॥ जय श्री राधे कृष्णा ॥

ULTIMATE MATHEMATICS

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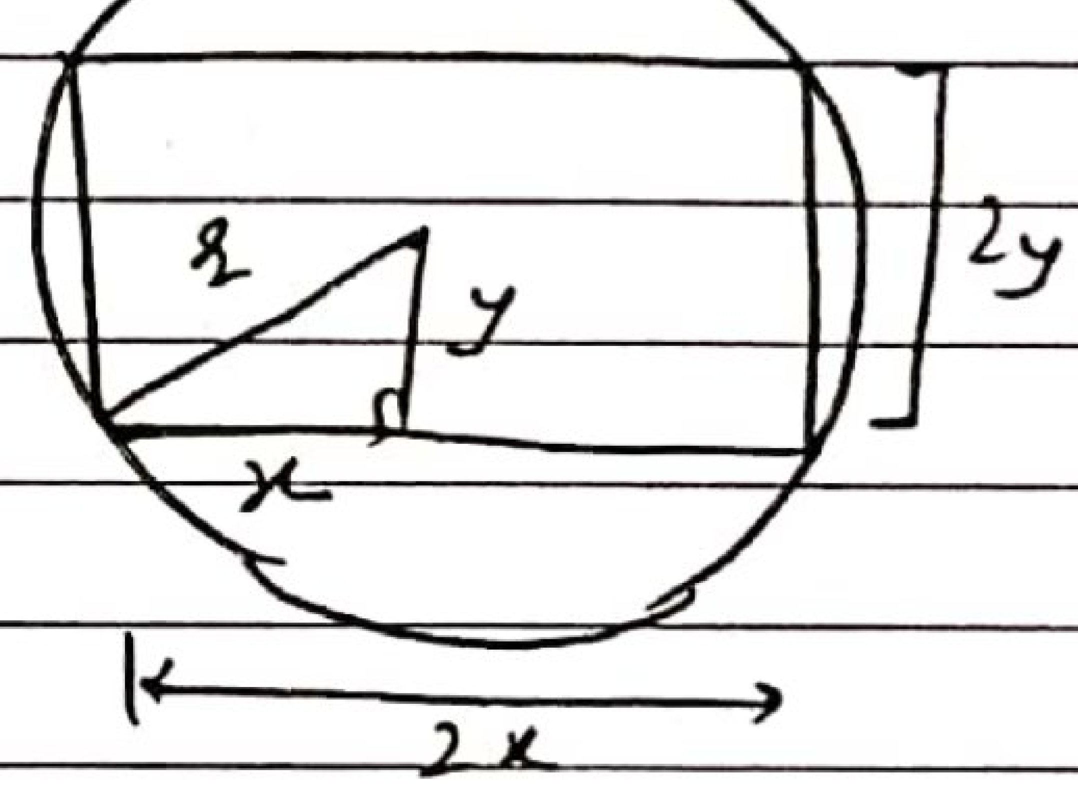
Chapter: A.OD CLASS NO: 6

Topic Maxima - Minima (Continued...)

Ques. 1 Show that of all the rectangles "inscribed" in a given circle, the square has the maximum area.

Solution (i) let $2x \to$ length of rectangle

$2y \to$ breadth of rectangle



(·) $r^2 = x^2 + y^2$ ...(given) ---(1)

(·) let $A \to$ area of rectangle

$A = (2x)(2y) = 4xy$ --- (to be Max)

$A = x\sqrt{r^2 - x^2}$ -- (from eq(1))

S(quaring)

$A^2 = x^2(r^2 - x^2)$

$A^2 = x^2r^2 - x^4$

let $A^2 = z$

then A is Max/Min as according to z is Max/Min

$z = x^2r^2 - x^4$

Diff wrt x

$\frac{dz}{dx} = 2xr^2 - 4x^3$

for Max/Min

put $\frac{dZ}{dx} = 0$

$$2xr^2 - 4x^3 = 0$$

$$\Rightarrow 2xr^2 = 4x^3$$

$$\Rightarrow \boxed{r^2 = 2x^2} \quad (\text{or}) \quad \boxed{x = \frac{r}{\sqrt{2}}}$$

Diff again wrt x

$$\frac{d^2Z}{dx^2} = 2r^2 - 12x^2$$

$$\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{r}{\sqrt{2}}} = 2r^2 - 6r^2 = -4r^2 < 0$$

$\therefore Z$ is Max.

