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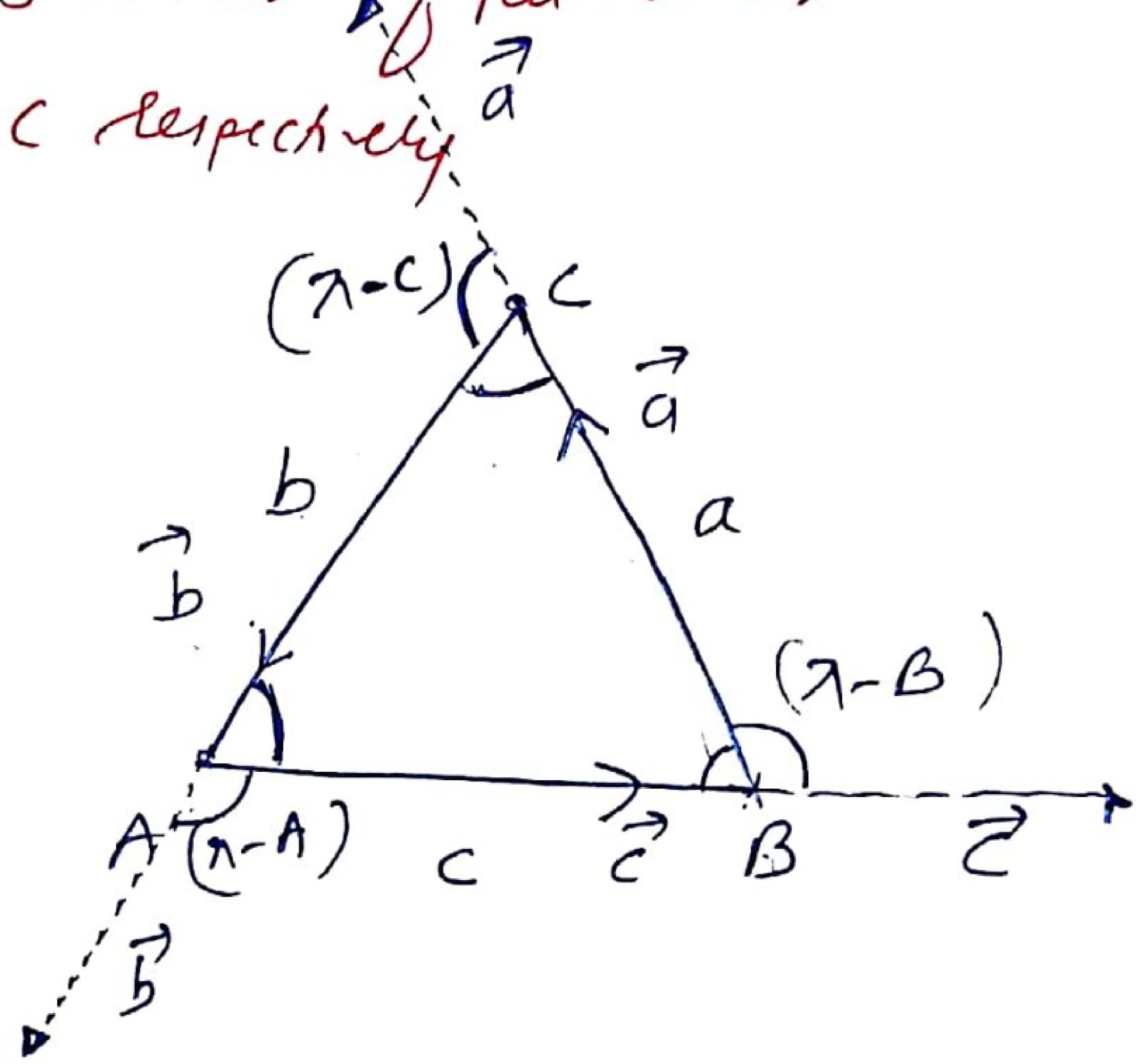
ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER:- VECTORS | CLASS No:- 6

Qns 1 Prove that in ΔABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ where a, b, c represents magnitudes of the sides opposite to the vertices A, B, C respectively.



