Supplementary Material for "**Continuous and discrete phasor analysis of binned or time-gated periodic decays**"

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Appendix A: Continuous phasor of PSEDs with Dirac IRF with offset

The continuous version of the phasor definition (Eq. (63)**Error! Reference source not found.**) assumes a periodic signal $S_T\left(t\right)$. The expressions derived here assume a decay offset $t_0 \in \left]0,T\right[$ (a negative offset $t_0 \in \left]-T,0\right]$ is equivalent to the positive offset $t_0{}' = t_0 + T \in \left]0,T\right[$).

A.1. Continuous phasor of ungated PSEDs with Dirac IRF with offset

Using Eq. (120) for $S_T\left(t\right)$, we obtain, after splitting the integral into two parts ($t \in \left[0,t_0\right[$ and $t \in \left[t_0,T\right[$) to account for the different form taken by Eq. (119):

$$\begin{cases} \left\| \Lambda_{\tau,T|t_0}\left(t\right) \right\|_T = 1 \\ \left\| \Lambda_{\tau,T|t_0}\left(t\right) e^{i2\pi ft} \right\|_T = \int_0^T dt \Lambda_{\tau,T|t_0}\left(t\right) e^{i2\pi ft} = \dfrac{e^{i2\pi ft_0}}{1-i2\pi f\tau} \end{cases} \tag{A1}$$

from which Eq. (126) follows:

$$z\left[\Lambda_{\tau,T|t_0}\right] = \frac{1}{1-i2\pi f\tau} e^{2\pi i f t_0} = \zeta_f\left(\tau\right) e^{2\pi i f t_0} \tag{A2}$$

A.2. Continuous phasor of square-gated PSEDs with Dirac IRF with offset

Using Eq. (121) for $S_T\left(t\right)$, we need to distinguish between the different possible forms taken by Eq. (121), depending on the value of $t' = t - t_0 - \lfloor \left(t-t_0\right)/T \rfloor T$:

$$\begin{aligned} &(i) \quad t' \in \left[0, T-\omega\right[\\ &(ii) \quad t' \in \left[T-\omega, T\right[\end{aligned} \tag{A3}$$

Within these two cases, the value of *t* depends on the location of *t* with respect to t_0:

$$\begin{aligned} &(a) \quad t \in \left[0,t_0\right[: \ \lfloor \left(t-t_0\right)/T \rfloor = 0 \\ &(b) \quad t \in \left[t_0,T\right[: \ \lfloor \left(t-t_0\right)/T \rfloor = 1 \end{aligned} \tag{A4}$$

It is convenient to distinguish between two cases:

A.2.1. Case $t_0 < \omega$:

$$\begin{aligned} \left\| \Lambda_{\tau,T,W|t_0}\left(t\right) \right\|_T &= \int_0^{t_0} dt \left(\frac{1-uy^{-1}}{1-y} e^{-\left(t-t_0+T\right)/\tau} + k + 1 \right) + \\ &+ \int_{t_0}^{t_0+T-\omega} dt \left(\frac{1-u}{1-y} e^{-\left(t-t_0\right)/\tau} + k \right) + \int_{t_0+T-\omega}^{T} dt \left(\frac{1-uy^{-1}}{1-y} e^{-\left(t-t_0\right)/\tau} + k + 1 \right) \\ &= W \end{aligned} \tag{A5}$$

where we have introduced the notations:

$$
\begin{aligned}
u &= e^{-\omega/\tau} \\
y &= e^{-T/\tau}
\end{aligned}
\tag{A6}
$$

Likewise, the numerator of the phasor expression is given by:

$$
\begin{aligned}
\left\| \Lambda_{\tau,T,W|t_0}(t) e^{i2\pi ft} \right\|_T &= \int_0^{t_0} dt \left(\frac{1-uy^{-1}}{1-y} e^{-(t-t_0+T)/\tau} + k + 1 \right) e^{i2\pi ft} + \\
&+ \int_{t_0}^{t_0+T-\omega} dt \left(\frac{1-u}{1-y} e^{-(t-t_0)/\tau} + k \right) e^{i2\pi ft} + \int_{t_0+T-\omega}^{T} dt \left(\frac{1-uy^{-1}}{1-y} e^{-(t-t_0)/\tau} + k + 1 \right) e^{i2\pi ft}
\end{aligned}
\tag{A7}
$$

A.2.2. Case $t_0 \geq \omega$:

$$
\begin{aligned}
\left\| \Lambda_{\tau,T,W|t_0}(t) \right\|_T &= \int_0^{t_0-\omega} dt \left(\frac{1-uy}{1-y} e^{-(t-t_0+T)/\tau} + k \right) + \\
&+ \int_{t_0-\omega}^{t_0} dt \left(\frac{1-uy^{-1}}{1-y} e^{-(t-t_0+T)/\tau} + k + 1 \right) + \int_{t_0}^{T} dt \left(\frac{1-u}{1-y} e^{-(t-t_0)/\tau} + k \right) \\
&= W
\end{aligned}
\tag{A8}
$$

and:

$$
\begin{aligned}
\left\| \Lambda_{\tau,T,W|t_0}(t) e^{i2\pi ft} \right\|_T &= \left\{ \int_0^{t_0-\omega} dt \left(\frac{1-u}{1-y} e^{-(t-t_0+T)/\tau} + k \right) e^{i2\pi ft} + \right. \\
&\left. + \int_{t_0-\omega}^{t_0} dt \left(\frac{1-uy^{-1}}{1-y} e^{-(t-t_0+T)/\tau} + k + 1 \right) e^{i2\pi ft} + \int_{t_0}^{T} dt \left(\frac{1-u}{1-y} e^{-(t-t_0)/\tau} + k \right) e^{i2\pi ft} \right\}
\end{aligned}
\tag{A9}
$$

After some basic calculations, we obtain, *in both cases*:

$$
\begin{aligned}
z\left[\Lambda_{\tau,T,W|t_0} \right] &= \frac{\sin \pi f \omega}{\pi fW} \frac{e^{-i\pi f\omega}}{1-i2\pi f\tau} e^{i2\pi ft_0} \\
&= z_{[W]}\left[\Lambda_{\tau,T} \right] e^{i2\pi ft_0} = z_{[W]}\left[\Lambda_{\tau,T|t_0} \right]
\end{aligned}
\tag{A10}
$$

which is the same as Eq. (75) for the continuous phasor of a square-gated PSED without offset, rotated by an angle $2\pi ft_0$.

Appendix B: Discrete phasor of *T*-periodic decays

In this Appendix, ***w***e will not make the distinction between original decay (hypothetically resulting from a Dirac excitation), IRF and recorded decay, and only consider the recorded decay. In this sense, it can either be viewed as a discussion of decays recorded with a Dirac IRF, or a discussion of recorded decays with arbitrary IRF, but with particular functional form (for instance, T-periodic single-exponential decays). The point of the Appendix is simply to derive useful mathematical relations used in different parts of this work.

We will first assume that the recorded decay $\left\{S_T\left(t_p\right)\right\}, 1 \leq p \leq N$ covers the whole period T ($N\theta = T$) and that phasor harmonic frequencies are multiple of the fundamental frequency $f_1 = 2\pi T^{-1}$. We will briefly discuss cases where it may makes sense to use different frequencies when the $\left\{S_T\left(t_p\right)\right\}_{1 \leq p \leq N}$ do not cover the whole period (Section B.2.3.b).

B.1. Discrete phasor of ungated PSEDs

We will first treat the case without decay offset (t_0 = 0) in some detail and only sketch the derivation in the case with decay offset in the interest of length.

B.1.1. No offset

For an ungated PSED with lifetime τ, $\Lambda_{\tau,T}\left(t\right)$ (Eq. (17)), we obtain:

$$\begin{cases} \left\|\Lambda_{\tau,T}\left(t_p\right)\right\|_N = \dfrac{\theta}{\tau\left(1-e^{-T/\tau}\right)} \displaystyle\sum_{p=1}^{N} e^{-(p-1)\theta/\tau} = \dfrac{\theta}{\tau\left(1-e^{-\theta/\tau}\right)} \\ \left\|\Lambda_{\tau,T}\left(t_p\right)e^{i2\pi f t_p}\right\|_N = \dfrac{\theta}{\tau\left(1-e^{-T/\tau}\right)} \displaystyle\sum_{p=1}^{N} e^{(p-1)\theta(-1/\tau + i2\pi f)} \\ \qquad = \dfrac{\theta}{\tau\left(1-e^{(-1/\tau+i2\pi f)\theta}\right)} \end{cases} \tag{B1}$$

Introducing:

$$\begin{cases} x\left(\tau\right) = e^{-\theta/\tau} \\ \alpha = 2\pi f \theta = 2\pi fT/N \end{cases} \tag{B2}$$

we get:

$$\zeta_{f,N}\left(\tau\right) \triangleq z_N\left[\Lambda_{\tau,T}\right] = \frac{1-x}{1-xe^{i\alpha}} = \frac{\left(1-x\right)\left(1-xe^{-i\alpha}\right)}{1-2x\cos\alpha + x^2} \tag{B3}$$

The functions $x\left(t\right)$ and discrete phasor of PSED $\zeta_{f,N}\left(\tau\right)$ will be used throughout this work.

Eq. (B3) can also be rewritten:

$$\begin{cases} z_N\left[\Lambda_{\tau,T}\right] = g\left(\tau\right) + is\left(\tau\right) = M_N\left(\tau\right)e^{i\Phi_N(\tau)} \\ M_N\left(\tau\right) = \dfrac{1-x}{\sqrt{1-2x\cos\alpha + x^2}} \\ \tan\Phi_N\left(\tau\right) = \dfrac{x\sin\alpha}{1-x\cos\alpha} \end{cases} \tag{B4}$$

For sin α = 0, (*i.e.* $\dfrac{n}{N} = \dfrac{q}{2},\ q \in \mathbb{N}$ where $f = \dfrac{n}{T}$), Eq. (B4) implies that $s(\tau) = 0$ for all τ. Specifically, if cos α = 1 (*i.e.* if q is even), the locus of discrete phasors of PSEDs is the single point z =

1. If cos α = -1 (*i.e.* if q is odd), the locus of discrete phasors of PSEDs is a line connecting 0 (τ = ∞) and 1 (τ = 0) with:

$$g(\tau) = \tanh\frac{\theta}{2\tau}. \tag{B5}$$

In all other cases, we can look for a quadratic relation linking g and s:

$$A_1 g^2 + 2A_2 gs + A_3 s^2 + 2A_4 g + 2A_5 s + A_6 = 0 \tag{B6}$$

where the coefficients $\{A_i\}$, $1 \le i \le 6$ are constants. We obtain the following equation:

$$\begin{aligned}
&A_1 + 2A_4 + A_6 + \\
&\left(-2(1+\cos\alpha)A_1 + 2\sin\alpha A_2 - 2(1+3\cos\alpha)A_4 + 2\sin\alpha A_5 - 4\cos\alpha A_6\right)x + \\
&\begin{pmatrix} \left(1+4\cos\alpha+\cos^2\alpha\right)A_1 - 2\sin\alpha(2+\cos\alpha)A_2 + \sin^2\alpha A_3 \\ +2\left(1+3\cos\alpha+2\cos^2\alpha\right)A_4 - 2\sin\alpha(1+2\cos\alpha)A_5 + 2\left(1+2\cos^2\alpha\right)A_6 \end{pmatrix} x^2 + \\
&\begin{pmatrix} -2\cos\alpha(1+\cos\alpha)A_1 + 2\sin\alpha(1+2\cos\alpha)A_2 - 2\sin^2\alpha A_3 \\ -2\left(1+\cos\alpha+2\cos^2\alpha\right)A_4 + 2\sin\alpha(1+2\cos\alpha)A_5 - 4\cos\alpha A_6 \end{pmatrix} x^3 + \\
&\left(\cos^2\alpha A_1 - 2\sin\alpha\cos\alpha A_2 + \sin^2\alpha A_3 + 2\cos\alpha A_4 - 2\sin\alpha A_5 + A_6\right)x^4 = 0
\end{aligned} \tag{B7}$$

where we have used the notation $x = e^{-\theta/\tau}$ introduced above. For this equation to be verified for all values of x (i.e. for all values of τ), all coefficients of powers of x in Eq. (B7) need to be equal to zero. This yields a system of 5 equations for the 6 unknowns $\{A_i\}_{1\le i\le 6}$ (it is clear from Eq. (B6) that only 5 of these parameters are independent, as a mere rescaling of all parameters by a constant preserve the identity). A solution of this equation can be found using a symbolic programming language such as Mathematica:

$$\begin{cases} A_1 = -2A_4 = \dfrac{2\sin\alpha}{1-\cos\alpha} \\ A_2 = A_6 = 0 \\ A_3 = \dfrac{2(1+\cos\alpha)}{\sin\alpha} \\ A_5 = 1 \end{cases} \tag{B8}$$

Eq. (B6) can then be simplified into:

$$\begin{cases} (g-g_c)^2 + (s-s_c)^2 = r^2 \\ g_c = \dfrac{1}{2} \\ s_c = -\dfrac{1}{2}\tan(\alpha/2) \\ r = \dfrac{1}{2\left|\cos(\alpha/2)\right|} \end{cases} \tag{B9}$$

showing that, unless $\cos\frac{\alpha}{2} = 0$ (*i.e.* $\frac{n}{N} = q + \frac{1}{2},\ q \in \mathbb{N}$), the locus of the discrete phasor of PSEDs is an arc of circle centered on (g_c, s_c), with radius r (Fig. 4).

The special case $\cos(\alpha/2) = 0$ is encountered for instance for $n = 1$, $N = 2$, in which case the locus of phasor is a line connecting 0 and 1, as discussed above.

In all other cases, as the number of gates N increases, $\alpha = 2\pi f \theta$ decreases and a tends to ½, while s_c tends to 0, showing that the locus of the phasors of PSEDs approaches the standard $\mathcal{L}_\infty$. In all cases, $z_N(0) = 1$ and $z_N(\infty) = 0$ as for the standard $\mathcal{L}_\infty$.

The two possible values of s_* such that $g_* = \frac{1}{2}$ are given by:

$$\begin{cases} s_*^{(1)} = \dfrac{1}{2}\dfrac{\cos\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}} \\ s_*^{(2)} = -\dfrac{1}{2}\dfrac{\cos\frac{\alpha}{2}}{1-\sin\frac{\alpha}{2}} \end{cases} \tag{B10}$$

which tend to $\pm\frac{1}{2}$ when $\alpha \to 0$ (*i.e.* $N \to \infty$). The corresponding solutions for x are given by:

$$\begin{cases} x_*^{(1)} = \left(1 + 2\sin\frac{\alpha}{2}\right)^{-1} \\ x_*^{(2)} = \left(1 - 2\sin\frac{\alpha}{2}\right)^{-1} \end{cases} \tag{B11}$$

only one of which corresponds to a lifetime τ_* given by $x_* = e^{-\theta/\tau_*}$ (the root such that $x_* \le 1$):

$$\tau_* = \frac{\theta}{\ln\left(1 + 2\left|\sin\frac{\alpha}{2}\right|\right)} \tag{B12}$$

τ_* is an increasing function of θ (or equivalently, a decreasing function of N), its minimum being obtained in the continuum limit ($\theta \to 0,\ N \to \infty$) where it takes the value $\tau_* = 1/(2\pi f)$ (Fig. S1). Its maximum is obtained for the largest meaningful interval $\theta = T/N$, *i.e.* the smallest meaningful number of samples, $N = 2$. In this latter case, however, we have seen that the locus of phasors of single exponential decays is either a single point or a straight segment (depending on the harmonic n).

