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ULTIMATE MATHEMATICS

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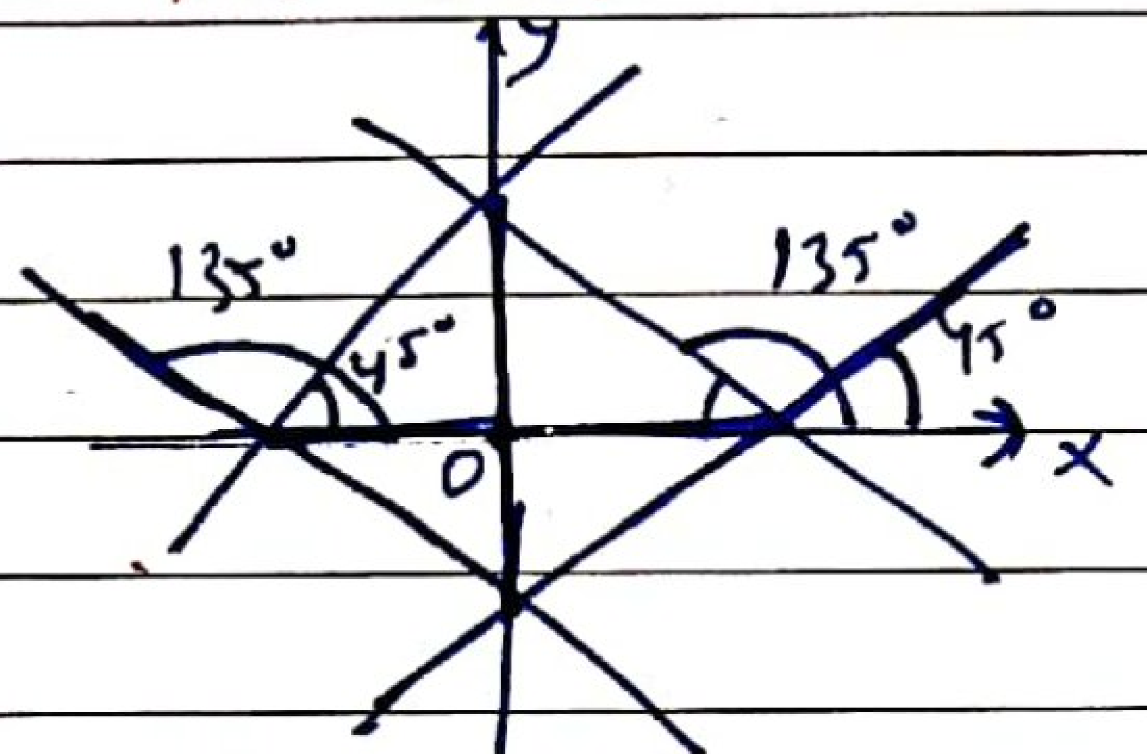
CHAPTER: A.O.D

Class No: 2

Topic Tangent & Normals.

Qn. 1 → Find the points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes.

Sol) x_1



$m = \tan(135) = -1$

$m = \tan(45) = 1$

$(m = \pm 1)$

(·) let the point of contact is (x_1, y_1)

(·) equation of curve

$$9y^2 = x^3$$

(·) Diff w.r.t x

$$18y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{x^2}{6y}$$

(·) slope of Tangent $(x_1, y_1) = \frac{x_1^2}{6y_1}$

(·) slope of Normal at $(x_1, y_1) = \frac{-6y_1}{x_1^2}$

(·) also slope of Normal $= \pm 1$ ($\because$ normal makes equal intercepts with the axes)

$$\Rightarrow \frac{-6y_1}{x_1^2} = \pm 1$$

$$\Rightarrow \boxed{y_1 = \mp \frac{x_1^2}{6}} \quad \text{--- (1)}$$

(·) also we have

$$9y_1^2 = x_1^3 \quad \text{--- (2)}$$

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A-O D (class No: 2) (2)

from (1) & (2)

