SOLUTIONS INTEGRATION ①

WORKSHEET NO: 6 (class No: 8)

Ans: 1 $I = \int \frac{1}{1+\cot x}\,dx$

$$I = \int \frac{1}{1 + \frac{\cos x}{\sin x}}\,dx$$

$$I = \int \frac{\sin x}{\sin x + \cos x}\,dx$$

let $\sin x = A \cdot \frac{d}{dx}(\sin x + \cos x) + B(\sin x + \cos x)$

$\sin x = A(\cos x - \sin x) + B(\sin x + \cos x)$

equating the cofficient of $\sin x$ and $\cos x$

$$\begin{aligned} 1 &= -\cancel{A} + B \\ 0 &= \cancel{A} + B \\ \hline 1 &= 2B \end{aligned}$$

$\boxed{B = 1/2}$ $\boxed{A = -1/2}$

$$\therefore I = \int \frac{-\frac{1}{2}(\cos x - \sin x) + \frac{1}{2}(\sin x + \cos x)}{\sin x + \cos x}\,dx$$

$$I = -\frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx + \frac{1}{2}\int dx$$

put $\sin x + \cos x = t$

$(\cos x - \sin x)dx = dt$

$$I = -\frac{1}{2}\int \frac{dt}{t} + \frac{1}{2}x$$

$$I = -\frac{1}{2}\log|t| + \frac{x}{2} + c$$

$$I = -\frac{1}{2}\log|\sin x + \cos x| + \frac{x}{2} + C$$ Ans

Qns. 2 → $I = \int \frac{4\sin x + 5\cos x}{5\sin x + 4\cos x}\, dx$

let $4\sin x + 5\cos x = A(5\cos x - 4\sin x) + B(5\sin x + 4\cos x)$

equaty coff of $\sin x$ & $\cos x$

$4 = -4A + 5B \quad \times 5$

$5 = 5A + 4B \quad \times 4$

$20 = -20A + 25B$

$20 = 20A + 16B$

$40 = 41B$

$\boxed{B = \frac{40}{41}} \Rightarrow 4A = \frac{200}{41} - 4$

$\cancel{4 + 4 \cdot \frac{160}{41} = 5B}$

$4A = \frac{36}{41}$

$A = \frac{9}{41}$

$\therefore I = \int \frac{\frac{9}{41}(5\cos x - 4\sin x) + \frac{40}{41}(5\sin x + 4\cos x)}{5\sin x + 4\cos x}\, dx$

$= \frac{9}{41}\int \frac{5\cos x - 4\sin x}{5\sin x + 4\cos x}\, dx + \frac{40}{41}\int dx$

put $5\sin x + 4\cos x = t$

$(5\cos x - 4\sin x)dx = dt$

$I = \frac{9}{41}\int \frac{dt}{t} + \frac{40}{41}x$

$I = \frac{9}{41}\log|5\sin x + 4\cos x| + \frac{40}{41}x + C$ Ans

Qns 3 → $I = \int \frac{1 + 8\cot x}{3\cot x + 2}\, dx$

$I = \int \frac{1 + \frac{8\cos x}{\sin x}}{\frac{3\cos x}{\sin x} + 2}\, dx$

$= \int \frac{\sin x + 8\cos x}{3\cos x + 2\sin x}\, dx$

let $\sin x + 8\cos x = A \cdot \frac{d}{dx}(3\cos x + 2\sin x) + B(3\cos x + 2\sin x)$

$\sin x + 8\cos x = A(-3\sin x + 2\cos x) + B(3\cos x + 2\sin x)$

equating coefficient of $\sin x$ & $\cos x$

$1 = -3A + 2B \quad \times 2$

$8 = 2A + 3B \quad \times 3$

$2 = -6A + 4B$

$24 = 6A + 9B$

$26 = 13B$

$\boxed{B = 2}$ $\boxed{A = 1}$

$\therefore I = \int \frac{(-3\sin x + 2\cos x) + 2(3\cos x + 2\sin x)}{3\cos x + 2\sin x} dx$

