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ULTIMATE MATHEMATICS: BY AJAY MITTAL

INTEGRATION : CLASS NO: 16

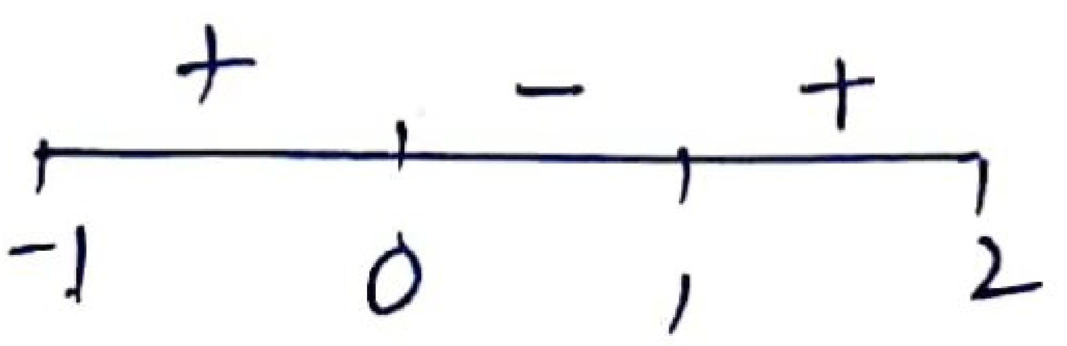
Qn. 1 → Evaluate $\int_{-1}^{2} |x^3 - x|\, dx$

Soln

$$I = \int_{-1}^{2} |x^3 - x|\, dx$$

$$= \int_{-1}^{2} |x(x+1)(x-1)|\, dx$$

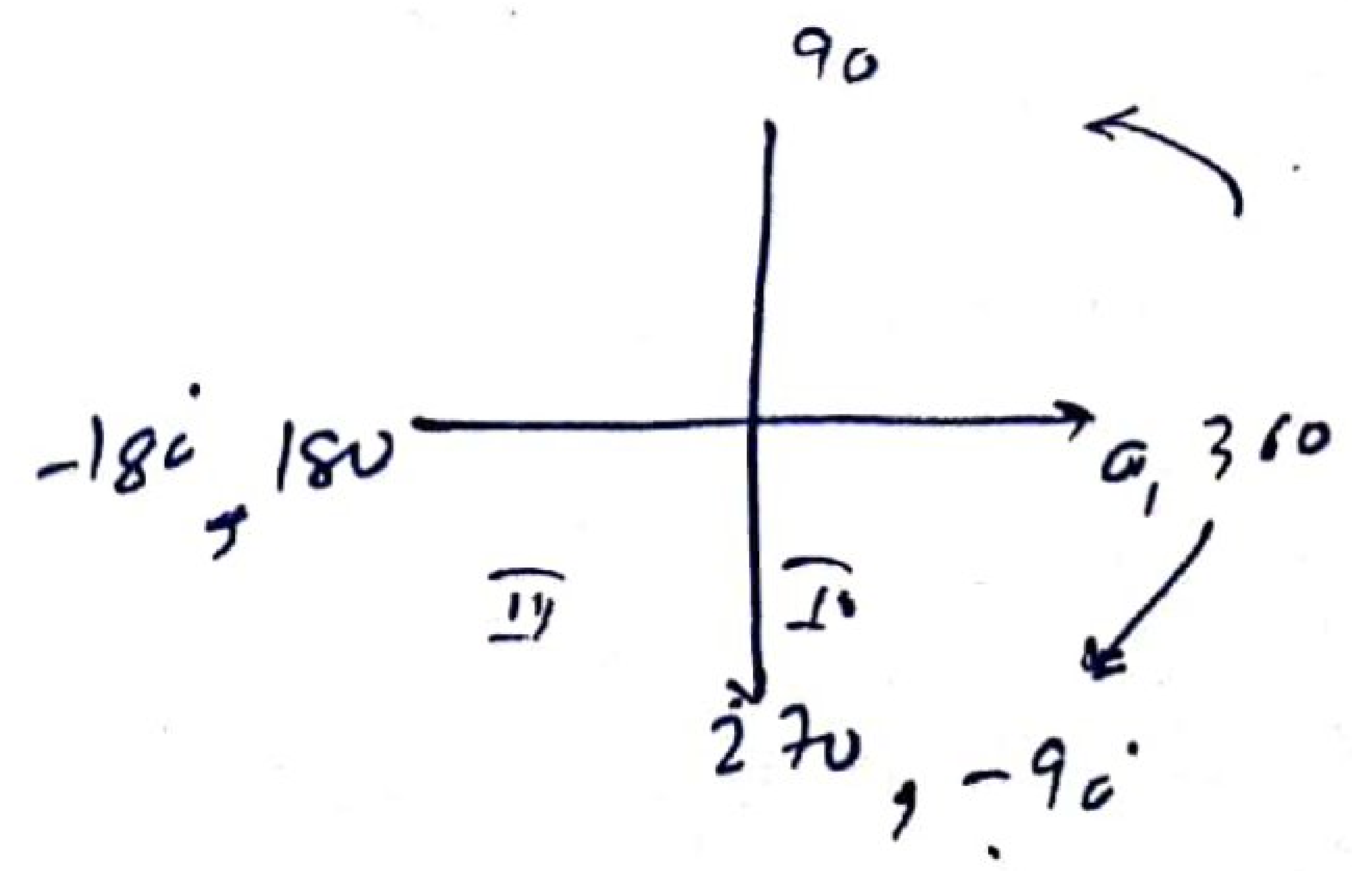
(zeros: $0, -1, 1$)



