SOLUTIONS : VECTORS | WORKSHEET No: 3

Ans 1 Given $\vec{a}$, $\vec{b}$ & $\vec{a}+\vec{b}$ is a unit vector

$\Rightarrow |\vec{a}|=1$; $|\vec{b}|=1$ & $|\vec{a}+\vec{b}|=1$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 1$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = 1$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = 1$

$\Rightarrow 1 + 1 + 2(1)(1)\cos\theta = 1$

$\Rightarrow \cos\theta = -1/2$

$\Rightarrow \theta = \pi - \pi/3$

$\boxed{\theta = 2\pi/3}$ Ans

Ans. 2 → Given $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors

$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

Given $\vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0$

we have $|\vec{a}+\vec{b}+\vec{c}| = 1$

Squaring $\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2 = 1$

$\Rightarrow (\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c}) = 1$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a}.\vec{b} + 2\vec{b}.\vec{c} + 2\vec{c}.\vec{a} = 1$

$\Rightarrow 1 + 1 + 1 + 0 + 2|\vec{b}||\vec{c}|\cos\beta + 2|\vec{c}||\vec{a}|\cos\alpha = 1$

$\Rightarrow 3 + 2(1)(1)\cos\beta + 2(1)(1)\cos\alpha = 1$

$\Rightarrow 3 + 2\cos\beta + 2\cos\alpha = 1$

$\Rightarrow 2(\cos\alpha + \cos\beta) = -2$

$\Rightarrow \boxed{\cos\alpha + \cos\beta = -1}$ Ans

Ans 3 → let $\vec{a} = x\hat{i} + 3\hat{j} - 7\hat{k}$ & $\vec{b} = x\hat{i} - x\hat{j} + 4\hat{k}$

given angle b/w $\vec{a}$ & $\vec{b}$ is acute

$\Rightarrow \theta < 90^\circ$ (I^{st} quadrant)

$\Rightarrow \cos\theta > 0$

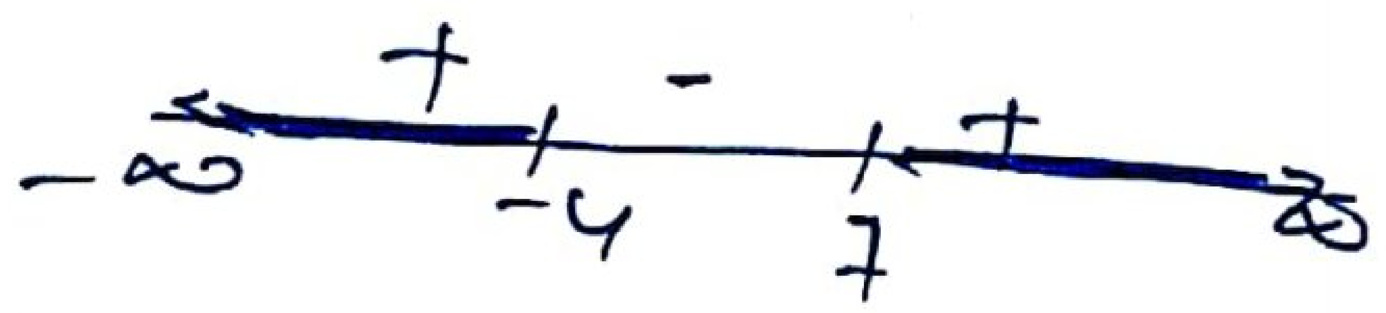
$\Rightarrow \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} > 0$

$\Rightarrow \vec{a}\cdot\vec{b} > 0$

$\Rightarrow (x\hat{i} + 3\hat{j} - 7\hat{k})\cdot(x\hat{i} - x\hat{j} + 4\hat{k}) > 0$

$\Rightarrow x^2 - 3x - 28 > 0$

$\Rightarrow (x-7)(x+4) > 0$



$\therefore \boxed{x \in (-\infty, -4) \cup (7, \infty)}$ Ans (Note: Misprint in worksheet Answer)

Ans 4 → Given $2\vec{a} + \vec{b} = \vec{p}$ $\times 1$

$\vec{a} + 2\vec{b} = \vec{q}$ $\times 2$

$\Rightarrow \quad 2\vec{a} + \vec{b} = \vec{p} \quad$ and $\quad 2\vec{a} + 4\vec{b} = 2\vec{q}$