$\therefore$ Area of rectangle is Max at $x = \frac{r}{\sqrt{2}}$

put $r^2 = 2x^2$ in eq (i)

$$2x^2 = x^2 + y^2$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x = y$$

$$\Rightarrow 2x = 2y$$

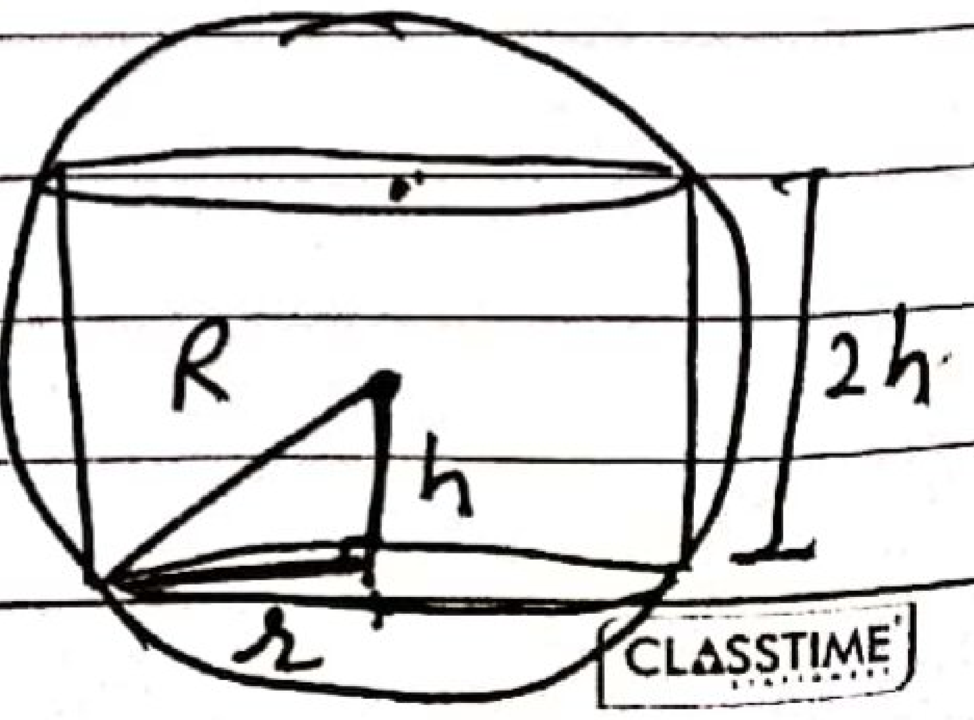
$$\Rightarrow l = b$$

$\therefore$ ~~A~~ rectangle = square

$\therefore$ Square has the Max. area **Ans**

Ques 2 → Find the volume of the largest cylinder that can be inscribed in a sphere of radius R cm.

Sol (·) let $r \to$ Radius of cylinder
$2h \to$ height of cylinder



(·) $R^2 = h^2 + r^2$ --- (given) ---(1)

(·) $V = \pi r^2 (2h)$ --- (to be Max)

A.OD Class No 6

$V = \pi (R^2 - h^2)(2h)$ ... (from eq (1))

