!! जय श्री राधेकृष्णा जय श्री गिरिराज जी महाराज !! (1)

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER: VECTORS : CLASS NO: 4

Ques 1 If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ & $\vec{c}$. Also find the angle.

Sol given $\vec{a} \perp \vec{b} \perp \vec{c}$

$\Rightarrow \vec{a}\cdot\vec{b}=0; \quad \vec{b}\cdot\vec{c}=0, \quad \vec{c}\cdot\vec{a}=0$

given $|\vec{a}|=|\vec{b}|=|\vec{c}| = \lambda$ (let)

we have $|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$= |\vec{a}|^2 + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{a} + |\vec{b}|^2 + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} + |\vec{c}|^2$

$= \lambda^2 + \lambda^2 + \lambda^2$

$|\vec{a}+\vec{b}+\vec{c}|^2 = 3\lambda^2$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\lambda$

let $\theta_1, \theta_2, \theta_3$ are the angles made by the $\vec{a}+\vec{b}+\vec{c}$ with $\vec{a}, \vec{b}$ & $\vec{c}$ respectively

$$\cos\theta_1 = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}|\,|\vec{a}|} = \frac{|\vec{a}|^2+0+0}{|\vec{a}+\vec{b}+\vec{c}|\,|\vec{a}|} = \frac{\lambda^2}{(\sqrt{3}\lambda)(\lambda)} = \frac{1}{\sqrt{3}}$$

$\Rightarrow \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$$\cos\theta_2 = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{b}}{|\vec{a}+\vec{b}+\vec{c}|\ |\vec{b}|} = \frac{|\vec{b}|^2+0+0}{|\vec{a}+\vec{b}+\vec{c}|\ |\vec{b}|} = \frac{\lambda^2}{(\sqrt{3}\lambda)(\lambda)} = \frac{1}{\sqrt{3}}$$

$\Rightarrow \theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Similarly $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Clearly $\theta_1 = \theta_2 = \theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ Ans

Ans 2 → If the sum of two unit vectors is also a unit vector. Find the magnitude of their difference.

Soln let $\vec{a}$ & $\vec{b}$ are two unit vectors

$\Rightarrow |\vec{a}|=1$ & $|\vec{b}|=1$

given $\vec{a}+\vec{b}$ is also a unit vector

$\Rightarrow$ $|\vec{a}+\vec{b}| = 1$

~~we~~ Squaring

$|\vec{a}+\vec{b}|^2 = 1$

$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = 1$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = 1$

$\Rightarrow 2 + 2\vec{a}\cdot\vec{b} = 1$

$\Rightarrow \boxed{\vec{a}\cdot\vec{b} = -\frac{1}{2}}$

To find $|\vec{a}-\vec{b}| = ?$

we have

$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$

$= 1 + 1 - 2\left(-\frac{1}{2}\right)$

$|\vec{a}-\vec{b}|^2 = 3$

$\boxed{|\vec{a}-\vec{b}| = \sqrt{3}}$ Ans

Que 3 → Find the values of 'a' for which the vector $\vec{r} = (a^2-4)\hat{i} + 2\hat{j} - (a^2-9)\hat{k}$ makes acute angles with the co-ordinate axes

Soln

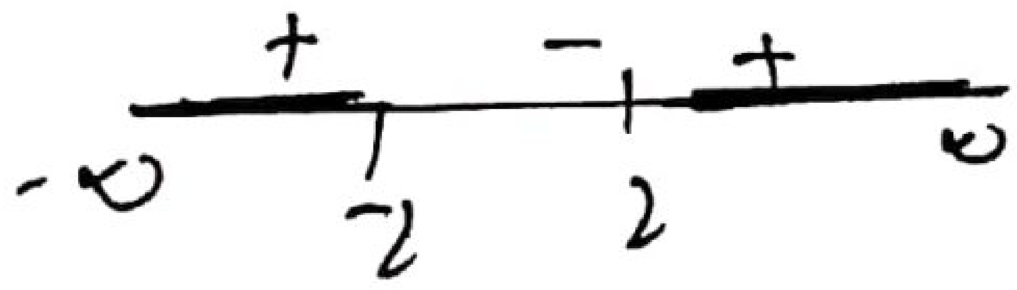
$\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along x, y, z-axis resp.

