|w| + i * arg(w)) * (1 + i * sn))

In any case, later he corrects and he does this, where now “lnr” is the modulo of the real numbers
exp((lnr + i * th) * (1 + i * sn))

What follows is just distributing with imaginary numbers, signs change as (i^2 = -1) (we don’t do this in our code, as it’s not really needed, he just does it for convenience)
= exp ((lnr - th * sn) + i * (lnr * sn + th)) =

Then he uses: e(a.b) = e^a.e^b
exp (lnr - th*sn) * exp (i *(lnr * sn + th))

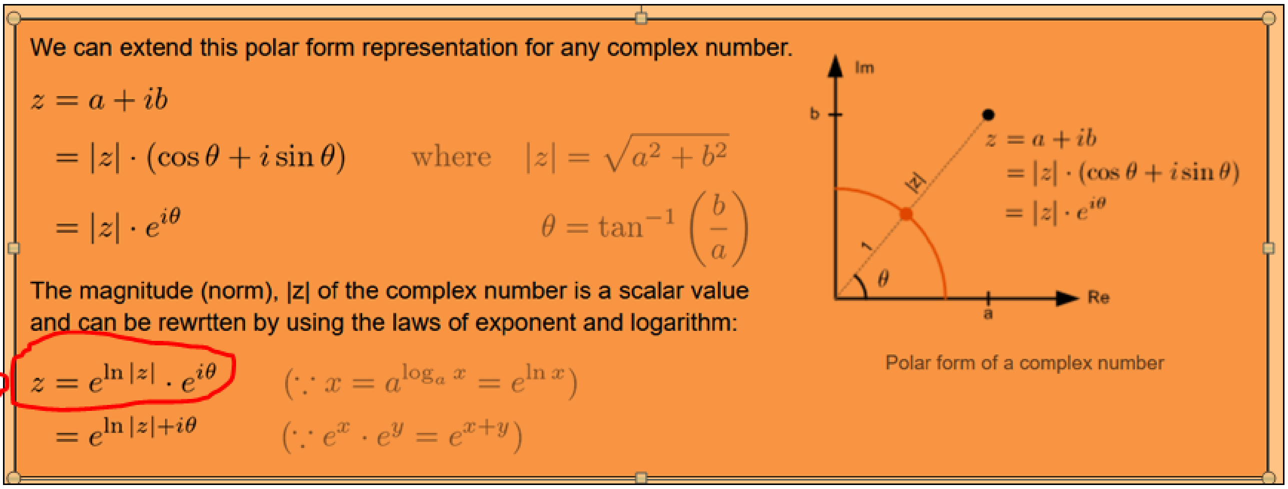
with:
lnr = log(length(uv))
sn = sn = -log(deformationScale)*(1./(2.*3.1415926))
th = atan(uv.y, uv.x)

The reason he wanted to express it like this is what he says below:

“….so when I want to go back to Cartesian coordinates I get a coordinate that corresponds with a vector of length exp(lnr - th*sn) rotated (sn * lnr + th) around the origin.

In other words, he wanted to have it expressed as below:

http://www.songho.ca/math/euler/euler.html


Im
b
|z|
1
θ
a
Re
$z = a + ib$
$= |z| \cdot (\cos\theta + i\sin\theta)$
$= |z| \cdot e^{i\theta}$

We can extend this polar form representation for any complex number.

$$z = a + ib$$
$$= |z| \cdot (\cos\theta + i\sin\theta) \qquad \text{where} \quad |z| = \sqrt{a^2 + b^2}$$
$$= |z| \cdot e^{i\theta} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

