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← ULTIMATE MATHEMATICS →

1

DIFFERENTIATION & CONTINUITY

CLASS NO: 9

Ques 1 → $f(x) = \begin{cases} x^2+3x+a & ; \quad x \le 1 \\ bx+2 & ; \quad x > 1 \end{cases}$

is everywhere differentiable. Find the values of a and b

Soln Since f(x) is differentiable, then it must be continuous.

Cont. at $x=1$

LHL = RHL = f(1)

$$\Rightarrow \lim_{x\to 1^-}\left(x^2+3x+a\right) = \lim_{x\to 1^+}(bx+2) = 1+3+a$$

put $x=1-h$, $h\to 0$ put $x = 1+h$, $h \to 0$

$$\Rightarrow \lim_{h\to 0}\left[(1-h)^2+3(1-h)+a\right] = \lim_{h\to 0}\left[b(1+h)+2\right] = 4+a$$

$$\Rightarrow 1+3+a = b+2 = 4+a$$

$$\Rightarrow 4+a = b+2$$

$$\Rightarrow \boxed{a = b-2} \quad \text{---} (1)$$

Differentiability at $x=1$

$$LHD = \lim_{x\to 1^-}\left(\frac{f(x)-f(1)}{x-1}\right)$$

$$= \lim_{x\to 1^-}\left(\frac{x^2+3x+a-(1+3+a)}{x-1}\right)$$

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$$LHD = \lim_{x\to 1^-}\left(\frac{x^2+3x-4}{x-1}\right)$$

$$= \lim_{x\to 1^-}\left(\frac{(x+4)\cancel{(x-1)}}{\cancel{x-1}}\right)$$

$$= \lim_{x\to 1^-}(x+4) \quad \text{put } x=1-h \ \& \ h\to 0$$

$$LHD = \lim_{h\to 0}(1-h+4) = 1+4 = 5$$

$$LHD = 5$$

$$RHD = \lim_{x\to 1^+}\left(\frac{f(x)-f(1)}{x-1}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{bx+2-(1+3+a)}{x-1}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{bx-2-a}{x-1}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{bx-2-(b-2)}{x-1}\right) \quad \text{-- \{from eq(1)\}}$$

$$= \lim_{x\to 1^+}\left(\frac{bx-b}{x-1}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{b\cancel{(x-1)}}{\cancel{x-1}}\right)$$

$$\boxed{RHD = b}$$

$$LHD = RHD$$

$$\Rightarrow \boxed{5 = b}$$

$$\therefore a = b-2 \Rightarrow a = 5-2 = 3$$

$$\therefore a=3,\ b=5 \quad \text{Ans}$$

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THEOREMS

(1) Rolle's theorem

given $f(x) =$ ------ ; defined on $[a,b]$

✓ $f(x)$ is continuous in $[a,b]$
✓ $f(x)$ is differentiable in (a,b)
✓ $f(a) = f(b)$

