॥ जय श्री राधे कृष्णा ॥ जय श्री गिरिराज जी महाराज ॥

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER: LIMITS & DERIVATIVES $\boxed{\text{CLASS No: 4}}$

Qn 1 $f(x) = x\sin x$

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{(x+h)\sin(x+h) - x\sin x}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{x\sin(x+h) + h\sin(x+h) - x\sin x}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{x\left(\sin(x+h)-\sin x\right)}{h} + \frac{\cancel{h}\sin(x+h)}{\cancel{h}}\right]$$

$$= \lim_{h\to 0}\left[\frac{x\cdot \cancel{2}\cos\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{\frac{h}{2}\times\cancel{2}} + \sin(x+h)\right]$$

$$= x\cos x \times 1 + \sin x \quad \cdots \left\{\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1\right\}$$

$$\boxed{f'(x) = x\cos x + \sin x} \text{ Ans}$$

Qn 2 $f(x) = x^2\cos x$

$$f'(x) = \lim_{h\to 0}\left[\frac{(x+h)^2\cdot\cos(x+h) - x^2\cos x}{h}\right]$$

$$f'(x) = \lim_{h\to 0}\left[\frac{(x^2+h^2+2hx)\cos(x+h) - x^2\cos x}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{x^2\cos(x+h) + (h^2+2hx)\cos(x+h) - x^2\cos x}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{x^2(\cos(x+h)-\cos x)}{h} + \frac{\cancel{h}(h+2x)\cos(x+h)}{\cancel{h}}\right]$$

$$= \lim_{h\to 0}\left(-x^2 \cdot \frac{\cancel{2}\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}\times\cancel{2}} + (h+2x)\cos(x+h)\right)$$

$$= -x^2\sin(x)\times 1 + 2x\cos x \quad \text{--} \left\{\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1\right\}$$

$$\boxed{f'(x) = -x^2\sin x + 2x\cos x} \text{ Ans}$$

Qu 3 $f(x) = \sin(x^2)$

$$f'(x) = \lim_{h\to 0}\left[\frac{\sin((x+h)^2) - \sin(x^2)}{h}\right]$$

$$= \lim_{h\to 0}\left(\frac{\sin(x^2+h^2+2hx) - \sin(x^2)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\cos\left(\frac{2x^2+h^2+2hx}{2}\right)\sin\left(\frac{h^2+2hx}{2}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\cos\left(\frac{2x^2+h^2+2hx}{2}\right)\sin\left(\frac{h^2+2hx}{2}\right)}{h\left(\frac{h^2+2hx}{2}\right)} \times \left(\frac{h^2+2hx}{2}\right)\right)$$

$$f'(x) = 2\cos(x^2) \times 1 \times \frac{2x}{2}$$

$$\boxed{f'(x) = 2x\cos(x^2)}$$

Q. 4 $\quad f(x) = \dfrac{2x+1}{3x+4}$ → Quotient rule check = Sol

$$\left\{ \frac{(3x+4)(2) - (2x+1)(3)}{(3x+4)^2} = \frac{5}{(3x+4)^2} \right. \text{ Ans}$$

$$f'(x) = \lim_{h \to 0} \left[\frac{\frac{2(x+h)+1}{3(x+h)+4} - \frac{2x+1}{3x+4}}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{(2x+2h+1)(3x+4) - (2x+1)(3x+3h+4)}{h(3x+3h+4)(3x+4)} \right]$$

$$= \lim_{h \to 0} \left[\frac{\cancel{6x^2} + \cancel{8x} + \cancel{6hx} + 8h + \cancel{3x} + \cancel{4} - \cancel{6x^2} - \cancel{6xh} - \cancel{8x} - \cancel{3x} - 3h - \cancel{4}}{h(3x+3h+4)(3x+4)} \right]$$

$$= \lim_{h \to 0} \left[\frac{5\cancel{h}}{\cancel{h}(3x+3h+4)(3x+4)} \right]$$

$$\boxed{f'(x) = \frac{5}{(3x+4)^2}} \text{ Ans}$$

Q. 5 $\quad f(x) = \sqrt{3x^2+1}$

$$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{3(x+h)^2+1} - \sqrt{3x^2+1}}{h} \right)$$

