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ULTIMATE MATHEMATICS

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Chapter: A.O.D

← CLASS No: 4 →

Topic: Derivative as a Rate measure (continued---)

Qn 1 → A car starts from a point 'P' at time $t=0$ seconds and stops at point Q. The distance 'x', in metres covered by it, in 't' seconds is given by

$$x = t^2\left(2-\frac{t}{3}\right)$$

Find the time taken by it to reach Q and also find distance between P and Q.

Sol:

x(m) (stop) (velocity = 0)

P ($t=0$) —— t(secon) —— Q ($t=t$)

$$x = 2t^2 - \frac{t^3}{3}$$

velocity → Rate of change of distance w.r.t time

$$\text{velocity} = \frac{dx}{dt} = 4t - t^2$$

at point Q, velocity $\left(\frac{dx}{dt}\right) = 0$ ← (Main concept)

$$\Rightarrow 4t - t^2 = 0$$

$$\Rightarrow t(4-t) = 0$$

$$t = 0 \text{ (or) } \boxed{t=4}$$

(Rejected)

$$\therefore x = 2(16) - \frac{64}{3} = \frac{96-64}{3} = \frac{32}{3}$$

$\therefore$ time taken = 4 seconds & distance covered = $\frac{32}{3}$ m

Ans

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A.O.D (Class No: 4) (2)

Qn. 2 → The total cost $C(x)$ in Rupees, associated with the production of 'x' units of an item is given by

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

Find the "marginal cost" when 3 units are produced.

Soln

$$\boxed{M.C = \frac{dC}{dx}}$$

Marginal Cost → cost of producing one extra unit

$$MC = \frac{dC}{dx} = 0.015x^2 - 0.04x + 30$$

$$\left(\frac{dC}{dx}\right)_{x=3} = (0.015)(9) - (0.04)(3) + 30$$

$$= 0.135 - 0.12 + 30$$

$$= 30.015$$

$$MC = ₹\, 30.02 \text{ (Approx)}$$

Qns: 3 → Total Revenue (₹) received from the sale of 'x' units of a product is given by

$$R(x) = 13x^2 + 26x + 15$$

Find the marginal Revenue when ~~x=7~~ $x = 7$

Soln

$$MR = \frac{dR}{dx}$$

$$MR = 26x + 26$$

$$(MR)_{x=7} = 26 \times 7 + 26$$

$$= \boxed{₹\quad}$$

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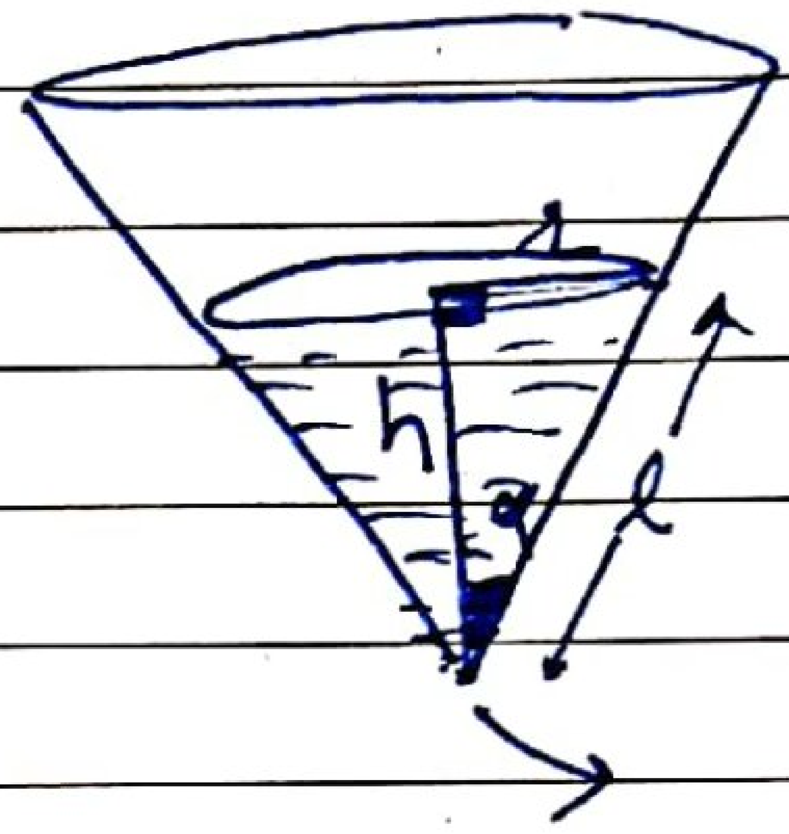
A.O.D (Class No: 4) (3)