for acute angle	for obtuse angle
$\vec{a} \cdot \vec{b} > 0$	$\vec{a} \cdot \vec{b} < 0$

Given $\vec{r} \cdot \hat{i} > 0$

$a^2 - 4 > 0$

$(a+2)(a-2) > 0$



$a \in (-\infty, -2) \cup (2, \infty)$

$\vec{r} \cdot \hat{j} > 0$

$2 > 0$

(true)

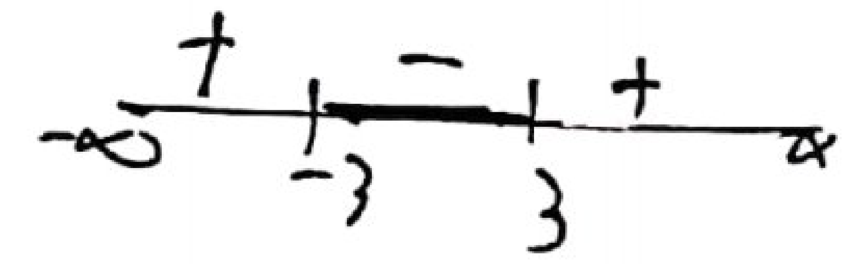
$a \in R$

$\vec{r} \cdot \hat{k} > 0$

$-(a^2 - 9) > 0$

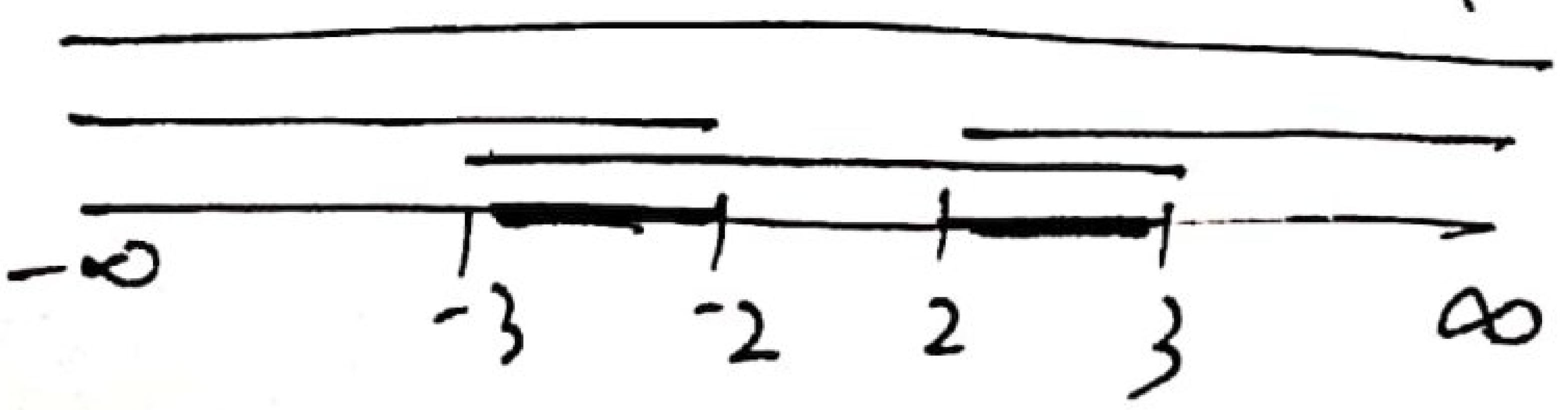
$a^2 - 9 < 0$

$(a+3)(a-3) < 0$



$a \in (-3, 3)$

Common



$\therefore \; a \in (-3, -2) \cup (2, 3)$ Ans

Que 4 → If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, then show that $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

Soln given $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$

$\Rightarrow \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 0$

$\Rightarrow \vec{a} \cdot (\vec{b} - \vec{c}) = 0$

$\Rightarrow$ either $\vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{0}$

(or) $\vec{a} \perp (\vec{b} - \vec{c})$

Ans

Qn. 5 → For any two vectors $\vec{a}$ & $\vec{b}$ show that

(i) $|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$ (Cauchy - Schwartz inequality)

(2) $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$ (Triangle Inequality)

Soln (i) $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow |\vec{a}\cdot\vec{b}| = \big||\vec{a}||\vec{b}|\cos\theta\big|$

