SOLUTIONS

WORKSHEET NO: 12 (CLASS : 16)

INTEGRATION

Ans: 1 $I = \int_{-\pi/2}^{\pi/2} x^3 + x\cos x + \tan^5 x + 1\, dx$

$$I = \int_{-\pi/2}^{\pi/2} x^3 + x\cos x + \tan^5 x\, dx + \int_{-\pi/2}^{\pi/2} 1\, dx$$

let $f(x) = x^3 + x\cos x + \tan^5 x$

$f(-x) = -x^3 + (-x)\cos(-x) + \tan^5(-x)$

$f(-x) = -x^3 - x\cos x - \tan^5 x$

$= -(x^3 + x\cos x + \tan^5 x)$

$f(-x) = -f(x)$ $\therefore$ $f(x)$ is an odd function

$\therefore I = 0 + \int_{-\pi/2}^{\pi/2} 1 \cdot dx$ $\quad \cdots \left\{ \int_{-a}^{a} f(x)\, dx = 0 \text{ if } f(-x) = -f(x) \right\}$

$I = (x)\Big|_{-\pi/2}^{\pi/2}$

$I = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$ $\quad \therefore \boxed{I = \pi}$ Ans

Ans 2. $I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ ---(1)

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right) dx \quad \cdots (P\ IV)$$

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx \quad \text{---}(2)$$

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① + ②

$$2I = \int_0^{\pi/2} \log\left(\frac{\cancel{4+3\sin x}}{\cancel{4+3\cos x}} \times \frac{\cancel{4+3\cos x}}{\cancel{4+3\sin x}} \right) dx$$

$$2I = \int_0^{\pi/2} \log(1)\, dx$$

$$2I = 0 \quad \text{---} \; \{\because \log 1 = 0\}$$

$$\boxed{I = 0}$$ Ans

Ques 3 → $I = \int_0^{2\pi} \cos^5 x\, dx$

$$I = 2\int_0^{\pi} \cos^5 x\, dx \quad \text{---(i)} \quad \left\{ \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x) = f(x) \right\}$$

$$I = 2\int_0^{\pi} \cos^5(\pi - x)\, dx \quad \text{---} \; (P\ IV)$$

$$I = 2\int_0^{\pi} -\cos^5 x\, dx \quad \text{---②} \quad \text{--} \; \{\cos(\pi - x) = -\cos x\}$$

$$I = -2\int_0^{\pi} \cos^5 x\, dx$$

① + ②

$$2I = 0$$

$$\boxed{I = 0}$$ Ans

Ques 4 → $I = \int_0^1 x(1-x)^n\, dx$

Two Methods

Method 1 $\quad I = \int_0^1 x(1-x)^n\, dx \quad \text{--①}$

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$$I = \int_0^1 (1-x)\,(1-(1-x))^n\,dx \quad \text{--} \quad (P\ \text{IV})$$ ③

$$I = \int_0^1 (1-x)\,x^n\,dx \quad \text{---} \ (2)$$

Note (Yahan (1) + (2) Nahi karna)

Just open the bracket

$$I = \int_0^1 x^n - x^{n+1}\,dx$$

$$I = \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right)_0^1$$

$$I = \left(\frac{1}{n+1} - \frac{1}{n+2}\right) - (0)$$

$$I = \frac{\cancel{n}+2 - \cancel{n} - 1}{(n+1)(n+2)}$$

$$\boxed{I = \frac{1}{(n+1)(n+2)}} \ \text{Ans}$$

Method 2 $\quad I = \int_0^1 x(1-x)^n\,dx$

put $1-x = t$ | when $x=0;\ t=1$

$dx = -dt$ | when $x=1;\ t=0$

$$\therefore I = -\int_1^0 (1-t)\cdot t^n\,dt$$

$$= -\int_1^0 (t^n - t^{n+1})\,dt$$

Ans

$$= -\left[\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2}\right]_1^0 = -\left[0 - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right] = \frac{1}{(n+1)(n+2)}$$

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Ans: 5 → $I = \int_{-\pi}^{\pi} \sin^3 x \cdot \cos^2 x \, dx$

here $f(x) = \sin^3 x \cdot \cos^2 x$

$f(-x) = \sin^3(-x) \cos^2(-x)$

$= -\sin^3 x \cdot \cos^2 x$

$= -f(x)$

$\therefore f(x)$ is an odd function

$\therefore \boxed{I = 0}$ Ans ---(P V)

Ans 6 → given $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$

$\Rightarrow \frac{1}{4} \int_0^a \frac{1}{(\frac{1}{2})^2 + x^2} dx = \frac{\pi}{8}$

$\Rightarrow \frac{1}{4} \times 2 \left[\tan^{-1}(2x) \right]_0^a = \frac{\pi}{8}$

