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ULTIMATE MATHEMATICS: BY AJAY MITTAL

REVISION : $\boxed{\text{LIMITS \& DERIVATIVES}}$ (CLASS No:2)

Q:1 Find derivative of $\sqrt{\cot x}$ using first principle method

Sol: Let $f(x) = \sqrt{\cot x}$

$$f'(x) = \lim_{h\to 0}\left[\frac{\sqrt{\cot(x+h)} - \sqrt{\cot x}}{h}\right]$$

Rationalize

$$f'(x) = \lim_{h\to 0}\left[\frac{\cot(x+h) - \cot x}{h\left(\sqrt{\cot(x+h)} + \sqrt{\cot x}\right)}\right]$$

$$\Rightarrow f'(x) = \lim_{h\to 0}\left[\frac{\overset{A}{\tan x} - \tan(\overset{B}{x+h})}{h\tan(x+h)\tan x\left(\sqrt{\cot(x+h)} + \sqrt{\cot x}\right)}\right]$$

$$= \lim_{h\to 0}\left[\frac{\tan(-h)\left(1+\tan(x)\tan(x+h)\right)}{h\tan(x+h)\tan x\left(\sqrt{\cot(x+h)} + \sqrt{\cot x}\right)}\right]$$

$$= -\lim_{h\to 0}\left(\frac{\tan h}{h}\right) \times \lim_{h\to 0}\left(\frac{1+\tan x\cdot\tan(x+h)}{\tan(x+h)\tan x\left(\sqrt{\cot(x+h)} + \sqrt{\cot x}\right)}\right)$$

$$= -1 \times \frac{\sec^2 x}{\tan^2 x\cdot 2\sqrt{\cot x}} \quad \cdots \left\{\lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1\right\}$$

$$\boxed{f'(x) = -\frac{\csc^2 x}{2\sqrt{\cot x}}}$$ Ans

Qn. 2 → Evaluate $\lim_{x\to a}\left(\frac{\sin x - \sin a}{\sqrt{x}-\sqrt{a}}\right)$

Soln

Rationalize

$$= \lim_{x\to a}\left(\frac{(\sin x - \sin a)(\sqrt{x}+\sqrt{a})}{x-a}\right)$$

put $x = a+h$ & $h\to 0$

$$= \lim_{h\to 0}\left(\frac{(\sin(\overset{A}{a+h}) - \sin \overset{B}{a})(\sqrt{a+h}+\sqrt{a})}{\cancel{a}+h-\cancel{a}}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cancel{2}\cos\left(\frac{2a+h}{2}\right).\sin\left(\frac{h}{2}\right)(\sqrt{a+h}+\sqrt{a})}{\frac{h}{2}\times\cancel{2}}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{h/2}\right) \times \lim_{h\to 0}\left(\cos\left(\frac{2a+h}{2}\right).(\sqrt{a+h}+\sqrt{a})\right)$$

$$= 1 \times (\cos a \times 2\sqrt{a}) \quad --- \left\{\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1\right\}$$

$$= 2\sqrt{a}\cos a \quad \text{Ans}$$

Qn 3 → If $\lim_{x\to 1}\left(\frac{x^4-1}{x-1}\right) = \lim_{x\to k}\left(\frac{x^3-k^3}{x^2-k^2}\right)$ Find value of 'k'

$$\Rightarrow \lim_{x\to 1}\left(\frac{x^4-1^4}{x-1}\right) = \lim_{x\to k}\left(\frac{\frac{x^3-k^3}{x-k}}{\frac{x^2-k^2}{x-k}}\right)$$

$$\Rightarrow 4(1)^3 = \frac{3(k)^2}{2(k)} \quad --- \left\{\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = na^{n-1}\right\}$$

$$\Rightarrow \frac{4\times 2}{3} = k \Rightarrow \boxed{k = 8/3} \text{ Ans}$$

Qn. 4 → If $f(x) = 1 - x + x^2 - x^3 \ldots\ldots - x^{99} + x^{100}$
find $f'(1)$

