

Frequency ទិន្នន័យកំណត់ global information គម្រោង f នៃរៀង



គេហទ័រ

# LECTURE 06 WAVELETS AND MULTIRESOLUTION PROCESSING

A diagram consisting of three blue-outlined squares of decreasing size from left to right, connected by a blue arrow pointing upwards towards the text "LECTURE 06 WAVELETS AND MULTIRESOLUTION PROCESSING". To the right of the text, there is a small blue wavy line icon.

Punnarai Siricharoen, Ph.D.

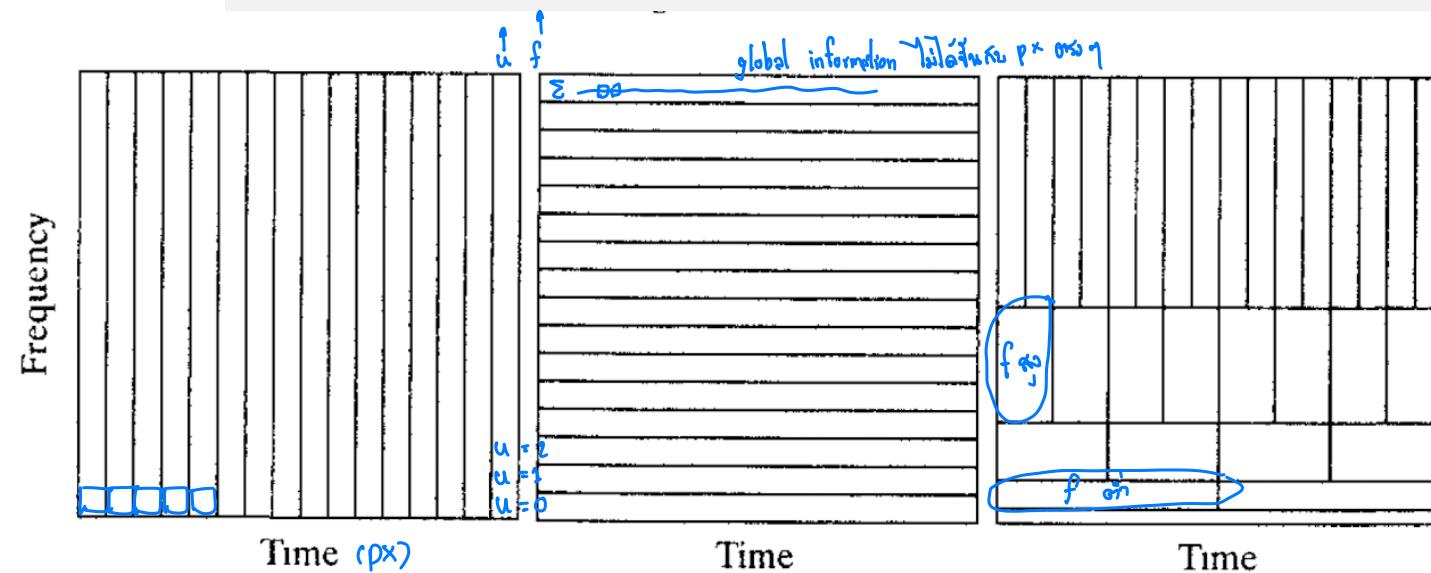
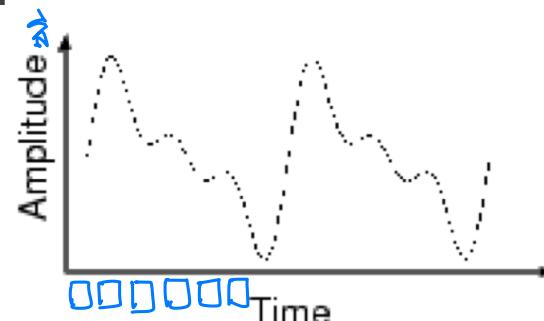
## OBJECTIVES

- To be able to understand the basic concept of wavelets (Haar wavelet)
- To be able to apply wavelets transform for multiresolution analysis

# CONTENT

- Introduction
- Image Pyramid
- Fast Wavelet Transform (1D & 2D)
- Python Exercises

# INTRODUCTION



a b c

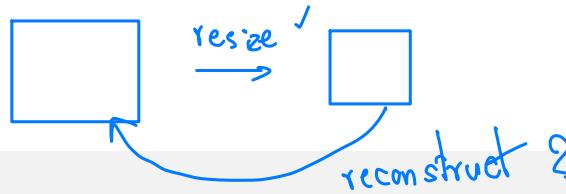
**FIGURE 7.21** Time-frequency tilings for (a) sampled data, (b) FFT, and (c) FWT basis functions.

# INTRODUCTION

- Small object, low contrast – make it higher resolution
- Large object, high contrast – look at coarse view
- Motivation for multiresolution processing
  - All of them together – study them in several resolutions



แฟนคลับมากกว่า

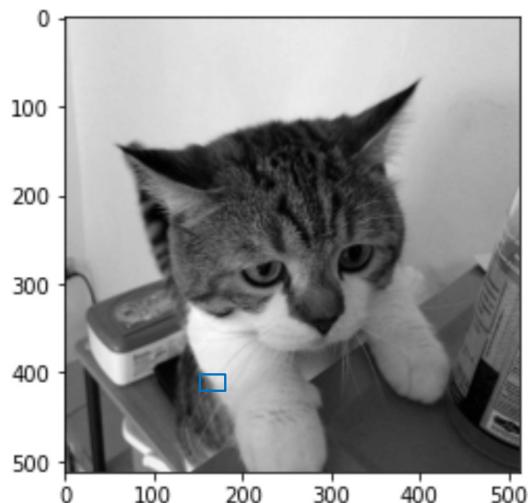


## INTRODUCTION

- Small object, low contrast – make it higher resolution
- Large object, high contrast – look at coarse view  
มองรวม
- Motivation for multiresolution processing  
ฟังก์ชันที่ต้องการ
- All of them together – study them in several resolutions

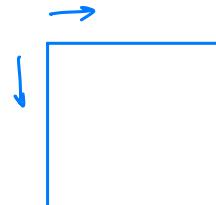
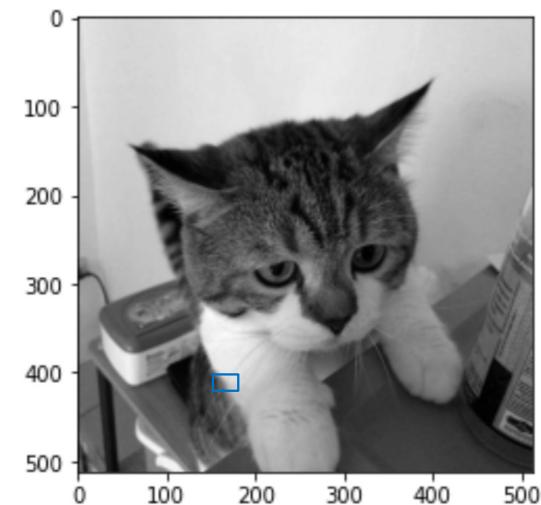


แฟนคลับแมวกล่อง



```
[ [ 67  66  70  87  98 114 138 147 151 166 180 183 180 181 186 185]
[ 56  61  75  87  96 110 127 132 138 155 174 179 178 180 184 184]
[ 57  68  85  88  89  96 108 115 124 140 156 168 174 177 178 176]
[ 55  65  77  77  77  84  98 111 122 132 149 163 171 176 179 179]
[ 48  53  63  63  70  83  97 106 114 122 136 149 157 165 172 175]
[ 43  50  57  57  65  80  89  90  98 112 118 134 149 162 171 172]
```

4



[ [ 67 66 70 87 88 114 138 147 151 166 180 183 180 181 186 185 ]
[ 56 61 75 87 86 110 127 132 138 155 174 179 178 180 184 184 ]
[ 57 68 85 88 89 96 108 115 124 140 156 168 174 177 178 176 ]
[ 55 65 77 77 77 84 98 111 122 132 149 163 171 176 179 179 ]
[ 48 53 63 63 70 83 97 106 114 122 136 149 157 165 172 175 ]
[ 43 50 57 57 65 80 89 90 98 112 118 134 149 162 171 172 ]

average

88.5

difference

88-89

-1

$$\frac{1}{2} + \frac{1}{2} = 1$$

ผลลัพธ์ = 1

convolution  
kernel downsampling  
kernel avg pooling

$[1 - 1] \rightarrow \text{downsampling } \downarrow 2$

$$\text{inner vector } 2 \neq 1$$

inner product

$$\left[ \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \right] \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} + \frac{1}{2} = 1 \checkmark$$

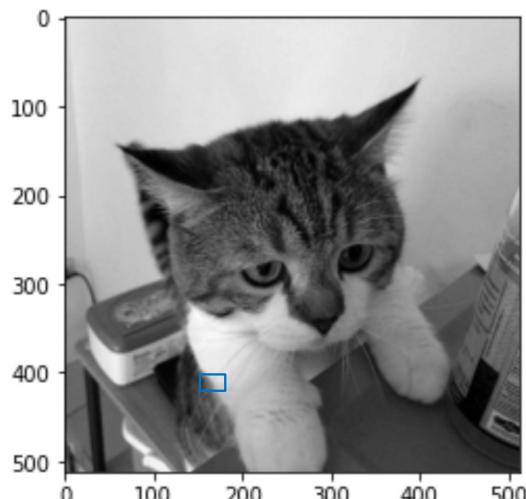
$$\left[ \frac{1}{2} \cdot \frac{1}{2} \right] \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

yellow unit vector

$$\left[ \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right] \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} - \frac{1}{2} = 0$$

white unit vector

(เดิมส่วนต่างๆ) ถ้าให้ไปหารด้วยขนาดของภาพจะได้ผลลัพธ์ที่เป็นบวก



```
[ [ 67  66  70  87  98 114 138 147 151 166 180 183 180 181 186 185]
[ 56  61  75  87  96 110 127 132 138 155 174 179 178 180 184 184]
[ 57  68  85  88  89  96 108 115 124 140 156 168 174 177 178 176]
[ 55  65  77  77  77  84  98 111 122 132 149 163 171 176 179 179]
[ 48  53  63  63  70  83  97 106 114 122 136 149 157 165 172 175]
[ 43  50  57  57  65  80  89  90  98 112 118 134 149 162 171 172]
```

