॥ जय श्री गिरिराज जी महाराज ॥

D & C → ULTIMATE MATHEMATICS →

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$\boxed{\text{TOPIC: } 5}$ $\boxed{\text{HIGHER ORDER DERIVATIVES}}$

$y = f(x)$

Diff $\frac{dy}{dx} \rightarrow \frac{d^2y}{dx^2} \rightarrow \frac{d^3y}{dx^3}$

$f(x) \rightarrow f'(x) \rightarrow f''(x) \rightarrow f'''(x)$

$y \rightarrow y_1 \rightarrow y_2 \rightarrow y_3$

Que 1 given $y = \tan^{-1}x$

Show $(1+x^2) \cdot \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

Method I (Direct) LCM

$y = \tan^{-1}x \quad \text{----(1)}$

Diff. wrt x

$$\frac{dy}{dx} = \frac{1}{1+x^2} \quad \text{--(2)}$$

Diff again wrt x

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)\cdot(0) - 1(2x)}{(1+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-2x}{(1+x^2)^2} \quad \text{--(3)}$$

taking LHS $(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}$

$$= \cancel{(1+x^2)}\left(\frac{-2x}{(1+x^2)^{\cancel{2}}}\right) + 2x \cdot \frac{1}{1+x^2}$$

$$= \frac{-2\cancel{x}}{1+x^2} + \frac{2\cancel{x}}{1+x^2} = 0 = RHS \quad \text{proved}$$

Method II

$y = \tan^{-1} x$ --- (1)

Show $(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)\frac{dy}{dx} = 1$$

Diff w.r.t x

$$(1+x^2)\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}(2x) = 0$$

Qn. 2 $y = A\cos(\log x) + B\sin(\log x)$

Show $x^2 \cdot y_2 + x\cdot\frac{dy}{dx} + y = 0$

Sol $y = A\cos(\log x) + B\sin(\log x)$ ---(i)

Diff w.r.t x

$$\frac{dy}{dx} = -A\frac{\sin(\log x)}{x} + B\cdot\frac{\cos(\log x)}{x}$$

$$\frac{dy}{dx} = \frac{-A\sin(\log x) + B\cos(\log x)}{x} \text{ ---(2)}$$

$$\Rightarrow x\frac{dy}{dx} = -A\sin(\log x) + B\cos(\log x)$$

Diff w.r.t x

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{A\cos(\log x)}{x} - \frac{B\sin(\log x)}{x}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -A\cos(\log x) - B\sin(\log x)$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\left(A\cos(\log x) + B\sin(\log x)\right)$$

$$\Rightarrow x^2 y_2 + x y_1 = -y \quad \ldots \{\text{from eq (1)}\}$$

$$\Rightarrow x^2 y_2 + x y_1 + y = 0 \quad \underline{\text{Proved}}$$

Q. 3 → $y = \left[\log\left(x+\sqrt{x^2+1}\right)\right]^2$ show that

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2$$

Sol. $y = \left[\log\left(x+\sqrt{x^2+1}\right)\right]^2 \quad \ldots (1)$

Diff. w.r.t x

$$\frac{dy}{dx} = 2\log\left(x+\sqrt{x^2+1}\right) \cdot \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\right)$$

$$\frac{dy}{dx} = 2\log\left(x+\sqrt{x^2+1}\right) \cdot \frac{1}{\cancel{x+\sqrt{x^2+1}}} \cdot \left(\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right)$$

$$\frac{dy}{dx} = \frac{2\log\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$$

$$\Rightarrow \sqrt{x^2+1}\cdot\frac{dy}{dx} = 2\log\left(x+\sqrt{x^2+1}\right)$$

Diff. again w.r.t x

$$\sqrt{x^2+1}\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{1}{\cancel{2}\sqrt{x^2+1}}\cdot(\cancel{2}x) = 2\cdot\frac{1}{x+\sqrt{x^2+1}}\cdot\left(1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\right.$$

$$\sqrt{x^2+1}\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{x}{\sqrt{x^2+1}} = 2\cdot\frac{1}{\ldots}\cdot\left(\cancel{\sqrt{x^2+1}+x}\right.$$

