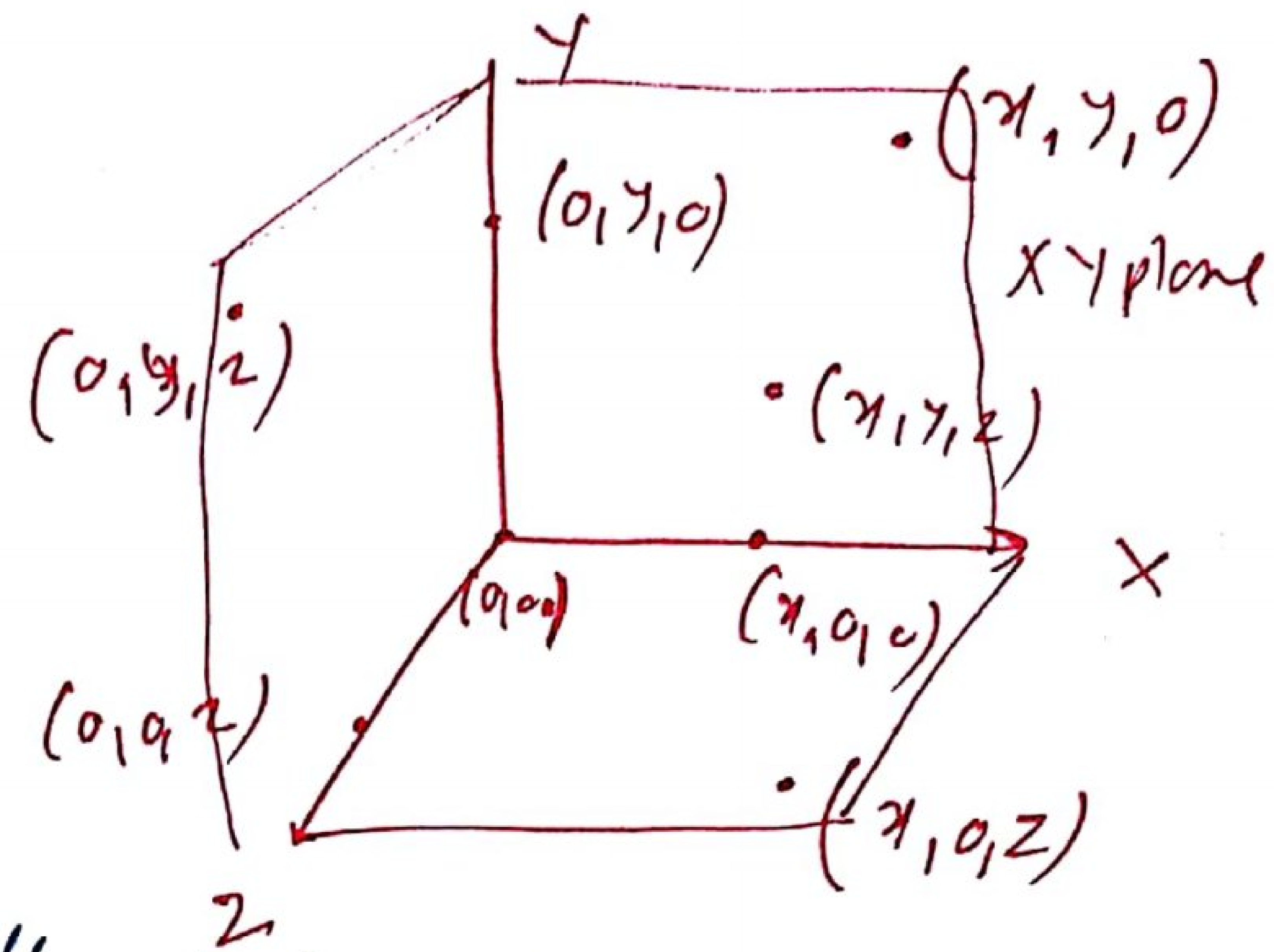
।। जय श्री गिरिराज जी महाराज जय श्री राधे कृष्णा ।।