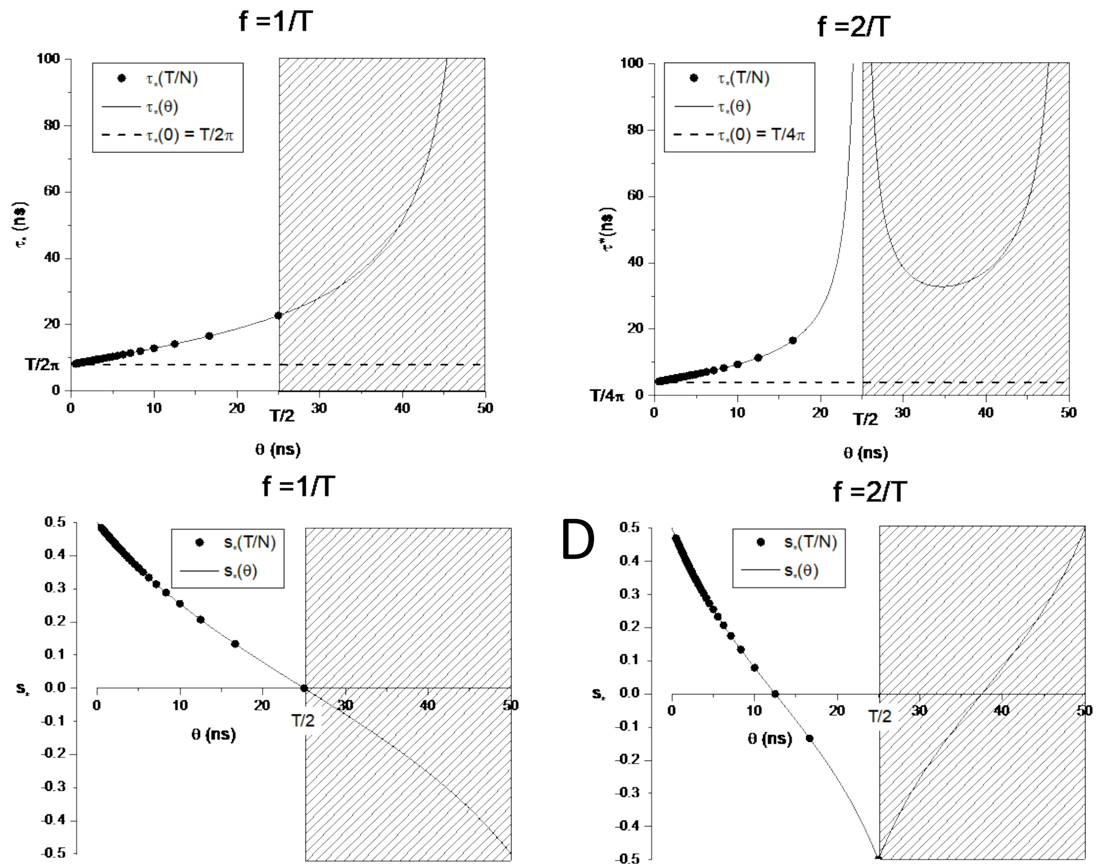

Fig. S1: Dependence of the lifetime τ_* and coordinate S_* of the discrete phasor characterized by g = ½ on the gate step θ. A laser period T = 50 ns is used for illustration. A, C: For a phasor harmonic $f = 1/T$, S_* increases monotonically up to its maximum value obtained at $T/2$ (N = 2 gates covering the laser period). The hatched area represents the region $\theta > T/2$ which does not have practical meaning. Plain curve in A, C corresponds to Eq. (B12), and symbols represent the data for actual gate step values T/N. The dashed line in A corresponds to the standard UC value. B, D: For a phasor harmonic $f = 2/T$, S_* still increases monotonically but diverges at its maximum value $T/2$ where the locus of phasors is reduced to a single point z = 1. The hatched areas represent the region $\theta > T/2$ which does not have practical meaning.

Note that for some choices of (n, N), $\mathcal{L}_N$ can be on the opposite side of the axis compared to $\mathcal{L}_\infty$ (*e.g.* $n = 2, N = 3 \Rightarrow x_*^{(1)}$ is the valid root, and the corresponding $s_*^{(1)}$ is negative, see Fig. 4B).

B.1.2. With offset

We use Eq. (120) for the definition of a PSED with offset t_0, $\Lambda_{\tau,T|t_0}(t)$ in Eq. (92). The denominator is:

$$\left\|\Lambda_{\tau,T|t_0}\left(t_p\right)\right\|_N = \frac{\theta}{\tau\left(1-e^{-T/\tau}\right)}\sum_{p=1}^{N} e^{-\left(t_p-t_0-\left\lfloor\left(t_p-t_0\right)/T\right\rfloor T\right)/\tau} \tag{B13}$$

where $t_p = (p-1)\theta$. To obtain a simpler expression for the sum, we need to split it into two parts, each with a single value for the integer part $n = \lfloor (t-t_0)/T \rfloor$. We can assume that $t_0 \in [0,T[$ for simplicity[1] and rewrite:

$$\begin{aligned} &t_0 = \theta_0 + r\theta \\ &\theta_0 \in [0,\theta[,\ r = \lfloor t_0/\theta \rfloor \in \mathbb{Z} \end{aligned} \tag{B14}$$

With this definition,

$$t - t_0 = (p-1-r)\theta - \theta_0 \tag{B15}$$

and the integer part takes the values:

$$\begin{aligned} &\theta_0 = 0 \Rightarrow \begin{cases} p \le r \Rightarrow n = -1 \\ p > r \Rightarrow n = 0 \end{cases} \\ &\theta_0 \ne 0 \Rightarrow \begin{cases} p \le r+1 \Rightarrow n = -1 \\ p > r+1 \Rightarrow n = 0 \end{cases} \end{aligned} \tag{B16}$$

It appears advantageous from Eq. (B16) to introduce the following integer:

$$q = \left\lceil \frac{t_0}{\theta} \right\rceil = \begin{cases} r & \text{if} \quad \theta_0 = 0 \\ r+1 & \text{if} \quad \theta_0 > 0 \end{cases} \tag{B17}$$

with which Eq. (B16) can be rewritten:

$$\begin{cases} p \le q \Rightarrow n = -1 \\ p > q \Rightarrow n = 0 \end{cases} \tag{B18}$$

We obtain the following expressions for the sum in Eq. (B13):

$$\sum_{p=1}^{N} e^{-\left(t_p - t_0 - \lfloor (t_p - t_0)/T \rfloor T\right)/\tau} = \frac{e^{t_0/\tau}}{1-x}\left(y\left(1-x^q\right) + x^q - x^N\right) \tag{B19}$$

where we have used the previous notations:

$$\begin{cases} x(t) = e^{-\theta/\tau} \\ y(t) = e^{-T/\tau} \end{cases} \tag{B20}$$

In the special case where the N gates cover the whole laser period exactly ($T = N\theta$), these expressions simplify into:

$$T = N\theta \Rightarrow \sum_{p=1}^{N} e^{-(t_p - t_0)[T]/\tau} = \frac{1-y}{1-x} x^q e^{t_0/\tau} \tag{B21}$$

Returning to the general case, the numerator in Eq. (92):

---

[1] If $t_0 \in [0,T[$, $t_0 - \lfloor t_0/T \rfloor T \in [0,T[$ can be used instead by periodicity.

$$\left\| \Lambda_{\tau,T|t_0}\left(t_p\right) e^{i2\pi f t_p} \right\|_N = \frac{\theta}{\tau\left(1-e^{-T/\tau}\right)} \sum_{p=1}^{N} e^{-\left(t_p - t_0\right)[T]/\tau} e^{i2\pi f t_p} . \tag{B22}$$

can be computed in the same manner. The sum in this expression reads:

$$\sum_{p=1}^{N} e^{-\left(t_p - t_0\right)[T]/\tau} e^{i2\pi f t_p} = \frac{e^{t_0/\tau}}{1-xe^{i\alpha}}\left(y\left(1-x^q e^{iq\alpha}\right) + x^q e^{iq\alpha} - x^N e^{iN\alpha}\right) \tag{B23}$$

where we have used the previous notation:

$$\alpha = 2\pi f \theta . \tag{B24}$$

If we assume that the N gates cover the whole laser period exactly ($T = N\theta$), these expressions simplify into:

$$T = N\theta \Rightarrow \quad \sum_{p=1}^{N} e^{-\left(t_p - t_0\right)[T]/\tau} e^{i2\pi f t_p} = \frac{1-y}{1-xe^{i\alpha}} x^q e^{iq\alpha} e^{t_0/\tau} \tag{B25}$$

The phasor is given by the ratio of Eq. (B23) and Eq. (B19) in the general case (discussed further below), or, if we assume that the N gates cover the whole laser period exactly, by the ratio of Eq. (B25) and Eq. (B21).

In that latter case ($T = N\theta$), this reads:

$$T = N\theta \Rightarrow \quad z_N\left[\Lambda_{\tau,T|t_0}\right] = \frac{1-x}{1-xe^{i\alpha}} e^{i\left\lceil \frac{t_0}{\theta} \right\rceil \alpha} = z_N\left[\Lambda_{\tau,T}\right] e^{i\left\lceil \frac{t_0}{\theta} \right\rceil \alpha} \tag{B26}$$

which shows that the discrete phasor of an ungated PSED with offset, is obtained by rotation of the phasor in the absence of offset (Eq. (B3)). It is easy to verify that this formula leads to Eq. (131).

For $T \neq N\theta$, we obtain:

$$z_N\left[\Lambda_{\tau,T|t_0}\right] = \frac{1-x}{1-xe^{i\alpha}} \frac{\left(1-y\right)x^q e^{iq\alpha} + y - x^N e^{iN\alpha}}{\left(1-y\right)x^q + y - x^N} \tag{B27}$$

Interestingly, if f is chosen such that $N\alpha = 2\pi n,\ n \in \mathbb{N}$, i.e. $f = n/\Theta$, where $\Theta = N\theta$, then some simplification of this expression is possible:

$$f = \frac{n}{N\theta} \Rightarrow z_N\left[\Lambda_{\tau,T|t_0}\right] = \frac{1-x}{1-xe^{i\alpha}} \frac{\left(1-y\right)x^q e^{iq\alpha} + y - x^N}{\left(1-y\right)x^q + y - x^N} \tag{B28}$$

The term $y - x^N = e^{-T/\tau} - e^{-N\theta/\tau}$ cannot be simplified unless $\Theta = T$, and its presence is due to the T-periodicity of the PSED. However, if we assume that there is no decay offset ($\theta_0 = 0,\ r = 0$), then we recover the expression for $z_N\left[\Lambda_{\tau,T}\right]$:

$$f = \frac{n}{N\theta}, t_0 = 0 \Rightarrow z_N\left[\Lambda_{\tau,T}\right] = \frac{1-x}{1-xe^{i\alpha}} \tag{B29}$$

In other words, in the special cases where there is no decay offset, and the N gates do not cover the whole period, it may be advantageous to use a non-standard phasor harmonic frequency f, defined as in Eq. (B29), in order to obtain a simple analytical form for the phasor.

B.2. Discrete phasor of square-gated PSEDs

B.2.1. General case (no offset)

For a square-gated PSED $\Lambda_{\tau,T,W}(t)$ with lifetime τ and gate width $W = \omega + kT$ (Eq. (47) **Error! Reference source not found.**):

$$\begin{cases} \left\| \Lambda_{\tau,T,W}(t_p) \right\|_N = \theta \sum_{p=1}^{N} \Lambda_{\tau,T,W}(t_p) \\ t_p = (p-1)\theta \\ \theta = \dfrac{T}{N} \end{cases} \tag{B30}$$

Because the expression of $\Lambda_{\tau,T,W}(t)$ depends on the location of t in [0, T], we need to find index r in [1, N] such that $t_r = (r-1)\theta < T - \omega$ and $t_{r+1} \geq T - \omega$. The result is:

$$r = \left\lceil \frac{T-\omega}{\theta} \right\rceil \tag{B31}$$

where $\lceil x \rceil$ designates the smallest integer larger than or equal to x (`ceil` function).

The sum in Eq. (B30) can be expanded as:

$$\left\| \Lambda_{\tau,T,W}(t_p) \right\|_N = \theta \left\{ \sum_{p=1}^{r} \left(\frac{1-e^{-\omega/\tau}}{1-e^{-T/\tau}} e^{-t_p/\tau} + k \right) + \sum_{p=r+1}^{N} \left(\frac{1-e^{-(\omega-T)/\tau}}{1-e^{-T/\tau}} e^{-t_p/\tau} + k + 1 \right) \right\} \tag{B32}$$

The final result, assuming $T = N\theta$, is:

$$\left\| \Lambda_{\tau,T,W}(t) \right\|_N = \theta \left\{ (k+1)N - r + \frac{1-ux^r y^{-1}}{1-x} \right\}. \tag{B33}$$

where we have used the notations:

$$\begin{cases} x(\tau) = e^{-\theta/\tau} \\ y(\tau) = e^{-T/\tau} \\ u(\tau) = e^{-\omega/\tau} \end{cases} \tag{B34}$$

The discrete phasor of a square-gated PSED is given by Eq. (92), where the denominator is given by Eq. (B33) and the numerator by:

$$\begin{aligned} \left\| \Lambda_{\tau,T,W}(t_p) e^{i2\pi f t_p} \right\|_N &= \theta \sum_{p=1}^{N} \Lambda_{\tau,T,W}(t_p) e^{2\pi i f t_p} \\ &= \theta \left\{ \sum_{p=1}^{r} \left(\frac{1-e^{-\omega/\tau}}{1-e^{-T/\tau}} e^{-t_p/\tau} + k \right) e^{2\pi i f t_p} + \sum_{p=r+1}^{N} \left(\frac{1-e^{-(\omega-T)/\tau}}{1-e^{-T/\tau}} e^{-t_p/\tau} + k + 1 \right) e^{2\pi i f t_p} \right\} \end{aligned} \tag{B35}$$

Assuming $T = N\theta$ and noting $\alpha = 2\pi f \theta$, we obtain:

$$\left\|\Lambda_{\tau,T,W}\left(t_p\right)e^{i2\pi f t_p}\right\|_N = \theta\left\{-\frac{1-e^{ir\alpha}}{1-e^{i\alpha}}+\frac{1-u}{1-y}\frac{1-x^r e^{ir\alpha}}{1-xe^{i\alpha}}+\frac{1-uy^{-1}}{1-y}\frac{x^r e^{ir\alpha}-y}{1-xe^{i\alpha}}\right\} \\ = \theta\left\{-\frac{1-e^{ir\alpha}}{1-e^{i\alpha}}+\frac{1-ux^r y^{-1}e^{ir\alpha}}{1-xe^{i\alpha}}\right\} \tag{B36}$$

and the expression for the discrete phasor of a square-gated PSED reads, assuming $T = N\theta$:

$$\begin{cases} z_{N[W]}\left[\Lambda_{\tau,T}\right] = \dfrac{-\dfrac{1-e^{ir\alpha}}{1-e^{i\alpha}}+\dfrac{1-\beta e^{ir\alpha}}{1-xe^{i\alpha}}}{(k+1)N-r+\dfrac{1-\beta}{1-x}} \\ \beta(\tau) = u(\tau)x(\tau)^r y(\tau)^{-1} \end{cases} \tag{B37}$$

B.2.2. Special cases (no offset)

<u>a. Gate width *W* proportional to gate step *θ*</u>

For cases where $T-\omega$ is proportional to θ, Eq. (B31) simplifies into:

$$r = \frac{T-\omega}{\theta}, \tag{B38}$$

and $uy^{-1}x^r = 1$, resulting in a simpler version of Eq. (B37):

$$z_{N[W]}\left[\Lambda_{\tau,T}\right] = \frac{1-e^{ir\alpha}}{(k+1)N-r}\left(\frac{1}{1-xe^{i\alpha}}-\frac{1}{1-e^{i\alpha}}\right) = -\frac{e^{i(r+1)\alpha/2}}{(k+1)N-r}\frac{\sin\dfrac{r\alpha}{2}}{\sin\dfrac{\alpha}{2}}z_N\left[\Lambda_{\tau,T}\right] \tag{B39}$$

which is a rotated and dilated version of the discrete phasor of ungated PSEDs ($z_N\left[\Lambda_{\tau,T}\right]$, Eq. (B3)), and thus is an arc of circle. Rewriting $W = q\theta$, Eq. (B39) takes the form:

$$z_{N[W]}\left[\Lambda_{\tau,T}\right] = \frac{\sin q\dfrac{\alpha}{2}}{q\sin\dfrac{\alpha}{2}}e^{-i(q-1)\frac{\alpha}{2}}\; z_N\left[\Lambda_{\tau,T}\right] \tag{B40}$$

<u>b. Value for *τ* = 0 and *τ* = ∞</u>

In the limit $\tau = 0$, one obtains:

If $r \neq \dfrac{T-\omega}{\theta}$:

$$z_{N[W]}\left[\Lambda_{0,T}\right] = \frac{-\dfrac{1-e^{ir\alpha}}{1-e^{i\alpha}}+1}{(k+1)N-r+1} = -\frac{1}{(k+1)N-r+1}\frac{\sin(r-1)\dfrac{\alpha}{2}}{\sin\dfrac{\alpha}{2}}e^{ir\frac{\alpha}{2}} \tag{B41}$$

and if $r = \dfrac{T-\omega}{\theta}$:

$$z_{N[W]}\left[\Lambda_{0,T}\right] = \frac{-\dfrac{1-e^{ir\alpha}}{1-e^{i\alpha}} + 1 - e^{ir\alpha}}{(k+1)N - r} = -\frac{1}{(k+1)N - r}\frac{\sin r\dfrac{\alpha}{2}}{\sin\dfrac{\alpha}{2}} e^{i(r+1)\frac{\alpha}{2}} \tag{B42}$$

In other words, the value of the phasor for $\tau = 0$ only depends on $r = \left\lceil \dfrac{T-\omega}{\theta} \right\rceil$.

In all cases, in the limit $\tau = \infty$ ($x = y = u = 1$), one obtains:

$$z_{N[W]}\left[\Lambda_{\infty,T}\right] = 0. \tag{B43}$$

Note that in the special cases were α is a multiple of 2π, *i.e.* $f = q\ N/T$, $q \in \mathbb{N}$, the above expressions can be further simplified using:

$$\frac{\alpha}{2} = q\pi,\ q \in \mathbb{N} \Rightarrow (\forall s \in \mathbb{N}), \frac{\sin s\dfrac{\alpha}{2}}{\sin\dfrac{\alpha}{2}} = s \text{ and } e^{is\frac{\alpha}{2}} = (-1)^{sq} \tag{B44}$$

This situation is however unlikely to be encountered, since it means that the harmonic $m = qN$, where the number of gates N is in general not very small, would be quite large.