$V = 2\pi (R^2 h - h^3)$

Diff wrt h

$\frac{dV}{dh} = 2\pi (R^2 - 3h^2)$

for Max / Min, put $\frac{dV}{dh} = 0$

$\boxed{R^2 = 3h^2}$ $\boxed{h = \frac{R}{\sqrt{3}}}$

Diff again wrt h

$\frac{d^2V}{dh^2} = 2\pi (-6h) = -12\pi h$

$\left(\frac{d^2V}{dh^2}\right)_{h=\frac{R}{\sqrt{3}}} = -12\pi \left(\frac{R}{\sqrt{3}}\right) < 0$

$\therefore$ vol of cylinder is Max at $h = \frac{R}{\sqrt{3}}$

put $h = \frac{R}{\sqrt{3}}$ in eq (1)

$R^2 = \frac{R^2}{3} + r^2$

$\boxed{r^2 = \frac{2R^2}{3}}$

Now $V_{max} = \pi r^2 (2h)$

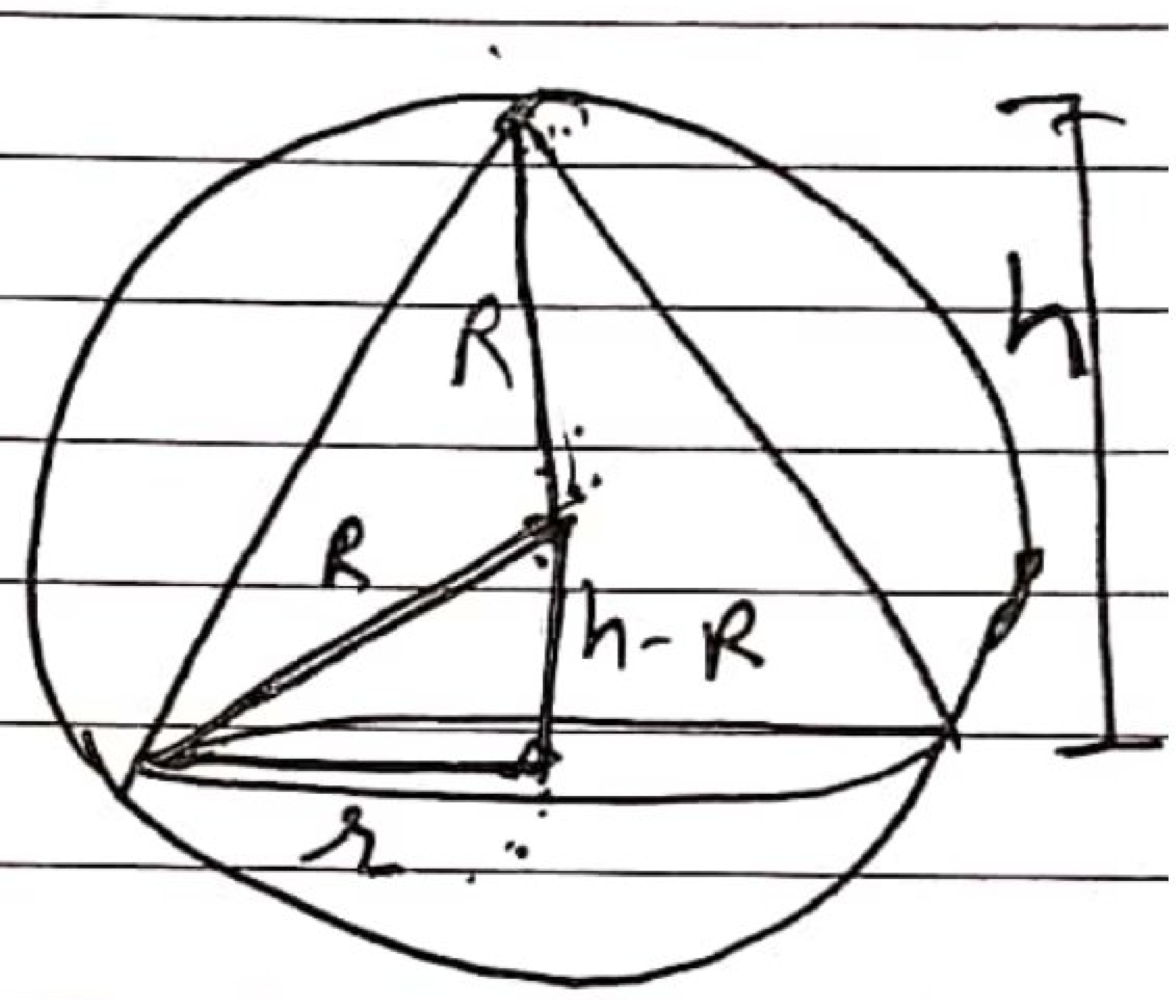
$= \pi \left(\frac{2R^2}{3}\right)\left(\frac{2R}{\sqrt{3}}\right)$