$\Rightarrow \quad 2\vec{a} + \vec{b} - 2\vec{a} - 4\vec{b} = \vec{p} - 2\vec{q}$

$\Rightarrow \quad -3\vec{b} = \vec{p} - 2\vec{q}$

$\Rightarrow \quad \boxed{\vec{b} = \frac{1}{3}(2\vec{q} - \vec{p})}$ put in eq (i)

$2\vec{a} + \frac{1}{3}(2\vec{q} - \vec{p}) = \vec{p}$

$\Rightarrow \quad 2\vec{a} = \vec{p} - \frac{1}{3}(2\vec{q} - \vec{p})$

$\Rightarrow \quad 2\vec{a} = \dfrac{3\vec{p} - 2\vec{q} + \vec{p}}{3}$

$\Rightarrow \quad \boxed{\vec{a} = \dfrac{4\vec{p} - 2\vec{q}}{6}}$

$\Rightarrow \therefore \quad \vec{b} = \frac{1}{3}(2\hat{i} - 2\hat{j} - \hat{i} - \hat{j})$

$\boxed{\vec{b} = \frac{1}{3}(\hat{i} - 3\hat{j})} \Rightarrow |\vec{b}| = \sqrt{\frac{1}{9} + 1} = \frac{\sqrt{10}}{3}$

& $\vec{a} = \frac{1}{6}(4\hat{i} + 4\hat{j} - 2\hat{i} + 2\hat{j})$

$\vec{a} = \frac{1}{6}(2\hat{i} + 6\hat{j})$

$\boxed{\vec{a} = \frac{1}{3}(\hat{i} + 3\hat{j})} \Rightarrow |\vec{a}| = \sqrt{\frac{1}{9} + 1} = \frac{\sqrt{10}}{3}$

$\vec{a} \cdot \vec{b} = \frac{1}{9}(1 - 9) = -\frac{8}{9}$

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{-\frac{8}{9}}{\frac{\sqrt{10}}{3}\frac{\sqrt{10}}{3}} = \dfrac{-\frac{8}{9}}{\frac{10}{9}} = -\frac{4}{5} \quad \therefore$ C Ans

Qn. 5 → given $|\vec{a}+\vec{b}| = |\vec{b}|$

(Squaring) ⇒ $|\vec{a}+\vec{b}|^2 = |\vec{b}|^2$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{b}|^2$$

$$\Rightarrow |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} = 0 \quad \text{---(1)}$$

Now $(\vec{a}+2\vec{b})\cdot\vec{a}$

$$= \vec{a}\cdot\vec{a} + 2\vec{b}\cdot\vec{a}$$

$$= |\vec{a}|^2 + 2\vec{a}\cdot\vec{b}$$

$$= 0 \quad \text{-- \{from eq (i)\}}$$

$\Rightarrow (\vec{a}+2\vec{b}) \perp \vec{a}$ Proved

Qn. 5 → let $\vec{\beta} = 2\hat{i} - \hat{j} + 3\hat{k}$ & $\vec{\alpha} = 2\hat{i} + 4\hat{j} - 2\hat{k}$

let $\vec{\beta} = \vec{\beta_1} + \vec{\beta_2}$

given $\vec{\beta_1}$ $||^{l}$ to $\vec{\alpha}$

$$\Rightarrow \vec{\beta_1} = \lambda\vec{\alpha}$$

$$\Rightarrow \vec{\beta_1} = 2\lambda\hat{i} + 4\lambda\hat{j} - 2\lambda\hat{k}$$

we have $\vec{\beta} = \vec{\beta_1} + \vec{\beta_2}$

$$\Rightarrow \vec{\beta_2} = \vec{\beta} - \vec{\beta_1}$$

$$\Rightarrow \vec{\beta_2} = (2\hat{i} - \hat{j} + 3\hat{k}) - (2\lambda\hat{i} + 4\lambda\hat{j} - 2\lambda\hat{k})$$

$$\Rightarrow \vec{\beta_2} = \hat{i}(2-2\lambda) + \hat{j}(-1-4\lambda) + \hat{k}(3+2\lambda)$$

given $\vec{\beta_2} \perp \vec{\alpha}$

$\Rightarrow \vec{\beta_2} \cdot \vec{\alpha} = 0$

$\Rightarrow \left(\hat{i}(2-2\lambda) + \hat{j}(-1-4\lambda) + \hat{k}(3+2\lambda)\right) \cdot (2\hat{i} + 4\hat{j} - 2\hat{k}) = 0$

