॥ जय श्री राधे-कृष्णा जय श्री गिरिराज जी महाराज ॥ (1)

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER: INTEGRATION: $\boxed{\text{CLASS NO: 13}}$

DEFINITE INTEGRATION

(·) $\int_a^b f(x)\,dx = \left[g(x)\right]_a^b = g(b) - g(a) = $ Definite value

Basic $I = \int_1^2 (x^2+1)\,dx = \left(\frac{x^3}{3} + x\right)_1^2 = \left(\frac{8}{3}+2\right) - \left(\frac{1}{3}+1\right)$

$= \frac{7}{3} + 1 = 10/3$

Basic $I = \int_1^2 \frac{x}{x^2+1}\,dx$

put $x^2+1 = t$

$x\,dx = \frac{dt}{2}$

limits change

when $x = 1 \Rightarrow t = 2$

when $x = 2 \Rightarrow t = 5$

$I = \frac{1}{2}\int_2^8 \frac{dt}{t}$

$= \frac{1}{2}\left(\log|t|\right)_2^5$

$= \frac{1}{2}\left[\log 5 - \log 2\right] = \frac{1}{2}\log(5/2)$ Ans

Type I → without properties

Type II → with property

Ans:- 1 $I = \int_0^{\pi/2} \sin(2x)\tan^{-1}(\sin x)\,dx$

$I = 2\int_0^{\pi/2} \sin x \cos x \cdot \tan^{-1}(\sin x)\,dx$

put $\sin x = t$ | when $x=0$ then; $t=0$

$\cos x\,dx = dt$ | when $x = \pi/2$, $t = 1$

$\therefore\ I = 2\int_0^1 \underset{II}{t}\ \underset{I}{\tan^{-1} t}\ dt$

$= 2\left[\left(\tan^{-1}t \cdot \frac{t^2}{2}\right)_0^1 - \frac{1}{2}\int_0^1 \frac{1}{1+t^2}\cdot t^2\,dt\right]$

$= 2\left[\left(\frac{\pi}{4}\cdot\frac{1}{2} - 0\right) - \frac{1}{2}\int_0^1 \frac{1+t^2-1}{1+t^2}\,dt\right]$

$= 2\left[\frac{\pi}{8} - \frac{1}{2}\int_0^1 1 - \frac{1}{1+t^2}\,dt\right]$

$= 2\left[\frac{\pi}{8} - \frac{1}{2}\left\{t - \tan^{-1}t\right\}_0^1\right]$

$= 2\left[\frac{\pi}{8} - \frac{1}{2}\left\{\left(1-\frac{\pi}{4}\right) - (0)\right\}\right]$

$= \frac{\pi}{4} - 1 + \pi/4$

$I = \frac{\pi}{2} - 1$ Ans

Ans: 3 → $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin(2x)}\,dx$

$$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\left(1 - 1 + \sin(2x)\right)}\,dx$$

$$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\left[1 - (1 - \sin(2x))\right]}\,dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\left[1 - (\sin x - \cos x)^2\right]}\,dx$$

put $\sin x - \cos x = t$ | when $x = 0$; $t = -1$

$(\cos x + \sin x)\,dx = dt$ | when $x = \pi/4$; $t = 0$

$$I = \int_{-1}^{0} \frac{dt}{9 + 16\left[1 - t^2\right]}$$

$$= \int_{-1}^{0} \frac{dt}{25 - 16t^2}$$

$$= \frac{1}{16}\int_{-1}^{0} \frac{1}{\left(\frac{5}{4}\right)^2 - t^2}\,dt$$

$$= \frac{1}{16} \times \frac{1}{2 \times \frac{5}{4}}\left(\log\left|\frac{\frac{5}{4} + t}{\frac{5}{4} - t}\right|\right)_{-1}^{0}$$