Soln By triangle law

$$\vec{c} + \vec{a} = -\vec{b}$$

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$$

$$\Rightarrow \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \quad \text{---}(1)$$

again

$$\vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times \vec{0}$$

$$\Rightarrow \vec{b} \times \vec{a} + \vec{0} + \vec{b} \times \vec{c} = \vec{0}$$

$$\Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \quad \text{--}(2) \text{---} \left\{ \begin{array}{l} \vec{a} \times \vec{a} = \vec{0} \\ \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \end{array} \right\}$$

from (1) & (2)

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$$

$$\Rightarrow |\vec{a}||\vec{b}| \sin(\pi - C) = |\vec{b}||\vec{c}| \sin(\pi - A) = |\vec{c}||\vec{a}| \sin(\pi - B)$$

$\Rightarrow ab\sin C = bc\sin A = ca\sin B$

divide by abc

$\Rightarrow \dfrac{\sin C}{c} = \dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$\Rightarrow \boxed{\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}}$ proved

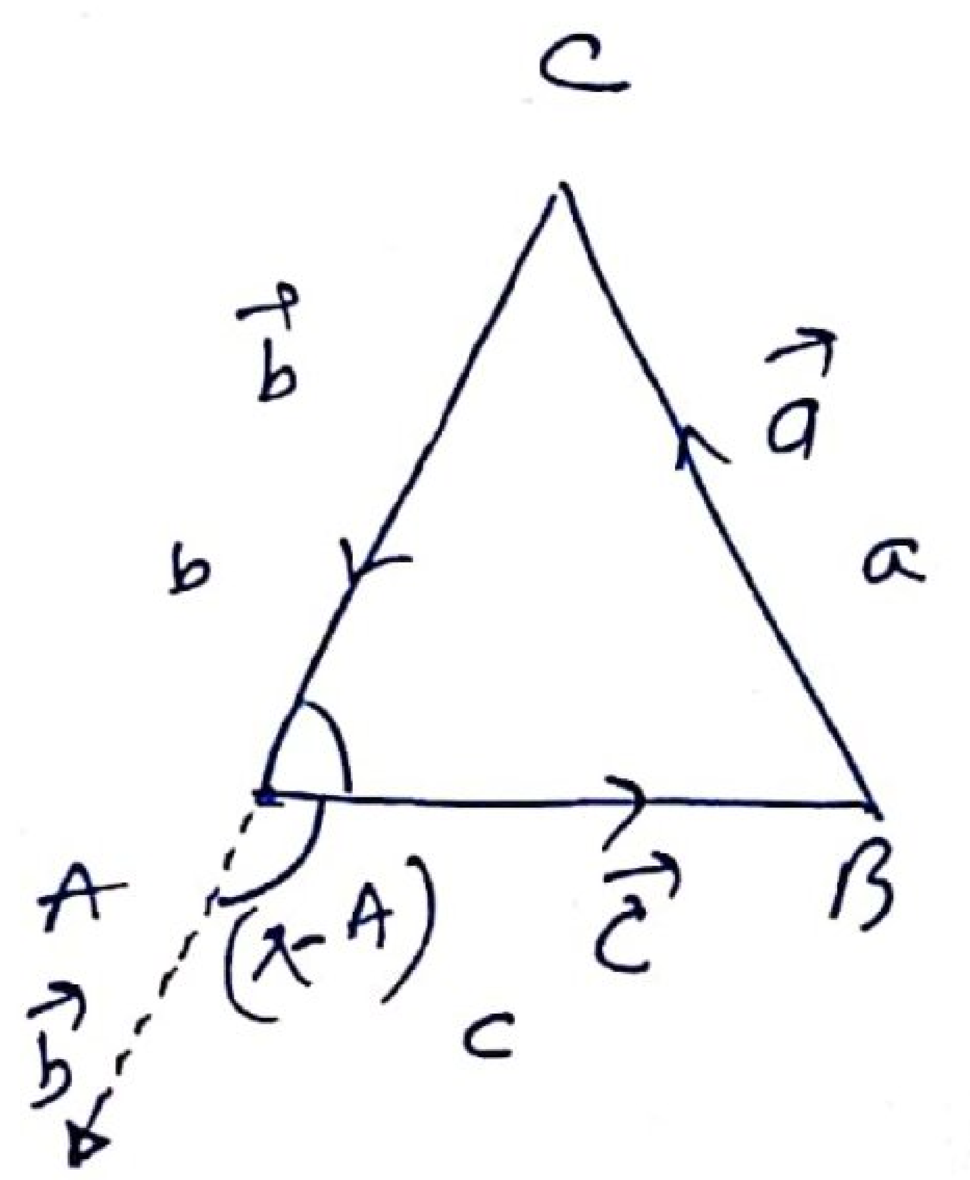
Ques 2 → Prove that in ΔABC, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, where a, b, c are the sides of the triangle opposite to the vertices A, B, C respectively.