Qn. 4 → Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of $2cm^2/sec$ in the surface area, through a tiny hole at the vertex of the bottom. When the slant height of cone is 4cm, find the rate of decrease of the slant height of water.

Soln:

Given $\alpha = \pi/4$



$$\frac{ds}{dt} = -2cm^2/sec$$

$l = 4cm$ (Initial)

To find: $\frac{dl}{dt} = ?$

here $S = C.S.A$

$S = \pi r l$

Relation → $\sin\left(\frac{\pi}{4}\right) = \frac{r}{l} \Rightarrow \frac{1}{\sqrt{2}} = \frac{r}{l} \Rightarrow \boxed{r = \frac{l}{\sqrt{2}}}$

$\therefore S = \pi\left(\frac{l}{\sqrt{2}}\right) \cdot l$

$S = \frac{\pi}{\sqrt{2}} \cdot l^2$

Diff. w.r.t 't'

$$\frac{ds}{dt} = \frac{\pi}{\sqrt{2}}(2l)\frac{dl}{dt}$$

$$\Rightarrow -\cancel{2} = \frac{\pi}{\sqrt{2}}(\cancel{2} \times 4)\frac{dl}{dt}$$

$$\Rightarrow \frac{dl}{dt} = -\frac{\sqrt{2}}{4\pi} \text{ cm/sec}$$

$\therefore$ slant hight is decreasing at the rate $= -\frac{\sqrt{2}}{4\pi}$ cm/sec Ans

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A.OD. (class No: 4) (4)

Qn. 5 → For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then how fast is the slope of curve changing when $x = 3$?

Soln

Given $y = 5x - 2x^3$; $\frac{dx}{dt} = 2$ unit/sec

$x = 3$

To find: $\frac{dm}{dt}$

Now $m = \frac{dy}{dx} = 5 - 6x^2$

Diff. w.r.t 't'

