!! जय श्री राधे कृष्णा !! जय श्री गिरिराज जी महाराज !! (1)

← ULTIMATE: MATHEMATICS: BY AJAY MITTAL →

← INTEGRATION: CLASS NO: 8 →

Type: 1 Form $\displaystyle\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$

✓ let $N^r = A \cdot \frac{d}{dx}(D^r) + B(D^r)$

✓ equate the coefficients of $\sin x$ & $\cos x$ both sides to find the values of A & B

Qn. 1 $\displaystyle I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$

let $3\sin x + 2\cos x = A(-3\sin x + 2\cos x) + B(3\cos x + 2\sin x)$

equating coefficients of $\sin x$ & $\cos x$

$3 = -3A + 2B \quad \times 2$

$2 = 2A + 3B \quad \times 3$

$\Rightarrow 6 = -6A + 4B$

$6 = 6A + 9B$

$12 = 13B$

$\Rightarrow B = \frac{12}{13}$

$A = -\frac{5}{13}$

$$\therefore I = \int \frac{-\frac{5}{13}(-3\sin x + 2\cos x) + \frac{12}{13}(3\cos x + 2\sin x)}{3\cos x + 2\sin x}\,dx$$

$$= -\frac{5}{13}\int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx + \frac{12}{13}\int dx$$

put $3\cos x + 2\sin x = t$

$(-3\sin x + 2\cos x)\,dx = dt$

$\therefore I = -\frac{5}{13}\int \frac{dt}{t} + \frac{12}{13}x$

$I = -\frac{5}{13}\log|3\cos x + 2\sin x| + \frac{12}{13}x + C$ Ans

(2) $I = \int \frac{1}{1+\tan x}\,dx$

$I = \int \frac{\cos x}{\cos x + \sin x}\,dx$

let $\cos x = A(-\sin x + \cos x) + B(\cos x + \sin x)$

proceed

Type-2

$$\int \frac{1}{\sin^2 x,\ \cos^2 x,\ \text{constant}\ ;\ \sin x \cdot \cos x,\ \sin(2x),\ \cos(2x)}\,dx$$

(Divide by $\cos^2 x$)

- N^r we get $\sec^2 x\,dx$
- D^r put $\tan x = t$
- D^r replace $\sec^2 x$ by $1+\tan^2 x$ (if any)

(3) $I = \int \frac{1}{2\sin^2 x + 3\cos^2 x - 1}\,dx$

divide by $\cos^2 x$

$I = \int \frac{\sec^2 x\,dx}{2\tan^2 x + 3 - (1+\tan^2 x)}\,dx$

$$I = \int \frac{\sec^2 x \, dx}{\tan^2 x + 2}$$

put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$I = \int \frac{dt}{t^2 + 2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$$

$$I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + c \quad \underline{\text{Ans}}$$

Qn. 4 $\quad I = \int \frac{1}{(3\sin x - 2\cos x)(\cos x + 3\sin x)} dx$

$$I = \int \frac{1}{3\sin x\cos x + 9\sin^2 x - 2\cos^2 x - 6\sin x\cos x}$$

divide by $\cos^2 x$

$$I = \int \frac{\sec^2 x \, dx}{9\tan^2 x - 2 - 3\tan x}$$

put $\tan x = t$

$\sec^2 x \, dx = dt$

$$I = \int \frac{dt}{9t^2 - 3t - 2} \quad \left(\int \frac{1}{\text{Quadratic}}\right)$$

Proceed

(5) $I = \int \frac{\sin x}{\sin(3x)} dx$

$$= \int \frac{\sin x}{3\sin x - 4\sin^3 x} dx$$

$$= \int \frac{1}{3 - 4\sin^2 x} dx$$

divide by $\cos^2 x$

$$I = \int \frac{\sec^2 x \, dx}{3\sec^2 x - 4\tan^2 x}$$

$$= \int \frac{\sec^2 x \, dx}{3(1+\tan^2 x) - 4\tan^2 x}$$

$$= \int \frac{\sec^2 x \, dx}{3 - \tan^2 x}$$

proceed

→ Type

Q.No. 6 $I = \int \frac{\sin x - \cos x}{\sqrt{\sin(2x)}} dx$

$$I = \int \frac{\sin x - \cos x}{\sqrt{1 - 1 + \sin(2x)}} dx$$

$$= \int \frac{\sin x - \cos x}{\sqrt{\sin^2 x + \cos^2 x - 1 + 2\sin x \cos x}} dx$$

