ANSWER KEY / Correct SOLUTIONS of EXAM No: 6

(1)

COMPLEX NUMBERS

Ans: 1 given

$$\frac{3-4ix}{3+4ix} = \alpha - i\beta \quad \cdots (1)$$

taking conjugate

$$\frac{3+4ix}{3-4ix} = \alpha + i\beta \quad - (2)$$

(1) × (2)

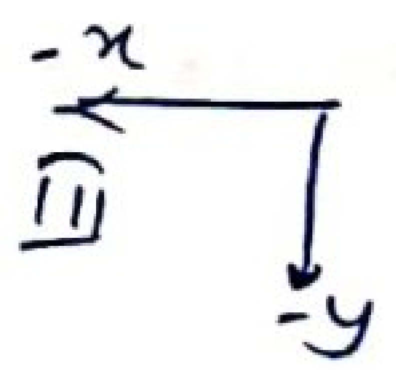
$$\frac{(\cancel{3-4ix})}{\cancel{3+4ix}} \times \frac{\cancel{3+4ix}}{\cancel{3-4ix}} = (\alpha - i\beta)(\alpha + i\beta)$$

$$\Rightarrow 1 = \alpha^2 + \beta^2$$

$\therefore$ $\boxed{\text{Ans: B}}$

Ans 2 → $z = x + iy$ is in 3^{rd} quadrant

$\Rightarrow x < 0, \; y < 0$



Now $\frac{\bar{z}}{z} = \frac{x-iy}{x+iy} \times \frac{x-iy}{x-iy}$

$$\frac{\bar{z}}{z} = \frac{(x^2-y^2)}{x^2+y^2} - \frac{2ixy}{x^2+y^2}$$

this will be in 3^{rd} quadrant if

Real part is -ve and imaginary part is -ve

$\frac{x^2-y^2}{x^2+y^2}$ should be -ve

$\Rightarrow \boxed{x < y}$

$\frac{-2xy}{x^2+y^2}$ should be -ve

given $x<0, \; y<0$

$\therefore \frac{-2xy}{x^2+y^2}$ is already -ve

$\therefore \quad x < y \quad$ and $\quad x < 0,\ y < 0$

$\Rightarrow \quad x < y < 0 \quad \therefore \boxed{\text{ANS} = B}$

Qn. 3 → let $z = \dfrac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$

$$z = \frac{1 - i\sin\alpha}{1 + 2i\sin\alpha} \times \frac{1 - 2i\sin\alpha}{1 - 2i\sin\alpha}$$

$$z = \frac{1 - 2i\sin\alpha - i\sin\alpha + 2i^2\sin^2\alpha}{1 - 4i^2\sin^2\alpha}$$

$$z = \frac{(1 - 2\sin^2\alpha)}{1 + 4\sin^2\alpha} - \frac{3i\sin\alpha}{1 + 4\sin^2\alpha}$$

for purely real $\quad Im(z) = 0$

$$\Rightarrow \frac{-3\sin\alpha}{1 + 4\sin^2\alpha} = 0$$

$$\Rightarrow \sin\alpha = 0$$

$\alpha = n\pi \quad \boxed{\text{ANS}: C}$

Qns. 4 $\quad \displaystyle\sum_{n=1}^{13} (i^n + i^{n+1})$

$$= (i + i^2) + (i^2 + i^3) + (i^3 + i^4) + (i^4 + i^5) + \cdots (i^{13} + i^{14})$$

$$= (i + i^2 + i^3 + i^4 + \cdots i^{13}) + (i^2 + i^3 + i^4 + i^5 + \cdots i^{14})$$

$\boxed{\text{Note}}$ Sum of 4 consecutive powers of i is always equal to zero

i.e $\quad i + i^2 + i^3 + i^4 = 0$

$\therefore = (0 + i^{13}) + (0 + i^{14}) = i - 1 = -1 + i \quad \therefore \boxed{\text{ANS} = C}$

