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$\boxed{\text{REVISION}}$: INTEGRATION: $\boxed{\text{CLASS NO: 2}}$

SPECIAL QNS

QN: 1 $I = \int \frac{1}{\sin x - \sin(2x)} dx$

$I = \int \frac{1}{\sin x - 2\sin x \cos x} dx$

$= \int \frac{1}{\sin x(1-2\cos x)} dx$

Main step $= \int \frac{\sin x}{\sin^2 x\,(1-2\cos x)} dx$

$= \int \frac{\sin x}{(1-\cos^2 x)(1-2\cos x)} dx$

put $\cos x = t \Rightarrow \sin x\,dx = -dt$

$\therefore I = -\int \frac{dt}{(1+t)(1-t)(1-2t)}$

Proceed

Ans $-\frac{1}{6}\log|1+\cos x| - \frac{1}{2}\log|1-\cos x| + \frac{2}{3}\log|1-2\cos x| + C$

—x—

QN: 2 $I = \int \frac{\sin x}{\sin(4x)} dx$

Sol: $I = \int \frac{\sin x}{2\sin(2x)\cos(2x)} dx$

$= \int \frac{\cancel{\sin x}}{4\cancel{\sin x}\cos x\cos(2x)} dx$

$= \frac{1}{4}\int \frac{1}{\cos x \cdot \cos(2x)} dx$

$$\Rightarrow I = \frac{1}{4}\int \frac{1}{\cos x\cdot(1-2\sin^2 x)}\,dx$$

$$= \frac{1}{4}\int \frac{\cos x}{\cos^2 x\,(1-2\sin^2 x)}\,dx$$

$$= \frac{1}{4}\int \frac{\cos x\,dx}{(1-\sin^2 x)(1-2\sin^2 x)}$$

$$I = \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)} \quad \cdots \left\{ \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \end{array} \right\}$$

Let $t^2 = y$ (temp)

$$\frac{1}{(1-t^2)(1-2t^2)} = \frac{1}{(1-y)(1-2y)}$$

Let $\frac{1}{(1-y)(1-2y)} = \frac{A}{1-y} + \frac{B}{1-2y}$

$$1 = A(1-2y) + B(1-y)$$

Proceed

Ans $-\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + C$

—x—

Ques 3 → $I = \int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\,d\theta$

Soln $I = \int \frac{\tan\theta(1+\tan^2\theta)}{1+\tan^3\theta}\,d\theta$

$$= \int \frac{\tan\theta\cdot\sec^2\theta}{1+\tan^3\theta}\,d\theta$$

$$= \int \frac{t\,dt}{1+t^3} \quad \cdots \left\{ \begin{array}{l} \tan\theta = t \\ \sec^2\theta\,d\theta = dt \end{array} \right\}$$

$$I = \int \frac{t\,dt}{(1+t)(1+t^2-t)}\,dt$$

L Q

let

$$\frac{t}{(t+1)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$$

$$t = A(t^2-t+1) + (Bt+C)(t+1)$$

$$t = A(t^2-t+1) + (Bt^2+Bt+Ct+C)$$

Proceed ANS

$$-\frac{1}{3}\log|\tan\theta+1| + \frac{1}{6}\log|\tan^2\theta-\tan\theta+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right)+C$$

— x —

Que 4 →

$$I = \int (3x-2)\sqrt{x^2+x+1}\,dx$$

$(2x+1)$

Sol

$$I = 3\int\left(x-\frac{2}{3}\right)\sqrt{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int\left(2x-\frac{4}{3}+1-1\right)\sqrt{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int\left(2x+1-\frac{7}{3}\right)\sqrt{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int(2x+1)\sqrt{x^2+x+1}\,dx - \frac{7}{2}\int\sqrt{x^2+x+1}\,dx$$

put $x^2+x+1=t$ in $f^{st}$ Integral

$(2x+1)dx = dt$

$$I = \frac{3}{2}\int \sqrt{t}\,dt + \frac{7}{2}\int\sqrt{\text{perfect square}}$$

Proceed

ANS $(x^2+x+1)^{3/2} - \frac{7}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x+1} + \frac{3}{8}\log\left|\left(\frac{2x+1}{2}\right)+\sqrt{x^2+x+1}\right|+c\right.$

Ans 5 → $I = \int e^x . \cos^2 x \, dx$

Sol $I = \int e^x . \left(\frac{1+\cos(2x)}{2}\right) dx$

$I = \frac{1}{2}\int e^x dx + \frac{1}{2}\int e^x . \cos(2x) dx$

$I = \frac{1}{2} e^x + \frac{1}{2} I_1 + C$

where $I_1 = \int e^x . \cos(2x) dx$

⋮

$\boxed{I_1 = \quad}$

Proceed ANS $I = \frac{e^x}{2} + \frac{e^x}{10}\left[\cos(2x) + 2\sin(2x)\right] + C$

—x—

Ans 6 → $I = \int \frac{1}{x^3} \sin(\log x) dx$

Sol put $\log x = t$

$x = e^t$

$dx = e^t dt$

$I = \int \frac{1}{e^{3t}} . \sin t . e^t dt$

$I = \int \underset{\text{II}}{e^{-2t}} . \underset{\text{I}}{\sin t} \, dt$

Proceed Ans $I = \frac{1}{10 x^3}\left[-3\sin(\log x) - \cos(\log x)\right] + C$

—x—

Qu. 7 → $I = \int \frac{1-\cos x}{\cos x\,(1+\cos x)}\,dx$

Soln

Method I    let $\cos x = y$ (temp)