average

88.5	102
77	91
66.5	90
61	84.5

difference

-1	-12
0	-14
-7	-14
-8	-9

$96 - 108$

$4 \times 4 \rightarrow 4 \times 2$

88	89	96	108
77	77	84	98
63	70	83	97
57	65	80	89

average

difference →

88.5	102
77	91
66.5	90
6	84.5

avg minus

average

difference ↓

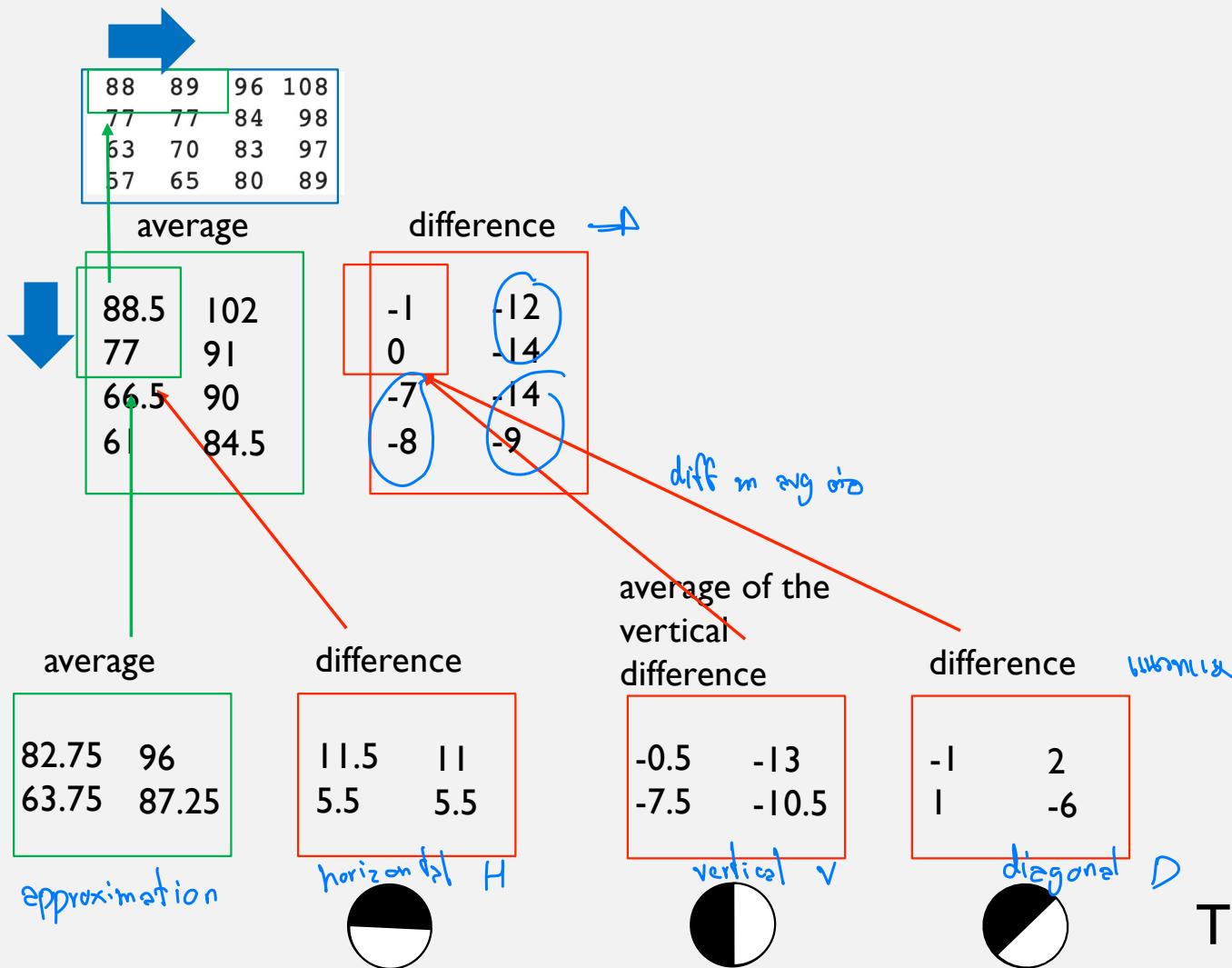
82.75	96
63.75	87.25

mini version அம்மொத்தங்கள்

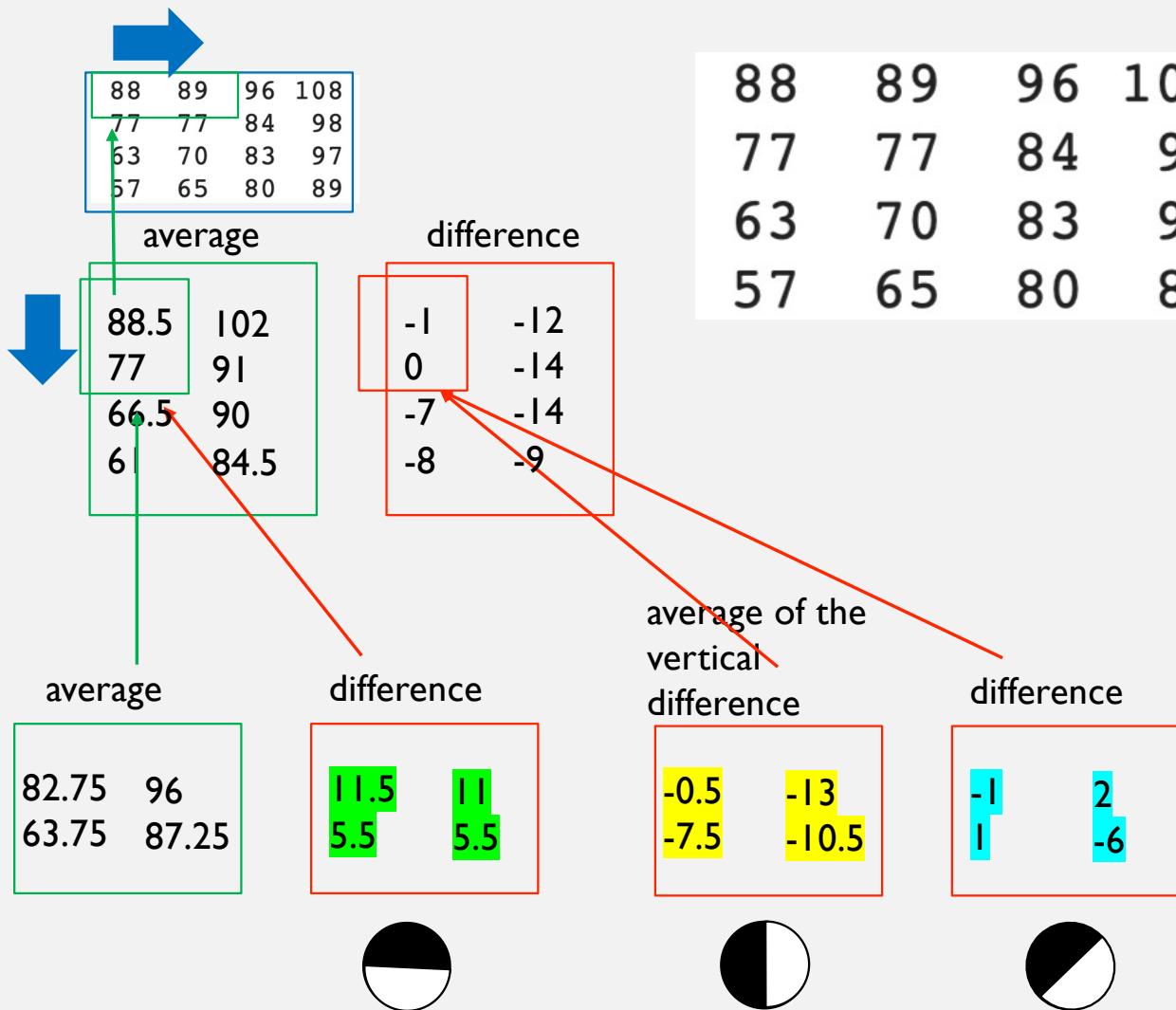
-1	-12
0	-14
-7	-14
-8	-9

horizontal difference

88.5 - 77	
11.5	11
5.5	5.5



Three species of differences

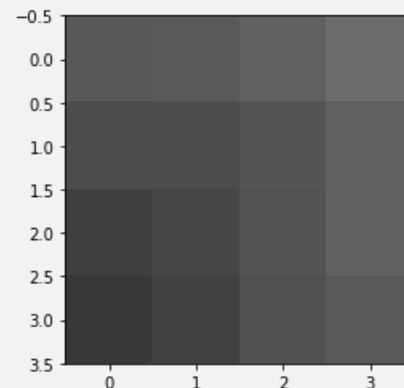


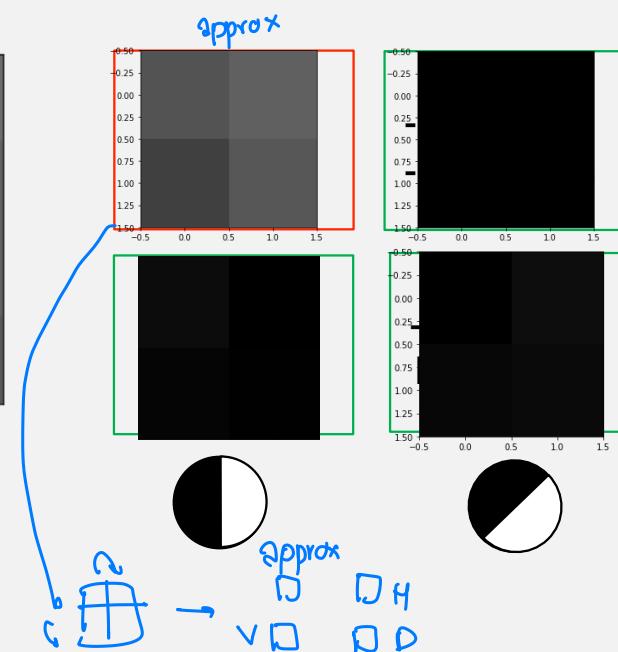
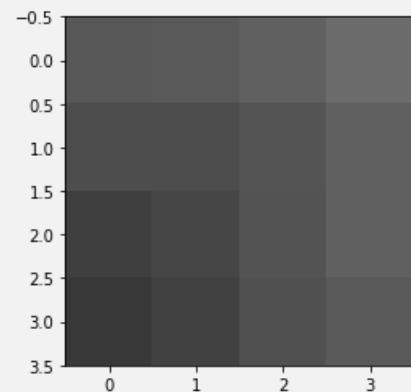
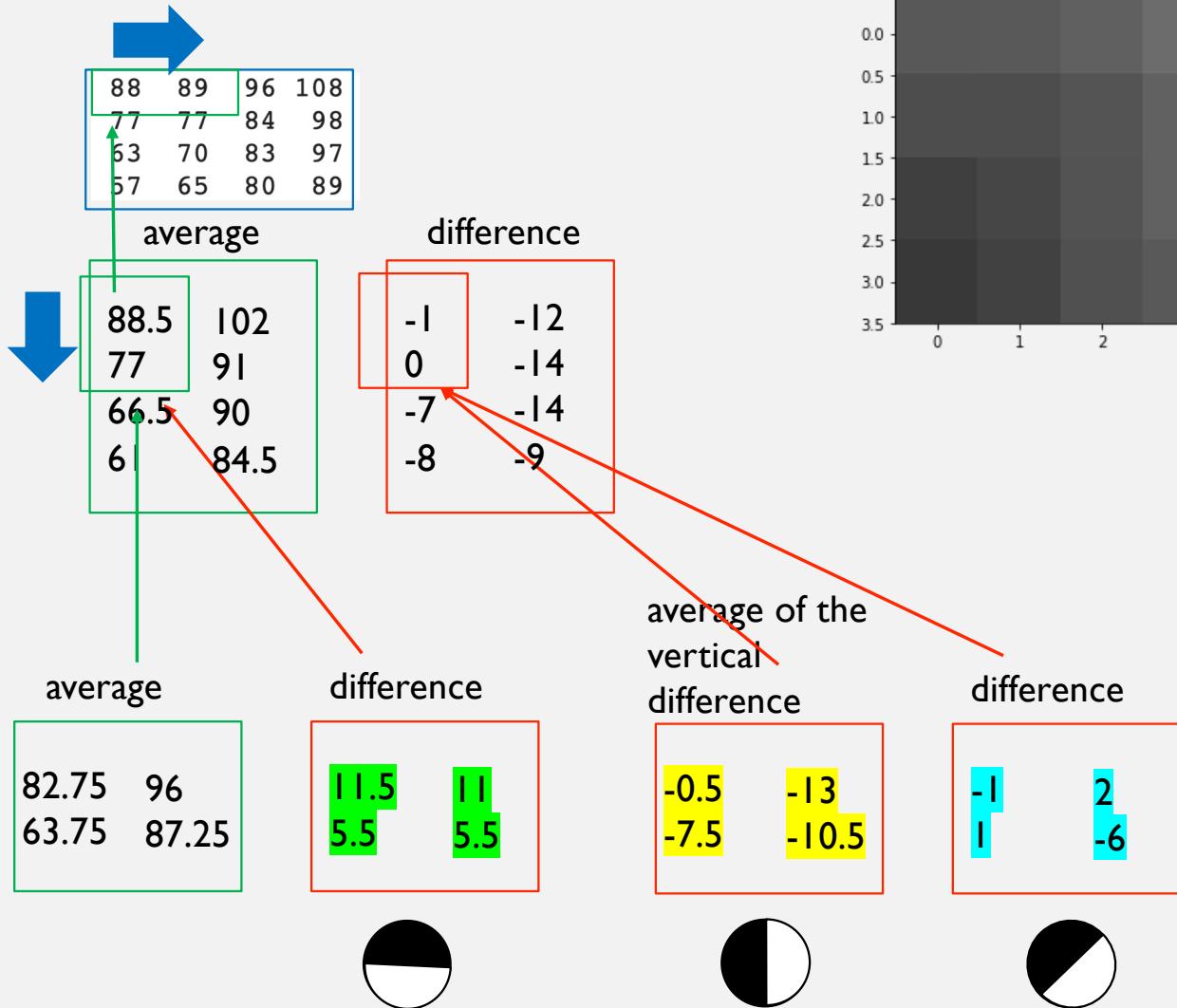
82.75	96
63.75	87.25

-0.5	-13
-7.5	-10.5

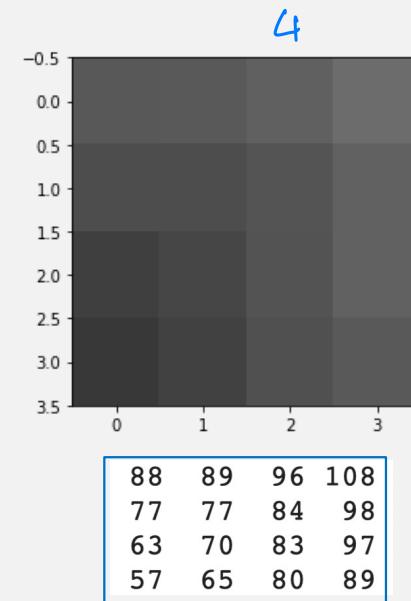
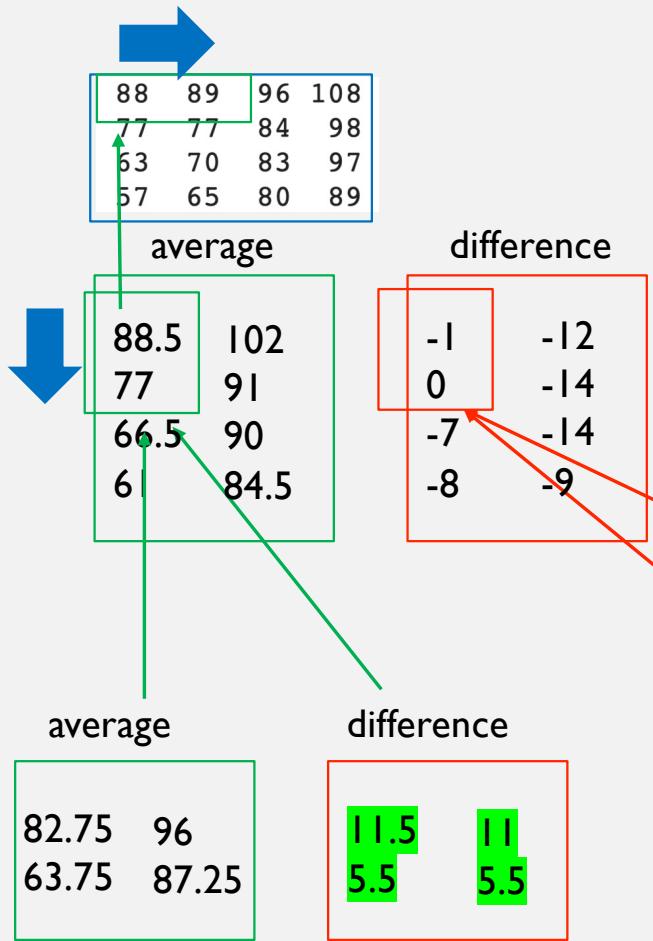
-1	2
1	-6

Three species of differences

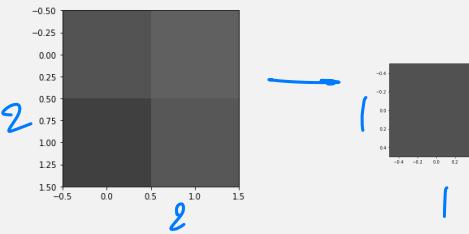




Three species of differences



model generalize မာက်။ compact အပို။



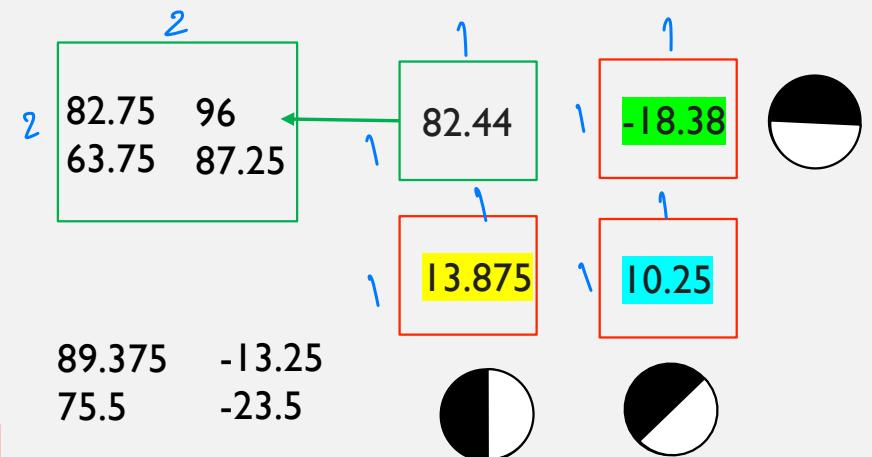
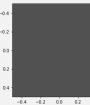
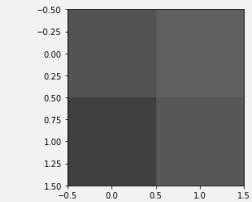
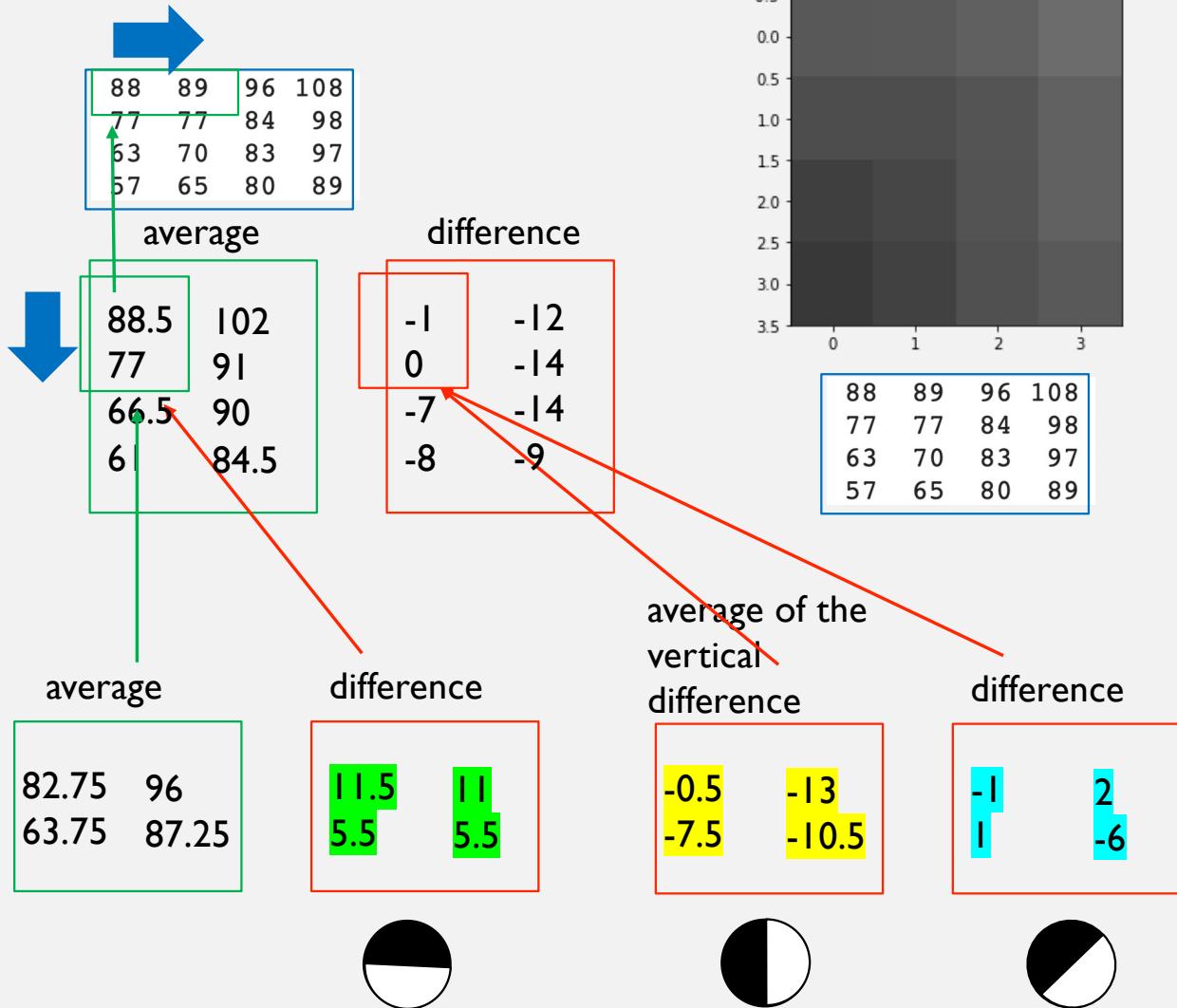
82.75  
63.75