Soln: Differentiate both sides w.r.t x

$$f'(x) = -1 + 2x - 3x^2 \ldots\ldots - 99x^{98} + 100x^{99}$$

$$f'(1) = -1 + 2 - 3 + \ldots\ldots - 99 + 100$$

APs ;

$$f'(1) = -\underbrace{(1+3+5+\ldots 99)}_{AP} + \underbrace{(2+4+8+\ldots 100)}_{AP}$$

$a=1,\ d=2,\ n=50$ $\qquad$ $a=2,\ d=2;\ n=50$

$$= -25(2+98) + 25(4+98)$$

$$= -2500 + 2550$$

$$= 50 \text{ Ans}$$

Qn. 5 → If $y = \dfrac{\sin(x+9)}{\cos x}$ find $\dfrac{dy}{dx}$ at $x=0$

Soln: Diff both sides w.r.t x

$$\frac{dy}{dx} = \frac{\cos x \cdot \frac{d}{dx}(\sin(x+9)) - \sin(x+9) \cdot \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos(x+9) + \sin(x+9)\sin x}{\cos^2 x}$$

$$= \frac{\cos(\cancel{x}+9-\cancel{x})}{\cos^2 x}$$

$$\frac{dy}{dx} = \frac{\cos(9)}{\cos^2 x} \Rightarrow \left(\frac{dy}{dx}\right)_{x=0} = \frac{\cos 9}{\cos^2 0} = \cos 9 \text{ Ans}$$

Qns 6 → evaluate $\lim_{x\to 0}\left(\dfrac{\cos(ax)-\cos(bx)}{\cos(cx)-1}\right)$

Soln

$$= -\lim_{x\to 0}\left(\frac{\cos(ax)-\cos(bx)}{1-\cos(cx)}\right)$$

$$= -\lim_{x\to 0}\left(\frac{-\cancel{2}\sin\left(\frac{a+b}{2}\right)x \cdot \sin\left(\frac{a-b}{2}\right)x}{\cancel{2}\sin^2\left(\frac{cx}{2}\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\dfrac{\sin\left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x}\cdot\left(\frac{a+b}{\cancel{2}}\right)\cancel{x} \cdot \dfrac{\sin\left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x}\cdot\left(\frac{a-b}{\cancel{2}}\right)\cancel{x}}{\dfrac{\sin^2\left(\frac{cx}{2}\right)}{\frac{c^2x^2}{4}}\times\frac{c^2\cancel{x^2}}{\cancel{4}}}\right)$$

$$= \frac{1(a+b)\cdot 1(a-b)}{c^2} \quad \cdots \left\{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1\right\}$$

$$= \frac{a^2-b^2}{c^2} \quad \text{Ans}$$

Qn. 7 → Find derivative of $f(x)=\tan\sqrt{x}$ using first principle method.

Soln

$$f'(x) = \lim_{h\to 0}\left(\frac{\overset{(A)}{\tan(\sqrt{x+h})} - \overset{(B)}{\tan\sqrt{x}}}{h}\right)$$

$$= \lim_{h\to 0}\left[\frac{\tan\left(\sqrt{x+h}-\sqrt{x}\right)\cdot\left(1+\tan(\sqrt{(x+h)})\tan\sqrt{x}\right)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\tan(\sqrt{x+h}-\sqrt{x})}{(\sqrt{x+h}-\sqrt{x})}\cdot\frac{(1+\tan\sqrt{x+h}\,\tan\sqrt{x})(\sqrt{x+h}-\sqrt{x})}{h}\right]$$

$$= 1 \times \lim_{x\to c}\left(1+\tan(\sqrt{x+h})\tan\sqrt{x}\right) \times \lim_{h\to c}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$$

Rationalize

$$= (1+\tan^2\sqrt{x}) \times \lim_{h\to c}\left(\frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})}\right)$$

$$= \sec^2(\sqrt{x}) \times \lim_{h\to c}\left(\frac{1}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$\boxed{f'(x) = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}}$$ Ans

Qn. 8 → If $y = \dfrac{\sin x - x\cos x}{x\sin x + \cos x}$. Find $\dfrac{dy}{dx}$