$\Rightarrow |\vec{a}\cdot\vec{b}| = |\vec{a}||\vec{b}|\,|\cos\theta|$

~~set~~ we always have $|\cos\theta| \le 1$

$\Rightarrow |\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$ proved

(2) we have $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$

$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$

where $\cos\theta \le 1$

$\Rightarrow |\vec{a}+\vec{b}|^2 \le |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|$

$\Rightarrow |\vec{a}+\vec{b}|^2 \le \left(|\vec{a}|+|\vec{b}|\right)^2$

$\Rightarrow |\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$ proved

Qn 6 → If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ a non-zero scalar then $\lambda\vec{a}$ is a unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = \frac{1}{|\lambda|}$

Soln Given $|\vec{a}| = a$

Given $\lambda\vec{a}$ is a unit vector

$\Rightarrow |\lambda\vec{a}| = 1$

$\Rightarrow |\lambda|\,|\vec{a}| = 1$

$\Rightarrow |\lambda|\, a = 1$

$\Rightarrow \boxed{a = \frac{1}{|\lambda|}}$ (D) Ans

Q. 7 → If $\vec{a}$ & $\vec{b}$ are unit vectors, find the greatest and least value of $\sqrt{3}\,|\vec{a}+\vec{b}| + |\vec{a}-\vec{b}|$

Sol Given $|\vec{a}| = 1$ & $|\vec{b}| = 1$

we have $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$

$= |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$

$= 2 + 2\cos\theta$

$= 2(1+\cos\theta)$

$|\vec{a}+\vec{b}|^2 = 2 \times 2\cos^2\theta/2 \Rightarrow \boxed{|\vec{a}+\vec{b}| = 2\cos\theta/2}$

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Similarly $|\vec{a}-\vec{b}| = 2\sin\theta/2$