$$= \int \frac{\sin x - \cos x \, dx}{\sqrt{(\sin x + \cos x)^2 - 1}}$$

put $\sin x + \cos x = t$

$(\cos x - \sin x)\,dx = dt$

$(\sin x - \cos x)\,dx = -dt$

$$I = -\int \frac{dt}{\sqrt{t^2-1}}$$

$$= -\log\left|t + \sqrt{t^2-1}\right| + C$$

$$= -\log\left|(\sin x + \cos x) + \sqrt{(\sin x + \cos x)^2 - 1}\right| + C$$

$$= -\log\left|(\sin x + \cos x) + \sqrt{\sin(2x)}\right| + C$$

Qn. 7 $I = \int \frac{\sin x + \cos x}{9 + 16\sin(2x)}\,dx$

$$I = \int \frac{\sin x + \cos x}{9 + 16\left(1 - 1 + \sin(2x)\right)}\,dx$$

$$= \int \frac{\sin x + \cos x}{9 + 16\left[1 - (1 - \sin(2x))\right]}\,dx$$

$$= \int \frac{\sin x + \cos x}{9 + 16\left[1 - (\sin x - \cos x)^2\right]}\,dx$$

put $\sin x - \cos x = t$

$(\cos x + \sin x)\,dx = dt$

$$\therefore I = \int \frac{dt}{9+16(1-t^2)}$$

$$= \int \frac{dt}{25-16t^2}$$

$$= \frac{1}{16}\int \frac{1}{\left(\frac{5}{4}\right)^2 - t^2}\,dt$$

$$= \frac{1}{16} \times \frac{1}{2\times\frac{5}{4}} \log\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right| + c$$

$$= \frac{1}{40} \log\left|\frac{5+4t}{5-4t}\right| + c$$

$$= \frac{1}{40} \log\left|\frac{5+4(\sin x - \cos x)}{5-4(\sin x - \cos x)}\right| + c$$

Q.8 $I = \int \sqrt{\tan x} + \sqrt{\cot x}\, dx$

$$I = \int \frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\, dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cdot \cos x}}\, dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\frac{\sin(2x)}{2}}}\, dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{\sin(2x)}}\, dx$$ Proceed

(x) $\int \sqrt{\text{Quadratic}}\, dx$

(x) $\int \text{linear} \sqrt{\text{Quadratic}}\, dx$

Topic : $\boxed{\text{BY PARTS}}$

$$I = \int \underset{\text{I}}{f(x)} \cdot \underset{\text{II}}{g(x)}\, dx$$

$$= I^{st} \cdot \int II^{nd}\, dx - \int \left(\frac{d}{dx}(I^{st}) \cdot \int II^{nd}\, dx \right) dx$$

$\boxed{\text{ILATE}}$

I → Inverse Trig

L → logarithm

A → algebraic

T → Trig

E → exponential

Basic $\quad I = \int \underset{\text{I}}{x} \underset{\text{II}}{\sin x}\, dx$

$$= x(-\cos x) - \int 1 \cdot (-\cos x)\, dx$$

$$= -x\cos x + \int \cos x\, dx$$

$$I = -x\cos x + \sin x + C \quad \text{Ans}$$

Basic $\quad I = \int \underset{\text{II}}{x^3} \cdot \underset{\text{I}}{\log x}\, dx$

$$I = \log x \cdot \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4}\, dx$$

$$= \frac{x^4}{4} \cdot \log x - \frac{1}{4} \cdot \frac{x^4}{4} + C \quad \text{Ans}$$

Ans 9 $I = \int \log x \cdot dx$

$$I = \int \underset{I}{\log x} \cdot \underset{II}{1}\, dx$$

$$= \log x \cdot x - \int \frac{1}{\cancel{x}} \cdot \cancel{x}\, dx$$

$$I = x \log x - x + c$$

Ans 10 → $I = \int \sin^{-1} x \, dx$

$$= \int \underset{I}{\sin^{-1} x} \cdot \underset{II}{1}\, dx$$

$$= \sin^{-1} x \cdot x - \int \frac{1}{\sqrt{1-x^2}} \cdot x\, dx$$

put $1 - x^2 = t$

$x\,dx = -\frac{dt}{2}$

$$I = x \sin^{-1} x + \frac{1}{2} \int \frac{dt}{\sqrt{t}}$$

$$I = x \sin^{-1} x + \sqrt{1-x^2} + c \quad \text{Ans}$$

Ans 11 → $I = \int \underset{I}{x^3} \cdot \underset{II}{e^x}\, dx$

$$I = x^3 \cdot e^x - 3 \int \underset{I}{x^2} \cdot \underset{II}{e^x}\, dx$$

$$I = x^3 \cdot e^x - 3\left[x^2 \cdot e^x - 2 \int \underset{I}{x} \cdot \underset{II}{e^x}\, dx \right]$$

$$I = x^3 \cdot e^x - 3\left[x^2 e^x - 2\left\{ x e^x - \int 1 \cdot e^x\, dx \right\}\right] + c \quad \text{Ans}$$