$= \int \frac{-3\sin x + \cos x}{3\cos x + 2\sin x} dx + 2\int dx$

put $3\cos x + 2\sin x = t$

$(-3\sin x + 2\cos x)dx = dt$

$I = \int \frac{dt}{t} + 2x$

$I = \log|3\cos x + 2\sin x| + 2x + C$ Ans

Ques 4 → $I = \int \frac{\sin x - \cos x}{\sqrt{\sin(2x)}} dx$

$I = \int \frac{\sin x - \cos x}{\sqrt{1 - 1 + \sin(2x)}} dx$

$= \int \frac{\sin x \cancel{+} \cos x}{\sqrt{(\sin^2 x + \cos^2 x + 2\sin x \cos x) - 1}} dx$

$$I = \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 1}} \, dx$$

put $\sin x + \cos x = t$

$$(\cos x - \sin x) dx = dt$$

$$(\sin x - \cos x) dx = -dt$$

$$I = -\int \frac{dt}{\sqrt{t^2 - 1}}$$

$$= -\log \left| t + \sqrt{t^2 - 1} \right| + C$$

$$I = -\log \left| (\sin x + \cos x) + \sqrt{(\sin x + \cos x)^2 - 1} \right| + C$$

$$I = -\log \left| \sin x + \cos x + \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x - 1} \right| + C$$

$$I = -\log \left| \sin x + \cos x + \sqrt{\sin(2x)} \right| + C \quad \text{Ans}$$

Ans = 5 → $I = \int \sqrt{\tan x} + \sqrt{\cot x} \, dx$

$$I = \int \frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}} \, dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} \, dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\frac{\sin(2x)}{2}}} \, dx$$

$$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - 1 + \sin(2x)}} \, dx$$

$$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (1 - \sin(2x))}} \, dx$$

$$I = \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\,dx$$

put $\sin x - \cos x = t$

$(\cos x + \sin x)\,dx = dt$

$$I = \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\sin^{-1}(t) + C$$

$$I = \sqrt{2}\sin^{-1}(\sin x - \cos x) + C \quad \underline{\text{Ans}}$$

Ans 6 → $I = \int \frac{1}{(\sin x - 2\cos x)(2\sin x + \cos x)}\,dx$

$$I = \int \frac{1}{2\sin^2 x + \sin x\cos x - 4\sin x\cos x - 2\cos^2 x}\,dx$$

$$= \int \frac{1}{2\sin^2 x - 2\cos^2 x - 3\sin x\cos x}\,dx$$

divide by $\cos^2 x$

$$I = \int \frac{\sec^2 x \,dx}{2\tan^2 x - 2 - 3\tan x}$$

put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$I = \int \frac{dt}{2t^2 - 3t - 2}$$

$$= \frac{1}{2}\int \frac{1}{t^2 - \frac{3t}{2} - 1}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\left(t - \frac{3}{4}\right)^2 - \frac{9}{16} - 1}\,dt$$

$$I = \frac{1}{2}\int \frac{dt}{\left(t-\frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2}$$

$$= \frac{1}{2} \times \frac{1}{2\times\frac{5}{4}} \log\left|\frac{t-\frac{3}{4}-\frac{5}{4}}{t-\frac{3}{4}+\frac{5}{4}}\right| + C$$

$$= \frac{1}{5}\log\left|\frac{4t-8}{4t+2}\right| + C$$

$$= \frac{1}{5}\log\left|\frac{2t-4}{2t+1}\right| + C$$ Ans $\longrightarrow$ replace t

OR

$$= \frac{1}{5}\log\left|\frac{2(t-2)}{2t+1}\right| + C$$

$$= \frac{1}{5}\log 2 + \frac{1}{5}\log\left|\frac{t-2}{2t+1}\right| + C$$

both are constant sum $\rightarrow$ = new constant

$$I = \frac{1}{5}\log\left|\frac{t-2}{2t+1}\right| + C_1$$

Replace $\quad I = \frac{1}{5}\log\left|\frac{\tan x - 2}{2\tan x + 1}\right| + C$ Ans