$$= \frac{1}{40}\left(\log\left|\frac{5 + 4t}{5 - 4t}\right|\right)_{-1}^{0}$$

$$= \frac{1}{40}\left[\log|1| - \log\left|\frac{1}{9}\right|\right]$$

$$= \frac{1}{40}\left[-\log\frac{1}{9}\right]$$

$$= \frac{1}{40}\log(9) \quad \text{Ans}$$

$$= \frac{1}{40}\log(3)^2$$

$$= \frac{1}{20}\log 3 \quad \text{Ans}$$

Ans: 3 → $I = \int_{\pi/2}^{\pi} e^{x}\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$$I = \int_{\pi/2}^{\pi} e^{x}\left(\frac{1-2\sin(x/2)\cos(x/2)}{2\sin^2(x/2)}\right)dx$$

$$I = \int_{\pi/2}^{\pi} e^{x}\left(\frac{1}{2}\operatorname{cosec}^2(x/2) - \cot(x/2)\right)dx$$

$$I = -\int_{\pi/2}^{\pi} \underset{\text{II}}{e^{x}}\,\underset{\text{I}}{\cot(x/2)}\,dx + \int_{\pi/2}^{\pi} e^{x}\cdot\frac{1}{2}\operatorname{cosec}^2(x/2)\,dx$$

$$= -\left[\left(\cot(x/2)\cdot e^{x}\right)_{\pi/2}^{\pi} + \cancel{\frac{1}{2}\int_{\pi/2}^{\pi}\operatorname{cosec}^2(x/2)\cdot e^{x}dx}\right] + \cancel{\frac{1}{2}\int_{\pi/2}^{\pi} e^{x}\operatorname{cosec}^2(x/2)\,dx}$$

$$= -\left[\cot\left(\frac{\pi}{2}\right)\cdot e^{\pi} - \cot(\pi/4)\cdot e^{\pi/2}\right]$$

$$= -\left[0 - e^{\pi/2}\right]$$

$$\boxed{I = e^{\pi/2}}$$ Ans

Ans. 4 → $I = \int_{0}^{\pi/4} \frac{\sin x\cdot\cos x}{\sin^4 x + \cos^4 x}dx$

divide N & D by $\cos^4 x$

$$I = \int_{0}^{\pi/4} \frac{\tan x\cdot\sec^2 x\,dx}{\tan^4 x + 1}$$

put $\tan^2 x = t$

$\tan x\cdot\sec^2 x\,dx = \frac{dt}{2}$

| $x = 0$; $t = 0$
| $x = \pi/4$, $t = 1$

$$\therefore I = \frac{1}{2}\int_0^1 \frac{dt}{t^2+1}$$

$$= \frac{1}{2}\left(\tan^{-1} t\right)_0^1$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$

$$\boxed{I = \pi/8}$$ Ans

Qn: 5 → $$I = \int_0^{\pi/2} \frac{\cos^2 x \, dx}{\cos^2 x + 4\sin^2 x}$$

Divide N & D by $\cos^4 x$

$$I = \int_0^{\pi/2} \frac{\sec^2 x \, dx}{\sec^2 x + 4\tan^2 x \cdot \sec^2 x}$$

$$I = \int_0^{\pi/2} \frac{\sec^2 x \, dx}{\sec^2 x \left(1 + 4\tan^2 x\right)}$$

$$I = \int_0^{\pi/2} \frac{\sec^2 x \, dx}{\left(1+\tan^2 x\right)\left(1+4\tan^2 x\right)}$$

put $\tan x = t$ | $x = 0$; $t = 0$

$\sec^2 x \, dx = dt$ | $x = \pi/2$; $t = \infty$

$$I = \int_0^{\infty} \frac{dt}{\left(1+t^2\right)\left(1+4t^2\right)}$$

let $t^2 = y$ (temp

$$\therefore \frac{1}{\left(1+t^2\right)\left(1+4t^2\right)} = \frac{1}{(1+y)(1+4y)}$$

let $\frac{1}{(1+y)(1+4y)} = \frac{A}{1+y} + \frac{B}{1+4y}$

$1 = A(1+4y) + B(1+y)$

$0 = 4A + \cancel{B}$

$1 = A + \cancel{B}$

$-1 = 3A$ $\Rightarrow$ $A = -1/3$ $B = 4/3$

$\therefore I = \int_0^\infty \frac{-1}{3(1+t^2)} + \frac{4}{3(1+4t^2)} dt$

$I = -\frac{1}{3}\int_0^\infty \frac{1}{1+t^2} dt + \frac{4}{3}\int_0^\infty \frac{1}{1+4t^2} dt$