$V_{max} = \frac{4\pi R^3}{3\sqrt{3}}$ (~~[illegible]~~ cm^3

Que 3 → Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Solution (·) $r \to$ Radius of cone

$h \to$ height of cone



(·) $R^2 = (h-R)^2 + r^2$ --- (given) (i)

(·) $V = \frac{1}{3}\pi r^2 h$ --- (to be Max)

$V = \frac{1}{3}\pi\left(R^2 - (h-R)^2\right)h$ --- {from eq (i)}

$V = \frac{1}{3}\pi\left(-h^2 + 2hR\right)h$

$V = \frac{1}{3}\pi\left(-h^3 + 2h^2R\right)$

$\frac{dV}{dh} = \frac{1}{3}\pi\left(-3h^2 + 4hR\right)$

for Max/Min; put $\frac{dV}{dh} = 0$

$3h^2 = 4hR$

$\boxed{h = \frac{4R}{3}}$ or $R = \frac{3h}{4}$

Diff again

$\frac{d^2V}{dh^2} = \frac{1}{3}\pi\left(-6h + 4R\right)$

$\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4R}{3}} = \frac{1}{3}\pi\left(-8R + 4R\right)$

$= -\frac{4R\pi}{3} < 0$

∴ vol. of cone is max at $h = \frac{4R}{3}$

A.OD class No 6

put $h = \frac{4R}{3}$ in eq (i)

