!! जय श्री राधे कृष्णा जय श्री गिरिराज जी महाराज !!

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER:- VECTORS | CLASS No: 2

Operation on vectors

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

(1) addition of two ~~two~~ vectors

$$\vec{a}+\vec{b} = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k}$$

(2) Subtraction $\vec{a}-\vec{b} = (a_1-b_1)\hat{i} + (a_2-b_2)\hat{j} + (a_3-b_3)\hat{k}$

(3) equality of two vectors $\vec{a}=\vec{b}$

$$a_1 = b_1 \ ; \ a_2 = b_2 \ ; \ a_3 = b_3$$

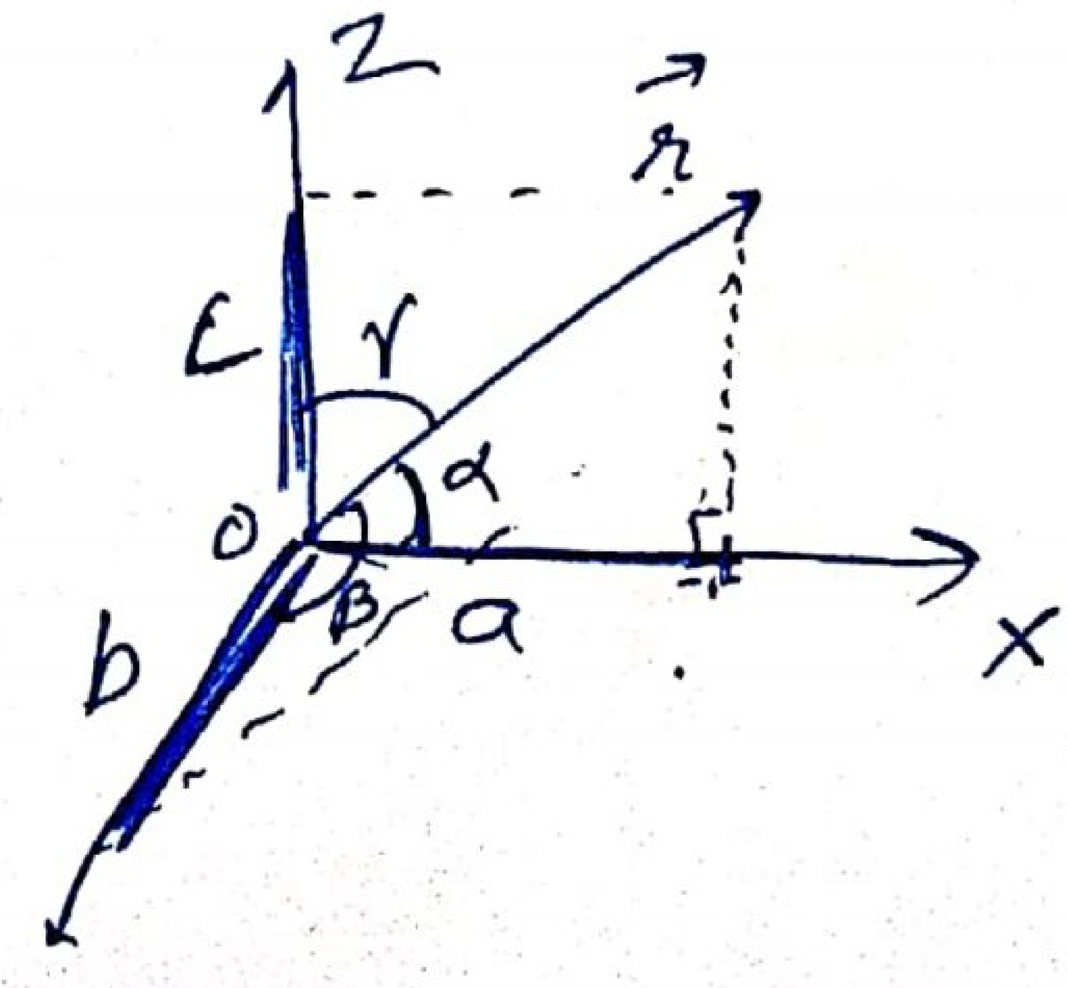
(4)
$$\lambda\vec{a} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = \lambda a_1\hat{i} + \lambda a_2\hat{j} + \lambda a_3\hat{k}$$