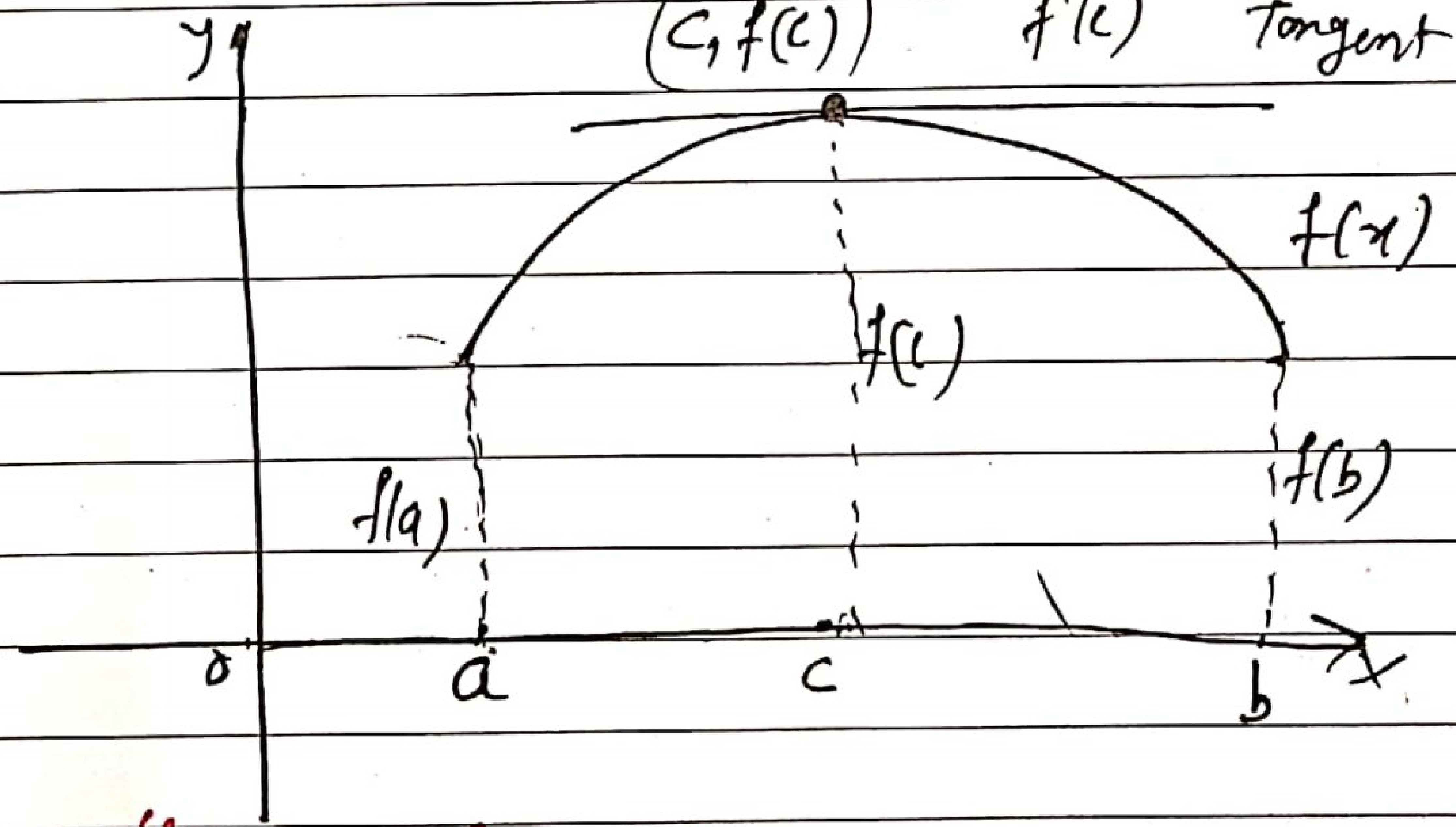
then there exists at least one value $c \in (a,b)$ such that

$$f'(c) = 0$$

then Rolle's theorem is verified.

✓ $(c, f(c))$ is a point on the curve where tangent is parallel to x-axis

geometrically



(2) Mean value theorem (LMV)

↳ lagrange's

given $f(x) =$ - - - - defined on $[a,b]$

✓ $f(x)$ is continuous in $[a,b]$
✓ $f(x)$ is differentiable in (a,b)

