! जय श्री राधे कृष्णा जय श्री गिरिराज जी महाराज !

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER : LIMITS & DERIVATIVES

← CLASS NO: 2 →

LIMITS

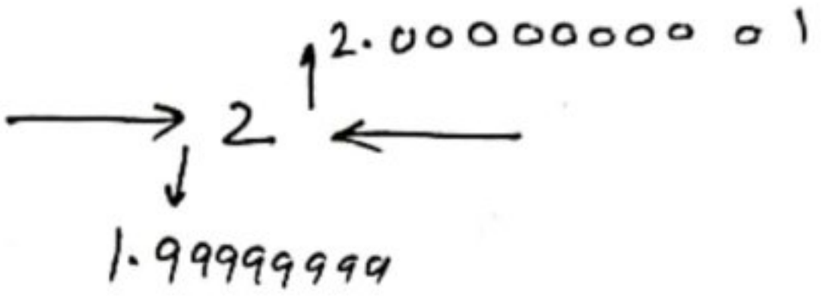
(·) $\lim_{x \to a} f(x)$

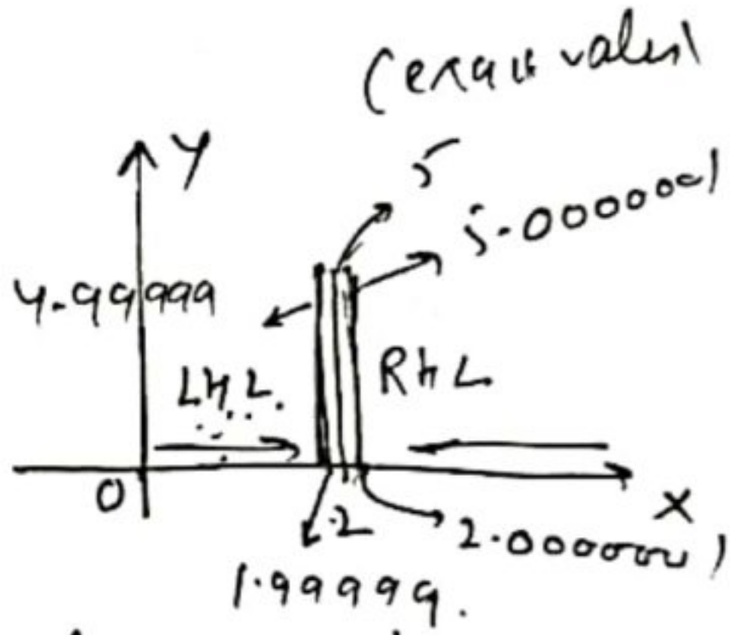
(·) $\lim_{x \to 2} (x+3) = 5$ (limit value)

(·) when $x \to 2$ then $f(x) \to 5$

(·) exact value $f(2) = 5$

(·) $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty$ (Indeterminate form)





TRIGONOMETRY LIMITS

① $\lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1$ ... {Reason when θ is very very small then $\sin\theta \approx \theta$ $x \to 0$ then $\sin x \simeq x$

② $\lim_{x \to 0} \left(\frac{\tan x}{x}\right) = 1$

③ $\lim_{x \to 0} (\cos x) = 1$

④ $\lim_{x \to 0} \left(\frac{\sin^2 x}{x^2}\right) = 1 = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 = (1)^2 = 1$

(·) $1 - \cos x = 2\sin^2(x/2)$

(1) evaluate $\lim_{x\to 0}\left(\frac{\sin(3x)}{x}\right)$

$= \lim_{x\to 0}\left(\frac{\sin(3x)}{3x}\right)\times 3$

$= 1 \times 3$ --- {when $x\to 0$, $3x\to 0$}

$= 3$ .. {$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$}

(2) $\lim_{x\to c}\left(\frac{\sin(ax)}{\sin(bx)}\right)$

$= \lim_{x\to c}\left(\frac{\frac{\sin(ax)}{ax}\times ax}{\frac{\sin(bx)}{bx}\times bx}\right)$

$= \frac{1\times a}{1\times b}$ --- {when $x\to 0$ then $ax\to 0$ and $bx\to 0$}

$= a/b$ Ans --- {$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$}

(3) $\lim_{x\to 0}(\operatorname{cosec}x - \cot x)$ $\uparrow 1\times 0 = 0$

$= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$ $= \lim_{x\to 0}\left(\frac{\tan(x/2)}{x/2}\times\frac{x}{2}\right)$