Direction Ratios of a vector (D.R's)

let $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$

a, b, c are the direction Ratios of $\vec{r}$

Direction Cosines of a vector (D.C's)



let $\vec{r}$ makes α, β, γ ~~are~~ ~~the~~ angles with X, Y, Z-axis respectively

$l, m, n \rightarrow$ denoted

$l = \cos\alpha$ | $l = \frac{a}{\sqrt{a^2+b^2+c^2}}$
$m = \cos\beta$ | $m = \frac{b}{\sqrt{a^2+b^2+c^2}}$
$n = \cos\gamma$ | $n = \frac{c}{\sqrt{a^2+b^2+c^2}}$

$\boxed{l^2+m^2+n^2=1}$ Property

$\Rightarrow \boxed{\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1}$

—x—

Que. 1 Let $\vec{a} = \hat{i}+2\hat{j}$ and $\vec{b} = 2\hat{i}+\hat{j}$
Is $|\vec{a}| = |\vec{b}|$? Are the vectors $\vec{a}$ & $\vec{b}$ are equal?

Sol. $|\vec{a}| = \sqrt{1+4} = \sqrt{5}$

$|\vec{b}| = \sqrt{4+1} = \sqrt{5}$

$|\vec{a}| = |\vec{b}|$

$\vec{a} \neq \vec{b}$ because their Corresponding components are not equal.

Point

If $\vec{a} = \vec{b}$ then $|\vec{a}| = |\vec{b}|$ | If $|\vec{a}| = |\vec{b}|$ then it is not necessary that $\vec{a} = \vec{b}$

Que. 2 → Find a unit vector in the direction of Sum of the vectors $\vec{a} = 2\hat{i}+2\hat{j}-5\hat{k}$ & $\vec{b} = 2\hat{i}+\hat{j}+3\hat{k}$ (Resultant)