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER: 3-D | CLASS NO: 1

## LINES

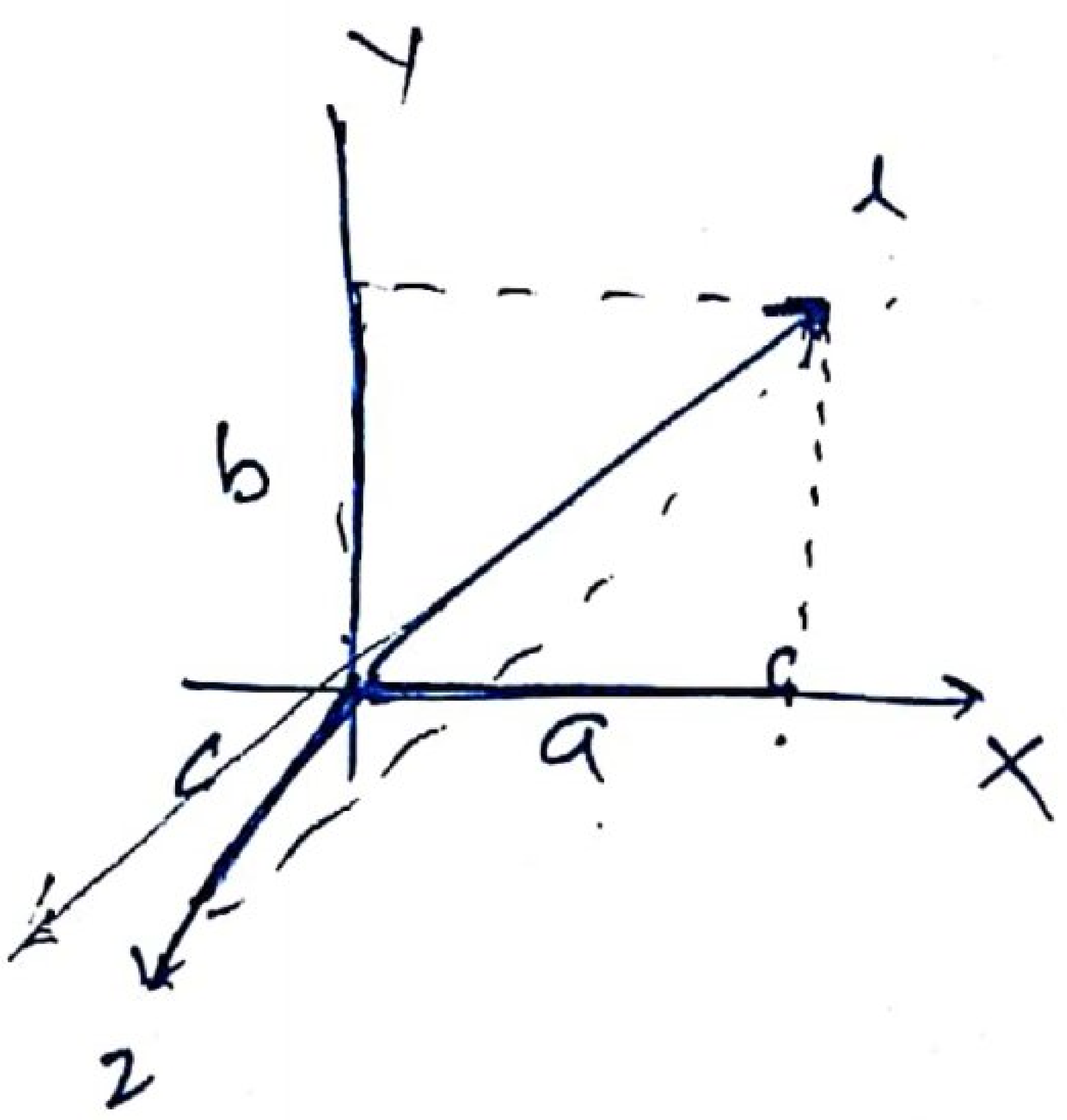
(·) Direction Cosines and Direction Ratios




(·) $\alpha, \beta, \gamma$ are the angles made by the line with X, Y, Z axis respect.