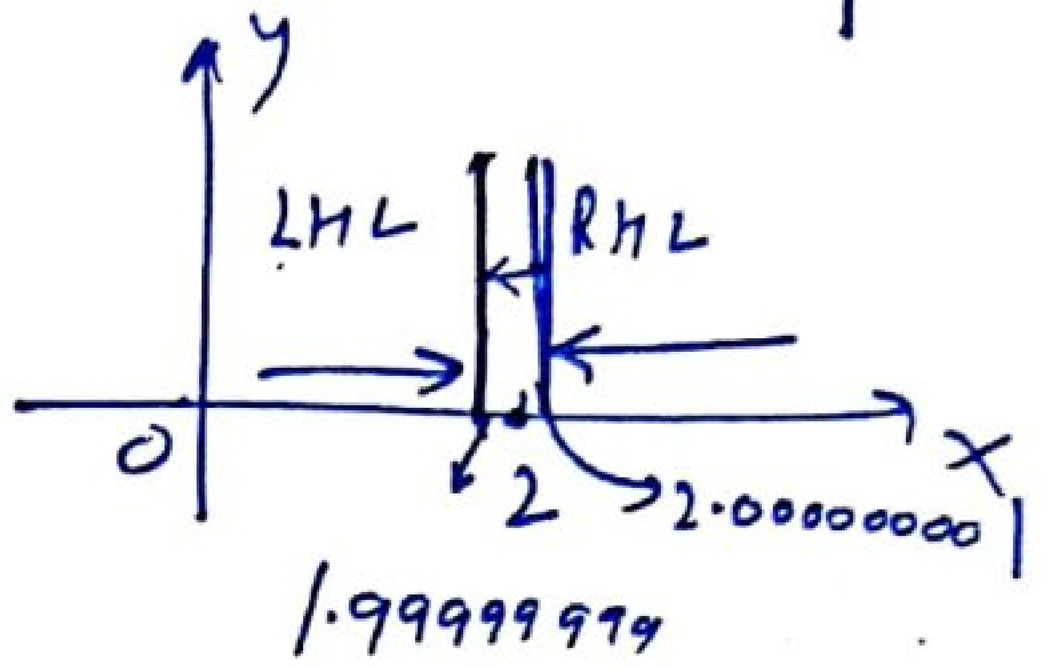
$$= \lim_{h \to 0} \left(\frac{\cancel{3x^2} + 3h^2 + 6xh + \cancel{1} - \cancel{3x^2} - \cancel{1}}{h\left(\sqrt{3(x+h)^2+1} + \sqrt{3x^2+1}\right)} \right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{\cancel{h}(3h+6x)}{\cancel{h}\left(\sqrt{3(x+h)^2+1}+\sqrt{3x^2+1}\right)}\right)$$

$$f'(x) = \frac{6x}{2\sqrt{3x^2+1}} = \frac{3x}{\sqrt{3x^2+1}} \quad \underline{\text{Ans}}$$

— x —

$\boxed{\text{LIMITS}}$



$\lim_{x\to a} f(x)$ exists

when $\boxed{LHL = RHL}$

$LHL = \lim_{x\to a^-}(f(x))$ put $x = a-h$, $h\to 0$; $x<a$

$h = 0.0000001$

$RHL = \lim_{x\to a^+}(f(x))$ put $x = a+h$ & $h\to 0$; $x>a$

Q-1
$$f(x) = \begin{cases} 3x^2-1 & ; x\le 2 \\ 2-3x^3 & ; x>2 \end{cases}$$

evaluate $\lim_{x\to 2} f(x)$ if exists.

