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॥ जय श्री गिरिराज जी महाराज ॥ (1)

← ULTIMATE MATHEMATICS →

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Differentiation & continuity

CLASS NO: 6

Qn 1 → If $y = x\log\left(\frac{x}{a+bx}\right)$ show that

$$x^3 \cdot \frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$

Solution: given $y = x\log\left(\frac{x}{a+bx}\right)$

$$\Rightarrow y = x\left[\log x - \log(a+bx)\right] \quad \text{----(i)}$$

Diff wrt x

$$\frac{dy}{dx} = x\left(\frac{1}{x} - \frac{1}{a+bx}\cdot b\right) + \left(\log x - \log(a+bx)\right)\cdot 1$$

$$\frac{dy}{dx} = \cancel{x}\left[\frac{a+\cancel{bx}-\cancel{bx}}{\cancel{x}(a+bx)}\right] + \left(\log x - \log(a+bx)\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{a+bx} + \log x - \log(a+bx) \quad \text{----(2)}$$

Diff again wrt x

$$\frac{d^2y}{dx^2} = \frac{(a+bx)(0) - (a)(b)}{(a+bx)^2} + \frac{1}{x} - \frac{b}{a+bx}$$

$$\frac{d^2y}{dx^2} = \frac{-ab}{(a+bx)^2} + \frac{1}{x} - \frac{b}{a+bx}$$

$$\frac{d^2y}{dx^2} = \frac{-abx + a^2+b^2x^2+2abx - bx(a+bx)}{x(a+bx)^2}$$

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$$= \frac{-\cancel{abx} + a^2 + \cancel{b^2x^2} + \cancel{2abx} - \cancel{abx} - \cancel{b^2x^2}}{x(a+bx)^2}$$

$$\frac{d^2y}{dx^2} = \frac{a^2}{x(a+bx)^2}$$

Taking LHS

$$x^3 \cdot \frac{d^2y}{dx^2}$$

$$= x^3 \cdot \frac{a^2}{x(a+bx)^2}$$

$$= \frac{x^2a^2}{(a+bx)^2}$$

$$= \left(\frac{ax}{a+bx}\right)^2$$

Now taking RHS

$$\left(x\frac{dy}{dx} - y\right)^2$$

$$\Rightarrow \left[x\left(\frac{a}{a+bx} + \log x - \log(a+bx)\right) - \left(x\log x - x\log(a+bx)\right)\right]^2$$

$$= \left[\frac{ax}{a+bx} + \cancel{x\log x} - \cancel{x\log(a+bx)} - \cancel{x\log x} - \cancel{x\log(a+bx)}\right]^2$$

$$= \left(\frac{ax}{a+bx}\right)^2$$

LHS = RHS Proof

D & C Class No = 6 (3)

Qn 2 → $x = a\cos\theta + b\sin\theta$

$y = a\sin\theta - b\cos\theta$

Show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Sol:

Diff wrt θ

$\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta$ | $\frac{dy}{d\theta} = a\cos\theta + b\sin\theta$

$\frac{dx}{d\theta} = -y$ | $\frac{dy}{d\theta} = x$

$\frac{dy}{dx} = -\frac{x}{y}$ --- (1)

$\Rightarrow y\frac{dy}{dx} = -x$ —

Diff wrt x

$y\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{dy}{dx} = -1$

$\Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -1$

$\Rightarrow y\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{dy}{dx} = -1$

$\Rightarrow y\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(-\frac{x}{y}\right) = -1$ --- {from eq (1)}

$\Rightarrow y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ Ans

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Ques 3 → If $(x-a)^2 + (y-b)^2 = c^2$

Show that $\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ is a constant and independent of a and b

Sol: Given $(x-a)^2 + (y-b)^2 = c^2$ --- (1)

Diff w.r.t x

$$2(x-a).(1) + 2(y-b).\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow (x-a) + (y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x-a)}{(y-b)} \quad \text{--- (2)}$$

Diff again

$$\frac{d^2y}{dx^2} = -\left[\frac{(y-b)(1) - (x-a)\left(\frac{dy}{dx}\right)}{(y-b)^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\left[(y-b) + \frac{(x-a)^2}{(y-b)}\right]}{(y-b)^2}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-c^2}{(y-b)^3} \quad \text{--- (from eq (1))}$$

Consider

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

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$$= \frac{\left[1 + \frac{(x-a)^2}{(y-b)^2}\right]^{3/2}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{\left[\frac{(y-b)^2 + (x+a)^2}{(y-b)^2}\right]^{3/2}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{\left[\frac{c^2}{(y-b)^2}\right]^{3/2}}{\frac{-c^2}{(y-b)^3}} \quad \text{--- (from eq (1)}$$

$$= \frac{\frac{c^3}{\cancel{(y-b)^3}}}{\frac{-c^2}{\cancel{(y-b)^3}}}$$

$$= \frac{c^3}{-c^2}$$

$= -c$ which is a constant and independent of a & b

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TOPIC: 6 IMPLICIT FUNCTIONS

(general Diff.)