$= \frac{1}{2} \left[\tan^{-1}(2a) - 0 \right] = \frac{\pi}{8}$

$\Rightarrow \tan^{-1}(2a) = \frac{\pi}{4}$

$\Rightarrow 2a = \tan(\pi/4)$

$\Rightarrow 2a = 1$

$\Rightarrow \boxed{a = 1/2}$ Ans

Ans 7 → $I = \int_{-\pi/4}^{\pi/4} \frac{1}{1+\cos(2x)} dx$

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$$I = \int_{-\pi/4}^{\pi/4} \frac{1}{2\cos^2 x}\, dx$$

$$I = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x \, dx$$

$$I = \frac{1}{2}\left(\tan x\right)_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}\left(\tan\left(\frac{\pi}{4}\right) - \tan^{-1}\left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2}\left(1-(-1)\right) = \frac{1}{\cancel{2}}(\cancel{2})$$

$$\boxed{I = 1}$$ ANS

Ans. 8 $\rightarrow$

$$I = \int_0^{\pi} x \log(\sin x)\, dx \quad \text{---} (1)$$

$$I = \int_0^{\pi} (\pi - x) \log(\sin(\pi - x))\, dx \quad \text{---} (P\ IV)$$

$$I = \int_0^{\pi} (\pi - x) \log(\sin x)\, dx \quad \text{--} (2)$$

(1) + (2)

$$2I = \pi \int_0^{\pi} \log(\sin x)\, dx$$

$$\cancel{2} I = \cancel{2}\pi \int_0^{\pi/2} \log(\sin x)\, dx \quad \text{---} (P\ VI)$$

$$\left\{ \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad f(2a-x) = f(x) \right\}$$

$$I = \pi \int_0^{\pi/2} \log(\sin x)\, dx \quad \text{--} (3)$$

$$I = \pi \int_0^{\pi/2} \log\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx \quad \text{---} (P\ IV)$$

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$$I = \pi \int_0^{\pi/2} \log(\cos x)\,dx \quad \cdots (4)$$

(3) + (4)

$$2I = \pi \int_0^{\pi/2} \log(\sin x \cdot \cos x)\,dx$$

$$2I = \pi \int_0^{\pi/2} \log\left(\frac{\cancel{2}\sin x (2x)}{2}\right) dx$$

$$2I = \pi \int_0^{\pi/2} \log(\sin(2x))\,dx - \pi \int_0^{\pi/2} \log 2\,dx$$

$$2I = \pi \int_0^{\pi/2} \log(\sin(2x))\,dx - \pi (x \log 2)_0^{\pi/2}$$

$$2I = \pi \int_0^{\pi/2} \log(\sin(2x))\,dx - \pi \left(\frac{\pi}{2} \log 2\right)$$

$$2I = \pi I_1 - \frac{\pi^2}{2} \log 2 \quad \cdots (5)$$

when $I_1 = \int_0^{\pi/2} \log(\sin(2x))\,dx$

put $2x = t$, $dx = \frac{dt}{2}$ | when $x = 0$; $t = 0$; when $x = \pi/2$; $t = \pi$

$$\therefore I_1 = \frac{1}{2} \int_0^{\pi} \log(\sin t)\,dt$$

$$I_1 = \frac{1}{\cancel{2}} \times \cancel{2} \int_0^{\pi/2} \log(\sin t)\,dt \quad \cdots (P\ VI)$$

$$I_1 = \int_0^{\pi/2} \log(\sin x)\,dx \quad \cdots (P\ I)$$

$\boxed{I_1 = \frac{I}{\pi}}$ (from equation No: 3)

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∴ equation (5) becomes

$$2I = \pi\left(\frac{I}{\pi}\right) - \frac{\pi^2}{2}\log 2$$

$$2I = I - \frac{\pi^2}{2}\log 2$$

$$\Rightarrow \boxed{I = -\frac{\pi^2}{2}\log 2}$$ Ans

Ans 9 → $I = \int_{-2}^{2} |x\cos(\pi x)|\,dx$

here $f(x) = |x\cos(\pi x)|$

$f(-x) = |-x\cos(-\pi x)|$

$f(-x) = |x\cos(\pi x)| = f(x)$

∴ $f(x) \to$ even function

$$\therefore I = 2\int_{0}^{2} |x\cos(\pi x)|\,dx \quad \cdots \left\{ \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ when } f(-x) = f(x) \right\}$$

4 cases

(1) $0 \le x < 1/2$; $0 \le \pi x < \pi/2$

$|\overset{(+)}{x}\,\overset{(+)}{\cos(\pi x)}| = x\cos(\pi x)$

(2) $\frac{1}{2} \le x < 1$; $\frac{\pi}{2} \le \pi x < \pi$ (2nd quad)