Sol $LHL = \lim_{x\to 2^-}(3x^2-1)$

put $x = 2-h$ & $h\to 0$

$LHL = \lim_{h\to 0}\left(3(2-h)^2-1\right) = 12-1 = 11$ $\boxed{LHL = 11}$

$RHL = \lim_{x \to 2^+} (2 - 3x^3)$

put $x = 2+h$ & $h \to 0$

$RHL = \lim_{h \to 0} [2 - 3(2+h)^3]$

$= 2 - 3(8)$
$= -22$

$LHL \neq RHL$
$\therefore \lim_{x \to 2} (f(x))$ does not exist <u>Ans</u>

<u>Qn-2</u> $f(x) = \begin{cases} \frac{|x|}{x} ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}$

Check whether $\lim_{x \to 0} (f(x))$ exists?

<u>Method I</u>

$LHL = \lim_{x \to 0^-} \left(\frac{|x|}{x} \right)$

put $x = 0 - h = -h$ & $h \to 0$

$LHL = \lim_{h \to 0} \left(\frac{|-h|}{-h} \right)$

$= \lim_{h \to 0} \left(\frac{h}{-h} \right)$

$= \lim_{h \to 0} (-1)$

$\boxed{LHL = -1}$

$RHL = \lim_{x \to 0^+} \left(\frac{|x|}{x} \right)$

put $x = 0 + h = h$
$h \to 0$

$RHL = \lim_{h \to 0} \left(\frac{|h|}{h} \right)$

$= \lim_{h \to 0} \left(\frac{h}{h} \right)$

$= \lim_{h \to 0} (1)$

$RHL = 1$

$LHL \neq RHL$
$\therefore \lim_{x \to 0} f(x)$ does not exist

Method II

$$f(x) = \begin{cases} \frac{x}{x} & : x > 0 \\ \frac{-x}{x} & : x < 0 \\ 0 & . \; x = 0 \end{cases}$$

$$f(x) = \begin{cases} 1 & : x > 0 \\ -1 & x < 0 \\ 0 & : x = 0 \end{cases}$$

$$LHL = \lim_{x \to 0^-} (-1) = -1$$

$$RHL = \lim_{x \to 0^+} (1) = 1$$

$LHL \neq RHL$

$\therefore \lim_{x \to 0} f(x)$ does not exist

-x-

Que 3 $f(x) = \begin{cases} 5x - \lambda & : 0 < x \le 1 \\ 4x^3 - 3x & ; 1 < x \le 2 \end{cases}$ find value of λ

so that $\lim_{x \to 1} (f(x))$ exists.

Sol Since $\lim_{x \to 1} f(x)$ exists

$\Rightarrow LHL = RHL$

$$\Rightarrow \lim_{x \to 1^-} (5x - \lambda) = \lim_{x \to 1^+} (4x^3 - 3x)$$

put $x = 1 - h$ & $h \to 0$ put $x = 1 + h$ $h \to 0$

$$\Rightarrow \lim_{h \to 0} (5(1-h) - \lambda) = \lim_{h \to 0} (4(1+h)^3 - 3(1+h))$$

$\Rightarrow 5 - \lambda = 4 - 3$

$\Rightarrow -\lambda = -4$

$\boxed{\lambda = 4}$ Ans

Qn. 4 → $f(x) = \begin{cases} a+bx & ; \quad x<1 \\ 4 & ; \quad x=1 \\ b-ax & ; \quad x>1 \end{cases}$

and if $\lim_{x\to 1} f(x) = f(1)$. Find the values of a & b.