Sol: $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{b} + \vec{c} = -\vec{a}$

$\Rightarrow |\vec{b} + \vec{c}| = |-\vec{a}| = |\vec{a}|$

$\Rightarrow |\vec{b} + \vec{c}|^2 = |\vec{a}|^2$



$\Rightarrow (\vec{b} + \vec{c})\cdot(\vec{b} + \vec{c}) = |\vec{a}|^2$

$\Rightarrow |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c} = |\vec{a}|^2$

$\Rightarrow |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos(\pi - A) = |\vec{a}|^2$

$\Rightarrow b^2 + c^2 + 2bc(-\cos A) = a^2$

$\Rightarrow b^2 + c^2 - a^2 = 2bc\cos A$

$\Rightarrow \boxed{\cos A = \dfrac{b^2+c^2-a^2}{2bc}}$ Ans

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Ans 3 → Using vector, show that

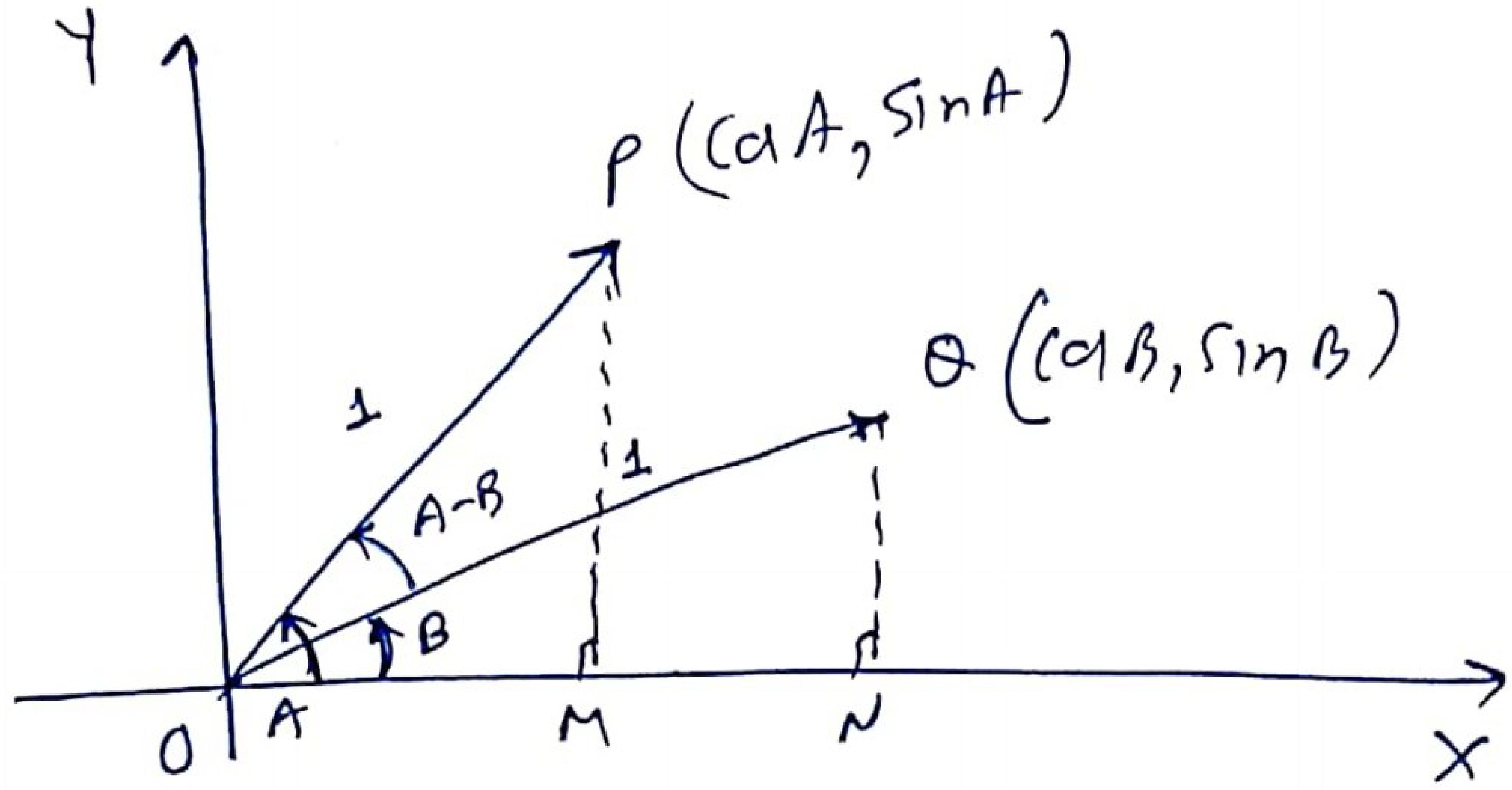
$\cos(A-B) = \cos A \cos B + \sin A \sin B$