<u>c. Gate width *W* smaller than the gate step *θ*</u>

In this case, Eq. (104) reads:

$$r = \left\lceil \frac{T-\omega}{\theta} \right\rceil = \left\lceil N - \frac{W}{\theta} \right\rceil = N \tag{B45}$$

from which it results that Eq. (103) reads:

$$z_{N[W]}\left[\Lambda_{\tau,T}\right] = \frac{-\dfrac{1-e^{ir\alpha}}{1-e^{i\alpha}} + \dfrac{1-e^{ir\alpha}}{1-xe^{i\alpha}}}{\dfrac{1}{1-x}} = z_N\left[\Lambda_{\tau,T}\right] \tag{B46}$$

In other words, in these cases, the discrete phasor of square-gated PSEDs does not depend on the gate width and is identical to the discrete phasor of ungated PSEDs.

<u>d. Gate with reduced width *ω* smaller than the gate step *θ*</u>

This case encompasses the previous one, but is a bit more general (if not common). Returning to Eq. (B31) defining r, the point at which the expression for $\Lambda_{\tau,T,W}(t)$ changes form, it has no solution in [1, N[if the gate step $\theta > \omega$, or in other words, if the gates are not overlapping or contiguous but instead are separated by a finite gap. In this case, only the first sum in Eqs. (B32) and (B35) is involved and:

$$\begin{aligned}\left\|\Lambda_{\tau,T,W}\left(t_p\right)\right\|_N &= \theta\sum_{p=1}^{N}\left(\frac{1-e^{-\omega/\tau}}{1-e^{-T/\tau}}e^{-t_p/\tau}+k\right)\\ &= \theta\left\{kN+\frac{1-u}{1-y}\frac{1-x^N}{1-x}\right\}\\ &= \theta\left(kN+\frac{1-u}{1-x}\right)\end{aligned}\tag{B47}$$

where we have used $x^N = y$, since we are assuming $T = N\theta$, to obtain the last identity. Similarly, the numerator of *z* is given by:

$$\begin{aligned}\left\|\Lambda_{\tau,T,W}\left(t_p\right)e^{i2\pi ft_p}\right\|_N &= \theta\sum_{p=1}^{N}\Lambda_{\tau,nT,W}\left(t_p\right)e^{2\pi ift_p} = \theta\sum_{p=1}^{N}\left(\frac{1-u}{1-y}e^{-t_p/\tau}+k\right)e^{i(p-1)\alpha}\\ &= \theta\left\{k\frac{1-e^{iN\alpha}}{1-e^{i\alpha}}+\frac{1-u}{1-y}\frac{1-x^N e^{iN\alpha}}{1-xe^{i\alpha}}\right\} = \theta\frac{1-u}{1-xe^{i\alpha}}\end{aligned}\tag{B48}$$

where we have used $x^N = y$ and $e^{iN\alpha} = 1$ to obtain the last identity. Finally, we obtain the discrete phasor of a square-gated PSED when the gate step θ is larger than the reduced gate width ω as:

$$\theta > \omega \Rightarrow\ z_{N[W]}\left[\Lambda_{\tau,T}\right] = \frac{1}{kN+\dfrac{1-u}{1-x}}\frac{1-u}{1-xe^{i\alpha}} = \frac{\dfrac{1-u}{1-x}}{kN+\dfrac{1-u}{1-x}}z_N\left[\Lambda_{\tau,T}\right]\tag{B49}$$

where $z_N\left[\Lambda_{\tau,T}\right]$ is the discrete phasor of ungated PSEDs given by Eq. (B3). Note that because the prefactor of $z_N\left[\Lambda_{\tau,T}\right]$ depends on τ, Eq. (B49) does *not* represent an arc of circle unless $k = 0$, in which case the phasor of square-gated decays is identical to that of the ungated decays as seen above. $k = 0$ is in fact the most likely case, as there is hardly any justification for choosing a gate width $W > T$ (unless the hardware prevents obtaining gates with $W < T$).

It is easy to verify that in the special case $\theta = \omega$, for which $x = u$, the previous formula also applies and:

$$\theta = \omega\ \Rightarrow\ z_{N[W]}\left[\Lambda_{\tau,T}\right] = \frac{1}{kN+1}\frac{1-x}{1-xe^{i\alpha}}\tag{B50}$$

This formula is identical to Eq. (B3) obtained for the discrete phasor of an ungated PSED, except for the prefactor $(1+kN)^{-1}$, showing that Eq. (B50) describes an arc of circle centered on $\left(g_c, s_c\right)$ given by:

$$\begin{cases} g_c = \dfrac{1}{2}\dfrac{1}{kN+1} \\ s_c = -\dfrac{1}{2}\dfrac{1}{kN+1}\tan\left(\alpha/2\right)\end{cases}\tag{B51}$$

and with radius *r* equal to:

$$r=\frac{1}{kN+1}\frac{1}{2\left|\cos\left(\alpha/2\right)\right|}, \tag{B52}$$

e. $N = 2$ gates

A last special case of interest is when the number of gates $N = 2$ and the gates cover the whole laser period, $\theta = T/2$. Here again, the final formula depends on the gate width W (or more exactly its reduced width, ω):

$$z_{N[W]}\left[\Lambda_{\tau,T}\right]=\begin{cases}\dfrac{\dfrac{1-u}{1-y}\left(1+(-1)^n x\right)+k\left(1+(-1)^n\right)}{2k+\dfrac{1-u}{1-y}(1+x)}, \ \omega<\dfrac{T}{2}\\[2ex] \dfrac{\dfrac{1-u}{1-y}+k+(-1)^n\left(\dfrac{1-uy^{-1}}{1-y}x+k+1\right)}{2k+1+\dfrac{1-u+\left(1-uy^{-1}\right)x}{1-y}}, \omega\geq\dfrac{T}{2}\end{cases} \tag{B53}$$

where we have used the definition $f = nT^{-1}$ for the harmonic frequency.

The main cases of interest are obtained for $k = 0$ (gate of width $W = \omega < T$). Depending on whether the harmonic is an odd (e.g. $n = 1$) or even (e.g. $n = 2$) multiple of the fundamental frequency T^{-1}, one obtains:

$$z_{N[W]}\left[\Lambda_{\tau,T}\right]=\begin{cases}\tanh\dfrac{\theta}{2\tau}, & W<\dfrac{T}{2},\ k=0,\ n \text{ odd}\\[1ex] 1, & W<\dfrac{T}{2},\ k=0,\ n \text{ even}\\[1ex] \dfrac{u\left(xy^{-1}-1\right)+y-x}{2-u\left(xy^{-1}+1\right)-y+x}, & W\geq\dfrac{T}{2},\ k=0,\ n \text{ odd}\\[1ex] 1, & W\geq\dfrac{T}{2},\ k=0,\ n \text{ even}\end{cases} \tag{B54}$$

In all cases, the phasor is a real number, meaning the locus of the phasor of square-gated PSEDs is a segment of the $s = 0$ axis (or a single point $z = 1$, in case n is even). However, in case $W<\dfrac{T}{2}$, the whole [0,1] segment is covered ($z_{N[W]}\left[\Lambda_{0,T}\right]=1$ and $z_{N[W]}\left[\Lambda_{\infty,T}\right]=0$), whereas in case $W\geq\dfrac{T}{2}$, both $z_{N[W]}\left[\Lambda_{0,T}\right]=0$ and $z_{N[W]}\left[\Lambda_{\infty,T}\right]=0$, which shows that the locus of the phasor of square-gated PSEDs is a segment occupying only *a fraction* of [0,1] segment, as illustrated in Fig. 5. The phasor first increases with τ, to reach a value z_{max} for a particular value τ_{max}, before decreasing back to 0 as $\tau\to\infty$.

B.2.3. General case (with offset)

As in the ungated PSED case, the presence of an offset t_0 introduces an additional subtlety in the calculation, with the expression $(t-t_0)[T]$ intervening in definition (121). As indicated in Fig. S2, depending on the gates location t_p in the period ($p = 1,\dots, N$), this expression takes either one of the two forms:

$$q = \left\lceil \frac{t_0}{\theta} \right\rceil \rightarrow \begin{cases} p \le q \Rightarrow (t_p - t_0)[T] = t_p - t_0 - T \\ p > q \Rightarrow (t_p - t_0)[T] = t_p - t_0 \end{cases} \tag{B55}$$

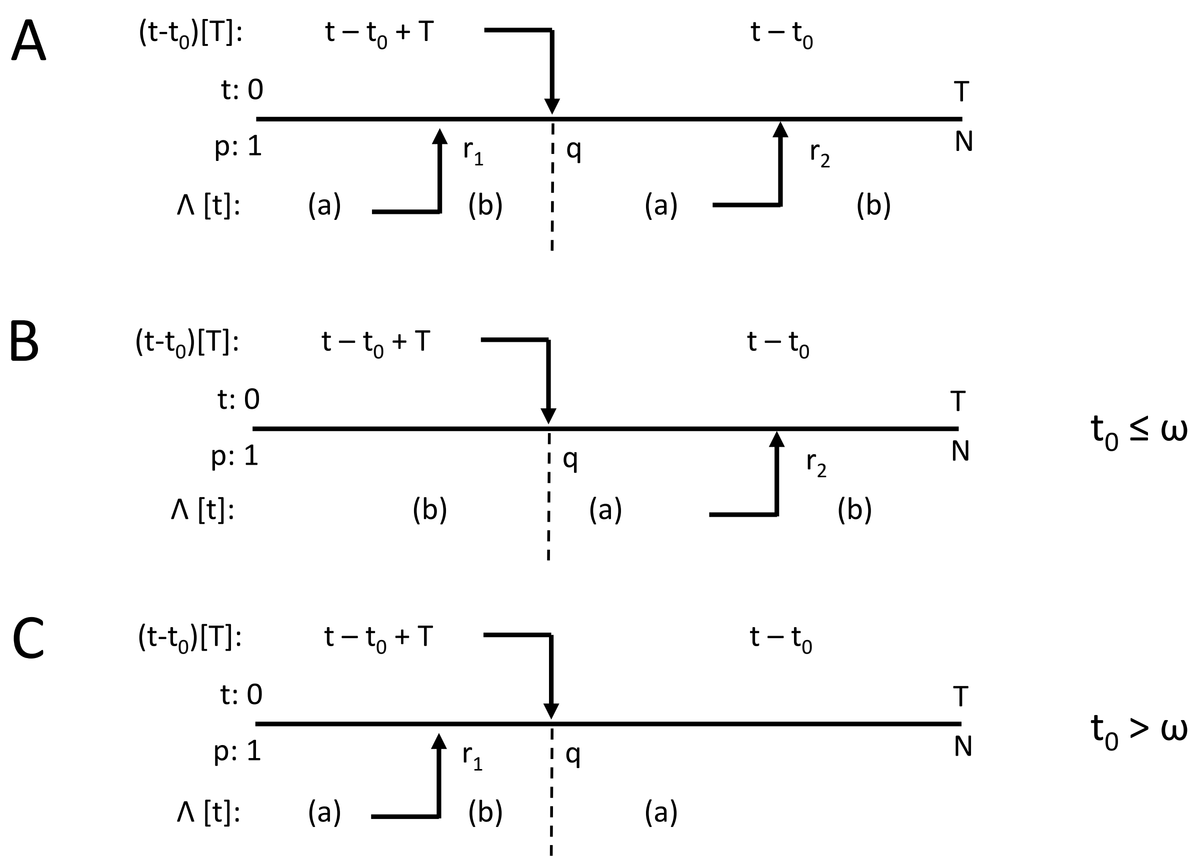

Fig. S2: A: Depending on the gate index p, the expression $(t_p - t_0)[T]$ takes the form $t_p - t_0 + T\ (p \le q)$ or $t_p - t_0\ (p > q)$, where q is defined in the text. Likewise, depending on the gate index p position with respect to index r_1 and r_2 defined in the text, the square-gated PSED expression to use is form (a) or (b) of Eq. (121). Specifically, (B) for $t_0 \le \omega$, a single index, $r_2 \ge q$ is necessary, while (C) when $t_0 > \omega$, a single index $r_1 \le q$ is needed.

To determine which form (a) or (b) of Eq. (121) to use for $\Lambda_{\tau,T,W,t_0}(t)$, we need to look into each of the two domains defined in Eq. (B55) and determine index r_1 and r_2 such that:

$$
\begin{aligned}
&p \le q \Rightarrow \begin{cases} p \le r_1 \Rightarrow \text{form } (a) \\ p > r_1 \Rightarrow \text{form } (b) \end{cases} \\
&p > q \Rightarrow \begin{cases} p \le r_2 \Rightarrow \text{form } (a) \\ p > r_2 \Rightarrow \text{form } (b) \end{cases}
\end{aligned}
\tag{B56}
$$

It is straightforward to establish that there are two main situations, $t_0 \le \omega,\ \ r_2 = N + \left\lceil \frac{t_0 - \omega}{\theta} \right\rceil$ and $t_0 > \omega,\ r_1 = \left\lceil \frac{t_0 - \omega}{\theta} \right\rceil$ (see Fig. S2B & C), with $t_0 = \omega$ being a special case of $t_0 \le \omega$ for which $r_2 = N$. Calculation of the denominator of Eq. (92) thus involves:

$$
\begin{aligned}
&t_0 \le \omega \Rightarrow \\
&\quad \left\| \Lambda_{\tau,T,W|t_0}\left(t_p\right) \right\|_N = \theta \left\{ \sum_{p=1}^{q} (b)\left(t_p - t_0 + T\right) + \sum_{p=q+1}^{r_2} (a)\left(t_p - t_0\right) + \sum_{p=r_2+1}^{N} (b)\left(t_p - t_0\right) \right\} \\
&t_0 > \omega \Rightarrow \\
&\quad \left\| \Lambda_{\tau,T,W|t_0}\left(t_p\right) \right\|_N = \theta \left\{ \sum_{p=1}^{r_1} (a)\left(t_p - t_0 + T\right) + \sum_{p=r_1+1}^{q} (b)\left(t_p - t_0 + T\right) + \sum_{p=q+1}^{N} (a)\left(t_p - t_0\right) \right\}
\end{aligned}
\tag{B57}
$$

where the notation $(b)\left(t_p - t_0 + T\right)$ means that form (*b*) of Eq. (121) needs to be used, with $\left(t - t_0\right)\left[T\right] = t - t_0 + T$ and so on. Similar sums are involved for the numerator of Eq. (92).

After some lengthy but straightforward calculations, one obtains:

$$
z_N\left[\Lambda_{\tau,T,W|t_0}\right] = \frac{\dfrac{e^{ir_2\alpha} - e^{iq\alpha}}{1 - e^{i\alpha}} + \dfrac{x^q e^{iq\alpha} - u y^{-1} x^{r_2} e^{ir_2\alpha}}{1 - x e^{i\alpha}} e^{t_0/\tau}}{(k+1)N + q - r_2 + \dfrac{x^q - u y^{-1} x^{r_2}}{1 - x} e^{t_0/\tau}},\ t_0 \le \omega \tag{B58}
$$

and:

$$
z_N\left[\Lambda_{\tau,T,W|t_0}\right] = \frac{\dfrac{e^{ir_1\alpha} - e^{iq\alpha}}{1 - e^{i\alpha}} + \dfrac{x^q e^{iq\alpha} - u x^{r_1} e^{ir_1\alpha}}{1 - x e^{i\alpha}} e^{t_0/\tau}}{kN + q - r_1 + \dfrac{x^q - u x^{r_1}}{1 - x} e^{t_0/\tau}},\ t_0 > \omega \tag{B59}
$$

which can be combined into a single form:

$$
\begin{cases}
z_{N[W]}\left[\Lambda_{\tau,T|t_0}\right] = z_N\left[\Lambda_{\tau,T,W|t_0}\right] = \dfrac{\dfrac{e^{ir\alpha} - e^{iq\alpha}}{1 - e^{i\alpha}} + \dfrac{x^q e^{iq\alpha} - u x^{r} e^{ir\alpha}}{1 - x e^{i\alpha}} e^{t_0/\tau}}{kN + q - r + \dfrac{x^q - u x^{r}}{1 - x} e^{t_0/\tau}} \\
q = \left\lceil \dfrac{t_0}{\theta} \right\rceil, \quad r = \left\lceil \dfrac{t_0 - \omega}{\theta} \right\rceil
\end{cases}
\tag{B60}
$$

As discussed in the main text (Section 4), this equation does not in general represent an arc of circle.