Soln $f(1) \to$ value of function when $x=1$

$\Rightarrow f(1) = 4$

$\Rightarrow \lim_{x\to 1} f(x) = 4$

$\to$ LHL = RHL = 4

$\Rightarrow \lim_{x\to 1^-} (a+bx) = \lim_{x\to 1^+} (b-ax) = 4$

put $x = 1-h$, $h\to 0$ put $x = 1+h$, $h \to 0$

$\Rightarrow \lim_{h\to 0} (a+b(1-h)) = \lim_{h\to 0} (b-a(1+h)) = 4$

$\Rightarrow a+b = b-a = 4$

$\Rightarrow$
$$\begin{array}{r} \cancel{a}+b=4 \\ b-\cancel{a}=4 \\ \hline 2b=8 \end{array}$$

$\Rightarrow b=4$ $a=0$ Ans

Qn. 5 $f(x) = \begin{cases} mx^2+n & : \quad x<0 \\ nx+m & : \quad 0 \le x \le 1 \\ nx^3+m & ; \quad x>1 \end{cases}$

For what integers 'm' and 'n' does both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist ?

Soln Given $\lim_{x\to 0} f(x)$ exists

$\Rightarrow$ LHL = RHL

$\Rightarrow \lim_{x\to 0^-}(mx^2+n) = \lim_{x\to 0^+}(nx+m)$

put $x = 0-h$ $= -h$, $h\to 0$ put $x = 0+h$ $= h$, $h\to 0$

$\Rightarrow \lim_{h\to 0}\left(m(-h)^2+n\right) = \lim_{h\to 0}(nh+m)$

$\Rightarrow \boxed{n = m}$ — (1)

& also $\lim_{x\to 1} f(x)$ exists

$\Rightarrow$ LHL = RHL

$\Rightarrow \lim_{x\to 1^-}(nx+m) = \lim_{x\to 1^+}(nx^3+m)$

put $x = 1-h$, $h\to 0$ put $x = 1+h$, $h\to 0$

$\Rightarrow \lim_{h\to 0}\left(n(1-h)+m\right) = \lim_{h\to 0}\left(n(1+h)^3+m\right)$

$\Rightarrow \boxed{n+m = n+m}$ — (2)

$\therefore$ m and n can be any integers such that $m=n$ Ans

Qn. 6 → $f(x) = \begin{cases} |x|+1 & ; x<0 \\ 0 & ; x=0 \\ |x|-1 & ; x>0 \end{cases}$

For what value of 'a' does $\lim_{x \to a} f(x)$ exists ?

Sol $LHL = \lim_{x \to 0^-} (|x|+1)$

put $x = 0-h = -h$ & $h \to 0$

$LHL = \lim_{h \to 0} (|-h|+1)$

$LHL = 1$

$RHL = \lim_{x \to 0^+} (|x|-1)$

put $x = 0+h = h$ & $h \to 0$

$RHL = \lim_{h \to 0} (|h|-1)$

$RHL = 0-1$ $\quad$ $RHL = -1$

$LHL \neq RHL$

$\Rightarrow \lim_{x \to 0} (f(x))$ does not exists ...(1)

but we are given that

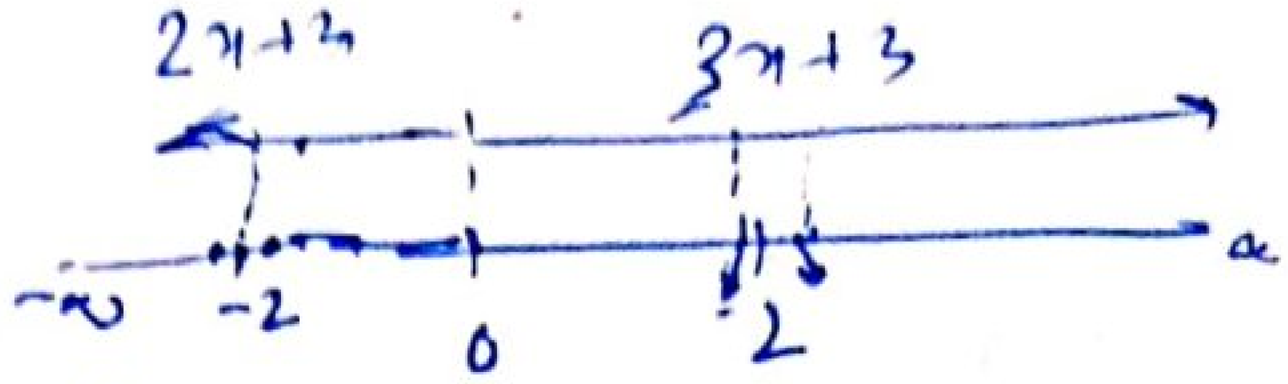
$\lim_{x \to a} f(x)$ exists ...(2)