Soln

Let $\vec{OP}$ & $\vec{OQ}$ are two unit vectors in the X-Y plane making angles A & B with X-axis respectively

$\vec{OP} = \cos A\,\hat{i} + \sin A\,\hat{j}$

$\vec{OQ} = \cos B\,\hat{i} + \sin B\,\hat{j}$



Now angle b/w $\vec{OP}$ & $\vec{OQ}$

$$\cos(A-B) = \frac{\vec{OP}\cdot\vec{OQ}}{|\vec{OP}|\,|\vec{OQ}|}$$

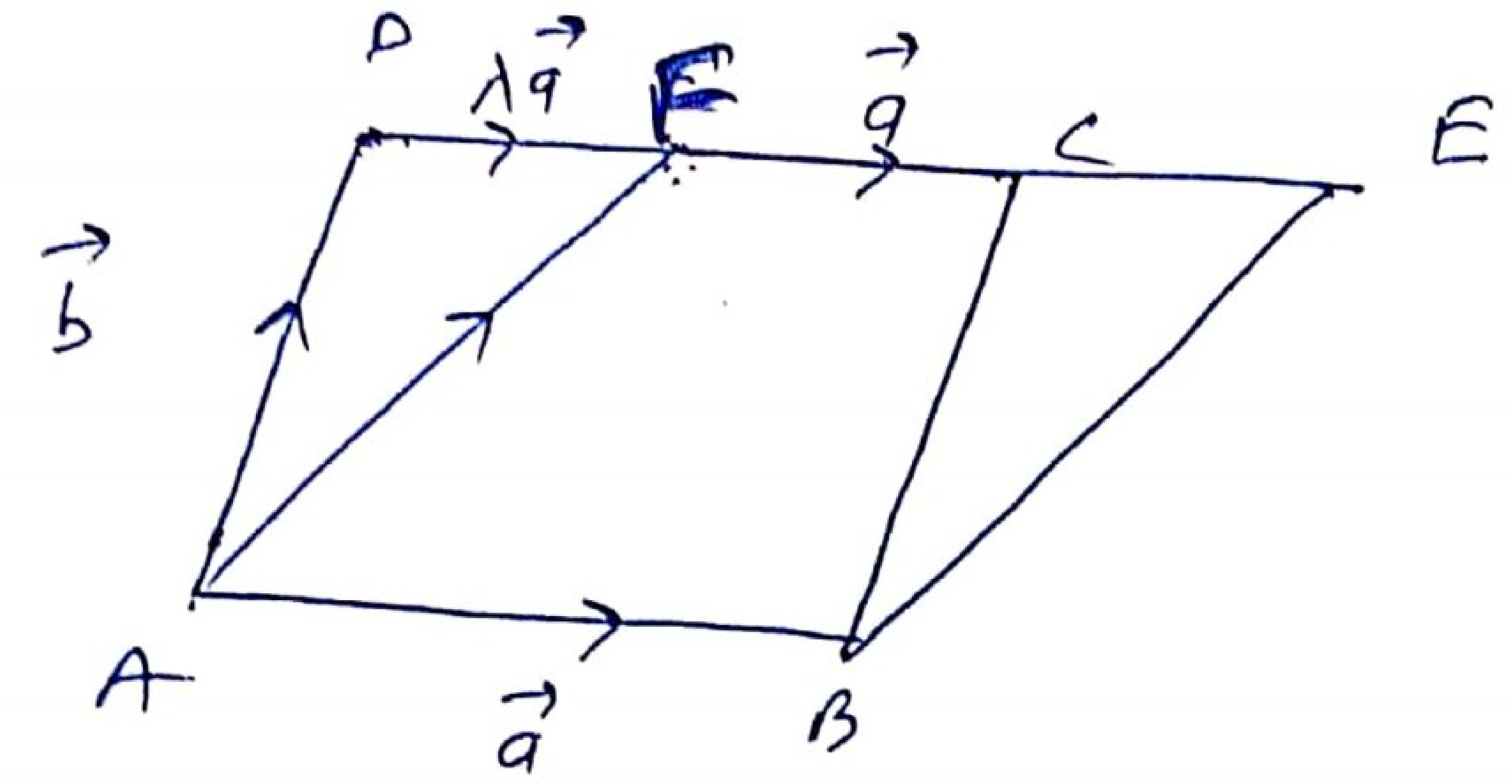
$$\Rightarrow \cos(A-B) = \frac{(\cos A\,\hat{i} + \sin A\,\hat{j})\cdot(\cos B\,\hat{i} + \sin B\,\hat{j})}{(1)(1)}$$

$$\Rightarrow \boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B}$$

Ans

Qn 4 → Using vectors, prove that the parallelogram on the same base and between the same parallels are equal in Area.