(·) direction angles: $\alpha, \beta, \gamma$

(·) direction cosines: $\cos\alpha, \cos\beta, \cos\gamma$
$= l, m, n$




(·) $l^2 + m^2 + n^2 = 1$

(·) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(·) Direction Ratios of a line: $a, b, c$

(·) $l = \dfrac{a}{\sqrt{a^2+b^2+c^2}}$, $m = \dfrac{b}{\sqrt{a^2+b^2+c^2}}$, $n = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$

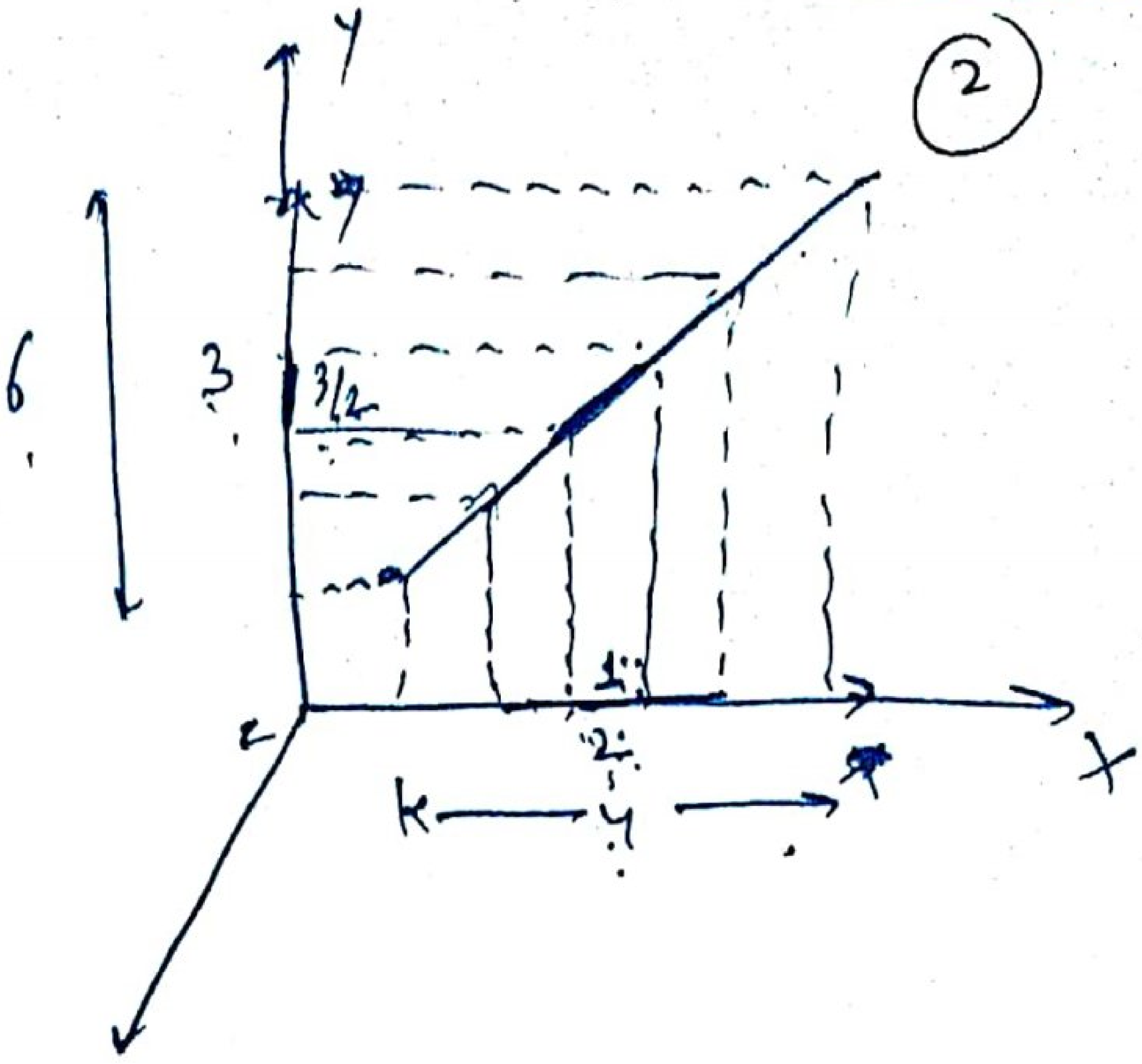
(·) $\boxed{\text{D.R's \& D.c's are proportional}}$