$|\overset{(+)}{x}\,\overset{(-)}{\cos(\pi x)}| = -x\cos(\pi x)$

(3) $1 \le x < 3/2$; $\pi \le \pi x < 3\pi/2$ (3rd quad)

$|\overset{(+)}{x}\,\overset{(-)}{\cos(\pi x)}| = -x\cos(\pi x)$

(4) $\frac{3}{2} \le x < 2$; $\frac{3\pi}{2} \le \pi x < 2\pi$ (4th quad)

$|\overset{(+)}{x}\,\overset{(+)}{\cos(\pi x)}| = x\cos(\pi x)$

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$$\therefore I = 2\left[\int_0^{1/2} x\cos(\lambda x)\,dx - \int_{1/2}^{3/2} x\cos(\lambda x)\,dx + \int_{3/2}^{2} x\cos(\lambda x)\,dx \right.$$

Let $I_1 = \int \underset{\mathrm{I}}{x}\, \underset{\mathrm{II}}{\cos(\lambda x)}\, dx$

$$= x\frac{\sin(\lambda x)}{\lambda} - \frac{1}{\lambda}\int \sin(\lambda x)\,dx$$

$$I_1 = \frac{x\sin(\lambda x)}{\lambda} + \frac{\cos(\lambda x)}{\lambda^2}$$

$$\therefore I = 2\left[\left(\frac{x\sin(\lambda x)}{\lambda} + \frac{\cos(\lambda x)}{\lambda^2} \right)_0^{1/2} - \left(\frac{x\sin(\lambda x)}{\lambda} + \frac{\cos(\lambda x)}{\lambda^2} \right)_{1/2}^{3/2} + \left(\frac{x\sin(\lambda x)}{\lambda} + \frac{\cos(\lambda x)}{\lambda^2} \right)_{3/2}^{2} \right]$$

$$= 2\left[\left(\frac{1}{2\lambda} + 0 \right) - \left(0 + \frac{1}{\lambda^2} \right) - \left\{ \left(-\frac{3}{2\lambda} + 0 \right) - \left(\frac{1}{2\lambda} + 0 \right) \right\} + \left\{ \left(0 + \frac{1}{\lambda^2} \right) - \left(-\frac{3}{2\lambda} + 0 \right) \right\} \right]$$

$$= 2\left[\frac{1}{2\lambda} - \cancel{\frac{1}{\lambda^2}} + \frac{3}{2\lambda} + \frac{1}{2\lambda} + \cancel{\frac{1}{\lambda^2}} + \frac{3}{2\lambda} \right]$$

$$= \cancel{2}\left[\frac{1+3+1+3}{\cancel{2}\lambda} \right]$$

$$\boxed{I = \frac{8}{\lambda}}$$ Ans

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Ans 10 → $I = \int_{-1}^{2} f(x)\,dx$

$f(x) = |x+1| + |x| + |x-1|$

$\circled{-1} \quad \circled{0} \quad \circled{1}$

$$f(x) = \begin{cases} (x+1) - x - (x-1) & : -1 \le x < 0 \\ (x+1) + (x) - (x-1) & : 0 \le x < 1 \\ (x+1) + (x) + (x-1) & ; 1 \le x < 2 \end{cases}$$

$$f(x) = \begin{cases} -x + 2 & : -1 \le x < 0 \\ x + 2 & : 0 \le x < 1 \\ 3x & : 1 \le x < 2 \end{cases}$$

$$\therefore I = \int_{-1}^{0} -x+2\,dx + \int_{0}^{1} x+2\,dx + \int_{1}^{2} 3x\,dx$$

$$= \left(-\frac{x^2}{2} + 2x\right)_{-1}^{0} + \left(\frac{x^2}{2} + 2x\right)_{0}^{1} + \left(3\frac{x^2}{2}\right)_{1}^{2}$$

$$= (0) - \left(-\frac{1}{2} - 2\right) + \left(\frac{1}{2} + 2\right) - (0) + (6) - \left(\frac{3}{2}\right)$$

$$= \frac{5}{2} + \frac{5}{2} + \frac{9}{2}$$

$$\therefore \boxed{I = \frac{19}{2}} \text{ Ans}$$

Ans 11 → $I = \int_{0}^{\pi/4} \sqrt{1+\sin(2x)}\,dx$

$$I = \int_{0}^{\pi/4} \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}\,dx$$

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(10)

$$I = \int_0^{\pi/4} \sqrt{(\sin x + \cos x)^2}\, dx$$

$$I = \int_0^{\pi/4} (\sin x + \cos x)\, dx$$

$$= \left(-\cos x + \sin x\right)_0^{\pi/4}$$

$$= \left(-\cancel{\tfrac{1}{\sqrt{2}}} + \cancel{\tfrac{1}{\sqrt{2}}}\right) - (-1 + 0)$$

$\boxed{I = 1}$ Ans

—x—

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