82.44

average of the  
vertical  
difference



Three species of differences



Three species of differences



A

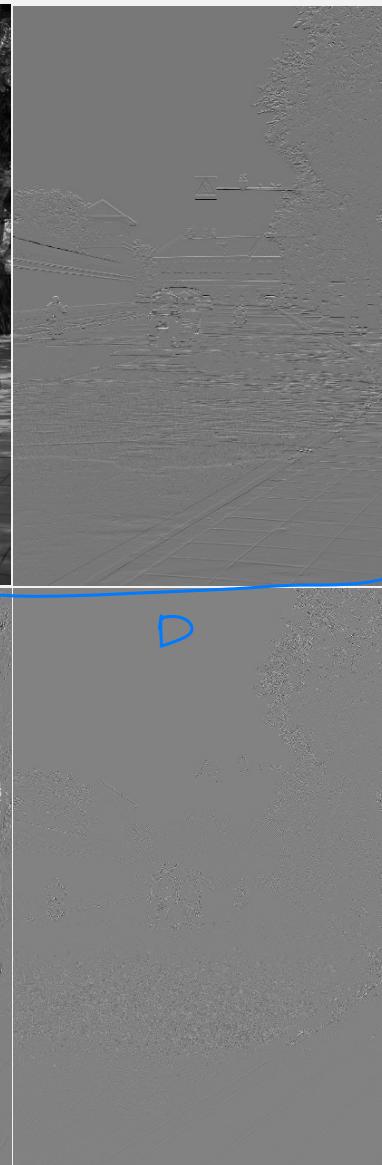


V



H

olini ของมนุษย์ในภาพ

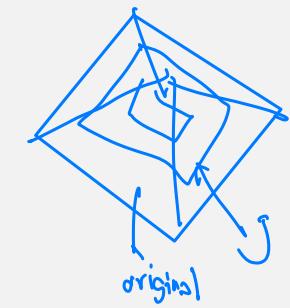
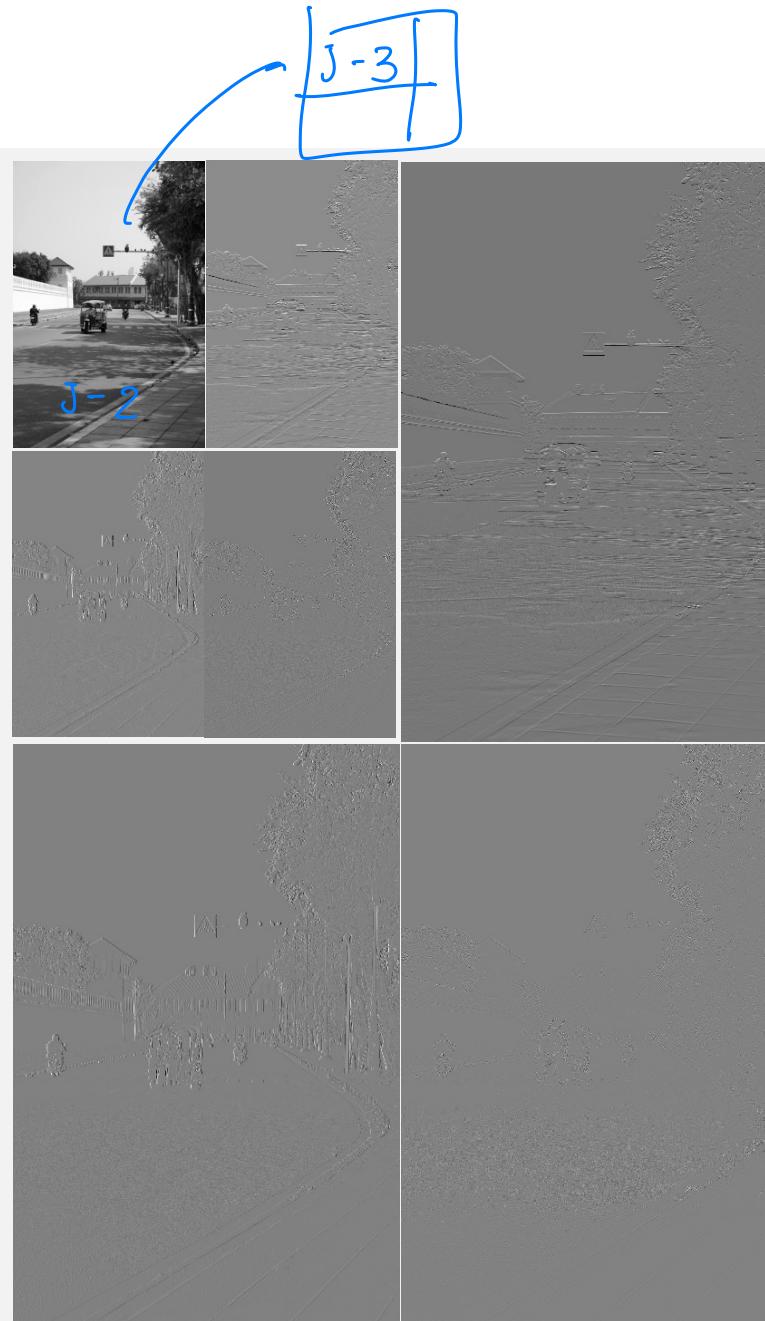


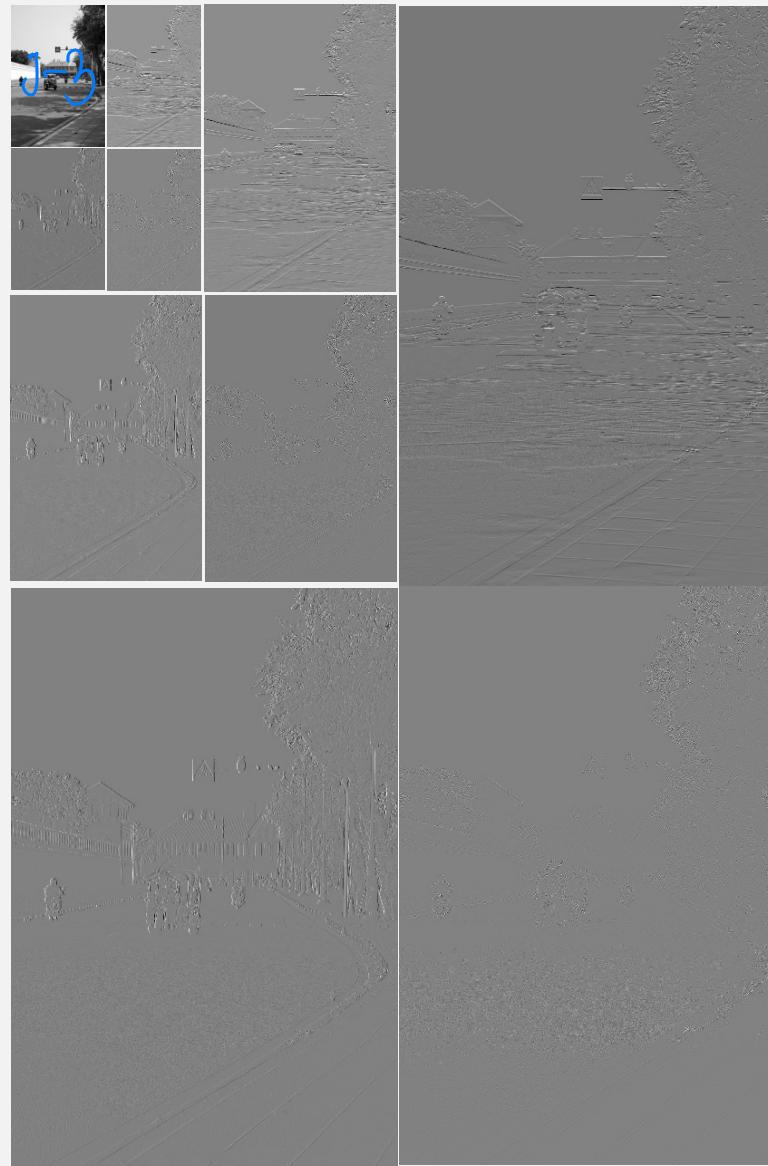
+

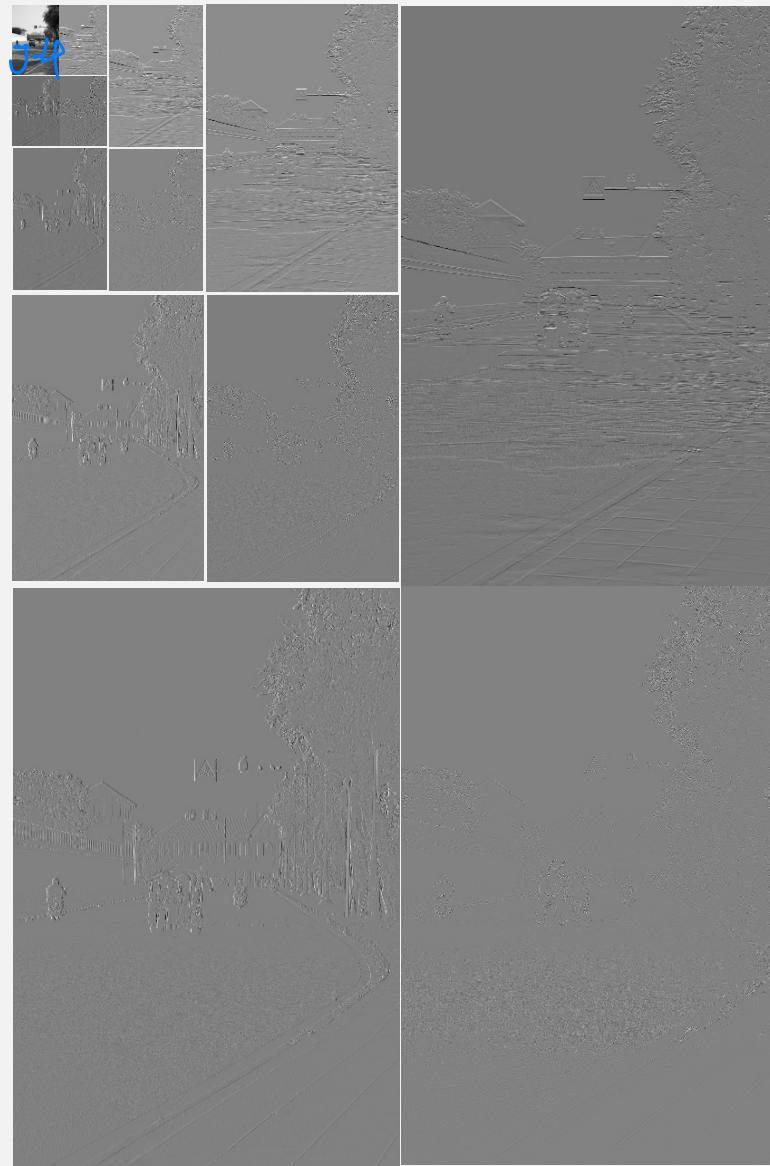
ผลลัพธ์ที่ได้ H V  
เพื่อให้มีรูปแบบของมนุษย์

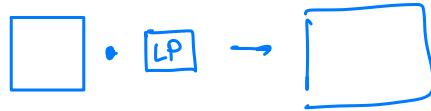
กรอง Sobel G<sub>x</sub> G<sub>y</sub>

มนุษย์

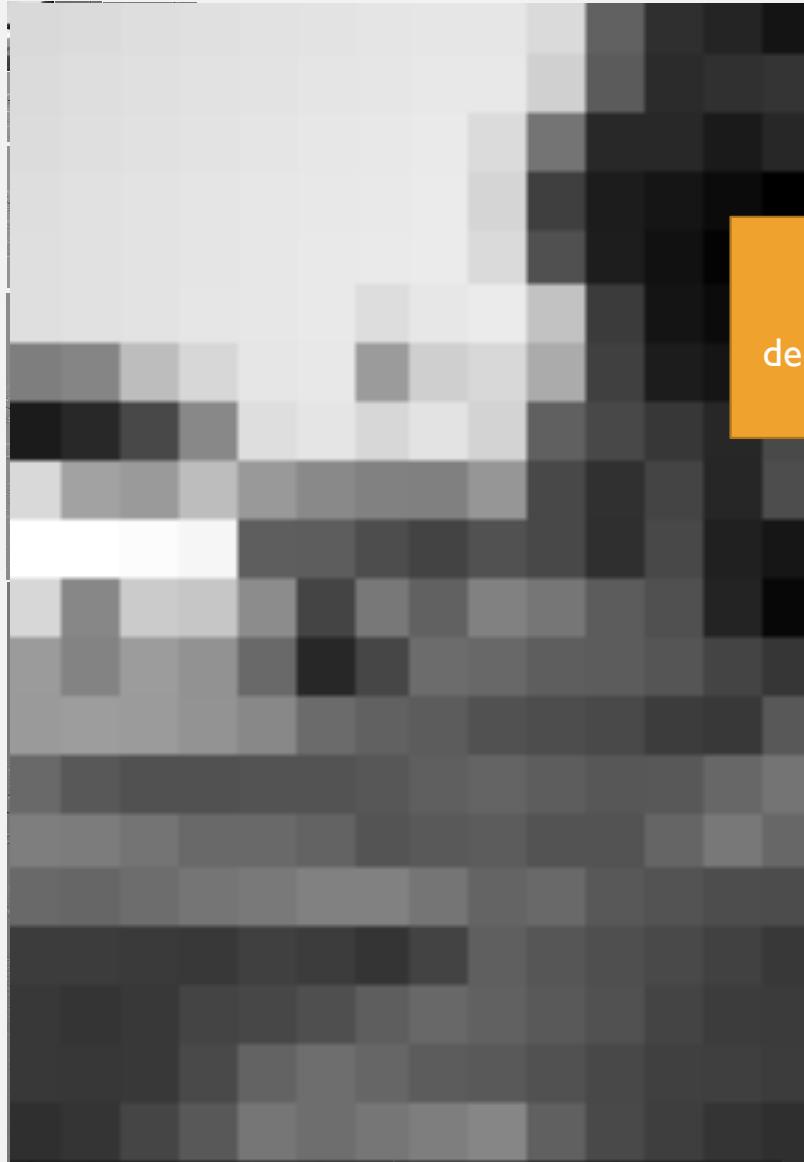








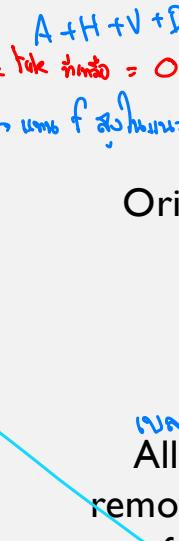
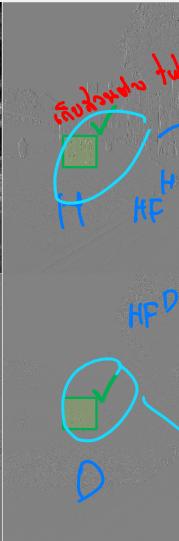
เปลี่ยนกัน กัน รากทุกอย่าง ให้ได้ ลักษณะ convol