we have $\sqrt{3}|\vec{a}+\vec{b}| + |\vec{a}-\vec{b}|$

$= \sqrt{3}\left(2\cos\frac{\varphi}{2}\right) + 2\sin\frac{\varphi}{2}$

$= 2\sqrt{3}\cos\frac{\varphi}{2} + 2\sin\frac{\varphi}{2}$

Concept

$a\cos\theta + b\sin\theta$

Max value $= \sqrt{a^2+b^2}$

Min value $= -\sqrt{a^2+b^2}$

Max value $= \sqrt{(2\sqrt{3})^2+(2)^2}$

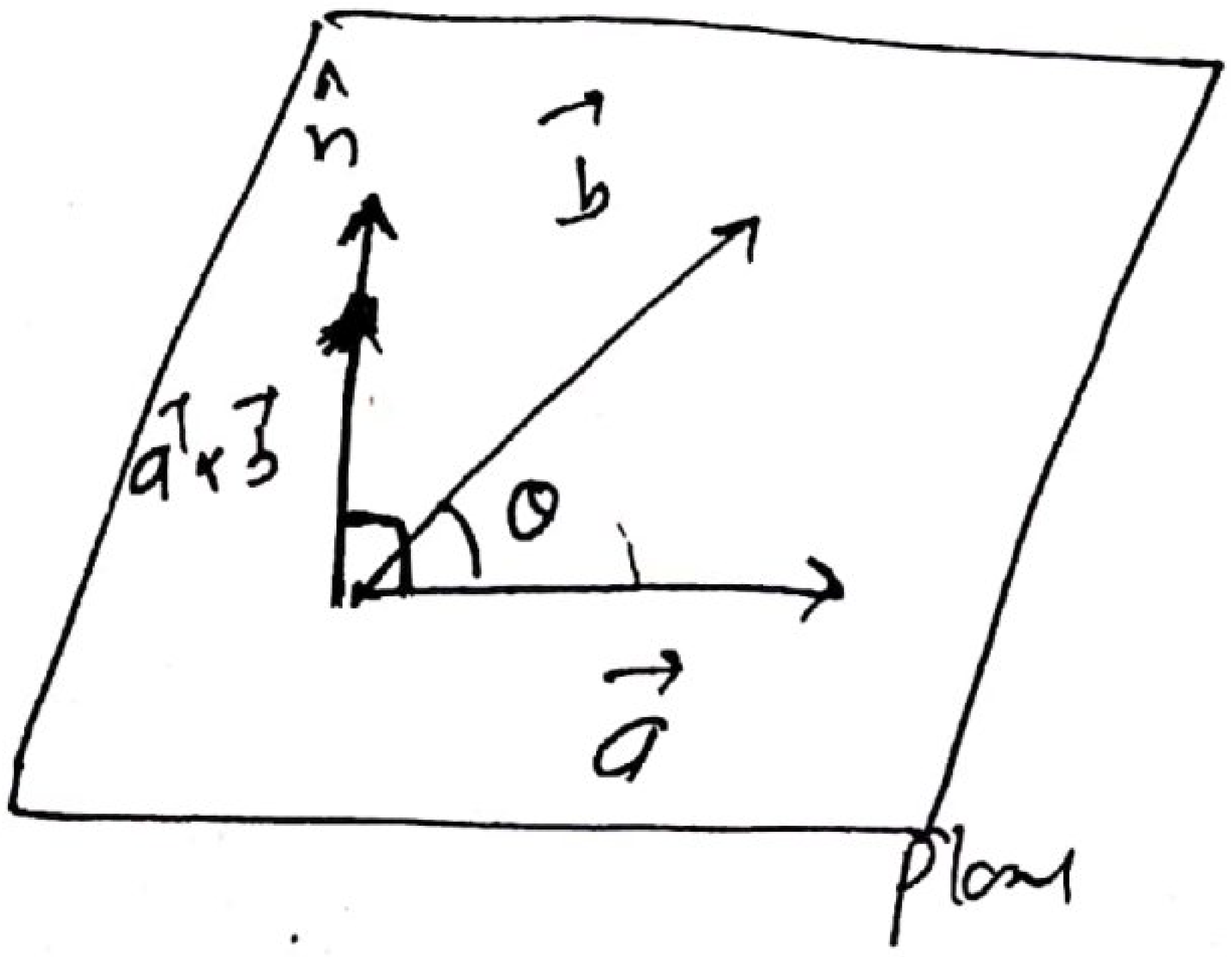
$= \sqrt{12+4} = 4$

Min value $= -\sqrt{12+4} = -4$

Ans

Topic Vector product / CROSS PRODUCT of Two vectors.

(·) $\vec{a}\times\vec{b}$ is a vector of magnitude $|\vec{a}||\vec{b}|\sin\theta$ and direction $\hat{n}$ (unit vector perpendicular to the plane containing $\vec{a}$ & $\vec{b}$)



(·) $\vec{a}\times\vec{b} = \left(|\vec{a}||\vec{b}|\sin\theta\right)\hat{n}$

(·) $\boxed{|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta}$ .. $\{|\hat{n}| = 1\}$

(·) $\vec{a}\times\vec{b}$ is a vector which is $\perp^r$ to both $\vec{a}$ & $\vec{b}$

(·) If $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$
then $\vec{a}\times\vec{b}=\vec{0}$

(·) If $\vec{a}$ & $\vec{b}$ are parallel or collinear
i.e $\theta=0$ or $\theta=\pi$

$$\boxed{\vec{a}\times\vec{b}=\vec{0}}$$

(·) If $\vec{a}\times\vec{b}=\vec{0}$ then either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$
or $\vec{a}\parallel\vec{b}$

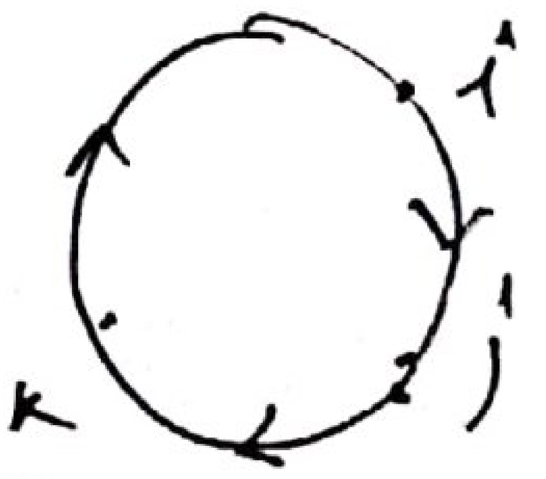
(·)