$$R^2 = \left(\frac{4R}{3} - R\right)^2 + r^2$$

$$R^2 = \left(\frac{R}{3}\right)^2 + r^2$$

$$R^2 = \frac{R^2}{9} + r^2$$

$$\boxed{r^2 = \frac{8R^2}{9}}$$

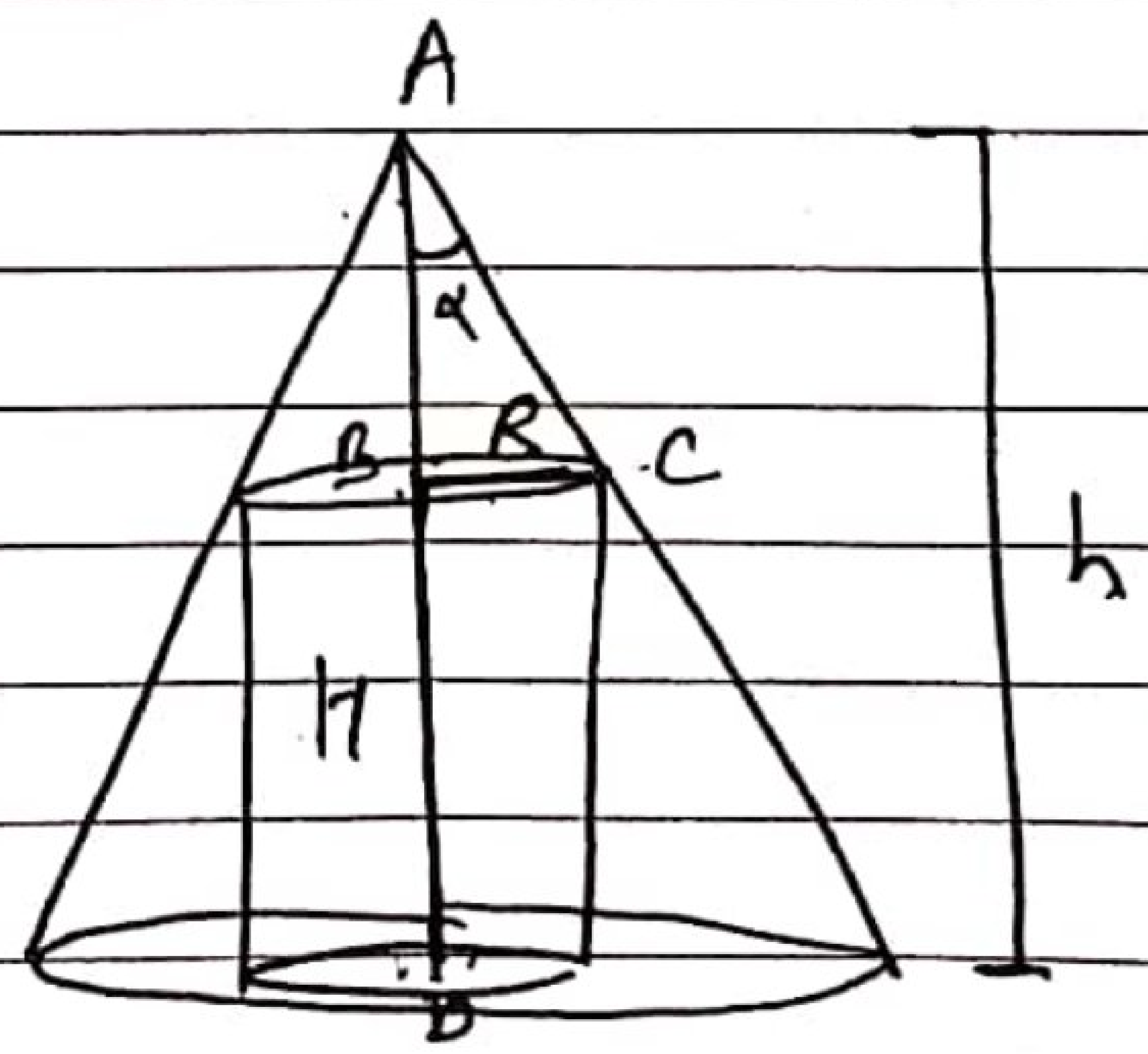
vol. of cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\frac{8R^2}{9}\right)\left(\frac{4R}{3}\right)$$

$$= \frac{8}{27} \times \frac{4}{3}\pi R^3$$

vol. of cone $= \frac{8}{27} \times$ volume of sphere Proved

Qn. 4 → Show that the volume of the greatest cylinder that can be inscribed in a cone of height 'h' and semi-vertical angle 'α' is $\frac{4}{27}\pi h^3 \tan^2\alpha$. Also show that height of the cylinder is $\frac{h}{3}$.



Sol.

(i) $R \to$ Radius of cylinder

$H \to$ height of cylinder

(i) $H = h - AB$

ΔABC $\tan\alpha = \frac{R}{AB} \Rightarrow AB = \frac{R}{\tan\alpha}$

Imp

$\therefore \boxed{H = h - \frac{R}{\tan\alpha}}$

(·) $V = \pi R^2 H$ - - - (to be Max)

Topic : .... Date. : .... Page No. : ....