Six-scale  
decomposition

J

J-1 (resize + avg) ເພີ້ນ LPF ດ້ວຍ f ທັງໝົດ



$$A + H + V + D = \text{original}$$

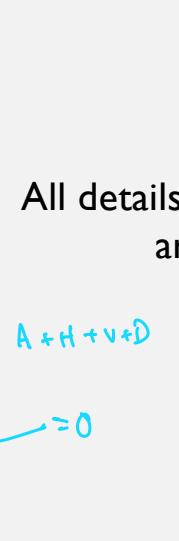
$$H + V + D = 0$$

ໃຫຍ່ທີ່ໄດ້ກຳລົງ

Original image



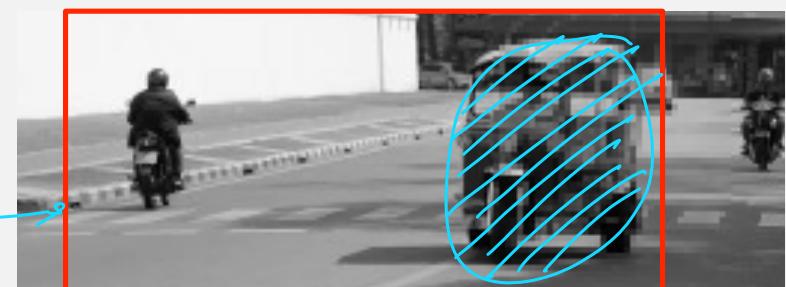
ເນັດ ມາພູດຕະ ກົງກອນ  
All details are  
removed except  
for Tuk-Tuk



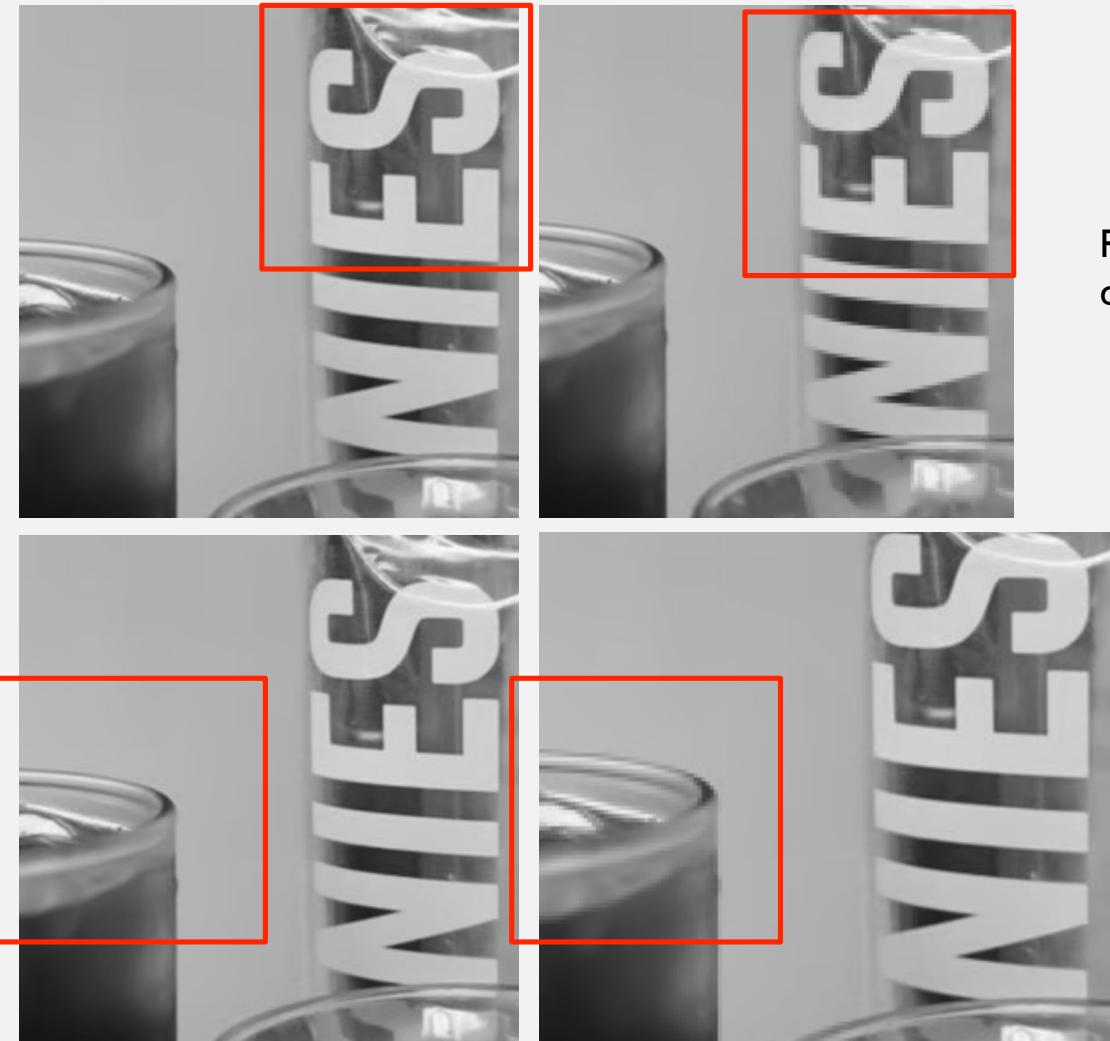
All details of Tuk-Tuk  
are removed

$$A + H + V + D =$$

$$= 0$$



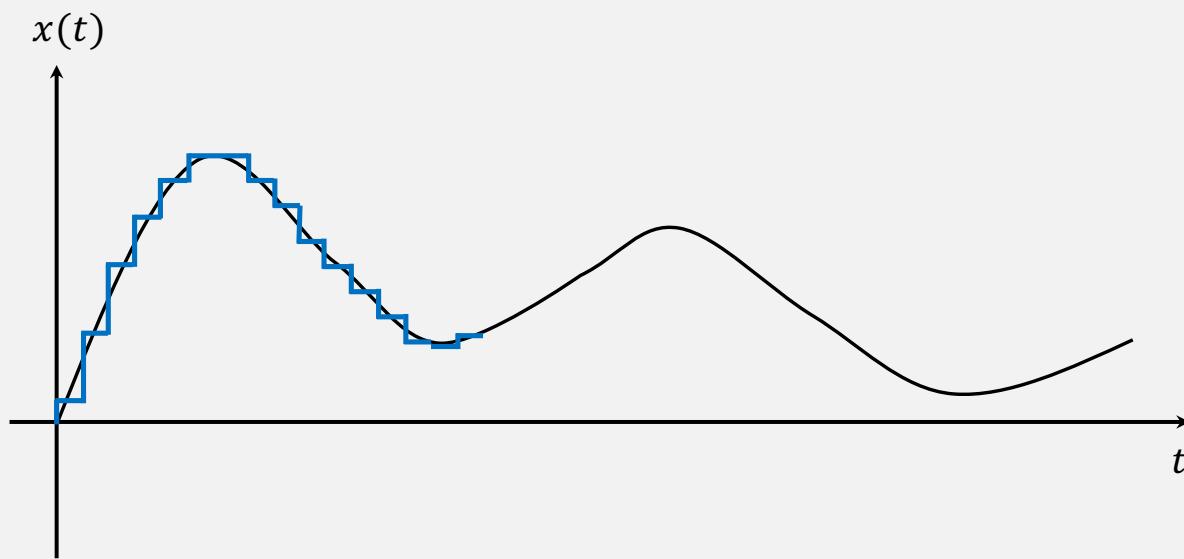
ស្ម័គ្របែនការ decompose នាយកឱតនាំវិញ 4 component A H V D

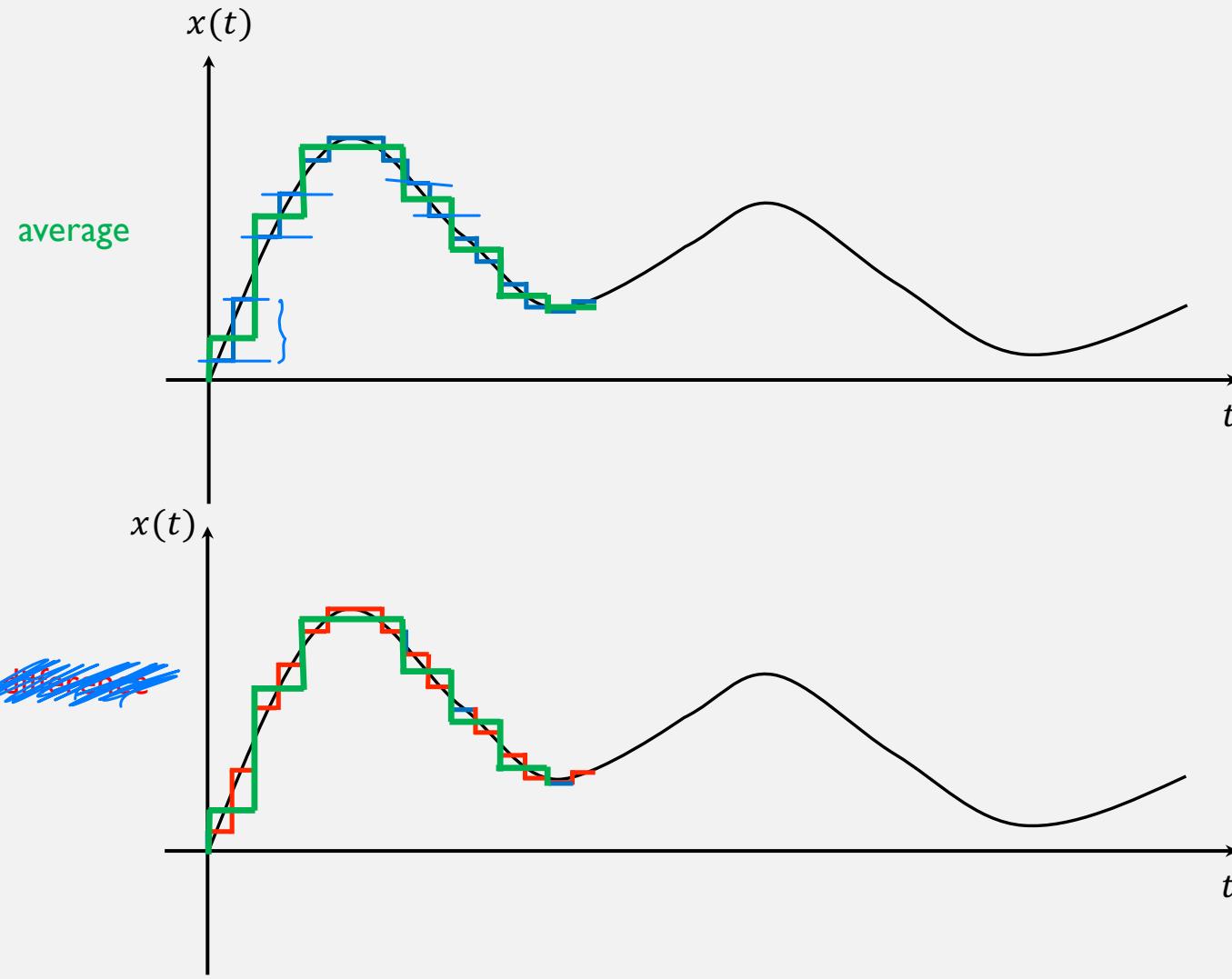


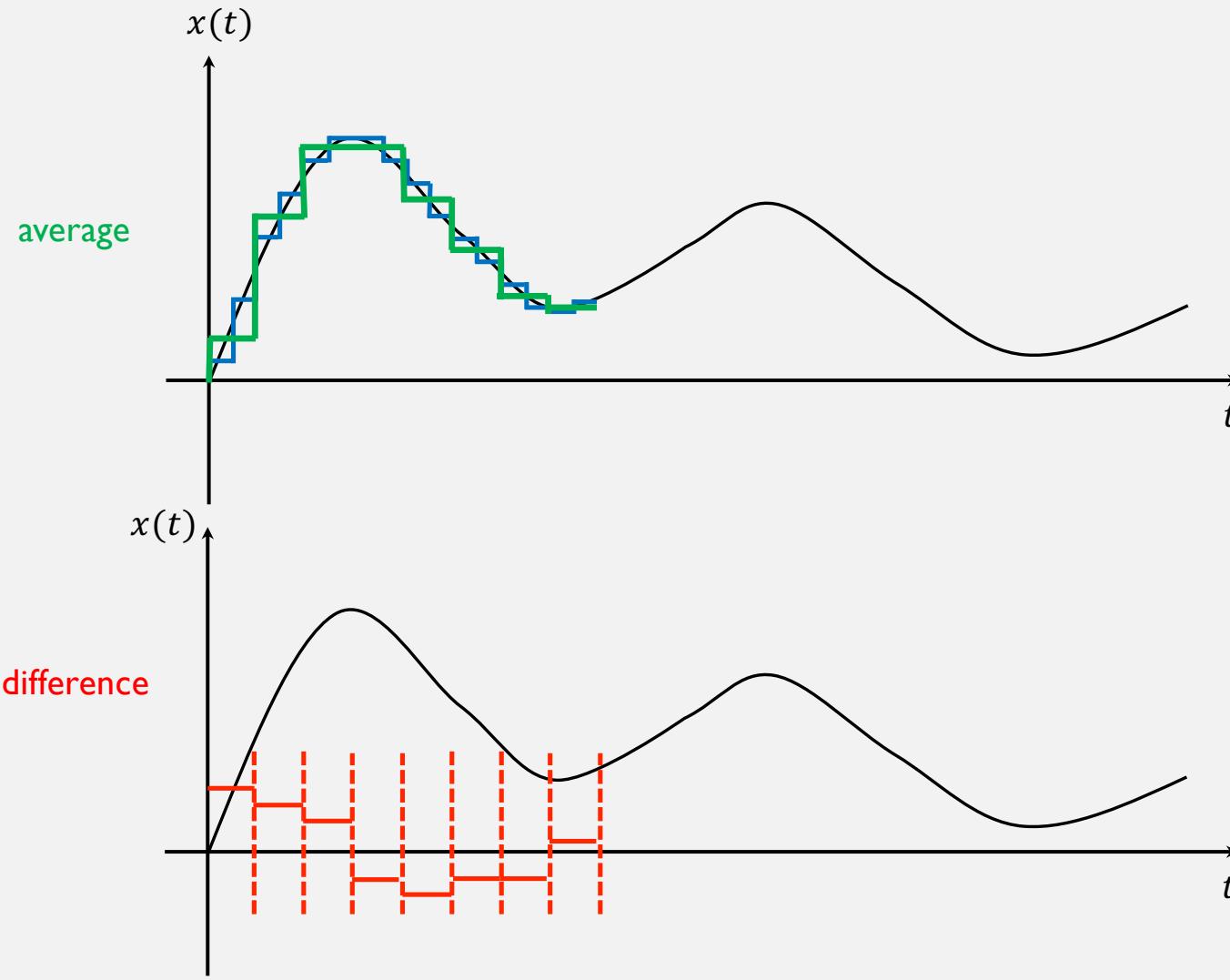
Remove "ES" details

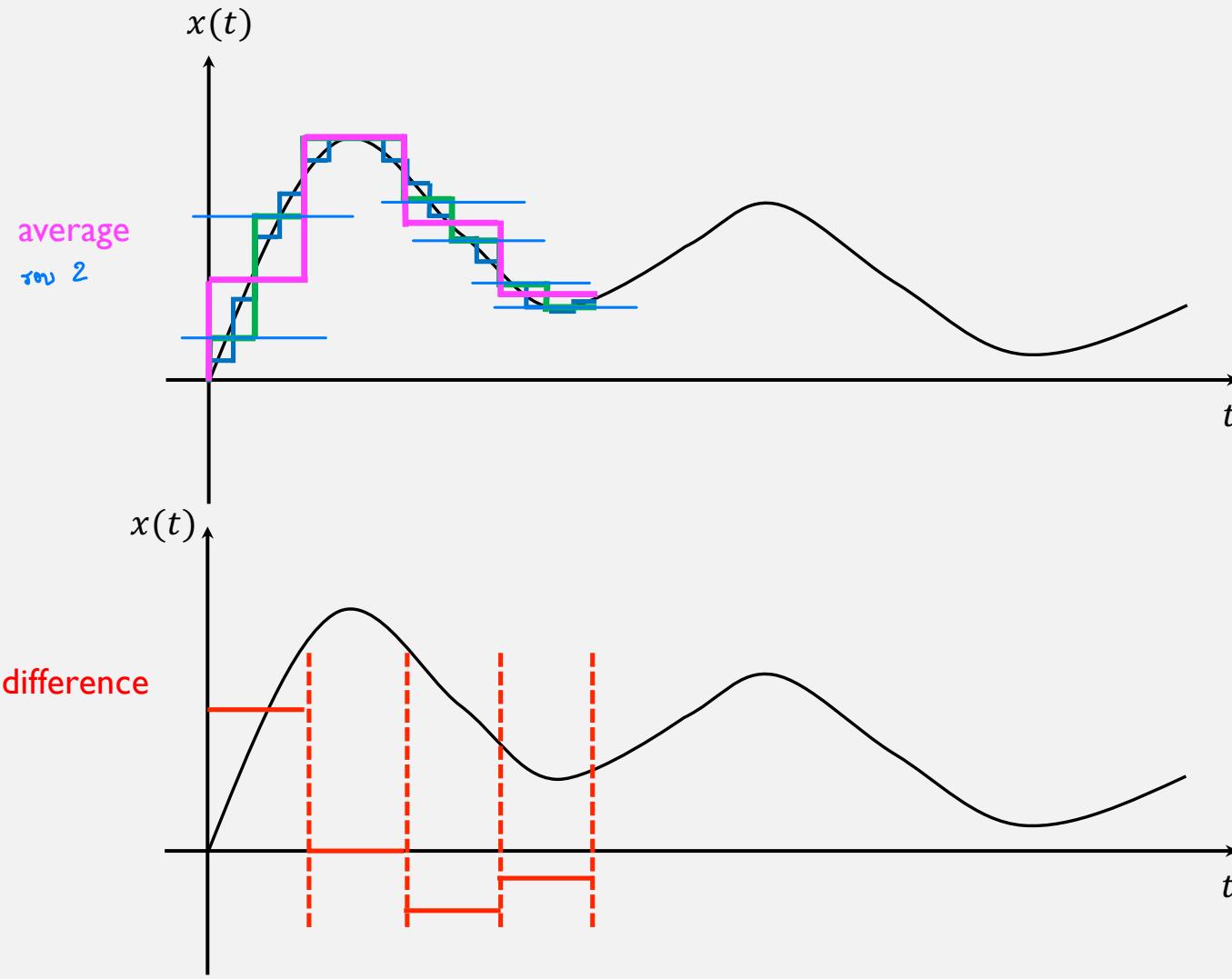
Remove edge of the glass details

S ເກືອກງານ  
Q ເອກະ









# IMAGE PYRAMIDS

- Representing an image more than one resolution
  - Base – high resolution
  - Apex – low resolution
  - Base level  $J$  is size  $2^J \times 2^J$  (or  $N \times N$ ) ( $J = \log_2 N$ )
  - Level  $j$  is size  $2^j \times 2^j$ , where  $0 \leq j \leq J$
  - Fully populated pyramid – from  $2^J \times 2^J$  to  $2^0 \times 2^0$  ( $|x|$  too small)
  - Most pyramid truncated to  $P+I$  levels ( $j = J-P, J-P+1, \dots, J$ )