$$9\left(\mp\frac{x_1^2}{6}\right)^2 = x_1^3$$

$$\Rightarrow 9\frac{x_1^4}{36} = x_1^3$$

$$\Rightarrow x_1^4 = 4x_1^3$$

$$\Rightarrow x_1^4 - 4x_1^3 = 0$$

$$\Rightarrow x_1^3(x_1-4) = 0$$

$$\boxed{x_1 = 0} \qquad \boxed{x_1 = 4}$$

put in eq (2) $\quad 9y_1^2 = x_1^3$

for $x_1 = 0 \qquad 9y_1^2 = 0 \Rightarrow y_1 = 0$

for $x_1 = 4 \qquad 9y_1^2 = 64 \Rightarrow y_1^2 = \frac{64}{9}$

$$y_1 = \pm\frac{8}{3}$$

$\therefore$ points $(0,0)$ $(4, 8/3)$ $(4, -8/3)$

reject

($\because$ normal makes equal intercepts with the axes, it can't pass through the origin)

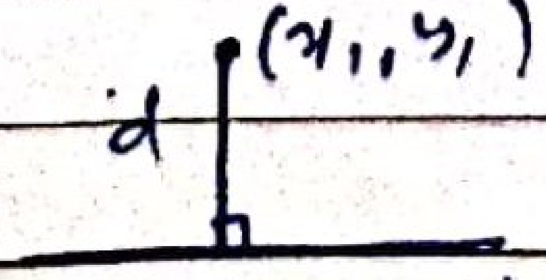
$\therefore$ required points are $(4, 8/3)$ $(4, -8/3)$ Ans

Que 2 → Show that the normal at any point θ to the curve

$x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$

is at a constant distance from the origin

sol: $\rightarrow$



$ax+by+c=0$

$$d = \frac{|ax_1+by_1-c|}{\sqrt{a^2+b^2}}$$

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A.O.D class No= 2

(∵) Point of Contact $\left((a\cos\theta + a\theta\sin\theta),\ a\sin\theta - a\theta\cos\theta\right)$

(∵) Diff wrt θ

$$\frac{dx}{d\theta} = -a\sin\theta + a(\theta\cos\theta + \sin\theta) = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\cos\theta - a(-\theta\sin\theta + \cos\theta) = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{\cancel{a\theta}\sin\theta}{\cancel{a\theta}\cos\theta} = \tan\theta$$

(∵) Slope of Tangent at θ = $\tan\theta$

(∵) Slope of Normal = $-\frac{1}{\tan\theta} = -\cot\theta$

(∵) equation of Normal

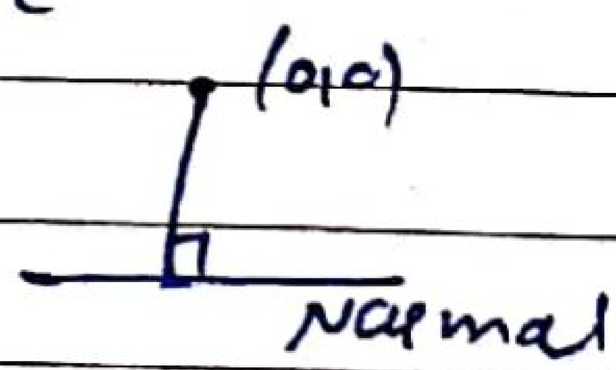
$$y - (a\sin\theta - a\theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta - a\theta\sin\theta)$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + \cancel{a\theta\sin\theta\cos\theta} = -x\cos\theta + a\cos^2\theta + \cancel{a\theta\sin\theta\cos\theta}$$

$$\Rightarrow x\cos\theta + y\sin\theta - a\sin^2\theta - a\cos^2\theta = 0$$

$$\Rightarrow \boxed{x\cos\theta + y\sin\theta - a = 0}$$ equation of Normal

Now distance b/w origin & the Normal



$$= \frac{|0 + 0 - a|}{\sqrt{\cos^2\theta + \sin^2\theta}}$$

$$|-2| = 2$$
$$|-x| = |x|$$

$= |a|$ units which is a constant

Proved

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A·OD (class No: 2) (4)

Ques: 3 → Find the equation of tangents to the curve $y = \cos(x+y)$; $-2\pi \le x \le 2\pi$ that are parallel to the line $x + 2y = 0$

Sol: (·) Let the point of contact is (x_1, y_1)