Sol $\vec{a} + \vec{b} = 4\hat{i} + 3\hat{j} - 2\hat{k}$

let $\vec{c} = 4\hat{i} + 3\hat{j} - 2\hat{k}$

to find $\hat{c}$

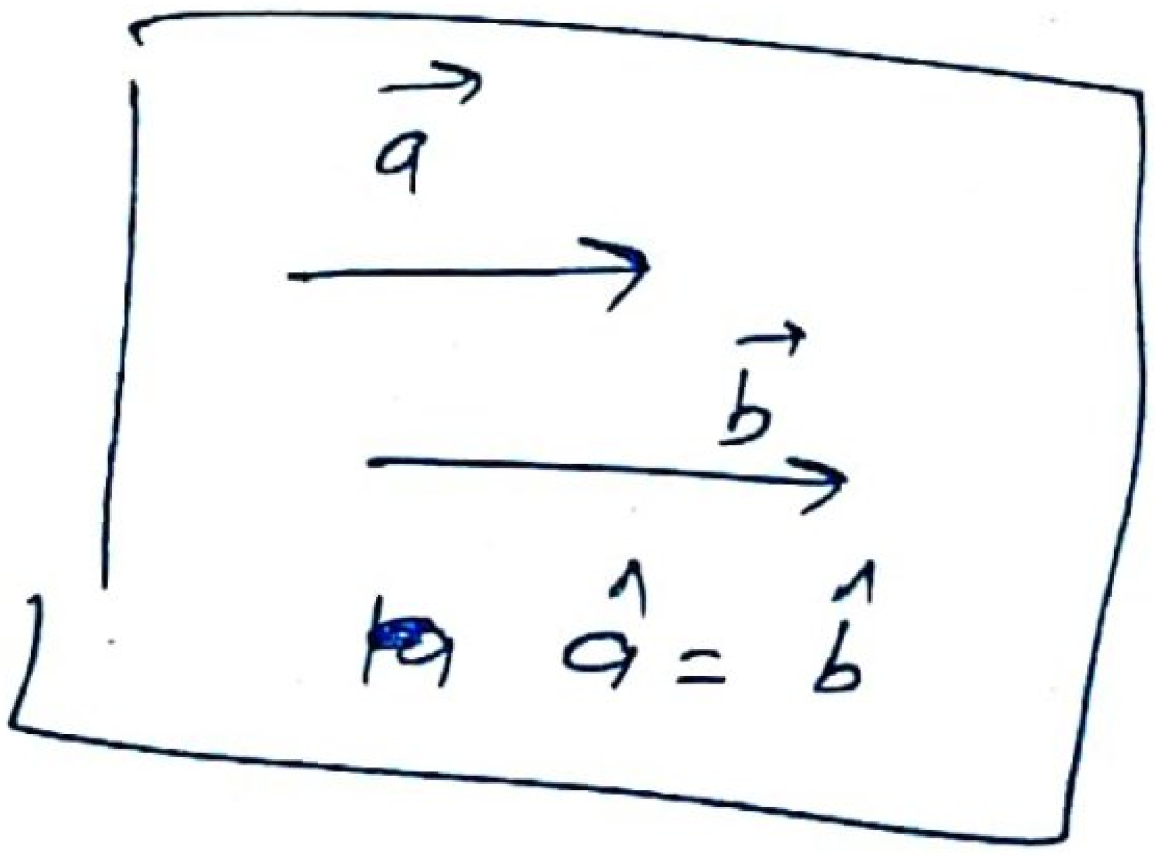
$$\hat{c} = \frac{\vec{c}}{|\vec{c}|}$$

$$\hat{c} = \frac{4\hat{i} + 3\hat{j} - 2\hat{k}}{\sqrt{16+9+4}}$$

$$\hat{c} = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$ Ans

Qn 3 → Find a vector in the direction of the vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units

Soln let $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$



let $\vec{b}$ is an required vector

given $|\vec{b}| = 8$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{25+1+4}} = \frac{5}{\sqrt{30}}\hat{i} - \frac{1}{\sqrt{30}}\hat{j} + \frac{2}{\sqrt{30}}\hat{k}$$

given $\hat{a} = \hat{b}$

Now vector = (Mag) (unit vector)

$$\vec{b} = |\vec{b}|\,\hat{b} \Rightarrow \vec{b} = \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}$$

Ans

Qn. 4 → Find the direction ratios and direction cosines of the vector joining the points $A(1,2,-3)$ and $B(-1,-2,1)$, directed from A to B

Soln $A(1,2,-3)$ & $B(-1,-2,1)$ $\qquad A \longrightarrow B$

$$\vec{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + (1+3)\hat{k}$$

$$\vec{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$$

Direction Ratios: $a = -2$, $b = -4$, $c = 4$

Direction Cosines $l = \dfrac{a}{\sqrt{a^2+b^2+c^2}} = \dfrac{-2}{\sqrt{4+16+16}} = \dfrac{-2}{6} = \dfrac{-1}{3}$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{-4}{6} = \frac{-2}{3}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{4}{6} = \frac{2}{3}$$

$\therefore$ $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$ are the D c's **Ans**

Qn 5 → Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively in the ratio 2:1

(i) Internally & (ii) Externally.