$= \lim_{x\to c}\left(\frac{1-\cos x}{\sin x}\right)$ OR

$= \lim_{x\to 0}\left(\frac{2\sin^2(x/2)}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$ $= \lim_{x\to 0}\left(\tan(x/2)\right) = 0.$

(4) $$\lim_{x\to 0}\left(\frac{1-\cos(mx)}{1-\cos(nx)}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin^2\left(\frac{mx}{2}\right)}{2\sin^2\left(\frac{nx}{2}\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin^2\left(\frac{mx}{2}\right)\times\frac{m^2x^2}{4}}{\frac{m^2x^2}{4}}\Big/\frac{\sin^2(nx/2)\times\frac{n^2x^2}{4}}{\frac{n^2x^2}{4}}\right)$$

$$= \frac{1\times m^2}{1\times n^2} \quad \text{--- when } x\to 0 \text{ then } \frac{mx}{2}\to 0 \text{ \& } \frac{nx}{2}\to 0,\ \lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$$

$$= \frac{m^2}{n^2} \quad \underline{\text{Ans}}$$

Q5 $$\lim_{x\to 0}\left(\frac{\sin(ax)+bx}{ax+\sin(bx)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin(ax)}{ax}\times ax+bx}{ax+\frac{\sin(bx)}{bx}\times bx}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin(ax)}{ax}\times a+b}{a+\frac{\sin(bx)}{bx}\times b}\right)$$

$$= \frac{1\times a+b}{a+1\times b} \quad \text{--- when } x\to 0 \text{ then } ax\to 0 \text{ \& } bx\to 0,\ \lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$$

$$= 1 \quad \underline{\text{Ans}}$$

QN:- 6 → evaluate $\lim_{x\to 0}\left(\frac{\sec(4x)-\sec(2x)}{\sec(3x)-\sec(x)}\right)$

$$= \lim_{x\to 0}\left(\frac{\frac{1}{\cos(4x)}-\frac{1}{\cos(2x)}}{\frac{1}{\cos(3x)}-\frac{1}{\cos x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\cos(2x)-\cos(4x)}{\cos x-\cos(3x)}\times\frac{\cos(3x)\cdot\cos x}{\cos(4x)\cdot\cos(2x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{-2\sin(3x)\cdot\sin(-x)}{-2\sin(2x)\cdot\sin(-x)}\times\frac{\cos(3x)\cos x}{\cos(4x)\cdot\cos(2x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin(3x)}{3x}\times 3x}{\frac{\sin(2x)}{2x}\times 2x}\times\frac{\cos(3x)\cdot\cos x}{\cos(4x)\cdot\cos(2x)}\right)$$

$$= \frac{1\times 3}{1\times 2}\times\frac{1}{1}$$

--- { when $x\to 0$, then $3x\to 0$ & $2x\to 0$, $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$ }

$$= 3/2 \quad \underline{\text{Ans}}$$

QN:- 7 → $\lim_{x\to 0}\left(\frac{\sin(6x)-\sin(3x)}{\sin(2x)-\sin(4x)}\right)$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin(6x)}{6x}\times 6x-\frac{\sin(3x)}{3x}\times 3x}{\frac{\sin(2x)}{2x}\times 2x-\frac{\sin(4x)}{4x}\times 4x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin(6x)}{6x}\times 6 - \frac{\sin(3x)}{3x}\times 3}{\frac{\sin(2x)}{2x}\times 2 - \frac{\sin(4x)}{4x}\times 4}\right)$$

$$= \frac{1\times 6 - 1\times 3}{1\times 2 - 1\times 4} \quad \left\{\begin{array}{l}\text{when } x\to 0\\ 6x\to 0,\ 4x\to 0,\ 3x\to 0,\ 2x\to 0\\ \text{and } \lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1\end{array}\right.$$

$$= \frac{3}{-2} = -3/2 \ \underline{\text{Ans}}$$

Q No: 8 → $\lim_{x\to 0}\left(\frac{\tan x - \sin x}{x^3}\right)$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin x}{\cos x} - \sin x}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x - \sin x\cos x}{x^3\cdot\cos x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x(1-\cos x)}{x^3\cdot\cos x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x\cdot 2\sin^2(x/2)}{x^3\cdot\cos x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{2\sin^2(x/2)}{\frac{x^2}{4}\times 4}\cdot\frac{1}{\cos x}\right)$$

