!! जय श्री गिरिराज जी महाराज जय श्री राधे कृष्णा !!

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LIMITS & DERIVATIVES : $\boxed{\text{REVISION}}$ CLASS No: 4

Q.No.1 Evaluate $\lim_{x\to 1} \frac{(x+x^2+x^3+\cdots x^n)-n}{x-1}$

Sol. $= \lim_{x\to 1}\left[\frac{(x+x^2+x^3+\cdots x^n)-(1+1+1+\cdots n\ \text{times})}{x-1}\right]$

$= \lim_{x\to 1}\left(\frac{(x-1)+(x^2-1)+(x^3-1)+\cdots(x^n-1)}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{\cancel{x-1}}{\cancel{x-1}}\right) + \lim_{x\to 1}\left(\frac{x^2-1^2}{x-1}\right) + \lim_{x\to 1}\left(\frac{x^3-1^3}{x-1}\right) + \cdots \lim_{x\to 1}\left(\frac{x^n-1^n}{x-1}\right)$

$= 1 + 2(1)^1 + 3(1)^2 + \cdots\cdots n(1)^{n-1}$

$= 1+2+3+\cdots n$

$= \frac{n(n+1)}{2}$ Ans

Q.No.2 Evaluate (extra)/optional $\lim_{x\to\infty}\left(\frac{\sqrt{3x^2+x+1}-\sqrt{x^2+3}}{4x+5}\right)$

Sol. Divide N & D by x

$= \lim_{x\to\infty}\left(\frac{\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}-\sqrt{1+\frac{3}{x^2}}}{4+\frac{5}{x}}\right) = \frac{\sqrt{3}-1}{4}$ Ans

Qn 3 → evaluate $\lim_{x \to \infty}\left(\sqrt{x^2+x+1} - \sqrt{x^2+3}\right)$

Sol) Rationalize

$$= \lim_{x \to \infty}\left(\frac{\left(\sqrt{x^2+x+1} - \sqrt{x^2+3}\right) \times \left(\sqrt{x^2+x+1} + \sqrt{x^2+3}\right)}{\left(\sqrt{x^2+x+1} + \sqrt{x^2+3}\right)}\right)$$

$$= \lim_{x \to \infty}\left(\frac{\cancel{x^2}+x+1-\cancel{x^2}-3}{\sqrt{x^2+x+1}+\sqrt{x^2+3}}\right)$$

$$= \lim_{x \to \infty}\left(\frac{x-2}{\sqrt{x^2+x+1}+\sqrt{x^2+3}}\right)$$

Divide N & D by x

$$= \lim_{x \to \infty}\left(\frac{1-\frac{2}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{3}{x^2}}}\right)$$

$$= \frac{1-0}{\sqrt{1+0+0}+\sqrt{1+0}} = \frac{1}{2} \text{ Ans}$$

Qn 4 → evaluate $\lim_{x \to 0}\left(\frac{\tan x + 4\tan(2x) - 3\tan(3x)}{x^2 \tan x}\right)$

Sol $\lim_{x \to 0}\left[\frac{\tan x + 4 \times \left(\frac{2\tan x}{1-\tan^2 x}\right) - 3\left(\frac{3\tan x - \tan^3 x}{1-3\tan^2 x}\right)}{x^2 \tan x}\right]$

$$= \lim_{x\to 0}\left[\frac{\tan x(1-\tan^2 x)(1-3\tan^2 x) + 8\tan x(1-3\tan^2 x) - 3(3\tan x - \tan^3 x)(1-\tan^2 x)}{x^2 \tan x(1-\tan^2 x)(1-3\tan^2 x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{\cancel{1} - 4\tan^2 x + \cancel{3\tan^4 x} + \cancel{8} - 24\tan^2 x - \cancel{9} + 12\tan^2 x - \cancel{3\tan^4 x}}{x^2(1-\tan^2 x)(1-3\tan^2 x)}\right]$$