$\frac{dm}{dt} = (-12x)\frac{dx}{dt}$

$= (-12)(3)(2)$

$= -72$ unit/sec

∴ Slope of curve is ↓ at the rate -72 unit/sec

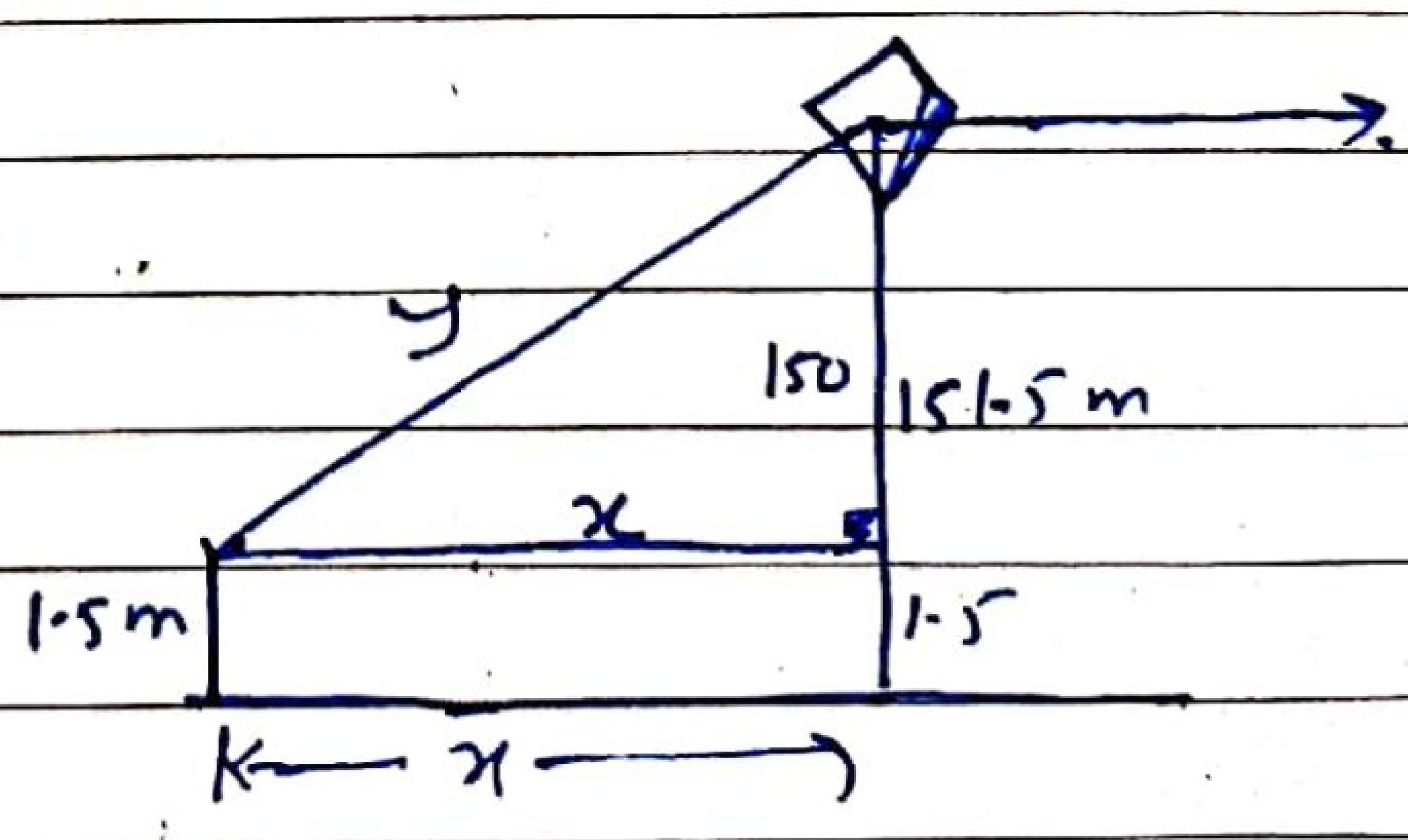
Ans

Qn. 6 → A kite is moving horizontally at a height of 151.5 meters. If the speed of kite is 10m/sec, how fast is the string being let out; when the kite is 250m away from the boy who is flying the kite? The height of boy is 1.5m.

Soln

Given $\frac{dx}{dt} = 10$ m/sec

$y = 250$ m



to find $\frac{dy}{dt}$

$x^2 + 22500 = y^2$

$x^2 + 22500 = 62500$

$x^2 = 40000$

$\boxed{x = 200}$

A·O.D (class No. 4) (5)

given $x^2 + 22500 = y^2$

Diff. w.r.t. 't'

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

$$(200)(10) = (250)\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{2000}{250} = 8\ m/sec$$

∴ ~~length~~ String is cut at the rate of 8 m/sec Ans

Qn. 7 → The volume of cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

Soln let $x \to$ side/edge of the cube

given $\frac{dv}{dt} = k$

Now $V = x^3$

diff. w.r.t. t

$$\frac{dv}{dt} = 3x^2 \cdot \frac{dx}{dt}$$

$$k = 3x^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{k}{3x^2}$$

Now $S = 6x^2$

Diff. w.r.t. 't'