$\therefore \frac{1-\cos x}{\cos x(1+\cos x)} = \frac{1-y}{y(1+y)}$

let $\frac{1-y}{y(1+y)} = \frac{A}{y} + \frac{B}{1+y}$

$\Rightarrow 1-y = A(1+y) + By$

$-1 = A + B$

$1 = A$    $A = 1$    $B = -2$

$\therefore I = \int \frac{1}{\cos x} - \frac{2}{1+\cos x}\,dx$

$= \int \sec x\,dx - 2\int \frac{1}{1+\cos x}\,dx$

$= \log|\sec x + \tan x| - 2\int \frac{1-\cos x}{\sin^2 x}\,dx$

$I = \log|\sec x + \tan x| - 2(-\cot x + \operatorname{cosec} x) + C$

Method II    $I = \int \frac{1-\cos x}{\cos x(1+\cos x)}\,dx$

$= \int \frac{\tan^2(x/2)}{\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}\,dx$

$= \int \frac{\tan^2(x/2)\cdot\sec^2(x/2)}{1-\tan^2(x/2)}\,dx$

put $\tan(x/2) = t$

$\sec^2(x/2)\,dx = 2\,dt$

$I = 2\int \frac{t^2}{1-t^2}\,dt$

$= -2\int \frac{1-t^2-1}{1-t^2}\,dt$

Proceed

⑥

Q.N 8 → $I = \int \sqrt{\cot\theta}\, d\theta$

Sol: $I = \int \sqrt{\cot\theta}\, d\theta$

put $\cot\theta = t^2$

$$-\operatorname{cosec}^2\theta\, d\theta = 2t\,dt$$

$$d\theta = -\frac{2t\,dt}{\operatorname{cosec}^2\theta}$$

$$d\theta = \frac{-2t\,dt}{1+\cot^2\theta}$$

$$d\theta = \frac{-2t\,dt}{t^4+1}$$

$$\therefore I = -2\int \frac{t\cdot t\,dt}{t^4+1}$$

$$I = -2\int \frac{t^2}{t^4+1}\,dt$$

Divide by $t^2$

$$I = -2\int \frac{1}{t^2+\frac{1}{t^2}}\,dt$$

$$I = -\int \frac{1+1+\frac{1}{t^2}-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt$$

$$I = -\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt - \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt$$

$$I = -\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}\,dt - \int \frac{1-\frac{1}{t^2}\,dt}{\left(t+\frac{1}{t}\right)^2-2}$$

Procced

$u$

Ans $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot\theta-1}{\sqrt{2\cot\theta}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{\cot\theta+1-\sqrt{2\cot\theta}}{\cot\theta+1+\sqrt{2\cot\theta}}\right| + C$

Ans 9 → $I = \int \sqrt{\operatorname{cosec} x - 1}\, dx$

Soln $I = \int \sqrt{\frac{1}{\sin x} - 1}\, dx$

$= \int \sqrt{\frac{1-\sin x}{\sin x}}\, dx$

$= \int \sqrt{\frac{1-\sin x}{\sin x} \times \frac{1+\sin x}{1+\sin x}}\, dx$

$= \int \frac{\cos x}{\sqrt{\sin x + \sin^2 x}}\, dx$

$= \int \frac{dt}{\sqrt{t^2+t}}$ ... $\left\{ \begin{array}{l} \sin x = t \\ \cos x\, dx = dt \end{array} \right\}$

Perfect square

Proceed Ans $\log\left| \left(\sin x + \frac{1}{2}\right) + \sqrt{\sin^2 x + \sin x} \right| + C$

—x—

Ques 10 → $I = \int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\, dx$

Soln $I = \int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)} \times \frac{\sin(x-\alpha)}{\sin(x-\alpha)}}\, dx$

$= \int \frac{\sin(x-\alpha)}{\sqrt{\sin^2 x - \sin^2 \alpha}}$ ... $\{ \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B \}$

$= \int \frac{\sin x \cos \alpha - \cos x \cdot \sin \alpha}{\sqrt{\sin^2 x - \sin^2 \alpha}}\, dx$

$$I = \cos\alpha \int \frac{\sin x}{\sqrt{\sin^2 x - \sin^2\alpha}}\,dx - \sin\alpha \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\alpha}}\,dx$$

$$= \cos\alpha \int \frac{\sin x\,dx}{\sqrt{1-\cos^2 x - 1 + \cos^2\alpha}} - \sin\alpha \int \frac{\cos x\,dx}{\sqrt{\sin^2 x - \sin^2\alpha}}$$

put $\cos x = t$
$\sin x\,dx = -dt$

$\sin x = z$
$\cos x\,dx = dz$

$$= -\cos\alpha \int \frac{dt}{\sqrt{\cos^2\alpha - t^2}} - \sin\alpha \int \frac{dz}{\sqrt{z^2 - \sin^2\alpha}}$$

$$= -\cos\alpha \cdot \sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right) - \sin\alpha \log\left|\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right| + C$$