Ques 7 $\rightarrow$ $I = \int \frac{1}{\sin^2 x + \sin(2x)}\,dx$

$$I = \int \frac{1}{\sin^2 x + 2\sin x\cos x}\,dx$$

Divide by $\cos^2 x$

$$I = \int \frac{\sec^2 x}{\tan^2 x + 2\tan x}$$

put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\therefore I = \int \frac{dt}{t^2+2t}$

$I = \int \frac{1}{(t+1)^2 - 1} dt$

$I = \frac{1}{2} \log \left| \frac{t+1-1}{t+1+1} \right| + c$

$I = \frac{1}{2} \log \left| \frac{\tan x}{\tan x + 2} \right| + c$ Ans

Ques. 8 → $I = \int \frac{\cos x}{\cos(3x)} dx$

$I = \int \frac{\cos x}{4\cos^3 x - 3\cos x} dx$

$I = \int \frac{1}{4\cos^2 x - 3} dx$

divide by $\cos^2 x$

$I = \int \frac{\sec^2 x}{4 - 3\sec^2 x} dx$

$I = \int \frac{\sec^2 x}{4 - 3(1+\tan^2 x)} dx$ put $\tan x = t$
$\sec^2 x\, dx = dt$

$I = \int \frac{dt}{4-3(1+t^2)}$

$= \int \frac{1}{1-3t^2} dt$

$= \frac{1}{3} \int \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2 - t^2}$

$= \frac{1}{3} \times \frac{1}{2 \times \frac{1}{\sqrt{3}}} \log \left| \frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t} \right| + C$

$$I = \frac{1}{2\sqrt{3}} \log\left|\frac{1+\sqrt{3}t}{1-\sqrt{3}t}\right| + C$$

$$I = \frac{1}{2\sqrt{3}} \log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right| + C \qquad \text{Ans}$$

Ans. 9 → $$I = \int \frac{1}{(2\sin x + 3\cos x)^2}\,dx$$

$$I = \int \frac{1}{4\sin^2 x + 9\cos^2 x + 12\sin x\cos x}\,dx$$

divide by $\cos^2 x$

$$I = \int \frac{\sec^2 x\,dx}{4\tan^2 x + 9 + 12\tan x}$$

put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$I = \int \frac{dt}{4t^2 + 12t + 9}$$

$$I = \frac{1}{4}\int \frac{1}{t^2 + 3t + \frac{9}{4}}\,dt$$

$$= \frac{1}{4}\int \frac{1}{\left(t+\frac{3}{2}\right)^2 \cancel{-\frac{9}{4}} + \cancel{\frac{9}{4}}}\,dt$$

$$= \frac{1}{4}\int \frac{1}{\left(t+\frac{3}{2}\right)^2}\,dt$$

$$I = -\frac{1}{4}\cdot\frac{1}{\left(t+\frac{3}{2}\right)} + C \qquad \text{---}\left\{\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C \text{ shortcut}\right.$$

$$I = -\frac{1}{2}\cdot\frac{1}{(2t+3)} + C$$

$$I = -\frac{1}{2}\cdot\frac{1}{(2\tan x + 3)} + C \qquad \text{Ans}$$

Ans 10 → $I = \int (\log x)^2 \, dx$

$$I = \int \underset{I}{(\log x)^2} \cdot \underset{II}{1} \, dx$$

$$= (\log x)^2 \cdot x - \int 2\log x \cdot \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx$$

$$= x(\log x)^2 - 2\int \underset{I}{\log x} \cdot \underset{II}{1} \, dx$$

$$= x(\log x)^2 - 2\left[\log x \cdot x - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx\right]$$

$$I = x(\log x)^2 - 2\left[x\log x - x\right] + c \quad \text{Ans}$$

Ans 11 → $I = \int \underset{II}{x} \underset{I}{\tan^{-1} x} \, dx$

$$= \tan^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2} \, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int 1 - \frac{1}{1+x^2} \, dx$$

$$I = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x\right) + c \quad \text{Ans}$$