$$I = \int_{-1}^{0} (x^3 - x)\, dx - \int_{0}^{1} x^3 - x\, dx + \int_{1}^{2} x^3 - x\, dx$$

Proceed Ans $= \frac{11}{4}$

Qn. 2 $I = \int_{-1}^{3/2} |x \sin(\pi x)|\, dx$



Case I $-1 \le x < -\frac{1}{2}$; $-\pi \le \pi x < -\frac{\pi}{2}$ (III)

$|x \sin(\pi x)| = x \sin(\pi x)$

Case II $-\frac{1}{2} \le x < 0$; $-\frac{\pi}{2} \le \pi x < 0$ (IV)

$|x \sin(\pi x)| = x \sin(\pi x)$

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case III $\quad 0 \le x < \frac{1}{2} \quad ; \quad 0 \le \overset{I}{\pi x} < \frac{\pi}{2}$

$$|x\sin(\pi x)| = x\sin(\pi x)$$

case IV $\quad \frac{1}{2} \le x < 1 \quad ; \quad \frac{\pi}{2} \le \overset{II}{\pi x} < \pi$

$$|x\sin(\pi x)| = x\sin(\pi x)$$

case V $\quad 1 \le x < \frac{3}{2} \quad ; \quad \pi \le \overset{III}{\pi x} < \frac{3\pi}{2}$

$$|\overset{(+)}{x}\overset{(-)}{\sin(\pi x)}| = -x\sin(\pi x)$$

$$\therefore I = \int_{-1}^{1} x\sin(\pi x)\,dx - \int_{1}^{3/2} x\sin(\pi x)\,dx$$

let $\quad I_1 = \int \underset{I}{x}\,\underset{II}{\sin(\pi x)}\,dx$

$$I_1 = -\frac{x\cos(\pi x)}{\pi} + \frac{1}{\pi}\int \cos(\pi x)\,dx$$

$$I_1 = -\frac{x\cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2}$$

$$\therefore I = \left[-\frac{x\cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2}\right]_{-1}^{1} - \left[-\frac{x\cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2}\right]_{1}^{3/2}$$

$$= \left[\left(\frac{1}{\pi}+0\right) - \left(\frac{-1}{\pi}+0\right)\right] - \left[\left(0 - \frac{1}{\pi^2}\right) - \left(\frac{1}{\pi}+0\right)\right]$$

$$= \frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi}$$

$$\therefore \boxed{I = \frac{3}{\pi} + \frac{1}{\pi^2}} \quad \text{Ans}$$

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Qn. 3 $I = \int_{-2}^{2} |x \cos(\pi x)|\, dx$