Qn. 5

$$|z+1-i| = |z-1+i|$$

let $z = x+iy$

$$|x+iy+1-i| = |x+iy-1+i|$$

$$\Rightarrow |(x+1)+i(y-1)| = |(x-1)+i(y+1)|$$

$$\Rightarrow \sqrt{(x+1)^2+(y-1)^2} = \sqrt{(x-1)^2+(y+1)^2}$$

Squaring

$$\cancel{x^2}+1+2x+\cancel{y^2}+1-2y = \cancel{x^2}+1-2x+\cancel{y^2}+1+2y$$

$$\Rightarrow 4x-4y=0$$

$\Rightarrow x-y=0$ clearly it represents a straight line

$\therefore$ $\boxed{\text{Ans} = D}$

Qn 6 +

$$\frac{i^{4n+1}-i^{4n-1}}{2}$$

$$= \frac{i^{4n}\cdot i - \frac{i^{4n}}{i}}{2}$$

we know that $i^{4n} = 1$

$$= \frac{i-\frac{1}{i}}{2}$$

$$= \frac{i+i}{2} \quad \text{-- } \left\{\because \frac{1}{i} = -i\right\}$$

$$= \frac{2i}{2} = i \quad \therefore \boxed{\text{Ans} = D}$$

Qn.7 →

$$z = \frac{11-3i}{1+i}$$

$$z = \frac{11-3i}{1+i} \times \frac{1-i}{1-i}$$

$$z = \frac{11-11i-3i+3i^2}{1-i^2}$$

$$z = \frac{8-14i}{2} = 4-7i$$

Now Given $|z-i\alpha| = 4$

$$\Rightarrow |4-7i-i\alpha| = 4$$

$$\Rightarrow |4-i(7+\alpha)| = 4$$

$$\Rightarrow \sqrt{16+(7+\alpha)^2} = 4$$

Squaring

$$16+49+\alpha^2+14\alpha = 16$$

$$\Rightarrow \alpha^2+14\alpha+49=0$$

$$\Rightarrow (\alpha+7)^2=0$$

$$\Rightarrow \alpha+7=0$$

$$\Rightarrow \alpha=-7 \quad \therefore \boxed{\text{Ans} = A}$$

Qn.8 →

$$|z|^2 + 7\bar{z} = 0$$

let $z = x+iy$

$$\left(\sqrt{x^2+y^2}\right)^2 + 7(x-iy) = 0$$

$$x^2+y^2+7x-7iy=0$$

$\Rightarrow (x^2+y^2+7x) - 7iy = 0+0i$

$\Rightarrow x^2+y^2+7x=0 \quad \Big| \quad -7y=0$

$y=0$

$\Rightarrow x^2+7x=0$

$x(x+7)=0$

$x=0, \; x=-7$

$\therefore z=0+0i$

$z=-7+0i$

$\boxed{\text{Ans}=A}$

$\therefore$ two solutions possible

Qn 9 →

$$(-\sqrt{-1})^{4n+3}$$

$= (-i)^{4n+3}$ --- {since $\sqrt{-1}=i$}

$= (-1)^{4n+3} \cdot i^{4n+3}$ --- {$(ab)^n = a^n b^n$}

$= (-1)\, i^{4n} \cdot i^3$ --- {$(-ve)^{\text{odd power}} = -ve$}

$= (-1)(1)\, i^3$ -- {$i^{4n}=1$}

$= (-1)(-i)$

$= i \quad \therefore \boxed{\text{Ans}=B}$

Qn 10 →

$$x^3+7x^2-x+16$$

given $x=1+2i$

$\Rightarrow x-1=2i$

Sq. $x^2-2x+1=4i^2$

$\Rightarrow x^2-2x+1=-4$

$\Rightarrow x^2-2x+5=0$

$$
\begin{array}{r|l}
 & x+9 \\
x^2-2x+5 & x^3+7x^2-x+16 \\
 & -(x^3-2x^2+5x) \\
\hline
 & 9x^2-6x+16 \\
 & -(9x^2-18x+45) \\
\hline
 & 12x-29
\end{array}
$$