then there exists a value $c \in (a,b)$ such that

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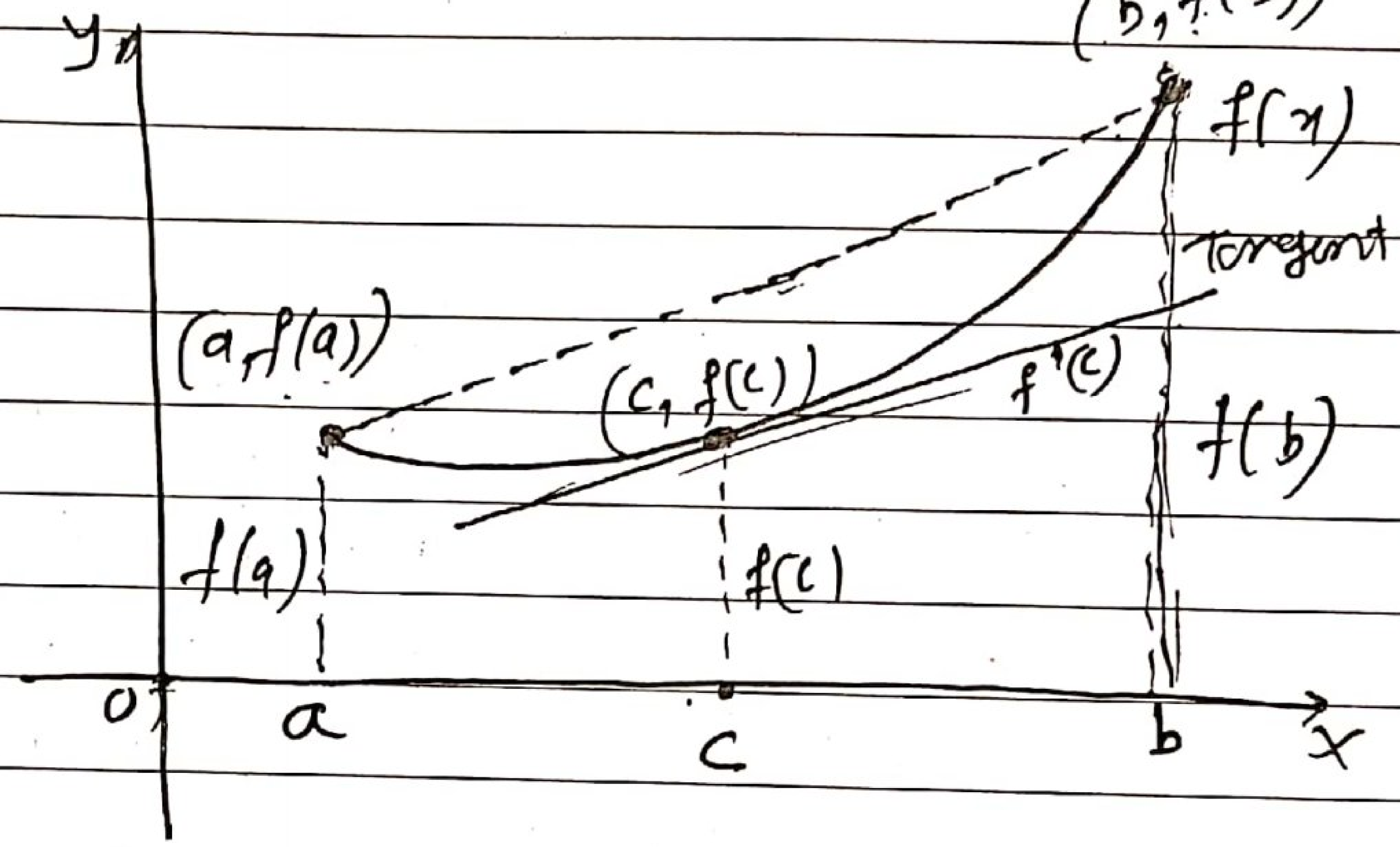
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$(c, f(c))$ is a point on the curve where tangent is parallel to the chord joining the points $(a, f(a))$ and $(b, f(b))$

Geometrically



Ans 1→ verify Rolle's theorem for the function
$f(x) = x(x-3)^2 \; ; \; 0 \le x \le 3$

Solution $f(x) = x^3 - 6x^2 + 9x \quad ; \; x \in [0,3]$

(1) Since $f(x)$ is a polynomial function which is everywhere continuous $\therefore f(x)$ is also continuous in $[0,3]$

(2) Diff wrt x

$$f'(x) = 3x^2 - 12x + 9$$

Clearly $f'(x)$ exists for all values of $x \in (0,3)$
$\therefore f(x)$ is differentiable in $(0,3)$

(3) $f(0) = 0 \quad ; \; f(3) = 0$

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$f(0) = f(3)$

$\therefore$ the three conditions of Rolle's theorem are satisfied there there exists atleast one value $c \in (0,3)$ such that

$$f'(c) = 0$$
$$\Rightarrow 3c^2 - 12c + 9 = 0$$
$$\Rightarrow c^2 - 4c + 3 = 0$$
$$\Rightarrow (c-3)(c-1) = 0$$
$$\Rightarrow c = 3, \quad c = 1$$

Clearly $c = 1 \in (0,3)$; $c = 3 \notin (0,3)$

$\therefore$ Rolle's theorem is verified Ans

$f(x) = x(x-3)^2$

$f(1) = 1(1-3)^2 = 4$

$\therefore (1,4)$ is the point on the curve where tangent is parallel to X-axis.