$\boxed{\{|-x| = |x|\}}$

Sol: here $f(x) = |x \cos(\pi x)|$

$f(-x) = |(-x)\cos(-\pi x)| = |-x \cos(\pi x)|$

even function $= |x \cos(\pi x)| = f(x)$

$\therefore I = 2\int_{0}^{2} |x \cos(\pi x)|\, dx$

Case I $\quad 0 \le x < 1/2 \quad ; \quad 0 \le \pi x < \pi/2$

$|x \cos(\pi x)| = x \cos(\pi x)$

Case II $\quad \frac{1}{2} \le x < 1 \quad ; \quad \frac{\pi}{2} \le \pi x < \pi$

$|\overset{+}{x} \overset{-}{\cos}(\pi x)| = -x\cos(\pi x)$

Case III $\quad 1 \le x < 3/2 \quad ; \quad \pi \le \pi x < 3\pi/2$

$|\overset{+}{x} \overset{-}{\cos}(\pi x)| = -x\cos(\pi x)$

Case IV $\quad \frac{3}{2} \le x < 2 \quad ; \quad \frac{3\pi}{2} \le \pi x \le 2\pi$

$|\overset{+}{x} \overset{+}{\cos}(\pi x)| = x\cos(\pi x)$

$$I = 2\left[\int_{0}^{1/2} x\cos(\pi x)\,dx - \int_{1/2}^{3/2} x\cos(\pi x)\,dx + \int_{3/2}^{2} x\cos(\pi x)\,dx\right]$$

proceed $\boxed{\text{Ans} = 8/\pi}$

Qn 4 * $I = \int_{-1}^{1} \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1}$

$\frac{\cancel{-x^3 + |x| + 1}}{x^2 + 2|x| + 1}$

$$I = \int_{-1}^{1} \frac{x^3}{x^2 + 2|x| + 1}\,dx + \int_{-1}^{1} \frac{|x| + 1}{x^2 + 2|x| + 1}$$

$f(x) = \frac{x^3}{x^2 + 2|x| + 1}$

$f(-x) = -f(x)$

odd funch

$g(x) = \frac{|x| + 1}{x^2 + 2|x| + 1}$

$g(-x) = g(x)$

even funch

$$I = 0 + 2\int_{0}^{1} \frac{|x| + 1}{x^2 + 2|x| + 1}\,dx$$

$$I = 2\int_{0}^{1} \frac{x + 1}{x^2 + 2x + 1}\,dx$$

$$I = 2\int_{0}^{1} \frac{\cancel{x+1}}{(x+1)^{\cancel{2}}}\,dx$$

$$I = 2\int_{0}^{1} \frac{1}{x+1}\,dx$$

$$I = 2\left(\log|x+1|\right)_0^1$$

$$I = 2(\log 2 - \log 1)$$

$\therefore \boxed{I = 2\log 2}$ Ans

Qn. 5 → If $x = \int_0^y \frac{dt}{\sqrt{1+9t^2}}$ and $\frac{d^2y}{dx^2} = ay$

find value of a

Soln

$$x = \int_0^y \frac{1}{\sqrt{1+9t^2}}\, dt$$

$$x = \frac{1}{3}\int_0^y \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 + t^2}}\, dt$$

$$x = \frac{1}{3}\left(\log\left| t + \sqrt{t^2+\frac{1}{9}}\right|\right)_0^y$$

$$x = \frac{1}{3}\left[\log\left|y + \sqrt{y^2+\frac{1}{9}}\right| - \log\left|\frac{1}{3}\right|\right]$$

$$\frac{dx}{dy} = \frac{1}{3}\left(\frac{1}{y+\sqrt{y^2+\frac{1}{9}}} \cdot \left(1 + \frac{\cancel{2}y}{\cancel{2}\sqrt{y^2+\frac{1}{9}}}\right)\right)$$

$$= \frac{1}{3}\left(\frac{1}{\cancel{y+\sqrt{y^2+\frac{1}{9}}}} \cdot \frac{\cancel{\left(\sqrt{y^2+\frac{1}{9}}+y\right)}}{\sqrt{y^2+\frac{1}{9}}}\right)$$

$$\frac{dx}{dy} = \frac{1}{3} \cdot \frac{1}{\sqrt{y^2+\frac{1}{9}}}$$

$$= \frac{1}{\cancel{3}} \times \frac{\cancel{3}}{\sqrt{9y^2+1}}$$

$$\frac{dy}{dx} = \sqrt{9y^2+1}$$

Now

$$\frac{d^2y}{dx^2} = \frac{1}{2\sqrt{9y^2+1}} \cdot \left(18y\frac{dy}{dx}\right) = \frac{1}{2\cancel{\sqrt{9y^2+1}}} \cdot \left(18y \cdot \cancel{\sqrt{9y^2+1}}\right)$$