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$$

(·) There are two sets of D.C's of a line

$\frac{2}{1} = \frac{\cancel{x}}{4} = \frac{1}{2}$

$\cancel{\frac{2}{1}} = \frac{2}{1} = \frac{3}{\frac{3}{2}} = 2$




(·) there are Infinite set of D·R's of a line

(·) If two lines are parallel

$a_1, b_1, c_1 \rightarrow$ D·R's of 1st line

$a_2, b_2, c_2 \rightarrow$ DR's of 2nd line

then $\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}} = \lambda$ (Or) $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$

(·) If two lines are $\perp^r$

$$\boxed{a_1 a_2 + b_1 b_2 + c_1 c_2 = 0}$$

(·) D·R's of a line joining two points $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$

then $a = x_2 - x_1$ ; $b = y_2 - y_1$ ; $c = z_2 - z_1$

**Ques 1** Find the D.c's of a line passing through two points $(-2,4,-5)$ & $(1,2,3)$

Sol: let $P(-2,4,-5)$ & $Q(1,2,3)$

$x_1 = -2,\ y_1 = 4,\ z_1 = -5$

$x_2 = 1,\ y_2 = 2,\ z_2 = 3$

$a = x_2 - x_1 = 3$ ; $b = y_2 - y_1 = -2$ ; $c = z_2 - z_1 = 8$

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{3}{\sqrt{9+4+64}} = \frac{3}{\sqrt{77}}$$