$\therefore x^3+7x^2-x+16$

$=(x^2-2x+5)(x+9)+(12x-29)$

$=(0)(x+9)+12x-29$

$=12x-29$

$=12(1+2i)-29$ --- {given $x=1+2i$}

$=12+24i-29$

$=-17+24i$ Ans

$\boxed{\text{Ans:- B}}$

Qn 11 →

$\cancel{z(2i)}$ $z(2-i)=3+i$

$\Rightarrow z=\frac{3+i}{2-i}$

$\Rightarrow$

$z=\frac{(3+i)\times(2+i)}{(2-i)\times(2+i)}$

$z=\frac{6+3i+2i+i^2}{4-i^2}$

$z=\frac{5}{5}+\frac{5i}{5}=1+i$

$z^{20}=(1+i)^{20}$

$=[(1+i)^2]^{10}=(\cancel{1}+\cancel{i^2}+2i)^{10}=(2i)^{10}$

$=2^{10}i^{10}$

$= 2^{10} i^2$

$= 2^{10}(-1)$

$= -2^{10}$ $\boxed{\text{Ans:- A}}$

Qn 12 →

$$|z^2-1| = |z|^2+1$$

let $z = x+iy$

$\Rightarrow |(x+iy)^2-1| = \left(\sqrt{x^2+y^2}\right)^2+1$

$\Rightarrow |x^2-y^2+2ixy-1| = x^2+y^2+1$

$\Rightarrow |(x^2-y^2-1)+2ixy| = x^2+y^2+1$

$\Rightarrow \sqrt{(x^2-y^2-1)^2+(2xy)^2} = x^2+y^2+1$

Squaring

$\cancel{x^4}+\cancel{y^4}+\cancel{1}-2x^2y^2+\cancel{2y^2}-2x^2+4x^2y^2 = \cancel{x^4}+\cancel{y^4}+2x^2y^2+\cancel{1}+\cancel{2y^2}+2x^2$

$\Rightarrow 2x^2y^2-2x^2 = \cancel{2x^2y^2}+2x^2$

$\Rightarrow 4x^2=0$

$\Rightarrow x^2=0$

$\Rightarrow x=0$

$\therefore$ z lies on Imaginary Axis $\boxed{\text{Ans:- A}}$

Qn:- 13 → let $z = \dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$

Rationalize

$$= \frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}} \times \frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} + \sqrt{5-12i}}$$

$$= \frac{(5+\cancel{12i}) + (5-\cancel{12i}) + 2\sqrt{(5+12i)(5-12i)}}{(\cancel{5}+12i) - (\cancel{5}-12i)}$$

$$= \frac{10 + 2\sqrt{25+144}}{24i}$$

$$= \frac{10 + 2(13)}{24i}$$

$$= \frac{36}{24i}$$

$$= -\frac{36}{24}i \quad \cdots \left\{\because \frac{1}{i} = -i\right\}$$

$$z = -\frac{3}{2}i$$

(or) $z = 0 - \frac{3}{2}i$

Now $\bar{z} = 0 + \frac{3}{2}i$ $\quad \therefore$ $\boxed{\text{ANS} = B \text{ or } D}$

Q.No 14 → $z = (1+i)^4 \left(1+\frac{1}{i}\right)^4$

$$z = (1+i)^4 (1-i)^4 \quad \cdots \left\{\because \frac{1}{i} = -i\right\}$$

$$= ((1+i)(1-i))^4 \quad \cdots \{a^n b^n = (ab)^n\}$$

$$= (1-i^2)^4$$

$$= (1+1)^4$$

$$= 2^4 = 16$$

$\therefore$ $\boxed{\text{ANS} = B}$

Qn. 15 → given

$$(x^4+2xi)-(3x^2+iy)=(3-5i)+(1+2iy)$$

$$\Rightarrow (x^4-3x^2)+i(2x-y)=(4)+i(2y-5)$$

$x^4-3x^2=4$

$x^4-3x^2-4=0$

$(x^2-4)(x^2+1)=0$

$x^2=4$; $x^2=-1$

$\boxed{x=\pm 2}$; $x=\pm i$

Rejected (real values are required)