$$\begin{aligned} N &= 512 \text{ px} \\ 2^J &= 2^9 \\ J &= 9 \text{ ระดับ} \\ &\quad 10 \text{ level} \end{aligned}$$

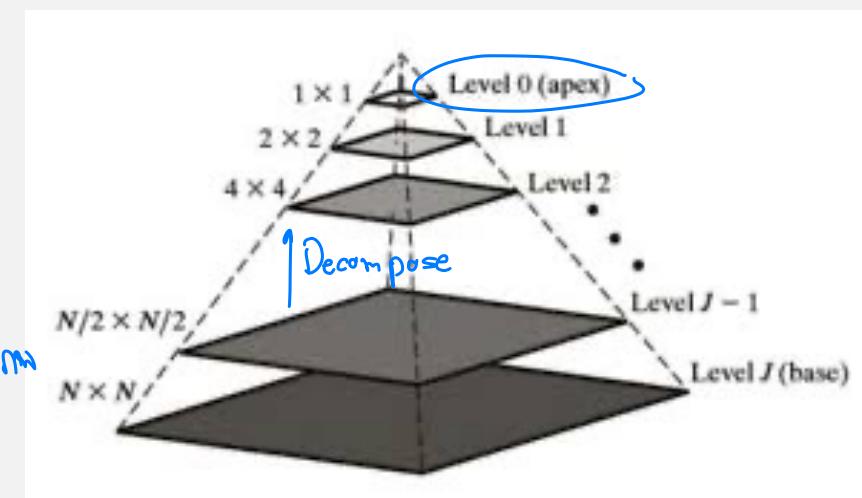
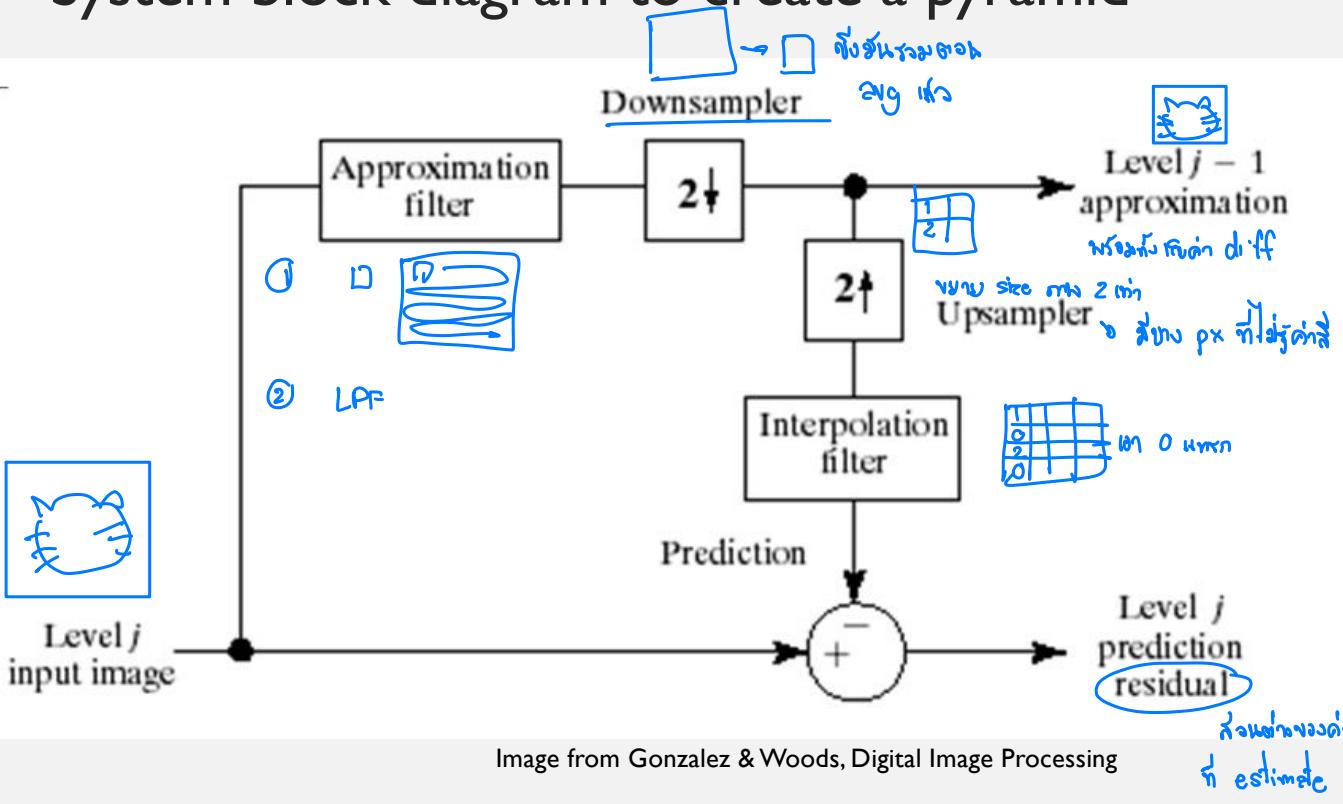


Image from Gonzalez & Woods, Digital Image Processing

พีระมิด pyramid หรือ ภาพที่มีขนาดฐาน pyramid

# IMAGE PYRAMIDS

- System block diagram to create a pyramid



**Filtering** *box filter / weight / Gaussian filter*

- averaging filtering creates a mean pyramid

- lowpass Gaussian filtering creates a Gaussian pyramid

Without filtering, aliasing can be occurred in the upper levels. (subsampling pyramid)

# IMAGE PYRAMIDS

J

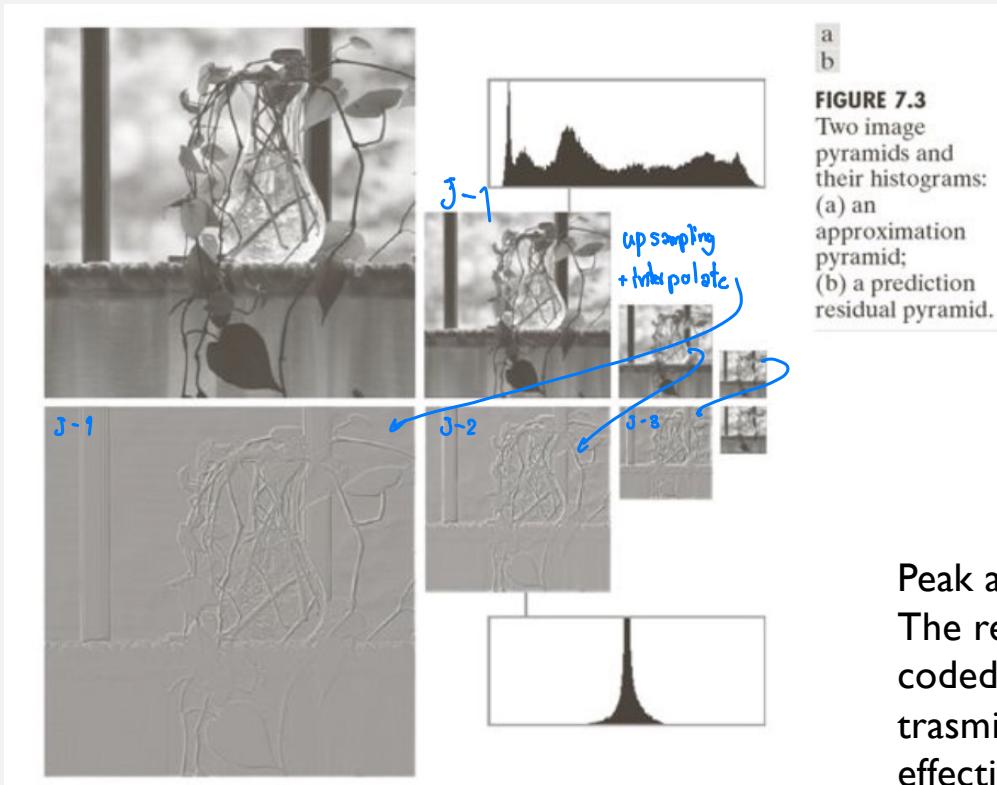


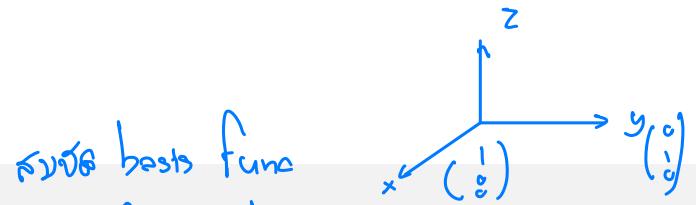
Image from Gonzalez & Woods, Digital Image Processing

Peak at around zero.  
The residuals can be  
coded, stored, and  
transmitted more  
effectively than  
approximation

Use 5x5 low pass Gaussian convolution kernel (for filtering)  
-Original image  $512 \times 512$  ( $J = 9$ )  
 $-P+1 = 4$  so  $j = 6, 7, 8, 9$

Laplacian level 6 ( $64 \times 64$ ) predicts Gaussian pyramid level 7 ( $128 \times 128$  approximation) by upsampling and filtering and add the Laplacian level 7.

cartesian coordinate, complex expo filter basis function



សុវត្ថបែនភាព

① ឯកសារឱ្យជាមួយ

② រៀងចាំ និងចំណេះដឹង

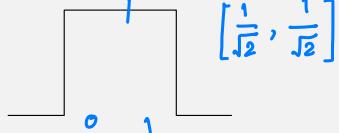
$$\langle \phi_1, \phi_2 \rangle = 0$$

## BASIS FUNCTIONS

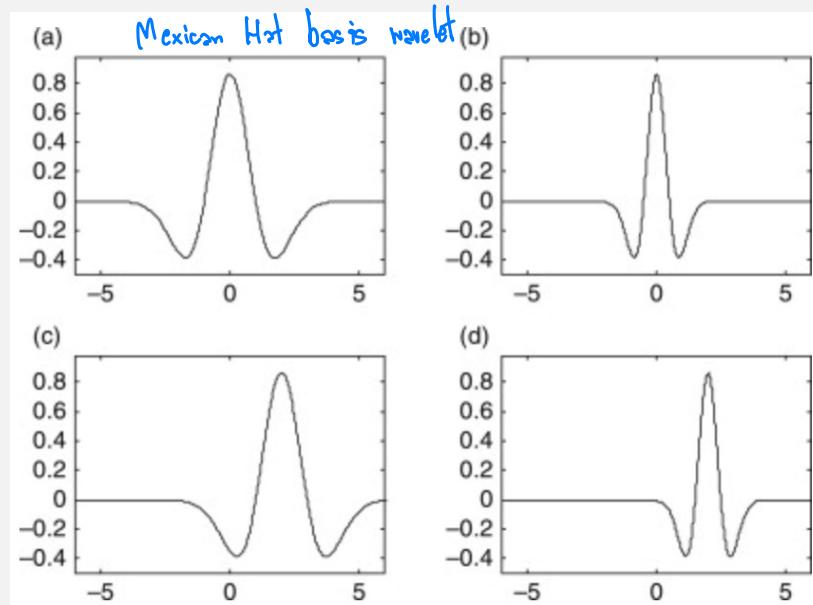
- Basis functions used by the wavelet transform are all generated from one initial function called the mother wavelet. The simplest mother wavelet is the Haar wavelet. The basis functions are orthonormal wavelets.

- $h_\varphi(n)$  - scaling function coefficients <sup>avg</sup>
- $h_\psi(n)$  - wavelet function coefficients <sup>mother wavelet  $\rightarrow$  m difference</sup>

$$h_\varphi(n) = \begin{cases} 1/\sqrt{2} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$



$$h_\psi(n) = \begin{cases} 1/\sqrt{2} & n = 0 \\ -1/\sqrt{2} & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$



The Mexican Hat wavelet. (a) Mother wavelet, (b) dilation by a factor of 2, (c) translation of 4 to the right, and (d) combination of (b) and (c).

