!! जय श्री गिरिराज जी महाराज : जय श्री राधे कृष्णा !!

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER: VECTORS : $\boxed{\text{CLASS NO: 5}}$

Ques 1 Find λ and μ, if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=\vec{0}$

Soln

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0\hat{i}+0\hat{j}+0\hat{k}$$

$$\Rightarrow \hat{i}(6\mu-27\lambda)-\hat{j}(2\mu-27)+\hat{k}(2\lambda-6)=0\hat{i}+0\hat{j}+0\hat{k}$$

$\Rightarrow$ $6\mu-27\lambda=0$ | $-2\mu+27=0$, $\mu=27/2$ | $2\lambda-6=0$, $\lambda=3$

$$\therefore \boxed{\lambda=3,\ \mu=27/2} \text{ Ans}$$

Ques 2 Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

Soln

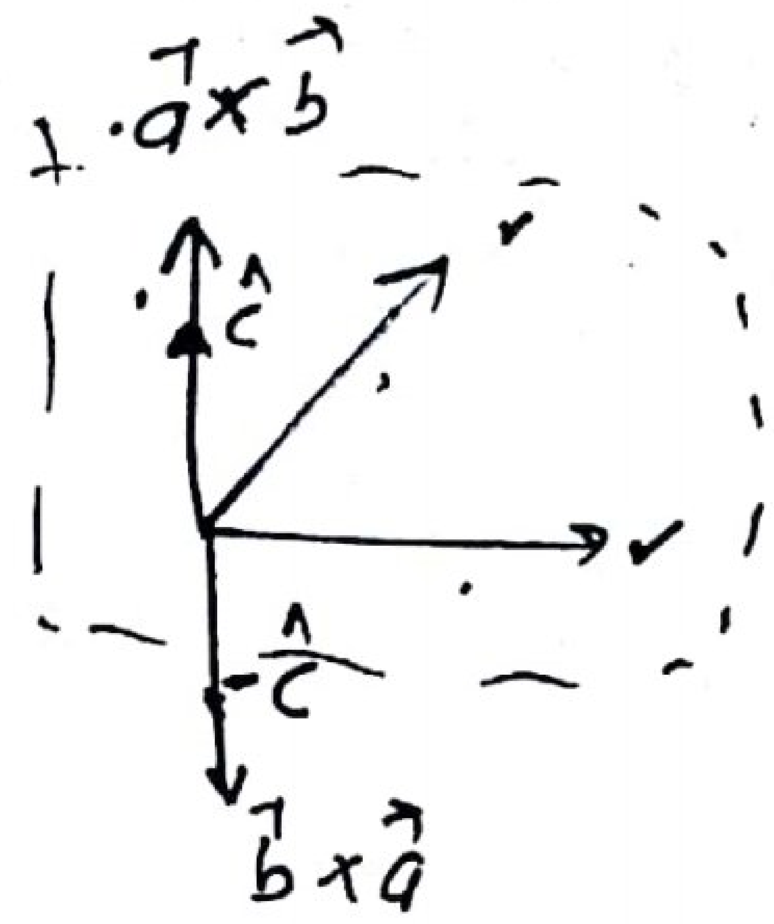
$$\vec{a}+\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$$
$$\vec{a}-\vec{b}=-\hat{j}-2\hat{k}$$

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i}+4\hat{j}-2\hat{k}$$

let $\vec{c}=-2\hat{i}+4\hat{j}-2\hat{k}$

$$|\vec{c}|=\sqrt{4+16+4}=2\sqrt{6}$$

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{-2\hat{i} + 4\hat{j} - 2\hat{k}}{2\sqrt{6}}$$



$$\boxed{\hat{c} = -\frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}}$$

Imp also $\hat{c} = \frac{1}{\sqrt{6}}\hat{i} - \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$

finally $\hat{c} = \mp\frac{1}{\sqrt{6}}\hat{i} \pm \frac{2}{\sqrt{6}}\hat{j} \mp \frac{1}{\sqrt{6}}\hat{k}$ Ans

Qn. 3 → Find the area of triangle having the points $A(1,1,1)$ $B(1,2,3)$, $C(2,3,1)$ as its vertices

Soln

$$\vec{AB} = \hat{j} + 2\hat{k}$$

$$\vec{AC} = \hat{i} + 2\hat{j}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4\hat{i} + 2\hat{j} - \hat{k}$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{16+4+1} = \sqrt{21}$$

Now Area of $\triangle ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{21}$ square units