$$= \lim_{x\to 0}\left(\frac{-16\tan^2 x}{x^2(1-\tan^2 x)(1-3\tan^2 x)}\right)$$

$$= \frac{-16\times 1}{(1-0)(1-0)} \quad \cdots \left\{\lim_{x\to 0}\left(\frac{\tan x}{x}\right)=1\right\}$$

$$= -16 \text{ Ans}$$

Qn 5 → Evaluate $\lim_{x\to 0}\left(\frac{1-\cos x\sqrt{\cos(2x)}}{x^2}\right)$

Soln Rationalize

$$= \lim_{x\to 0}\left[\frac{\left(1-\cos x\sqrt{\cos(2x)}\right)}{x^2}\frac{\left(1+\cos x\sqrt{\cos(2x)}\right)}{\left(1+\cos x\sqrt{\cos(2x)}\right)}\right]$$

$$= \lim_{x\to 0}\left(\frac{1-\cos^2 x\cdot\cos(2x)}{x^2\left(1+\cos x\sqrt{\cos(2x)}\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-\cos^2 x\cdot(2\cos^2 x - 1)}{x^2}\right)\times\lim_{x\to 0}\left(\frac{1}{1+\cos x\sqrt{\cos(2x)}}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-2\cos^4 x+\cos^2 x}{x^2}\right)\times\frac{1}{2}$$

$$\Rightarrow \quad -\lim_{x\to 0}\left(\frac{2\cos^4 x-\cos^2 x-1}{x^2}\right)\times\frac{1}{2}$$

$$= -\frac{1}{2}\lim_{x\to 0}\left(\frac{2\cos^4 x-2\cos^2 x+\cos^2 x-1}{x^2}\right)$$

$$= -\frac{1}{2}\lim_{x\to 0}\left(\frac{2\cos^2 x(\cos^2 x-1)+1(\cos^2 x-1)}{x^2}\right)$$

$$= -\frac{1}{2}\lim_{x\to 0}\left(\frac{(2\cos^2 x+1)(\cos^2 x-1)}{x^2}\right)$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{(2\cos^2 x+1)\cdot\sin^2 x}{x^2}\right)$$

$$= \frac{1}{2}\times(2+1)\times 1 \quad \ldots\left\{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1\right\}$$

$$= 3/2 \quad \text{Ans}$$

Qn 6 : Evaluate $\lim_{x\to\pi/4}\left(\frac{\sqrt{\cos x}-\sqrt{\sin x}}{x-\pi/4}\right)$

Sol: Rationalize

$$= \lim_{x\to\pi/4}\left(\frac{\cos x-\sin x}{\left(x-\frac{\pi}{4}\right)\left(\sqrt{\cos x}+\sqrt{\sin x}\right)}\right)$$

$$= \lim_{x\to\frac{\pi}{4}}\left(\frac{\cos x-\sin x}{x-\pi/4}\right)\times\lim_{x\to\frac{\pi}{4}}\left(\frac{1}{\sqrt{\cos x}+\sqrt{\sin x}}\right)$$

put $x = \frac{\pi}{4} + h$ in I^st limit.

& $h \to 0$

$$\Rightarrow \lim_{h \to 0} \left(\frac{\cos(\frac{\pi}{4}+h) - \sin(\frac{\pi}{4}+h)}{\frac{\pi}{4}+h-\frac{\pi}{4}} \right) \times \frac{1}{\sqrt{\frac{1}{\sqrt{2}}} + \sqrt{\frac{1}{\sqrt{2}}}}$$

$$= \lim_{h \to 0} \left(\frac{\left(\cancel{\frac{1}{\sqrt{2}}\cos h} - \frac{1}{\sqrt{2}}\sin h\right) - \left(\cancel{\frac{1}{\sqrt{2}}\cos h} + \frac{1}{\sqrt{2}}\sin h\right)}{h} \right) \times \frac{1}{2\sqrt{\frac{1}{\sqrt{2}}}}$$

$$= \lim_{h \to 0} \left(-\sqrt{2}\frac{\sin h}{h} \right) \times \frac{1}{2 \times \frac{1}{(2)^{1/4}}}$$

