॥ जय श्री राधे कृष्णा। जय श्री गिरिराज जी महाराज ॥

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER: INTEGRATION : CLASS NO: 14

(DEFINITE INTEGRALS)

PROPERTIES of Definite Integrals:

(1) $$\boxed{\int_a^b f(x)\,dx = \int_a^b f(t)\,dt}$$

eg $$\int_0^{\pi/2} \sin x\,dx = -(\cos x)_0^{\pi/2} = -(0-1) = 1$$

$$\int_0^{\pi/2} \sin t\,dt = -(\cos t)_0^{\pi/2} = -(0-1) = 1$$

(2) $$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

eg $$\int_a^b f(x)\,dx = (\phi(x))_a^b = \phi(b) - \phi(a)$$

$$-\int_b^a f(x)\,dx = -(\phi(x))_b^a = -(\phi(a) - \phi(b)) = \phi(b) - \phi(a)$$

(3) $$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

RHS $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = (\phi(x))_a^c + (\phi(x))_c^b$$

$$= \cancel{\phi(c)} - \phi(a) + \phi(b) - \cancel{\phi(c)}$$

$$= \phi(b) - \phi(a) = \int_a^b f(x)\,dx$$

4 $$\boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$$

Proof RHS $\int_0^a f(a-x)\,dx$

put $a-x=t$ | $x=0\,;\ t=a$

$-dx = dt \Rightarrow dx = -dt$ | $x=a\,;\ t=0$

$$\therefore \text{RHS} = -\int_a^0 f(t)\,dt$$

$$= \int_0^a f(t)\,dt \quad \cdots (P\ II)$$

$$= \int_0^a f(x)\,dx \quad \cdots (P-I)$$

$= \text{LHS}$ **proved**

5 $$\boxed{\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx}$$

Hint RHS put $a+b-x=t$

6 $$\boxed{\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; f(2a-x) = f(x) \\ 0 & ; f(2a-x) = -f(x) \end{cases}}$$

Proof LHS $$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \quad \cdots (P-II)$$

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put $x = 2a - t$ in 2^{nd} Integral

$dx = -dt$

when $x = a$; $t = a$

when $x = 2a$; $t = 0$

$$\therefore \int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_a^0 f(2a-t)\,dt$$

$$= \int_0^a f(x)dx + \int_0^a f(2a-t)dt \quad \text{-- } (P\text{-}II)$$

$$= \int_0^a f(x)dx + \int_0^a f(2a-x)dx \quad \text{--- } (P\ I)$$

$$\int_0^{2a} f(x)dx = \int_0^a \left(f(x) + f(2a-x)\right)dx$$

$$\int_0^{2a} f(x)dx = \begin{cases} \int_0^a (f(x) + f(x))\,dx & \text{; when } f(2a-x) = f(x) \\ \int_0^a (f(x) - f(x))dx & \text{when } f(2a-x) = -f(x) \end{cases}$$

$$= \begin{cases} 2\int_0^a f(x)dx \ ; & f(2a-x) = f(x) \\ 0 \ ; & f(2a-x) = -f(x) \end{cases}$$

(P-XII) even-odd function   proof

$$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{; when } f(x) \text{ is an even function } f(-x) = f(x) \\ 0 & \text{; when } f(x) \to \text{odd func. } f(-x) = -f(x) \end{cases}$$

Qn: 1 Evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ --- (b)

Sol: let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ ---- (1)

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)}+\sqrt{\cos(\frac{\pi}{2}-x)}}\,dx \quad --- \left\{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right\}$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad --- (2)$$

(1) + (2)

$$2I = \int_0^{\pi/2} \frac{\cancel{\sqrt{\sin x}+\sqrt{\cos x}}}{\cancel{\sqrt{\sin x}+\sqrt{\cos x}}}\,dx$$

$$= \int_0^{\pi/2} 1\cdot dx$$

$$= (x)_0^{\pi/2}$$

$$2I = \frac{\pi}{2}$$

$$\boxed{I = \pi/4} \text{ Ans}$$

Qn: 2 → $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$

Qn: 3 → $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$

Sol: $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$ -- (1)

$$I = \int_0^{2\pi} \frac{1}{1+e^{\sin(2\pi-x)}}\,dx \quad --- (P\ \text{IV})$$

$$I = \int_0^{2\pi} \frac{1}{1+e^{-\sin x}}\,dx$$

$$I = \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x}+1}\,dx \quad -- (2)$$