A·O·D class no: 6 (6)

$$V = \pi R^2\left(h - \frac{R}{\tan\alpha}\right)$$

$$V = \pi\left(R^2 h - \frac{R^3}{\tan\alpha}\right)$$

Diff w.r.t R

$$\frac{dV}{dR} = \pi\left(2Rh - \frac{3R^2}{\tan\alpha}\right)$$

for Max/Min; put $\frac{dV}{dR} = 0$

$$2Rh = \frac{3R^2}{\tan\alpha}$$

$$h = \frac{3R}{2\tan\alpha} \qquad \boxed{R = \frac{2h\tan\alpha}{3}}$$

Diff again

$$\Rightarrow \frac{d^2V}{dR^2} = \pi\left(2h - \frac{6R}{\tan\alpha}\right)$$

$$\left(\frac{d^2V}{dR^2}\right)_{R = \frac{2h\tan\alpha}{3}} = \pi(2h - 4h) = -2\pi h < 0$$

$\therefore$ vol. of cylinder is Max at $R = \frac{2h\tan\alpha}{3}$

put $R = \frac{2h\tan\alpha}{3}$ in $H = h - \frac{R}{\tan\alpha}$

$$\Rightarrow H = h - \frac{2h}{3}$$

$$\Rightarrow H = h/3 \quad \text{proved}$$

Vol of cylinder $= \pi R^2 H = \pi\left(\frac{4h^2\tan^2\alpha}{9}\right)\left(\frac{h}{3}\right)$

$$= \frac{4}{27}\pi h^3\tan^2\alpha$$

proved

CLASSTIME

Topic : .... Date. : .... Page No. : ....

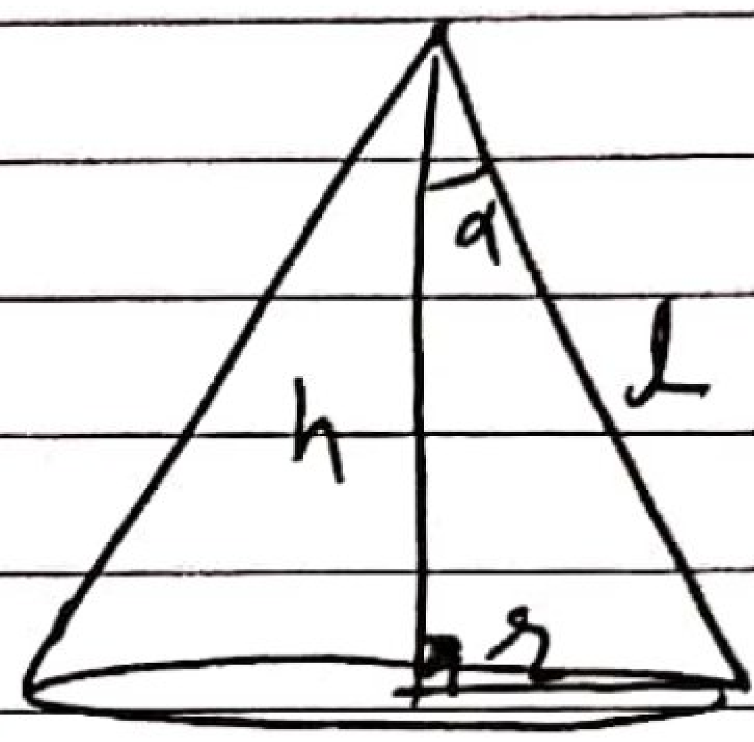
AOD class No: 6

(7)

Ques 5 → Show that the semi-vertical angle of a right circular cone of given surface area and maximum volume is $\sin^{-1}(1/3)$

3 variable

Soln (·) $h \rightarrow$ height of cone
$r \rightarrow$ Radius " "
$l \rightarrow$ slant height "



(·) $S \rightarrow$ T.S.A of cone

$$S = \pi r l + \pi r^2 \quad \text{... (given) ... (i)}$$

(·) $V = \frac{1}{3}\pi r^2 h$ ... (to be Max)

