!! जय श्री गिरिराज जी महाराज जय श्री राधे कृष्णा !! (1)

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER: INTEGRATION : CLASS NO: 15

(DEFINITE INTEGRALS)

Q.No: 1 $I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$ ... (1)

$I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2\cos^2(\pi-x) + b^2\sin^2(\pi-x)}dx$ ... (P IV)

$I = \int_0^{\pi} \frac{(\pi-x)}{a^2\cos^2 x + b^2\sin^2 x}dx$ ... (2)

(1) + (2)

$2I = \pi\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx$

Divide by $\cos^2 x$

$2I = \pi\int_0^{\pi} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}dx$

Special

$\tan x = t$
$\sec^2 x\,dx = dt$
$x = 0;\ t = 0$
$x = \pi;\ t = 0$ ✗

{Check here $f(x) = \frac{\sec^2 x}{a^2 + b^2\tan^2 x}$

$f(\pi - x) = \frac{\sec^2 x}{a^2+b^2\tan^2 x} = f(x)$}

$2I = \pi \times \cancel{2}\int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}dx$ ... $\left\{\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \quad f(2a-x) = f(x)\right\}$

put $\tan x = t$
$\sec^2 x\,dx = dt$

| $x = 0,\ t = 0$
| $x = \pi/2;\ t = \infty$

$$\therefore \quad I = \pi \int_0^\infty \frac{dt}{a^2+b^2t^2}$$

$$I = \frac{\pi}{b^2} \int_0^\infty \frac{dt}{\left(\frac{a}{b}\right)^2+t^2}$$

$$= \frac{\pi}{b^2} \times \frac{b}{a}\left[\tan^{-1}\left(\frac{tb}{a}\right)\right]_0^\infty$$

$$= \frac{\pi}{ab}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{\pi}{ab}\left(\frac{\pi}{2} - 0\right)$$

$$\boxed{I = \frac{\pi^2}{2ab}} \text{ Ans}$$

Qn 2 $\quad I = \int_0^{\pi/2} \log(\sin x)\,dx \quad \ldots (1) \quad$ (or) $\quad I = \int_0^{\pi/2} \log(\cos x)\,dx$

$$I = \int_0^{\pi/2} \log\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx \quad \ldots (P\,IV)$$

$$I = \int_0^{\pi/2} \log(\cos x)\,dx \quad \ldots (2)$$

(1) + (2)

$$2I = \int_0^{\pi/2} \log(\sin x \cdot \cos x)\,dx$$

$$2I = \int_0^{\pi/2} \log\left(\frac{\sin(2x)}{2}\right)dx$$

$$2I = \int_0^{\pi/2} \log(\sin(2x)) - \log 2 \; dx$$

(3)

$$2I = \int_0^{\pi/2} \log(\sin(2x))dx - \int_0^{\pi/2} \log 2\, dx$$

$$2I = \int_0^{\pi/2} \log(\sin(2x))dx - \left(x\log 2\right)_0^{\pi/2}$$

$$2I = \int_0^{\pi/2} \log(\sin(2x))dx - \frac{\pi}{2}\log 2$$

$$2I = I_1 - \frac{\pi}{2}\log 2 \quad \text{--- (3)}$$

when $I_1 = \int_0^{\pi/2} \log(\sin(2x))dx$

put $2x = t$, $dx = \frac{dt}{2}$ | when $x = 0$; $t = 0$, $x = \pi/2$; $t = \pi$

$$I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\, dt$$

{ Check $f(t) = \log(\sin t)$
$f(\pi - t) = \log(\sin(\pi - t)) = \log(\sin t) = f(t)$ }

$$I_1 = \frac{1}{\cancel{2}} \times \cancel{2} \int_0^{\pi/2} \log(\sin t)\, dt \quad \text{--- (P VI)}$$

$$I_1 = \int_0^{\pi/2} \log(\sin x)dx \quad \text{--- (P I)}$$

$$I_1 = I$$

$$\therefore 2I = I - \frac{\pi}{2}\log 2$$

$$\boxed{I = -\frac{\pi}{2}\log 2} \text{ Ans}$$

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Qn. 3 $I = \int_0^\pi \log(1+\cos x)\,dx$ --- ①

$I = \int_0^\pi \log\left(1+\cos(\pi - x)\right)dx$ --- (P IV)