$$= -\sqrt{2} \times \frac{1}{2 \times \frac{1}{2^{1/4}}}$$

$$= -\frac{(2)^{1/2} \times 2^{1/4}}{2}$$

$$= -(2)^{1/2+\frac{1}{4}-1}$$

$$= -(2)^{-1/4} \text{ Ans}$$

Qu 7 → evaluate $\lim_{x \to 0} \left(\frac{\log(5+x) - \log(5-x)}{x} \right)$

Sol: $\lim_{x \to 0} \left(\frac{\log\left(\frac{5+x}{5-x}\right)}{x} \right)$ .... $\{\log A - \log B = \log(\frac{A}{B})\}$

$$= \lim_{x \to 0} \left(\frac{\log\left(1 + \frac{5+x}{5-x} - 1\right)}{x} \right)$$

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$$= \lim_{x\to 0}\left(\frac{\log\left(1+\frac{2x}{5-x}\right)}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\log\left(1+\frac{2x}{5-x}\right)}{x\times\frac{2}{5-x}}\times\frac{2}{5-x}\right)$$

$$= 1\times\frac{2}{5-0}$$

$$= 2/5 \text{ Ans}$$

Qn. 5 $\lim_{x\to 5}\left(\frac{\log x-\log 5}{x-5}\right)$

Sol put $x = 5+h$ & $h\to 0$

$$= \lim_{h\to 0}\left(\frac{\log(5+h)-\log 5}{5+h-\cancel{5}}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(\frac{5+h}{5}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(1+\frac{h}{5}\right)}{\frac{h}{5}\times 5}\right)$$

$$= \frac{1}{5} \quad \cdots \left\{\lim_{x\to 0}\left(\frac{\log(1+x)}{x}\right)=1\right\} \text{ Ans}$$

Qn- 9 → Show that $\lim_{x\to 0}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$ does not exists.

IMP

Soln

$$LHL = \lim_{x\to 0^-}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$$

put $x = 0-h = -h$ & $h \to 0$

$$= \lim_{h\to 0}\left(\frac{e^{-\frac{1}{h}}-1}{e^{-\frac{1}{h}}+1}\right)$$

$$\lim_{x\to 0}\left(\frac{e^{x}-1}{x}\right) = 1$$

$$= \frac{e^{-\infty}-1}{e^{-\infty}+1}$$

$e^0 = 1$	$e^{\infty} = \infty$	$e^{-\infty} = 0$ $\frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$$= \frac{0-1}{0+1} \quad LHL = -1$$

RHL $\lim_{x\to 0^+}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$ put $x = 0+h = h$ & $h\to 0$

$$= \lim_{h\to 0}\left(\frac{e^{\frac{1}{h}}-1}{e^{1/h}+1}\right)$$

Divide by $e^{1/h}$

$$= \lim_{h\to 0}\left(\frac{1-e^{-1/h}}{1+e^{-1/h}}\right) = \frac{1-e^{-\infty}}{1+e^{-\infty}} = \frac{1-0}{1+0} = 1$$

$LHL \neq RHL$ $\therefore \lim_{x\to 0} f(x)$ does not exist

Qn 10 → Diff using first principle $f(x) = (-x)^{-3}$

Sol

$$f(x) = (-x)^{-3}$$

$$f(x) = \frac{1}{(-x)^3}$$

$$f(x) = \frac{1}{-x^3} = -\frac{1}{x^3}$$

$$f'(x) = \lim_{h \to 0} \left(\frac{\frac{-1}{(x+h)^3} - \left(-\frac{1}{x^3}\right)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{-\frac{1}{(x+h)^3} + \frac{1}{x^3}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{-x^3 + (x+h)^3}{h (x+h)^3 \cdot x^3} \right)$$

$$= \lim_{h \to 0} \left(\frac{-\cancel{x^3} + \cancel{x^3} + h^3 + 3x^2h + 3xh^2}{h (x+h)^3 \cdot x^3} \right)$$