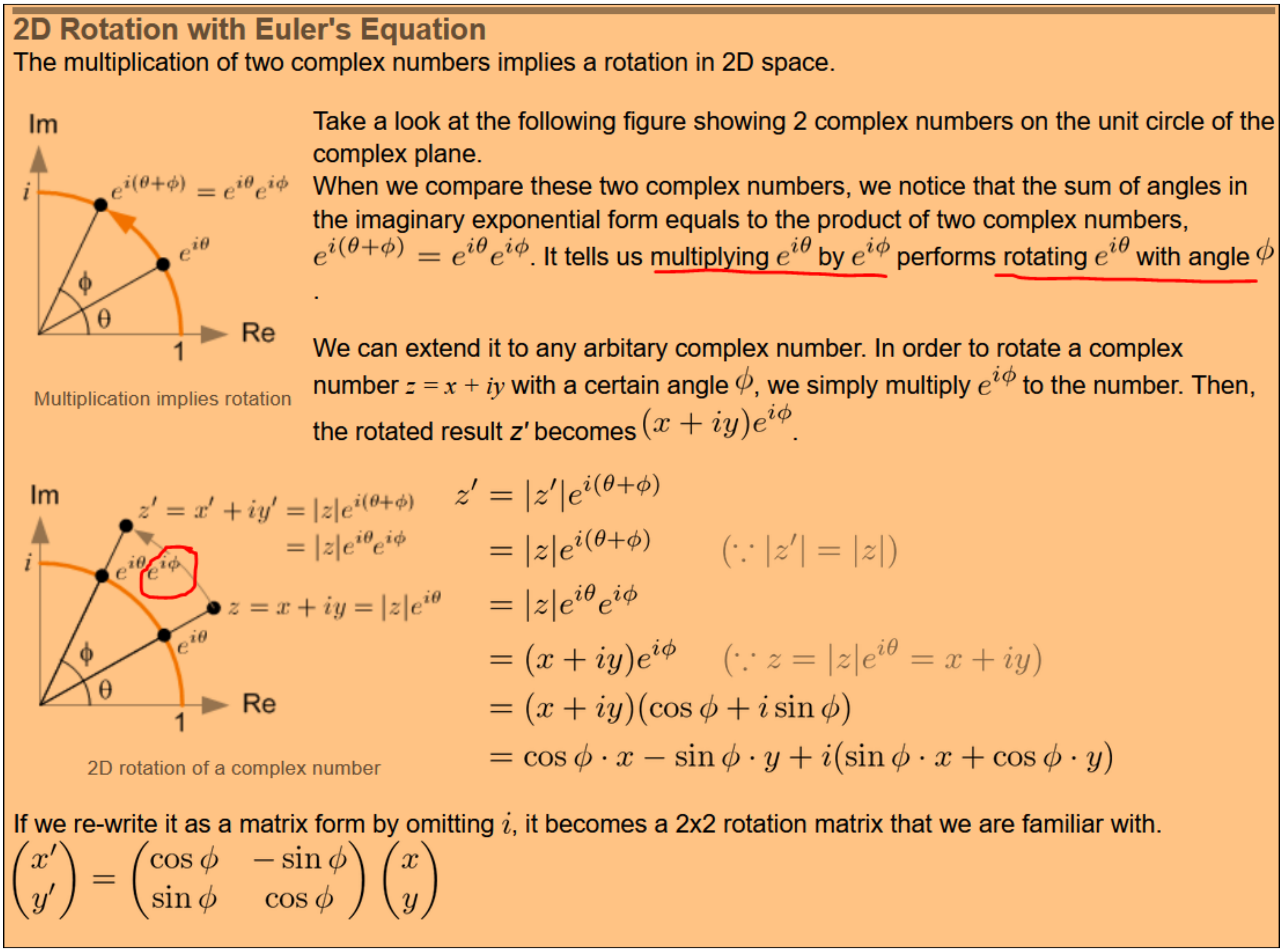
The magnitude (norm), |z| of the complex number is a scalar value and can be rewrtten by using the laws of exponent and logarithm:

$$\boxed{z = e^{\ln|z|} \cdot e^{i\theta}} \qquad (\because x = a^{\log_a x} = e^{\ln x})$$
$$= e^{\ln|z| + i\theta} \qquad (\because e^x \cdot e^y = e^{x+y})$$

Polar form of a complex number

Explanation: what he has on the last expression is the typical “rotation of a complex number”

http://www.songho.ca/math/euler/euler.html



2D Rotation with Euler's Equation

The multiplication of two complex numbers implies a rotation in 2D space.

Multiplication implies rotation

Take a look at the following figure showing 2 complex numbers on the unit circle of the complex plane.
When we compare these two complex numbers, we notice that the sum of angles in the imaginary exponential form equals to the product of two complex numbers, $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$. It tells us multiplying $e^{i\theta}$ by $e^{i\phi}$ performs rotating $e^{i\theta}$ with angle ϕ .

We can extend it to any arbitary complex number. In order to rotate a complex number $z = x + iy$ with a certain angle ϕ, we simply multiply $e^{i\phi}$ to the number. Then, the rotated result z′ becomes $(x + iy)e^{i\phi}$.

Im
i
$z' = x' + iy' = |z|e^{i(\theta+\phi)}$
$= |z|e^{i\theta}e^{i\phi}$
$e^{i\theta}e^{i\phi}$
$z = x + iy = |z|e^{i\theta}$
$e^{i\theta}$
ϕ
θ
1
Re

2D rotation of a complex number

$$
\begin{aligned}
z' &= |z'|e^{i(\theta+\phi)} \\
&= |z|e^{i(\theta+\phi)} \qquad (\because |z'| = |z|) \\
&= |z|e^{i\theta}e^{i\phi} \\
&= (x + iy)e^{i\phi} \qquad (\because z = |z|e^{i\theta} = x + iy) \\
&= (x + iy)(\cos\phi + i\sin\phi) \\
&= \cos\phi \cdot x - \sin\phi \cdot y + i(\sin\phi \cdot x + \cos\phi \cdot y)
\end{aligned}
$$

If we re-write it as a matrix form by omitting i, it becomes a 2x2 rotation matrix that we are familiar with.

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
$$