Ans

Que 11 $I = \int \frac{x^2}{(x\sin x + \cos x)^2}\,dx$

Sol put $x\sin x + \cos x = t$

$(x\cos x + \cancel{\sin x} - \cancel{\sin x})\,dx = dt$

$\Rightarrow x\cos x\,dx = dt$

$$I = \int \underbrace{\frac{x\cos x}{(x\sin x + \cos x)^2}}_{II} \cdot \underbrace{x\sec x}_{I}\,dx$$

$$= x\sec x \int \frac{x\cos x\,dx}{(x\sin x + \cos x)^2} - \int \left( (x\sec x\tan x + \sec x) \cdot \int \frac{x\cos x\,dx}{(x\sin x + \cos x)^2} \right) dx$$

$$= x\sec x \left(-\frac{1}{x\sin x + \cos x}\right) + \int (x\sec x\tan x + \sec x) \cdot \frac{1}{x\sin x + \cos x}\,dx$$

$\left\{ \text{put } x\sin x + \cos x = t,\ x\cos x\,dx = dt \quad \therefore \int \frac{x\cos x\,dx}{(\ )^2} = \int \frac{dt}{t^2} = -\frac{1}{t} = \frac{-1}{x\sin x + \cos x} \right\}$

$$I = \frac{-x\sec x}{x\sin x+\cos x} + \int \frac{\sec x(x\tan x+1)}{x\sin x+\cos x}\,dx$$

$$= -\frac{x\sec x}{x\sin x+\cos x} + \int \frac{\sec x\cdot\left(\frac{\cancel{x\sin x+\cos x}}{\cos x}\right)}{\cancel{x\sin x+\cos x}}\,dx$$

$$= \frac{-x\sec x}{x\sin x+\cos x} + \int \sec^2 x\,dx$$

$$\underline{I} = \frac{-x\sec x}{x\sin x+\cos x} + \tan x + C \quad \underline{\text{Ans}}$$

Ques $\underline{12}$ → $I = \int \frac{\log x}{(1+\log x)^2}\,dx$

$\underline{\text{Sol}}$, put $\log x = t$

$x = e^t$

$dx = e^t\,dt$

$$I = \int \frac{t}{(1+t)^2}\cdot e^t\,dt$$

$$= \int e^t\cdot\frac{t}{(t+1)^2}\,dt$$

$$= \int e^t\cdot\left(\frac{t+1-1}{(t+1)^2}\right)dt$$

$$= \int e^t\cdot\left(\frac{1}{t+1} - \frac{1}{(t+1)^2}\right)dt$$

$$= e^t\cdot\frac{1}{t+1} + C$$

$$= x\cdot\frac{1}{\log x+1} + C \quad \underline{\text{Ans}}$$

Qn. 13 →
$$I = \int \frac{x^2 \sin^{-1}x}{(1-x^2)^{3/2}}\,dx$$

Sol

$$I = \int \frac{x^2 \sin^{-1}x}{(1-x^2)\sqrt{1-x^2}}\,dx$$

$\sin^{-1}x = t \longrightarrow x = \sin t$

$\frac{1}{\sqrt{1-x^2}}\,dx = dt$

$$\therefore I = \int \frac{\sin^2 t \cdot t}{1-\sin^2 t}\,.dt$$

$$= \int t \cdot \tan^2 t\,dt$$

$$= \int t \cdot (\sec^2 t - 1)\,dt$$

$$= \int \underset{\text{I}}{t}\, \underset{\text{II}}{\sec^2 t}\,dt - \int t\,dt$$

Proceed

Ans $\frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \frac{1}{2}\log|1-x^2| - \frac{(\sin^{-1}x)^2}{2} + C$

— x —

Qn. 14 →
$$I = \int \frac{\sin^{-1}x}{x^2}\,dx$$

Soln

$$I = \int \underset{\text{II}}{\frac{1}{x^2}} \cdot \underset{\text{I}}{\sin^{-1}x}\,dx$$

$$I = \sin^{-1}x \cdot \left(-\frac{1}{x}\right) + \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{x}\,dx$$

put $1-x^2 = t^2$

$-2x\,dx = 2t\,dt$

$dx = \frac{-t\,dt}{x}$

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$$I = -\frac{\sin^{-1}x}{x} - \int \frac{1}{\cancel{t}} \cdot \frac{1}{x} \cdot \frac{\cancel{t}\,dt}{x}$$

$$= -\frac{\sin^{-1}x}{x} - \int \frac{dt}{1-t^2}$$

$$= -\frac{\sin^{-1}x}{x} - \frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + C$$