$I = \int_0^\pi \log\left(1-\cos x\right)dx$ --- ②

① + ②

$2I = \int_0^\pi \log\left((1+\cos x)(1-\cos x)\right)dx$

$2I = \int_0^\pi \log\left(\sin^2 x\right)dx$

$2I = 2\int_0^\pi \log\left(\sin x\right)dx$

$I = \int_0^\pi \log(\sin x)\,dx$

$I = 2\int_0^{\pi/2} \log(\sin x)\,dx$ --- (P - VI)

$\frac{I}{2} = \int_0^{\pi/2} \log(\sin x)\,dx$ --- ③

$\frac{I}{2} = \int_0^{\pi/2} \log(\cos x)\,dx$ --- ④

③ + ④

$I = \int_0^{\pi/2} \log(\sin x \cdot \cos x)\,dx$

$I = \int_0^{\pi/2} \log\left(\sin(2x)\right) - \log 2\,dx$

$I = \int_0^{\pi/2} \log\left(\sin(2x)\right)dx - \frac{\pi}{2}\log 2$

$$I = I_1 - \frac{\pi}{2}\log 2$$

when $I_1 = \int_0^{\pi/2} \log(\sin(2x))\,dx$

proceed

$$I_1 = \int_0^{\pi/2} \log(\sin x)\,dx$$

$$I_1 = \frac{I}{2}$$

$$\therefore\ I = \frac{I}{2} - \frac{\pi}{2}\log 2$$

$$\frac{I}{2} = -\frac{\pi}{2}\log 2$$

$$\boxed{I = -\pi\log 2}\ \underline{\text{Ans}}$$

Ans: 4 $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}}\,dx$ --- (1)

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\tan\sqrt{\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}\,dx \quad \cdots \left\{ \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \right.$$

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+1}\,dx \text{ --- (2)}$$

(1) + (2)

$$2I = \int_{\pi/6}^{\pi/3} \cancel{\frac{1+\sqrt{\tan x}}{1+\sqrt{\tan x}}}\,dx$$

$$2I = (x)_{\pi/6}^{\pi/3}$$

$$2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\boxed{I = \pi/12}$$ Ans

Qn. 5 $$I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x} + \sqrt{x}}\,dx \quad \text{--- (1)}$$

$$I = \int_1^2 \frac{\sqrt{1+2-x}}{\sqrt{3-(1+2-x)} + \sqrt{1+2-x}}\,dx \quad \text{--- (P V)}$$

$$I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x} + \sqrt{3-x}}\,dx \quad (2)$$

(1) + (2)

$$2I = \int_1^2 \frac{\cancel{\sqrt{x} + \sqrt{3-x}}}{\cancel{\sqrt{3-x} + \sqrt{x}}}\,dx$$

$$2I = (x)_1^2$$

$$2I = 2 - 1$$

$$I = 1/2$$ Ans

Qn. 6 $$I = \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx$$

here $f(x) = x^3 \sin^4 x$

$f(-x) = (-x)^3 \cdot \sin^4(-x) = -x^3 \sin^4 x = -f(x)$

$f(x) \rightarrow$ odd function

$\therefore I = 0$ ... $\left\{ \int_{-a}^{a} f(x)\,dx = 0 \right.$ when $f(-x) = -f(x) \}$

Ans

Qn. 7 $I = \int_{-\pi/2}^{\pi/2} \sin^4 x \, dx$

here $f(x) = \sin^4 x$

$f(-x) = \sin^4(-x) = \sin^4 x = f(x)$

$\therefore f(x) \rightarrow$ even function

$\therefore I = 2\int_{0}^{\pi/2} \sin^4 x \, dx$ ... $\left\{ \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \right.$ when $f(-x) = f(x) \}$

$I = 2\int_{0}^{\pi/2} (\sin^2 x)^2 dx$

$I = 2\int_{0}^{\pi/2} \left(\frac{1-\cos(2x)}{2}\right)^2 dx$

$I = \frac{1}{2}\int_{0}^{\pi/2} 1 + \cos^2(2x) - 2\cos(2x)\, dx$

$I = \frac{1}{2}\int_{0}^{\pi/2} 1 + \frac{1+\cos(4x)}{2} - 2\cos(2x)\, dx$

$I = \frac{1}{4}\int_{0}^{\pi/2} 3 + \cos(4x) - 4\cos(2x)\, dx$

$= \frac{1}{4}\left[3x + \frac{\sin(4x)}{4} - 2\sin(2x)\right]_0^{\pi/2}$

$= \frac{1}{4}\left[\left(\frac{3\pi}{2} + 0 - 0\right) - (0)\right] \Rightarrow \boxed{I = \frac{3\pi}{8}}$ Ans