Main point

$$V = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$$

$$V = \frac{1}{3}\pi r^2 \sqrt{\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2} \quad \text{... \{from eq (i)\}}$$

$$V = \frac{1}{3}\pi r^2 \sqrt{\frac{S^2 + \pi^2 r^4 - 2S\pi r^2}{\pi^2 r^2} - r^2}$$

$$V = \frac{1}{3\cancel{\pi}}\cancel{\pi} r^{\cancel{2}} \frac{\sqrt{S^2 + \cancel{\pi^2 r^4} - 2S\pi r^2 - \cancel{\pi^2 r^4}}}{\cancel{\pi}\cancel{r}}$$

$$V = \frac{1}{3}\pi r \sqrt{S^2 - 2S\pi r^2}$$

$$V^2 = \frac{1}{9}\pi^2 r^2 (S^2 - 2S\pi r^2)$$

$$V^2 = \frac{1}{3}\pi (r^2 S^2 - 2S\pi r^4)$$

let $V^2 = Z$

then V is Max/Min as according to Z a Max/min

A.O.D class No: 6

$$z = \frac{1}{3}\pi\left(r^2 s^2 - 2s\pi r^4\right)$$

Diff wrt r

$$\frac{dz}{dr} = \frac{1}{3}\pi\left(2rs^2 - 8s\pi r^3\right)$$

for Max/Min put $\frac{dz}{dr} = 0$

$$2rs^2 = 8s\pi r^3$$

$$\boxed{s = 4\pi r^2}$$

$$r = \sqrt{\frac{s}{4\pi}}$$

Diff again

$$\frac{d^2z}{dr^2} = \frac{1}{3}\pi\left(2s^2 - 24s\pi r^2\right)$$

$$\left(\frac{d^2z}{dr^2}\right)_{r=\sqrt{\frac{s}{4\pi}}} = \frac{1}{3}\pi\left(2s^2 - 6s^2\right)$$

$$= -\frac{4s^2}{3} < 0$$

$\therefore$ z is Max

$\therefore$ volume of cone is Max. at $r = \sqrt{\frac{s}{4\pi}}$

put $s = 4\pi r^2$ in eq (1)

$$4\pi r^2 = \pi rl + \pi r^2$$

$$3\pi r^2 = \pi r l$$

$$3r = l$$

Now $\sin\alpha = \frac{r}{l}$.