$$\frac{d^2y}{dx^2} = 9y \quad \therefore \boxed{a=9} \text{ Ans}$$

Q.N: 6 →

If $\int_0^1 \frac{e^t}{1+t}\,dt = a$

find $\int_0^1 \frac{e^t}{(1+t)^2}\,dt$

Sol we have $\int_0^1 \frac{e^t}{1+t}\,dt = a$

$$\Rightarrow \int_0^1 \underset{II}{e^t} \cdot \underset{I}{\frac{1}{1+t}}\,dt = a$$

$$\Rightarrow \left(\frac{1}{1+t}\cdot e^t\right)_0^1 + \int_0^1 \frac{1}{(1+t)^2}\cdot e^t\,dt = a$$

$$\Rightarrow \left(\frac{1}{2}\cdot e - 1\right) + \int_0^1 \frac{1}{(1+t)^2}\cdot e^t\,dt = a$$

$$\Rightarrow \int_0^1 \frac{1}{(1+t)^2}\,e^t\,dt = a - \frac{e}{2} + 1 \quad \text{Ans}$$

Q.N 7 If $\int \frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}}\,dx = ax + b\log|4e^x + 5e^{-x}| + c$

find value of a & b

Soln let $3e^x - 5e^{-x} = A(D^r) + B\,\frac{d}{dx}(D^r)$

$3e^x - 5e^{-x} = A(4e^x + 5e^{-x}) + B(4e^x - 5e^{-x})$

Equate the coefficient of e^x & e^{-x}

$\boxed{3 = 4A + 4B}$ & $\boxed{-5 = 5A - 5B}$ solve

Solⁿ get $A = \frac{-1}{8}$ & $B = \frac{7}{8}$

$$\therefore \int \frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}}\,dx = \int \frac{-\frac{1}{8}\cdot(4e^x+5e^{-x})}{4e^x+5e^{-x}} + \frac{\frac{7}{8}\cdot(4e^x-5e^{-x})}{4e^x+5e^{-x}}\,dx$$

$$= \frac{-1}{8}\int 1\cdot dx + \frac{7}{8}\int \frac{4e^x-5e^{-x}}{\underbrace{4e^x+5e^{-x}}_{t}}\,dx$$

$$= \frac{-1}{8}x + \frac{7}{8}\int \frac{dt}{t}$$

$$= \frac{-1}{8}x + \frac{7}{8}\log\left|4e^x+5e^{-x}\right| + C$$

Comp with $ax + b\log|4e^x+5e^{-x}| + C$

we get $\boxed{a = \frac{-1}{8} \ \& \ b = 7/8}$ Ans

Qn 8 → Evaluate $I = \int_0^1 \underset{\text{II}}{x} \underset{\text{I}}{(\tan^{-1}x)^2}\,dx$

Solⁿ: $I = \left((\tan^{-1}x)^2 \cdot \frac{x^2}{2}\right)_0^1 - \frac{2}{2}\int_0^1 \frac{\tan^{-1}x}{1+x^2}\cdot x^2\,dx$

$$= \left(\frac{\pi^2}{16}\cdot\frac{1}{2}\right) - 0 - \int_0^1 \frac{\tan^{-1}x}{1+x^2}\cdot x^2\,dx$$

put $\tan^{-1}x = t$ | $x=0$; $t=0$

$\frac{1}{1+x^2}\,dx = dt$ | $x=1$, $t=\pi/4$

$$\therefore I = \frac{\pi^2}{32} - \int_0^{\pi/4} t\tan^2 t\,dt$$

$$I = \frac{\pi^2}{32} - \int_0^{\pi/4} \underset{I}{t}\,\underset{II}{(\sec^2 t - 1)}\, dt$$

$$= \frac{\pi^2}{32} - \left[\left(t(\tan t - t)\right)_0^{\pi/4} - \int_0^{\pi/4} (\tan t - t)\, dt \right]$$