It is easy to verify that for $\tau = \infty$, $z\left[\Lambda_{\infty,T,W|t_0}\right] = 0$

For $\tau = 0$, we need to distinguish between 4 cases depending on the respective values of:

$$\begin{cases} \theta_q = q\theta - t_0 \\ \theta_r = r\theta - (t_0 - \omega) \end{cases} \tag{B61}$$

The final result for $z\left[\Lambda_{0,T,W|t_0}\right]$ is easy to verify and can be tabulated as:

	$\theta_r = 0$	$\theta_r > 0$
$\theta_q = 0$	$\frac{1}{(kN+(q+1)-(r+1))}\frac{e^{i(r+1)\alpha}-e^{i(q+1)\alpha}}{1-e^{i\alpha}}$	$\frac{1}{(kN+q+1-r)}\frac{e^{ir\alpha}-e^{i(q+1)\alpha}}{1-e^{i\alpha}}$
$\theta_q > 0$	$\frac{1}{(kN+q-(r+1))}\frac{e^{i(r+1)\alpha}-e^{iq\alpha}}{1-e^{i\alpha}}$	$\frac{1}{(kN+q-r)}\frac{e^{ir\alpha}-e^{iq\alpha}}{1-e^{i\alpha}}$

(B62)

These formulas are undefined if $k = 0$ and $q = r$ ($\theta_q = \theta_r = 0$ or $\theta_q > 0$ and $\theta_r > 0$), or $k = 0$ and $q = r + 1$ ($\theta_q > 0$ and $\theta_r = 0$), or $k = 0$ and $q = r - 1$ ($\theta_q = 0$ and $\theta_r > 0$). In those cases, the result is instead:

	$\theta_r = 0$	$\theta_r > 0$
$\theta_q = 0$	$k = 0,\ q = r \to e^{ir\alpha}$	$k = 0,\ q = r - 1 \to e^{ir\alpha}$
$\theta_q > 0$	$k = 0,\ q = r + 1 \to e^{ir\alpha}$	$k = 0,\ q = r \to e^{ir\alpha}$

(B63)

In other words, in these situations (which also includes $\theta_q = \theta_r = 0$, which strictly speaking is not a square-gated case), the phasor of the square-gated decay with offset at $\tau = 0$ is equal to $e^{ir\alpha}$.

B.2.4. Special cases (with offset)

<u>a. Gate width equal to gate step: $W = \theta$</u>

In the particular case where the gate width W equals the gate step θ (a case relevant for instance to binned TCSPC data where the time bin can be identified to a gate), the analytical expression (B62) takes the simpler form:

$$z_N\left[\Lambda_{\tau,T,\theta|t_0}\right] = \frac{\left(1 - xe^{i\alpha} - \left(1 - e^{i\alpha}\right)x^q e^{t_0/\tau}\right)}{1 - xe^{i\alpha}} e^{i(q-1)\alpha} \tag{B64}$$

We can distinguish two general subcases: (i) $t_0 = q\theta$ and (ii) $t_0 = q\theta - \theta_0,\ \theta_0 \in \left]0, \theta\right[$.

(i) In the first case, Eq. (B64) simplifies into:

$$t_0 = q\theta \quad \Rightarrow \quad z_{N[W]}\left[\Lambda_{\tau,T|t_0}\right] = \frac{1-x}{1-xe^{i\alpha}} e^{i2\pi f t_0} = z_N\left[\Lambda_{\tau,T|t_0}\right] \tag{B65}$$

In other words, for a square-gated ($W = \theta$) PSED with offset t_0 proportional to the gate step θ, the discrete phasor is equal to the discrete phasor of the ungated PSED with the same offset.

(ii) In the second case, Eq. (B64) can be rewritten:

$$t_0 = q\theta - \theta_0,\ \theta_0 \in \left]0,\theta\right[\ \Rightarrow\ z_N\left[\Lambda_{\tau,T,\theta|t_0}\right] = \frac{\left(1-x+\left(1-e^{i\alpha}\right)\left(x-e^{-\theta_0/\tau}\right)\right)}{1-xe^{i\alpha}} e^{i(q-1)\alpha} \tag{B66}$$

We recognize the term $z_N\left[\Lambda_{\tau,T}\right] = (1-x)/(1-xe^{i\alpha})$ in this expression, but this is where the similarity with the previous case ends, since there are additional terms as well, all depending on τ. The term $e^{i(q-1)\alpha}$ introduces a constant rotation depending on the offset t_0, which is constant for sets of offsets t_0 within θ from one another. However, because of the additional term involving θ_0 in Eq. (B66), this does not mean that the corresponding SEPLs are identical, contrary to the case of the ungated decays with offset.

b. Truncated decay with offset: $t_1 = t_0$ and $t_N < T - \omega$

Because of the assumption that all t_ps are such that $t_0 \leq t_p = t_0 + (p-1)\theta < T - \omega$, only the first expression in Eq. (121) for the square-gated PSED needs to be used, and the time argument reads:

$$t_p' = t_p - t_0 - \lfloor (t_p - t_0)/T \rfloor T = (p-1)\theta \tag{B67}$$

This simplifies the calculation of the numerator and denominator in the phasor formula (Eq.(92)). We obtain:

$$\left\|\Lambda_{\tau,T,W|t_0}\left(t_p\right)\right\|_N = \theta\left\{\frac{1-u}{1-y}\sum_{p=1}^{N} e^{-t_p'/\tau} + kN\right\} = \theta\left\{kN + \frac{1-u}{1-y}\frac{1-x^N}{1-x}\right\} \tag{B68}$$

and:

$$\begin{aligned}\left\|\Lambda_{\tau,T,W|t_0}\left(t_p\right)e^{i2\pi f t_p}\right\|_N &= \theta\sum_{p=1}^{N}\left(\frac{1-u}{1-y}e^{-t_p'/\tau} + k\right)e^{i2\pi f t_p} \\ &= \theta\left(k\frac{1-e^{iN\alpha}}{1-e^{i\alpha}} + \frac{1-u}{1-y}\frac{1-x^N e^{iN\alpha}}{1-xe^{i\alpha}}\right)e^{i2\pi f t_0}\end{aligned} \tag{B69}$$

This expression simplifies greatly if we assume that the phasor frequency f is chosen such that:

$$f = \frac{n}{N\theta} = \frac{n}{D} \tag{B70}$$

In that case:

$$\left\|\Lambda_{\tau,T,W|t_0}\left(t_p\right)e^{i2\pi f t_p}\right\|_N = \theta\frac{1-u}{1-y}\frac{1-x^N}{1-xe^{i\alpha}}e^{i2\pi f t_0} \tag{B71}$$

and the phasor reads:

$$z_{N[W]}\left[\Lambda_{\tau,T|t_0}\right]=\frac{\left\|\Lambda_{\tau,T,W}\left(t_p\right)e^{i2\pi ft_p}\right\|_N}{\left\|\Lambda_{\tau,T,W,t_0}\left(t_p\right)\right\|_N}=\frac{\dfrac{1-x^N}{1-xe^{i\alpha}}}{kN\dfrac{1-y}{1-u}+\dfrac{1-x^N}{1-x}}e^{i2\pi ft_0} \tag{B72}$$

Unless $k = 0$ (i.e. $W < T$), this expression does not describe a simple curve. However, $k = 0$ (*i.e.* $W < T$) is the most common case, for which Eq. (B72) simplifies into:

$$z_{N[W]}\left[\Lambda_{\tau,T|t_0}\right]=\frac{1-x}{1-xe^{i\alpha}}e^{i2\pi ft_0}=z_N\left[\Lambda_{\tau,T|t_0}\right] \tag{B73}$$

which is the same formula as for a discrete phasor of an ungated PSED with offset t_0, and corresponds to an arc of circle rotated about the origin.

B.3. Discrete phasor of arbitrary T-periodic decays

B.3.1. General case

By analogy with the continuous phasor case discussed in Section 3.2.4, the discrete phasor of an arbitrary *T*-periodic decay $S_T\left(t\right)$ can be written as a weighted sum of the discrete phasors of PSEDS, $\zeta_{f,N}\left(\tau\right)$. As argued in that discussion, it is advantageous to look at the $\left\| \; \right\|_N$*-normalized* decay $\sigma_T\left(t\right)$ now defined (because we are dealing with discrete phasors) as:

$$\sigma_T\left(t\right)=\frac{S_T\left(t\right)}{\left\|S_T\left(t_p\right)\right\|_N} \tag{B74}$$

By definition, we shall call $\phi_0\left(\tau\right)$ the weight function such that:

$$\sigma_T\left(t\right)=\int_0^\infty d\tau\phi_0\left(\tau\right)\Lambda_{\tau,T}\left(t\right) \tag{B75}$$

The discrete phasor of $\sigma_T\left(t\right)$ is given by:

$$z_N\left[\sigma_T\right]=\frac{\left\|\sigma_T\left(t_p\right)e^{i2\pi ft_p}\right\|_N}{\left\|\sigma_T\left(t_p\right)\right\|_N} \tag{B76}$$

The two terms in this ratio are:

$$\begin{cases} \left\|\sigma_T\left(t_p\right)\right\|_N = \int_0^\infty d\tau \phi_0(\tau)\left\|\Lambda_{\tau,T}\left(t_p\right)\right\|_N = 1 \\ \left\|\sigma_T\left(t_p\right)e^{i2\pi f t_p}\right\|_N = \int_0^\infty d\tau \phi_0(\tau)\left\|\Lambda_{\tau,T}\left(t_p\right)e^{i2\pi f t_p}\right\|_N = \int_0^\infty d\tau \phi_0(\tau)\frac{\theta}{\tau\left(1-xe^{i\alpha}\right)} \\ \qquad = \int_0^\infty d\tau \phi_0(\tau)\frac{\theta}{\tau(1-x)}\zeta_{f,N}(\tau) \\ x = 1-e^{-\theta/\tau}; \alpha = 2\pi f\theta \end{cases} \tag{B77}$$

The first result comes from the definition of $\sigma_T(t)$(Eq. (B74)) and the second from Eq. (B1) and the definition of the discrete PSED phasor (Eq. (B3)).

Introducing the weight function $\mu_0(\tau)$ and the $\| \; \|_N$-normalized basis $\left\{\Lambda_{\tau,T,N}(t)\right\}_{\tau>0}$:

$$\begin{cases} \mu_0(\tau) = \frac{\theta}{\tau(1-x)}\phi_0(\tau) \\ \Lambda_{\tau,T,N}(t) = \frac{\tau(1-x)}{\theta}\Lambda_{\tau,T}(t) \\ \left\|\Lambda_{\tau,T,N}\left(t_p\right)\right\| = 1 \end{cases} \tag{B78}$$

we can rewrite:

$$\begin{cases} \sigma_T(t) = \int_0^\infty d\tau \mu_0(\tau)\Lambda_{\tau,T,N}(t) \\ z_N\left[\sigma_T\right] = \int_0^\infty d\tau \mu_0(\tau)\zeta_{f,N}(\tau) \\ z_N\left[\Lambda_{\tau,T,N}\right] = \zeta_{f,N}(\tau) \end{cases} \tag{B79}$$

The last identity in Eq. (B79) follows from Eq. (B78) and the discrete phasor invariance by dilation (Eq. (89)).

Eq. (B79) shows that when decomposing a normalized *T*-periodic function in the basis of $\left\{\Lambda_{\tau,T,N}(t)\right\}_{\tau>0}$, its discrete phasor takes the same functional form in terms of the discrete phasors $\zeta_{f,N}(\tau)$.

B.3.2. Special case: linear combination of exponentials

Starting from the same definition for decay $S(t)$and its weight function $\phi_0(\tau)$as in the continuous case (Eq. (85)), we obtain for its *T*-periodic version:

$$S_T(t) = \sum_{i=1}^{n} a_i \tau_i \Lambda_{\tau_i,T}(t) \tag{B80}$$

Eq. (B78) for $\mu_0(\tau)$ yields:

$$\begin{cases} \mu_0(t) = \sum_{i=1}^{n} \mu_i \delta(\tau - \tau_i) \\ \mu_i = \dfrac{a_i}{\left(1 - e^{-\theta/\tau_i}\right)} \Bigg/ \sum_{j=1}^{n} \dfrac{a_j}{\left(1 - e^{-\theta/\tau_j}\right)} \end{cases} \tag{B81}$$

Note that for $N \to \infty$ (*i.e.* $\theta \to 0$), the μ_i s defined by Eq. (B81) are identical to those obtained in the continuous case (Eq. (86)), as expected.

With these definitions, the $\| \|_N$-normalized decay and its discrete phasor are given by:

$$\begin{cases} \sigma_T(t) = \sum_{i=1}^{n} \mu_i \Lambda_{\tau_i,T,N}(t) \\ z_N[\sigma_T] = \sum_{i=1}^{n} \mu_i \zeta_{f,N}(\tau_i) \end{cases} \tag{B82}$$

In other words, the discrete phasor of a linear combination of single-exponential decays can be expressed as a linear combination of phasors, but, in order for the same functional form to be used, the discrete decay needs to be expressed in the basis of $\left\{\Lambda_{\tau,T,N}(t)\right\}_{\tau>0}$ (which are proportional to the pure exponentials). The discrete phasor of the total decay is then expressed in the same manner as a function of their individual phasors.

Appendix C: Phasor of the convolution product of periodic decays

C.1. Relation between convolution $*$ and cyclic convolution $\underset{T}{*}$

Let *f*(*t*) be a non-periodic function. The *T-periodic summation* of *f* is defined as the infinite sum of shifted versions of the original function:

$$f_T(t) = \sum_{i=-\infty}^{+\infty} f(t - iT) \tag{C1}$$

Let $g_T(t)$ be another *T*-periodic function obtained as the *T-periodic summation* of *g*, a non-periodic function, similarly to the process described for *f* in Eq. (C1). The convolution of *f* and g_T verifies:

$$
\begin{aligned}
f * g_T(t) &= \int_{-\infty}^{+\infty} du\; f(u)\, g_T(t-u) \\
&= \sum_{i=-\infty}^{+\infty} \int_{iT}^{iT+T} du\; f(u)\, g_T(t-u) = \sum_{i=-\infty}^{+\infty} \int_{iT}^{iT+T} du\; f(u)\, g_T(t-u+iT) \\
(u = v + iT) &= \sum_{i=-\infty}^{+\infty} \int_{0}^{T} dv\; f(v+iT)\, g_T(t-v) = \int_{0}^{T} dv \sum_{i=-\infty}^{+\infty} f(v+iT) g_T(t-v) \\
&= \int_{0}^{T} dv\; f_T(v)\, g_T(t-v) = f_T \underset{T}{*} g_T(t)
\end{aligned}
\tag{C2}
$$

This establishes the identity between the convolution product involving an integral with infinite bounds of a non-periodic function with a *T*-periodic one and the *cyclic* (or *circular*) *convolution product*, denoted by a $\underset{T}{*}$ symbol, involving an integral of two *T*-periodic functions over a single period *T*.

Note that the convolution product as defined in the last line of Eq. (C2) is commutative:

$$
\begin{aligned}
& f_T \underset{T}{*} g_T(t) = g_T \underset{T}{*} f_T(t) \\
& \quad = f * g_T(t) = f_T * g(t) = g_T * f(t) = g * f_T(t)
\end{aligned}
\tag{C3}
$$

as can be easily verified with the example of the cyclic convolution of two PSEDs defined by Eq. (17):

$$
\Lambda_{\tau_0,T} \underset{T}{*} \Lambda_{\tau,T}(t) = \Psi_{\tau,\tau_0,T}(t) \tag{C4}
$$

Finally, it is useful to notice that the convolution product of two non-periodic functions is unrelated to those discussed above (as can be easily verified using two exponential function such as I_0 and F_0 defined in Eq. (19)):

$$
\begin{aligned}
f_T * g(t) &= (f * g)_T(t) = (g * f)_T(t) \\
&\neq f * g(t)
\end{aligned}
\tag{C5}
$$

C.2. Cyclic convolution of the mirrored gate function and detected periodic decay

In the case of a *T*-periodic decay $S_T(t)$ (given by Eq. (14)) convolved with a *nT*-periodic function ($n \geq 1$) $\overline{\Gamma}_{W,nT}(t)$ (Eq. (38)):

$$
\begin{aligned}
S_{W,T}(s) = \overline{\Gamma}_{W,nT} \underset{nT}{*} S_T(t) &= \int_0^{nT} du\, \overline{\Gamma}_{W,nT}(t-u) \int_0^T dv\, I_T(v)\, f_{0,T}(u-v) \\
(w = u - v) \qquad &= \int_0^{nT} du\, \overline{\Gamma}_{W,nT}(t-u) \int_{u-T}^{u} dw\, I_T(u-w)\, f_{0,T}(w) \\
(x = u - w) \qquad &= \int_0^{T} dw\, f_{0,T}(w) \int_{-w}^{nT-w} dx\, \overline{\Gamma}_{W,nT}(t-w-x)\, I_T(x) \\
&= \int_0^{T} dw\, f_{0,T}(w)\, \overline{\Gamma}_{W,nT} \underset{T}{*} I_T(t-w) \\
&= \int_0^{T} dw\, f_{0,T}(w)\, I_{T,W}(t-w) = f_{0,T} \underset{T}{*} I_{T,W}(t) \\
&= I_{T,W} \underset{T}{*} f_{0,T}(t)
\end{aligned}
\tag{C6}
$$

where we have used the T-periodicity of $I_T(t)$ and $f_{0,T}(t)$ and the nT-periodicity of $\overline{\Gamma}_{W,nT}(t)$ and $I_T(t)$ to change the bounds of integration. The cyclic convolution product in the next-to-last line involves an integral over [0, nT], but the resulting gated instrument response function $I_{T,W}(t)$ is T-periodic:

$$
I_{T,W}(t) = \int_0^{nT} du\, \overline{\Gamma}_{W,nT}(u)\, I_T(t-u)
\tag{C7}
$$

The last cyclic convolution product in Eq. (C6) involves the standard integration over [0,T].