$= -\frac{1}{3}\left(\tan^{-1} t\right)_0^\infty + \frac{4}{3} \times \frac{1}{4}\int_0^\infty \frac{1}{(\frac{1}{2})^2 + t^2} dt$

$= -\frac{1}{3}\left[\frac{\pi}{2} - 0\right] + \frac{1}{3}\left[2\tan^{-1}(2t)\right]_0^\infty$

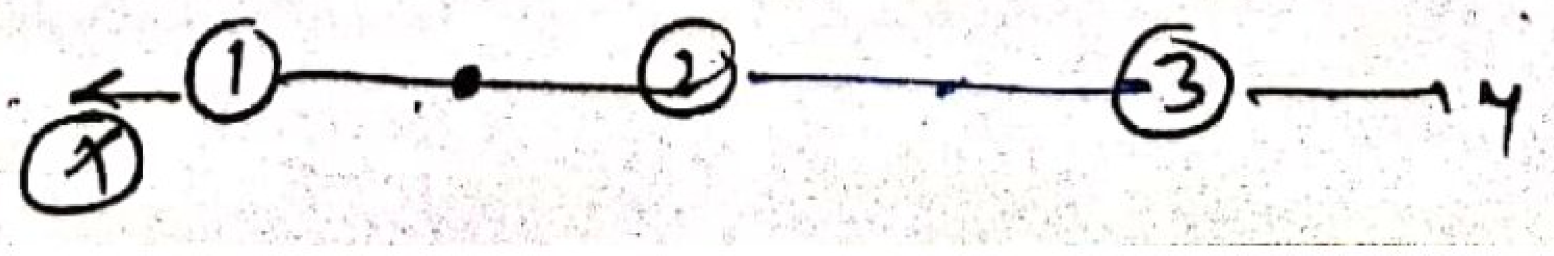
$= \frac{-\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2} - 0\right]$

$= -\pi/6 + \pi/3$

$\boxed{I = \pi/6}$ Ans

Q.No. 6 $I = \int_1^4 |x-2| + |x-1| + |x-3| \, dx$

$I = \int_1^4 |x-1| + |x-2| + |x-3| \, dx$



$$= I = \int_1^2 (x-1)-(x-2)-(x-3)\,dx + \int_2^3 (x-1)+(x-2)-(x-3)\,dx$$

$$+ \int_3^4 (x-1)+(x-2)+(x-3)\,dx$$

$$I = \int_1^2 (-x+4)\,dx + \int_2^3 x\,dx + \int_3^4 3x-6\,dx$$

$$= \left(-\frac{x^2}{2}+4x\right)_1^2 + \left(\frac{x^2}{2}\right)_2^3 + \left(\frac{3x^2}{2}-6x\right)_3^4$$

$$= \left[(-2+8)-\left(-\frac{1}{2}+4\right) + \left(\frac{9}{2}-2\right) + (24-24)-\left(\frac{27}{2}-18\right)\right.$$

$$= 6 - \frac{7}{2} + \frac{5}{2} + \frac{9}{2}$$

$$= \frac{12-7+5+9}{2}$$

$$\boxed{I = \frac{21}{2}} \quad \underline{\underline{\text{Ans}}}$$

$\underline{\underline{\text{Q No: 7}}} \rightarrow$

$$I = \int_{\pi/4}^{\pi/2} \sqrt{1-\sin(2x)}\,dx$$

$$I = \int_{\pi/4}^{\pi/2} \sqrt{\sin^2 x + \cos^2 x - 2\sin x\cos x}\,dx$$

$$= \int_{\pi/4}^{\pi/2} \sqrt{(\sin x - \cos x)^2}\,dx$$

$$= \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$= (-\cos x + \sin x)_{\pi/4}^{\pi/2}$$

$$= (-1) - \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = -1 + (\sqrt{2}) = \sqrt{2} - 1 \text{ Ans}$$

Scanned with CamScanner

Q-8 → $I = \int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$

$I = 5\int_1^2 \frac{x^2}{x^2+4x+3}\,dx$

$$\begin{array}{r|l} & 1 \\ x^2+4x+3 & x^2 \\ & -(x^2+4x+3) \\ \hline & -4x-3 \end{array}$$

