← ULTIMATE MATHEMATICS →

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Chapter: PMI (3rd class)

Type: 3 Inequalities

Ques 1 Using PMI, show that $(2n+7) < (n+3)^2$

Soln let $P(n): (2n+7) < (n+3)^2$

$P(1): \quad 2+7 < (1+3)^2$

$9 < 16$

clearly $P(1)$ is true

let $P(k)$ be true

$P(k): (2k+7) < (k+3)^2$

T.P $P(k+1)$ is true

$P(k+1): \quad (2k+9) < (k+4)^2$ (or) $k^2+8k+16$

we have $\quad (2k+7) < (k+3)^2$

adding 2

Compare

$(2k+7)+2 < (k+3)^2+2$

$\Rightarrow (2k+9) < k^2+6k+11 <$

$\Rightarrow (2k+9) < k^2+6k+11 < k^2+8k+16$

$\Rightarrow (2k+9) < k^2+8k+16$

$\Rightarrow (2k+9) < (k+4)^2$

$\therefore P(k+1)$ is true

$\therefore$ By PMI, $P(n)$ is true for all $n \in N$

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PMI Class No: 3

Qn 1 → Show using PMI

$1^2+2^2+\cdots\cdots n^2 > \frac{n^3}{3}$ for all $n \in N$.

Sol:

$P(n): 1^2+2^2+\cdots\cdots n^2 > \frac{n^3}{3}$

$P(1): 1^2 > \frac{1^3}{3}$

$1 > \frac{1}{3}$ clearly $P(1)$ is true

Let $P(k)$ be true

$P(k)= 1^2+2^2+\cdots k^2 > \frac{k^3}{3}$

To prove $P(k+1)$ is true

$P(k+1): 1^2+2^2+\cdots k^2+(k+1)^2 > \frac{(k+1)^3}{3}$ (or) $\frac{k^3+1+3k^2+3k}{3}$

We have $1^2+2^2+\cdots k^2 > \frac{k^3}{3}$

adding $(k+1)^2$

$1^2+2^2+\cdots k^2+(k+1)^2 > \frac{k^3}{3}+(k+1)^2$

$\Rightarrow 1^2+2^2+3^2\cdots k^2+(k+1)^2 > \frac{k^3+3k^2+3+6k}{3}$

$\Rightarrow 1^2+2^2+\cdots k^2+(k+1)^2 > \frac{k^3+3k^2+3+6k}{3} > \frac{k^3+1+3k^2+3k}{3}$

$\Rightarrow 1^2+2^2+\cdots k^2+(k+1)^2 > \frac{k^3+1+3k^2+3k}{3}$

$\Rightarrow 1^2+2^2+\cdots k^2+(k+1)^2 > \frac{(k+1)^3}{3}$

$\therefore P(k+1)$ is true

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PMI (Class No: 3) (3)

Ans 3 → Show using PMI

(self) $1+2+3 \ldots n < \frac{(2n+1)^2}{8}$

Qn. 4 → Prove that $(1+x)^n \geq (1+nx)$; where $x > -1$ using PMI

Sol $P(n): (1+x)^n \geq (1+nx)$

$P(1): 1+x \geq 1+x$ ∴ $P(1)$ is true

let $P(k)$ be true

$P(k): (1+x)^k \geq (1+kx)$

T.P $P(k+1)$ is true

$P(k+1): (1+x)^{k+1} \geq (1+(k+1)x)$ or $(1+kx+x)$

we have

$(1+x)^k \geq (1+kx)$

multiply by $(1+x)$

$(1+x)^k \cdot (1+x) \geq (1+kx)(1+x)$

$\Rightarrow (1+x)^{k+1} \geq 1+x+kx+kx^2$

$\Rightarrow (1+x)^{k+1} \geq (1+x+kx+kx^2) \geq 1+kx+x$

$\Rightarrow (1+x)^{k+1} \geq 1+kx+x \ldots$ { ~~when $x > -1$~~ given / when $x=0$ / $kx^2 = 0$ }

$\Rightarrow (1+x)^{k+1} \geq 1+(k+1)x$

∴ $P(k+1)$ is true

PMI Class No. 3

Qn. 5 → Using PMI show that $2^n > n$ for all $n \in N$. (self)

Qn. 6 → using PMI, show that

$$\cos\theta \cdot \cos(2\theta) \cdot \cos(4\theta) \cdots \cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n \sin\theta}$$

Sol

$$P(n): \cos\theta \cdot \cos(2\theta) \cdots \cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n\sin\theta}$$

$$P(1): \cos\theta = \frac{\sin(2\theta)}{2\sin\theta} = \frac{\cancel{2}\cancel{\sin\theta}\cos\theta}{\cancel{2}\cancel{\sin\theta}}$$

$$\Rightarrow \cos\theta = \cos\theta \quad \therefore P(1) \text{ is true}$$

let $P(k)$ be true

$$P(k): \boxed{\cos\theta \cdot \cos(2\theta) \cdots \cos(2^{k-1}\theta)} = \frac{\sin(2^k\theta)}{2^k\sin\theta}$$

$$P(k+1): \cos\theta \cdot \cos(2\theta) \cdots \cos(2^{k-1}\theta) \cdot \cos(2^k\theta) = \frac{\sin(2^{k+1}\theta)}{2^{k+1}\sin\theta}$$