C.3. Continuous phasor convolution rule

With this definition, the numerator of the continuous phasor expression for a T-periodic function (Eq. **Error! Reference source not found.**(63)) reads:

$$
\begin{aligned}
\left\| f_T \underset{T}{*} g_T(t) e^{i2\pi ft} \right\|_T &= \int_0^T dt\, f_T \underset{C}{*} g_T(t) e^{i2\pi ft} \\
&= \int_0^T dt \int_0^T du\, f_T(u)\, g_T(t-u)\, e^{i2\pi ft} \\
&= \int_0^T du\, f_T(u)\, e^{i2\pi fu} \int_0^T dt\, g_T(t-u)\, e^{i2\pi f(t-u)}
\end{aligned}
\tag{C8}
$$

Due to the T-periodicity of g_T and the exponential term, we can rewrite:

$$
\begin{aligned}
\int_0^T dt\, g_T(t-u)e^{i2\pi f(t-u)} &= \int_{-u}^{T-u} dv\, g_T(v)e^{i2\pi fv} \\
&= \int_{-u}^{0} dv\, g_T(v)e^{i2\pi fv} + \int_{0}^{T-u} dv\, g_T(v)e^{i2\pi fv} \\
(w = v+T) \qquad &= \int_{T-u}^{T} dw\, g_T(T+w)e^{i2\pi f(T+w)} + \int_{0}^{T-u} dv\, g_T(v)e^{i2\pi fv} \\
&= \int_{T-u}^{T} dw\, g_T(w)e^{i2\pi fw} + \int_{0}^{T-u} dv\, g_T(v)e^{i2\pi fv} = \int_0^T dv\, g_T(v)e^{i2\pi fv}
\end{aligned}
\tag{C9}
$$

from which it follows that:

$$
\begin{aligned}
\left\| f_T \underset{T}{*} g_T(t)e^{i2\pi ft} \right\|_T &= \int_0^T dt\,{}_T g_T(t)e^{i2\pi ft}\int_0^T du\, f_T(u)e^{i2\pi fu} \\
&= \left\| f_T(t)e^{i2\pi ft} \right\|_T \left\| g_T(t)e^{i2\pi ft} \right\|_T
\end{aligned}
\tag{C10}
$$

Similarly, we obtain for the numerator:

$$
\left\| f_T \underset{T}{*} g_T \right\|_T = \| f_T \|_T \, \| g_T \|_T \tag{C11}
$$

from which it results that the continuous phasor of the convolution of two *T*-periodic decays is the product of their individual continuous phasors (‘*continuous phasor convolution rule*’):

$$
z\left[f_T \underset{T}{*} g_T \right] = z\left[f_T \right] z\left[g_T \right] \tag{C12}
$$

C.4. Discrete phasor convolution rule

As mentioned in Section 3, the discrete phasor is related to the DFT of the *N*-periodic sequence $\left\{ S_T\left(t_p\right)\right\}, 1 \le p \le N$:

$$
z_N\left[S_T \right](f) = \frac{\mathcal{F}^*\left[S_T \right](f)}{\mathcal{F}^*\left[S_T \right](0)} \tag{C13}
$$

and as such inherits the properties of DFTs, in particular that related to the convolution of two *discrete* periodic functions. However, the convolution products involved in time-resolved spectroscopy are for the most part not involving discrete functions (discretization applies only at the data recording level), and thus, this property is of little use. Instead, it is replaced by a different phasor convolution rule. Because this distinction is important, we will first recall the convolution rule as it applies to discrete periodic functions and their DFT before examining the phasor itself.

C.4.1. DFT convolution rule

Lets start with the definition of the *discrete cyclic convolution* of two *T*-periodic functions f_T and g_T sampled at *N* equidistant time points $\left\{ t_p = \left(p-1\right)\theta \right\}, 1 \le p \le N$:

$$f_T *_N g_T(t_p) = \sum_{m=1}^{N} f_T(t_m) g_T(t_{p-m}) = \sum_{m=1}^{N} f_m g_{p-m} \tag{C14}$$

with obvious definitions for f_m and g_m and the index N placed next to the convolution product symbol ($*_N$) indicates a discrete cyclic convolution. In the above definition, $p-m \leq 0 \Rightarrow g_{p-m} = g_{N+p-m}$ accounts for the T-periodicity (or equivalently, in terms of indices, N-periodicity) of g_T.

The convolution theorem for DFTs is easily verified by plugging Eq. (C14) in the definition of the DFT (Eq. (94)**Error! Reference source not found.**) and reads:

$$\mathcal{DF}\left[f_T *_N g_T\right] = D\mathcal{F}\left[f_T\right]\mathcal{DF}\left[g_T\right] \tag{C15}$$

where the frequency $f_n = n/T$, $0 \leq n \leq N-1$ (or equivalently the harmonic n) is omitted in the DFT notation, but is implicit.

C.4.2. Discrete phasor convolution rule

Eq. (C15) results in the following identity:

$$z_N\left[f_T *_N g_T\right] = z_N\left[f_T\right] z_N\left[g_T\right] \tag{C16}$$

which is not particularly useful, because the discrete cyclic convolution product does not intervene in any physical process. Instead, one is usually interested in the discrete phasor of a *continuous* cyclic convolution product, for which we have in general ('negative *discrete phasor convolution rule*'):

$$z_N\left[f_T \underset{T}{*} g_T\right] \neq z_N\left[f_T\right] z_N\left[g_T\right]. \tag{C17}$$

because the cyclic convolution products in Eqs. (C16) & (C17) are in general different:

$$f_T \underset{T}{*} g_T(t_p) \neq f_T *_N g_T(t_p) \tag{C18}$$

This 'negative' discrete phasor convolution rule (Eq. (C17)) has a direct implication for the phasor of samples measured with a setup that results in a measured signal equal to a *continuous* convolution product (e.g. Eq. (14) or Eq. **Error! Reference source not found.**(42)):

$$S_T(t) = I_T \underset{T}{*} F_{0,T}(t) \quad \text{or} \quad S_{T,W}(t) = I_{T,W} \underset{T}{*} F_{0,T}(t) \tag{C19}$$

Indeed, the discrete phasor convolution rule states that, in general, the discrete phasor of the convolution products in Eq. (C19) *cannot* be written as a product of the respective phasors:

$$z_N\left[I_T \underset{T}{*} F_{0,T}\right] \neq z_N\left[I_T\right] z_N\left[F_{0,T}\right] \quad \text{or} \quad z_N\left[I_{T,W} \underset{T}{*} F_{0,T}\right] \neq z_N\left[I_{T,W}\right] z_N\left[F_{0,T}\right] \tag{C20}$$

This has the very important implication that standard phasor calibration, as implemented for continuous phasors, in general *does not work*.

However, in some particular cases, the discrete phasor convolution rule can be rewritten:

$$z_N\left[I_T \underset{T}{*} F_{0,T}\right] = \kappa\, z_N\left[I_T\right] z_N\left[F_{0,T}\right] \tag{C21}$$

where κ is constant for a family of decays $\left\{F_{0,T,\lambda}(t)\right\}$, $\lambda \in \Omega$, where Ω is a subset of $\mathbb{R}$ (generally the family of PSEDs) and $I_T(t)$ represents the instrument response function. This '*weak discrete phasor convolution rule*' allows using a modified phasor calibration approach, as is discussed in Section 8.

We shall review a few examples of the weak as well as the negative discrete phasor convolution rule in the following sub-sections.

C.4.3. Examples of the discrete phasor convolution rule

C.4.3.1. Discrete phasor of ungated PSEDs with single-exponential IRF

To make this clear, lets consider the case of a *T*-periodic single-exponential IRF (time constant τ_0) convolved with a single-exponential decay with lifetime τ (sample emitted signal $\Psi_{\tau,\tau_0,T}(t)$ defined in Section 2.1.6). The convolution product defining this sample emitted signal has to be a continuous convolution product, as this is how the physics of the process is defined:

$$\Psi_{\tau,\tau_0,T}(t) = \Lambda_{\tau_0,T} \underset{T}{*} \Lambda_{\tau,T}(t) \tag{C22}$$

As shown in Appendix D.8, the discrete phasor of $\Psi_{\tau,\tau_0,T}(t)$, is given by (Eq. (D32)):

$$\begin{aligned} z_N\left[\Psi_{\tau,\tau_0,T}\right] &= z_N\left[\Lambda_{\tau,T}\right] z_N\left[\Lambda_{\tau_0,T}\right] e^{i2\pi f\theta} \\ &\neq z_N\left[\Lambda_{\tau,T}\right] z_N\left[\Lambda_{\tau_0,T}\right] \end{aligned} \tag{C23}$$

While the discrete phasor of the continuous convolution product is not equal to the product of the respective discrete phasors in this case, the result shows that it is *proportional* to the product, with a constant proportionality factor $\kappa = e^{i2\pi f\theta}$ where $\theta = T/N$ is the gate step. This result is at the basis of the phasor calibration strategy for ungated decays discussed in Section 8.3. Note however that Eq. (C23) does not apply to decays other than PSEDs

By contrast, it is easy to verify directly that Eq. (C16) holds for $\Lambda_{\tau_0,T}$ and $\Lambda_{\tau,T}$, but once again, the discrete convolution product:

$$\begin{aligned} \Lambda_{\tau_0,T} *_N \Lambda_{\tau,T}(t_p) &= \frac{1}{\tau_0\tau\left(e^{-\theta/\tau} - e^{-\theta/\tau_0}\right)}\left(\frac{e^{-p\theta/\tau}}{1-e^{-T/\tau}} - \frac{e^{-p\theta/\tau_0}}{1-e^{-T/\tau_0}}\right) \\ &= \frac{1}{x-x_0}\left(\frac{x}{\tau_0}\Lambda_{\tau,T}(t_p) - \frac{x_0}{\tau}\Lambda_{\tau_0,T}(t_p)\right) \end{aligned} \tag{C24}$$

(where $\theta = T/N;\ x(\tau) = e^{-\theta/\tau};\ x_0 = x(\tau_0) = e^{-\theta/\tau_0}$) defined at discrete points $\left\{t_p = (p-1)\theta\right\}, 1 \leq p \leq N$ only, is different from the physical signal incident on the detector, $\Psi_{\tau,\tau_0,T}(t)$ (Eq. (20)), and has in fact no physical interpretation.

C.4.3.2. Discrete phasor of square-gated PSEDs

The proportionality observed in Eq. (C23) is an exception rather than the rule. For instance, as soon as a square-gate of width W is added to the acquisition scheme (with a Dirac IRF), the discrete phasor of a square-gated PSED (convolution product of a mirrored square-gate function and a PSED) defined by:

$$\Lambda_{\tau,nT,W}(t) = \bar{\Pi}_{W,nT} \underset{T}{*} \Lambda_{\tau,T}(t) \tag{C25}$$

is given by Eq. (103), which, in general cannot be expressed as a product of terms involving the discrete phasor of a PSED ($z_N\left[\Lambda_{\tau,T}\right]$) and that of a square gate ($z_N\left[\bar{\Pi}_{W,nT}\right]$). However, in the particular case where the gate width W is proportional to the gate step θ, Eq. (106) or (B39) for the discrete phasor of a square-gated PSED can be rewritten:

$$\begin{aligned} r = \frac{T-\omega}{\theta} \Rightarrow z_N\left[\bar{\Pi}_{W,nT} \underset{T}{*} \Lambda_{\tau,T}\right] &= z_{N[W]}\left[\Lambda_{\tau,T}\right] = -\frac{e^{i\alpha}}{(k+1)N-r}\frac{1-e^{ir\alpha}}{1-e^{i\alpha}}\frac{1-x}{1-xe^{i\alpha}} \\ &= e^{i\alpha} z_N\left[\bar{\Pi}_{W,nT}\right] z_N\left[\Lambda_{\tau,T}\right] \end{aligned} \tag{C26}$$

where we have used Eq. (C36) for the expression of $z_N\left[\bar{\Pi}_{W,nT}\right]$ and $\alpha = 2\pi f\theta$.
Eq. (C26) is clearly of the form defined in Eq. (C21) with $\kappa = e^{i2\pi f\theta}$.

C.4.3.3. Discrete phasor of square-gated PSED with single-exponential IRF

The phasor of a decay given by Eq. (D19) was calculated in Section D.9, and even in the special case where the width W is proportional to the gate step θ (eq. (D49), the result shows that:

$$z_N\left[\Psi_{\tau,\tau_0,T,W}\right] \neq \kappa\, z_N\left[\bar{\Pi}_{W,nT}\right] z_N\left[\Psi_{\tau,\tau_0,T}\right] \tag{C27}$$

C.5. Phasors of mirrored square gates

Here we derive a couple of expressions used in the previous sections.

C.5.1. Continuous phasor of a mirrored square-gate

The continuous phasor of the mirrored square-gate defined by Eq. (36) is given by:

$$z\left[\bar{\Pi}_{W,nT}\right] = \frac{\left\|\bar{\Pi}_{W,nT}(t)e^{i2\pi ft}\right\|_{nT}}{\left\|\bar{\Pi}_{W,nT}(t)\right\|_{nT}} \tag{C28}$$

where $\bar{\Pi}_{W,nT}(t)$ is defined in Eq. (37). It follows:

$$\begin{cases} \left\|\bar{\Pi}_{W,nT}(t)\right\|_{nT} = \int_0^{nT} dt\bar{\Pi}_{W,nT}(t) = \int_{nT-W}^{nT} dt = W \\ \left\|\bar{\Pi}_{W,nT}(t)e^{i2\pi ft}\right\|_{nT} = \int_0^{nT} dt\bar{\Pi}_{W,nT}(t)e^{i2\pi ft} = \int_{nT-W}^{nT} dt\, e^{i2\pi ft} = \frac{1-e^{-i2\pi fW}}{i2\pi f} \end{cases} \tag{C29}$$

and the continuous phasor:

$$z\left[\overline{\Pi}_{W,nT}\right]=\frac{\sin\pi fW}{\pi fW}e^{-i\pi fW}=M_W e^{-i\varphi_W} \tag{C30}$$

using the definitions of Eq. (75).

C.5.2. Discrete phasor of a mirrored square-gate

The discrete phasor of the mirrored square-gate defined by Eq. (36) is given by:

$$z_N\left[\overline{\Pi}_{W,nT}\right]=\frac{\left\|\overline{\Pi}_{W,nT}\left(t_p\right)e^{i2\pi ft_p}\right\|_N}{\left\|\overline{\Pi}_{W,nT}\left(t_p\right)\right\|_N} \tag{C31}$$

To simplify, we will first assume that $n = 1$, $W < T$. In that case,

$$\left\|\overline{\Pi}_{W,T}\left(t_p\right)\right\|_N=\theta\sum_{p=1}^{N}\overline{\Pi}_{W,T}\left(\left(p-1\right)\theta\right) \tag{C32}$$

where:

$$\overline{\Pi}_{W,T}\left(\left(p-1\right)\theta\right)=\Pi_{0,W,T}\left(T-\left(p-1\right)\theta\right)=\begin{cases}0 \text{ if } T-\left(p-1\right)\theta<0\\1 \text{ if } 0\leq T-\left(p-1\right)\theta\leq W\\0 \text{ if } T-\left(p-1\right)\theta>W\end{cases} \tag{C33}$$

Calling $r=\lceil\left(T-W\right)/\theta\rceil$, $\overline{\Pi}_{W,T}\left(\left(p-1\right)\theta\right)=1$ for $p\in\left[r+1,N+1\right]$ and we obtain:

$$\begin{cases}\left\|\overline{\Pi}_{W,T}\left(t\right)\right\|_N=\left(N-r\right)\theta\\\left\|\overline{\Pi}_{W,T}\left(t\right)e^{i2\pi ft}\right\|_N=-\theta\dfrac{1-e^{ir\alpha}}{1-e^{i\alpha}}\end{cases} \tag{C34}$$

where $\alpha=2\pi f\theta$. If follows that the discrete phasor of the mirrored square-gate is given by:

$$z_N\left[\overline{\Pi}_{W,T}\right]=-\frac{1}{N-r}\frac{1-e^{ir\alpha}}{1-e^{i\alpha}}=-\frac{1}{N-r}\frac{\sin\left(r\alpha/2\right)}{\sin\left(\alpha/2\right)}e^{i(r-1)\alpha/2} \tag{C35}$$

If we dont assume $W < T$, then the sum in Eq. (C32) needs to be extended up to nN and it is easy to verify that $\overline{\Pi}_{W,nT}\left(\left(p-1\right)\theta\right)=1$ only for $p\in\left[\left(n-k-1\right)N+r+1,nN+1\right]$ where we have used the definition $k=\lfloor W/T\rfloor, W=\omega+kT$ introduced in Eq. (47) and the modified definition $r=\lceil\left(T-\omega\right)/\theta\rceil$. After some simple algebra, we obtain:

$$z_N\left[\overline{\Pi}_{W,nT}\right]=-\frac{1}{\left(k+1\right)N-r}\frac{1-e^{ir\alpha}}{1-e^{i\alpha}}=-\frac{1}{\left(k+1\right)N-r}\frac{\sin\left(r\alpha/2\right)}{\sin\left(\alpha/2\right)}e^{i(r-1)\alpha/2} \tag{C36}$$

When W is proportional to θ ($W=q\theta$), $r=\left(k+1\right)N-q$, Eq. (C36) can be rewritten:

$$z_N\left[\overline{\Pi}_{W,nT}\right]=\frac{\sin\left(q\alpha/2\right)}{q\sin\left(\alpha/2\right)}e^{-i(q+1)\alpha/2} \tag{C37}$$

Appendix D: Some properties of the convolution of two *T*-periodic single-exponential functions

Decays of the type discussed in Section 2.1.6 (convolution of a periodic single-exponential IRF with characteristic time τ_* and a single-exponential emission with lifetime τ) are interesting as they showcase a number of properties of *T*-periodic decays that are lost when the periodicity is not considered. Here, we review some useful properties of this family of decays and derive a few results used in the main text. For simplicity, we will refer to this type of decays as *convolution of PSEDs*.