$$\frac{ds}{dt} = 12x\frac{dx}{dt}$$

$$\frac{ds}{dt} = 12(x)\left(\frac{k}{3x^2}\right) = \frac{4k}{x}$$

$$\frac{ds}{dt} \propto \frac{1}{x}$$ proof

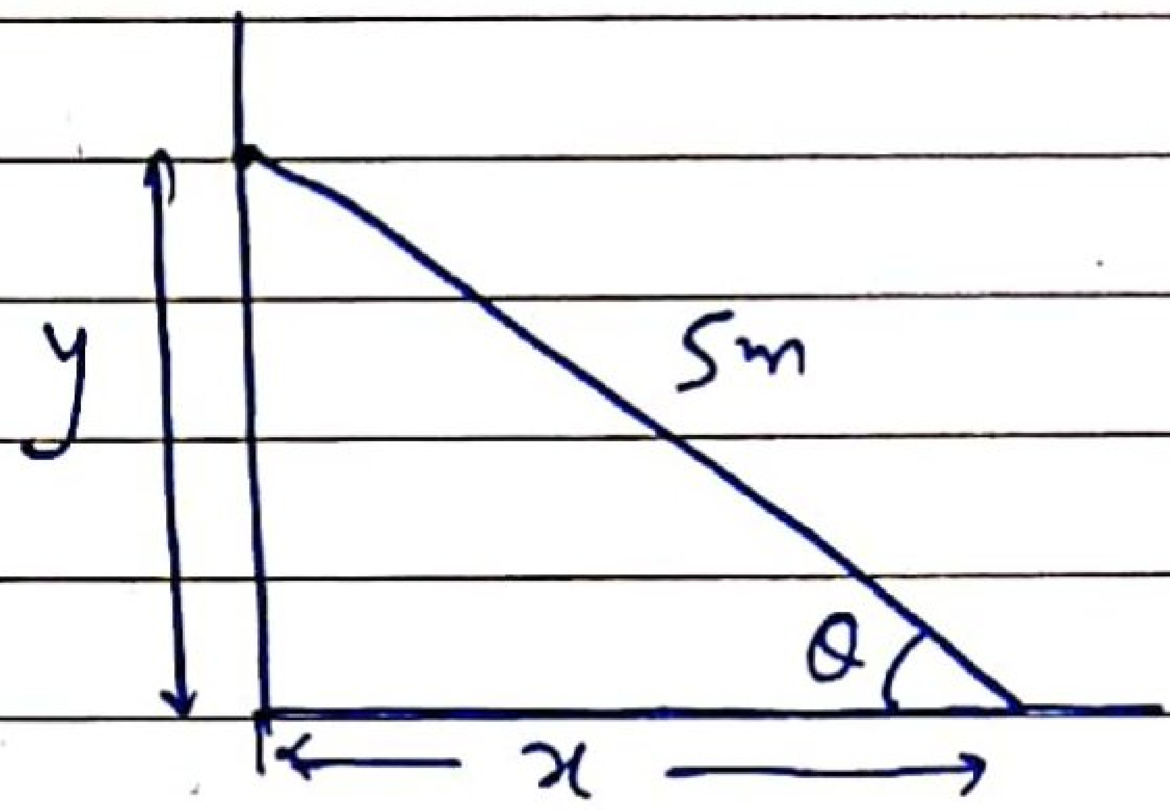
A.OD (Class No. 4) (6)

Qn 8 → A ladder 5 m long, standing on a horizontal floor leans against a wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then find the rate at which the angle b/w the floor and the ladder is decreasing when lower end of ladder is 2 m from the wall.

