SOLUTIONS | WORKSHEET NO: 10 (Class No=13) ①

INTEGRATION

Qn: 1 $I = \int_0^{\pi/4} 2\tan^3 x \, dx$

$$I = 2\int_0^{\pi/4} \tan x \cdot \tan^2 x \, dx$$

$$= 2\int_0^{\pi/4} \tan x \left(\sec^2 x - 1\right) dx$$

$$I = 2\int_0^{\pi/4} \tan x \sec^2 x \, dx - 2\int_0^{\pi/4} \tan x \, dx$$

put $\tan x = t$ | when $x = 0$; $t = 0$

$\sec^2 x \, dx = dt$ | when $x = \pi/4$; $t = 1$

$$\therefore I = 2\int_0^1 t \, dt - 2\left(\log|\sec(x)|\right)_0^{\pi/4}$$

$$= \cancel{2}\left[\frac{t^2}{\cancel{2}}\right]_0^1 - 2\left[\log|\sqrt{2}| - \log|1|\right]$$

$$= 1 - 2\left(\frac{1}{2}\log 2 - 0\right)$$

$$\boxed{I = 1 - \log 2} \quad \text{Ans}$$

Qn 2 → $I = \int_0^1 \sin^{-1} x \, dx$

$$I = \int_0^1 \underset{\mathrm{I}}{\sin^{-1} x} \cdot \underset{\mathrm{II}}{1} \, dx$$

$$= \left(\sin^{-1} x \cdot x\right)_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot x \, dx$$

put $1 - x^2 = t$ | when $x = 0$, $t = 1$

$x \, dx = -\frac{dt}{2}$ | when $x = 1$, $t = 0$

$$I = \left(\frac{7}{2}\cdot 1 - 0\right) + \frac{1}{2}\int_1^0 \frac{dt}{\sqrt{t}}$$

$$= \frac{7}{2} + \frac{1}{\cancel{2}} \times \cancel{2}\left(\sqrt{t}\right)_1^0$$

$$= \frac{7}{2} + (0-1)$$

$$\boxed{I = \frac{7}{2} - 1}$$ Ans

Ans. 3 → $I = \int_0^1 xe^x\,dx$

$$I = \int_0^1 \underset{I}{x}\,\underset{II}{e^x}\,dx$$

$$= \left(x\cdot e^x\right)_0^1 - \int_0^1 e^x\,dx$$

$$= (e - 0) - \left(e^x\right)_0^1$$

$$= \cancel{e} - (\cancel{e} - e^0)$$

$$\boxed{I = 1}$$ Ans

Ans 4 → $I = \int_1^3 \frac{1}{x^2(x+1)}\,dx$

let $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$\Rightarrow \quad 1 = A(x)(x+1) + B(x+1) + Cx^2$

$\Rightarrow 1 = A(x^2+x) + B(x+1) + Cx^2$

$0 = A + C$

$0 = A + B \quad \Rightarrow \boxed{B=1} \quad \boxed{A=-1} \quad \boxed{C=+1}$

$1 = B$

$\therefore I = \int_1^3 \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\, dx$

$= \left[-\log|x| - \frac{1}{x} + \log|x+1|\right]_1^3$

$= \left[-\log 3 - \frac{1}{3} + \log|4|\right] - \left[-\log|1| - 1 + \log 2\right]$

$= -\log 3 - \frac{1}{3} + 2\log 2 + 0 + 1 - \log 2$

$= 1 - \frac{1}{3} + \log 2 - \log 3$

$\boxed{I = \frac{2}{3} + \log\left(\frac{2}{3}\right)}$ Ans

Qn: 5 $\quad I = \int_0^{\pi/2} \sin^3 x\, dx$

$I = \frac{1}{4}\int_0^{\pi/2} 3\sin x - \sin(3x)\, dx$

Scanned with CamScanner

$$I = \frac{1}{4}\left[-3\cos x + \frac{\cos(3x)}{3}\right]_0^{\pi/2}$$

$$I = \frac{1}{4}\left[(0+0) - \left(-3 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[-\left(\frac{-8}{3}\right)\right]$$

$$\boxed{I = \frac{2}{3}}$$ Ans

Q.No. 6 → $I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin(2x)}}\,dx$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - 1 + \sin(2x)}}\,dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - (1 - \sin(2x))}}\,dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\,dx$$

put $\sin x - \cos x = t$

$(\cos x + \sin x)\,dx = dt$

when $x = \pi/6$

$t = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2}$

when $x = \pi/3$

$t = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$

$$\therefore I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$= \left(\sin^{-1} t\right)_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}$$

$$I = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$$

$$I = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

$$\boxed{I = 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)}$$ ANS

Ans 7 → $I = \int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$

Rationalize

$$I = \int_0^1 \frac{1}{\sqrt{1+x} - \sqrt{x}} \times \frac{(\sqrt{1+x} + \sqrt{x})}{(\sqrt{1+x} + \sqrt{x})}\, dx$$