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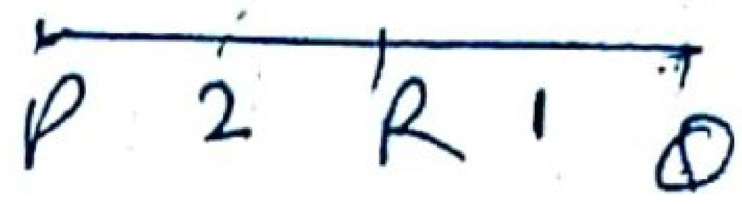
Soln given $\vec{OP} = \hat{i} + 2\hat{j} - \hat{k}$

$\vec{OQ} = -\hat{i} + \hat{j} + \hat{k}$



let $\vec{OR}$ be the p.v of point R

given ratio : 2:1

(i) Internally $\vec{OR} = \dfrac{2\vec{OQ} + \vec{OP}}{2+1}$

$$\vec{OR} = \frac{-2\hat{i} + 2\hat{j} + 2\hat{k} + \hat{i} + 2\hat{j} - \hat{k}}{3}$$

$$\vec{OR} = -\frac{\hat{i}}{3} + \frac{4\hat{j}}{3} + \frac{\hat{k}}{3} \quad \text{Ans}$$

(2) Externally $\vec{OR} = \dfrac{2\vec{OQ} - \vec{OP}}{2-1}$

$$\vec{OR} = \frac{-2\hat{i} + 2\hat{j} + 2\hat{k} - \hat{i} - 2\hat{j} + \hat{k}}{1}$$

$$\vec{OR} = -3\hat{i} + 3\hat{k} \quad \text{Ans}$$

Ques 6 → Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$ respectively, form the vertices of a right angled triangle

Soln given $\vec{OA} = \vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

$\vec{OB} = \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{OC} = \vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$

Side vectors

$\vec{AB} = \vec{OB} - \vec{OA} = -\hat{i} + 3\hat{j} + 5\hat{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\vec{CA} = \vec{OA} - \vec{OC} = 2\hat{i} - \hat{j} + \hat{k}$

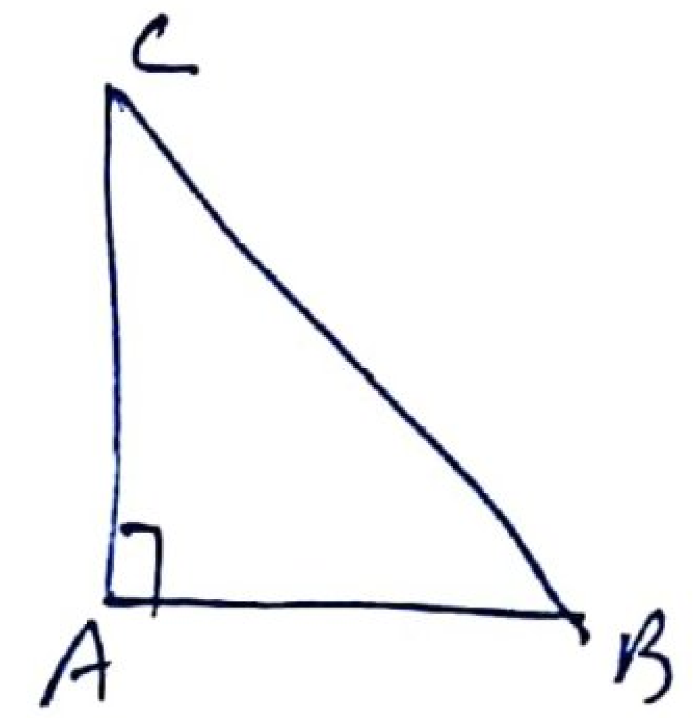
$|\vec{AB}| = \sqrt{1+9+25} = \sqrt{35} \Rightarrow |\vec{AB}|^2 = 35$