$$= \frac{\pi^2}{32} - \left[\frac{\pi}{4}\left(1 - \frac{\pi}{4}\right) - \left(\log|\sec t| - \frac{t^2}{2}\right)_0^{\pi/4} \right]$$

$$= \frac{\pi^2}{32} - \left[\frac{\pi}{4} - \frac{\pi^2}{16} - \left(\left(\log\sqrt{2} - \frac{\pi^2}{32}\right) - (\log 1 - 0) \right) \right]$$

$$= \cancel{\frac{\pi^2}{32}} - \frac{\pi}{4} + \frac{\pi^2}{16} + \frac{1}{2}\log 2 - \cancel{\frac{\pi^2}{32}}$$

$$I = \frac{1}{2}\log 2 - \frac{\pi}{4} + \frac{\pi^2}{16} \quad \underline{\text{Ans}}$$

Qn 9 Evaluate $\displaystyle\int_0^{\pi/2} \frac{\tan x\, dx}{1 + m^2 \tan^2 x}$

Sol $\displaystyle I = \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1 + m^2 \frac{\sin^2 x}{\cos^2 x}}\, dx$

$$= \int_0^{\pi/2} \frac{\sin x \cos x\, dx}{\cos^2 x + m^2 \sin^2 x}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin(2x)\, dx}{\cos^2 x + m^2 \sin^2 x}$$

Put $\cos^2 x + m^2 \sin^2 x = t$

$-\sin(2x) + m^2 \cdot \sin(2x)\, dx = dt$

$\sin(2x)\, dx = \frac{dt}{m^2 - 1}$

when $x = 0;\ t = 1$

when $x = \frac{\pi}{2}$ $t = m^2$

$$\therefore I = \frac{1}{2(m^2-1)} \int_1^{m^2} \frac{dt}{t}$$

$$= \frac{1}{2(m^2-1)} \left(\log|t|\right)_1^{m^2}$$

$$= \frac{1}{2(m^2-1)} \left(\log m^2 - \log 1 \right)$$

$$= \frac{1}{\cancel{2}(m^2-1)} \cdot \cancel{2} \log m$$

$$\boxed{I = \frac{\log m}{m^2-1}}$$ Ans

Q.10 → $I = \displaystyle\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}\, dx}{(1-\cos x)^{5/2}}$

Soln $I = \displaystyle\int_{\pi/3}^{\pi/2} \frac{\sqrt{2\cos^2(x/2)}\, dx}{\left(2\sin^2(x/2)\right)^{5/2}}$

$$= \int_{\pi/3}^{\pi/2} \frac{\sqrt{2} \cdot \cos(x/2)\, dx}{2^{5/2} \cdot \sin^5(x/2)}$$

$$= \frac{1}{4} \int_{\pi/3}^{\pi/2} \frac{\cos(x/2)\, dx}{\sin^5(x/2)}$$

put $\sin\left(\frac{x}{2}\right) = t$

$\frac{1}{2} \cos(x/2)\, dx = dt$

$\cos\left(\frac{x}{2}\right) dx = 2\, dt$

when $x = \pi/3$; $t = 1/2$

$x = \pi/2$; $t = 1/\sqrt{2}$

$$\therefore I = \frac{1}{4}\cdot 2\int_{1/2}^{1/\sqrt{2}} \frac{dt}{t^5}$$

$$= \frac{1}{2}\int_{1/2}^{1/\sqrt{2}} t^{-5}\,dt$$

$$= \frac{1}{2}\left(\frac{t^{-4}}{-4}\right)_{1/2}^{1/\sqrt{2}}$$

$$= -\frac{1}{8}\left(\frac{1}{t^4}\right)_{1/2}^{1/\sqrt{2}}$$

$$= -\frac{1}{8}(4-16)$$

$$= -\frac{1}{8}(-12) = \frac{3}{2} \qquad \therefore \; I = 3/2 \text{ Ans}$$

Q. 11 → $I = \int_{-\pi/4}^{\pi/4} \log(\cos x + \sin x)\,dx$

Soln

$$I = \int_{-\pi/4}^{\pi/4} \log(\cos x + \sin x)\,dx \text{ --- (1)}$$

$$I = \int_{-\pi/4}^{\pi/4} \log(\cos(-x) + \sin(-x))\,dx \text{ -- (P V)}$$

$$I = \int_{-\pi/4}^{\pi/4} \log(\cos x - \sin x)\,dx \text{ -- (2)}$$

(1) + (2)

$$2I = \int_{-\pi/4}^{\pi/4} \log(\cos^2 x - \sin^2 x)\,dx$$

$$2I = \int_{-\pi/4}^{\pi/4} \log(\cos(2x))\,dx$$

here $f(x) = \log(\cos(2x))$

$f(-x) = \log(\cos(-2x)) = \log(\cos(2x)) = f(x)$
(even)