Ans

Qn. 4 → If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then $\vec{a} \times \vec{b} = \vec{0}$

Is the converse true? Justify your answer with an example

Soln Converse: If $\vec{a} \times \vec{b} = \vec{0}$ then either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

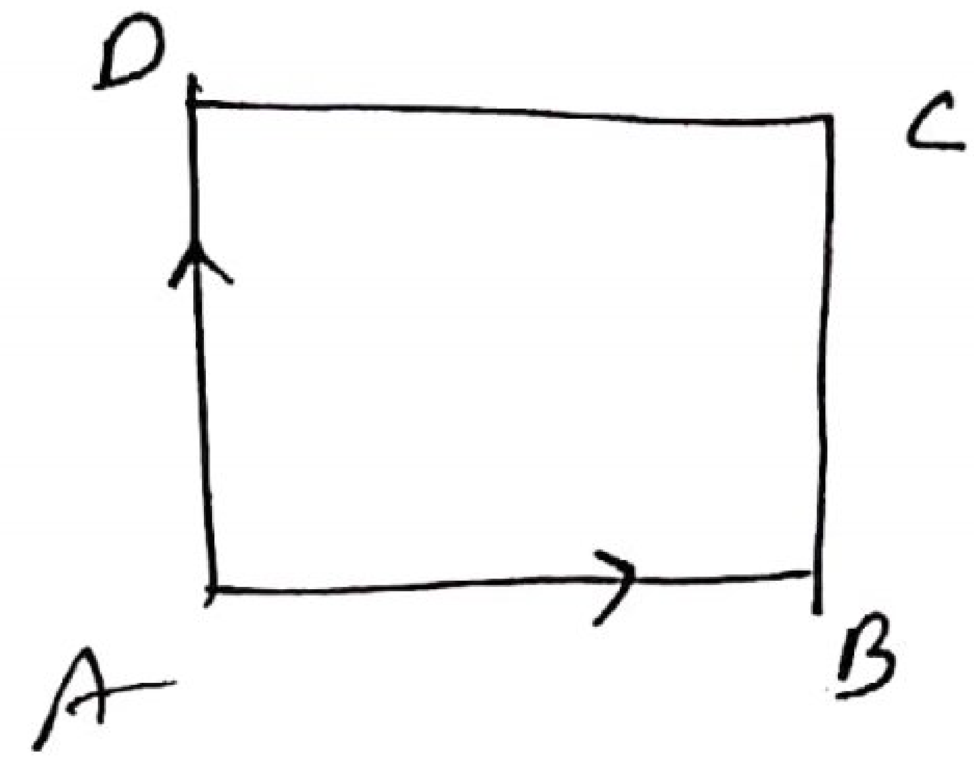
False : Reason $\vec{a}$ can be parallel to $\vec{b}$

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ & $\vec{b} = 2\hat{i} + 2\hat{j} + 2\hat{k}$ ⇒ $\vec{a} \times \vec{b} = \vec{0}$

Qn. 5 → Find the area of the rectangle having vertices A, B, C and D with position vectors $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$, $\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$, $\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ resp.

Soln $\overrightarrow{AB} = 2\hat{i}$

$\overrightarrow{AD} = -\hat{j}$



$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= 0\hat{i} - 0\hat{j} - 2\hat{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{4} = 2$$

Area of rectangle ABCD = $|\overrightarrow{AB} \times \overrightarrow{AD}|$ = 2 square units. Ans

Qn. 6 → Given that $\vec{a}\cdot\vec{b} = 0$ and $\vec{a}\times\vec{b} = \vec{0}$ what can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

Soln: Given $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow$ either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$

Given $\vec{a}\times\vec{b} = \vec{0}$

$\Rightarrow$ either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b}$

$\Rightarrow$ $\boxed{\vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0}}$ Ans

Qn. 7 → If a unit vector $\vec{a}$ makes $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an obtuse angle θ with $\hat{k}$, then find θ and hence components of $\vec{a}$.