$$= 1\times\frac{2\times 1}{4}\times\frac{1}{1} \quad \left\{\begin{array}{l}\text{when } x\to 0\\ x/2\to 0\\ \lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1\end{array}\right.$$

$$= 1/2 \ \underline{\text{Ans}}$$

QN:= 9 → Evaluate $\lim_{x\to 0}\left(\frac{1-\cos x\cdot\sqrt{\cos(2x)}}{x^2}\right)$

Rationalize

Rationalize
√ - √
√ - □
□ - √

$$= \lim_{x\to 0}\left(\frac{1-\cos x\sqrt{\cos(2x)}}{x^2} * \frac{1+\cos x\sqrt{\cos(2x)}}{1+\cos x\sqrt{\cos(2x)}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1-\cos^2 x\cdot\cos(2x)}{x^2(1+\cos x\sqrt{\cos(2x)})}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-\cos^2 x\cdot(2\cos^2 x-1)}{x^2(1+\cos x\sqrt{\cos(2x)})}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-2\cos^4 x+\cos^2 x}{x^2(\quad)}\right)$$

$$= -\lim_{x\to 0}\left(\frac{2\cos^4 x-\cos^2 x-1}{x^2(\quad)}\right)$$

$$= -\lim_{x\to 0}\left(\frac{2\cos^4 x-2\cos^2 x+\cos^2 x-1}{x^2(\quad)}\right)$$

$$= -\lim_{x\to 0}\left(\frac{2\cos^2 x(\cos^2 x-1)+1(\cos^2 x-1)}{x^2(\quad)}\right)$$

$$= -\lim_{x\to 0}\left(\frac{(\cos^2 x-1)\cdot(2\cos^2 x+1)}{x^2(\quad)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x\cdot(2\cos^2 x+1)}{x^2(1+\cos x\sqrt{\cos(2x)})}\right)$$

$$= 1\times\frac{(3)}{2} = 3/2 \quad \text{Ans}$$

Trigo limits type = 2

when $\lim_{x \to a} f(x)$ put $x = a+h$ (or) $x = a-h$

$h \to 0$

(10) $\lim_{x \to \pi/2} \left(\frac{1 - \sin x}{(\frac{\pi}{2} - x)^2} \right)$

$x \to 2$

$x \to 2+h$

$\Rightarrow 2 + 0.000000001$

$\Rightarrow 2.0000$

put $x = \pi/2 + h$ & $h \to 0$

$= \lim_{h \to 0} \left(\frac{1 - \sin(\frac{\pi}{2} + h)}{(\frac{\pi}{2} - (\frac{\pi}{2} + h))^2} \right)$

$= \lim_{h \to 0} \left(\frac{1 - \cos h}{(\frac{\pi}{2} - \frac{\pi}{2} - h)^2} \right)$

$= \lim_{h \to 0} \left(\frac{1 - \cos h}{h^2} \right)$

$= \lim_{x \to 0} \left(\frac{2 \sin^2(h/2)}{\frac{h^2}{4} \times 4} \right)$

$= \frac{2 \times 1}{4}$ -- { when $x \to 0$, $h/2 \to 0$, $\lim_{x \to 0} \left(\frac{\sin x}{x} \right) = 1$ }

$= 1/2$ Ans

Q.N 4

$$\lim_{x \to \frac{\pi}{6}} \left(\frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2} \right)$$

put $x = \pi/6 + h$ & $h \to 0$

$$= \lim_{h \to 0} \left(\frac{2 - \sqrt{3}\cos\left(\frac{\pi}{6} + h\right) - \sin\left(\frac{\pi}{6} + h\right)}{\left(6\left(\frac{\pi}{6} + h\right) - \pi\right)^2} \right)$$

$$= \lim_{h \to 0} \left[\frac{2 - \sqrt{3}\left(\frac{\sqrt{3}}{2}\cos h - \frac{1}{2}\sin h\right) - \left(\frac{1}{2}\cos h + \frac{\sqrt{3}}{2}\sin h\right)}{(\pi + 6h - \pi)^2} \right]$$