(·) $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$

(·) $\vec{a}\times\vec{a}=\vec{0}$ (angle $\theta=0$)

(·) $\hat{i}\times\hat{i}=\vec{0}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}$

(·) $\hat{i}\times\hat{j}=\hat{k}$; $\hat{j}\times\hat{k}=\hat{i}$; $\hat{k}\times\hat{i}=\hat{j}$
$\hat{j}\times\hat{i}=-\hat{k}$; $\hat{i}\times\hat{k}=-\hat{j}$



(·) let $\vec{a}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$
$\vec{b}=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$ → expand vector quantity

(·) $m\vec{a}\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times m\vec{b}$

(·) $m\vec{a}\times n\vec{b} = mn(\vec{a}\times\vec{b})$

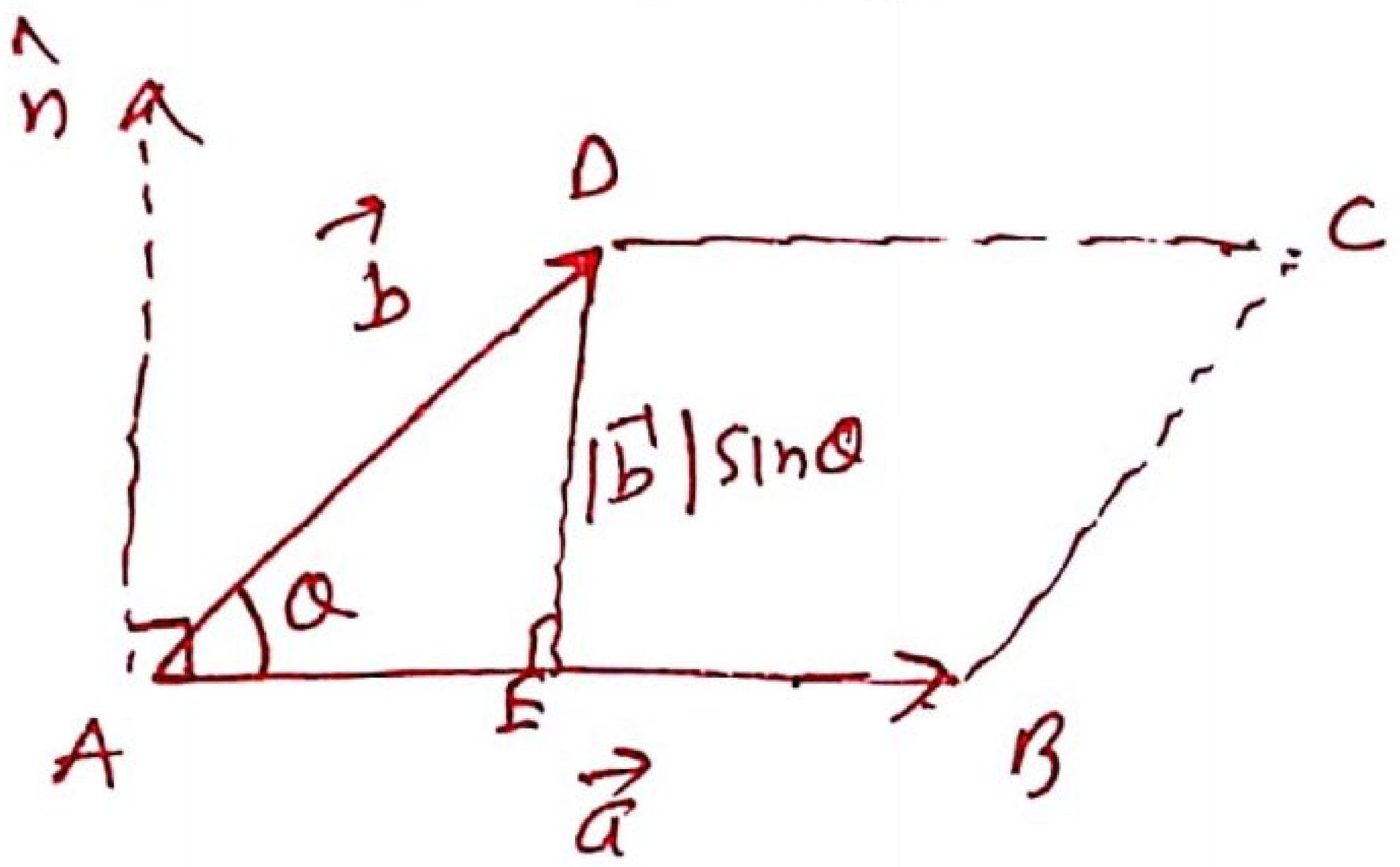
(·) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c} = \vec{a}\times(\vec{b}+\vec{c})$