$|\vec{BC}| = \sqrt{1+4+36} = \sqrt{41} \Rightarrow |\vec{BC}|^2 = 41$

$|\vec{CA}| = \sqrt{4+1+1} = \sqrt{6} \Rightarrow |\vec{CA}|^2 = 6$

Clearly $|\vec{BC}|^2 = |\vec{AB}|^2 + |\vec{CA}|^2$

$\therefore$ A, B, C are the vertices of a right angled triangle Ans



Qn. 7 → Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear

Sol: let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ & $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$

$\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$

$\vec{b} = -2(2\hat{i} - 3\hat{j} + 4\hat{k})$

$\vec{b} = -2\vec{a}$

$\therefore \vec{a}$ & $\vec{b}$ are collinear

OR

$\frac{2}{-4} = \frac{-3}{6} = \frac{4}{-8}$

$-\frac{1}{2} = -\frac{1}{2} = -\frac{1}{2}$

Clearly corresponding components are in equal ratio

$\therefore \vec{a}$ & $\vec{b}$ are collinear

Qn 8 → Using vectors, show that the points $A(1,-2,-8)$, $B(5,0,-2)$ & $C(11,3,7)$ are collinear

Sol

$\vec{AB} = 4\hat{i} + 2\hat{j} + 6\hat{k}$

$\vec{BC} =$

$\vec{CA} =$

$|\vec{AB}| =$

$|\vec{BC}| =$

$|\vec{CA}| =$

Check $\square = \square + \square$ $\therefore$ A, B, C points are collinear

OR

$A(1,-2,-8)$ $B(5,0,-2)$ $C(11,3,7)$

$\vec{AB} = 4\hat{i} + 2\hat{j} + 6\hat{k}$

$\vec{BC} = 6\hat{i} + 3\hat{j} + 9\hat{k}$

$\frac{4}{6} = \frac{2}{3} = \frac{6}{9}$

$\frac{2}{3} = \frac{2}{3} = \frac{2}{3}$ Clearly correspondg. components are in equal ratio

$\therefore \vec{AB} \parallel \vec{BC}$

$A \longrightarrow B \longrightarrow C$

But point B is common

$B \longrightarrow C$

$\therefore$ A, B, C must be collinear

Ans

Qn 9 → Find the value of x for which $x(\hat{i}+\hat{j}+\hat{k})$ is a unit vector