Q.N-2 → Verify Rolle's theorem

$$f(x) = \sin x + \cos x - 1 \quad ; \quad x \in [0, \pi/2]$$

Sol: (1) Since sine function, cosine function, constant function are everywhere continuous.
also, addition and subtraction of two continuous functions is also continuous.
$\therefore f(x)$ is continuous in $[0, \frac{\pi}{2}]$

(2) $f'(x) = \cos x - \sin x$

Clearly $f'(x)$ exists for all $x \in (0, \frac{\pi}{2})$

$\therefore f(x)$ is differentiable in $(0, \pi/2)$

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(3) $f(0) = \sin(0) + \cos(0) - 1 = 0 + 1 - 1 = 0$

$f(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2}) - 1 = 1 + 0 - 1 = 0$

$f(0) = f(\pi/2)$

the three conditions are satisfied then there exists a value $c \in (0, \frac{\pi}{2})$ such that

$f'(c) = 0$

$\cos c - \sin c = 0$

$\cos c = \sin c$

$\Rightarrow \tan c = 1$

$c = \frac{\pi}{4} \in (0, \frac{\pi}{2})$; $c = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \notin (0, \frac{\pi}{2})$

$\therefore$ Rolle's theorem is verified Ans

QN: 3 → Verify LMV for $f(x) = 2\sin x + \sin(2x)$ on $[0, \pi]$

Sol (1) Since sine function is everywhere continuous and addition of two continuous function is also continuous

$\therefore f(x)$ is continuous in $[0, \pi]$

(2) $f'(x) = 2\cos x + 2\cos(2x)$

Clearly $f'(x)$ exists for all values of $x \in (0, \pi)$

$\therefore f(x)$ is differentiable in $(0, \pi)$

the two conditions of LMV get satisfied, then there exists atleast one value $c \in (0, \pi)$

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such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$2\cos c + 2\cos(2c) \cancel{\cos c - \sin c} = \frac{f(\pi)-f(0)}{\pi - 0}$$

$$\cancel{\cos c - \sin c} =$$

$$2\cos c + 2\cos(2c) = \frac{(2\sin\pi + \sin(2\pi)) - (2\sin 0 + \sin 0)}{\pi}$$

$$2\cos c + 2\cos(2c) = 0$$

$$\Rightarrow \cos c + \cos(2c) = 0$$

$$\Rightarrow \cos c + 2\cos^2 c - 1 = 0$$

$$\Rightarrow 2\cos^2 c + \cos c - 1 = 0$$

$$\Rightarrow 2\cos^2 c + 2\cos c - \cos c - 1 = 0$$

$$\Rightarrow 2\cos c(\cos x + 1) - 1(\cos c + 1) = 0$$

$$\Rightarrow (2\cos c - 1)(\cos c + 1) = 0$$

$$\Rightarrow \cos c = \frac{1}{2} \quad \Big| \quad \cos c = -1$$

$$c = \frac{\pi}{3} \in (0,\pi) \quad \Big| \quad c = \pi \notin (0,\pi)$$

$\therefore$ LMV is satisfied Ans

Qn 4 → $f(x) = 3 + (x-2)^{2/3}$; $x \in [1,3]$

Check the applicability of Rolle's theorem

Sol

$$f(1) = 3 + (1-2)^{2/3} = 3 + (-1)^{2/3} = 3 + 1 = 4$$

$$f(3) = 3 + (3-2)^{2/3} = 3 + 1 = 4$$

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$f(x) = \cancel{3+(x-2)}\ 3+(x-2)^{2/3}$

$f'(x) = \frac{2}{3}(x-2)^{-1/3}$

$f'(x) = \frac{2}{3(x-2)^{1/3}}$

$f'(x)$ does not exists when $x=2 \in (1,3)$

$f(x)$ is not differentiable in $(1,3)$

$\therefore$ R.T is not applicable

Qn 5 → If $f:[-5,5] \to R$ is differentiable and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$