$$m = \frac{-2}{\sqrt{77}} \quad ; \quad n = \frac{8}{\sqrt{77}}$$ Ans

**Ques 2** → Show that the points $A(2,3,-4)$ $B(1,-2,3)$ & $C(3,8,-11)$ are collinear

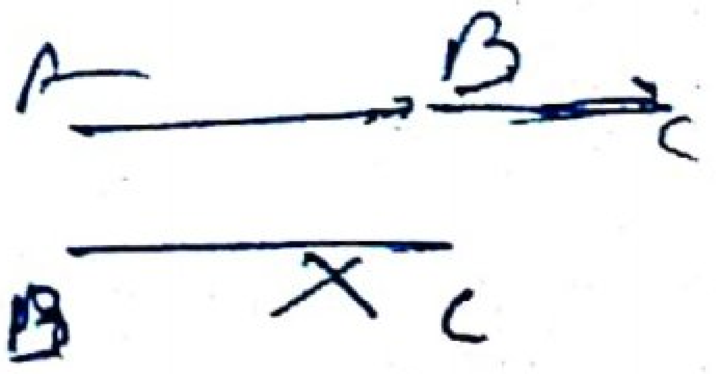
Sol: DR's of line AB = $-1, -5, 7$

DR's of line BC = $2, 10, -14$

$$\frac{-1}{2} = \frac{-5}{10} = \frac{7}{-14}$$

Clearly DR's are in same ratio

$\therefore$ line AB is parallel to line BC




But point B is common

$\therefore$ A, B, C must be collinear Ans

**Qu 3 →** Find the D.C's of X, Y & Z axis

Sol$^n$ for X-axis : $\alpha = 0°$, $\beta = 90°$, $\gamma = 90°$

$l = \cos 0° = 1$ ; $m = \cos 90° = 0$ ; $n = \cos 90° = 0$

$\therefore$ 1, 0, 0 are D.C of X-axis

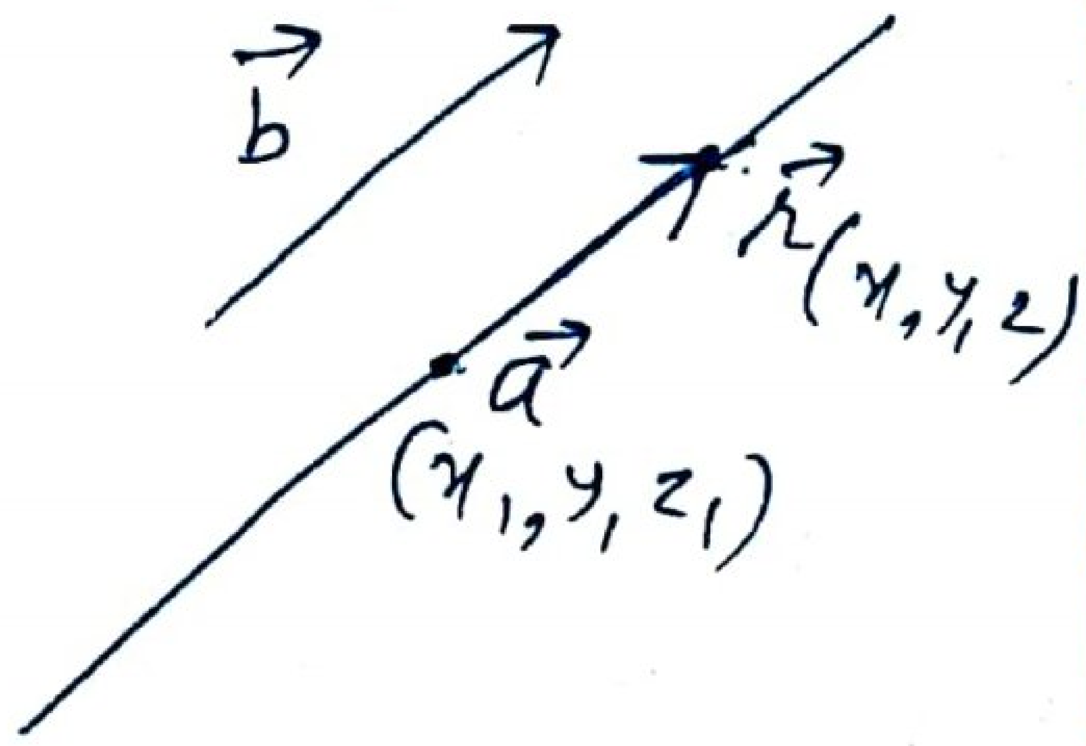
for Y-axis : $\alpha = 90°$, $\beta = 0°$, $\gamma = 90°$

$l = 0$, $m = 1$, $n = 0$

for Z-axis = 0, 0, 1 Ans

## LINES

(1) equation of a line passing through a given point and parallel to a given vector




vector : $\boxed{\vec{r} = \vec{a} + \lambda \vec{b}}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$

Here $(\vec{r} - \vec{a}) \parallel \vec{b}$.