Sol let $\vec{a} = x(\hat{i}+\hat{j}+\hat{k})$

$\vec{a} = x\hat{i} + x\hat{j} + x\hat{k}$

Given $\vec{a}$ is a unit vector

$\Rightarrow |\vec{a}| = 1$

$\Rightarrow \sqrt{x^2+x^2+x^2} = 1$

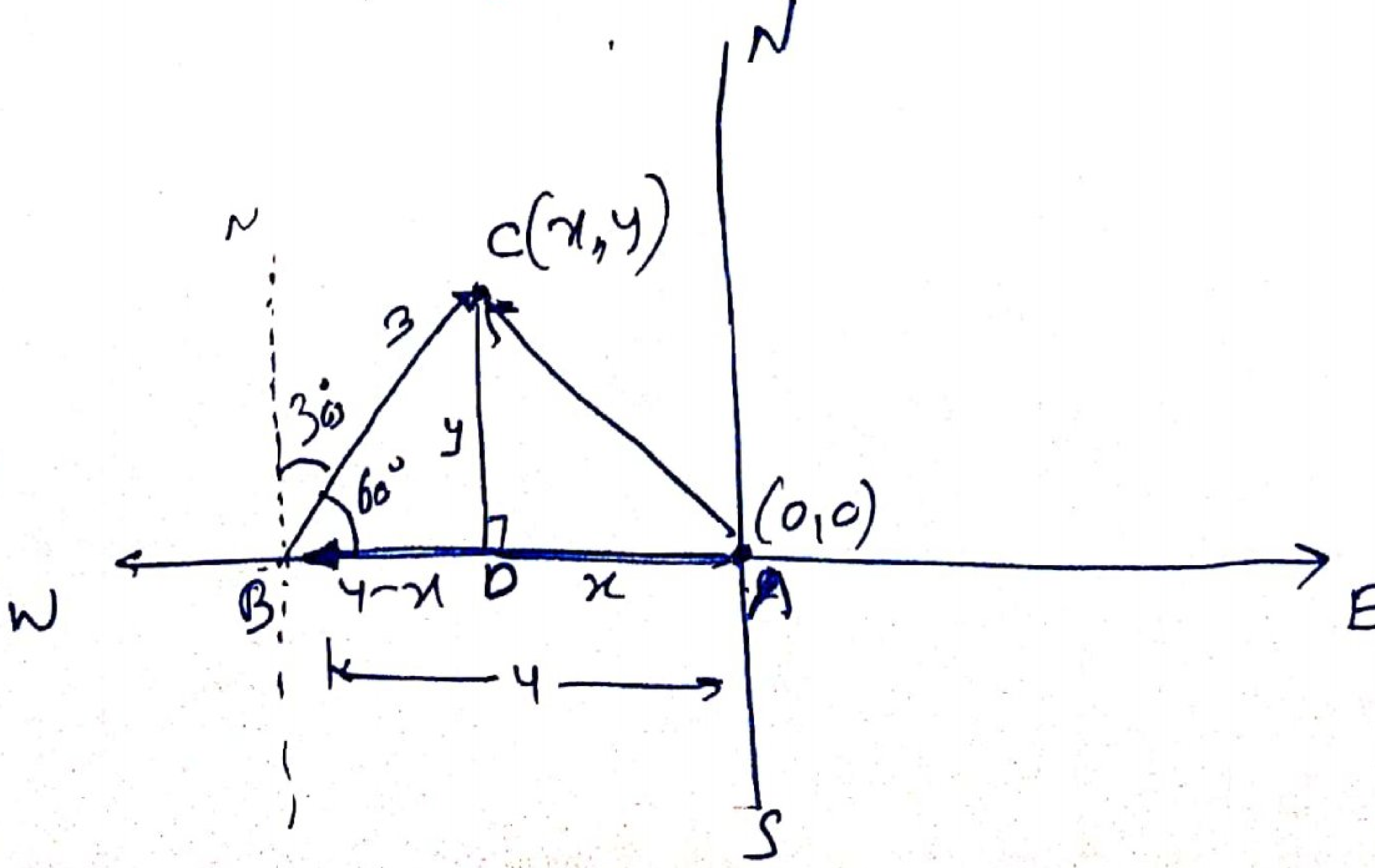
$\Rightarrow \sqrt{3x^2} = 1$

Squaring $3x^2 = 1$

$x^2 = 1/3 \Rightarrow x = \pm 1/\sqrt{3}$ Ans

Qn. 10 → A girl walks 4 km towards west, then she walks 3 km in direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Soln



ΔBCD

$\sin(60^\circ) = \frac{y}{3}$ | $\cos(60^\circ) = \frac{4-x}{3}$

$\frac{\sqrt{3}}{2} = \frac{y}{3}$ | $\frac{1}{2} = \frac{4-x}{3}$

$\Rightarrow y = \frac{3\sqrt{3}}{2}$ | $3 = 8 - 2x$

$2x = 5$

$x = 5/2$

$\therefore C\left(-\frac{5}{2}, \frac{3\sqrt{3}}{2}\right)$

Required ~~dista~~ displacement

$\vec{AC} = \left(-\frac{5}{2} - 0\right)\hat{i} + \left(\frac{3\sqrt{3}}{2} - 0\right)\hat{j}$

$\vec{AC} = -\frac{5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

$|\vec{AC}| = \sqrt{\frac{25}{4} + \frac{27}{4}} = \sqrt{\frac{52}{4}} = \sqrt{13}$ units

Qn. 11 → Find the direction cosines of a vector which is equally inclined to the Axes.