$2x-y=2y-5$

$2x=3y+5$

$y=\frac{2x+5}{3}$

for $x=2$; $y=\frac{9}{3}=3$

$x=-2$; $y=1/3$

$\therefore x=2, y=3$ (or) $x=-2, y=1/3$ $\therefore$ $\boxed{\text{Ans} = C}$

Qn 16 → $|z+1|=z+2+2i$

let $z=x+iy$

$|x+iy+1|=x+iy+2+2i$

$\Rightarrow |(x+1)+iy|=(x+2)+i(y+2)$

$\Rightarrow \sqrt{(x+1)^2+y^2}=(x+2)+i(y+2)$

$\Rightarrow \sqrt{x^2+y^2+2x+1} + 0i = (x+2) + i(y+2)$

equating real & imaginary parts

$\sqrt{x^2+y^2+2x+1} = x+2$ and $0 = y+2$

$y = -2$

$\Rightarrow \sqrt{x^2+4+2x+1} = x+2$

squaring

$\cancel{x^2}+2x+5 = \cancel{x^2}+4+4x$

$\Rightarrow 2x = +1$

$\Rightarrow x = 1/2$

$\therefore z = \frac{1}{2} - 2i$ $\boxed{\text{Ans} = C}$

Qn-17

given $z^2 + |z|^2 = 0$

let $z = x+iy$

$(x+iy)^2 + \left(\sqrt{x^2+y^2}\right)^2 = 0$

$\Rightarrow x^2 - \cancel{y^2} + 2ixy + x^2 + \cancel{y^2} = 0$

$\Rightarrow 2x^2 + 2ixy = 0 + 0i$

$\Rightarrow 2x^2 = 0$ | $2xy = 0$

$\Rightarrow \boxed{x=0}$ | Since $x = 0$

$\therefore y$ can take any value

$\therefore$ there are Infinite many solutions

$\boxed{\text{Ans} = C}$

Ans 18

$$|1-i|^x = 2^x$$

$$\Rightarrow (\sqrt{1+1})^x = 2^x$$

$$\Rightarrow (\sqrt{2})^x = 2^x$$

$$\Rightarrow (2)^{x/2} = 2^x$$

$$\Rightarrow \frac{x}{2} = x$$

$$\Rightarrow x = 2x$$

$$\Rightarrow 2x - x = 0$$

$$\Rightarrow x = 0$$

$\boxed{\text{Ans}= D}$

$\therefore$ there is no Non-zero solution.

Ans. 19 $\rightarrow$

$$z = \frac{1+i}{1-i} - \frac{1-i}{1+i}$$

$$z = \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}$$

$$z = \frac{\cancel{1}+\cancel{i^2}+2i - \cancel{1}-\cancel{i^2}+2i}{1-i^2}$$

$$z = \frac{4i}{2}$$

$$z = 2i$$

$$z = 0 + 2i$$

$\therefore \boxed{\text{Ans}= A}$

$$|z| = \sqrt{0+4} = 2$$

Ans. 20 $\rightarrow$

$$z = \frac{(1-i\sqrt{3})(\cos\theta + i\sin\theta)}{(2-2i)(\cos\theta - i\sin\theta)}$$

$$\Rightarrow |z| = \left| \frac{(1-i\sqrt{3})\ (\cos\theta + i\sin\theta)}{(2-2i)(\cos\theta - i\sin\theta)} \right|$$

$$= \frac{|1-i\sqrt{3}|\ |\cos\theta + i\sin\theta|}{|2-2i|\ |\cos\theta - i\sin\theta|} \quad --- \left\{ \begin{array}{l} |z_1 z_2| = |z_1||z_2| \\ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \end{array} \right.$$

$$= \frac{(\sqrt{1+3})(\sqrt{\cos^2\theta + \sin^2\theta})}{(\sqrt{4+4})(\sqrt{\cos^2\theta + \sin^2\theta})}$$

$$= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \text{ Ans}$$

$\boxed{\text{Ans = A}}$

Qn 21 →

$$(\sqrt{5} + \sqrt{3}i)^{33} = 2^{49} z$$

$$\Rightarrow |(\sqrt{5} + \sqrt{3}i)^{33}| = |2^{49} z|$$

$$\Rightarrow |\sqrt{5} + i\sqrt{3}|^{33} = 2^{49}|z| \quad --- \{\because |z^n| = |z|^n\}$$