Soln



$\vec{DF}$ & $\vec{a}$ are collinear

$\Rightarrow \vec{DF} = \lambda\vec{a}$

Δ ADF (by triangle law)

$\vec{b} + \lambda\vec{a} = \vec{AF}$

Area of parallelogram ABCD $= |\vec{a}\times\vec{b}|$

Area of parallelogram ABEF $= |\vec{a}\times(\vec{AF})|$

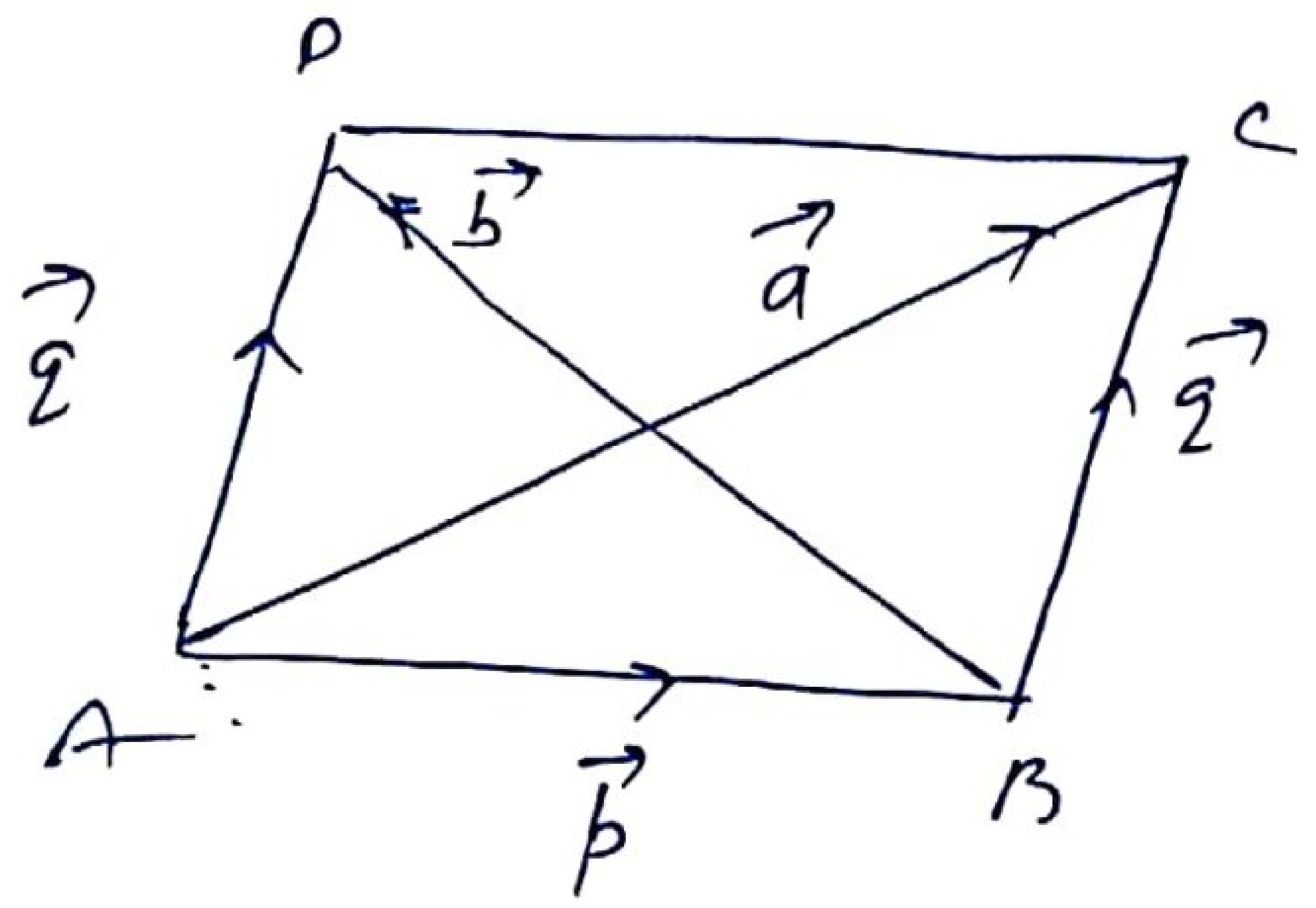
$= |\vec{a}\times(\vec{b}+\lambda\vec{a})|$