QN 12 → $I = \int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$

$I = \int \frac{\sin^{-1}x}{\sqrt{1-x^2}\cdot(1-x^2)}\,dx$

put $\sin^{-1}x = t \longrightarrow x = \sin t$

$\frac{1}{\sqrt{1-x^2}}\,dx = dt$

$I = \int \frac{t\,dt}{1-\sin^2 t}$

$= \int \underset{I}{t}\,\underset{II}{\sec^2 t}\,dt$

$= t\tan t - \int \tan t\,dt$

$= t\tan t + \log|\cos t| + C$

$= \cancel{\sin^{-1}x \tan(\sin^{-1}x)}$

$= \sin^{-1}x \cdot \frac{x}{\sqrt{1-x^2}} + \log|\sqrt{1-x^2}| + C$

$I = \frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \frac{1}{2}\log|1-x^2| + C$ Ans

QNS 13 → $I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

put $x = \cos(2\theta)$

$dx = -2\sin(2\theta)\,d\theta$

$I = -2\int \tan^{-1}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\cdot\sin(2\theta)\,d\theta$

$I = -2\int \underset{I}{\theta}\cdot\underset{II}{\sin(2\theta)}\,d\theta$

Proceed

INTEGRATION : WORKSHEET No: 6

(Class No: 8)

Qn. 1 $I = \int \frac{1}{1+\cot x}\,dx$ **ANS** $-\frac{1}{2}\log|\sin x+\cos x| + \frac{1}{2}x + C$

Qn. 2 → $I = \int \frac{4\sin x + 5\cos x}{5\sin x + 4\cos x}\,dx$ **Ans** $\frac{40}{41}x + \frac{9}{41}\log|5\sin x + 4\cos x| + C$

Qn. 3 → $I = \int \frac{1+8\cot x}{3\cot x + 2}\,dx$ **Ans** $2x + \log|2\sin x + 3\cos x| + C$

Qn. 4 → $I = \int \frac{\sin x - \cos x}{\sqrt{\sin(2x)}}\,dx$ **Ans** $-\log\left|(\sin x+\cos x) + \sqrt{\sin(2x)}\right| + C$

Qn. 5 → $I = \int \sqrt{\tan x} + \sqrt{\cot x}\,dx$ **Ans** $\sqrt{2}\sin^{-1}(\sin x - \cos x) + C$

Qn. 6 → $I = \int \frac{1}{(\sin x - 2\cos x)(2\sin x + \cos x)}\,dx$ **Ans** $\frac{1}{5}\log\left|\frac{\tan x + 2}{2\tan x + 1}\right| + C$

Qn. 7 → $I = \int \frac{1}{\sin^2 x + \sin(2x)}\,dx$ **Ans** $\frac{1}{2}\log\left|\frac{\tan x}{\tan x + 2}\right| + C$

Qn. 8 → $I = \int \frac{\cos x}{\cos(3x)}\,dx$ **ANS** $\frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right| + C$

Qn. 9 → $I = \int \frac{1}{(2\sin x + 3\cos x)^2}\,dx$ **Ans** $-\frac{1}{2(2\tan x + 3)} + C$

Qn. 10 → $I = \int (\log x)^2\,dx$ **Ans** $x(\log x)^2 - 2(x\log x - x) + C$

Qn. 11 → $\int x\tan^{-1}x\,dx$ **Ans** $\frac{x^2}{2}\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x) + C$

Qn 12 → $I = \int (\sin^{-1}x)^2 \, dx$ **Ans** $x(\sin^{-1}x)^2 - 2\left[-\sin^{-1}x \cdot \sqrt{1-x^2} + x\right] + C$

Qn. 13 → $I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}} \, dx$ **Ans** $\frac{1}{2} x \cos^{-1}x - \frac{1}{2}\sqrt{1-x^2} + C$

Qn 14 → $I = \int \frac{x^2 \sin^{-1}x}{(1-x^2)^{3/2}} \, dx$ **Ans** $\frac{x \sin^{-1}x}{\sqrt{1-x^2}} - \frac{1}{2}(\sin^{-1}x)^2 + \frac{1}{2}\log(1-x^2) + C$

Qn 15 → $\int \frac{x \tan^{-1}x}{(1+x^2)^{3/2}} \, dx$ **Ans** $-\frac{\tan^{-1}x}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + C$

Qn 16 → $\int x \sin x \cdot \cos(2x) \, dx$ **Ans** $\frac{1}{2}\left[-\frac{x\cos(3x)}{3} + \frac{\sin(3x)}{9} + x\cos x - \sin x\right] + C$

Qn 17 → $\int \cos^{-1}(4x^3 - 3x) \, dx$ **Ans** $3x\cos^{-1}x - 3\sqrt{1-x^2} + C$

Qn 18 → $\int (x+1)\log x \, dx$ **Ans** $\left(x + \frac{x^2}{2}\right)\log x - \left(x + \frac{x^2}{4}\right) + C$

— x —