Sol Given $\frac{dy}{dt} = -0.1$ m/sec

$x = 2$ m.

to find $\frac{d\theta}{dt} = ?$



$\sin\theta = \frac{P}{H} = \frac{y}{5}$

Diff wrt 't'

$\cos\theta \cdot \frac{d\theta}{dt} = \frac{1}{5}\frac{dy}{dt}$

Now $\cos\theta = \frac{x}{5} = \frac{2}{5}$

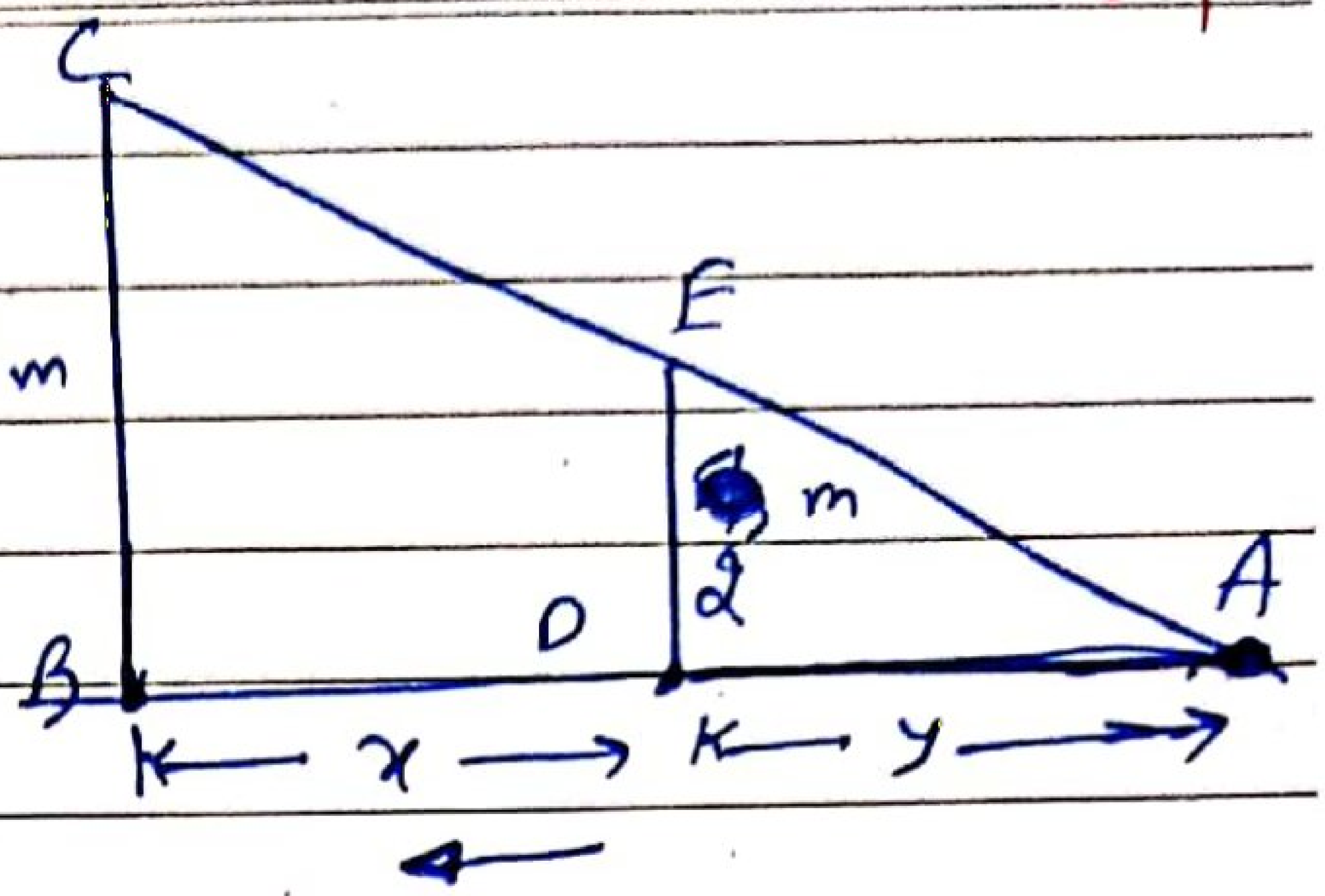
$\therefore \frac{2}{\cancel{5}}\frac{d\theta}{dt} = \frac{1}{\cancel{5}}(-0.1)$

$\frac{d\theta}{dt} = -\frac{0.1}{2} = -\frac{1}{20}$ radians/sec

$\therefore$ angle is ↓ at the rate $\frac{1}{20}$ rad/sec

Qns 9 → A man, 2m tall, walks at the rate of $1\frac{2}{3}$ m/sec towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is the "tip" of his shadow moving? At what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of light?