$$\sin\alpha = \frac{r}{3r}$$

$$\sin\alpha = 1/3$$

$\Rightarrow \alpha = \sin^{-1}(1/3)$ Ans

CLASSTIME

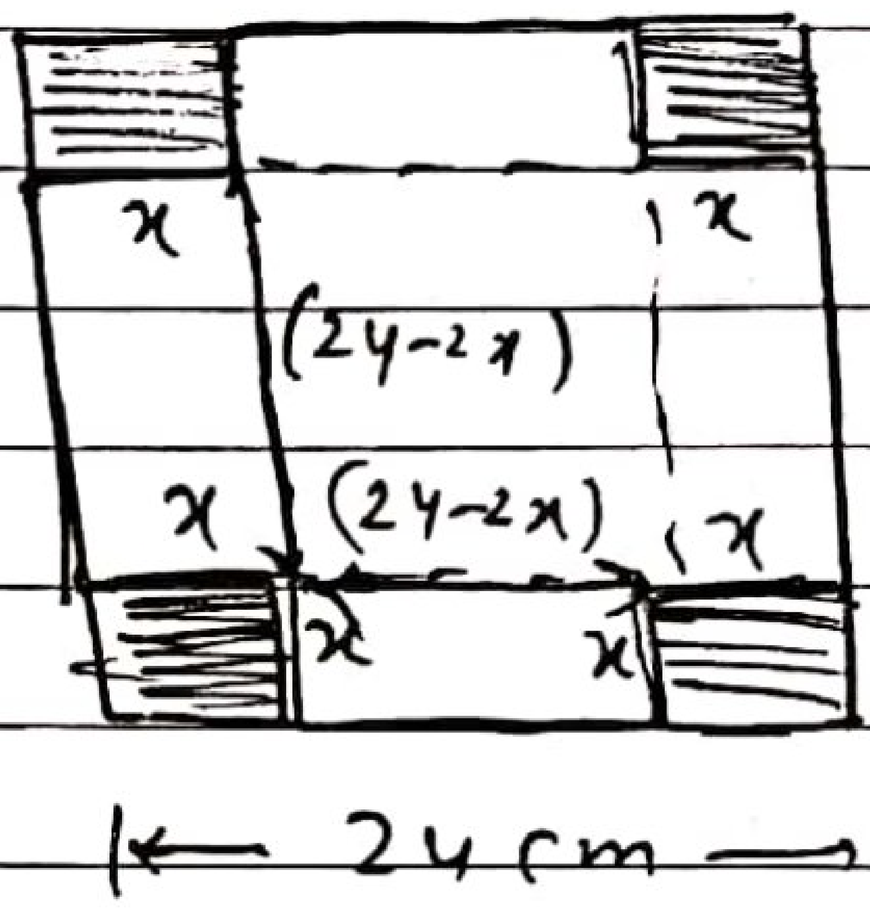
Topic : .... Date. : .... Page No. : 9

A·OD class No: 6

Qn. 6 → A square piece of tin of side 24cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also find the maximum volume.

Single variable

Sol. let $x \to$ side of the square to be cut off



$V \to$ volume of the box/cuboid

$l = 24-2x$; $b = 24-2x$, $h = x$

$V = lbh$

$V = (24-2x)^2 \cdot x$ ..... (to be Max)

$V = (576 + 4x^2 - 96x)x$

$V = 576x + 4x^3 - 96x^2$

$\frac{dv}{dx} = 576 + 12x^2 - 192x$

put $\frac{dv}{dx} = 0$

$12x^2 - 192x + 576 = 0$

$x^2 - 16x + 48 = 0$

$\Rightarrow (x-4)(x-12) = 0$

$\boxed{x=4}$; $(x=12) \to$ Rejected

Diff again

$\frac{d^2v}{dx^2} = 24x - 192$

CLASSTIME

Topic : .... Date. : .... Page No. : ....

A.I.O.D class No: 6 (10)

$$\left(\frac{d^2v}{dx^2}\right)_{x=4} = \cancel{20}\ 96-192$$

$$= -96 < 0$$

$\therefore$ vol. of cuboid/box is Maximum at $x=4$

$$\therefore V_{max} = (24-8)^2 4$$

$$= (16)^2 \times 4$$

$$= 256 \times 4$$

$$= 1024 \text{ cm}^3 \text{ Ans}$$

CLASSTIME

A.O.D (Maxima-Minima) (Class No: 6)

← Worksheet No: 5 →

Qn: 1 → Find two positive numbers x & y such that $x+y=60$ and xy^3 is Maximum

Ans $x=15$ & $y=45$

Qn 2 → Find two +ve numbers x & y such that their sum is 35 and the product x^2y^5 is maximum

Ans $x=10$, $y=25$

Qn 3 → Show that all the rectangles with a given perimeter, the square has the largest area

Qn 4 → Show that of all the rectangles of given area, the square has the smallest perimeter

Qn 5 → Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

Qn 6 → Prove that the area of right angled triangle of given hypotenuse is maximum when the triangle is isosceles.

Qn 7 Show that a cylinder of a given volume which is open at the top, has minimum total surface area, provided its height is equal to the radius of its base

Qn 8 → Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $\frac{2a}{\sqrt{3}}$

Topic : .... Date. : .... Page No. : ....

Qn 9 → Prove that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone of height 'h' & semi-vertical angle 'α' is half of that of the cone

(3 variable)

Qn 10 → Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius 12cm is 16cm.

Qn 11 → Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is $500\pi \text{ cm}^3$

Qn 12 → A closed cylinder has volume 2156 cm^3. What will be the radius of its base so that its total surface area is minimum

Ans $r = 7$ cm

Qn 13 → A rectangular sheet of tin 45cm by 24 cm is made in to a box without top by cutting off squares from each corners and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum? Also find the maximum volume.

Ans Side = 3cm & volume = 432 cm^3