Escher and the Droste effect.

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Outline of talk

1. Escher and his Print Gallery
2. Paintings and their transformations
3. The Droste effect
4. Completing the Print Gallery

M.C.Escher (1898 – 1972) was one of the most famous graphic artists of the 20th century. The “Print Gallery” is one of his more popular works. His idea was to capture self similarity but in an annular way.

Unfortunately Escher left the center of the image blank and was unable to complete it.

Question

How might we exploit the maths behind the picture to fill in the mystery gap?

This question intrigued the famous number theorist H. Lenstra and (joint with B. de Smit) eventually led to an interesting solution back in 2002.

Solving this problem itself is not only interesting but leads to incredible mathematical tools for manipulating images.

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How to ruin art

Throughout this talk $\mathcal{C}$ will be a fixed set of colours.

Definition

Let $D \subseteq \mathbb{C}$. A painting on D is simply a function $f : D \to \mathcal{C}$. The set of paintings on D will be denoted $\mathcal{F}_D$.

A convenient choice is $D = \mathbb{C}^\times$, since this is a group under multiplication (allowing us to study pictures under scaling).

Note that this choice allows us to consider paintings that extend indefinitely outwards from the origin. While this may seem silly it will often allow us to visualize the mathematical structure of some paintings.

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Back to basics

Recall the complex exponential function:

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}^\times .$$

It is a surjective group homomorphism with kernel $2\pi i\mathbb{Z}$ and so it induces an isomorphism $\mathbb{C}/2\pi i\mathbb{Z} \cong \mathbb{C}^\times$.

The inverse of this isomorphism is given by the complex logarithm (which is a multi valued map if the codomain is $\mathbb{C}$ rather than $\mathbb{C}/2\pi i\mathbb{Z}$).

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Now consider $f \in \mathcal{F}_{\mathbb{C}^\times}$. Then we can view this as a $2\pi i$ periodic function $g \in \mathcal{F}_{\mathbb{C}}$ via:

$$\mathbb{C} \to \mathbb{C}/2\pi i\mathbb{Z} \xrightarrow{exp} \mathbb{C}^\times \xrightarrow{f} \mathcal{C}.$$

Conversely a $2\pi i$ periodic function $g \in \mathcal{F}_{\mathbb{C}}$ can be viewed as $f \in \mathcal{F}_{\mathbb{C}^\times}$ via:

$$\mathbb{C}^\times \xrightarrow{log} \mathbb{C}/2\pi i\mathbb{Z} \xrightarrow{\bar{g}} \mathcal{C}.$$

The advantage of these processes is that we can turn paintings with multiplicative scaling into ones with additive scaling, and vice versa.

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Making spirals

The space $\mathbb{C}/2\pi i\mathbb{Z}$ is naturally in bijection with the infinite strip $S = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Im}(z) < 2\pi\}$.

Pick a point w on the upper boundary of S. Then rotate the strip and scale so that w becomes the point $2\pi i$. The image of the transformed strip S' under the exponential function is an infinite spiral.

Moving to the right on S is equivalent to moving “inwards” on the spiral.

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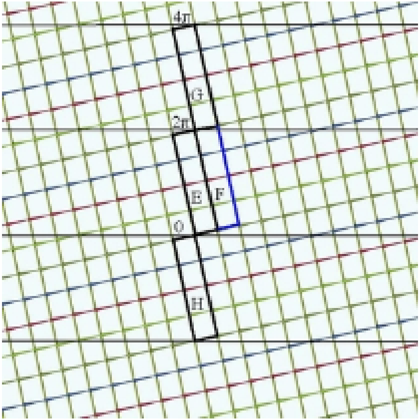
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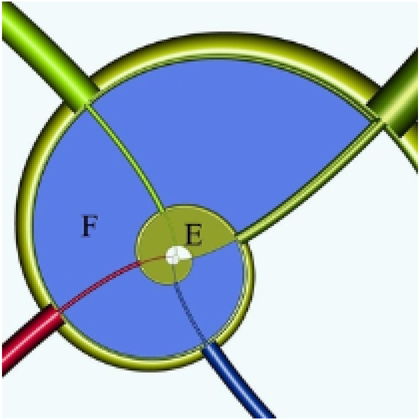
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Mathematically the rotation and scaling of the strip is just given by multiplication by $\alpha_w = \frac{2\pi i}{w}$. One can easily work out the angle of rotation and scaling factor from this.

Moral: If we start with $f \in \mathcal{F}_{\mathbb{C}^\times}$ we can create new images $f_w \in \mathcal{F}_{\mathbb{C}^\times}$ that are spiral versions of f. These are called Escherizations of f.