(·) equation of curve

$$y = \cos(x+y)$$

(·) Diff wrt x

$$\frac{dy}{dx} = -\sin(x+y)\cdot\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(x+y) - \sin(x+y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(1 + \sin(x+y)\right) = -\sin(x+y)$$

$$\frac{dy}{dx} = \frac{-\sin(x+y)}{1+\sin(x+y)}$$

(·) Slope of tangent at (x_1, y_1)

$$= \frac{-\sin(x_1+y_1)}{1+\sin(x_1+y_1)}$$

(·) Slope of given ~~of~~ line $(x+2y=0) = -\frac{1}{2}$

(·) Since Tangent is parallel to the line

$$\Rightarrow \frac{-\sin(x_1+y_1)}{1+\sin(x_1+y_1)} = \frac{-1}{2}$$

$$\Rightarrow 2\sin(x_1+y_1) = 1 + \sin(x_1+y_1)$$

$$\Rightarrow \sin(x_1+y_1) = 1$$

$x_1 + y_1 = \pi/2$; $x_1 + y_1 = -\frac{3\pi}{2}$

Reason
$$\boxed{\begin{array}{l} \sin\left(\frac{3\pi}{2}\right) = -1 \\ \sin\left(-\frac{3\pi}{2}\right) = 1 \end{array}}$$

(·) also we have

$$y_1 = \cos(x_1 + y_1)$$

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$\Rightarrow y_1 = \cos\left(\frac{\pi}{2}\right)$

$\Rightarrow y_1 = 0$

$\Rightarrow x_1 + 0 = \pi/2$

$\Rightarrow x_1 = \pi/2$

$\therefore \left(\frac{\pi}{2}, 0\right)$

$\Rightarrow y_1 = \cos\left(-\frac{3\pi}{2}\right)$

$y_1 = \cos\left(\frac{3\pi}{2}\right) = 0$

$\therefore x_1 + 0 = -3\pi/2$

$x_1 = -3\pi/2$

$\therefore$ point $\left(-\frac{3\pi}{2}, 0\right)$

✓ equation of Tangent at $\left(\frac{\pi}{2}, 0\right)$

$$y - 0 = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$$

$$2y = -x + \pi/2$$

$$x + 2y = \frac{\pi}{2} \quad \underline{\underline{\text{Ans}}}$$

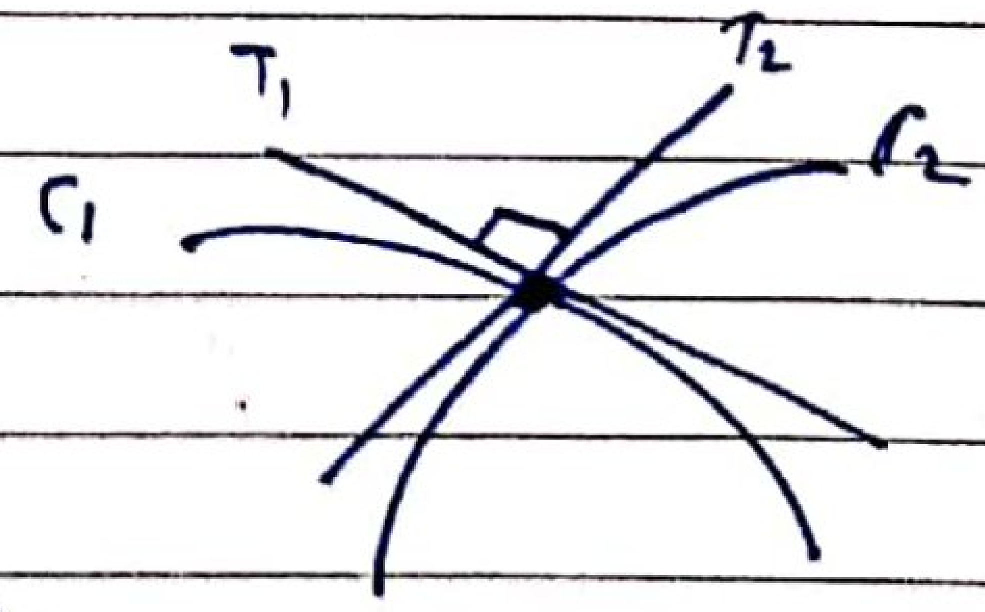
✓ eq. of Tangent at $\left(-\frac{3\pi}{2}, 0\right)$

$$y - 0 = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$$

$$\Rightarrow 2y = -x - \frac{3\pi}{2}$$

$$\Rightarrow x + 2y + \frac{3\pi}{2} = 0 \quad \underline{\underline{\text{Ans}}}$$