OR

put $x = \sin t$

$dx = \cos t\, dt$

$$I = \int \frac{t \cdot \cos t\, dt}{\sin^2 t}$$

$$= \int \underset{I}{t} \cdot \underbrace{\operatorname{cosec} t \cdot \cot t}_{II}\, dt$$

proceed

Qn 15 → $I = \int \sec^{-1}\sqrt{x}\, dx$

Sol: put $x = t^2$

$dx = 2t\,dt$

$$I = 2\int \underset{I}{\sec^{-1}t} \cdot \underset{II}{t}\, dt$$

$$= 2\left[\sec^{-1}x \cdot \frac{t^2}{2} - \int \frac{1}{t\sqrt{t^2-1}} \cdot \frac{t^2}{2}\, dt\right]$$

$$= \sec^{-1}t \cdot t^2 - \int \frac{t}{\sqrt{t^2-1}}\, dt$$

put $t^2 - 1 = z$

proceed

Ans $x\sec^{-1}\sqrt{x} - \sqrt{x-1} + C$

Qn 16 → $I = \int \log_{10} x + \sec(x^\circ)\, dx$

Sol: $I = \int \log_{10} x\, dx + \int \sec(x^\circ)\, dx$

$= \int \frac{\log x}{\log 10}\, dx + \int \sec\left(\frac{\pi}{180}x\right) dx$

$= \frac{1}{\log 10} \int \log x \cdot 1\, dx + \frac{180}{\pi} \log\left|\sec\left(\frac{\pi x}{180}\right) + \tan\left(\frac{\pi x}{180}\right)\right|$

$= \frac{1}{\log 10} \cdot \left(x \log x - x\right) + \frac{180}{\pi} \log\left|\sec(x^\circ) + \tan(x^\circ)\right| + c$

Ans

Qn. 17 → $I = \int \frac{dx}{2e^{2x} + 3e^{x} + 1}$

Sol: $I = \int \frac{1}{\frac{2}{e^{-2x}} + \frac{3}{e^{-x}} + 1}\, dx$

$= \int \frac{e^{-2x}}{2 + 3e^{-x} + e^{-2x}}\, dx$

$= \int \frac{e^{-x}\, e^{-x}\, dx}{2 + 3e^{-x} + e^{-2x}}$

$e^{-x} = t \Rightarrow e^{-x} dx = -dt$

$I = -\int \frac{t\, dt}{t^2 + 3t + 2} \rightarrow \int \frac{\text{linear}}{\text{Quadratic}}$ or partial fraction

Ans $-\frac{1}{2}\log\left|e^{-2x}+3e^{-x}+2\right| + \frac{3}{2}\log\left|\frac{e^{-x}+1}{e^{-x}+2}\right| + C$

—x—

Qn 18 → $I = \int \frac{ax^3+bx}{x^4+c^2}\,dx$

Soln

$$I = a\int \frac{x^3}{x^4+c^2}\,dx + b\int \frac{x}{x^4+c^2}\,dx$$

put. $x^4+c^2 = t$ | $x^2 = z$

$4x^3dx = dt$ | $x\,dx = \frac{dz}{2}$

$x^3dx = \frac{dt}{4}$

$$I = \frac{a}{4}\int \frac{dt}{t} + \frac{b}{2}\int \frac{dz}{z^2+c^2}$$

$$= \frac{a}{4}\log\left|x^4+c^2\right| + \frac{b}{2}\times\frac{1}{c}\tan^{-1}\left(\frac{z}{c}\right) + C$$

$$= \frac{a}{4}\log\left|x^4+c^2\right| + \frac{b}{2c}\tan^{-1}\left(\frac{x^2}{c}\right) + C \quad \text{Ans}$$

Qn. 19 → $I = \int \frac{x^2}{(a+bx)^2}\,dx$

Soln

put $a+bx = t$

$b\,dx = dt \Rightarrow dx = \frac{dt}{b}$

$$I = \frac{1}{b}\int \frac{\left(\frac{t-a}{b}\right)^2 . dt}{t^2}$$

$$= \frac{1}{b^3}\int \frac{t^2-2at+a^2}{t^2}\,dt$$

$$= \frac{1}{b^3}\int 1 - \frac{2a}{t} + \frac{a^2}{t^2}\,dt$$

$$= \frac{1}{b^3}\left[t - 2a\log|t| - \frac{a^2}{t}\right] + C$$

Ans

Qn. 20 → $I = \int 2^{2^{2^x}} \cdot 2^{2^x} \cdot 2^x \, dx$

Soln

put $2^{2^{2^x}} = t$

$2^{2^{2^x}} \cdot \log 2 \cdot 2^{2^x} \cdot \log 2 \cdot 2^x \cdot \log 2 \, dx = dt$

$\Rightarrow 2^{2^{2^x}} \cdot 2^{2^x} \cdot 2^x \, dx = \dfrac{dt}{(\log 2)^3}$

$I = \dfrac{1}{(\log 2)^3} \int dt$

$= \dfrac{1}{(\log 2)^3} \cdot 2^{2^{2^x}} + c$ Ans

Qn. 21 → special

$I = \int \dfrac{1}{\sqrt[4]{(x-1)^3 (x+2)^5}} \, dx$

Soln $I = \int \dfrac{1}{(x-1)^{3/4} \cdot (x+2)^{5/4}} \, dx$

$= \int \dfrac{1}{\dfrac{(x-1)^{3/4}}{(x+2)^{3/4}} \times (x+2)^{3/4} \times (x+2)^{5/4}} \, dx$