(·) Applications of vector product

(·) Area of parallelogram with $\vec{a}$ & $\vec{b}$ as its adjacent sides

$$\boxed{\text{Area of } \|gm = |\vec{a}\times\vec{b}|}$$

Proof



$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

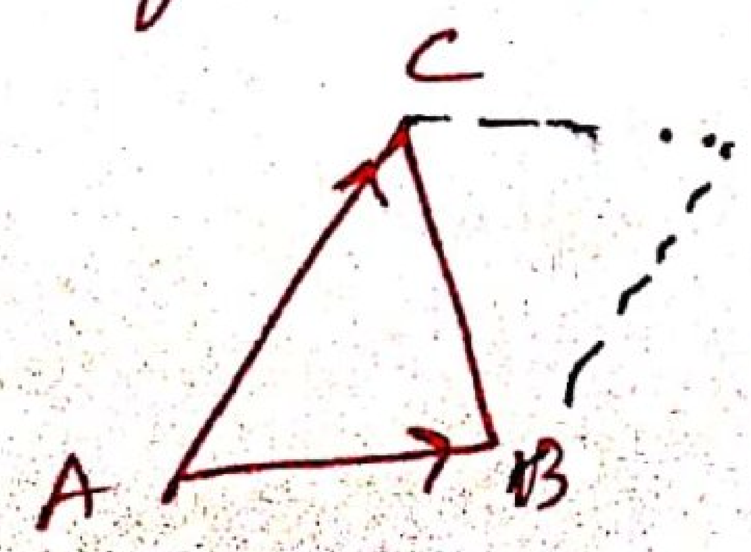
$\vec{a}\times\vec{b} = |\vec{a}|(DE)\hat{n}$ $\left(\because \triangle ADE,\ \sin\theta = \frac{DE}{AD} = \frac{DE}{|\vec{b}|}\right)$