$$= \lim_{h \to 0} \left(\frac{\cancel{h}(h^2 + 3x^2 + 3xh)}{\cancel{h}(x+h)^3 \cdot x^3} \right)$$

$$= \frac{3x^2}{x^3 \cdot x^3}$$

$$f'(x) = \frac{3}{x^4} \quad \text{Ans}$$

Qn. 11 → Diff using first principle method $f(x) = \frac{\cos x}{x^2}$

Sol. $f(x) = \frac{\cos x}{x^2}$

$$f'(x) = \lim_{h \to 0} \left(\frac{\frac{\cos(x+h)}{(x+h)^2} - \frac{\cos x}{x^2}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{x^2 \cos(x+h) - (x+h)^2 \cdot \cos x}{h \cdot (x+h)^2 \cdot x^2} \right)$$

$$= \lim_{h \to 0} \left(\frac{x^2 \cos(x+h) - (x^2 + h^2 + 2hx) \cos x}{h \cdot (x+h)^2 x^2} \right)$$

$$= \lim_{h \to 0} \left(\frac{x^2 \cos(x+h) - x^2 \cos x - (h^2 + 2hx) \cos x}{h (x+h)^2 x^2} \right)$$

$$= \lim_{h \to 0} \left(\frac{x^2 (\cos(x+h) - \cos x)}{h (x+h)^2 x^2} - \frac{\cancel{h}(h+2x) \cos x}{\cancel{h}(x+h)^2 x^2} \right)$$

$$= \lim_{h \to 0} \left[\frac{-x^2 \cdot \cancel{2} \sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{\cancel{2} \times \frac{h}{2} (x+h)^2 x^2} - \frac{(h+2x) \cos x}{(x+h)^2 x^2} \right]$$

$$= \frac{-x^2 \cdot \sin x \times 1}{x^4} - \frac{2x \cdot \cos x}{x^4}$$

$$= \frac{-x^2 \sin x - 2x \cos x}{x^4} = \frac{-x \sin x - 2 \cos x}{x^3} \quad \text{Ans}$$

Qn. 12 → If $y = \sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}$

Special Show that $2xy\frac{dy}{dx} = \left(\frac{x}{a} - \frac{a}{x}\right)$

Sol $y = \sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}$

Squaring.

$y^2 = \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}\right)^2$

$y^2 = \frac{x}{a} + \frac{a}{x} + 2\cancel{\sqrt{\frac{x}{a}}}\cancel{\sqrt{\frac{a}{x}}}$

$y^2 = \frac{x}{a} + \frac{a}{x} + 2$

$y \to \frac{dy}{dx}$

$\sin y \to \cos y \cdot \frac{dy}{dx}$

$\log y \to \frac{1}{y} \cdot \frac{dy}{dx}$

$y^2 \to 2y \cdot \frac{dy}{dx}$

$y^n = n\,y^{n-1} \cdot \frac{dy}{dx}$

Diff w.r.t x

$2y \cdot \frac{dy}{dx} = \frac{1}{a} - \frac{a}{x^2}$

$\boxed{\frac{1}{x} \to -\frac{1}{x^2}}$

multiply both sides by x

$2xy\frac{dy}{dx} = \frac{x}{a} - \frac{a}{x}$ Ans

Qn. 13 → $y = \frac{x}{\sin^n x}$. Diff w.r.t x

Soln $y = \frac{x}{\sin^n x}$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{(\sin^n x) \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(\sin^n x)}{(\sin^n x)^2}$$

$$\frac{dy}{dx} = \frac{\sin^n x \cdot 1 - x \cdot n \sin^{n-1} x \cdot \cos x}{\sin^{2n} x}$$

$$= \frac{\sin^{n-1} x \left(\sin x - xn \cdot \cos x \right)}{\sin^{2n} x}$$

$$= \frac{\sin x - nx \cos x}{\sin^{2n-n+1} x}$$

$$\frac{dy}{dx} = \frac{\sin x - nx \cos x}{\sin^{n+1} x}$$ Ans