Sol. given $a=-5$, $b=5$

given $f(x)$ is differentiable

$\therefore$ $f(x)$ must be continuous

$\therefore$ the two condition of LMV are satisfied, then there exists a value $c \in (-5,5)$ such that

$$f'(c) = \frac{f(5)-f(-5)}{5-(-5)} \quad \text{--- (1)}$$

given that $f'(c) \neq 0$ --- (2)

from (1) & (2)

$\Rightarrow$

$$\frac{f(5)-f(-5)}{10} \neq 0$$

$\Rightarrow f(5)-f(-5) \neq 0 \Rightarrow f(5) \neq f(-5)$

Ans

WORKSHEET NO: 7

Differentiation & continuity

Qn 1 → Show that the function $f(x) = x - [x]$ is discontinuous at all the Integral points.

Qn 2 → If $f(x) = \dfrac{\tan(\frac{\pi}{4} - x)}{\cot(2x)}$, find the value which can be assigned to $f(x)$ at $x = \pi/4$ so that the function $f(x)$ becomes continuous everywhere in $[0, \pi/2]$

ANS $= \dfrac{1}{2}$ HINT take $\lim_{x \to \frac{\pi}{4}} f(x) = f(\frac{\pi}{4})$

Qn 3 → Determine $f(0)$ so that the function $f(x)$ defined by
$$f(x) = \frac{(4^x - 1)^3}{\sin(\frac{x}{4}) \log(1 + \frac{x^2}{3})}$$
becomes continuous at $x = 0$

ANS $f(0) = 12(\log 4)^3$ HINT take $\lim_{x \to 0} (f(x)) = f(0)$

Qn 4 → $f(x) = \begin{cases} x - 1 ; & x < 2 \\ 2x - 3 ; & x \geq 2 \end{cases}$ Show that $f(x)$ is not differentiable at $x = 2$

Qn 5 → $f(x) = x^2$, Show that $f(x)$ is differentiable at $x = 1$

Qn 6 → $f(x) = x|x|$. Show that $f(x)$ is differentiable at $x = 0$

Worksheet No: 7 (2)

Ques 7 → Discuss the differentiability of the function $f(x) = |x-1| - |x-2|$ at $x=1$ & $x=2$

Ques 8 → $f(x) = \begin{cases} x^2 & ; \; x \le 2 \\ ax+b & ; \; x > 2 \end{cases}$

find value of a and b so that $f(x)$ is differentiable at $x=2$

Ans $a=4$; $b=-4$ HINT $f(x)$ is also cont. at $x=2$

THEOREMS :-

Qn 9 $f(x) = \sin^2 x$; $0 \le x \le \pi$

verify Rolle's theorem

Ans $c = \pi/2$

Qn 10 → $f(x) = x^3 - 6x^2 + 11x - 6$ verify Rolle's theorem in the Interval $[1,3]$

Ans $c = 2 \pm \frac{1}{\sqrt{3}}$

Ques 11 → $f(x) = e^x(\sin x - \cos x)$; on $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

verify Rolle's theorem

Ans $c = \pi$

Qn 12 → $f(x) = x(x+3)e^{-x/2}$; on $x \in [-3, 0]$

verify Rolle's theorem

Ans $c = -2$

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Worksheet No: 7 3

Qn. 13 → $f(x) = \sin^4 x + \cos^4 x$; $x \in \left[0, \frac{\pi}{2}\right]$

verify Rolle's theorem

Ans $c = \pi/4$

Qn. 14 → Find the points on the curve $f(x) = 12(x+1)(x-2)$ on $[-1, 2]$ where tangent is parallel to x-axis

Ans $\left(\frac{1}{2}, -27\right)$

Qn. 15 → verify LMV for the $f(x) = x(x-1)(x-2)$ on $\left[0, \frac{1}{2}\right]$

Ans $c = 1 - \frac{\sqrt{21}}{6}$

Qn. 16 → find the points on the curve $y = (x-3)^2$ where the tangent is parallel to the chord joining ~~(1, 2) and (2, 2)~~ $(3, 0)$ and $(4, 1)$

Ans $\left(\frac{7}{2}, \frac{1}{4}\right)$

Qn. 17 → It is given that for the function $f(x) = x^3 + bx^2 + ax$; $x \in [1, 3]$

Rolle's theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. Find the values of a & b

Ans $a = 11$, $b = -6$

— x —