LHS

$$\boxed{\cos\theta \cdot \cos(2\theta) \cdots \cos(2^{k-1}\theta)}\,\cos(2^k\theta)$$

$$= \frac{\sin(2^k\theta)}{2^k\sin\theta} \cdot \cos(2^k\theta) \quad \cdots \left\{ 2\sin\theta \cdot \cos\theta = \frac{1}{2}\sin(2\theta) \right\}$$

$$= \frac{\frac{1}{2}\left[2\sin(2^k\theta) \cdot \cos(2^k\theta)\right]}{2^k\sin\theta}$$

$$= \frac{\sin(2 \cdot 2^k\theta)}{2^{k+1} \cdot \sin\theta} = \frac{\sin(2^{k+1}\theta)}{2^{k+1}\sin\theta} = \text{RHS}$$

$\therefore P(k+1)$ is true

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PMI (3rd class)

Qu. 7 → Show using PMI

$$\sin\theta + \sin(3\theta) + \cdots \sin(n\theta) = \frac{\sin\left(\frac{n\theta}{2}\right)\cdot\sin\left(\frac{n+1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}$$

Soln let $P(n): \sin\theta + \sin(3\theta) + \cdots \sin(n\theta) = \frac{\sin\left(\frac{n\theta}{2}\right)\cdot\sin\left(\frac{n+1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}$

$P(1): \sin\theta = \frac{\cancel{\sin\left(\frac{\theta}{2}\right)}\cdot\sin\left(\frac{1+1}{2}\right)\theta}{\cancel{\sin\frac{\theta}{2}}} = \sin\left(\frac{2\theta}{2}\right) = \sin\theta$

$\therefore P(1)$ is true

let $P(k)$ be true

$P(k): \sin\theta + \sin(3\theta) + \cdots \sin(k\theta) = \frac{\sin\left(\frac{k\theta}{2}\right)\cdot\sin\left(\frac{k+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$

$P(k+1)$:

$\sin\theta + \sin(3\theta) + \cdots \sin(k\theta) + \sin(k+1)\theta = \frac{\sin\left(\frac{k+1}{2}\right)\theta\cdot\sin\left(\frac{k+2}{2}\right)\theta}{\sin\frac{\theta}{2}}$

LHS $\boxed{\sin\theta + \sin(3\theta) + \cdots \sin(k\theta)} + \sin(k+1)\theta$

$= \frac{\sin\left(\frac{k\theta}{2}\right)\cdot\sin\left(\frac{k+1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)} + \sin(k+1)\theta$

$= \left[\frac{\sin\left(\frac{k\theta}{2}\right)\cdot\sin\left(\frac{k+1}{2}\right)\theta + \sin(k+1)\theta\cdot\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right]$

$= \frac{2\sin\left(\frac{k\theta}{2}\right)\cdot\sin\left(\frac{k+1}{2}\right)\theta + 2\sin(k+1)\theta\cdot\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$ (A) (B) (A) (B)

$= \frac{\cos\left[\frac{k\theta}{2} - \frac{(k+1)\theta}{2}\right] - \cos\left[\frac{k\theta}{2} + \frac{(k+1)\theta}{2}\right] + \cos\left[(k+1)\theta - \frac{\theta}{2}\right] - \cos\left((k+1)\theta + \frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$

PMI (class no: 3)

$$= \frac{\cos\left(-\frac{\theta}{2}\right) - \cos\left(\frac{2k\theta+\theta}{2}\right) + \cos\left(\frac{2k\theta+\theta}{2}\right) - \cos\left(\frac{2k\theta+3\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{\overset{(A)}{\cos\left(\frac{\theta}{2}\right)} - \cos\overset{(B)}{\left(\frac{2k\theta+3\theta}{2}\right)}}{2\sin\frac{\theta}{2}}$$

$$= \frac{-2\sin\left(\frac{\frac{\theta}{2}+\frac{2k\theta+3\theta}{2}}{2}\right)\cdot\sin\left(\frac{\frac{\theta}{2}-\frac{(2k\theta+3\theta)}{2}}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{-\cancel{2}\sin\left(\frac{2k\theta+4\theta}{4}\right)\cdot\sin\left(-\frac{(2k\theta+2\theta)}{4}\right)}{\cancel{2}\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(\frac{(k+2)\theta}{2}\right)\cdot\sin\left(\left(\frac{k+1}{2}\right)\theta\right)}{\sin\frac{\theta}{2}} = \text{RHS}$$

$\therefore P(k+1)$ is true.

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Qn. 8 → Using PMI, show that
(self)

$$\sin\alpha + \sin(\alpha+\beta) + \cdots\cdots \sin(\alpha+(n-1)\beta) = \frac{\sin\left(\alpha+\frac{(n-1)}{2}\beta\right)\cdot\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Qn 9 → If $10^n + 3\cdot 4^{n+2} + k$ is divisible by 9 for all $n \in N$, then find the least positive integral value of k

Ans $k = 5$