Ans 12 → $I = \int (\sin^{-1} x)^2 \, dx$

$$I = \int \underset{I}{(\sin^{-1} x)^2} \cdot \underset{II}{1} \, dx$$

$$= (\sin^{-1} x)^2 \cdot x - \int \frac{2\sin^{-1} x}{\sqrt{1-x^2}} \cdot x \, dx$$

$$I = x(\sin^{-1}x)^2 - 2\int \frac{x \sin^{-1}x}{\sqrt{1-x^2}}\,dx$$

put $\sin^{-1}x = t \rightarrow x = \sin t$

$$\frac{1}{\sqrt{1-x^2}}\,dx = dt$$

$$I = x(\sin^{-1}x)^2 - 2\int \underset{\text{II}}{\sin t} \cdot \underset{\text{I}}{t}\,dt$$

$$= x(\sin^{-1}x)^2 - 2\left[t\cdot(-\cos t) + \int \cos t\,dt\right]$$

$$= x(\sin^{-1}x)^2 - 2\left[-t\cos t + \sin t\right] + c$$

$$= x(\sin^{-1}x)^2 - 2\left[-\sin^{-1}x\sqrt{1-\sin^2 t} + x\right] + c$$

$$I = x(\sin^{-1}x)^2 - 2\left[-\sin^{-1}x.\sqrt{1-x^2} + x\right] + c \quad \underline{\text{Ans}}$$

$\underline{\underline{\text{Ans}=13}} \rightarrow$ $$I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$$

put $x = \cos(2\theta)$

$dx = -2\sin(2\theta)\,d\theta$

$$I = \int \tan^{-1}\sqrt{\frac{1-\cos(2\theta)}{1+\cos(2\theta)}}\,.\left(-2\sin(2\theta)\,d\theta\right)$$

$$I = -2\int \tan^{-1}\sqrt{\frac{\cancel{2}\sin^2\theta}{\cancel{2}\cos^2\theta}}\,.\sin(2\theta)\,d\theta$$

$$= -2\int \tan^{-1}(\tan\theta)\sin(2\theta)\,d\theta$$

$$= -2\int \underset{\text{I}}{\theta}\cdot\underset{\text{II}}{\sin(2\theta)}\,d\theta$$

$$= -2\left[-\theta\frac{\cos(2\theta)}{2} + \frac{1}{2}\int \cos(2\theta)\,d\theta\right]$$

$$= -2\left[-\frac{\theta\cos(2\theta)}{2} + \frac{\sin(2\theta)}{4}\right] + C$$

$$= \theta\cos(2\theta) - \frac{\sin(2\theta)}{2} + C$$

$$= \frac{1}{2}\cos^{-1}x \cdot x - \frac{\sqrt{1-\cos^2(2\theta)}}{2} + C$$

$$I = \frac{1}{2}x\cos^{-1}x - \frac{\sqrt{1-x^2}}{2} + C \quad \underline{\text{ANS}}$$

Ans 14 → $I = \int \frac{x^2\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$

$$I = \int \frac{x^2\sin^{-1}x}{(1-x^2)\sqrt{1-x^2}}\,dx$$

put $\sin^{-1}x = t \Rightarrow x = \sin t$

$$\frac{1}{\sqrt{1-x^2}}\,dx = dt$$

$$I = \int \frac{\sin^2 t \cdot t}{(1-\sin^2 t)}\,dt$$

$$I = \int t\tan^2 t\,dt$$

$$= \int \underset{I}{t}\,\underset{II}{(\sec^2 t - 1)}\,dt$$

$$= t(\tan t - t) - \int(\tan t - t)\,dt$$

$$= t(\tan t - t) - \left[\log|\sec t| - \frac{t^2}{2}\right] + C$$

$$= t\tan t - t^2 + \log|\cos t| + \frac{t^2}{2} + C \quad \cdots \left\{ \log(A/B) = -\log(B/A) \right\}$$

$$= \sin^{-1}x \cdot \frac{\sin t}{\cos t} - (\sin^{-1}x)^2 + \log\left|\sqrt{1-\sin^2 t}\right| + \frac{(\sin^{-1}x)^2}{2} + C$$

$$I = \sin^{-1}x \cdot \frac{x}{\sqrt{1-x^2}} - \frac{(\sin^{-1}x)^2}{2} + \frac{1}{2}\log|1-x^2| + C \quad \underline{\text{Ans}}$$