$$\Rightarrow (\sqrt{5+3})^{33} = 2^{49}|z|$$

$$(2^{3/2})^{33} = 2^{49}|z| \quad --- \{\sqrt{8} = 2\sqrt{2} = 2^{3/2}\}$$

$$\Rightarrow (2)^{\frac{99}{2}} = 2^{49}|z|$$

$$\Rightarrow \frac{2^{\frac{99}{2}}}{2^{49}} = |z| \Rightarrow |z| = 2^{\frac{99}{2} - 49}$$

$$|z| = 2^{1/2} = \sqrt{2}$$

$\boxed{\text{Ans = B}}$

Ans 22 → $|1-\bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1-|z_1|^2)(1-|z_2|^2)$

Consider LHS

$$|1-\bar{z}_1 z_2|^2 - |z_1-z_2|^2$$

$$\Rightarrow (1-\bar{z}_1 z_2)\overline{(1-\bar{z}_1 z_2)} - (z_1-z_2)\overline{(z_1-z_2)} \quad \cdots \{\because |z|^2 = z\bar{z}\}$$

$$= (1-\bar{z}_1 z_2)(1-z_1\bar{z}_2) - (z_1-z_2)(\bar{z}_1-\bar{z}_2)$$

$$= 1 - \cancel{z_1\bar{z}_2} - \cancel{\bar{z}_1 z_2} + z_1\bar{z}_1 z_2\bar{z}_2 - z_1\bar{z}_1 + \cancel{z_1\bar{z}_2} + \cancel{z_2\bar{z}_1} + z_2\bar{z}_2$$

$$= 1 + |z_1|^2|z_2|^2 - |z_1|^2 - |z_2|^2$$

$$= 1(1-|z_1|^2) + |z_2|^2(1-|z_1|^2)$$

$$= (1-|z_1|^2)(1-|z_2|^2)$$

Comp with RHS $k(1-|z_1|^2)(1-|z_2|^2)$

we get $\boxed{k=1}$ $\therefore \boxed{\text{Ans} = 1}$

Qn 23 → let $z = 3i^3 - 2ai^2 + (1-a)i + 5$

$$z = -3i + 2a + i - ai + 5$$

$$z = (2a+5) - 2i - ai$$

$$z = (2a+5) - i(2+a)$$

for Real $\text{Im}(z) = 0$

$\Rightarrow -(2+a) = 0$

$\Rightarrow 2+a = 0$

$\Rightarrow \boxed{a = -2}$

$\boxed{\text{Ans} = C}$

Qn 24 → Given $\dfrac{(a^2+1)^2}{2a-i} = x - iy$ --- (1)

taking conjugate

$$\frac{(a^2+1)^2}{2a+i} = x+iy \quad \text{--- (2)}$$

(1) × (2)

$$\frac{(a^2+1)^2}{2a-i} \times \frac{(a^2+1)^2}{2a+i} = (x+iy)(x-iy)$$

$$\Rightarrow \frac{(a^2+1)^4}{4a^2-i^2} = x^2 - i^2y^2$$

$$\Rightarrow \frac{(a^2+1)^2}{4a^2+1} = x^2+y^2$$

$\therefore \boxed{\text{Ans} = C}$

Qn 25 $z = 1+2i$ $\quad \boxed{|z| = \sqrt{1+4} = \sqrt{5}}$

Now $\left|\dfrac{7-z}{1-z^2}\right| = \dfrac{|7-z|}{|1-z^2|}$ --- $\left\{ \because \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \right.$

$$= \frac{|7-1-2i|}{|1-(1-4+4i)|}$$

$$= \frac{|6-2i|}{|4-4i|} = \frac{\sqrt{36+4}}{\sqrt{16+16}} = \frac{\sqrt{40}}{\sqrt{32}} = \sqrt{\frac{40}{32}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$= \frac{|z|}{2}$$

$\therefore \boxed{\text{Ans} = D}$