Soln: let α, β, γ are the Angles made by the vector with X, Y, Z axis resp.

we have $\alpha = \beta = \gamma$

$\Rightarrow \cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow l = m = n$

we have $l^2 + m^2 + n^2 = 1$

$\Rightarrow 3l^2 = 1$

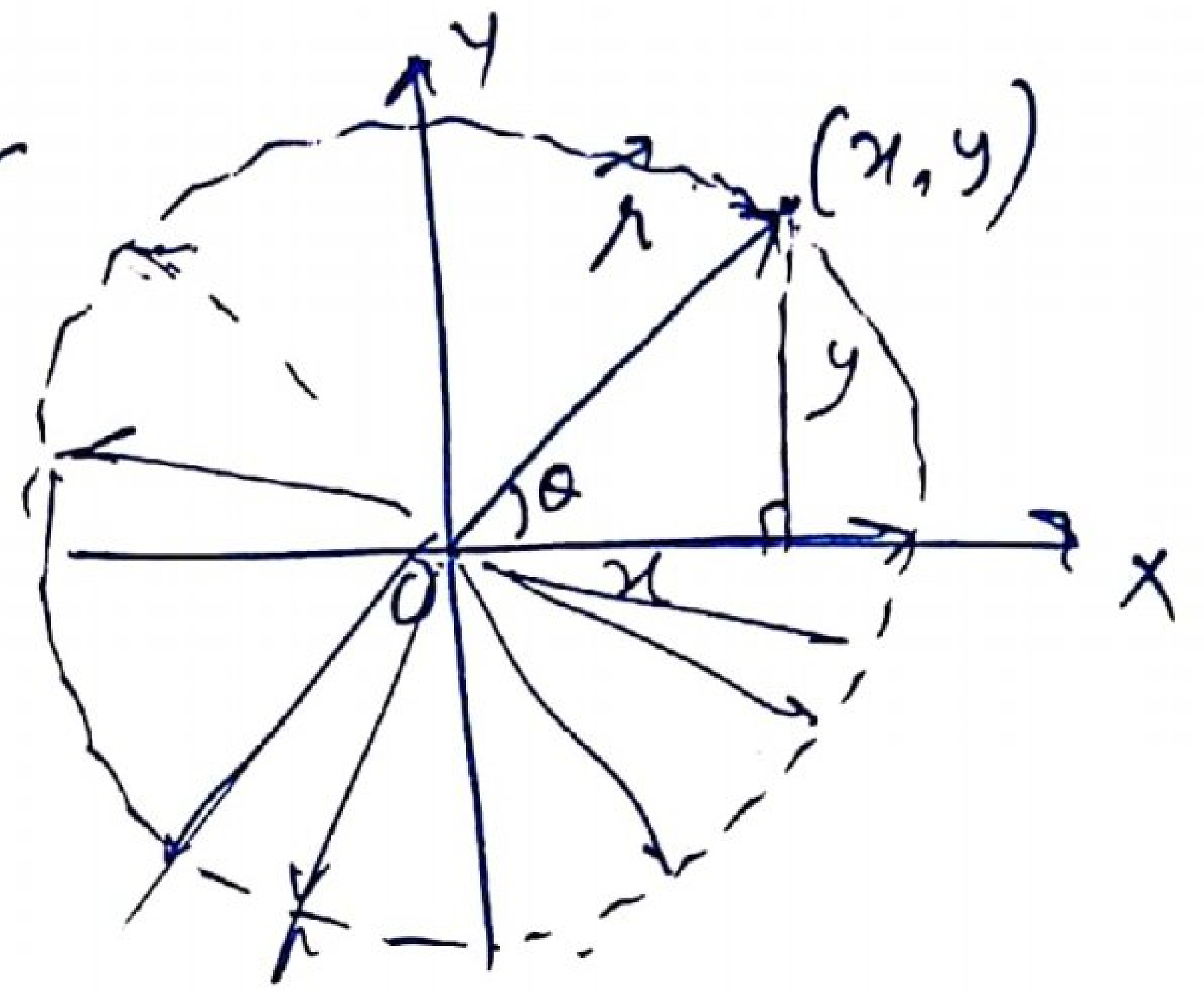
$\rightarrow \quad l = \pm \frac{1}{\sqrt{3}}$