Soln given $\alpha = \pi/3$; $\beta = \pi/4$; $\gamma = \theta$

we know that $l^2 + m^2 + n^2 = 1$

$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$

$\Rightarrow \cos^2\theta = 1 - \frac{1}{2} - \frac{1}{4}$

$\Rightarrow \cos^2\theta = \frac{1}{4}$

$\Rightarrow \cos\theta = \pm\frac{1}{2}$

$\Rightarrow \cos\theta = -\frac{1}{2}$ ... $\{\because \theta$ is obtuse angle$\}$

$\Rightarrow \theta = \pi - \pi/3 = 2\pi/3$

$\boxed{\theta = 2\pi/3}$

Concept

let $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\hat{a} = \frac{2}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}$

$a = 2$, $b = 3$, $c = 4$

$l = \frac{2}{\sqrt{29}}$, $m = \frac{3}{\sqrt{29}}$ $n = \frac{4}{\sqrt{29}}$

$l = \cos\alpha = \cos\pi/3 = 1/2$

$m = \cos\beta = \cos\pi/4 = 1/\sqrt{2}$

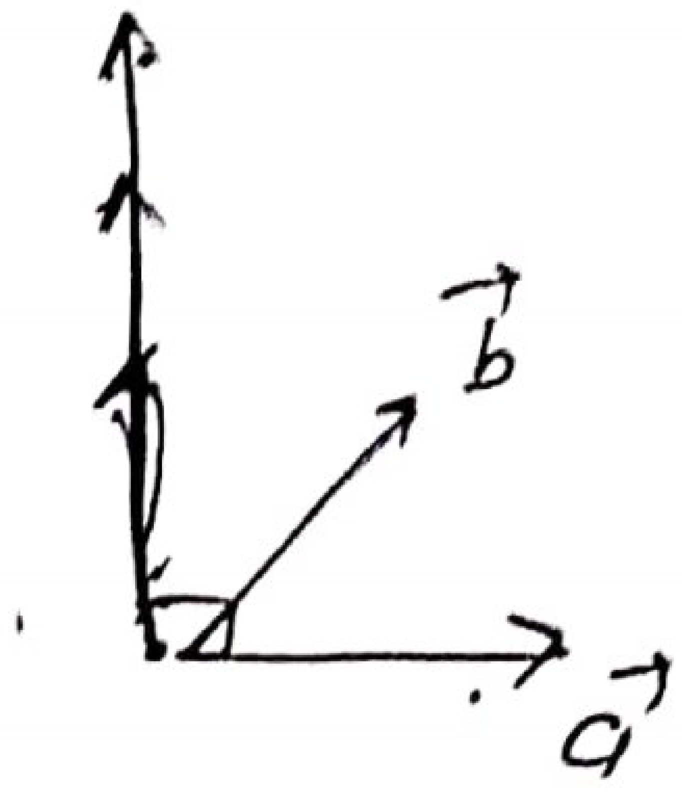
$n = \cos\gamma = \cos(2\pi/3) = -1/2$

$\boxed{\hat{a} = l\hat{i} + m\hat{j} + n\hat{k}}$ $\therefore$ components of $\hat{a}$ are $1/2$, $1/\sqrt{2}$ & $-1/2$ Ans

Qn 8 → Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = 3\hat{j} - \hat{k}$ and $\vec{c} = 7\hat{i} - \hat{k}$

Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ & $\vec{b}$ and $\vec{c} \cdot \vec{d} = 1$

Soln Let $\vec{d} \perp$ to $\vec{a}$ & $\vec{b}$



then $\boxed{\vec{d} = \lambda(\vec{a} \times \vec{b})}$

$$\Rightarrow \vec{d} = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 3 & -1 \end{vmatrix} = \lambda(\hat{i} + \hat{j} + 3\hat{k})$$

Given $\vec{c} \cdot \vec{d} = 1$

$$\Rightarrow (7\hat{i} + 0\hat{j} - \hat{k}) \cdot (\lambda\hat{i} + \lambda\hat{j} + 3\lambda\hat{k}) = 1$$

$$\Rightarrow 7\lambda - 3\lambda = 1$$

$$\Rightarrow \lambda = 1/4$$

$$\therefore \vec{d} = \frac{1}{4}\hat{i} + \frac{1}{4}\hat{j} + \frac{3}{4}\hat{k} \quad \text{Ans}$$

Qn. 9 → If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ then show that

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Soln we have $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\Rightarrow \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$$

$$\Rightarrow \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0} \quad \cdots \left\{ \begin{array}{l} \vec{a} \times \vec{a} = \vec{0} \\ \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \end{array} \right\}$$

$$\Rightarrow \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \quad \text{---(i)}$$

Q(ii) $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{b}\times(\vec{a}+\vec{b}+\vec{c})=\vec{b}\times\vec{0}$

$\Rightarrow \vec{b}\times\vec{a}+\vec{0}+\vec{b}\times\vec{c}=\vec{0}$

$\Rightarrow \vec{b}\times\vec{c}=-\vec{b}\times\vec{a}$

$\Rightarrow \vec{b}\times\vec{c}=\vec{a}\times\vec{b}$ - - - - (2)

from (1) & (2)