$\vec{a}\times\vec{b} = (AB)(DE)\hat{n}$ $|\vec{b}|\sin\theta = DE$

$\vec{a}\times\vec{b} = (\text{Base})(\text{height})\hat{n}$

$|\vec{a}\times\vec{b}| = (\text{Base})(\text{height}) =$ Area of parallelogram

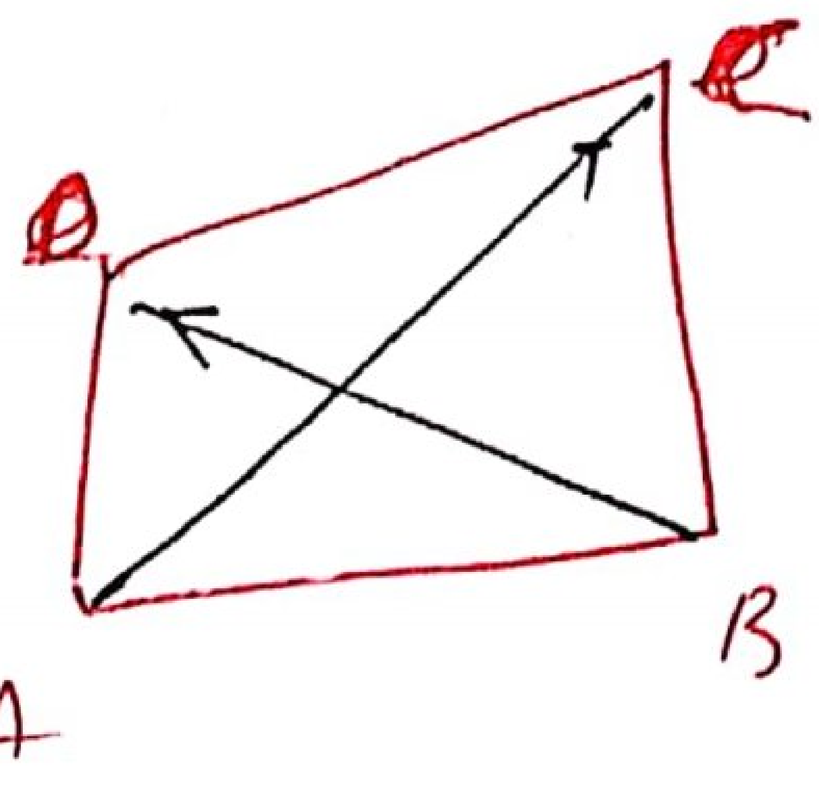
(·) Area of $\triangle$ (Triangle) ABC



$\text{Area} = \frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}|\vec{AB}\times\vec{CA}|$

$= \frac{1}{2}|\vec{BA}\times\vec{BC}|$

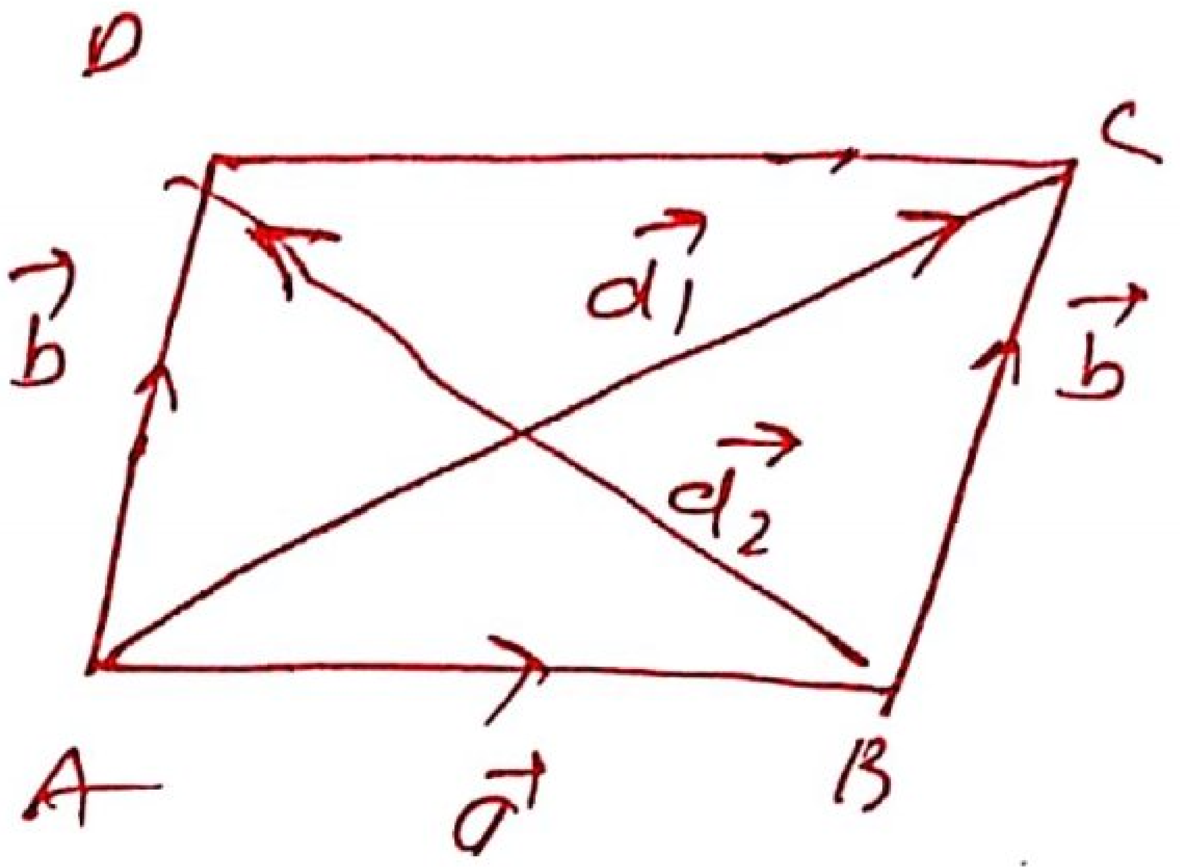
Area of Quadrilateral



$$\text{Area} = \frac{1}{2}\left|\vec{AC} \times \vec{BD}\right|$$

diagonals.