$\therefore$

$$\cancel{2}I = \cancel{2}\int_{0}^{\pi/4} \log(\cos(2x))\,dx$$

$$I = \int_{0}^{\pi/4} \log(\cos(2x))\,dx$$

put $2x = t$, $dx = \frac{dt}{2}$ | $x = 0$, $t = 0$; $x = \pi/4$; $t = \pi/2$

$$I = \frac{1}{2}\int_{0}^{\pi/2} \log(\cos t)\,dt$$

$$2I = \int_{0}^{\pi/2} \log(\cos t)\,dt \quad \text{--- (3)}$$

$$2I = \int_{0}^{\pi/2} \log(\sin t)\,dt \quad \text{--- (4)}$$

(3) + (4)

$$4I = \int_{0}^{\pi/2} \log(\sin t \cos t)\,dt$$

$$4I = \int_{0}^{\pi/2} \log(\sin(2t)) - \log(2)\,dt$$

$$4I' = \int_{0}^{\pi/2} \log(\sin(2t)) - (t\log 2)_{0}^{\pi/2}$$

$$4I = I_1 - \frac{\pi}{2}\log 2$$

when $I_1 = \int_0^{\pi/2} \log(\sin(2t))\,dt$

put $2t = z$ | $t = 0,\ z = 0$

$dt = \frac{dz}{2}$ | $t = \pi/2;\ z = \pi$

$\therefore I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin z)\,dz$

$I_1 = \frac{1}{2} \times 2\int_0^{\pi/2} \log(\sin z)\,dz$ -- {P VI}

$I_1 = \int_0^{\pi/2} \log(\sin t)\,dt$ -- (P I)

$I_1 = I$

$\therefore 4I = 2I - \pi \lg 2$

$2I = -\frac{\pi}{2} \lg 2$

$I = -\frac{\pi}{4} \lg 2$ Ans

← WORKSHEET NO: 12 →

Q. 1 $\displaystyle\int_{-\pi/2}^{\pi/2} x^3 + x\cos x + \tan^5 x + 1 \, dx$ $\boxed{\text{ANS} = \pi}$

Q. 2 $\displaystyle\int_{0}^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ $\boxed{\text{ANS} = 0}$

Qn. 3 → $\displaystyle\int_{0}^{2\pi} \cos^5 x \, dx$ $\boxed{\text{ANS} = 0}$

Qn. 4 → $\displaystyle\int_{0}^{1} x(1-x)^n \, dx$ $\boxed{\text{ANS} = \frac{1}{(n+1)(n+2)}}$

Qn. 5 → $\displaystyle\int_{-\pi}^{\pi} \sin^3 x \cdot \cos^2 x \, dx$ $\boxed{\text{ANS} = 0}$

Qn. 6 → If $\displaystyle\int_{0}^{a} \frac{1}{1+4x^2} dx = \frac{\pi}{8}$, then find a $\boxed{\text{ANS: } a = 1/2}$

Qn. 7 → $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos(2x)}$ $\boxed{\text{ANS: } 1}$

Qn. 8 → $\displaystyle I = \int_{0}^{\pi} x \log(\sin x) \, dx$ $\boxed{\text{ANS } -\frac{\pi^2}{2}\log 2}$

Qn. 9 → $\displaystyle\int_{-2}^{2} |x\cos(\pi x)| \, dx$ $\boxed{\text{ANS: } \frac{8}{\pi}}$

Qn. 10 → evaluate $\displaystyle\int_{-1}^{2} f(x) \, dx$ where

$f(x) = |x+1| + |x| + |x-1|$ ANS $\frac{19}{2}$

Q. 11 $\displaystyle\int_{0}^{\pi/4} \sqrt{1+\sin(2x)} \, dx$ ANS 1

— x —