$\Rightarrow \vec{r} - \vec{a} = \lambda \vec{b}$

$\Rightarrow \vec{r} = \vec{a} + \lambda \vec{b}$ Pr

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Cartesian form

$$\boxed{\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}}$$

Proof

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$$

equate corresponding components

$x = x_1 + a\lambda$ | $y = y_1 + b\lambda$ | $z = z_1 + \lambda c$

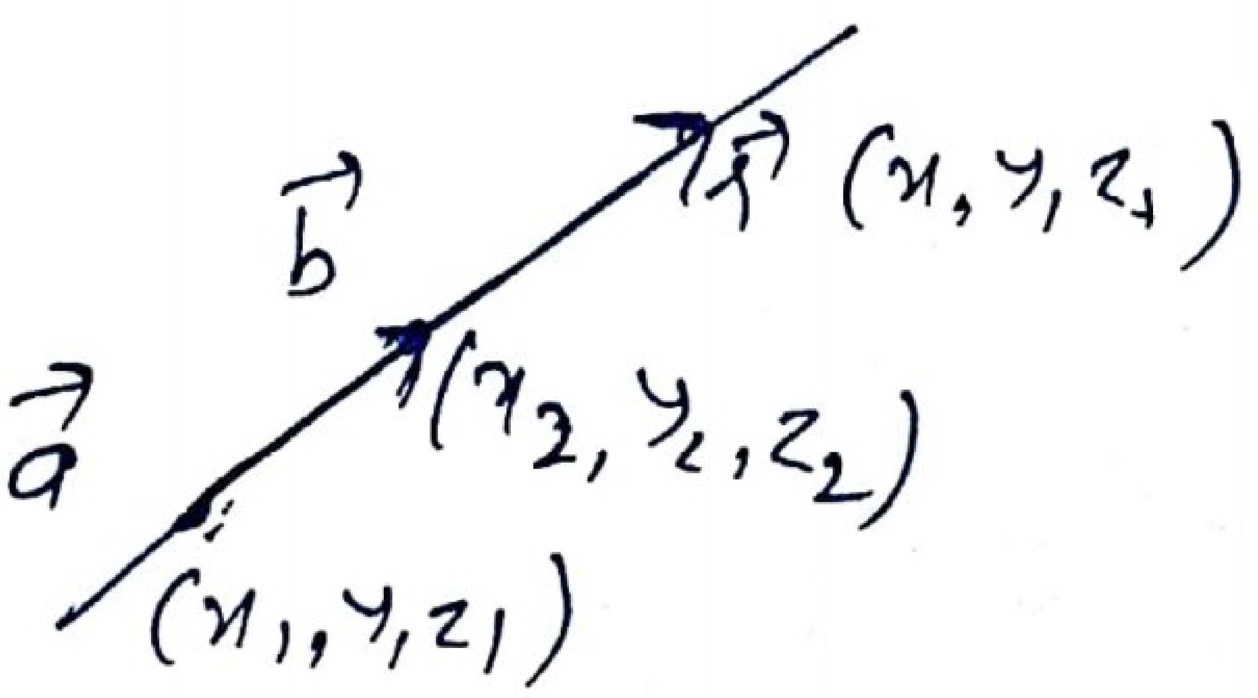
$\frac{x-x_1}{a} = \lambda$ | $\frac{y-y_1}{b} = \lambda$ | $\frac{z-z_1}{c} = \lambda$

$$\Rightarrow \boxed{\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \lambda}$$

## (2) Equation of a line passing through two points

vector equation

$$\boxed{\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})}$$




Proof

$(\vec{r} - \vec{a})$ collinear or parallel to $(\vec{b} - \vec{a})$

$$\Rightarrow (\vec{r} - \vec{a}) = \lambda(\vec{b} - \vec{a})$$

$$\Rightarrow \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \quad \text{Proved}$$