$= \int \dfrac{1}{\left(\dfrac{x-1}{x+2}\right)^{3/4} \cdot (x+2)^2} \, dx$

$= \int \dfrac{\left(\dfrac{x-1}{x+2}\right)^{-3/4}}{(x+2)^2} \, dx$

put $\frac{x-1}{x+2}=t$

$$\frac{(x+2)(1)-(x-1)(1)}{(x+2)^2}\,dx=dt$$

$$\frac{3}{(x+2)^2}\,dx=dt$$

$$\frac{dx}{(x+2)^2}=\frac{dt}{3}$$

$$\therefore I=\frac{1}{3}\int t^{-3/4}\,dt$$

$$=\frac{1}{3}(t)^{1/4}\times 4+C$$

$$I=\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+C \quad \text{Ans}$$

Qn 22 → $I=\int \tan(x-\theta)\tan(x+\theta)\tan(2x)\,dx$

solution we have $2x=(x-\theta)+(x+\theta)$

$$\Rightarrow \tan(2x)=\tan\left((x-\theta)+(x+\theta)\right)$$

$$\Rightarrow \tan(2x)=\frac{\tan(x-\theta)+\tan(x+\theta)}{1-\tan(x-\theta)\tan(x+\theta)}$$

$$\Rightarrow \tan(2x)-\tan(2x)\tan(x-\theta)\tan(x+\theta)=\tan(x-\theta)+\tan(x+\theta)$$

$$\Rightarrow \tan(2x)\tan(x-\theta)\tan(x+\theta)=\tan(x-\theta)+\tan(x+\theta)-\tan(2x)$$

$$\therefore I=\int \tan(x-\theta)+\tan(x+\theta)-\tan(2x)\,dx$$

$$=\log|\sec(x-\theta)|+\log|\sec(x+\theta)|-\frac{1}{2}\log|\sec(2x)|+C \quad \text{Ans}$$

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Qn- 23 →  $I = \int \frac{1}{\sin^4 x + \cos^4 x} dx$

Sol: ~~Divide~~ Divide by $\cos^4 x$

$$I = \int \frac{\sec^4 x}{\tan^4 x + 1} dx$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x\, dx}{\tan^4 x + 1}$$

$$= \int \frac{t^2+1}{t^4+1} dt$$

$x^4$ वाले type

Proceed

Ans $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right) + C$

Qn 24 →  $I = \int \frac{1}{x^4 - a^2x^2 + a^4} dx$

Sol: Divide by $x^2$

$$I = \int \frac{\frac{1}{x^2}}{x^2 - a^2 + \frac{a^4}{x^2}} dx$$

D & M by $2a^2$

$\sqrt{\text{Constant}}$ → D & M

$$= \frac{1}{2a^2} \int \frac{\frac{2a^2}{x^2}}{x^2 + \frac{a^4}{x^2} - a^2} dx$$

$$= \frac{1}{2a^2} \int \frac{\frac{a^2}{x^2} + \frac{a^2}{x^2} + 1 - 1}{x^2 + \frac{a^4}{x^2} - a^2} dx$$

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$$= \frac{1}{2a^2}\int \frac{1+\frac{a^2}{x^2}}{x^2+\frac{a^4}{x^2}-a^2}\,dx - \frac{1}{2a^2}\int \frac{1-\frac{a^2}{x^2}}{x^2+\frac{a^4}{x^2}-a^2}\,dx$$

$$= \frac{1}{2a^2}\int \frac{1+\frac{a^2}{x^2}}{\underbrace{\left(x-\frac{a^2}{x}\right)}_{u}{}^2+2a^2-a^2}\,dx - \frac{1}{2a^2}\int \frac{1-\frac{a^2}{x^2}}{\underbrace{\left(x+\frac{a^2}{x}\right)}_{v}{}^2-2a^2-a^2}\,dx$$

$$= \frac{1}{2a^2}\int \frac{du}{u^2+a^2} - \frac{1}{2a^2}\int \frac{dv}{v^2-(\sqrt{3}a)^2}$$

Proceed by self

Q.n. 25 → $I = \int \frac{e^{\tan^{-1}x}}{(1+x^2)^2}\,dx$

Sol: Put $\tan^{-1}x = t$

$$\frac{1}{1+x^2}\,dx = dt$$

$$I = \int \frac{e^t\,dt}{1+x^2}$$

$$= \int \frac{e^t}{1+\tan^2 t}\,dt$$

$$= \int \frac{e^t}{\sec^2 t}\,dt$$

$$= \int e^t \cdot \cos^2 t\,dt$$

$$= \int e^t \cdot \frac{(1+\cos(2t))}{2}\,dt$$

Proceed Ans $\frac{1}{2}e^{\tan^{-1}x} + \frac{1}{10}e^{\tan^{-1}x}\left(\frac{1-x^2}{1+x^2} + \frac{4}{1+x^2}\right) + C$

Qn 26 → $I = \int \frac{\sin x - x\cos x}{x(x+\sin x)}\,dx$

Soln $I = \int \frac{\sin x - x\cos x + x - x}{x(x+\sin x)}\,dx$

$= \int \frac{(x+\sin x) - x(1+\cos x)}{x(x+\sin x)}\,dx$

$= \int \frac{1}{x}\,dx - \int \frac{1+\cos x}{\underbrace{x+\sin x}_{=t}}\,dx$

$= \log|x| - \log|x+\sin x| + C$ Ans

Q. 27 $I = \int \frac{1-\sin x}{\sin x(1+\sin x)}\,dx$

Sol $I = \int \frac{1-\sin x + \sin x - \sin x}{\sin x(1+\sin x)}\,dx$

$= \int \frac{(1+\sin x) - 2\sin x}{\sin x(1+\sin x)}\,dx$

$= \int \frac{1}{\sin x}\,dx - 2\int \frac{1}{\underbrace{1+\sin x}_{\text{rationalize}}}\,dx$