$$= \lim_{h \to 0} \left[\frac{2 - \frac{3}{2}\cos h + \frac{\sqrt{3}}{2}\sin h - \frac{1}{2}\cos h - \frac{\sqrt{3}}{2}\sin h}{36h^2} \right]$$

$$= \lim_{h \to 0} \left(\frac{2 - 2\cos h}{36h^2} \right)$$

$$= \lim_{h \to 0} \left(\frac{1 - \cos h}{18h^2} \right)$$

$$= \lim_{h \to 0} \left(\frac{2\sin^2(h/2)}{18 \frac{h^2}{4} \times 4} \right)$$

$$= \frac{2 \times 1}{18 \times 4} \quad \text{-- } \{ \text{When } h \to 0 \text{ then } \tfrac{h}{2} \to 0 \text{ and } \lim_{x \to 0} \left(\tfrac{\sin x}{x}\right) = 1 \}$$

$$= \frac{1}{36} \quad \text{Ans}$$

← LIMITS & DERIVATIVES →

WORKSHEET NO: 1

QN: 1 Evaluate $\lim_{x \to 0} \left(\frac{\tan x - \sin x}{\sin^3 x} \right)$ ANS: ½

QN: 2 Evaluate $\lim_{x \to 0} \left(\frac{\sin(2x) + 3x}{2x + \sin(3x)} \right)$ ANS: 1

QN: 3 → Evaluate $\lim_{x \to 0} \left(\frac{\sin(2x) + \sin(6x)}{\sin(5x) - \sin(3x)} \right)$ ANS: 4

QN: 4 → Evaluate $\lim_{x \to 0} \left(\frac{\cot(2x) - \operatorname{cosec}(2x)}{x} \right)$ ANS: −1

QN: 5 → Evaluate $\lim_{x \to 0} \left(\frac{\sin x - 2\sin(3x) + \sin(5x)}{x} \right)$ ANS: 0

QN 6 → Evaluate $\lim_{x \to 0} \left(\frac{\cos(2x) - 1}{\cos x - 1} \right)$ ANS: 4

QN 7 → Evaluate $\lim_{x \to 0} \left(\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x} \right)$ ANS: 1

QN 8 → If $\lim_{x \to 0} (k x \operatorname{cosec}(x)) = \lim_{x \to 0} (x \operatorname{cosec}(kx))$

Show that $k = \pm 1$

QN 9 → Evaluate $\lim_{x \to 0} \left(\frac{\sec(5x) - \sec(3x)}{\sec(3x) - \sec x} \right)$ ANS: 2

QN 10 → Evaluate $\lim_{x \to 0} \left(\frac{\sin(2x) \cdot (\cos(3x) - \cos x)}{x^3} \right)$ ANS: −8

QN 11 → Evaluate $\lim_{x \to 0} \left(\frac{\sin(a+x) + \sin(a-x) - 2\sin a}{x \sin x} \right)$ ANS: $-\sin a$

Qn 12 evaluate $\lim_{x \to \frac{\pi}{6}} \left(\frac{\sqrt{3}\sin x - \cos x}{x - \frac{\pi}{6}} \right)$ Ans = 2

Qn. 13 evaluate $\lim_{x \to \frac{\pi}{2}} \left(\frac{\tan(2x)}{x - \pi/2} \right)$ Ans = 2

Qn 14 → evaluate $\lim_{x \to \frac{\pi}{2}} \left(\frac{1 + \cos(2x)}{(\pi - 2x)^2} \right)$ Ans = 1/2

Qn 15 → evaluate $\lim_{x \to \frac{\pi}{2}} \left(\frac{\cot x - \cos x}{(\pi - 2x)^3} \right)$ Ans = $\frac{1}{16}$

Qn 16 → evaluate $\lim_{x \to \frac{\pi}{2}} \left(\frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} \right)$ Ans = 2

Qn 17 → evaluate $\lim_{x \to \frac{\pi}{4}} \left(\frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} \right)$ Ans $\frac{1}{16\sqrt{2}}$

Qn 18 → evaluate $\lim_{x \to \frac{\pi}{6}} \left(\frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1} \right)$ Ans = −3

(HINT: don't put $x = \frac{\pi}{6} + h$

(Make factors in both N & D)

(Split the Middle term)

— x —