Qn. 4 → Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$ (orthogonal)



Soln given curve cut at right angle

T.P. $8k^2 = 1$

$C_1: x = y^2$ & $C_2: xy = k$

Solve. C_1 & C_2

$$y^3 = k$$

$$\Rightarrow \boxed{y = k^{1/3}} \Rightarrow \boxed{x = k^{2/3}}$$

ACD (class No: 2) (5)

$\therefore$ point of Intersection | Point of Contact is $(k^{2/3}, k^{1/3})$

$C_1: \quad x = y^2$

Diff $\quad 1 = 2y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2y}$

$m_1 = \frac{1}{2k^{1/3}}$

$C_2: \quad xy = k$

Diff $\quad x\frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = \frac{-y}{x}$

$m_2 = -\frac{k^{1/3}}{k^{2/3}}$

Since the curves cut at right angles

$\therefore \quad m_1 m_2 = -1$

$$\left(\frac{1}{2\cancel{k^{1/3}}}\right)\left(\frac{-\cancel{k^{1/3}}}{k^{2/3}}\right) = -1$$

$$\frac{1}{2k^{2/3}} = 1$$

$$\Rightarrow \quad 1 = 2k^{2/3}$$

Cubing both sides

$\boxed{1 = 8k^2}$ proved

Qn 5 $\to$ The curve $y = ax^3 + bx^2 + cx + 5$ touches the
special
X-axis at $P(-2, 0)$ and cuts the Y-axis at the point Q where the gradient is 3. ~~Fin~~ Find the equation of curve completely.

Soln

$P(-2, 0)$ lies on the curve

$$0 = -8a + 4b - 2c + 5$$

$$8a - 4b + 2c = 5 \quad \text{---} \quad (1)$$

Diff equation of curve

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

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A-012 (Class No:-7) (7)