$\Rightarrow a \neq 0$

$a \in R - \{0\}$ Ans

Q.7 $f(x) = \begin{cases} 2x+3 & : x \le 0 \\ 3x+3 & : x > 0 \end{cases}$

evaluate $\lim_{x\to 2} (f(x))$



Sol. $LHL = \lim_{x\to 2^-} (3x+3)$

put $x = 2-h$, $h \to 0$

$RHL = \lim_{x\to 2^+} (3x+3)$

put $x = 2+h$, $h \to 0$

process

Q.8 → evaluate $\lim_{x\to 5} (f(x))$ when $f(x) = |x| - 5$

Sol $f(x) = \begin{cases} x-5 & : x \ge 0 \\ -x-5 & : x < 0 \end{cases}$

process

Q.9 $\lim_{x\to 1}\left(\frac{f(x)-2}{x^2-1}\right) = \lambda$ evaluate $\lim_{x\to 1}(f(x))$

$\Rightarrow \dfrac{\lim_{x\to1}(f(x)) - \lim_{x\to1}(2)}{\lim_{x\to1}(x^2-1)} = \lambda$

$\Rightarrow \lim_{x\to1} f(x) - 2 = \lambda\left(\lim_{x\to1}(x^2-1)\right)$

$\Rightarrow \lim_{x\to1}(f(x)) - 2 = 0$

$\Rightarrow \lim_{x\to1}(f(x)) = 2$ Ans

← WORKSHEET No:- 3 → (Class No-4)

Limits & Derivatives

Qn 1 $f(x) = x\tan x$ (using first principle) Ans $f'(x) = x\sec^2 x + \tan x$

Qn 2 $f(x) = x^2\sin x$ (first principle) Ans $f'(x) = x^2\cos x + 2x\sin x$

Qn 3 $f(x) = \cot(x^2)$ (first principle) Ans $-2x\,\mathrm{cosec}(x^2)$

Qn 4 $f(x) = \mathrm{cosec}(x^2)$ (first principle) Ans $f'(x) = -2x\,\mathrm{cosec}(x^2).\cot(x^2)$

Qn 5 → $f(x) = \dfrac{3x-5}{2x+4}$ (first principle) Ans $f'(x) = \dfrac{22}{(2x+4)^2}$

Qn 6 $f(x) = \dfrac{1}{\sqrt{2x+3}}$ (first principle) Ans $-\dfrac{1}{(2x+3)^{3/2}}$

Qn 7 → Evaluate $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} \dfrac{x}{|x|} ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}$

Ans limit does not exist

Qn 8 → Given $\lim_{x\to 2} (f(x))$ exists find value of 'λ'

where $f(x) = \begin{cases} 4x-5 : & x \le 2 \\ x-\lambda ; & x > 2 \end{cases}$ Ans $\lambda = -1$

Qn 9 → $f(x) = \begin{cases} \dfrac{3x}{|x|+2x} ; & x \neq 0 \\ 0 : & x = 0 \end{cases}$ Show that $\lim_{x\to 0} f(x)$ does not exist.

Qn 10 → $f(x) = \begin{cases} x - [x] : & x < 2 \\ 4 : & x = 2 \\ 3x-5 : & x > 2 \end{cases}$

Evaluate $\lim_{x\to 2} f(x)$ Ans $= 1$

—x—