$\Rightarrow 4 - 4\lambda - 4 - 16\lambda - 6 - 4\lambda = 0$

$\Rightarrow -6 - 24\lambda = 0$

$\Rightarrow \boxed{\lambda = \frac{-1}{4}}$

$\therefore \vec{\beta_1} = -\frac{1}{2}\hat{i} - \hat{j} + \frac{1}{2}\hat{k}$ & $\vec{\beta_2} = \frac{5}{2}\hat{i} + \frac{5}{2}\hat{k}$

Now $\vec{\beta_1} + \vec{\beta_2} = \left(-\frac{1}{2}\hat{i} - \hat{j} + \frac{1}{2}\hat{k}\right) + \left(\frac{5}{2}\hat{i} + \frac{5}{2}\hat{k}\right)$

$= 2\hat{i} - \hat{j} + 3\hat{k}$

$= \vec{\beta}$ verified Ans

Qn. 7 →

given $|\vec{a}| = 1$ & $|\vec{b}| = 1$

given $|\vec{a} + \vec{b}| = \sqrt{3}$

$\Rightarrow |\vec{a} + \vec{b}|^2 = 3$

$\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = 3$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = 3$

$\Rightarrow 1 + 1 + 2\vec{a} \cdot \vec{b} = 3$

$\Rightarrow \vec{a} \cdot \vec{b} = \frac{1}{2}$

Now $(2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b})$

$= (2\vec{a} \cdot 3\vec{a}) + 2\vec{a} \cdot \vec{b} - (5\vec{b} \cdot 3\vec{a}) - 5\vec{b} \cdot \vec{b}$

$= 6|\vec{a}|^2 - 13\vec{a} \cdot \vec{b} - 5|\vec{b}|^2$

$= 6(1) - 13\left(\frac{1}{2}\right) - 5(1)$

$= 6 - \frac{13}{2} - 5$

$= \frac{-11}{2}$ Ans

Qn. 8 → Given $|\vec{a} + \vec{b}| = 60$ | $|\vec{a} - \vec{b}| = 40$

$\Rightarrow |\vec{a} + \vec{b}|^2 = 3600$ | $\Rightarrow |\vec{a} - \vec{b}|^2 = 1600$

adding these equations

$|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 3600 + 1600$

$\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) + (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = 5200$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + \cancel{2\vec{a} \cdot \vec{b}} + |\vec{a}|^2 + |\vec{b}|^2 - \cancel{2\vec{a} \cdot \vec{b}} = 5200$

$\Rightarrow 2(|\vec{a}|^2 + |\vec{b}|^2) = 5200$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 = 2600$

$\Rightarrow |\vec{a}|^2 + (46)^2 = 2600$

$\Rightarrow |\vec{a}|^2 = 2600 - 2116 = 484$

$\Rightarrow \boxed{|\vec{a}| = 22}$ Ans

Q.N. 9 → let given vectors are

$\vec{a} = \hat{i} + \hat{j} - 3\hat{k}$; $\vec{b} = \hat{i} + 3\hat{j} - 2\hat{k}$; $\vec{c} = 2\hat{i} + \hat{j} + 4\hat{k}$

let Required vector is $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

given $\vec{d} \cdot \vec{a} = 0$; $\vec{d} \cdot \vec{b} = 5$ & $\vec{d} \cdot \vec{c} = 8$

$\Rightarrow$ $x + y - 3z = 0 \Rightarrow y = 3z - x$

$x + 3y - 2z = 5$

$2x + y + 4z = 8$

$\Rightarrow$ $x + 9z - 3x - 2z = 5 \Rightarrow -2x + 7z = 5 \quad \times 1$

and $2x + 3z - x + 4z = 8 \Rightarrow x + 7z = 8 \quad \times 2$

$$\Rightarrow \begin{array}{r} -\cancel{2x} + 7z = 5 \\ \cancel{2x} + 14z = 16 \\ \hline 21z = 21 \end{array}$$

$\Rightarrow$ $z = 1$ $\quad x = 1$ $\quad y = 2$

$\therefore$ $\boxed{\vec{d} = \hat{i} + 2\hat{j} + \hat{k}}$ Ans

Q.N. 10 → let $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$

$\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$

let θ be the angle b/w them

given θ is obtuse

$\Rightarrow$ $\theta > 90°$ (2^{nd} quadrant)