$= |\vec{a}\times\vec{b} + \vec{0}|$

$= |\vec{a}\times\vec{b}|$

= Area of parallelogram ABCD Proved

Ques 5 → Show that the area of the parallelogram whose diagonals are given by $\vec{a}$ & $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$
Also find area of the parallelogram whose diagonals are $2\hat{i}-\hat{j}+\hat{k}$ & $\hat{i}+3\hat{j}-\hat{k}$



Sol → Let $\vec{p}$ & $\vec{q}$ are the adjacent sides of ABCD

In ΔABC

$$\boxed{\vec{p}+\vec{q}=\vec{a}} \quad \text{--- (1)}$$

In ΔABD

$$\vec{p}+\vec{b}=\vec{q}$$

$$\boxed{\vec{b}=\vec{q}-\vec{p}} \quad \text{--- (2)}$$

(1) + (2)

$$2\vec{q}=\vec{a}+\vec{b}$$

✓ $\vec{q}=\frac{1}{2}(\vec{a}+\vec{b})$

(1) − (2)

✓ $\vec{p}=\frac{1}{2}(\vec{a}-\vec{b})$

Area Parallelogram ABCD $= |\vec{p}\times\vec{q}|$

$$=\left|\frac{1}{2}(\vec{a}-\vec{b})\times\frac{1}{2}(\vec{a}+\vec{b})\right|$$

$$=\frac{1}{4}\left|(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})\right|$$

$$= \frac{1}{4}\left|\vec{a}\times\vec{b} - \vec{b}\times\vec{a} + \vec{0} + \vec{0}\right|$$

$$= \frac{1}{4}\left|\vec{a}\times\vec{b} + \vec{a}\times\vec{b}\right|$$

$$= \frac{2}{4}\left|\vec{a}\times\vec{b}\right|$$

Area of ||gm $= \frac{1}{2}|\vec{a}\times\vec{b}|$ Proved

(ii) Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$

& $\vec{b} = \hat{i} + 3\hat{j} - \hat{k}$

Self Proceed

Qn. 6 If $|\vec{a}| = 4$ and $-3 \le \lambda \le 2$, then find the range of $|\lambda\vec{a}|$

Soln We have $-3 \le \lambda \le 2$

$$\Rightarrow 0 \le |\lambda| \le 3$$

$$\Rightarrow 0 \le |\lambda|\,|\vec{a}| \le 3|\vec{a}|$$

$$\Rightarrow 0 \le |\lambda\vec{a}| \le 3\times 4 \quad \text{--- } \{|\lambda\vec{a}| = |\lambda||\vec{a}|\}$$

$$\Rightarrow 0 \le |\lambda\vec{a}| \le 12$$

$\therefore$ Range of $|\lambda\vec{a}| = [0, 12]$ Ans

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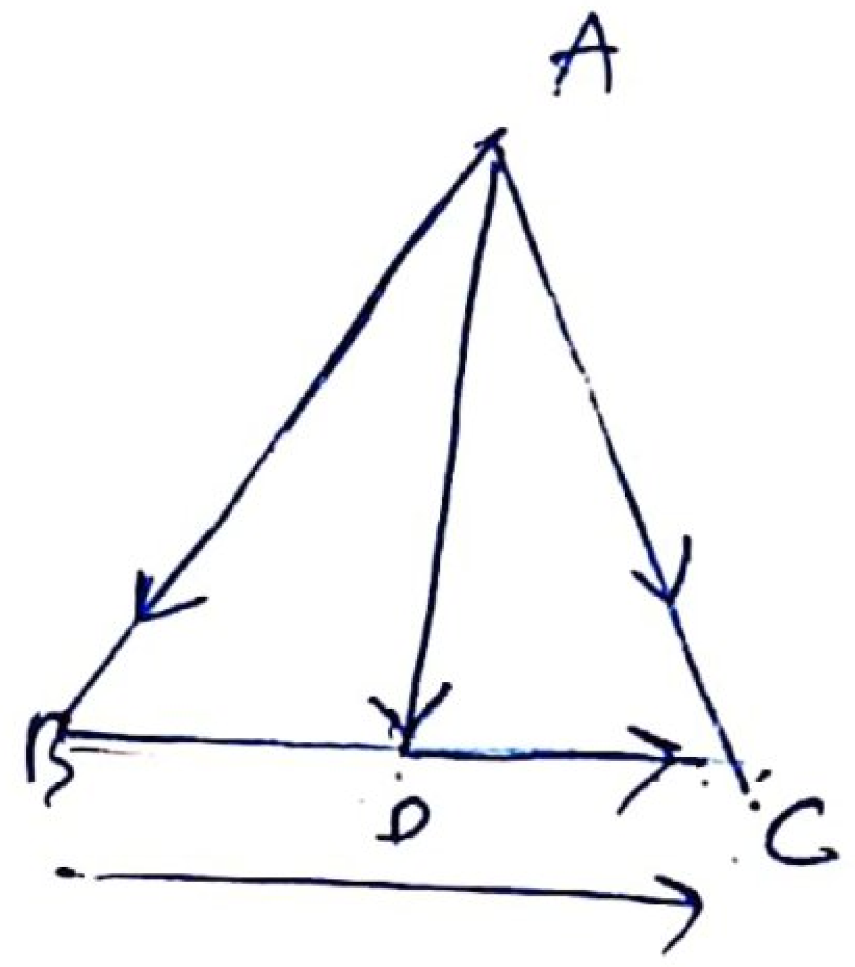
Ans 7 → Two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represents the two sides AB and AC respectively of a ΔABC. Find the length of Median through A