$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$ Proved

Q No. 10 → let $\vec{a},\vec{b},\vec{c}$ be unit vectors such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}=0$ and the angle b/w $\vec{b}$ & $\vec{c}$ is $\pi/6$. Prove that $\vec{a}=\pm 2(\vec{b}\times\vec{c})$

Soln given $|\vec{a}|=1$; $|\vec{b}|=1$; $|\vec{c}|=1$

given $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\cdot\vec{c}=0$

$\Rightarrow \vec{a}\perp\vec{b}$ & $\vec{a}\perp\vec{c}$ --- $\{\because \vec{a}\neq\vec{0}$, $\vec{b}\neq\vec{0}$, $\vec{c}\neq\vec{0}\}$

$\Rightarrow \vec{a}=\lambda(\vec{b}\times\vec{c})$

given angle b/w $\vec{b}$ & $\vec{c}=\pi/6$

$|\vec{a}|=|\lambda(\vec{b}\times\vec{c})|$

$\Rightarrow |\vec{a}|=|\lambda|\,|\vec{b}\times\vec{c}|$

$\Rightarrow |\vec{a}|=|\lambda|\,|\vec{b}|\,|\vec{c}|\sin(\pi/6)$

$\Rightarrow 1=|\lambda|\times\frac{1}{2}$ $\Rightarrow |\lambda|=2$ $\Rightarrow \lambda=\pm 2$

$\boxed{\vec{a}=\pm 2(\vec{b}\times\vec{c})}$ Ans

Qu 11 * If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$. Prove that $\vec{a}, \vec{b}, \vec{c}$ are mutually at right angles such that $|\vec{b}| = 1$ and $|\vec{c}| = |\vec{a}|$

Sol given $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$

$\Rightarrow$ $\vec{c} \perp \vec{a}$ & $\vec{c} \perp \vec{b}$ and $\vec{a} \perp \vec{b}$ & $\vec{a} \perp \vec{c}$

$\Rightarrow$ $\vec{a} \perp \vec{b}$, $\vec{b} \perp \vec{c}$, $\vec{c} \perp \vec{a}$

$\therefore$ $\vec{a}, \vec{b}, \vec{c}$ are mutually at right angles.

again $\vec{a} \times \vec{b} = \vec{c}$ & $\vec{b} \times \vec{c} = \vec{a}$

$\Rightarrow$ $|\vec{a} \times \vec{b}| = |\vec{c}|$ and $|\vec{b} \times \vec{c}| = |\vec{a}|$

$\Rightarrow$ $|\vec{a}||\vec{b}| \sin\left(\frac{\pi}{2}\right) = |\vec{c}|$ and $|\vec{b}||\vec{c}| \sin\left(\frac{\pi}{2}\right) = |\vec{a}|$

$\Rightarrow$ $|\vec{a}||\vec{b}| = |\vec{c}|$ and $|\vec{b}||\vec{c}| = |\vec{a}|$

$\Rightarrow$ $|\vec{b}||\vec{c}||\vec{b}| = |\vec{c}|$

$\Rightarrow$ $|\vec{b}|^2 \cancel{|\vec{c}|} = \cancel{|\vec{c}|}$

$\Rightarrow$ $|\vec{b}|^2 = 1$

$\Rightarrow$ $|\vec{b}| = 1$ $\Rightarrow$ $|\vec{c}| = |\vec{a}|$ proved

Qn. 12 → Show that $|\vec{a}\times\vec{b}|^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$

Soln LHS

$$|\vec{a}\times\vec{b}|^2$$

$$= \left(|\vec{a}||\vec{b}|\sin\theta\right)^2$$

$$= |\vec{a}|^2\,|\vec{b}|^2\sin^2\theta$$

$$= |\vec{a}|^2\,|\vec{b}|^2\left(1-\cos^2\theta\right)$$

$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$= (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b}) - (|\vec{a}||\vec{b}|\cos\theta)(|\vec{a}||\vec{b}|\cos\theta)$$