Soln



$$\frac{dx}{dt} = -\frac{5}{3}\ m/sec$$

$$x = \frac{10}{3}\ m$$

(i) to find $\frac{d}{dt}(x+y) = ?$

(ii) to find $\frac{dy}{dt} = ?$

$\Delta ABC \sim \Delta ADE$

$$\frac{\frac{16}{3}}{2} = \frac{x+y}{y}$$

$$\Rightarrow \frac{8}{3} = \frac{x+y}{y}$$

$$\Rightarrow 8y = 3x + 3y$$

$$\Rightarrow 5y = 3x$$

Diff. w.r.t t

$$5\frac{dy}{dt} = 3\frac{dx}{dt}$$

$$5\frac{dy}{dt} = \cancel{3}\left(\frac{-5}{\cancel{3}}\right)$$

$$\frac{dy}{dt} = -1\ m/sec$$

(i) $\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt} = \frac{-5}{3} - 1 = \frac{-8}{3}\ m/sec$

$\therefore$ (i) $\therefore$ tip of shadow moving at the rate $\frac{8}{3}$ m/sec

(ii) $\therefore$ length of shadow is changing at the rate 1 m/sec

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WORKSHEET NO: 4
(A.O.D)

Qns 1 → Water is dripping out at a steady rate of 1 cubic cm/sec through a tiny hole at the vertex of the conical vessel. When the slant height of the water in the vessel is 4cm, find the rate of decrease of slant height, where vertical angle of the conical vessel is $\frac{\pi}{3}$.

ANS $-\frac{1}{2\sqrt{3}\pi}$ cm/sec

HINT here $2\alpha = \frac{\pi}{3}$ $\therefore$ $\alpha = \pi/6$

Qn. 2 → Find the rate of change of volume of a sphere with respect to its surface area when the radius is 2cm

ANS $\frac{dv}{ds} = 1 \text{ cm}^3/\text{sec}$

Qn 3 → A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface area. Prove that the radius is decreasing at a constant rate

Qn. 4 → If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius

Qn. 5 → A kite is 120m high and 130m of string is out. If the kite is moving away horizontally at the rate of 52 m/sec. Find the rate at which the string is being paid out

ANS 20m/sec

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(2)

Q. 6 → A man is moving away from a tower 41.6 m high at the rate of 2 m/sec. Find the rate at which the angle of elevation of the top of tower is changing, when he ~~at~~ is at a distance of 30 m from the foot of the tower. Assume that the eye level (height) of the man is 1.6 m from the ground

Ans $\frac{d\theta}{dt} = \frac{-4}{125}$ radian/sec

Q. 7 → Prove that the curves $y^2 = 4x$ and $x^2+y^2-6x+1=0$ "touch" each other at the point $(1,2)$

Tangent & Normal

Hint Show $m_1 = m_2$ (for touch)

Q. 8 → Find the coordinates of the point on the curve $\sqrt{x}+\sqrt{y}=4$ at which tangent is equally inclined to the axes

T & N

Ans $(4,4)$ Hint $m = \pm 1$ (equally inclined)

Q. 9 → Prove that the curves $xy=4$ and $x^2+y^2=8$ touch each other

T & N

Hint $m_1 = m_2$ (show)

Q. 10 → Find the slope of the tangent to the curve $x = t^2+3t-8$ & $y = 2t^2-2t-5$ at the point $(2,-1)$

T & N

Ans: 6/7

Q. 11 → Show that the two curves $x^3-3xy^2+2=0$ and $3x^2y-y^3-2=0$ cut orthogonally

T & N

Hint Show $m_1 m_2 = -1$

— x —

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