Given
$\vec{AB} = \hat{j}+\hat{k}$

$\vec{AC} = 3\hat{i}-\hat{j}+4\hat{k}$

to find $|\vec{AD}|$

$$\boxed{\text{Short cut} \quad \vec{AD} = \frac{1}{2}(\vec{AB}+\vec{AC})}$$

detail)

$$\vec{DC} = \frac{1}{2}(\vec{BC})$$

ΔADC (triangle law) $\qquad$ In ΔABC

$\vec{AD}+\vec{DC} = \vec{AC}$ $\qquad$ $\vec{AB}+\vec{BC} = \vec{AC}$

$\Rightarrow \vec{AD} = \vec{AC} - \frac{1}{2}(\vec{BC})$ $\qquad$ $\vec{BC} = \vec{AC} - \vec{AB}$

$\Rightarrow \vec{AD} = \vec{AC} - \frac{1}{2}(\vec{AC}-\vec{AB})$

$\Rightarrow \vec{AD} = \dfrac{\vec{AC}+\vec{AB}}{2}$

$\Rightarrow \vec{AD} = \frac{1}{2}(3\hat{i}+5\hat{k})$

$\Rightarrow |\vec{AD}| = \frac{1}{2}\sqrt{9+25}$

$|\vec{AB}| = \dfrac{\sqrt{34}}{2}$ units Ans

(9)