(1) + (2)

$$2I = \int_0^{2\pi} \frac{\cancel{1+e^{\sin x}}}{\cancel{1+e^{\sin x}}}\,dx$$

$$2I = (x)_0^{2\pi}$$

$$2I = 2\pi - 0$$

$$\boxed{I = \pi} \text{ Ans}$$

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Que 4 → $\int_0^{\pi/2} \log(\tan x)\,dx$

Sol. let $I = \int_0^{\pi/2} \log(\tan x)\,dx$ --- (1)

$I = \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx$ -- (P IV)

$I = \int_0^{\pi/2} \log(\cot x)\,dx$ -- (2)

(1) + (2)

$2I = \int_0^{\pi/2} \log(\tan x) + \log(\cot x)\,dx$

$2I = \int_0^{\pi/2} \log(\tan x \cdot \cot x)\,dx$

$2I = \int_0^{\pi/2} \log(1)\,dx$

$2I = \int_0^{\pi/2} 0 \cdot dx$

$2I = 0$

$\boxed{I = 0}$ Ans

Que 5 $\int_0^{\pi/4} \log(1+\tan x)\,dx$

Soln let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$ ... (1)

$I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$ -- (P IV)

$I = \int_0^{\pi/4} \log\left(1 + \frac{1-\tan x}{1+\tan x}\right)dx$

6

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right) dx \quad \text{--- (2)}$$

(1) + (2)

$$2I = \int_0^{\pi/4} \log\left(\cancel{(1+\tan x)} \times \frac{2}{\cancel{1+\tan x}}\right) dx$$

$$2I = \int_0^{\pi/4} \log(2)\, dx$$

$$= \left(x \log 2\right)_0^{\pi/4}$$

$$2I = \frac{\pi}{4}\log 2 - 0$$

$$\boxed{I = \frac{\pi}{8}\log 2} \text{ Ans}$$

Q.No:6 →
$$\int_0^{\pi/2} \left(2\log(\sin x) - \log(\sin(2x))\right) dx$$

Soln:
$$I = \int_0^{\pi/2} 2\log(\sin x) - \log(\sin(2x))\, dx$$

$$I = \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{\sin(2x)}\right) dx$$

$$I = \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{2\cancel{\sin x}\cos x}\right) dx$$

$$I = \int_0^{\pi/2} \log\left(\frac{\tan x}{2}\right) dx \quad \text{--- (1)}$$

$$I = \int_0^{\pi/2} \log\left(\frac{\tan(\frac{\pi}{2}-x)}{2}\right) dx \quad \text{--- (P IV)}$$

$$I = \int_0^{\pi/2} \log\left(\frac{\cot x}{2}\right) dx \quad \cdots (2)$$

(1) + (2)

$$2I = \int_0^{\pi/2} \log\left(\frac{\tan x}{2} \times \frac{\cot x}{2}\right) dx$$

$$2I = \int_0^{\pi/2} \log\left(\frac{1}{4}\right) dx$$

$$2I = -\int_0^{\pi/2} \log(4)\, dx$$

$$2I = -\left(x \log 4\right)_0^{\pi/2}$$

$$2I = -\left(\frac{\pi}{2} \log 4 - 0\right)$$

$$\boxed{I = -\frac{\pi}{4} \log 4} \quad \text{Ans}$$

(or) $I = -\frac{\pi}{4} \times 2 \log 2$

$$I = -\frac{\pi}{2} \log 2 \quad \underline{\text{Ans}}$$

Ques. 7 → $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$

Sol $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx \quad - (1)$

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx \quad - (2)$$

(1) + (2)

$$2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

$$2I = \int_0^{\pi/2} \frac{1}{\frac{2\tan(x/2)}{1+\tan^2(x/2)} + \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}\, dx$$

$$2I = \int_0^{\pi/2} \frac{\sec^2(x/2)\, dx}{2\tan(x/2) + 1 - \tan^2(x/2)}$$

Put $\tan\left(\frac{x}{2}\right) = t$

$\frac{1}{2}\sec^2(x/2)\,dx = dt$

$\sec^2(x/2)\,dx = 2dt$

$x = 0 \;;\; t = 0$

$x = \pi/2 \;;\; t = 1$

$$\therefore \cancel{2}I = \cancel{2}\int_0^1 \frac{dt}{2t+1-t^2}$$

$$I = -\int_0^1 \frac{1}{t^2-2t-1}\,dt$$

$$I = -\int_0^1 \frac{1}{(t-1)^2 - (\sqrt{2})^2}$$

$$I = \int_0^1 \frac{1}{(\sqrt{2})^2 - (t-1)^2}\,dt$$

$$I = \frac{1}{2\sqrt{2}}\left(\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right)_0^1$$