D.1. Calculation of the convolution product

Note: In the following discussion, we will assume that one of the time constants is not equal to zero (namely the time constant representing the excitation function, $\tau_\times \neq 0$).

D.1.1. Case $\tau \neq \tau_x$

a. No offset

The *T*-periodic single-exponential IRF $I_T(t)$ is given by:

$$\begin{aligned} I_T(t) &= \frac{1}{\tau_\times}\sum_{i=-\infty}^{+\infty} e^{-(t-iT)/\tau_\times} H(t-iT) = \frac{1}{\tau_\times}\sum_{i=-\infty}^{k=\lfloor t/T \rfloor} e^{-(t-iT)/\tau_\times} \\ &= \frac{1}{\tau_\times\left(1-e^{-T/\tau_\times}\right)} e^{-\left(t-\lfloor t/T \rfloor T\right)/\tau_\times} = \frac{1}{\tau_\times\left(1-e^{-T/\tau_\times}\right)} e^{-t[T]/\tau_\times} = \Lambda_{\tau_\times,T}(t) \end{aligned} \tag{D1}$$

and the convolution with the single-exponential decay $F_0(t)$ with lifetime τ is given by:

$$\begin{aligned} I_T * F_0(t) &= \Lambda_{\tau_\times,T} * \Lambda_\tau(t) \\ &= \int_{-\infty}^{+\infty} du\, I_T(u) F_0(t-u) = \frac{1}{\tau\tau_\times}\int_{-\infty}^{t} du\, e^{-(t-u)/\tau} \sum_{i=-\infty}^{k} e^{-(u-iT)/\tau_\times} H(u-iT) \\ &= \frac{e^{-t/\tau}}{\tau\tau_\times}\sum_{i=-\infty}^{k} e^{iT/\tau_\times}\int_{-\infty}^{t} du\, e^{-u\left(\frac{1}{\tau_\times}-\frac{1}{\tau}\right)} H(u-iT) \\ &= \frac{e^{-t/\tau}}{\tau\tau_\times}\sum_{i=-\infty}^{k} e^{iT/\tau_\times}\int_{iT}^{t} du\, e^{-u\left(\frac{1}{\tau_\times}-\frac{1}{\tau}\right)} = \frac{1}{\tau-\tau_\times}\sum_{i=-\infty}^{k}\left(e^{-(t-iT)/\tau} - e^{-(t-iT)/\tau_\times}\right) \end{aligned} \tag{D2}$$

This immediately gives:

$$\begin{aligned} \Psi_{\tau,\tau_\times,T}(t) &= \Lambda_{\tau_\times,T} * \Lambda_\tau(t) = \Lambda_{\tau_\times,T} \underset{T}{*} \Lambda_{\tau,T}(t) \\ &= \frac{1}{\tau-\tau_\times}\left(\frac{e^{-t[T]/\tau}}{1-e^{-T/\tau}} - \frac{e^{-t[T]/\tau_\times}}{1-e^{-T/\tau_\times}}\right) = \frac{\tau\Lambda_{\tau,T}(t)-\tau_\times\Lambda_{\tau_\times,T}(t)}{\tau-\tau_\times} \end{aligned} \tag{D3}$$

Its integral over [0, *T*] is equal to 1.

b. With offset

In the presence of an offset t_0, the T-periodic single-exponential IRF is given by:

$$\Lambda_{\tau_\times,T|t_0}(t) = \Lambda_{\tau_\times,T}(t - t_0) \tag{D4}$$

where $\Lambda_{\tau,T|t_0}(t)$ is defined in Eq. (120). It is straightforward to verify that the convolution of this IRF and the single-exponential decay $F_0(t)$ with lifetime τ is given by:

$$\begin{aligned}\Psi_{\tau,\tau_\times,T|t_0}(t) &= \Lambda_{\tau_\times,T|t_0} * F_0(t) = \frac{1}{\tau - \tau_\times}\left(\frac{e^{-t'/\tau}}{1-e^{-T/\tau}} - \frac{e^{-t'/\tau_\times}}{1-e^{-T/\tau_\times}}\right) = \frac{\tau\Lambda_{\tau,T}(t') - \tau_\times\Lambda_{\tau_\times,T}(t')}{\tau - \tau_\times} \\ &= \frac{\tau\Lambda_{\tau,T|t_0}(t) - \tau_\times\Lambda_{\tau_\times,T|t_0}(t)}{\tau - \tau_\times}\end{aligned} \tag{D5}$$

which is the formula obtained in the absence of offset with the replacement $t \mapsto t' = t - t_0$.

D.1.2. Case τ = τx

a. No offset

When $\tau = \tau_\times$, Eq. (D3) is replaced by:

$$\Lambda_{\tau_\times,T} * \Lambda_{\tau_\times}(t) = \Psi_{\tau_\times,\tau_\times,T}(t) = \frac{1}{\tau_\times}\frac{e^{-t[T]/\tau_\times}}{1-e^{-T/\tau_\times}}\left(\frac{t[T]}{\tau_\times} + \frac{e^{-T/\tau_\times}}{1-e^{-T/\tau_\times}}\frac{T}{\tau_\times}\right) \tag{D6}$$

which reaches its maximum at $t = t_M$ given by:

$$t_M = \tau_\times - \frac{e^{-T/\tau_\times}}{1-e^{-T/\tau_\times}}T \tag{D7}$$

and its minimum at $t = 0$.

b. With offset

Similarly, Eq. (D5) is replaced by:

$$\Psi_{\tau_\times,\tau_\times,T|t_0}(t) = \frac{1}{\tau_\times}\frac{e^{-t'/\tau_\times}}{1-e^{-T/\tau_\times}}\left(\frac{t'}{\tau_\times} + \frac{e^{-T/\tau_\times}}{1-e^{-T/\tau_\times}}\frac{T}{\tau_\times}\right) \tag{D8}$$

We will not consider this case any further in the remainder, but it should be reminded that it needs to be treated separately, as formulas for $\tau \neq \tau_\times$ will not apply when $\tau = \tau_\times$.

D.2. Position of the maximum and minimum of the decay

One verify easily that the maximum of the convolution of PSEDs defined by Eq. (D3) and represented in Fig. S3A for a few values of τ is located at t_M given by ($\tau \neq \tau_\times$):

$$t_M = \frac{1}{1/\tau_\times - 1/\tau} \ln \frac{\tau}{\tau_\times} \frac{\left(1-e^{-T/\tau}\right)}{\left(1-e^{-T/\tau_\times}\right)} \tag{D9}$$

t_M is an increasing function of τ and tends to $t_{M,\infty}$ when $\tau \to \infty$ given by:

$$t_{M,\infty} = -\tau_\times \ln\left[\frac{\tau_\times}{T}\left(1-\mathrm{e}^{-T/\tau_\times}\right)\right] \tag{D10}$$

In other words, as the lifetime of the emitting species increases, the location of the peak of the recorded signal is shifted towards larger values, but remains bounded by $t_{M,\infty}$.

The decays minimum is attained at $t = 0$ and is equal to:

$$F_T(0) = \frac{1}{\tau - \tau_\times}\left(\frac{1}{1-e^{-T/\tau}} - \frac{1}{1-e^{-T/\tau_\times}}\right) \tag{D11}$$

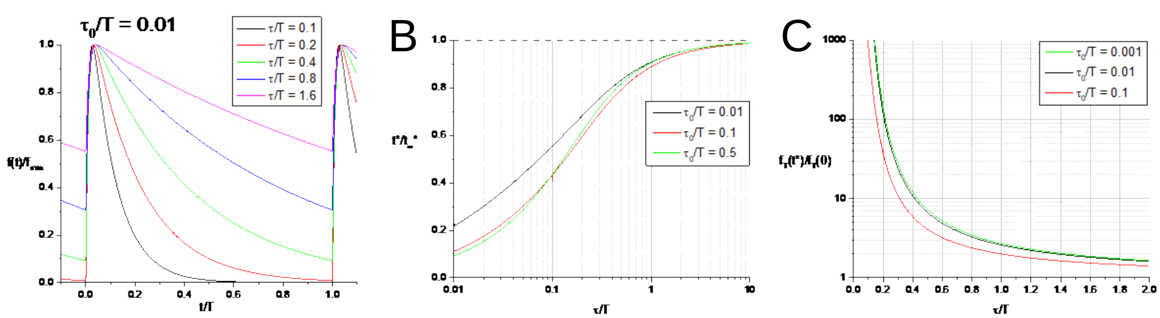

Fig. S3: Convolution of an exponential excitation function (τ_0) with an exponential emission function (τ). A: Representative curves for an excitation function characterized with a decay time τ_0 equal to 1% of the period duration T. The decays are normalized to their maximum value, set to 1. As the lifetime τ increases, the maximum location, t^*, increases (see panel B), while the contrast ratio (max/min) decreases (see panel C). B: Peak location relative to the asymptotic peak location (t^*_∞) as a function of lifetime τ for different values of τ_0 / T. C: Contrast ratio as a function of lifetime τ for different values of τ_0 / T. As the lifetime increases, the apparent baseline (minimum decay value) increases, making the decay look like it comprises a constant background component (see panel A).

D.3. Contrast ratio

An interesting feature of convolution of PSEDs is the fact that, as τ increases, the ratio between the maximum and minimum of these functions, which we can call a *contrast ratio*, decreases:

$$\frac{\Psi_{\tau,\tau_\times,T}(t_M)}{\Psi_{\tau,\tau_\times,T}(0)} = \frac{\left(1-e^{-T/\tau_\times}\right)e^{-t_M/\tau} - \left(1-e^{-T/\tau}\right)e^{-t_M/\tau_\times}}{e^{-T/\tau} - e^{-T/\tau_\times}} \tag{D12}$$

This property is clearly visible in Fig. S3 in which different versions of Eq. (D12) **Error! Reference source not found.** for a fixed ratio $\tau_\times / T$ and variable lifetime τ, normalized to their maximum, are represented. The minimum of the decays with larger lifetimes are getting closer to the maximum and away from 0, as τ increases. This is in the absence of additional background, and therefore shows that there is information in the minimum of a periodic decay function.

D.4. Continuous phasor of the decay

D.4.1. No offset

Plugging in Eq. (D3)**Error! Reference source not found.** in Eq. (63) **Error! Reference source not found.**, we obtain:

$$\begin{cases} \left\| \Psi_{\tau,\tau_\times,T}(t) e^{i2\pi ft} \right\|_T = \dfrac{1}{1-i2\pi f\tau} \dfrac{1}{1-i2\pi f\tau_\times} \\ \left\| \Psi_{\tau,\tau_\times,T}(t) \right\|_T = 1 \end{cases} \tag{D13}$$

from which is results that the continuous phasor of $F_T(t)$ is given by:

$$z\left[\Psi_{\tau,\tau_\times,T}\right] = \frac{1}{1-i2\pi f\tau} \frac{1}{1-i2\pi f\tau_\times} = \zeta_f(\tau_\times)\zeta_f(\tau) \tag{D14}$$

which also follows from the continuous phasor convolution rule (Eq. (69)) and the definition of $\Psi_{\tau,\tau_\times,T}(t)$ as a convolution product (Eq. (D3)).

As a result, the locus of continuous phasors of convolution of PSEDs is a scaled and rotated semicircle, the scaling factor and rotation angle being given by (Eq. (71)):

$$\begin{cases} m(\tau_\times) = \dfrac{1}{\sqrt{1+(2\pi f\tau_\times)^2}} \\ \varphi(\tau_\times) = \tan^{-1}(2\pi f\tau_\times) \end{cases} \tag{D15}$$

D.4.2. With offset

It is straightforward to verify that, in the presence of an offset t_0, Eq. (D14) is replaced by:

$$z\left[\Psi_{\tau,\tau_\times,T|t_0}\right] = \frac{1}{1-i2\pi f\tau} \frac{1}{1-i2\pi f\tau_\times} e^{i2\pi ft_0} = \zeta_f(\tau_\times)\zeta_f(\tau) e^{i2\pi ft_0} \tag{D16}$$

which allows writing:

$$z\left[\Psi_{\tau,\tau_\times,T|t_0}\right] = z\left[\Lambda_{\tau_\times,T|t_0}\right]\zeta_f(\tau) \tag{D17}$$

where $z\left[\Lambda_{\tau_\times,T|t_0}\right]$, the calibration phasor, is given by Eq. (126).

As a result, the locus of continuous phasors of convolution of PSEDs with offset is also a scaled and rotated semicircle, the scaling factor and rotation angle being given by:

$$\begin{cases} m(\tau_\times, t_0) = \dfrac{1}{\sqrt{1+(2\pi f \tau_\times)^2}} \\ \varphi(\tau_\times, t_0) = \tan^{-1}(2\pi f \tau_\times) + 2\pi f t_0 \end{cases} \tag{D18}$$

D.5. Square-gated periodic single-exponential decay

D.5.1. No offset

Using Eq. (D3) for $\Psi_{\tau,\tau_\times,T}(t)$, the signal accumulated during a square gate of width W is given by Eq. (33) with $\Gamma_{s,W,nT}(t)$ replaced by the boxcar function $\Pi_{s,W}(t)$ (Eq. (29)):

$$\begin{aligned} \Psi_{\tau,\tau_\times,T,W}(s) &= \int_s^{s+W} dt \Psi_{\tau,\tau_\times,T}(t) \Pi_{s,W}(t) = \int_s^{s+W} dt \Pi_{s,W}(t) \frac{\tau\Lambda_{\tau,T}(t) - \tau_\times \Lambda_{\tau_\times,T}(t)}{\tau - \tau_\times} \\ &= \frac{1}{\tau - \tau_\times}\left(\tau\Lambda_{\tau,T,W}(s) - \tau_\times \Lambda_{\tau_\times,T,W}(s)\right) \end{aligned} \tag{D19}$$

where we have used the notation $\Lambda_{\tau,T,W}(t)$ of Section 2.2.3 for the square-gated integral of $\Lambda_{\tau,T}(t)$. Using the results of Eq. (47), we obtain the explicit formula:

$$\Psi_{\tau,\tau_\times,T,W}(t) = \begin{cases} \dfrac{1}{\tau-\tau_\times}\left(\tau \dfrac{1-u}{1-y} e^{-t'/\tau} - \tau_\times \dfrac{1-u_\times}{1-y_\times} e^{-t'/\tau_\times}\right) + k, & \text{if } t' \in [0, T-\omega[\\ \dfrac{1}{\tau-\tau_\times}\left(\tau \dfrac{1-uy^{-1}}{1-y} e^{-t'/\tau} - \tau_\times \dfrac{1-u_\times y_\times^{-1}}{1-y_\times} e^{-t'/\tau_\times}\right) + k+1, & \text{if } t' \in [T-\omega, T[\end{cases}$$

$$\text{where} \quad \begin{cases} k = \lfloor W/T \rfloor \\ \omega = W[T] = W - kT \\ t' = t[T] = t - \lfloor t/T \rfloor T \\ y = e^{-T/\tau};\ y_\times = e^{-T/\tau_\times};\ u = e^{-\omega/\tau};\ u_\times = e^{-\omega/\tau_\times} \end{cases} \tag{D20}$$

D.5.2. With offset

As in the case of the ungated decay, the expression for the square-gated decay in the presence of offset is obtained by a simple replacement $t \mapsto t - t_0$ in the previous expression (Eq. (D20)):

$$\Psi_{\tau,\tau_\times,T,W|t_0}(t) = \Psi_{\tau,\tau_\times,T,W}(t - t_0) \tag{D21}$$

D.6. Continuous phasor of the square-gated decay

D.6.1. No offset

The continuous phasor of $\Psi_{\tau,\tau_\times,T,W}(t)$ given by Eq. (D19) is given by definition by (Eq. (63)):

$$z\left[\Psi_{\tau,\tau_\times,T,W}\right] = \frac{\left\|\Psi_{\tau,\tau_\times,T,W}(t)e^{i2\pi ft}\right\|_T}{\left\|\Psi_{\tau,\tau_\times,T,W}(t)\right\|_T} \tag{D22}$$

Using Eq. (D19) and the fact (used to derive Eq. (75)) that:

$$\begin{cases} \left\|\Lambda_{\tau,T,W}(t)\right\|_T = W \\ \left\|\Lambda_{\tau,T,W}(t)e^{i2\pi ft}\right\|_T = WM_W e^{-i\varphi_W}\zeta_f(\tau) \end{cases} \tag{D23}$$

where M_W and φ_W where defined in Eq. (75), we obtain:

$$z\left[\Psi_{\tau,\tau_\times,T,W}\right] = z\left[\bar{\Pi}_{W,nT}\right]\zeta_f(\tau)\zeta_f(\tau_\times) = z_{[W]}\left[\Lambda_{\tau_\times,T}\right]\zeta_f(\tau) \tag{D24}$$

$z_{[W]}\left[\Lambda_{\tau_\times,T}\right] = z\left[I_{T,W}\right]$ is the continuous phasor of the square-gated instrument response function (calibration phasor), while $\zeta_f(\tau) = z\left[\Lambda_{\tau,T}\right]$ is the continuous phasor of the samples response to a periodic Dirac IRF. We thus recover by a direct calculation, that the locus of phasors of square-gated PSEDs convolved with an exponential IRF is a semicircle rotated by an angle $\varphi_\times - \varphi_W = \tan^{-1}(2\pi f\tau_\times) - \pi fW$ and dilated by a factor $m_* M_W$ (given in Eqs. (74) & (75)).