$$I = \int_0^1 \frac{\sqrt{1+x} + \sqrt{x}}{1 + \cancel{x} - \cancel{x}}\, dx$$

$$= \int_0^1 \sqrt{1+x} + \sqrt{x}\, dx$$

$$= \left[\frac{2}{3}(1+x)^{3/2} + \frac{2}{3}(x)^{3/2}\right]_0^1$$

$$= \frac{2}{3}\left[(2)^{3/2} + \cancel{(1)} - \cancel{1}\right]$$

$$= \frac{2}{3}(2\sqrt{2})$$

$$\boxed{I = \frac{4\sqrt{2}}{3}}$$ ANS

Ans. 8 → $I = \int_2^8 |x-5|\, dx$

5

$$I = -\int_2^5 (x-5)\, dx + \int_5^8 (x-5)\, dx$$

$$I = -\left(\frac{x^2}{2} - 5x\right)_2^5 + \left(\frac{x^2}{2} - 5x\right)_5^8$$

$$= -\left[\left(\frac{25}{2} - 25\right) - (2 - 10)\right] + (32 - 40) - \left(\frac{25}{2} - 25\right)$$

$$= -\left(-\frac{25}{2} + 8\right] - 8 + \frac{25}{2}$$

$$= \frac{25}{2} - 8 - 8 + \frac{25}{2}$$

$$= 25 - 16$$

$$\Rightarrow \boxed{I = 9} \text{ Ans}$$

Q.N. 9 → $I = \int_{-5}^{5} |x+2| \, dx$ (−2)

$$I = -\int_{-5}^{-2} (x+2)\,dx + \int_{-2}^{5} (x+2)\,dx$$

$$= -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\left[(2-4) - \left(\frac{25}{2} - 10\right)\right] + \left[\left(\frac{25}{2} + 10\right) - (2-4)\right]$$

$$= -\left[-2 - \frac{5}{2}\right] + \frac{45}{2} + 2$$

$$= 2 + \frac{5}{2} + \frac{45}{2} + 2$$

$$= 4 + 25$$

$$\boxed{I = 29} \text{ Ans}$$

Q.N. 10 $I = \int_{-\pi/4}^{\pi/4} \sin^2 x \, dx$

here $f(x) = \sin^2(x)$

Scanned with CamScanner

$f(-x) = \sin^2(-x) = \sin^2 x = f(x)$

$\therefore f(x) \rightarrow$ even function

$$I = 2\int_0^{\pi/4} \sin^2 x \, dx \quad \text{--- (P I)}$$

$$I = \frac{\cancel{2}}{\cancel{2}}\int_0^{\pi/4} 1 - \cos(2x) \, dx$$

$$I = \left(x - \frac{\sin(2x)}{2}\right)\Bigg|_0^{\pi/4}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\right) - (0)$$

$$\boxed{I = \frac{\pi}{4} - \frac{1}{2}} \quad \text{Ans.}$$

Ans. 11 $\quad I = \int_{-1}^{1} \frac{1}{x^2+2x+5} \, dx$

$$I = \int_{-1}^{1} \frac{1}{(x+1)^2 - 1 + 5} \, dx$$

$$= \int_{-1}^{1} \frac{1}{(x+1)^2 + (2)^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$\boxed{I = \frac{\pi}{8}} \quad \text{Ans}$$

Qno 12 → $I = \int_{1/3}^{1} \frac{(x-x^3)^{1/3}}{x^4} dx$

$I = \int_{1/3}^{1} \frac{x\left(\frac{1}{x^2}-1\right)^{1/3}}{x^4} dx$

$I = \int_{1/3}^{1} \frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3} dx$

put $\frac{1}{x^2} - 1 = t$

$\frac{-2}{x^3} dx = dt \Rightarrow \frac{dx}{x^3} = -\frac{dt}{2}$

when $x = 1/3$, $t = 8$

when $x = 1$, $t = 0$

$\therefore I = \frac{-1}{2}\int_{8}^{0} \cancel{dt}\,(t)^{1/3} dt$

$= -\frac{1}{2}\left(\frac{3}{4} t^{4/3}\right)_8^0$

$= -\frac{3}{8}\left(0 - (8)^{4/3}\right)$

$= -\frac{3}{8}(0 - 16)$

$= \frac{3}{8} \times 16$

$\boxed{I = 6}$ Ans

Note. (Misprint in worksheet Answer)