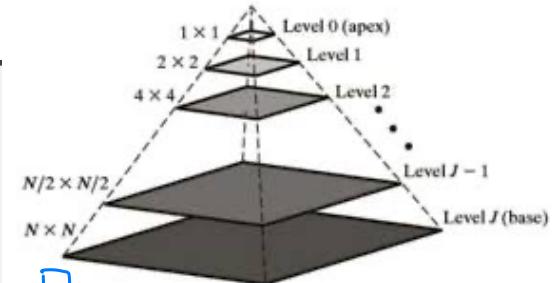
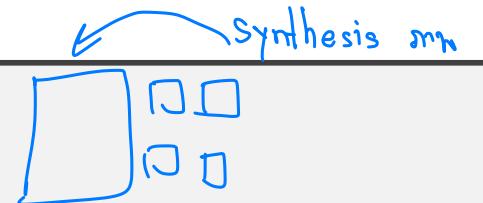
$$\psi(t) = \frac{2}{\sqrt{3}} \pi^{-\frac{1}{4}} (1 - t^2) e^{-\frac{t^2}{2}}$$



FNT ການແກ່ໄລ

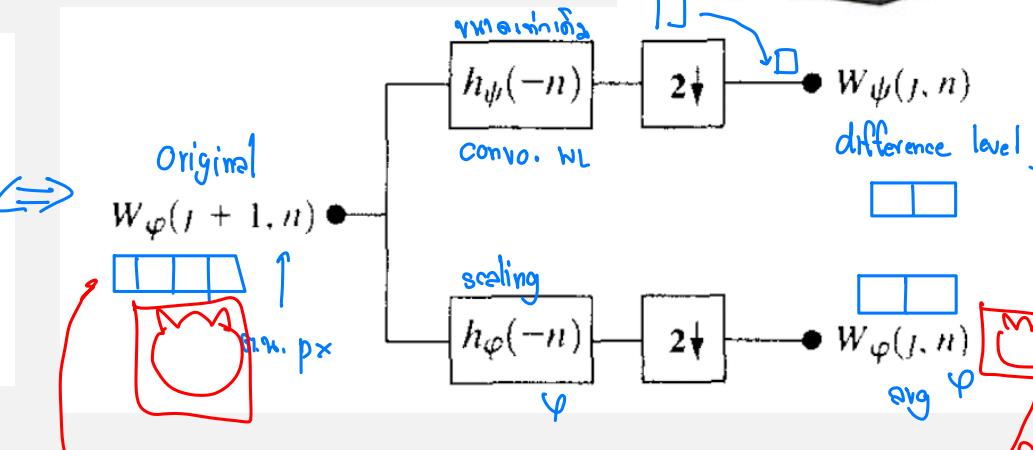
## ONE-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)

- FWT Analysis bank
- Efficient implementation of discrete wavelet transform



$$\text{detail } \psi \\ W_\psi(j, k) = h_\psi(-n) * W_\varphi(j + 1, n) \Big|_{n=2k, k \geq 0}$$

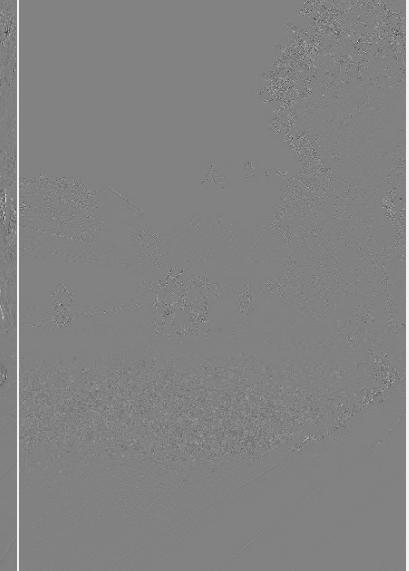
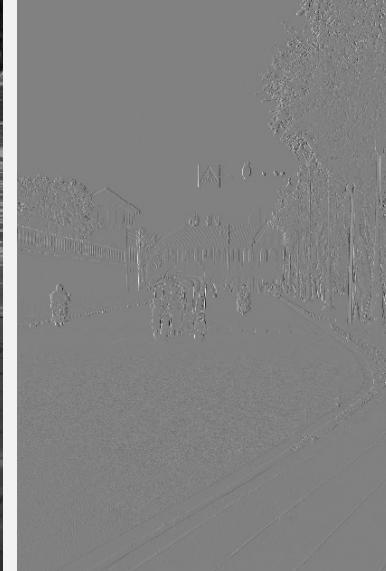
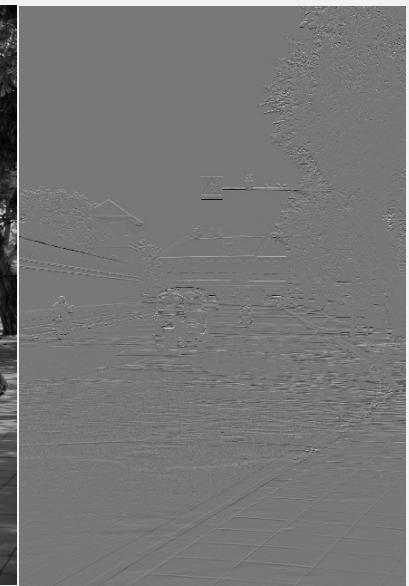
$$\text{approx } \varphi \\ W_\varphi(j, k) = h_\varphi(-n) * W_\varphi(j + 1, n) \Big|_{n=2k, k \geq 0}$$



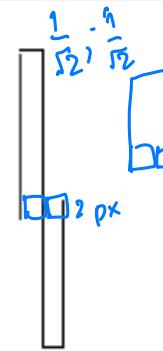
$W_\varphi(j, k)$  - approximation coefficients with scale level  $j$ , translating it by  $k$ .  $W_\psi(j, k)$  - details coefficients. The image in level  $j + 1$  is subsampled from level  $j$  so  $2k = n$ .

$$W_{\psi}(j, k) = h_{\psi}(-n) * W_{\varphi}(j + 1, n) \Big|_{n=2k, k \geq 0}$$

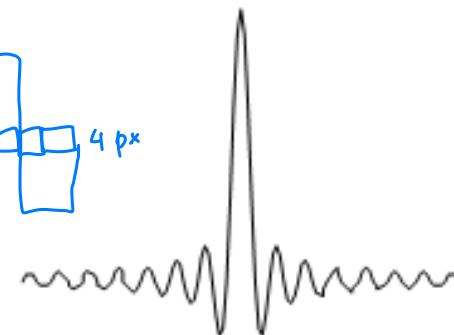
$$W_{\varphi}(j, k) = h_{\varphi}(-n) * W_{\varphi}(j + 1, n) \Big|_{n=2k, k \geq 0}$$



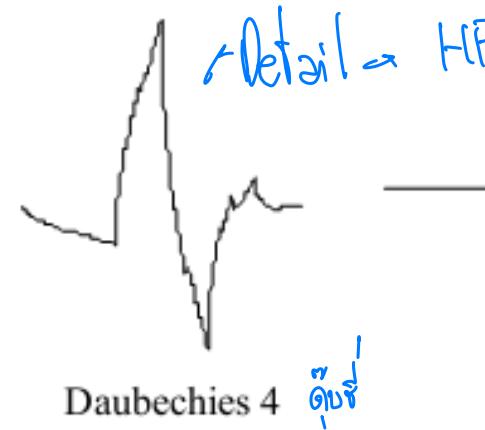
## OTHER WAVELETS



Haar



Shannon or Sinc



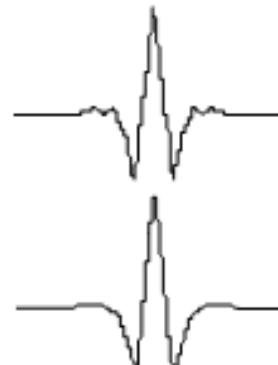
Daubechies 4



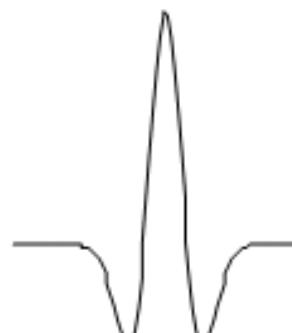
Daubechies 20



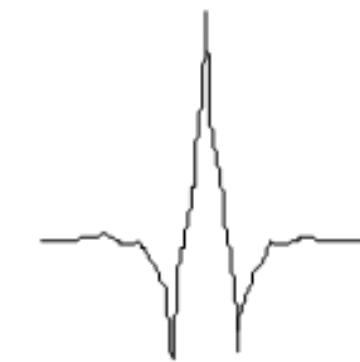
Gaussian or Spline



Biorthogonal



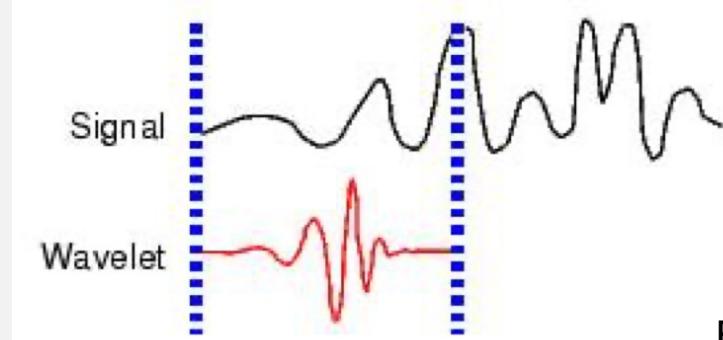
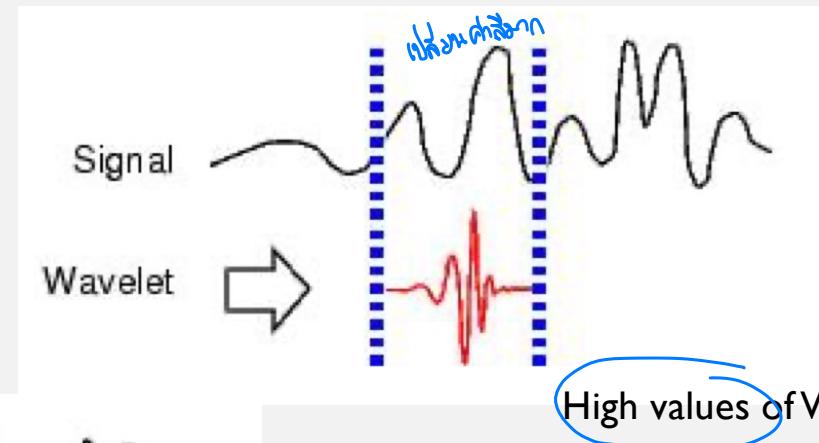
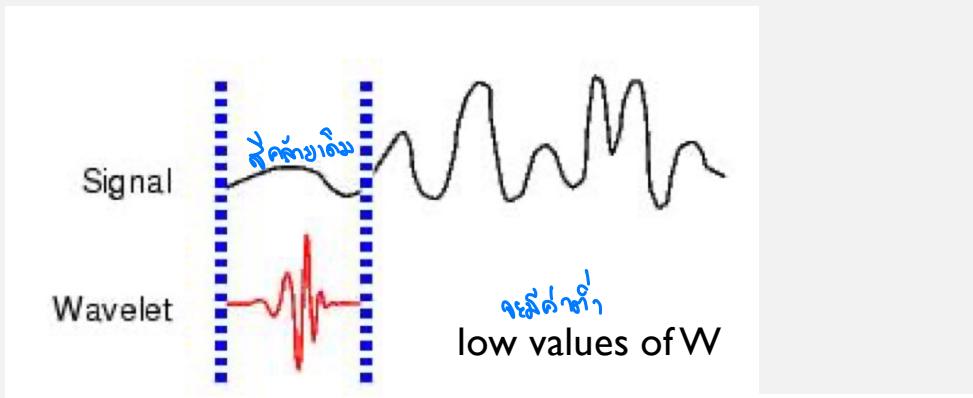
Mexican Hat



Coiflet

# WAVELET TRANSFORM IN ONE DIMENSION

- Continuous wavelet transform



Different scales

Reference: Colorado school of Mines

# ONE-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)

- **FWT Analysis bank**

- $f(n) = \{1, 4, -3, 0\}$  find DWT corresponding scaling and wavelet vectors (Haar)

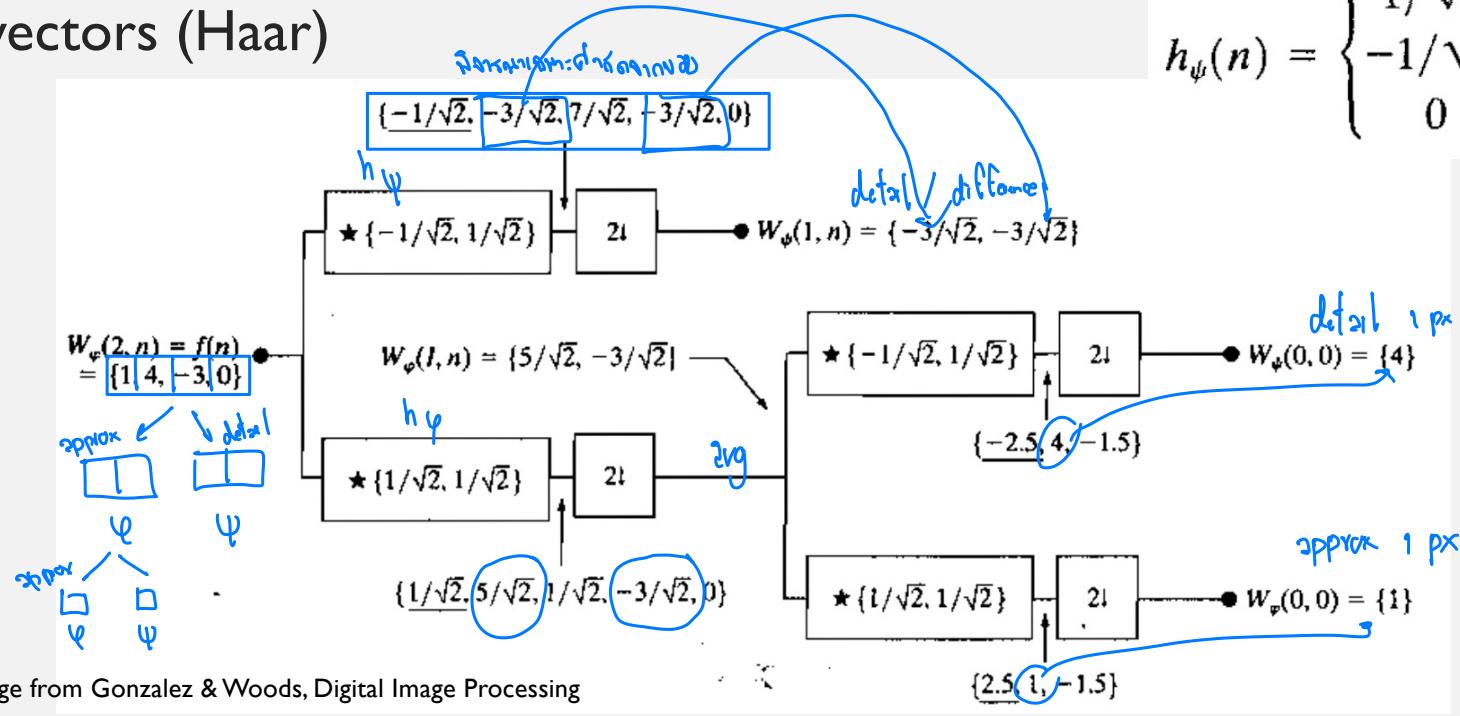
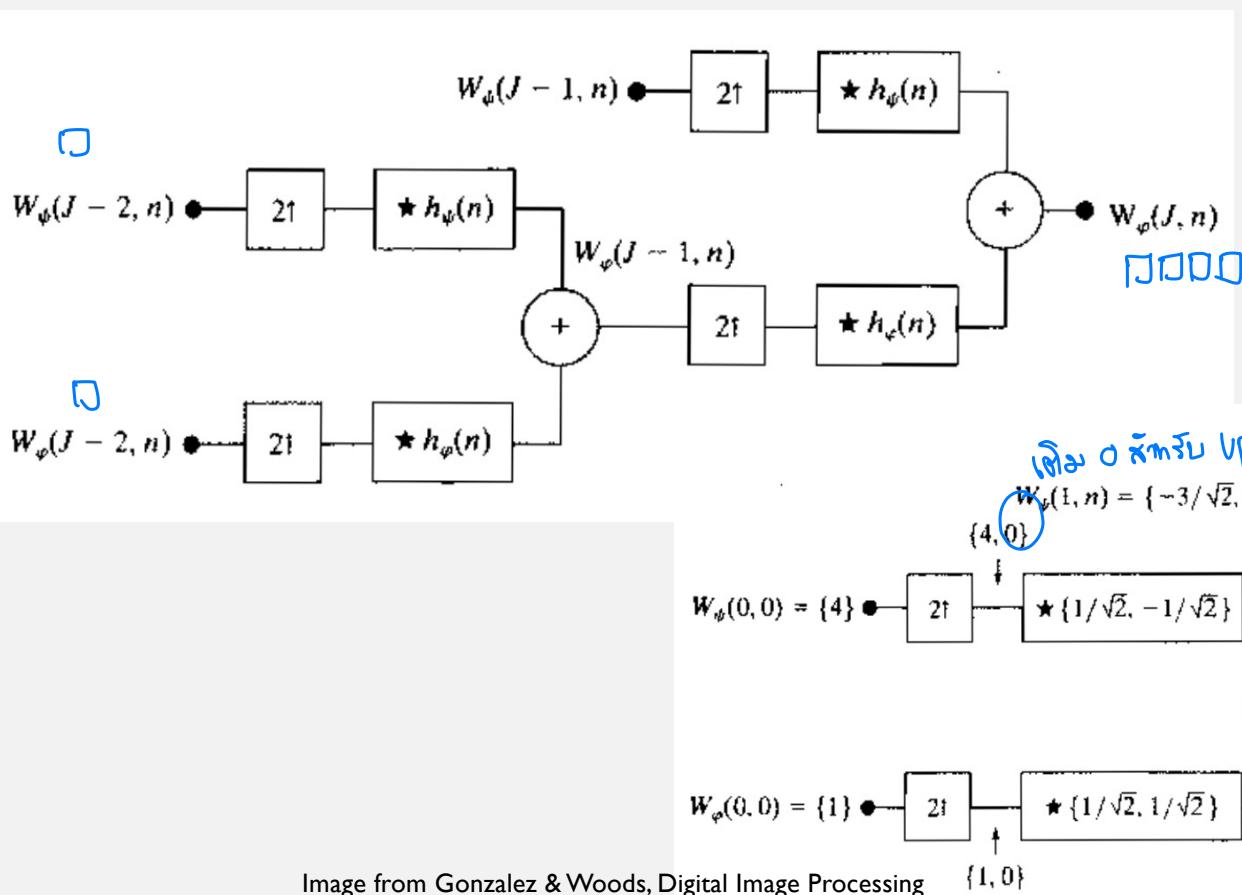