Soln Diff. w.r.t x

$$\frac{dy}{dx} = \frac{(x\sin x+\cos x)\cdot\frac{d}{dx}(\sin x - x\cos x) - (\sin x - x\cos x)\cdot\frac{d}{dx}(x\sin x+\cos x)}{(x\sin x+\cos x)^2}$$

$$= \frac{(x\sin x+\cos x)\cdot(\cancel{\cos x} - \{-x\sin x + \cancel{\cos x}\}) - (\sin x - x\cos x)(x\cos x + \cancel{\sin x} - \cancel{\sin x})}{(x\sin x+\cos x)^2}$$

$$= \frac{(x\sin x+\cos x)(x\sin x) - (\sin x - x\cos x)(x\cos x)}{(x\sin x+\cos x)^2}$$

$$= \frac{x^2\sin^2 x + \cancel{x\sin x\cos x} - \cancel{x\sin x\cos x} + x^2\cos^2 x}{(x\sin x+\cos x)^2}$$

$$= \frac{x^2(\sin^2 x + \cos^2 x)}{(x\sin x + \cos x)^2}$$

$$\boxed{f'(x) = \frac{x^2}{(x\sin x + \cos x)^2}}$$ Ans

Q.N 9 → Evaluate $\lim_{x \to 0} f(x)$

where $f(x) = \begin{cases} \frac{3x}{|x| + 2x} ; & x \neq 0 \\ 0 : & x = 0 \end{cases}$

Soln Redefine the function

$$f(x) = \begin{cases} \frac{3x}{x + 2x} ; & x > 0 \\ \frac{3x}{-x + 2x} : & x < 0 \\ 0 : & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 1 ; & x > 0 \\ 3 : & x < 0 \\ 0 : & x = 0 \end{cases}$$

$LHL = \lim_{x \to 0^-} (3) = 3$

$RHL = \lim_{x \to 0^+} (1) = 1.$

Clearly $LHL \neq RHL$

$\therefore \lim_{x \to 0} f(x)$ does not exists

Ans

Ans 10 → evaluate $\lim_{x\to 1}\left(\frac{3^{x-1}-1}{\sin(\pi x)}\right)$

Soln put $x = 1+h$ & $h\to 0$

$$= \lim_{h\to 0}\left(\frac{3^h-1}{\sin(\pi(1+h))}\right)$$

$$= \lim_{h\to 0}\left(\frac{3^h-1}{\sin(\underbrace{\pi+\pi h}_{3^{rd}\text{ quad}})}\right)$$

$$= \lim_{h\to 0}\left(\frac{3^h-1}{-\sin(\pi h)}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{3^h-1}{h}\times \cancel{h}}{-\frac{\sin(\pi h)}{\pi h}\times \pi \cancel{h}}\right)$$

$$= \frac{\log 3}{-1\times\pi}$$

$$= -\frac{\log 3}{\pi} \quad \text{Ans}$$

Ans 11 → evaluate $\lim_{x\to \frac{\pi}{8}}\left(\frac{\cot(4x)-\cos(4x)}{(\pi-8x)^3}\right)$

Soln put $x = \frac{\pi}{8}+h$ & $h\to 0$

$$= \lim_{h\to 0}\left(\frac{\cot\left(4\left(\frac{\pi}{8}+h\right)\right)-\cos\left(4\left(\frac{\pi}{8}+h\right)\right)}{\left(\pi-8\left(\frac{\pi}{8}+h\right)\right)^3}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cot\overset{\text{II}}{\left(\frac{3\pi}{2}+4h\right)} - \cos\overset{\text{II}}{\left(\frac{3\pi}{2}+4h\right)}}{(\cancel{\pi}-\cancel{\pi}-8h)^3}\right)$$

$$= \lim_{h\to 0}\left(\frac{-\tan(4h) + \sin(4h)}{-512h^3}\right)$$

$$= \frac{\cancel{-}1}{512}\lim_{h\to 0}\left(\frac{\tan(4h)-\sin(4h)}{h^3}\right)$$

$$= \frac{1}{512}\lim_{h\to 0}\left(\frac{\frac{\sin(4h)}{\cos(4h)} - \sin(4h)}{h^3}\right)$$

$$= \frac{1}{512}\lim_{h\to 0}\left(\frac{\sin(4h) - \sin(4h)\cos(4h)}{h^3\cdot\cos(4h)}\right)$$