$y = f(x)$ (Explicit function)

$f(x, y) = c$ (Implicit function)

Q:1 → If $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

Show that $\frac{dy}{dx} = \frac{x+y}{x-y}$

Sol: Diff wrt x

$$\frac{1}{x^2+y^2}\cdot\left(2x + 2y\frac{dy}{dx}\right) = 2\cdot\frac{1}{1+\frac{y^2}{x^2}}\cdot\left(\frac{x\frac{dy}{dx} - y\cdot 1}{x^2}\right)$$

$$\Rightarrow \frac{\cancel{2}\left(x + y\frac{dy}{dx}\right)}{\cancel{x^2+y^2}} = \cancel{2}\cdot\frac{\cancel{x^2}}{\cancel{x^2+y^2}}\left(\frac{x\frac{dy}{dx} - y}{\cancel{x^2}}\right)$$

$$\Rightarrow x + y\frac{dy}{dx} = x\frac{dy}{dx} - y$$

$$\Rightarrow x + y = \frac{dy}{dx}(x-y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y} \quad \underline{\text{Ans}}$$

Q. 2 If $\sin y = x\sin(a+y)$ Show that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Sol: $\sin y = x\sin(a+y)$ ---- (1)

Diff $\cos y \cdot \frac{dy}{dx} = x\cos(a+y)\cdot\frac{dy}{dx} + \sin(a+y)\cdot 1$

$\frac{dy}{dx}$ ... $\frac{dy}{dx}$

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$$\frac{dy}{dx}\left(\cos y - x\cos(a+y)\right) = \sin(a+y)$$

$$\frac{dy}{dx} = \frac{\sin(a+y)}{\cos y - x\cos(a+y)}$$

$$= \frac{\sin(a+y)}{\cos y - \frac{\sin y}{\sin(a+y)}\cdot\cos(a+y)} \quad \text{--- (from eq(1))}$$

$$= \frac{\sin^2(a+y)}{\sin(a+y)\cos y - \sin y\cdot\cos(a+y)}$$

$$= \frac{\sin^2(a+y)}{\sin(a+\cancel{y}-\cancel{y})}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a} \quad \text{Ans}$$

Ques 3 → If $x\sqrt{1+y} + y\sqrt{1+x} = 0$
Show that $\frac{dy}{dx} = \frac{-1}{(x+1)^2}$

$\cancel{y = \frac{-x}{x+1}}$

Sol $x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow x^2 + x^2y = y^2 + y^2x$$

$$\Rightarrow x^2y - y^2x = y^2 - x^2 \quad (-1)$$

$$\Rightarrow xy(x-y) = (y+x)(y-x)$$

$\Rightarrow \quad xy(\cancel{x-y}) = (y+x)(\cancel{y-x})$

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$xy = -(y+x)$

$xy = -y - x$

$y + xy = -x$

$y(1+x) = -x$

$y = \dfrac{-x}{1+x}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{(1+x)(-1) - (-x)(1)}{(1+x)^2}$$

$$= \frac{-1 - \cancel{x} + \cancel{x}}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(1+x)^2} \quad \underline{\underline{\text{Ans}}}$$

Qn. 4 If $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$

Show that $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$

Sol^n put $x^3 = \sin A$ and $y^3 = \sin B$

$\Rightarrow \sqrt{1-\sin^2 A} + \sqrt{1-\sin^2 B} = a(\sin A - \sin B)$

$\Rightarrow \cos A + \cos B = a(\sin A - \sin B)$

$\Rightarrow \cancel{2}\cancel{\cos\left(\frac{A+B}{2}\right)}\cos\left(\frac{A-B}{2}\right) = a \cdot \cancel{2}\cancel{\cos\left(\frac{A+B}{2}\right)}\sin\left(\frac{A-B}{2}\right)$

$\Rightarrow \cos\left(\frac{A-B}{2}\right) = a \cdot \sin\left(\frac{A-B}{2}\right)$

$\Rightarrow \quad \cot\left(\frac{A-B}{2}\right) = a$

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$$\cot\left(\frac{A-B}{2}\right) = a$$

$$\Rightarrow \left(\frac{A-B}{2}\right) = \cot^{-1} a$$

$$\Rightarrow A-B = 2\cot^{-1} a$$

Replace A & B $\left\{ \begin{array}{l} x^3 = \sin A \\ y^3 = \sin B \end{array} \right\}$

$$\Rightarrow \sin^{-1}(x^3) - \sin^{-1}(y^3) = 2\cot^{-1} a$$

constant

Diff wrt x

$$\frac{1}{\sqrt{1-x^6}} \cdot 3x^2 - \frac{1}{\sqrt{1-y^6}} \cdot 3y^2 \frac{dy}{dx} = 0$$

$$\frac{x^2}{\sqrt{1-x^6}} - \frac{y^2 \frac{dy}{dx}}{\sqrt{1-y^6}} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2}\frac{\sqrt{1-y^6}}{\sqrt{1-x^6}}$$

Ans