proceed

Ans $\log|\operatorname{cosec} x - \cot x| - 2(\tan x - \sec x) + C$

Qn. 28 → $I = \int \frac{x^2-1}{(x^2+1)\cdot\sqrt{x^3+x^2+x}}\,dx$

Soln Trick Divide by $x^2$

$$I = \int \frac{1-\frac{1}{x^2}}{\left(\frac{x^2+1}{x}\right)\cdot\frac{\sqrt{x^3+x^2+x}}{x}}\,dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)\sqrt{x+1+\frac{1}{x}}}$$

put $x+\frac{1}{x} = t$

$\left(1-\frac{1}{x^2}\right)dx = dt$

$$I = \int \frac{dt}{t\sqrt{t+1}}$$

put $t+1 = z^2$

Process

# REVISION WORKSHEET NO: 2 INTEGRATION

**Ans 1** $I = \int \frac{\sin x}{\sin(4x)}\,dx$ **Ans** $-\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + C$

**Ans 2** → $\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\,d\theta$ **Ans** $-\frac{1}{3}\log|\tan\theta+1| + \frac{1}{6}\log|\tan^2\theta - \tan\theta + 1|$
$+ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C$

**Ans 3** → $\int (x-2)\sqrt{2x^2-6x+5}\,dx$ **Ans** $\frac{1}{6}(2x^2-6x+5)^{3/2} - \frac{1}{\sqrt{2}}\left[\frac{2x-3}{4}\right.$
$\left.\sqrt{x^2-3x+\frac{5}{2}} + \frac{1}{8}\log\left|\frac{2x-3}{2} + \sqrt{x^2-3x+\frac{5}{2}}\right|\right]$
$+C$

**Ans 4** → $\int e^x \cos^2 x\,dx$ **Ans** $\frac{e^x}{2} + \frac{e^x}{10}\left[\cos(2x) + 2\sin(2x)\right] + C$

**Ans 5** → $\int e^x \sin^2 x\,dx$ **Ans** $\frac{e^x}{2} - \frac{e^x}{10}\left[\cos(2x) + 2\sin(2x)\right] + C$

**Ans. 6** → $\int \frac{1}{x^3}\cdot\sin(\log x)\,dx$ **Ans** $\frac{1}{10x^3}\left[-3\sin(\log x) - \cos(\log x)\right] + C$

**Ans 7** → $\int (3x-2)\sqrt{x^2+x+1}\,dx$ **Ans** $(x^2+x+1)^{3/2} - \frac{7}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x+1}\right.$
$\left.+ \frac{3}{8}\log\left|\frac{2x+1}{2} + \sqrt{x^2+x+1}\right|\right] + C$

**Ans. 8** → $\int e^{2x}\cdot\sin x\cos x\,dx$ **Ans** $\frac{e^{2x}}{8}\left(\sin(2x) - \cos(2x)\right) + C$

Hint: $\sin x \cos x = \frac{\sin(2x)}{2}$

**Ans. 9** → $\int x^2 \cdot e^{x^3} \cdot \cos(x^3)\,dx$ **Ans** $\frac{e^{x^3}}{6}\left(\sin(x^3) + \cos(x^3)\right) + C$

**Ans 10** → $\int \frac{1}{\sin x - \sin(2x)}\,dx$ **Ans** $-\frac{1}{6}\log|1+\cos x| - \frac{1}{2}\log|1-\cos x|$
$+ \frac{2}{3}\log|1-2\cos x| + C$

Qu. 11 → $I = \int \frac{1-\cos x}{\cos x(1+\cos x)}\,dx$ Ans $\log|\sec x + \tan x| - 2\tan(x/2) + C$

Qu. 12 → $I = \int \sqrt{\cot\theta}\,d\theta$ Ans $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot\theta - 1}{\sqrt{2\cot\theta}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{\cot\theta + 1 - \sqrt{2\cot\theta}}{\cot\theta + 1 + \sqrt{2\cot\theta}}\right| + C$

Qu. 13 → $I = \int \sqrt{\sec x - 1}\,dx$ Ans $-\log\left|\left(\cos x + \frac{1}{2}\right) + \sqrt{\cos^2 x + \cos x}\right| + C$

Qu. 14 → $\int \sqrt{\operatorname{cosec} x - 1}\,dx$ Ans $\log\left|\left(\sin x + \frac{1}{2}\right) + \sqrt{\sin^2 x + \sin x}\right| + C$

Qu. 15 → $\int \frac{1}{\sqrt{1-e^{2x}}}\,dx$ Ans $-\log\left|e^{-x} + \sqrt{e^{-2x} - 1}\right| + C$

Qu. 16 → $\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$ Ans $-\cos\alpha\,\sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right) - \sin\alpha.$
$\log\left|\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right| + C$

Qu. 17 → $\int \operatorname{cosec}^3 x\,dx$ Ans $-\frac{\operatorname{cosec} x \cot x}{2} + \frac{1}{2}\log|\cot x + \operatorname{cosec} x| + C$

Qu. 18 → $\int \frac{x^2}{(x\sin x + \cos x)^2}\,dx$ Ans $\frac{-x\sec x}{x\sin x + \cos x} + \tan x + C$