$$= \frac{1}{512}\lim_{h\to 0}\left(\frac{\sin(4h)\cdot(1-\cos(4h))}{h^3\cdot\cos(4h)}\right)$$

$$= \frac{1}{512}\lim_{h\to 0}\left(\frac{\sin(4h)\cdot 2\sin^2(2h)}{h^3\cdot\cos(4h)}\right)$$

$$= \frac{1}{512}\lim_{h\to 0}\left(\frac{\sin(4h)}{h}\cdot\frac{2\sin^2(2h)}{h^2}\cdot\frac{1}{\cos(4h)}\right)$$

$$= \frac{1}{512}\lim_{h\to 0}\left(\frac{\sin(4h)}{4h}\times 4\cdot\frac{2\sin^2(2h)}{4h^2}\times 4\times\frac{1}{\cos(4h)}\right)$$

$$= \frac{1}{512}\times 1\times 4\times 1\times 8\times\frac{1}{1} \quad \ldots \left\{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1\right\}$$

$$= \frac{32}{512} \quad \text{Ans}$$

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Q.No. 12 → Find derivative of $f(x) = x\cos x$ using first principle

Soln $f'(x) = \lim_{h \to 0} \left(\frac{(x+h)\cos(x+h) - x\cos x}{h} \right)$

$= \lim_{h \to 0} \left(\frac{x\cos(x+h) + h\cos(x+h) - x\cos x}{h} \right)$

$= \lim_{h \to 0} \left(\frac{x(\cos(x+h) - \cos x)}{h} + \frac{\cancel{h}\cos(x+h)}{\cancel{h}} \right)$

$= \lim_{h \to 0} \left(\frac{-x \cdot \cancel{2}\sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{\frac{h}{2} \times \cancel{2}} + \cos(x+h) \right)$

$= -x\sin x \times 1 + \cos x \quad \cdots \left\{ \lim_{x \to 0} \frac{\sin x}{x} = 1 \right\}$

$\boxed{f'(x) = -x\sin x + \cos x}$ Ans

Qn 13 → evaluate $\lim_{x \to 0} \left(\frac{10^x - 2^x - 5^x + 1}{x\tan x} \right)$

Soln $\lim_{x \to 0} \left(\frac{2^x(5^x - 1) - 1(5^x - 1)}{x\tan x} \right)$

$= \lim_{x \to 0} \left(\frac{(2^x - 1)(5^x - 1)}{x\tan x} \right)$

$= \lim_{x \to 0} \left(\frac{\left(\frac{2^x - 1}{x}\right) \times \cancel{x} \times \left(\frac{5^x - 1}{x}\right) \times \cancel{x}}{\cancel{x}\left(\frac{\tan x}{x}\right) \times \cancel{x}} \right) = \log 2 \cdot \log 5$ Ans

Qn 14 $\lim_{x \to 0} \left(\frac{a^x - b^x - c^x + 1}{\tan x} \right)$

$= \lim_{x \to 0} \left(\frac{a^x - b^x - c^x + 1 - 1 + 1}{\tan x} \right)$

$= \lim_{x \to 0} \left(\frac{(a^x - 1) - 1(b^x - 1) - 1(c^x - 1)}{\tan x} \right)$

$= \lim_{x \to 1} \left(\frac{\frac{(a^x - 1)}{x} - \frac{(b^x - 1)}{x} - \frac{(c^x - 1)}{x}}{\frac{\tan x}{x}} \right)$

$= \frac{\log a - \log b - \log c}{1}$ .. $\left\{ \lim_{x \to 0} \left(\frac{\tan x}{x} \right) = 1, \ \lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \log a \right\}$

$= \log\left(\frac{a}{b}\right) - \log c$

$= \log\left(\frac{a}{bc}\right)$ Ans.

Qn. 15 → $f(x) = \lambda x^2 + \mu x + 12$

$f'(4) = 15$ & $f'(2) = 11$ Find λ & μ

Sol: Diff wrt x

$f'(x) = 2\lambda x + \mu$

$f'(4) = 8\lambda + \mu = 15$

$f'(2) = 4\lambda + \mu = 11$

$4\lambda = 4$

$\lambda = 1$ $\mu = 7$ Ans