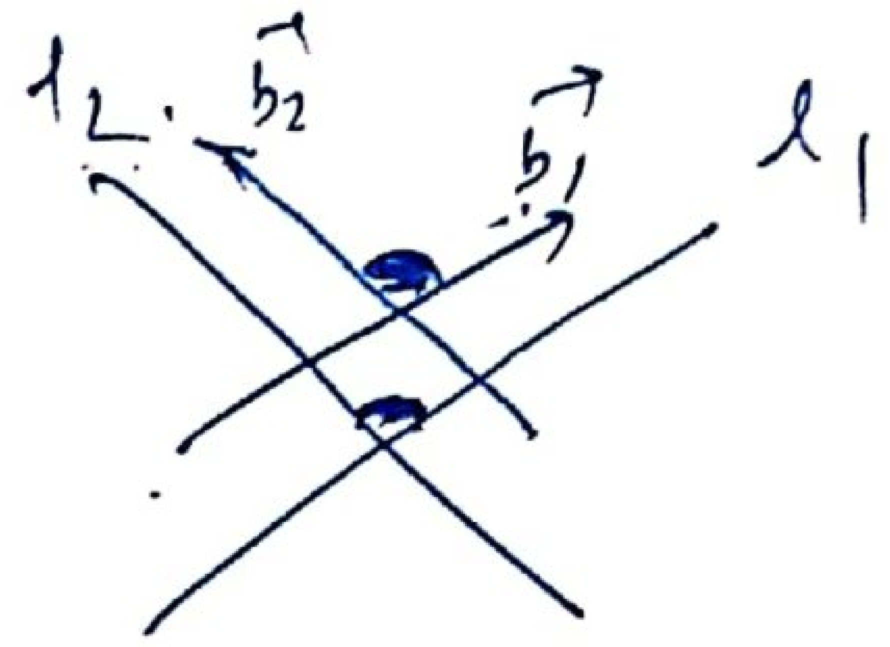
Cartesian

$$\boxed{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}}$$

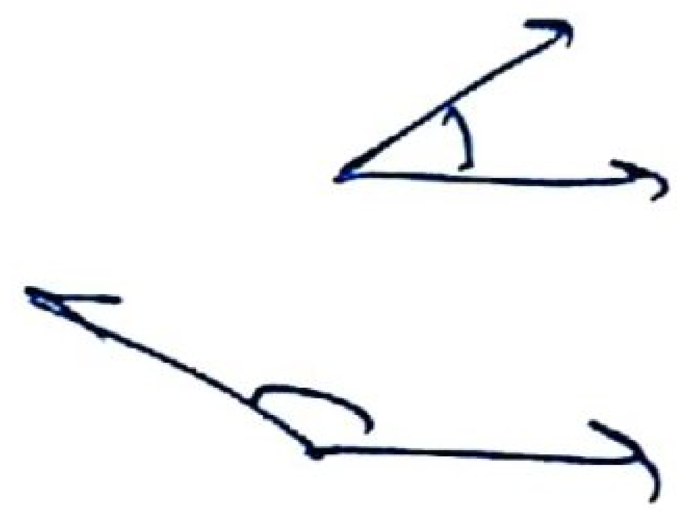
(3) Angle b/w two lines

Given lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$

$\vec{r} = \vec{a_2} + \mu \vec{b_2}$

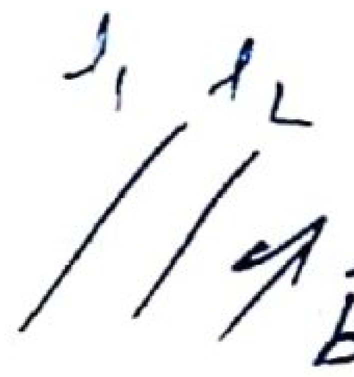



vector $\cos\theta = \left| \dfrac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|} \right|$



Cartesian $\cos\theta = \left| \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}} \right|$

$\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$




If two lines are $\perp^r$ then $\vec{b_1} \cdot \vec{b_2} = 0$

$\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

If two lines $\parallel$ then $\vec{b_1} = \vec{b_2}$ or $\vec{b_1} = \lambda \vec{b_2}$

$\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ or $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$

(4) Distance b/w two skew lines

lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ & $\vec{r} = \vec{a_2} + \mu \vec{b_2}$

Skew lines: non-parallel, non-intersecting, and lie in different planes

(5) Distance $= \left| \dfrac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$

Cartesian

$$\text{Distance} = \frac{\left|\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right|}{\sqrt{(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2}}$$

→ Modulus

→ determinant

③ Distance b/w two parallel lines

line

$$\vec{r} = \vec{a_1} + \lambda \vec{b}$$
$$\vec{r} = \vec{a_2} + \mu \vec{b}$$

vector Distance $= \dfrac{\left|(\vec{a_2}-\vec{a_1}) \times \vec{b}\right|}{|\vec{b}|}$

QUESTIONS

Q-1 Find the distance b/w the lines

$$\vec{r} = \hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$$
$$\vec{r} = 3\hat{i}+3\hat{j}-5\hat{k}+\mu(-12\hat{i}-18\hat{j}-36\hat{k})$$

Sol. 2nd line $\vec{r} = 3\hat{i}+3\hat{j}-5\hat{k}+\mu'(2\hat{i}+3\hat{j}+6\hat{k})$

where $\mu' = -6\mu$

given lines are parallel

here $\vec{a_1} = \hat{i}+2\hat{j}-4\hat{k}$ & $\vec{b} = 2\hat{i}+3\hat{j}+6\hat{k}$