Image from Gonzalez & Woods, Digital Image Processing

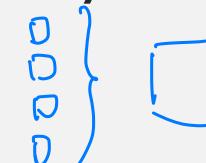
$$h_\varphi(n) = \begin{cases} 1/\sqrt{2} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_\psi(n) = \begin{cases} 1/\sqrt{2} & n = 0 \\ -1/\sqrt{2} & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

# ONE-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)



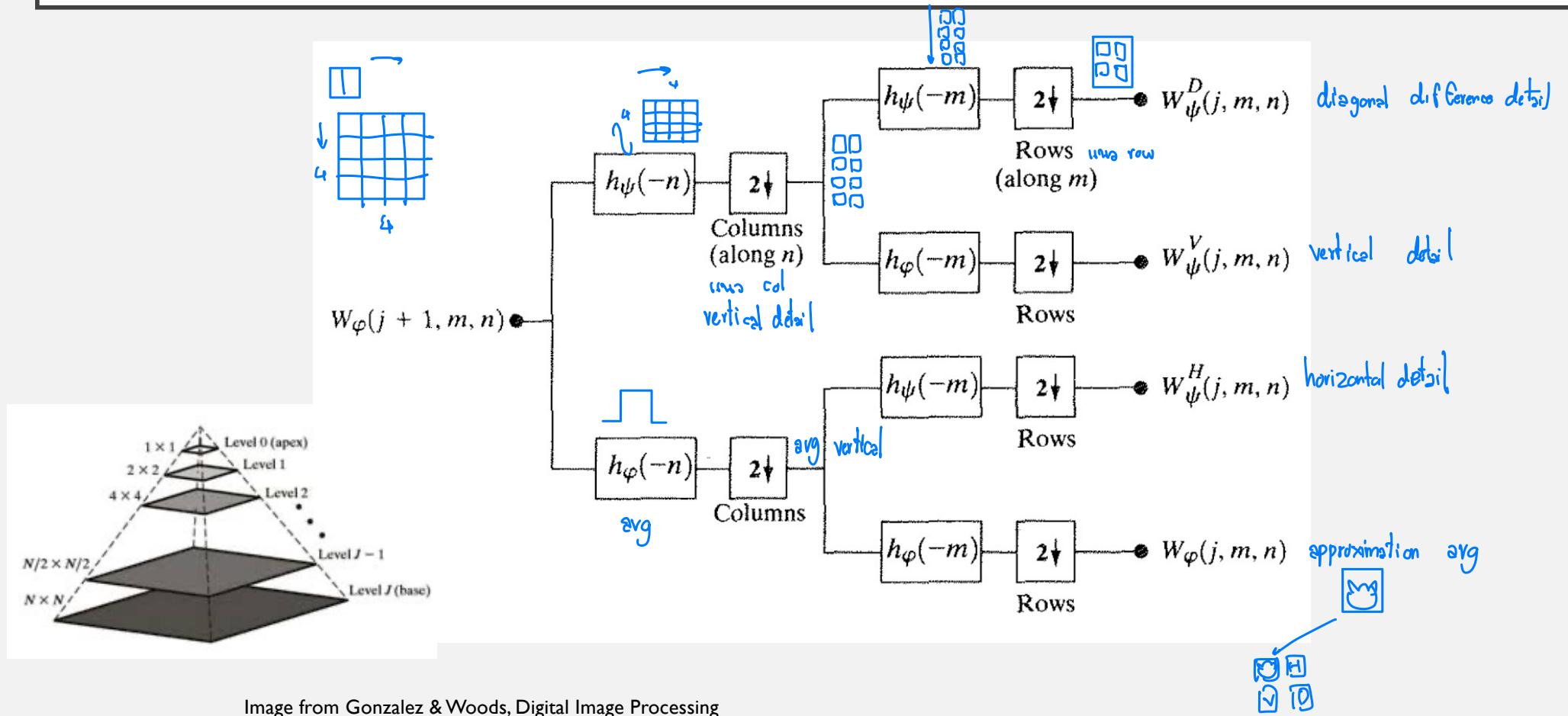
- FWT Synthesis bank



## ท้าสั่ว人格

Image from Gonzalez & Woods, Digital Image Processing

# TWO-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)

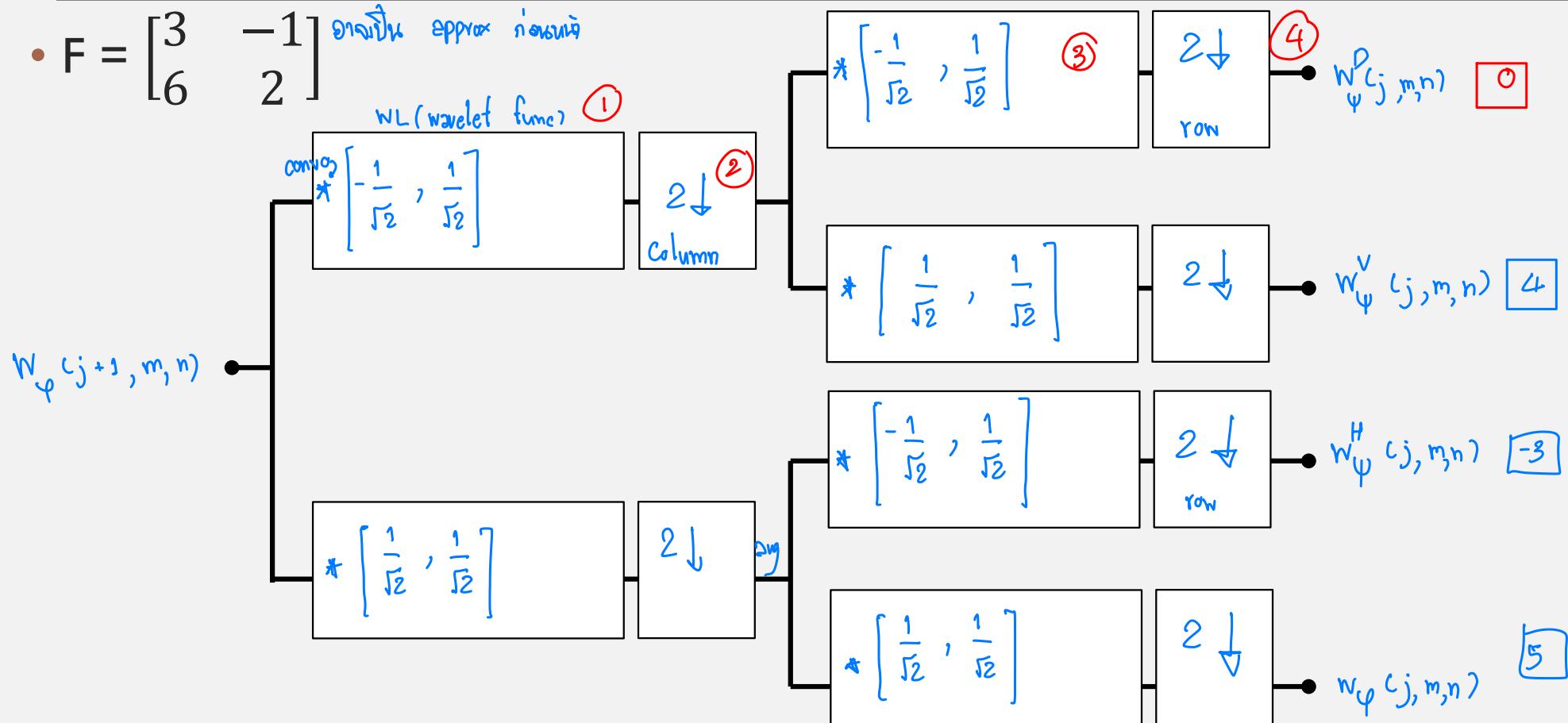


## TWO-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)

- Practice: a small image 2x2  $F = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$
- Compute 2D Wavelet transform with respect to Haar wavelets of this image. Draw the required filter bank and label all inputs and outputs with proper arrays.

## TWO-DIMENSIONAL FAST WAVELET TRANSFORM (FWT)

- $F = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$  ឧបតម្លៃនៃ approx ការគណន៍



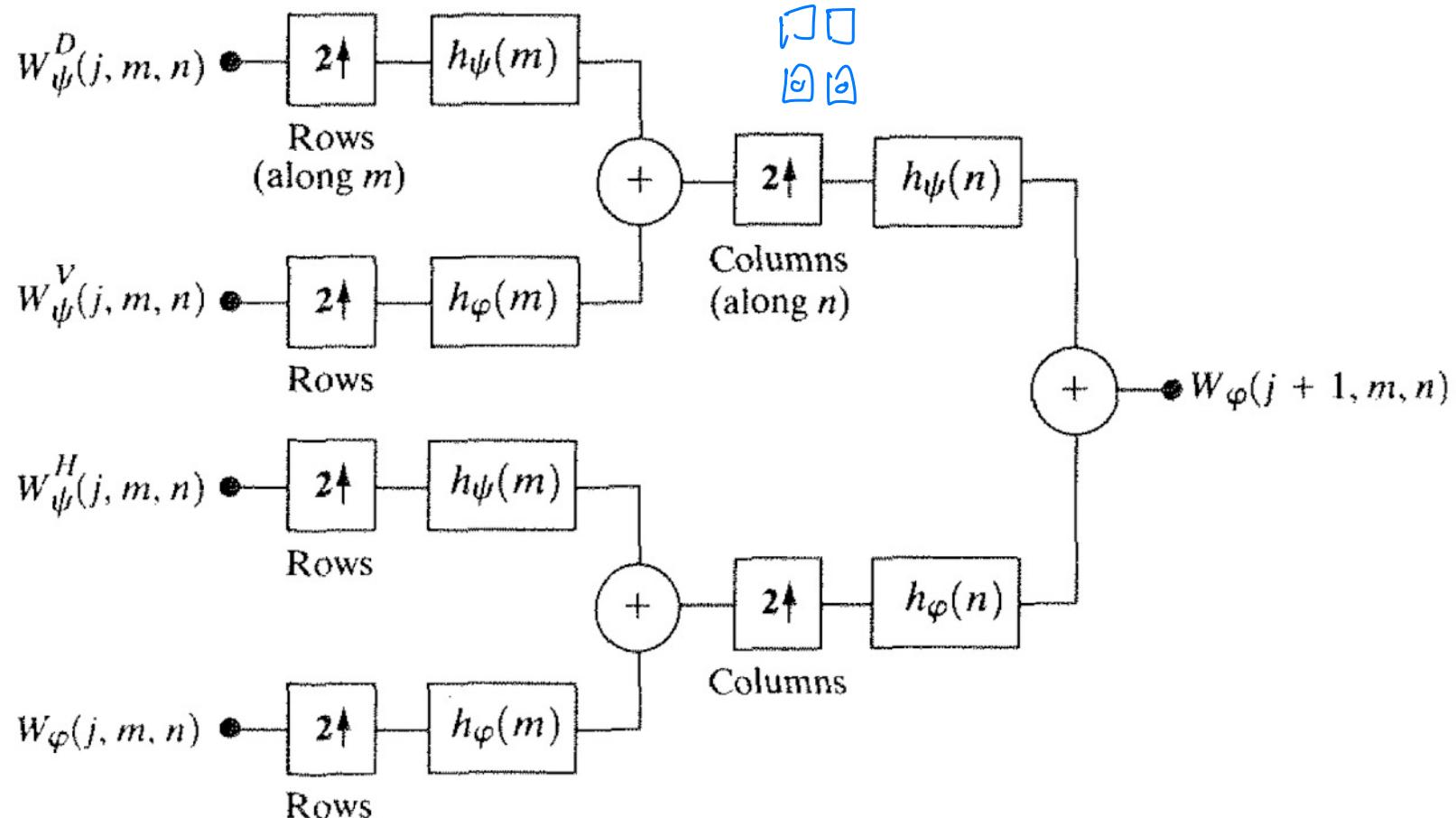
$$\begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix} * \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -3/\sqrt{2} & 4/\sqrt{2} & 1/\sqrt{2} \\ -6/\sqrt{2} & 4/\sqrt{2} & 2/\sqrt{2} \end{bmatrix} \xrightarrow{\text{①}} \begin{bmatrix} 1/\sqrt{2} & 4/\sqrt{2} \\ -1/\sqrt{2} & 4/\sqrt{2} \end{bmatrix} \xrightarrow{\text{②}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -4/\sqrt{2} \\ 0 \\ 4/2 \end{bmatrix} \xrightarrow{\text{③}} \boxed{0}$$

$$\begin{bmatrix} 4/\sqrt{2} \\ 4/2 \end{bmatrix} * \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4/2 \\ 8/2 \\ 4/2 \end{bmatrix} \xrightarrow[2\downarrow]{\longrightarrow} 4$$

$$\begin{array}{c} \left( \frac{1}{2}, \frac{1}{2} \right) \quad \left( \frac{1}{2}, -\frac{1}{2} \right) \\ \left( \frac{1}{2}, \frac{1}{2} \right) \end{array} \xrightarrow{\text{flip}} \begin{array}{c} \left[ \begin{smallmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{smallmatrix} \right] \\ \left[ \begin{smallmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{smallmatrix} \right] \end{array}$$

$$\rightarrow \boxed{5}$$



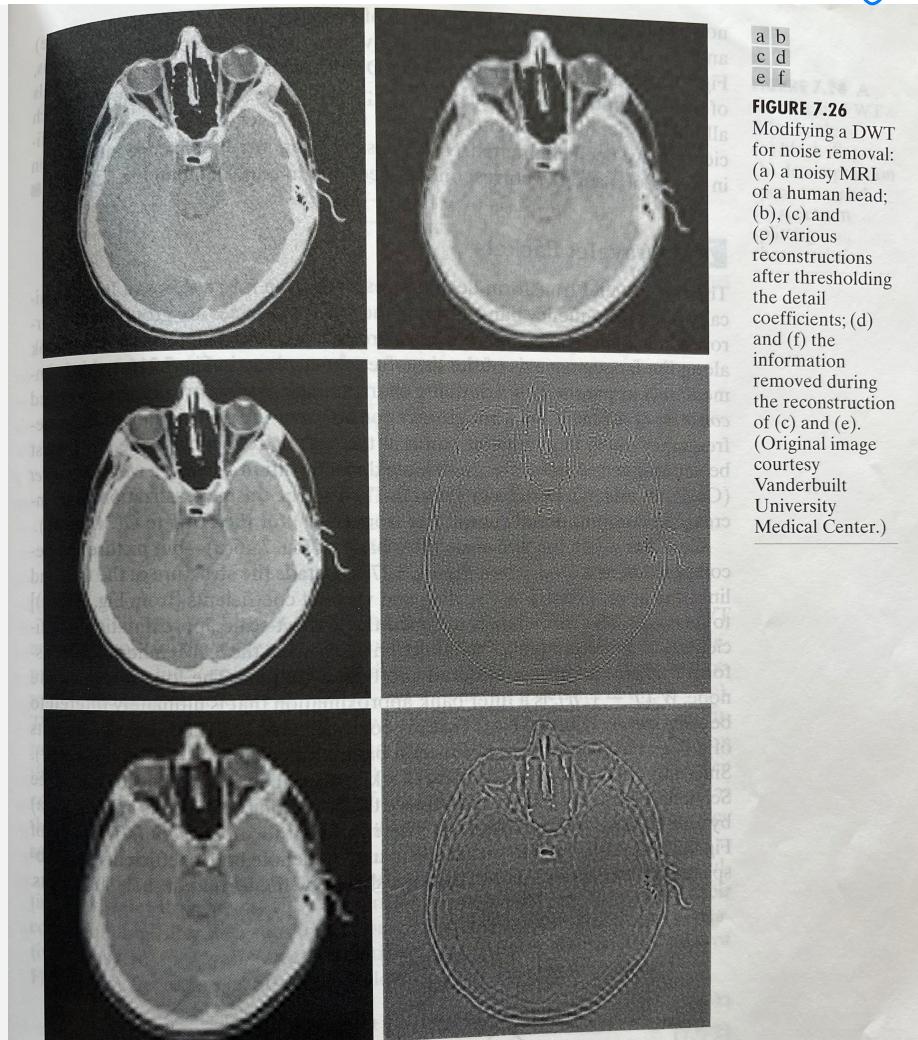


**FIGURE 7.22** The two-dimensional fast wavelet transform: (a) the analysis filter bank; (b) the resulting decomposition; and (c) the synthesis filter bank.