Qu. 19 → $\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx$ Ans $6\log|x| - \log|x+1| + \frac{9}{x+1} + C$

HINT $x$ common, then partial fraction (linear & linear repeating)

Qu. 20 → $\int \frac{\log x}{(1+\log x)^2}\,dx$ Ans $\frac{x}{\log x + 1} + C$

Qu. 21 → $\int e^x \cdot \left(\frac{2-x}{(1-x)^2}\right)dx$ Ans $\frac{e^x}{1-x} + C$

Qn. 22 → $\int e^{x}\cdot\left(\frac{x-4}{(x-2)^3}\right)dx$ Ans $\frac{e^{x}}{(x-2)^2}+C$

Qn. 23 → $\int \frac{\sin^{-1}x}{x^2}dx$ Ans $\frac{\sin^{-1}x}{x}-\frac{1}{2}\log\left|\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right|+C$

Qn. 24 → $\int \frac{x^2\cdot\sin^{-1}x}{(1-x^2)^{3/2}}dx$ Ans $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}+\frac{1}{2}\log|1-x^2|-\frac{(\sin^{-1}x)^2}{2}+C$

Qn. 25 → $\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx$ Ans $\frac{-\tan^{-1}x}{\sqrt{1+x^2}}+\frac{x}{\sqrt{1+x^2}}+C$

Qn. 26 → $\int \sec^{-1}\sqrt{x}\,dx$ Ans $x\sec^{-1}\sqrt{x}-\sqrt{x-1}+C$

Qn. 27 → $\int \sin^{-1}\sqrt{x}\,dx$ Ans $\frac{1}{2}(2x-1)\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}+C$

Qn. 28 → $\int \log_{10}x+\sec(x^{\circ})\,dx$ Ans $\frac{1}{\log(10)}(x\log x-x)+$
$\frac{180}{\pi}\log|\sec(x^{\circ})+\tan(x^{\circ})|+C$

Qn. 29 → $\int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$ Ans $3\left[x\tan^{-1}x-\frac{1}{2}\log|1+x^2|\right]+C$

Qn 30 → $\int x\sin x\cdot\cos(2x)\,dx$ Ans $\frac{1}{2}\left[-\frac{x\cos(3x)}{3}+\frac{\sin(3x)}{9}+x\cos x-\sin x\right]+C$

HINT: $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

Qn. 31 → $\int\sqrt{\frac{1+x}{x}}\,dx$ Ans $\sqrt{x^2+x}+\frac{1}{2}\log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C$

Hint: ऊपर Root पसंद नहीं है

Qn 32 → $\int\frac{1}{(2\sin x+3\cos x)^2}dx$ Ans $\frac{-1}{2(2\tan x+3)}+C$

Que 33 → $I = \int \frac{\cos x}{\cos(3x)} dx$ Ans $\frac{1}{2\sqrt{3}} \log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right| + C$

Que 34 → $I = \int \frac{1}{2-3\cos(2x)} dx$ Ans $\frac{1}{2\sqrt{5}} \left|\frac{\sqrt{5}\tan x - 1}{\sqrt{5}\tan x + 1}\right| + C$

HINT $\cos(2x) = \frac{1-\tan^2 x}{1+\tan^2 x}$ OR divide by $\cos^2 x$

Que 35 → $I = \int \frac{1}{1-2\sin x} dx$ Ans $\frac{1}{\sqrt{3}} \log\left|\frac{\tan(x/2) - 2 - \sqrt{3}}{\tan(x/2) - 2 + \sqrt{3}}\right| + C$

Que 36 → $I = \int \frac{1}{p+q\tan x} dx$ Ans $\frac{q}{p^2+q^2} \log|p\cos x + q\sin x|$
$+ \frac{p}{p^2+q^2} \cdot x + C$

HINT $\tan x = \frac{\sin x}{\cos x}$
then $N^r = A \cdot \frac{d}{dx}(D^r) + B(D^r)$

Que 37 → $I = \int \frac{8\cot x + 1}{3\cot x + 2} dx$ Ans $\frac{4}{25}x + \frac{3}{25} \log|4\cos x + 3\sin x| + C$

Que 38 → $\int \frac{1+\sin x}{\sin x(1+\cos x)} dx$ Ans $\frac{1}{4} \log|1-\cos x| - \frac{1}{4} \log|1-\cos x|$
$+ \frac{1}{2(1+\cos x)} + \tan(x/2) + C$

HINT Separate In $I^{st}$ Integral partial fraction & in $2^{nd}$ Rationalize

Que 39 → $\int \frac{dx}{2e^{2x} + 3e^{x} + 1}$ Ans $-\frac{1}{2} \log|e^{-2x} + 3e^{-x} + 2|$
$+ \frac{3}{2} \log\left|\frac{e^{-x}+1}{e^{-x}+2}\right| + C$

Que 40 → $\int \frac{2\sin(2\theta) - \cos\theta}{6 - \cos^2\theta - 4\sin\theta} d\theta$ Ans $2\log|\sin^2\theta - 4\sin\theta + 5|$
$+ 7\tan^{-1}(\sin\theta - 2) + C$

Qu. 41 → $I = \int \frac{ax^3 + bx}{x^4 + c^2}\,dx$ Ans. $\frac{a}{4}\log|x^4+c^2| + \frac{b}{2c}\tan^{-1}\left(\frac{x^2}{c}\right)+c$