Area of parallelogram



$$\boxed{\vec{a} + \vec{b} = \vec{d_1}}$$

$$\vec{a} + \vec{d_2} = \vec{b}$$

$$\boxed{\vec{d_2} = \vec{b} - \vec{a}}$$

$$\text{Area of } \| \text{gm} = \frac{1}{2}\left|\vec{d_1} \times \vec{d_2}\right|$$

– x –

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WORKSHEET No: 3 (VECTORS)

Qns 1 If $\vec{a}$ & $\vec{b}$ be two unit vectors and θ is the angle between them & $\vec{a}+\vec{b}$ is a unit vector. Find θ

Ans $\theta = 2\pi/3$

Qns 2 → Let $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $|\vec{a}+\vec{b}+\vec{c}| = 1$ and $\vec{a}$ is perpendicular to $\vec{b}$. If $\vec{c}$ makes angle α and β with $\vec{a}$ & $\vec{b}$ respectively, then what is the value of $\cos\alpha + \cos\beta$ Ans $= -1$

Qns 3 → If the angle b/w the vectors $x\hat{i}+3\hat{j}-7\hat{k}$ and $x\hat{i}-x\hat{j}+4\hat{k}$ is acute, then find the interval in which x lies. Ans $(-\infty, 4) \cup (7, \infty)$

Qns 4 → The vector $\vec{a}$ & $\vec{b}$ satisfy the equations $2\vec{a}+\vec{b} = \vec{p}$ and $\vec{a}+2\vec{b} = \vec{q}$ where $\vec{p} = \hat{i}+\hat{j}$ and $\vec{q} = \hat{i}-\hat{j}$. If θ is the angle b/w $\vec{a}$ & $\vec{b}$, then

(A) $\cos\theta = \frac{4}{5}$ (B) $\sin\theta = \frac{1}{\sqrt{2}}$ (C) $\cos\theta = -\frac{4}{5}$ (D) $\cos\theta = -\frac{3}{5}$

(Ans : c)

Qns 5 → If $\vec{a}$ & $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}| = |\vec{b}|$, then prove that $\vec{a}+2\vec{b}$ is $\perp^r$ to $\vec{a}$

Qn 6 → Express the vector $2\hat{i}-\hat{j}+3\hat{k}$ as the sum of a vector parallel and a vector perpendicular to $2\hat{i}+4\hat{j}-2\hat{k}$

Ans $-\frac{1}{2}\hat{i}-\hat{j}+\frac{1}{2}\hat{k}$; $\frac{5}{2}(\hat{i}+\hat{k})$

Qn 7 → If $\vec{a}$ & $\vec{b}$ are non-collinear unit vectors such that $|\vec{a}+\vec{b}|=\sqrt{3}$ find $(2\vec{a}-5\vec{b})\cdot(3\vec{a}+\vec{b})$

Ans $-11/2$

Qn 8 → If $|\vec{a}+\vec{b}|=60$ & $|\vec{a}-\vec{b}|=40$ and $|\vec{b}|=46$ find $|\vec{a}|$

Ans 22

Qn 9 → Dot product of a vector with $\hat{i}+\hat{j}-3\hat{k}$, $\hat{i}+3\hat{j}-2\hat{k}$ and $2\hat{i}+\hat{j}+4\hat{k}$ are 0, 5 & 8 respectively. Find the vector

Ans $\hat{i}+2\hat{j}+\hat{k}$

Qn 10 → Find the values of x for which the angle b/w the vectors $2x^2\hat{i}+4x\hat{j}+\hat{k}$ and $7\hat{i}-2\hat{j}+x\hat{k}$ is obtuse

Ans $x\in(0,1/2)$

Qn 11 → Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the sides of a right angled triangle

Qn 12 → Prove that $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=|\vec{a}|^2+|\vec{b}|^2$, if and only if $\vec{a}$ & $\vec{b}$ are ~~perp~~ perpendicular

Qn 13 → Find the value of $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{i}\times\hat{k})+\hat{k}\cdot(\hat{i}\times\hat{j})$

Ans 1

Qn 14 → Using vectors, find the area of the triangle with vertices $A(1,1,2)$ $B(2,3,5)$ $C(1,5,5)$

Ans $\frac{\sqrt{61}}{2}$ sq. units