$\underline{\text{Qn}}\cdot 15 \rightarrow$ $$I = \int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$$

$$I = \int \frac{x\tan^{-1}x}{(1+x^2)\sqrt{1+x^2}}\,dx$$

put $\tan^{-1}x = t \Rightarrow x = \tan t$

$$\frac{1}{1+x^2}\,dx = dt$$

$$I = \int \frac{\tan t \cdot t}{\sqrt{1+\tan^2 t}}\,dt$$

$$= \int \frac{\tan t}{\sec t}\cdot t\,dt$$

$$I = \int \underset{\mathrm{I}}{t}\,\underset{\mathrm{II}}{\sin t}\,dt$$

$$= -t\cos t + \int \cos t\,dt$$

$$I = -t\cos t + \sin t + C$$

$$= -t\,\frac{1}{\sqrt{1+\tan^2 t}} + \frac{\tan t}{\sec t} + C \quad \cdots \left\{ \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1+\tan^2\theta}},\ \sin\theta = \frac{\tan\theta}{\sec\theta} \right\}$$

$$I = -\frac{\tan^{-1}x}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + C \quad \underline{\text{Ans}}$$

$\underline{\text{Qn}}\ 16 \rightarrow$ $$I = \int x\sin x \cdot \cos(2x)\,dx$$

$$= \frac{1}{2}\int x\cdot\left(2\sin x\cdot\cos(2x)\right)dx$$

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$$= \frac{1}{2}\int x\left(\sin(3x) + \sin(-x)\right)dx$$

$$I = \frac{1}{2}\int \underset{\mathrm{I}}{x}\underbrace{(\sin(3x) - \sin x)}_{\mathrm{II}}\,dx$$

$$= \frac{1}{2}\left[x\left(-\frac{\cos(3x)}{3} + \cos x\right) - \int -\frac{\cos(3x)}{3} + \cos x\,dx\right]$$

$$I = \frac{1}{2}\left[-\frac{x\cos(3x)}{3} + x\cos x + \frac{\sin(3x)}{9} - \sin x\right] + C$$

Ans

Qn. 17 → $I = \int \cos^{-1}(4x^3 - 3x)\,dx$

put $x = \cos\theta$

$dx = -\sin\theta\,d\theta$

$$I = -\int \cos^{-1}(4\cos^3\theta - 3\cos\theta).\sin\theta\,d\theta$$

$$I = -\int \cos^{-1}(\cos(3\theta)).\sin\theta\,d\theta$$

$$I = -3\int \underset{\mathrm{I}}{\theta}\cdot\underset{\mathrm{II}}{\sin\theta}\,d\theta$$

$$= -3\left[-\theta\cos\theta + \int\cos\theta\,d\theta\right]$$

$$= -3\left[-\cos^{-1}x\,.\,x + \sin\theta\right] + C$$

$$I = 3x\cos^{-1}x - 3\sqrt{1-x^2} + C$$ Ans

Qn. 18 → $I = \int \underset{\mathrm{II}}{(x+1)}\underset{\mathrm{I}}{\log x}\,dx$

$$= (\log x)\left(\frac{x^2}{2} + x\right) - \int \frac{1}{x}\cdot\left(\frac{x^2}{2} + x\right)dx$$

$$I = \log x \cdot \left(\frac{x^2}{2} + x\right) - \int \left(\frac{x}{2} + 1\right) dx$$

$$I = \left(\frac{x^2}{2} + x\right) \log x - \left(\frac{x^2}{4} + x\right) + C$$ Ans

-x-