$\Rightarrow \cos\theta < 0$ { In 2^{n} quad $\cos\theta$ is $-ve$ }

$\Rightarrow \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} < 0$

$\Rightarrow \vec{a}\cdot\vec{b} < 0$

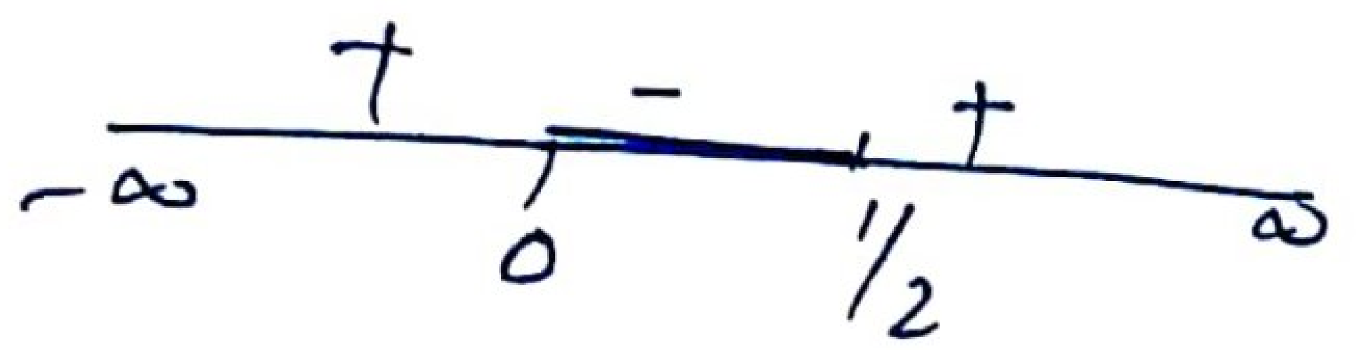
$\Rightarrow (2x^2\hat{i} + 4x\hat{j} + \hat{k})\cdot(7\hat{i} - 2\hat{j} + x\hat{k}) < 0$

$\Rightarrow 14x^2 - 8x + x < 0$

$\Rightarrow 14x^2 - 7x < 0$

$\Rightarrow 7x(2x-1) < 0$

$\Rightarrow x(2x-1) < 0$



$\therefore \boxed{x \in (0, 1/2)}$ Ans

Qn. 11 →

Given Side vectors of a triangle
(Note : not position vectors of vertices)

Let $\vec{AB} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{BC} = \hat{i} - 3\hat{j} - 5\hat{k}$

$\vec{CA} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

Now $\vec{AB}\cdot\vec{BC} = (2\hat{i} - \hat{j} + \hat{k})\cdot(\hat{i} - 3\hat{j} - 5\hat{k})$

$= 2 + 3 - 5 = 0$

$\therefore \vec{AB} \perp \vec{BC}$

$\therefore$ ABC is a Right angle triangle Ans

Qn 12 →

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$$

$$\Leftrightarrow \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} = |\vec{a}|^2 + |\vec{b}|^2$$

$$\Leftrightarrow \cancel{|\vec{a}|^2} + 2\vec{a}\cdot\vec{b} + \cancel{|\vec{b}|^2} = \cancel{|\vec{a}|} + \cancel{|\vec{b}|^2}$$

$$\Leftrightarrow 2\vec{a}\cdot\vec{b} = 0$$

$$\Leftrightarrow \vec{a}\cdot\vec{b} = 0$$

$$\Leftrightarrow \vec{a} \perp \vec{b}$$ -- { In Question it should be given $\vec{a} \neq 0$ & $\vec{b} \neq \vec{0}$)

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Qn. 13 →

$$\hat{i}\cdot(\hat{j}\times\hat{k}) + \hat{j}\cdot(\hat{i}\times\hat{k}) + \hat{k}\cdot(\hat{i}\times\hat{j})$$

$$= \hat{i}\cdot(\hat{i}) + \hat{j}\cdot(-\hat{j}) + \hat{k}\cdot(\hat{k})$$

$$= 1 - 1 + 1$$

$$= 1$$ Ans

Qn. 14 →

Given $A(1,1,2)$ $B(2,3,5)$ $C(1,5,5)$

$$\vec{AB} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{AC} = 0\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$|\vec{AB}\times\vec{AC}| = \sqrt{36+9+16} = \sqrt{61}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AC}|$$

$$= \frac{1}{2}\sqrt{61}$$ square units Ans