$\vec{a_2} = 3\hat{i}+3\hat{j}-5\hat{k}$

Now $\vec{a_2} - \vec{a_1} = 2\hat{i} + \hat{j} - \hat{k}$

$$(\vec{a_2} - \vec{a_1}) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix} = 9\hat{i} - 14\hat{j} + 4\hat{k}$$

$$|(\vec{a_2} - \vec{a_1}) \times \vec{b}| = \sqrt{81 + 196 + 16} = \sqrt{293}$$

$$|\vec{b}| = \sqrt{4 + 9 + 36} = 7$$

$\therefore$ Required distance $= \dfrac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|}$

$$= \frac{\sqrt{293}}{7} \text{ units Ans}$$

Q.2 Given equation of line $\dfrac{5-x}{3} = \dfrac{2y-8}{6} = z$

Find (1) Standard form (2) vector form (3) fixed point on the line (4) D.R's of line (5) D.c's of line (6) Any point on the line

Soln (1) Standard form $\dfrac{x-5}{-3} = \dfrac{y-4}{3} = \dfrac{z-0}{1}$

(2) vector form $\vec{r} = (5\hat{i} + 4\hat{j} + 0\hat{k}) + \lambda(-3\hat{i} + 3\hat{j} + \hat{k})$

(3) fixed point $(5, 4, 0)$

(4) DR $= -3, 3, 1$

(5) DC $= \dfrac{-3}{\sqrt{19}}, \dfrac{3}{\sqrt{19}}, \dfrac{1}{\sqrt{19}}$

(6) let $\dfrac{x-5}{-3} = \dfrac{y-4}{3} = \dfrac{z-0}{1} = \lambda$

$x = -3\lambda + 5$

$y = 3\lambda + 4$

$z = \lambda$

# WORKSHEET NO: 1 | 13-D

**Q.No 1.** Find the direction cosines of the sides of the triangle whose vertices are $(3, 5, -4)$, $(-1, 1, 2)$ & $(-5, -5, -2)$

Ans $\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$ ; $\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}$ ; $\frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}$

**Q.No 2** If a line makes $90°$, $135°$, $45°$ with X, Y, Z axes respectively. Find its direction cosines. Ans $0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

**Q.No. 3→** Find the direction cosines of a line which makes equal angles with the coordinate axes. Ans $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$

**Q.No. 4→** Find the vector equation of the line passing through the points $(-1, 0, 2)$ and $(3, 4, 6)$ Ans $\vec{r} = -\hat{i} + 2\hat{k} + \lambda(4\hat{i} + 4\hat{j} + 4\hat{k})$

**Q.No 5→** given equation of line : $\frac{3x-1}{6} = \frac{3-2y}{8} = z-1$

Find (1) Standard form
(2) fixed point on the line
(3) D.R's
(4) D.c's
(5) vector form
(6) any point on the line

**Q.No. 6→** Find angle b/w two lines $\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$ and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$ using cartesian form

(Ans $= \cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$)

**Qn 7** → Find the shortest distance b/w two lines

$$\vec{r} = \hat{i}+\hat{j} + \lambda(2\hat{i}-\hat{j}+\hat{k})$$

$$\vec{r} = 2\hat{i}+\hat{j}-\hat{k} + \mu(3\hat{i}-5\hat{j}+2\hat{k})$$

Ans $\frac{10}{\sqrt{59}}$ units

**Qn 8** → Find the distance b/w two lines

$$\vec{r} = (1-\lambda)\hat{i} + (\lambda-2)\hat{j} + (3-2\lambda)\hat{k}$$ and

$$\vec{r} = (\mu+1)\hat{i} + (2\mu-1)\hat{j} + (2\mu+1)\hat{k}$$

HINT: First arrange in standard form $\vec{r} = \vec{a}+\lambda\vec{b}$

Ans $\frac{8}{\sqrt{29}}$ units

**Qn 10** → Find the values of 'p' so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2} \quad \text{and} \quad \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

are at right angles

Ans $p = \frac{70}{11}$

HINT: arrange in ascending order & then $a_1a_2+b_1b_2+c_1c_2=0$

**Qn 11** → Show that the line passing through the points $(1,-1,2)$ & $(3,4,-2)$ is perpendicular to the line passing through the points $(0,3,2)$ & $(3,5,6)$

**Qn 12** → Find the value of 'k' so that the line passing through the points $(4,k,8)$, $(2,3,4)$ is parallel to the line through the points $(-1,-2,1)$ & $(1,2,5)$

Ans: $k=7$

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