Slope of tangent at $(-2, 0)$ $= 12a - 4b + c$

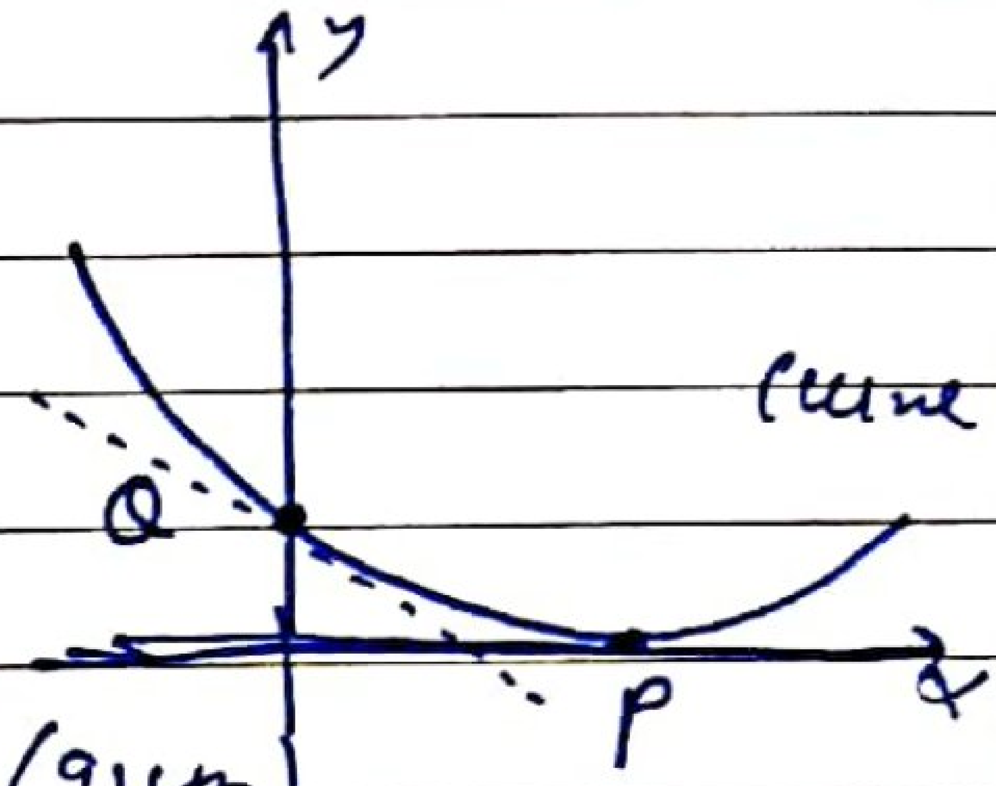
also slope of tangent (X-axis) $= 0$

$\Rightarrow$ $12a - 4b + c = 0$ --- (2)

Curve $y = ax^3 + bx^2 + cx + 5$ cut Y-axis at Q

put $x = 0$

$y = 5$ $\therefore$ $Q(0, 5)$



Slope of Tangent at $Q(0, 5)$

$= c$

also slope of Tangent at Q $= 3$ -- (given)

$\therefore$ $\boxed{c = 3}$ put in eq (2) also in eq (1)

$12a - 4b = -3$ -- (4)

$8a - 4b = -1$ -- (5)

Soln (4) & (5)

$4a = -2$

$a = -1/2$ $\Rightarrow$ $b = -3/4$ $c = 3$

$\therefore$ equation of curve $y = -\frac{1}{2}x^3 - \frac{3}{4}x^2 + 3x + 5$ Ans

Qn. 6 - Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where it crosses the Y-axis

Sol $y = be^{-x/a}$ cuts the Y axis

$\therefore$ put $x = 0$

$y = be^0$ $\Rightarrow$ $y = b$ $\therefore$ point $(0, b)$

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A.OD (class No: 2)

$y = be^{-x/a}$

Diff w.r.t x

$$\frac{dy}{dx} = be^{-x/a} \cdot \left(\frac{-1}{a}\right)$$

Slope of Tangent at $(0, b) = be^{0}\left(\frac{-1}{a}\right) = \frac{-b}{a}$

Now equation of tangent at $(0, b)$

$$y - b = -\frac{b}{a}(x - 0)$$

$$\Rightarrow ay - ab = -bx$$

$$\Rightarrow bx + ay = ab$$

divide by ab

$$\frac{x}{a} + \frac{y}{b} = 1$$ $\therefore$ the line

touches the curve.

← WORKSHEET No:- 2 →

Tangent Normals

Qns: 1 → Show that the curves $4x = y^2$ and $4xy = k$ cut at right angles, if $k^2 = 512$

Qns 2 → Show that the curves $y^2 = 8x$ and $2x^2 = y^2$ cut orthogonally at $(1, 2\sqrt{2})$

HINT Show $m_1 m_2 = -1$

Qn- 3 → Find the equation of all the lines of slope zero that are tangent to the curve $y = \frac{1}{x^2 - 2x + 3}$

ANS $2y - 1 = 0$

Qn 4 → Find the equation of all the lines having slope 2 which are tangent to the curve $y = \frac{1}{x-3}$

ANS There is no tangent to the curve having slope = 2

Qn 5 → Prove that $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ at the point (a, b)

Qn- 6 → The equation of the tangent at $(2,3)$ on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the values of a & b

ANS $a = 2$, $b = -7$

Qn- 7 → Find the equation of the tangent to the curve $x = \sin(3t)$, $y = \cos(2t)$ at $t = \pi/4$

ANS $2\sqrt{2}x - 3y - 2 = 0$

Qn. 8 → Find the coordinates of the points on the curve $y=x^2+3x+4$, the tangents at which pass through the origin

Ans $(2,14)$ $(-2,2)$

Ans 9 → Find the equation of the normal to the curve $y=2x^2+3\sin x$ when it crosses the Y axis

Ans $x+3y=0$

Qn 10 → Find the points on the curve $y=3x^2-9x+8$ at which the tangents are equally inclined with the axes

Hint slope $= \pm 1$

Ans $\left(\frac{5}{3},\frac{4}{3}\right)$ & $\left(\frac{4}{3},\frac{4}{3}\right)$

— x —