$$= (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{b})$$

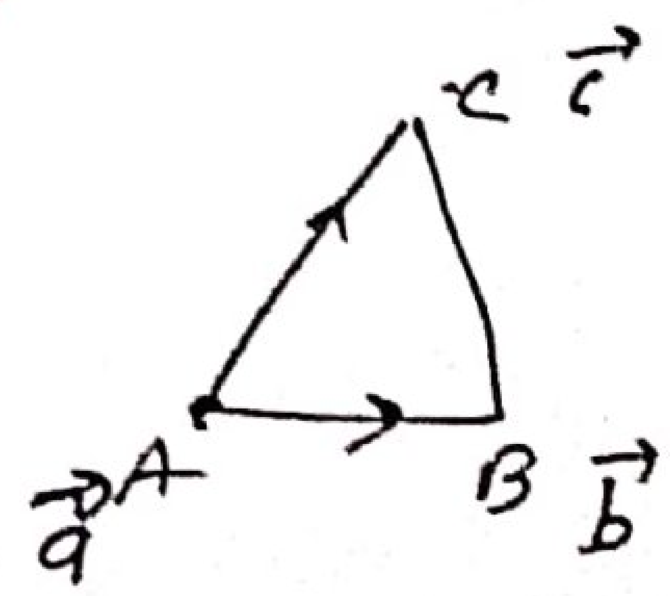
$$= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$$ Ans

Qn. 13 → If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C of a triangle ABC, Show that the area of triangle ABC is $\frac{1}{2}|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}|$. Deduce the condition for points $\vec{a}, \vec{b}, \vec{c}$ to be collinear

Sol

$$\vec{AB} = \vec{b} - \vec{a}$$

$$\vec{AC} = \vec{c} - \vec{a}$$



Area $\Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AC}|$

$$= \frac{1}{2} \left| (\vec{b}-\vec{a}) \times (\vec{c}-\vec{a}) \right|$$ (9)

$$= \frac{1}{2} \left| \vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{a}\times\vec{c} + \vec{a}\times\vec{a} \right|$$

$$= \frac{1}{2} \left| \vec{b}\times\vec{c} + \vec{a}\times\vec{b} + \vec{c}\times\vec{a} + \vec{0} \right| \quad \cdots \left\{ \begin{array}{l} \vec{a}\times\vec{a} = \vec{0} \\ \vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \end{array} \right\}$$

Area of ΔABC $= \frac{1}{2} \left| \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} \right|$

for points A, B, C to be collinear

Area of Δ ABC $= 0$

$$\Rightarrow \frac{1}{2} \left| \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} \right| = 0$$

$$\Rightarrow \left| \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} \right| = 0.$$

$$\Rightarrow \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$$

this is the required condition for which points A, B, C to be collinear **Ans**

WORKSHEET No: 4 VECTORS

Qn 1 The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to its diagonal. Also find its area

Ans $\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$; Area $= 11\sqrt{5}$ square units

Qn 2 Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$

Ans $\pm\frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k}$

Qn 3 Given $|\vec{a}|=3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$ and $\vec{a}\times\vec{b}$ is a unit vector. Then find the angle between $\vec{a}$ & $\vec{b}$

Ans $\pi/4$

Qn 4 Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as

$a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$.

Then show that $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

Qn 5 For any vector $\vec{a}$, prove that

$$|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = 2|\vec{a}|^2$$

Hint: let $\vec{a} = x\hat{i}+y\hat{j}+z\hat{k}$

Qn 6 Find a vector of magnitude 9, which is perpendicular to both the vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$

Ans $\mp 3\hat{i} \pm 6\hat{j} \pm 6\hat{k}$

Qn. 7 → If $A(0,1,1)$, $B(2,3,-2)$, $C(22,19,-5)$ & $D(1,-2,1)$ are the vertices of a quadrilateral ABCD. Find its Area. Ans $\sqrt{3160}$ square units

Qn 8 → Given $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 12$. Find $|\vec{a}\times\vec{b}|$ Ans 16

Qn 9 → If $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$. Show that $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$ where $\vec{a} \neq \vec{d}$ & $\vec{b} \neq \vec{c}$

Hint: Show $(\vec{a}-\vec{d})\times(\vec{b}-\vec{c}) = \vec{0}$

Qn. 10 → If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$; $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$; $\vec{a} \neq \vec{0}$ then show that $\vec{b} = \vec{c}$

Hint $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} \Rightarrow \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = 0$ and $\vec{a}\times\vec{b} - \vec{a}\times\vec{c} = \vec{0}$

Qn 11 → Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ & $\vec{b}$ and $\vec{c}\cdot\vec{d} = 15$

Ans $\frac{5}{3}(32\hat{i} - \hat{j} + 14\hat{k})$

Qn 12 → If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{j} - \hat{k}$ are given vectors then find a vector $\vec{b}$ satisfying equations $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{a}\cdot\vec{b} = 3$ Ans $\frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

—x—