$x+\sqrt{x^2+1}$ $\left(\sqrt{x^2+1}\right.$

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$$\sqrt{x^2+1}\cdot\frac{d^2y}{dx^2}+\frac{x}{\sqrt{x^2+1}}\cdot\frac{dy}{dx}=\frac{2}{\sqrt{x^2+1}}$$

$$\Rightarrow (x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=2 \quad \text{Ans}$$

Q.4 $y=e^{a\cos^{-1}x}$

Show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

Sol: $y=e^{a\cos^{-1}x}$ ---(1)

taking log

$\log y=a\cos^{-1}x\cdot\log e$

$\log y=a\cos^{-1}x$ --- $\{\log e=1\}$

Diff. w.r.t x

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{-a}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx}=-ay \quad \text{--(2)}$$

Diff again w.r.t x

$$\sqrt{1-x^2}\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)=-a\frac{dy}{dx}$$

$$(1-x^2)\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-a\frac{dy}{dx}\cdot\sqrt{1-x^2}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-a(-ay) \quad \text{--[from (2)]}$$

$\Rightarrow (1-x^2)\cdot y_2 - xy_1 - a^2y = 0$ Ans

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OR last question.

$x = \cos\left(\frac{1}{a}\log y\right)$

$\Rightarrow \cos^{-1}x = \frac{1}{a}\log y$

$\Rightarrow a\cos^{-1}x = \log y$

$\Rightarrow \log y = a\cos^{-1}x$

$\Rightarrow y = e^{a\cos^{-1}x}$

Qn. 5 $y = ae^{2x} + be^{-x}$

Show $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

Sol $y = ae^{2x} + be^{-x}$ ---(1)

Diff $\frac{dy}{dx} = ae^{2x}\cdot 2 + be^{-x}\cdot(-1)$

$\frac{dy}{dx} = 2ae^{2x} - be^{-x}$ --(2)

Diff again

$\frac{d^2y}{dx^2} = 2ae^{2x}\cdot(2) - be^{-x}(-1)$

$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$ --(3)

LHS $y_2 - y_1 - 2y$

$= \cancel{4ae^{2x}} + \cancel{be^{-x}} - \cancel{2ae^{2x}} + be^{-x} - 2ae^{2x}$

$= 0$

DEC class 5

Q.no. 6 $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

show $\dfrac{d^2y}{dx^2} = \dfrac{\sec^3\theta}{a\theta}$

Sol Diff w.r.t. θ

$$\frac{dx}{d\theta} = a(-\cancel{\sin\theta} + \theta\cos\theta + \cancel{\sin\theta})$$

$$\frac{dx}{d\theta} = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\{\cancel{\cos\theta} - (-\theta\sin\theta + \cancel{\cos\theta})\}$$

$$\frac{dy}{d\theta} = a\theta\sin\theta$$

Now $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\cancel{a\theta}\sin\theta}{\cancel{a\theta}\cos\theta}$

$$\frac{dy}{dx} = \tan\theta$$

Diff w.r.t. x

$$\frac{d^2y}{dx^2} = \sec^2\theta \left(\frac{d\theta}{dx}\right)$$ → Main step → reciprocal of $\dfrac{dx}{d\theta}$

$$= \sec^2\theta \cdot \frac{1}{a\theta\cos\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta} \quad \underline{\text{Ans}}$$