D.6.2. With offset

To compute the continuous phasor of $\Psi_{\tau,\tau_\times,T,W|t_0}(t)$, it is easiest to start from $\Psi_{\tau,\tau_\times,T|t_0}(t)$ (Eq. (D5)) and write:

$$\begin{aligned} \Psi_{\tau,\tau_\times,T,W|t_0}(s) &= \int_s^{s+W} dt \Psi_{\tau,\tau_\times,T|t_0}(t)\Pi_{s,W}(t) = \int_s^{s+W} dt \Pi_{s,W}(t) \frac{\tau\Lambda_{\tau,T|t_0}(t) - \tau_\times\Lambda_{\tau_\times,T|t_0}(t)}{\tau - \tau_\times} \\ &= \frac{1}{\tau - \tau_\times}\left(\tau\Lambda_{\tau,T,W|t_0}(s) - \tau_\times\Lambda_{\tau_\times,T,W|t_0}(s)\right) \end{aligned} \tag{D25}$$

The phasor:

$$z\left[\Psi_{\tau,\tau_\times,T,W|t_0}\right] = \frac{\left\|\Psi_{\tau,\tau_\times,T,W|t_0}(t)e^{i2\pi ft}\right\|_T}{\left\|\Psi_{\tau,\tau_\times,T,W|t_0}(t)\right\|_T} \tag{D26}$$

involves two quantities that are easily obtained from previous results (Eqs. (A5) & (A10)):

$$\left\|\Psi_{\tau,\tau_\times,T,W|t_0}(t)\right\|_T = \frac{1}{\tau-\tau_\times}\left(\tau\left\|\Lambda_{\tau,T,W|t_0}(t)\right\|_T - \tau_\times\left\|\Lambda_{\tau_\times,T,W|t_0}(t)\right\|_T\right) = W$$

$$\begin{aligned}\left\|\Psi_{\tau,\tau_\times,T,W|t_0}(t)e^{i2\pi ft}\right\|_T &= \frac{1}{\tau-\tau_\times}\left(\tau\left\|\Lambda_{\tau,T,W|t_0}(t)e^{i2\pi ft}\right\|_T - \tau_\times\left\|\Lambda_{\tau_\times,T,W|t_0}(t)e^{i2\pi ft}\right\|_T\right) \\ &= \frac{\sin \pi f\omega}{\pi f}e^{-i\pi f\omega}\left(\frac{\tau}{1-i2\pi f\tau} - \frac{\tau_\times}{1-i2\pi f\tau_\times}\right)e^{i2\pi ft_0}\end{aligned} \quad \text{(D27)}$$

The final result can be written in any of the following equivalent forms:

$$\begin{aligned} z\left[\Psi_{\tau,\tau_\times,T,W,t_0}\right] &= z\left[\overline{\Pi}_{W,nT}\right]\zeta_f(\tau)\zeta_f(\tau_\times)e^{i2\pi ft_0} \\ &= z_{[W]}\left[\Lambda_{\tau_\times,T}\right]\zeta_f(\tau)e^{i2\pi ft_0} \\ &= z_{[W]}\left[\Lambda_{\tau_\times,T|t_0}\right]\zeta_f(\tau) \\ &= z\left[\Psi_{\tau,\tau_\times,T,W}\right]e^{i2\pi ft_0} \end{aligned} \quad \text{(D28)}$$

Which shows that the locus of phasors of square-gated PSEDs convolved with a single-exponential IRF with offset is a semicircle rotated by an angle $\varphi_* - \varphi_W + 2\pi ft_0$ and dilated by a factor $m_* M_W$ (given in Eqs. (74) & (75)).

D.7. Discrete phasor of the ungated decay

D.7.1. No offset

The discrete phasor of $\Psi_{\tau,\tau_\times,T}(t)$ given by Eq. (D3) is obtained from definition (92):

$$z_N\left[\Psi_{\tau,\tau_\times,T}\right] = \frac{\left\|\Psi_{\tau,\tau_\times,T}(t_p)e^{i2\pi ft_p}\right\|_N}{\left\|\Psi_{\tau,\tau_\times,T}(t_p)\right\|_N} \quad \text{(D29)}$$

where $t_p = (p-1)\theta, 1 \le p \le N$ and we assume $T = N\theta$.

Using the results of Eq. (B1), we obtain:

$$\begin{cases}\left\|\Psi_{\tau,\tau_\times,T}(t_p)\right\|_N = \dfrac{1}{(\tau-\tau_\times)}\left(\tau\left\|\Lambda_{\tau,T}(t_p)\right\|_N - \tau_\times\left\|\Lambda_{\tau_\times,T}(t_p)\right\|_N\right) \\ \qquad = \dfrac{\theta}{(\tau-\tau_\times)}\left(\dfrac{1}{1-e^{-\theta/\tau}} - \dfrac{1}{1-e^{-\theta/\tau_\times}}\right) = \theta\dfrac{x-x_\times}{\tau-\tau_\times}\dfrac{1}{1-x}\dfrac{1}{1-x_\times} \\ \left\|\Psi_{\tau,\tau_\times,T}(t_p)e^{i2\pi ft_p}\right\|_N = \dfrac{1}{(\tau-\tau_\times)}\left(\tau\left\|\Lambda_{\tau,T}(t_p)e^{i2\pi ft_p}\right\|_N - \tau_\times\left\|\Lambda_{\tau_\times,T}(t_p)e^{i2\pi ft_p}\right\|_N\right) \\ \qquad = \dfrac{\theta}{(\tau-\tau_\times)}\left(\dfrac{1}{1-e^{(-1/\tau+i2\pi f)\theta}} - \dfrac{1}{1-e^{(-1/\tau_\times+i2\pi f)\theta}}\right) \\ \qquad = \theta e^{i\alpha}\dfrac{x-x_\times}{\tau-\tau_\times}\dfrac{1}{1-xe^{i\alpha}}\dfrac{1}{1-x_\times e^{i\alpha}}\end{cases} \quad \text{(D30)}$$

extending the notations of Eq. (B3) to encompass the terms involving $\tau_\times$. We get the final result:

$$\begin{cases} z_N\left[\Psi_{\tau,\tau_\times,T}\right] = \dfrac{1-x}{1-xe^{i\alpha}}\dfrac{1-x_\times}{1-x_\times e^{i\alpha}}e^{i\alpha} \\ x = x(\tau) = e^{-\theta/\tau};\; x_\times = x(\tau_\times) = e^{-\theta/\tau_\times};\; \alpha = 2\pi f\theta \end{cases} \tag{D31}$$

We can rewrite Eq. (D31) as:

$$z_N\left[\Psi_{\tau,\tau_\times,T}\right] = e^{i\alpha} z_N\left[\Lambda_{\tau,T}\right] z_N\left[\Lambda_{\tau_\times,T}\right] \tag{D32}$$

This identity states that the *discrete* phasor of the *continuous* cyclic convolution product of two single-exponential functions is *distinct* from the product of their individual *discrete* phasors but can be rewritten in the special form of Eq. (C21) (weak discrete phasor convolution rule).

Note: It will not escape the reader's attention that the limit when $\tau_\times \to 0$ of Eq. (D32) is:

$$\lim_{\tau_\times\to 0} z_N\left[\Psi_{\tau,\tau_\times,T}\right] = e^{i2\pi f\theta} z_N\left[\Lambda_{\tau,T}\right] \lim_{\tau_\times\to 0} z_N\left[\Lambda_{\tau_\times,T}\right] = e^{i2\pi f\theta} z_N\left[\Lambda_{\tau,T}\right] = e^{i2\pi f\theta}\zeta_{f,N}(\tau) \tag{D33}$$

However,

$$\lim_{\tau_\times\to 0}\Psi_{\tau,\tau_\times,T}(t) = \Lambda_{\tau,T}(t) \tag{D34}$$

which would suggest that:

$$\lim_{\tau_\times\to 0} z_N\left[\Psi_{\tau,\tau_\times,T}\right] = z_N\left[\lim_{\tau_\times\to 0}\Psi_{\tau,\tau_\times,T}\right] = z_N\left[\Lambda_{\tau,T}\right] = \zeta_{f,N}(\tau) \tag{D35}$$

Comparison of Eqs. (D33) and (D35) shows an additional phase term in the former. Its origin comes from the way $z_N\left[\Psi_{\tau,\tau_\times,T}\right]$ is calculated (see Eq. (D30) assuming $\tau_\times \neq 0$. Indeed this assumption results in the first term of the sum in $\left\|\Lambda_{\tau_*,T}(t_p)\right\|_N$ (resp. $\left\|\Lambda_{\tau_\times,T}(t_p)e^{i2\pi f t_p}\right\|_N$) to be equal to 1, because $t_1 = 0 \Rightarrow e^{-t_1/\tau_\times} = 1$. The remaining terms do tend to 0 when $\tau_\times \to 0$, but that single 1 value remains in the limit. In other words:

$$\lim_{\tau_\times\to 0} z_N\left[\Psi_{\tau,\tau_\times,T}\right] \neq z_N\left[\lim_{\tau_\times\to 0}\Psi_{\tau,\tau_\times,T}\right] \tag{D36}$$

which is due to the fact that $\Psi_{\tau,\tau_\times,T}(t)$ is a function with a value of 0 for $t=0$, except for $\tau_\times = 0$, in which case $\Psi_{\tau,0,T}(0) = \Lambda_{\tau,T}(0) = \tau^{-1}\left(1-e^{-T/\tau}\right)^{-1}$.

D.7.2. With offset

In the presence of offset, we start from Eq. (D5) for $\Psi_{\tau,\tau_\times,T|t_0}(t)$ and need to compute:

$$z_N\left[\Psi_{\tau,\tau_\times,T|t_0}\right] = \frac{\left\|\Psi_{\tau,\tau_\times,T|t_0}(t_p)e^{i2\pi f t_p}\right\|_N}{\left\|\Psi_{\tau,\tau_\times,T|t_0}(t_p)\right\|_N} \tag{D37}$$

The two quantities in Eq. (D37) are easily obtained from previous results for the discrete phasor of ungated PSEDs with offset (Appendix B.1.2):

$$\begin{aligned}\left\|\Psi_{\tau,\tau_\times,T|t_0}\left(t_p\right)\right\|_N &= \frac{1}{\tau-\tau_\times}\left(\tau\left\|\Lambda_{\tau,T|t_0}\left(t_p\right)\right\|_N - \tau_\times\left\|\Lambda_{\tau_\times,T|t_0}\left(t_p\right)\right\|_N\right)\\ &= \frac{\theta}{\tau-\tau_\times}\left(\frac{x^q e^{t_0/\tau}}{1-x} - \frac{x_\times^q e^{t_0/\tau_\times}}{1-x_\times}\right)\end{aligned} \tag{D38}$$

and similarly:

$$\begin{aligned}\left\|\Psi_{\tau,\tau_\times,T|t_0}\left(t_p\right)e^{i2\pi f t_p}\right\|_N &= \frac{1}{\tau-\tau_\times}\left(\tau\left\|\Lambda_{\tau,T|t_0}\left(t_p\right)e^{i2\pi f t_p}\right\|_N - \tau_\times\left\|\Lambda_{\tau_\times,T|t_0}\left(t_p\right)e^{i2\pi f t_p}\right\|_N\right)\\ &= \frac{\theta}{\tau-\tau_\times}\left(\frac{x^q e^{t_0/\tau}}{1-xe^{i\alpha}} - \frac{x_\times^q e^{t_0/\tau_\times}}{1-x_\times e^{i\alpha}}\right)e^{iq\alpha}\end{aligned} \tag{D39}$$

where $q = \left\lceil \frac{t_0}{\theta} \right\rceil$ (Eq. (B17)). We obtain:

$$\begin{cases} z_N\left[\Psi_{\tau,\tau_\times,T|t_0}\right] = z_N\left[\Lambda_{\tau,T}\right] z_N\left[\Lambda_{\tau_\times,T}\right]\Omega\left(\tau,\tau_\times,t_0\right)e^{iq\alpha}\\ \Omega\left(\tau,\tau_\times,t_0\right) = \dfrac{x^q e^{t_0/\tau}\left(1-x_\times e^{i\alpha}\right) - x_\times^q e^{t_0/\tau_\times}\left(1-xe^{i\alpha}\right)}{x^q e^{t_0/\tau}\left(1-x_\times\right) - x_\times^q e^{t_0/\tau_\times}\left(1-x\right)}\end{cases} \tag{D40}$$

which once again shows that the discrete phasor of the *continuous* cyclic convolution of two *T*-periodic functions is in general different from the product of their discrete phasors.

Note however that if $t_0 = q\theta$ (*i.e.* the offset corresponds to the start of a gate), the expression of factor Ω in Eq. (D40) simplifies into:

$$t_0 = q\theta \Rightarrow \Omega\left(\tau,\tau_\times,t_0\right) = e^{i\alpha} \tag{D41}$$

which yield the following simple result:

$$\begin{aligned} t_0 = q\theta \Rightarrow z_N\left[\Psi_{\tau,\tau_\times,T|t_0}\right] &= z_N\left[\Lambda_{\tau,T}\right] z_N\left[\Lambda_{\tau_\times,T}\right]e^{i(q+1)\alpha}\\ &= z_N\left[\Lambda_{\tau,T|t_0}\right] z_N\left[\Lambda_{\tau_\times,T}\right]e^{i\alpha}\\ &= z_N\left[\Lambda_{\tau,T}\right] z_N\left[\Lambda_{\tau_\times,T|t_0}\right]e^{i\alpha}\end{aligned} \tag{D42}$$

This result shows that when the decay offset corresponds to the start of one of the gates, the resulting discrete phasor is obtained as the product of the discrete phasor of the PSED, $z_N\left[\Lambda_{\tau,T}\right]$, and the discrete phasor of the 'instrument response function', $z_N\left[\Lambda_{\tau_\times,T|t_0}\right]$ multiplied by a constant, $\kappa = e^{i\alpha}$ (weak discrete phasor convolution rule). As a consequence, 'standard' calibration of the phasor of this type of decays will work as intended.