Qu. 42 → $\int \frac{x^3+x}{x^4-9}\,dx$ Ans $\frac{1}{4}\log|x^4-9| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right|+c$

Qu. 43 → $\int \frac{1-x^2}{x(1-2x)}\,dx$ Ans. $\frac{x}{2} + \log|x| - \frac{3}{4}\log|2x-1|+c$

Hint take (−) common, then divide, then partial fraction

Qus 44 → $\int \frac{\sin x\cdot\cos x}{\sqrt{\sin^4 x + 4\sin^2 x + 2}}\,dx$ Ans $\frac{1}{2}\log\left|(\sin^2 x + 2) + \sqrt{\sin^4 x + 4\sin^2 x + 2}\right|+c$

Qu. 45 → $\int \frac{\sin(2x)\cdot\cos(2x)}{\sqrt{9-\cos^4(2x)}}\,dx$ Ans $-\frac{1}{4}\sin^{-1}\left(\frac{\cos^2(2x)}{3}\right)+c$

Qus 46 → $\int \frac{1}{\sqrt{\sin^3 x\cdot\cos^5 x}}\,dx$ Ans $\frac{-2}{\sqrt{\tan x}} + \frac{2}{3}(\tan x)^{3/2}+c$

Qus 47 → $\int \sec^{4/3}x\cdot\operatorname{cosec}^{8/3}x\,dx$ Ans $-\frac{3}{5}\tan^{-5/3}x + 3\tan^{1/3}x+c$

Qu 48 → $\int \cot^6 x\,dx$ Ans $-\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x + c$

Qu 49 → $\int \frac{1}{x^4-5x^2+16}\,dx$ Ans $-\frac{1}{16\sqrt{3}}\log\left|\frac{x^2-\sqrt{13}\,x+4}{x^2+\sqrt{13}\,x+4}\right| + \frac{1}{8\sqrt{3}}\tan^{-1}\left(\frac{x^2-4}{\sqrt{3}\,x}\right)+c$

Qu. 50 → $\int \frac{1}{x^4-a^2x^2+a^4}\,dx$ Ans $\frac{1}{2a^2}\int\frac{1}{t^2+a^2}\,dt - \frac{1}{2a^2}\int\frac{dz}{z^2-(\sqrt{3}a)^2}$ (Proceed)

Qu 51 → $\int \frac{e^{\tan^{-1}x}}{(1+x^2)^2}\,dx$ Ans $\frac{1}{2}e^{\tan^{-1}x} + \frac{1}{10}e^{\tan^{-1}x}\left(\frac{1-x^2}{1+x^2}+\frac{4x}{1+x^2}\right)+c$

Ques 52 → $\int (2\sin(2x) - \cos x)\sqrt{6-\cos^2 x - 4\sin x}\, dx$

Ans $\frac{4}{3}(\sin^2 x - 4\sin x + 5)^{3/2} + 7\left[\left(\frac{\sin x - 2}{2}\right)\sqrt{\sin^2 x - 4\sin x + 5} + \log\left|(\sin x - 2) + \sqrt{\sin^2 x - 4\sin x + 5}\right|\right] + C$

Ques 53 → $\int \frac{1}{\sin^4 x + \cos^4 x}\, dx$ Ans $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right) + C$

Ques 54 → $\int \frac{1-\sin x}{\sin x(1+\sin x)}\, dx$ Ans $\log|\operatorname{cosec} x - \cot x| - 2(\tan x - \sec x) + C$

Ques 55 → $\int \frac{\sin x - x\cos x}{x(x+\sin x)}\, dx$ Ans $\log|x| - \log|x + \sin x| + C$

Ques 56 → $\int \frac{x^9}{(4x^2+1)^6}\, dx$ Ans $\frac{1}{10}\left(\frac{1}{\left(4+\frac{1}{x^2}\right)}\right)^5 + C$

Hint: Take common & then put t

Ques 57 → $\int \frac{\sec x}{1+\operatorname{cosec} x}\, dx$ Ans $-\frac{1}{4}\log|1-\sin x| + \frac{1}{4}\log|1+\sin x| - \frac{1}{2(1+\sin x)} + C$

Hint: आजो ऑके partial fraction

Ques 58 → $\int \frac{x^2-1}{(x^2+1)\sqrt{x^3+x^2+x}}\, dx$ Ans $\log\left|\frac{\sqrt{x+\frac{1}{x}+1} - 1}{\sqrt{x+\frac{1}{x}+1} + 1}\right| + C$

Ques 59 → $\int \tan(x+\theta)\tan(x-\theta)\tan(2x)\, dx$

Ans $\frac{\log|\sec(2x)|}{2} - \log|\sec(x-\theta)| - \log|\sec(x+\theta)| + C$

Ques 60 → $\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$ Ans $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$

Ques 61 → $\int 2^{2^{2^x}} \cdots 2^{2^x} \cdot 2^x\, dx$ Ans $\frac{1}{(\log 2)^3} \cdot 2^{2^{2^x}} + C$

Ques 62 → $\int \frac{x^2}{(a+bx)^2}\, dx$ Ans $\frac{1}{b^3}\left[(a+bx) - 2a\log|a+bx| - \frac{a^2}{a+bx}\right] + C$

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