$I = 5\int_1^2 1 - \frac{4x+3}{x^2+4x+3}\,dx$

$I = 5\int_1^2 1\,dx - 5\int_1^2 \frac{4x+3}{x^2+4x+3}\,dx$

$I = 5(x)_1^2 - 5\int_1^2 \frac{4x+3}{(x+1)(x+3)}\,dx$ ---- (4)

$I = 5 - 5I_1$ ---- (1)

$I_1 = \int_1^2 \frac{4x+3}{(x+1)(x+3)}\,dx$

let $\frac{4x+3}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

$4x+3 = A(x+3) + B(x+1)$

$4 = A + B$

$3 = 3A + B$

$1 = -2A$

$A = -1/2$ $\quad$ $B = \frac{9}{2}$

$\therefore I_1 = \int_1^2 -\frac{1}{2(x+1)} + \frac{9}{2(x+3)}\,dx$

$= \left(-\frac{1}{2}\log|x+1| + \frac{9}{2}\log|x+3|\right)_1^2$

$$I_1 = \left[-\frac{1}{2}\log 3 + \frac{9}{2}\log 5\right] - \left[-\frac{1}{2}\log 2 + \frac{9}{2}\log 4\right]$$

$$I_1 = -\frac{1}{2}\log 3 + \frac{9}{2}\log 5 + \frac{1}{2}\log 2 - \frac{9}{2}\log 4$$

$$I_1 = \frac{1}{2}\log\left(\frac{2}{3}\right) + \frac{9}{2}\log(5/4)$$

got on 461

$$\therefore I = 5 - \frac{5}{2}\log\left(\frac{5}{2}\right) - \frac{45}{2}\log(5/4) \quad \underline{\underline{Ans}}$$

Qu. 9 →

$$I = \int_0^{\pi/2} \sqrt{\tan x} + \sqrt{\cot x}\, dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\, dx$$

$$I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\, dx$$

$$I = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin(2x)}}\, dx$$

$$I = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1 - 1 + \sin(2x)}}\, dx$$

$$I = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\, dx$$

put $\sin x - \cos x = t$ | $x = 0$; $t = -1$

$(\cos x + \sin x)dx = dt$ | $x = \pi/2$; $t = 1$

$$I = \sqrt{2}\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\left(\sin^{-1} t\right)_{-1}^{1} = \sqrt{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \sqrt{2}\pi \quad \underline{\underline{Ans}}$$

Q No. 10 → $I = \int_0^{2\pi} e^x . \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx$
  II I

$$I = \left(\sin\left(\frac{\pi}{4}+\frac{x}{2}\right).e^x\right]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} \cos\left(\frac{\pi}{4}+\frac{x}{2}\right) . e^x \, dx$$
I II

$$I = -\frac{1}{\sqrt{2}}.e^{2\pi} - \frac{1}{\sqrt{2}} - \frac{1}{2}\left[\left(\cos\left(\frac{\pi}{4}+\frac{x}{2}\right).e^x\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} \sin\left(\frac{\pi}{4}+\frac{x}{2}\right)e^x dx\right]$$

$$I = -\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{2}\left[-\frac{1}{\sqrt{2}}e^{2\pi} - \frac{1}{\sqrt{2}}\right] - \frac{1}{4}I$$

$$I + \frac{I}{4} = \frac{-e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}}e^{2\pi} + \frac{1}{2\sqrt{2}}$$

$$\frac{5I}{4} = \frac{-2e^{2\pi} - 2 + e^{2\pi} + 1}{2\sqrt{2}}$$

$$\frac{5I}{4} = \frac{1}{2\sqrt{2}}\left[-e^{2\pi} - 1\right]$$

$$\boxed{I = -\frac{4}{10\sqrt{2}}\left(e^{2\pi}+1\right)}$$ Ans

Q. 11 → $I = \int_0^{\pi/2} \sqrt{\sin\phi}.\ \cos^5\phi\, d\phi$

$$I = \int_0^{\pi/2} \sqrt{\sin\phi}.\ \cos^4\phi.\ \cos\phi\, d\phi$$

$$= \int_0^{\pi/2} \sqrt{\sin\phi}.\ (1-\sin^2\phi)^2 . \cos\phi\, d\phi$$

put $\sin\phi = t$ | when $\phi = 0$; $t = 0$
$\cos\phi\, d\phi = dt$ | $\phi = \pi/2$; $t = 1$