In the general case, the values of factor Ω in Eq. (D40) when $\tau \to 0$ and $\tau \to \infty$ are given by:

$$\begin{cases}\Omega\left(0,\tau_\times,t_0\right) = 1\\ \Omega\left(\infty,\tau_\times,t_0\right) = \dfrac{\left(1-x_\times e^{i\alpha}\right) - x_\times^q e^{t_0/\tau_\times}\left(1-e^{i\alpha}\right)}{1-x_\times}\end{cases} \tag{D43}$$

D.8. Discrete phasor of the square-gated decay

D.8.1. No offset

The discrete phasor of $\Psi_{\tau,\tau_\times,T,W}(t)$ is given by:

$$z_{N[W]}\left[\Psi_{\tau,\tau_\times,T}\right] \triangleq z_N\left[\Psi_{\tau,\tau_\times,T,W}\right] = \frac{\left\|\Psi_{\tau,\tau_\times,T,W}\left(t_p\right)e^{i2\pi ft_p}\right\|_N}{\left\|\Psi_{\tau,\tau_\times,T,W}\left(t_p\right)\right\|_N} \tag{D44}$$

The two terms involved in this expression can be calculated based on the definition of $\Psi_{\tau,\tau_\times,T,W}(t)$ (Eq. (D19)):

$$\begin{cases} \left\|\Psi_{\tau,\tau_\times,T,W}\left(t_p\right)\right\|_N = \dfrac{1}{\tau-\tau_\times}\left(\tau\left\|\Lambda_{\tau,T,W}\left(t_p\right)\right\|_N - \tau_\times\left\|\Lambda_{\tau_\times,T,W}\left(t_p\right)\right\|_N\right) \\ \left\|\Psi_{\tau,\tau_\times,T,W}\left(t_p\right)e^{i2\pi ft_p}\right\|_N = \dfrac{1}{\tau-\tau_\times}\left(\tau\left\|\Lambda_{\tau,T,W}\left(t_p\right)e^{i2\pi ft_p}\right\|_N - \tau_\times\left\|\Lambda_{\tau_\times,T,W}\left(t_p\right)e^{i2\pi ft_p}\right\|_N\right) \end{cases} \tag{D45}$$

The necessary expressions have been calculated in Section B.2 (Eqs. (B32) & (B36)). We obtain:

$$\begin{cases} \left\|\Psi_{\tau,\tau_\times,T,W}\left(t_p\right)\right\|_N = \theta\left\{(k+1)N - r + \dfrac{1}{\tau-\tau_\times}\left(\tau\dfrac{1-\beta}{1-x} - \tau_\times\dfrac{1-\beta_\times}{1-x_\times}\right)\right\} \\ \left\|\Psi_{\tau,\tau_\times,T,W}\left(t_p\right)e^{i2\pi ft_p}\right\|_N = \theta\left\{-\dfrac{1-e^{ir\alpha}}{1-e^{i\alpha}} + \dfrac{1}{\tau-\tau_\times}\left(\tau\dfrac{1-\beta e^{ir\alpha}}{1-xe^{i\alpha}} - \tau_\times\dfrac{1-\beta_\times e^{ir\alpha}}{1-x_\times e^{i\alpha}}\right)\right\} \\ r = \left\lceil\dfrac{T-\omega}{\theta}\right\rceil \\ y = e^{-T/\tau};\ y_\times = e^{-T/\tau_\times};\ u = e^{-\omega/\tau};\ u_\times = e^{-\omega/\tau_\times};\ \beta = u\ x^r y^{-1};\quad \beta_\times = u_\times x_\times^r y_\times^{-1} \end{cases} \tag{D46}$$

which leads to the following expression for $z_{N[W]}\left[\Psi_{\tau,\tau_\times,T}\right]$:

$$\begin{cases} z_{N[W]}\left[\Psi_{\tau,\tau_\times,T}\right] = \dfrac{-\dfrac{1-e^{ir\alpha}}{1-e^{i\alpha}} + \dfrac{1}{\tau-\tau_\times}\left(\tau\dfrac{1-\beta e^{ir\alpha}}{1-xe^{i\alpha}} - \tau_\times\dfrac{1-\beta_\times e^{ir\alpha}}{1-x_\times e^{i\alpha}}\right)}{(k+1)N - r + \dfrac{1}{\tau-\tau_\times}\left(\tau\dfrac{1-\beta}{1-x} - \tau_\times\dfrac{1-\beta_\times}{1-x_\times}\right)} \\ k = \left\lfloor\dfrac{W}{T}\right\rfloor;\ r = \left\lceil\dfrac{T-\omega}{\theta}\right\rceil \\ y = e^{-T/\tau};\ y_\times = e^{-T/\tau_\times};\ u = e^{-\omega/\tau};\ u_\times = e^{-\omega/\tau_\times};\ \beta = u\ x^r y^{-1};\quad \beta_\times = u_\times x_\times^r y_\times^{-1} \end{cases} \tag{D47}$$

The phasor values for $\tau = 0$ and $\tau \to \infty$ are given by:

$$\begin{cases} z_{N[W]}\left[\Psi_{0,\tau_\times,T}\right] = \dfrac{-\dfrac{1-e^{ir\alpha}}{1-e^{i\alpha}} + \dfrac{1-\beta_\times e^{ir\alpha}}{1-x_\times e^{i\alpha}}}{(k+1)N - r + \dfrac{1-\beta_\times}{1-x_\times}} \\ z_{N[W]}\left[\Psi_{\infty,\tau_\times,T}\right] = 0 \end{cases} \tag{D48}$$

In the special case where W is a multiple of θ, we have seen that $\beta = \beta_\times = 1$, which leads to some simplifications for Eq. (D47):

$$z_{N[W]}\left[\Psi_{\tau,\tau_\times,T}\right] = -\frac{1}{(k+1)N - r}\frac{1-e^{ir\alpha}}{1-e^{i\alpha}}\left[1 - \frac{1-e^{i\alpha}}{\tau-\tau_\times}\left(\frac{\tau}{1-xe^{i\alpha}} - \frac{\tau_\times}{1-x_\times e^{i\alpha}}\right)\right] \tag{D49}$$

where we recognize the phasor $z_N\left[\overline{\Pi}_{W,nT}\right]$ (Eq. (C36)) as a prefactor, but the remainder of the expression is not factorizable with $z_N\left[\Psi_{\tau,\tau_\times,T}\right]$ (Eq. (D32)):

$$z_{N[W]}\left[\Psi_{\tau,\tau_\times,T}\right] = z_N\left[\overline{\Pi}_{W,nT}\right]\left[1 - \frac{1-e^{i\alpha}}{\tau-\tau_\times}\left(\frac{\tau}{1-xe^{i\alpha}} - \frac{\tau_\times}{1-x_\times e^{i\alpha}}\right)\right] \tag{D50}$$

D.8.2. With offset

The discrete phasor of $\Psi_{\tau,\tau_\times,T,W|t_0}(t)$ is given by:

$$z_{N[W]}\left[\Psi_{\tau,\tau_\times,T|t_0}\right] \triangleq z_N\left[\Psi_{\tau,\tau_\times,T,W|t_0}\right] = \frac{\left\langle \Psi_{\tau,\tau_\times,T,W|t_0}(t_p)e^{i2\pi f t_p}\right\rangle_N}{\left\langle \Psi_{\tau,\tau_\times,T,W|t_0}(t_p)\right\rangle_N} \tag{D51}$$

The two terms involved in this expression can be computed based on the definition of $\Psi_{\tau,\tau_\times,T,W|t_0}(t)$ (Eq. (D21)):

$$\begin{cases} \left\|\Psi_{\tau,\tau_\times,T,W|t_0}(t_p)\right\|_N = \dfrac{1}{\tau-\tau_\times}\left(\tau\left\|\Lambda_{\tau,T,W|t_0}(t_p)\right\|_N - \tau_\times\left\|\Lambda_{\tau_\times,T,W|t_0}(t_p)\right\|_N\right) \\ \left\|\Psi_{\tau,\tau_\times,T,W|t_0}(t_p)e^{i2\pi f t_p}\right\|_N = \dfrac{1}{\tau-\tau_\times}\left(\tau\left\|\Lambda_{\tau,T,W|t_0}(t_p)e^{i2\pi f t_p}\right\|_N - \tau_\times\left\|\Lambda_{\tau_\times,T,W|t_0}(t_p)e^{i2\pi f t_p}\right\|_N\right) \end{cases} \tag{D52}$$

The necessary expressions have been calculated in Section B.2 (Eq. (B60)). We obtain:

$$\begin{cases} z_{N[W]}\left[\Psi_{\tau,\tau_\times,T|t_0}\right] = \dfrac{\dfrac{e^{ir\alpha}-e^{iq\alpha}}{1-e^{i\alpha}} + \dfrac{1}{\tau-\tau_{*\times}}\left(\tau e^{t_0/\tau}\,\dfrac{x^q e^{iq\alpha} - ux^r e^{ir\alpha}}{1-xe^{i\alpha}} - \tau_\times e^{t_0/\tau_\times}\,\dfrac{x_\times^q e^{iq\alpha} - u_\times x_\times^r e^{ir\alpha}}{1-x_\times e^{i\alpha}}\right)}{kN + q - r + \dfrac{1}{\tau-\tau_\times}\left(\tau e^{t_0/\tau}\,\dfrac{x^q - ux^r}{1-x} - \tau_\times e^{t_0/\tau_\times}\,\dfrac{x_\times^q - u_\times x_\times^r}{1-x_\times}\right)} \\ q = \left\lceil \dfrac{t_0}{\theta} \right\rceil;\ r = \left\lceil \dfrac{t_0-\omega}{\theta} \right\rceil \\ y = y(\tau) = e^{-T/\tau};\ y_\times = y(\tau_\times) = e^{-T/\tau_\times};\ u = u(\tau) = e^{-\omega/\tau};\ u_\times = u(\tau_\times) = e^{-\omega/\tau_\times} \end{cases} \tag{D53}$$

Notice that there are several terms involving the offset t_0 in this expression, preventing any kind of simplification.
The phasor values for $\tau = 0$ and $\tau \to \infty$ are given by:

$$\begin{cases} z_{N[W]}\left[\Psi_{0,\tau_\times,T|t_0}\right] = \dfrac{\dfrac{e^{ir\alpha}-e^{iq\alpha}}{1-e^{i\alpha}} + e^{t_0/\tau_\times}\dfrac{x_\times^q e^{iq\alpha} - u_\times x_\times^r e^{ir\alpha}}{1-x_\times e^{i\alpha}}}{kN+q-r+e^{t_0/\tau_\times}\dfrac{x_*^q - u_\times x_\times^r}{1-x_\times}} \\ z_{N[W]}\left[\Psi_{\infty,\tau_\times,T|t_0}\right] = 0 \end{cases} \tag{D54}$$

Appendix E: SEPL computation from an experimental IRF curve

As discussed in Section 8.3.5, in the general experimental case, the exact functional form of the IRF might not be known, or if known, not match any of the examples detailed in this article. In these cases, it may be useful to compute pseudo-calibrated phasors as discussed in that section, and relate them to the loci of PSEDs. By definition, the pseudo-calibration factor is given by the computed phasor (which will assume to be a discrete phasor, since it is an experimental one), Eq. (88):

$$z_N\left[I_T \underset{T}{*} \Lambda_{0,T}\right] = z_N\left[I_T\right] = \frac{\left\| I_T\left(t_p\right) e^{i2\pi f t_p}\right\|_N}{\left\| I_T\left(t_p\right)\right\|_N} \tag{E1}$$

The phasors of PSEDs with nonzero lifetimes are then given by:

$$z_N\left[I_T \underset{T}{*} \Lambda_{\tau,T}\right] = \frac{\left\| I_T \underset{T}{*} \Lambda_{\tau,T}\left(t_p\right) e^{i2\pi f t_p}\right\|_N}{\left\| I_T \underset{T}{*} \Lambda_{\tau,T}\left(t_p\right)\right\|_N} \tag{E2}$$

where the continuous cyclic convolution product involved in Eq. (E2) is formally given by:

$$I_T \underset{T}{*} \Lambda_{\tau,T}\left(t_p\right) = \int_0^T du\, I_T(u)\, \Lambda_{\tau,T}\left(t_p - u\right) \tag{E3}$$

This integral requires knowledge of $I_T(t)$ over the whole interval [0, *T*], but in practice, only a finite number of values $\left\{I_T\left(t_p\right)\right\},\ p = 1,\ldots,N$ are known. $\Lambda_{\tau,T}(t)$ given by Eq. (17) can be computed for any value of *t*.

Although this lack of information is preventing an exact calculation of Eq. (E3), any numerical method to approximate this integral can be used instead. For instance, the Phasor Explorer software uses a Lobatto quadrature algorithm when choosing an adaptive integration option or a simple trapezoidal rule when using a fixed number of integration steps to compute the integral.

Computation of Eq. (E2) for a sufficient number of PSEDs (for instance the default in Phasor Explorer is to compute 1,000 PSED phasors with lifetimes logarithmically spaced between 1 ps and 1 μs) will provide a good representation of the SEPL.

Appendix F: Phasor Explorer and other software

F.1. Overview

This Appendix provides a brief overview of the Phasor Explorer software capabilities. Most calculations reported here were performed with it. A software installer can be found at https://sites.google.com/a/g.ucla.edu/phasor-explorer/. A detailed online manual is available on that website, as well as download and installation instructions. The LabVIEW source code is available on Github at https://doi.org/10.5281/zenodo.3884101 (reference number [38]).

Some of the curves presented in the figures were computed with *AlliGator*, a free Microsoft Windows 64 bit executable available at (https://sites.google.com/a/g.ucla.edu/alligator/) [15] together with an extensive online manual.

Publication quality graphs were generated with OriginPro 9.1 (OriginLab, Northampton, MA). The corresponding .opj file can be found in the Figshare repository mentioned in the Data Availability section (https://doi.org/10.6084/m9.figshare.11653182, reference number [37]) and opened with the free OriginViewer software (https://www.originlab.com/viewer/), which allows exporting data and figures.

The user interface consists in a main window with a top menu bar (File, Edit, Analysis, Windows). The user interface elements (graphs, buttons, numeric controls and indicators, etc.) displayed on the window can be modified with the help of a pull-down menu below the menu bar, offering the following options: 'Phasor Plot', 'Pseudo Phasor Lifetime', and 'Decay, Gate & Gated Decay'. They will be briefly discussed next.

F.2. Windows

In addition to the main window, the Windows menu gives access to 4 other windows: Notebook, Settings, Context Help and About.

F.2.1. Phasor Explorer main window

As mentioned earlier, the main windows appearance depends on the selected pull-down menu option located below the menu bar. Generally, it shows one graph and a few controls (numeric controls or indicators, check boxes). Each graph is a rich LabVIEW object allowing many interactions with each individual or groups of plots, including saving or loading them as ASCII files. Right-clicking a graph and choosing 'Copy Data' will copy a bitmap image of the Graph, which can then be pasted into the Notebook (or in another rich text format document) for documentation. Because of the finite resolution of such an image, it is generally recommended to save the raw data contained in each Graph (or individual plots) as ASCII files, and re-open them in a dedicated data analysis and visualization software for publication quality results (OriginPro was used for this purpose in this article).

F.2.2. Notebook

The Notebook window logs all user actions and allows user-typed notes to be recorded, even if the window is closed. Upon quitting the software, a reminder will be displayed, asking users whether or not they want to save the Notebook before quitting. This is the recommended choice, as it contains a lot of useful information which would be otherwise difficult if not impossible to recover from saved plot files, for instance. Graphs can be copied from the main window to the Notebook as bitmap. The Notebooks content can be saved at all times as a rich text format (RTF) file, which can then be opened by most text editors (as well as Phasor Explorer).

F.2.3. Settings

The Settings window gives access to all parameters/options controlling the different types of calculations: Display, Phasor, Decay, Gates, Detector. Once options and parameters have been set in the Settings window, calculations performed in the main window will use these parameters/options. The whole set of options/parameter is also printed in the Notebook.

F.2.4. Context Help and About

While each control and indicator generally shows a ‘tip strip’ next to it when the mouse hovers over them, a more extended description can be found in the floating Context Help window, which can be shown or hidden as needed.

The About dialog window provides legal disclaimer and copyright information.

F.3. Analysis

We provide here a brief summary of the type of analysis currently possible with *Phasor Explorer*. More details can be found in the online manual.

F.3.1. Phasor Plot

The different ‘Single-Exponential Phasor Loci’ (SEPL) discussed in this article can be reproduced by setting the gate parameters, laser period, phasor frequency and calibration lifetime in the respective panels of the Settings window, and then selecting Analysis>> Phasor Plot. This adds the corresponding plot to the graph.

Using a cursor linked to that plot, and moving it along the plotted SEPL, the three indicators labelled ‘tau_i’, ‘tau_ph’ and ‘tau_m’ will be updated according to the cursor location. ‘tau_i’ represents the actual lifetime used to compute the phasor at that location, while ‘tau_ph’ and ‘tau_m’ represent the calculated pseudo phase and modulus lifetimes corresponding to that phasor. Note that these values are computed only for the finite number of lifetime values used to build the SEPL (defined in the Display panel of the Settings window).

F.3.2. Pseudo Phase Lifetime

The dependence of the pseudo phase lifetime τ_{φ} of PSED as defined in this article can be studied using Analysis>>Pseudo Phase Lifetime Plot. The gate parameters, laser period, phasor frequency and calibration lifetime used in this calculation are those defined in the respective panels of the Settings window. A checkbox on the front panel (Compute Delta Tau) allows calculating the difference $\tau_{\varphi} - \tau$ instead.

F.3.3. Decay & Gates

The effect of gating, decay offset, truncation, etc. on the shape of the T-periodic decay (which is used to perform all other computations) can be studied by representing (i) the original PSED, (ii) the selected gate and (iii) the gated-decay using Analysis>>Decay, Gate & Gated Decay. Checkboxes on the front panel allow specifying which of the 3 curves to represent.