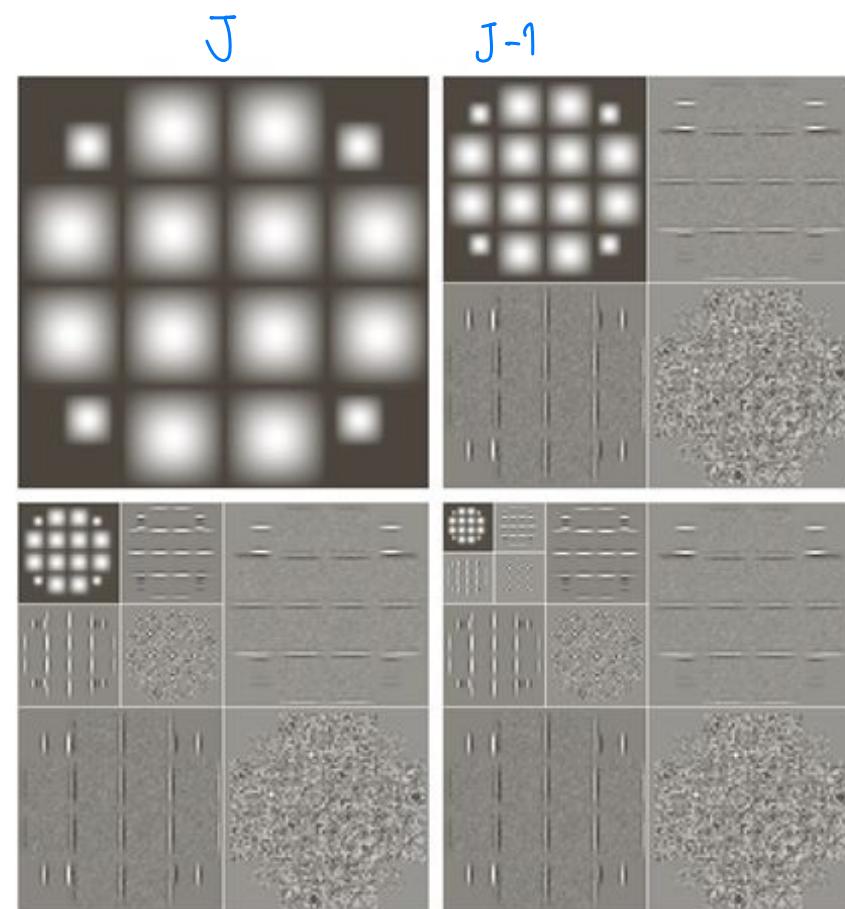
Synthesis filter banks

Image from Gonzalez & Woods, Digital Image Processing

Denoising



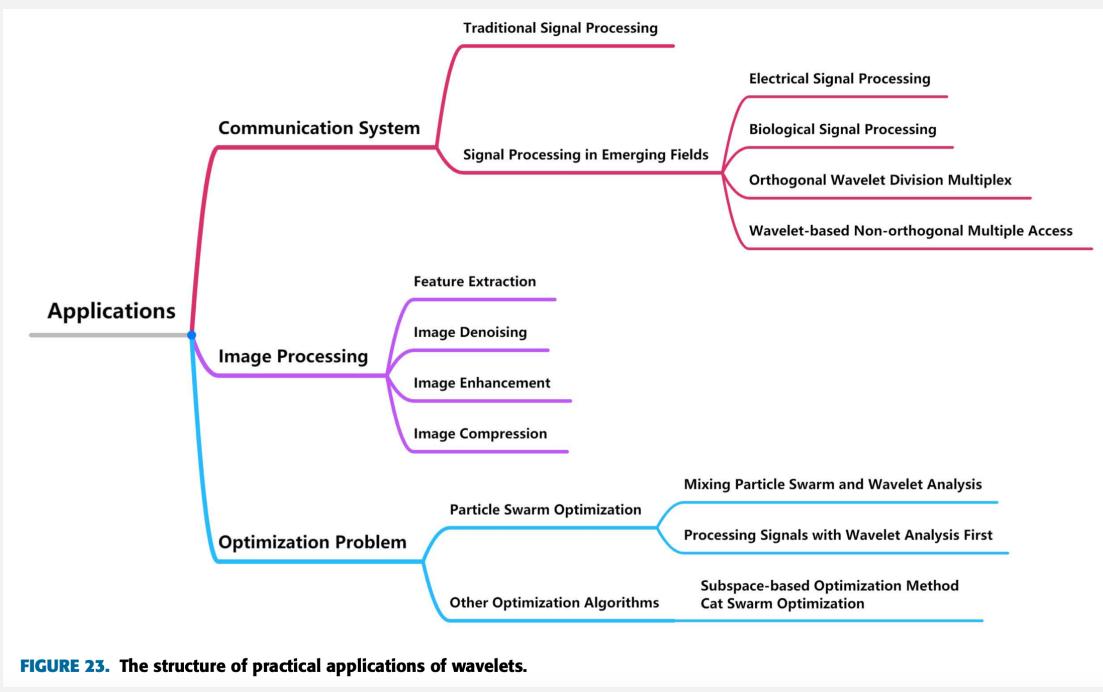
**FIGURE 7.26**  
Modifying a DWT for noise removal:  
(a) a noisy MRI of a human head;  
(b), (c) and  
(e) various  
reconstructions  
after thresholding  
the detail  
coefficients; (d)  
and (f) the  
information  
removed during  
the reconstruction  
of (c) and (e).  
(Original image  
courtesy  
Vanderbilt  
University  
Medical Center.)



**FIGURE 7.25**  
Computing a 2-D three-scale FWT:  
(a) the original  
image; (b) a one-  
scale FWT; (c) a  
two-scale FWT;  
and (d) a three-  
scale FWT.

tails) and reconstructing the image. Here, almost all of the background noise has been eliminated and the edges are only slightly disturbed. Figure 7.26(d) shows the information that is lost in the process. This result was generated by computing the inverse FWT of the decomposed image with all but the highest-resolution detail coefficients zeroed. As can be seen, it contains most of the

# APPLICATIONS OF WAVELETS

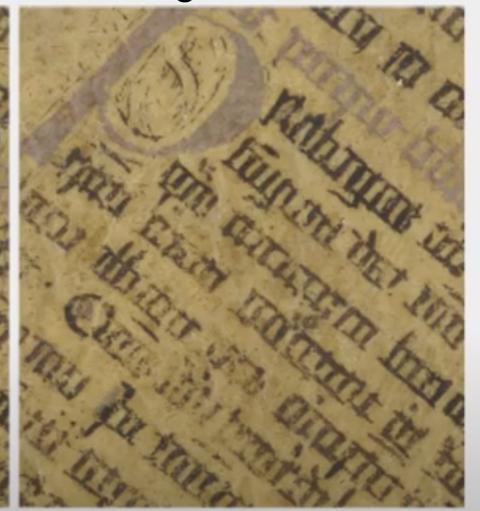
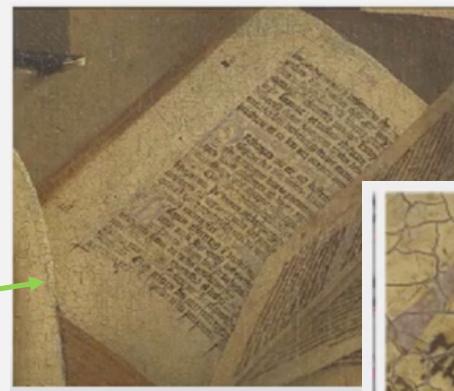


## Wavelets for image processing

- Optimally extract features in different sizes
  - Palm- print images after two-level wavelet decomposition & maintaining the uniqueness of each palmprint image and can be used for palmprint image classification
- Denoising, enhancement and compression in the image processing area
- Extracting structure and noise information through wavelet transform and generating high-quality images through GAN
- Wavelet kernel-based neural network
  - Initialization
  - Used as the activation function

T. Guo, T. Zhang, E. Lim, M. López-Benítez, F. Ma and L. Yu, "A Review of Wavelet Analysis and Its Applications: Challenges and Opportunities," in *IEEE Access*, vol. 10, pp. 58869-58903, 2022, doi: 10.1109/ACCESS.2022.3179517.

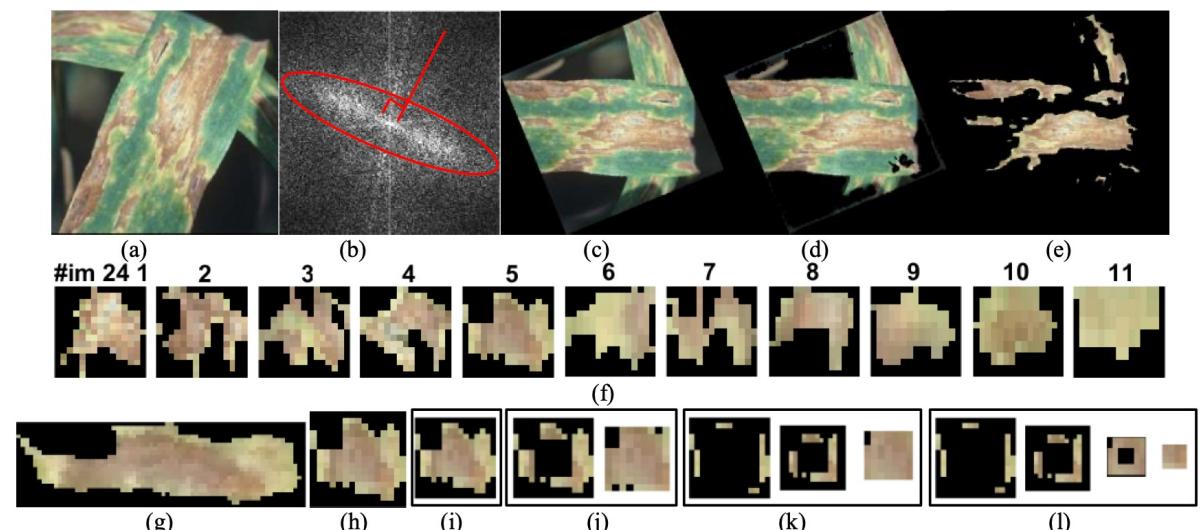
# WAVELETS FOR ART



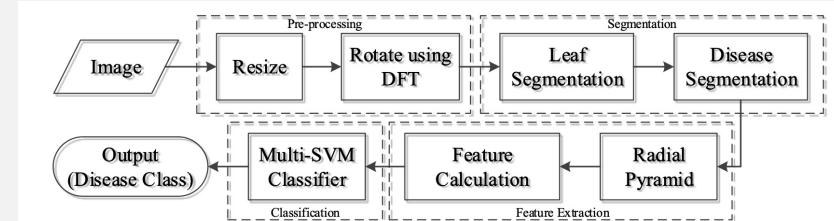
Ghent Altarpiece

Ref. <https://www.youtube.com/watch?v=jrF1SGPyF4g>  
Using Math to Understand Art | Ingrid Daubechies | TEDxDuke

# WAVELETS FOR AI



**Fig. 4.** Uncontrolled image processing in the system (a) original image, (b) DFT on Canny edge, (c) rotated image, (d) partially segmented image, (e) segmented disease, (f) constructed disease patches, (g) original disease blob (#5), (h) normalised disease patch (i) 1-level (j) 2-level (k) 3-level (l) 4-level radial pyramid patches



**Fig. 1.** Our proposed system for wheat disease classification

## 3.2 Leaf segmentation using multi-resolution discrete Wavelet transform

A single-level two-dimensional Wavelet transform (DWT) is exploited to decompose an image into coefficients of four different components, an approximation component ( $cA$ ), horizontal, vertical, and diagonal details ( $cH$ ,  $cV$ , and  $cD$ ). We combined horizontal, vertical and diagonal detail information based on Daubechies wavelets and then thresholded the low coefficient values out to remove part of the background from consideration in the subsequent stages of our process (see Fig. 4 (d)).

## EXERCISE #1 USE PYWAVELETS FOR IMAGE SPACE-FREQUENCY ANALYSIS

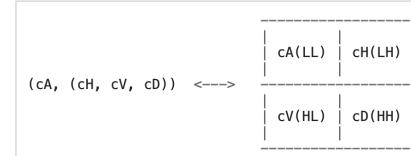
- Reference: <https://pywavelets.readthedocs.io/en/latest/>

```
import pywt
import cv2

original = cv2.imread("kitty.png", 0)
LL, (LH, HL, HH) = pywt.dwt2(original, 'haar')
```

```
# display
fig = plt.figure(figsize=(12, 3))
for i, a in enumerate([LL, LH, HL, HH]):
    ax = fig.add_subplot(1, 4, i + 1)
    ax.imshow(a, interpolation="nearest",
              cmap=plt.cm.gray)
    ax.set_title(titles[i], fontsize=10)
    ax.set_xticks([])
    ax.set_yticks([])
    fig.tight_layout()
plt.show()
```

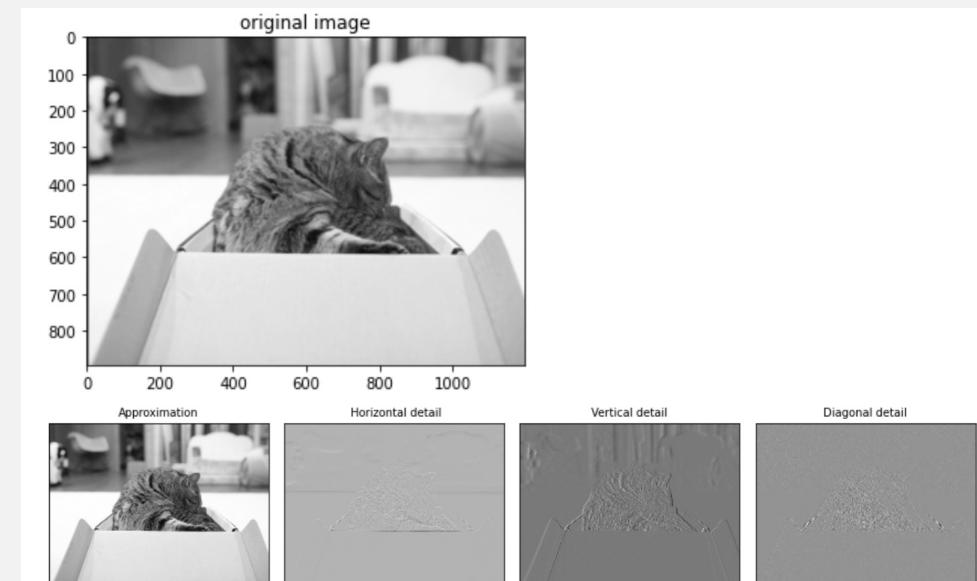
The relation to the other common data layout where all the approximation and details coefficients are stored in one big 2D array is as follows:



PyWavelets does not follow this pattern because of pure practical reasons of simple access to particular type of the output coefficients.

## **EXERCISE #1 USE PYWAVELETS FOR IMAGE SPACE-FREQUENCY ANALYSIS**

- Investigate the size of the original image, approximation, horizontal, vertical and diagonal details.
- Investigate the min/max values in the approximation, horizontal, vertical and diagonal details.



Reference:

<https://pywavelets.readthedocs.io/en/latest/>

## **EXERCISE #2 CAN YOU SEGMENT THE CAT?**

- 1) apply wavelet transform into "kitty.jpg" image
- 2) Again, investigate the min/max values in the approximation, horizontal, vertical and diagonal details.
- 3) Can you use a threshold to separate the high details from the low and segment the cat out?

## REFERENCES

- Rafael C. Gonzalez and Richard E.Woods, Digital Image Processing, Addison- Wesley
- Chapter 7 - Wavelets and multiresolution processing