$$I = \int_0^1 \sqrt{t}\,(1-t^2)^2\,dt$$

$$I = \int_0^1 \sqrt{t}\,(1+t^4-2t^2)\,dt$$

$$I = \int_0^1 \sqrt{t} + t^{9/2} - 2t^{5/2}\,dt$$

$$I = \left[\frac{2}{3}t^{3/2} + \frac{2}{11}t^{11/2} - 2\times\frac{2}{7}t^{7/2}\right]_0^1$$

$$I = \left(\frac{2}{3} + \frac{2}{11} - \frac{4}{7}\right) - (0)$$

$$I = \frac{154+42-132}{231} = \boxed{}$$ Ans

Q.N.-12 ⇒ If $f(x) = \int_0^x t \cdot \sin t\,dt$ find $f'(x)$

$\quad$ I $\quad$ II

Soln $\quad f(x) = (-t\cos t)_0^x + \int_0^x \cos t\,dt$

$$= (-x\cos x - 0) + (\sin t)_0^x$$

$$f(x) = -x\cos x + \sin x$$

Diff

$$f'(x) = -(-x\sin x + \cos x) + \cos x$$

$$= x\sin x - \cancel{\cos x} + \cancel{\cos x}$$

$$\boxed{f'(x) = x\sin x}$$ Ans

WORKSHEET NO: 10 (Class No: 13)

(DEFINITE INTEGRALS)

Qns: 1 $\int_0^{\pi/4} 2\tan^3 x \, dx$ ANS = $1-\log 2$

Qns: 2 → $\int_0^1 \sin^{-1}x \, dx$ Ans $\frac{\pi}{2}-1$

Qns: 3 → $\int_0^1 xe^x \, dx$ ANS = 1

Qn. 4 → $\int_1^3 \frac{dx}{x^2(x+1)}$ ANS = $\frac{2}{3}+\log\left(\frac{2}{3}\right)$

Qn. 5 → $\int_0^{\pi/2} \sin^3 x \, dx$ ANS = $2/3$

Qn. 6 $I = \int_{\pi/6}^{\pi/3} \frac{\sin x+\cos x}{\sqrt{\sin(2x)}} dx$ Ans $2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$

Qns. 7 → $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$ Ans $\frac{4\sqrt{2}}{3}$

Qn. 8 $\int_2^8 |x-5| \, dx$ Ans = 9

Qn. 9 → $\int_{-5}^{5} |x+2| \, dx$ Ans = 29

Qns 10 → $\int_{-\pi/4}^{\pi/4} \sin^2 x \, dx$ Ans = $\frac{\pi}{4}-\frac{1}{2}$

Qn. 11 → $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$ Ans $\frac{\pi}{8}$

Qn. 12 → $\int_{1/3}^{1} \frac{(x-x^3)^{1/3}}{x^4} dx$ Ans = 4

Qns 13 → $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

Ans $\frac{\pi}{2}-\log 2$

Qn: 14 → $\int_0^2 x\sqrt{x+2} \, dx$

Ans $\frac{16\sqrt{2}}{15}(\sqrt{2}+1)$

Qn 15 → $\int_0^2 \frac{6x+3}{x^2+4} dx$

Ans $3\log 2+\frac{3\pi}{8}$

HINT: Separate

Qn. 16 → $\int_0^{\pi/4} \sin^3(2t)\cdot\cos(2t) \, dt$

Ans = $\frac{1}{8}$

Qn 17 → $\int_1^2 \left(\frac{1}{x}-\frac{1}{2x^2}\right)\cdot e^{2x} dx$

Ans $\frac{e^2(e^2-2)}{4}$

Qn 18 → $\int_0^1 xe^x + \sin\left(\frac{\pi x}{4}\right) dx$

Ans $1+\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}$

—×—