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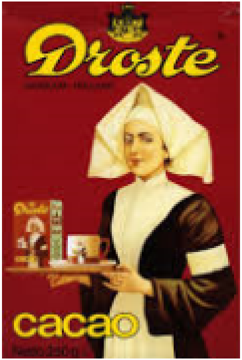
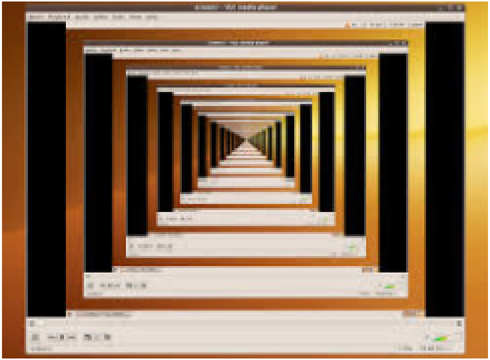
Outline of talk

What is the Droste effect?

A Droste picture is a picture that contains an exact copy of itself infinitely many times. The name stems back to a Dutch cocoa manufacturer.





Mathematically a Droste picture with scaling factor $q \in \mathbb{C}$ can be viewed as $f \in \mathcal{F}_{\mathbb{C}^\times}$ such that $f(qz) = f(z)$ for all $z \in \mathbb{C}^\times$. We may assume that $|q| > 1$.

For real values of C Droste pictures are simply pictures that contain themselves with scaling factor C. However complex values of C produce rotation as well as scaling (i.e. spiraling similarity).

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Fact

If $f \in \mathcal{F}_{\mathbb{C}^\times}$ is a Droste picture then f induces a well defined function $\mathbb{C}^\times/q^{\mathbb{Z}} \to \mathcal{C}$.

The exponential function again induces a surjective homomorphism:

$$\exp : \mathbb{C} \to \mathbb{C}^\times/q^{\mathbb{Z}},$$

but now the kernel is $2\pi i\mathbb{Z} + log(q)\mathbb{Z}$. So similar to earlier we have an isomorphism $\mathbb{C}/(2\pi i\mathbb{Z} + log(q)\mathbb{Z}) \cong \mathbb{C}^\times/q^{\mathbb{Z}}$.

Thus a Droste picture on $\mathbb{C}^\times$ can be viewed as a doubly periodic function on $\mathbb{C}$ with periods $2\pi i$ and $log(q)$ (which are $\mathbb{R}$-linearly independent since $|q| \neq 1$).

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The space $\mathbb{C}/(2\pi i\mathbb{Z} + \log(q)\mathbb{Z})$ is naturally in bijection with the parallelogram $R = \{z = 2\pi i a + \log(q)b \in \mathbb{C} \mid a, b \in [0, 1)\}$.

If $q \in \mathbb{R}$ then R is the rectangle passing through the points $0, 2\pi i, \log(q)$ and $2\pi i + \log(q)$.

We can still do Escherizations as before to transform the Droste picture in varying ways. In particular we may Escherize with respect to the corner $w = 2\pi i + \log(q)$ of the parallelogram.

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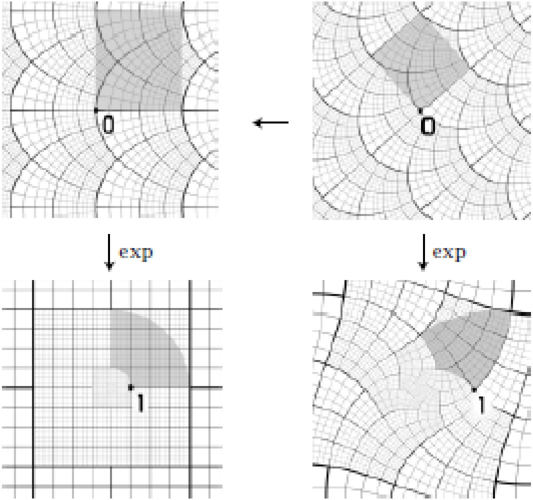
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So how did Lenstra et al complete the Print Gallery?

Fact

The completed version of the Print Gallery is a Droste picture with a complex scaling factor.

But how do we find the scaling factor? The idea is to consider the picture as the Escherization of a Droste picture with a real scaling factor q'.

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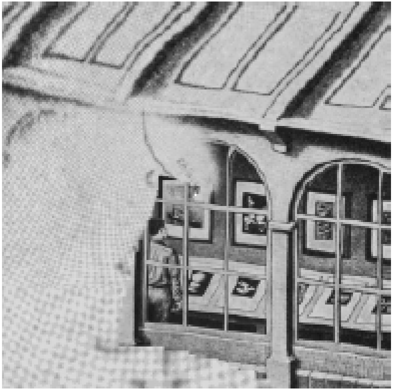
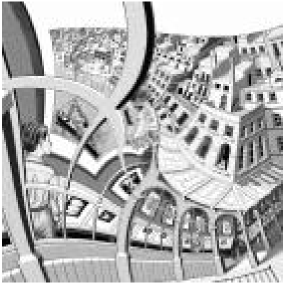
Stage 1: Undo the Escherization process, forcing the doubly periodic image to have a real period. Then apply the exponential map to produce a Droste picture with real scaling factor.



Stage 2: Zooming in on this new picture reveals the real scaling factor to be $q' = 256$. The spiral can be filled in by using Escher's original design.

Stage 3: Re-Escherizing gives the completed Escher picture.



Thanks for listening