Qno 13 → $I = \int_{0}^{1} \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

when $x = 0$, $\theta = 0$

when $x = 1$, $\theta = \pi/4$

$$\therefore\ I = \int_0^{\pi/4} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \cdot \sec^2\theta\, d\theta$$

$$I = \int_0^{\pi/4} \sin^{-1}(\sin(2\theta))\, \sec^2\theta\, d\theta$$

$$I = 2\int_0^{\pi/4} \underset{\mathrm{I}}{\theta} \cdot \underset{\mathrm{II}}{\sec^2\theta}\, d\theta$$

$$= 2\left[(\theta\tan\theta)_0^{\pi/4} - \int_0^{\pi/4} \tan\theta\, d\theta \right]$$

$$= 2\left[\left(\frac{\pi}{4}\cdot 1 - 0\right) - \left(\log|\sec\theta|\right)_0^{\pi/4} \right]$$

$$= 2\left[\frac{\pi}{4} - \{\log|\sqrt{2}| - \log 1\} \right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2 + 0 \right]$$

$$\boxed{I = \frac{\pi}{2} - \log 2}$$ Ans

Ans. 14 →
$$I = \int_0^2 x\sqrt{x+2}\, dx$$

put $x+2 = t^2$, $dx = 2t\,dt$ | when $x=0$; $t=\sqrt{2}$; when $x=2$, $t=2$

$$\therefore\ I = 2\int_{\sqrt{2}}^{2} (t^2-2)\cdot t\cdot t\, dt$$

$$= 2\int_{\sqrt{2}}^{2} (t^4 - 2t^2)\, dt$$

$$I = 2\left[\frac{t^5}{5} - \frac{2t^3}{3}\right]_{\sqrt{2}}^{2}$$

$$= 2\left[\left(\frac{32}{5} - \frac{16}{3}\right) - \left(\frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3}\right)\right]$$

$$= 2\left[\left(\frac{96-80}{15}\right) - \left(\frac{12\sqrt{2} - 20\sqrt{2}}{15}\right)\right]$$

$$= 2\left[\frac{16}{15} + \frac{8\sqrt{2}}{15}\right]$$

$$I = \frac{16}{15}\left(2 + \sqrt{2}\right) \quad \text{Ans}$$

or $\boxed{I = \frac{16\sqrt{2}}{15}\left(\sqrt{2}+1\right)}$ Ans

Qu. 15 → $I = \int_0^2 \frac{6x+3}{x^2+4}\,dx$

Separate

$$I = 6\int_0^2 \frac{x}{x^2+4}\,dx + 3\int_0^2 \frac{1}{x^2+4}\,dx$$

put $x^2+4 = t$, $x\,dx = \frac{dt}{2}$ | when $x=0$; $t=4$ | when $x=2$; $t=8$

$$\therefore I = \frac{6}{2}\int_4^8 \frac{dt}{t} + 3\times\frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 3\left(\log|t|\right)_4^8 + \frac{3}{2}\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$$

$$= 3\left(\log 8 - \log 4\right) + \frac{3}{2}\cdot\frac{\pi}{4}$$

$$\boxed{I = 3\log 2 + \frac{3\pi}{8}} \quad \text{Ans}$$

Qn 16 → $I = \int_0^{\pi/4} \sin^3(2t).\cos(2t)\,dt$

Put $\sin(2t) = z$
$2\cos(2t)\,dt = dz$
$\cos(2t)\,dt = \frac{dz}{2}$

when $t = 0;\ z = 0$
when $t = \pi/4;\ z = 1$

$$I = \frac{1}{2}\int_0^1 z^3.\,dz$$

$$= \frac{1}{2}\left(\frac{z^4}{4}\right)_0^1$$

$$= \frac{1}{8}(1-0)$$

$$\boxed{I = \frac{1}{8}}$$ Ans

Qn 17 → $I = \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right).e^{2x}\,dx$

$$I = \int_1^2 e^{2x}.\left(\frac{1}{x} - \frac{1}{2x^2}\right)dx$$

$$= \int_1^2 \underset{II}{e^{2x}}.\underset{I}{\frac{1}{x}}\,dx - \int_1^2 e^{2x}.\frac{1}{2x^2}\,dx$$

$$= \left(\frac{1}{x}.\frac{e^{2x}}{2}\right)_1^2 + \cancel{\int_1^2 \frac{1}{x^2}.\frac{e^{2x}}{2}} - \cancel{\int_1^2 e^{x}.\frac{1}{2x^2}\,dx}$$

$$= \left(\frac{1}{2}.\frac{e^4}{2}\right) - \left(\frac{e^2}{2}\right)$$

$$\boxed{I = \frac{e^2}{4}\left(e^2 - 2\right)}$$ Ans

Qn. 18 → $I = \int_0^1 xe^x + \sin\left(\frac{\pi x}{4}\right) dx$

$$I = \int_0^1 \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{e^x}\, dx + \int_0^1 \sin\left(\frac{\pi x}{4}\right) dx$$

$$= \left(xe^x\right)_0^1 - \int_0^1 e^x dx - \frac{4}{\pi}\left(\cos\left(\frac{\pi x}{4}\right)\right)_0^1$$

$$= (e^1 - 0) - (e^x)_0^1 - \frac{4}{\pi}\left(\cos\frac{\pi}{4} - \cos 0\right)$$

$$= e - (e - e^0) - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \cancel{e} - \cancel{e} + 1 - \frac{4}{\sqrt{2}\pi} + \frac{4}{\pi}$$

$$\boxed{I = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}}$$ Ans

– x –