$$I = \frac{1}{2\sqrt{2}}\left[\log|1| - \log\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right]$$

$$I = -\frac{1}{2\sqrt{2}}\log\left|\frac{(\sqrt{2}-1)^2}{2-1}\right|$$

$$I = -\frac{1}{\cancel{2}\sqrt{2}} \times \cancel{2}\log|\sqrt{2}-1|$$

$$I = -\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)$$

Removal of x

Q.No. 8 → $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$ ---(1)

$$I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx \quad \text{--- P IV}$$

$$I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx \quad \text{--- (2)}$$

(1) + (2)

$$2I = \int_0^{\pi} \frac{\cancel{x\sin x} + \pi \sin x - \cancel{x \sin x}}{1+\cos^2 x}\,dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

put $\cos x = t$ | $x = 0$; $t = 1$

$\sin x\,dx = -dt$ | $x = \pi$; $t = -1$

$$\therefore \quad 2I = -\pi \int_1^{-1} \frac{dt}{1+t^2}$$

$$2I = -\pi \left(\tan^{-1} t\right)_1^{-1}$$

$$= -\pi\left(\tan^{-1}(-1) - \tan^{-1}(1)\right)$$

$$= -\pi\left[-\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$2I = -\pi\left(-\pi/2\right)$$

$$\boxed{I = \frac{\pi^2}{4}} \quad \text{Ans}$$

Qu-9 → $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$

$$I = \int_0^{\pi} \frac{x \frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$I = \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx \quad \ldots (1)$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)} dx \ldots \quad P_{IV}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x} dx \quad \ldots (2)$$

(1)+(2)

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx$$

Rationalizing

$$2I = \pi \int_0^{\pi} \frac{\sin x (1-\sin x)}{\cos^2 x} dx$$

$$2I = \pi \int_0^{\pi} \tan x \sec x - \tan^2 x \, dx$$

$$2I = \pi \int_0^{\pi} \tan x \sec x - \sec^2 x + 1 \, dx$$

$$2I = \pi [\sec x - \tan x + x]_0^{\pi}$$

$$2I = \pi [(-1 + 0 + \pi) - (1 - 0 + 0)]$$

$$2I = \pi(\pi - 2) \Rightarrow \boxed{I = \frac{\pi}{2}(\pi - 2)} \text{ Ans}$$

WORKSHEET No: 11 (Class No: 14)

(Definite Integrals)

Qn: 1 $\displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cdot \cos x}\,dx$ $\boxed{\text{Ans} = 0}$

Qn: 2 $\displaystyle\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx$ $\boxed{\text{Ans} = \pi/2}$

Qn: 3 $\displaystyle\int_0^{\pi/2} \sin(2x)\cdot\log(\cot x)\,dx$ $\boxed{\text{Ans} = 0}$

Qn: 4 $\displaystyle\int_0^{1} \log\left(\frac{1}{x} - 1\right)dx$ $\boxed{\text{Ans} = 0}$

Qn: 5 → $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x\cdot\cos x}\,dx$ $\boxed{\text{Ans.} = \frac{\pi}{3\sqrt{3}}}$

Qn: 6 → $\displaystyle\int_0^{\pi} \frac{x}{1+\sin x}\,dx$ $\boxed{\text{Ans} = \pi}$

Qn: 7 → $\displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$ $\boxed{\text{Ans} \; -\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}-1)}$

Qn: 8 → $\displaystyle\int_0^{\pi} \frac{x\tan x}{\sec x\cdot\operatorname{cosec} x}\,dx$ $\boxed{\text{Ans} \; \frac{\pi^2}{4}}$

Qn: 9 → $\displaystyle\int_0^{\pi/2} 2\log(\cos x) - \log(\sin(2x))\,dx$ $\boxed{\text{Ans} = -\frac{\pi}{2}\log 2}$

Qn: 10 → $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}}\,dx$ $\boxed{\text{Ans} = \pi/4}$

Qn: 11 $\displaystyle\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx$ $\boxed{\text{Ans} = \pi/4}$

Qn: 12 → $\displaystyle\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx$ $\boxed{\text{Ans} \; \frac{\pi^2}{16}}$