$m = \pm \frac{1}{\sqrt{3}}$ Ans

$n = \pm \frac{1}{\sqrt{3}}$

Qn. 12 → Write all unit vectors in XY-plane

Soln let $\vec{r}$ is ~~in~~ a unit vector in X-Y plane



$\vec{r} = x\hat{i} + y\hat{j}$

$\cos\theta = \frac{x}{|\vec{r}|} = \frac{x}{1} = x$

$\sin\theta = \frac{y}{|\vec{r}|} = \frac{y}{1} = y$

$\therefore \boxed{\vec{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}}$ where θ varies from 0 to 2π

$|\vec{r}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

WORKSHEET NO: 1 (VECTORS)

Qn. 1 If the position vector $\vec{a}$ of a point $(12, n)$ is such that $|\vec{a}| = 13$, find the value of n

Ans: ± 5

Qn. 2 → If $\vec{a}$ & $\vec{b}$ are the position vectors of the points $(1, -1)$ & $(-2, m)$. Find the value of 'm' for which $\vec{a}$ & $\vec{b}$ are collinear

Ans: $m = 2$

Qn. 3 → Find unit vector in the direction of the vector joining the points $P(1,2,3)$ & $Q(4,5,6)$ directed from Q to P

Ans. $\hat{QP} = \frac{-1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

Qn. 4 → Using vectors, show that the points $A(-2,3,5)$, $B(1,2,3)$, $C(7,0,-1)$ are collinear

Qn. 5 → Find a vector of magnitude 5 units parallel to the resultant of the vectors

$$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k} \quad \& \quad \vec{b} = \hat{i} - 2\hat{j} + \hat{k}$$

Ans: $\sqrt{\frac{5}{2}}(3\hat{i} + \hat{j})$

Qn. 6 → If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$. Find a vector of magnitude 6 units which is parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$

Ans: $2\hat{i} - 4\hat{j} + 4\hat{k}$

Qn 7 Show that the points $A(2\hat{i}-\hat{j}+\hat{k})$, $B(\hat{i}-3\hat{j}-5\hat{k})$ and $C(3\hat{i}-4\hat{j}-4\hat{k})$ are the vertices of a right angled triangle

HINT given $\vec{OA}$, $\vec{OB}$ & $\vec{OC}$
find separate side vectors

Qn 8 → If a vector makes α, β, γ angles with OX, OY & OZ then show that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

Qn 9 → If a vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$. Find find θ Ans $\theta = \pi/3$

Hint $\alpha = \pi/3$, $\beta = \pi/4$

Qn 10 → If $\vec{a} = 3\hat{i}-\hat{j}-4\hat{k}$, $\vec{b} = -2\hat{i}+4\hat{j}-3\hat{k}$ and $\vec{c} = \hat{i}+2\hat{j}-\hat{k}$. Find $|3\vec{a}-2\vec{b}+4\vec{c}|$

Ans. $\sqrt{398}$

Qn 11 → If a vector makes $\frac{\pi}{3}$ and $\frac{\pi}{2}$ with $\hat{j}$ & $\hat{k}$ ~~respectively~~ respectively. Find an acute angle θ with $\hat{i}$

Ans $\theta = \pi/3$

Qn 12 → Find direction cosines of $\vec{a} = 6\hat{i}-2\hat{j}+3\hat{k}$

Ans $\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}$

Qn 13 → If vectors $x\hat{i}+2\hat{j}-z\hat{k}$ & $3\hat{i}-y\hat{j}+\hat{k}$ are equal. Find $x+y+z$ Ans 0