Q. No 7 If $x = \sin t$, $y = \sin(pt)$

Show $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$

Sol. Diff w.r.t t

$$\frac{dx}{dt} = \cos t \quad \Big| \quad \frac{dy}{dt} = \cos(pt) \cdot p$$

$$\frac{dy}{dx} = \frac{p\cos(pt)}{\cos t}$$

$$\Rightarrow \cos t \cdot \frac{dy}{dx} = p\cos(pt)$$

Diff w.r.t x

$$\Rightarrow \cos t \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot (-\sin t)\frac{dt}{dx} = -p\sin(pt) \cdot p\frac{dt}{dx}$$

$$\Rightarrow \cos t \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx}(-\sin t) \cdot \frac{1}{\cos t} = -\frac{p^2\sin(pt)}{\cos t}$$

$$\Rightarrow \cos^2 t\,\frac{d^2y}{dx^2} - \sin t \cdot \frac{dy}{dx} = -p^2\sin(pt)$$

$$\Rightarrow (1-\sin^2 t)\frac{d^2y}{dx^2} - \sin t \cdot \frac{dy}{dx} = -p^2\sin(pt)$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -p^2y$$

$\Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2y = 0$ Ans

CLASS 5

DEC

Q.8 → $y = \tan x + \sec x$

show $\dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1-\sin x)^2}$

Sol $y = \tan x + \sec x$

$\dfrac{dy}{dx} = \sec^2 x + \sec x \tan x$

$\dfrac{dy}{dx} = \dfrac{1}{\cos^2 x} + \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x}$

$\dfrac{dy}{dx} = \dfrac{1+\sin x}{\cos^2 x}$

$\dfrac{dy}{dx} = \dfrac{1+\sin x}{1-\sin^2 x}$

$\dfrac{dy}{dx} = \dfrac{\cancel{1+\sin x}}{\cancel{(1+\sin x)}(1-\sin x)}$

$\dfrac{dy}{dx} = \dfrac{1}{1-\sin x}$

Diff again wrt x (Proceed)

Q.9 → $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

Show $(1-x^2)y_2 - 3xy_1 - y = 0$

Hint Show $(1-x^2)y_2 - xy_1 - y = 0$

$\sqrt{1-x^2} \cdot y = \sin^{-1} x$

Diff

D & C | WORKSHEET No: 4

ULTIMATE MATHEMATICS

WORKSHEET No: 4 (Class No: 5)

Topic: Higher order derivatives

Qns 1 $y = \sin^{-1} x$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$

Qn2 → $y = \log\left(x + \sqrt{x^2+a^2}\right)$

Show that $(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Qns 3 → $y = \left\{x + \sqrt{x^2+1}\right\}^m$

Show that $(x^2+1)y_2 + xy_1 - m^2y = 0$

QN4 → $x = \tan\left(\frac{1}{a}\log y\right)$

Show $(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$

Hint $y = e^{a\tan^{-1}x}$

Qn 5 → $y = 500e^{7x} + 600e^{-7x}$ Show that

$$\frac{d^2y}{dx^2} = 49y$$

Qn 6 → $y = 3e^{2x} + 2e^{3x}$

Show that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Q. 7 → $y = (\tan^{-1}x)^2$

Show that $(1+x^2)^2 \cdot y_2 + 2x(1+x^2)y_1 = 2$

D & C worksheet = 4 (class-5)

(2)

Q. 8 → $y = (\sin^{-1}x)^2$

Show that $(1-x^2)y_2 - xy_1 - 2 = 0$

Q. 9 → $y = \operatorname{cosec}^{-1}x$

Show that $x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$

Q. 10 → $y = 2\sin t - \sin(2t)$

$x = 2\cos t - \cos(2t)$

Show $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$ $= -3/2$

Q. 11 → $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$

Show that $(1-x^2)y_2 - 3xy_1 - y = 0$

Q. 12 → $y = 3\cos(\log x) + 4\sin(\log x)$

Show that $x^2y_2 + xy_1 + y = 0$

Q. 13 → $x = a(\theta + \sin\theta)$

$y = a(1+\cos\theta)$

Show $\frac{d^2y}{dx^2} = \frac{-a}{y^2}$

Q.No-14 → $y = e^x(\sin x + \cos x)$

Show $y_2 - 2y_1 + 2y = 0$

— x —