Qn:8 $I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$

$f(x) = \sqrt{\frac{a-x}{a+x}}$

$f(-x) = \sqrt{\frac{a+x}{a-x}} \neq f(x) \neq -f(x)$

(neither even nor odd)

$I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}}\, dx$

$I = \int_{-a}^{a} \frac{a-x}{\sqrt{a^2-x^2}}\, dx$ (separate)

$I = a\int_{-a}^{a} \frac{1}{\sqrt{a^2-x^2}}\, dx - \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}\, dx$

$f(x) = \frac{1}{\sqrt{a^2-x^2}}$

$f(-x) = \frac{1}{\sqrt{a^2-x^2}} = f(x)$

even function

$g(x) = \frac{x}{\sqrt{a^2-x^2}}$

$g(-x) = \frac{-x}{\sqrt{a^2-x^2}} = -g(x)$

$g(x) \rightarrow$ odd funct.

$I = a \times 2\int_{0}^{a} \frac{1}{\sqrt{a^2-x^2}}\, dx - 0$ -- { P VII }

$I = 2a\left(\sin^{-1}\left(\frac{x}{a}\right)\right)_0^a$

$I = 2a\left(\frac{\pi}{2} - 0\right)$

$\boxed{I = \pi}$ Ans

Qn. 9 → $\int_{-\pi/4}^{\pi/4} \frac{2x + 2x\sin x}{1+\cos^2 x}\,dx$

$$f(x) = \frac{2x + 2x\sin x}{1+\cos^2 x}$$

$$f(-x) = \frac{-2x + 2x\sin x}{1+\cos^2 x} = -\left(\frac{2x - 2x\sin x}{1+\cos^2 x}\right) \neq f(x) \neq -f(x)$$

∴ neither

$$I = 2\int_{-\pi/4}^{\pi/4} \underbrace{\frac{x}{1+\cos^2 x}}_{f(x)}\,dx + 2\int_{-\pi/4}^{\pi/4} \underbrace{\frac{x\sin x}{1+\cos^2 x}}_{g(x)}\,dx$$

$f(-x) = -f(x)$ odd

$g(-x) = g(x)$ even

$$I = 0 + 4\int_0^{\pi/4} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$I = 4\int_0^{\pi/4} \frac{x\sin x}{1+\cos^2 x}\,dx \quad \text{—— (1)}$$

P IV removal of x

Ans $\boxed{I = \frac{\pi^2}{6\sqrt{3}}}$

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Que 10 → Show that $\int_0^a f(x) g(x)\, dx = 2\int_0^a f(x)\, dx$

given that $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Soln

let $I = \int_0^a f(x) g(x)\, dx$

$$I = \int_0^a f(a-x) \cdot g(a-x)\, dx \quad \text{-- (P IV)}$$

$$I = \int_0^a f(x) \cdot (4 - g(x))\, dx$$

$$I = 4\int_0^a f(x)\, dx - \int_0^a f(x) g(x)\, dx$$

$$I = 4\int_0^a f(x)\, dx - I$$

$$I = 2\int_0^a f(x)\, dx = \underline{\text{Rhs}}$$

Que. 11 → $I = \int_0^1 \tan^{-1}\left(\frac{2x - 1}{1 + x - x^2}\right) dx \quad \text{---} (1)$

$$I = \int_0^1 \tan^{-1}\left(\frac{2 - 2x - 1}{1 + 1 - x - (1-x)^2}\right) dx \quad \text{-- (P IV)}$$

$$I = \int_0^1 \tan^{-1}\left(\frac{1 - 2x}{2 - x - 1 - x^2 + 2x}\right) dx$$

$$I = \int_0^1 \tan^{-1}\left(\frac{1 - 2x}{1 + x - x^2}\right) dx \quad \text{---}$$

$$I = -\int_0^1 \tan^{-1}\left(\frac{2x - 1}{1 + x - x^2}\right) dx \quad \text{-} (2)$$

(1) + (2)

$2I = 0$

$I = 0$ $\underline{\text{Ans}}$

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