when $k = -1/2$

then $k\vec{a} + \frac{1}{2}\vec{a} =$ becomes $\vec{0}$

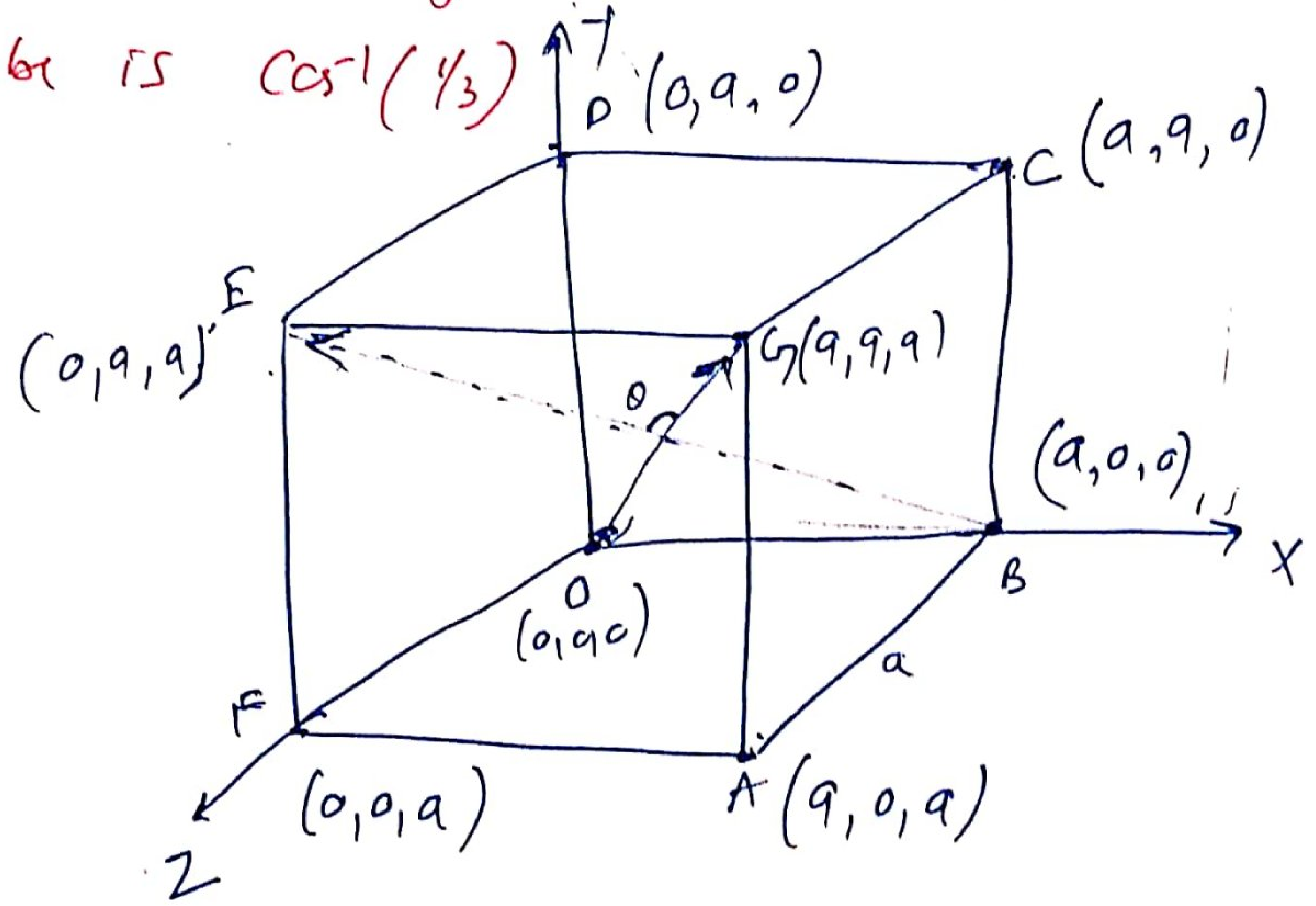
but $\vec{a}$ can't be $\parallel$ to $\vec{0}$

$\therefore k \neq -1/2$

$\therefore$ Required value of k $(-1,1) - \{-1/2\}$

$k \in (-1, -1/2) \cup (-1/2, 1)$ Ans

Qn 10 → Show that the angle b/w two diagonals of a cube is $\cos^{-1}(1/3)$



$\vec{BE} = -a\hat{i} + a\hat{j} + a\hat{k}$

$\vec{OG} = a\hat{i} + a\hat{j} + a\hat{k}$

let θ be the angle b/w them

$$\cos\theta = \frac{\vec{BE} \cdot \vec{OG}}{|\vec{BE}|\,|\vec{OG}|}$$

Qns 8 → If the points $(-1,-1,2)$, $(2,m,5)$ & $(3,11,6)$ are collinear. Find the value of m

Soln let $A(-1,-1,2)$ $B(2,m,5)$ $C(3,11,6)$

$\vec{OA} = -\hat{i}-\hat{j}+2\hat{k}$

$\vec{OB} = 2\hat{i}+m\hat{j}+5\hat{k}$

$\vec{OC} = 3\hat{i}+11\hat{j}+6\hat{k}$

A B C

$\vec{AB} = \vec{OB}-\vec{OA} = 3\hat{i}+\hat{j}(m+1)+3\hat{k}$

$\vec{BC} = \vec{OC}-\vec{OB} = \hat{i}+\hat{j}(11-m)+\hat{k}$

Since points A, B, C are collinear
∴ $\vec{AB}$ & $\vec{BC}$ are collinear

⇒ ~~Co~~ their correspondg. components are in equal Ratio

$$\frac{1}{3} = \frac{11-m}{m+1} = \frac{1}{3}$$

$$\Rightarrow 33-3m = m+1$$

$$\Rightarrow 32 = 4m$$

$$\Rightarrow m = 8$$ Ans

Qn. 9 → Find the values of k for which $|k\vec{a}| < |\vec{a}|$ and $(k\vec{a}+\frac{1}{2}\vec{a})$ is parallel to $\vec{a}$.

Soln we have $|k\vec{a}| < |\vec{a}|$

$\Rightarrow |k|\,\cancel{|\vec{a}|} < \cancel{|\vec{a}|} \Rightarrow |k| < 1 \Rightarrow \boxed{-1 < k < 1}$

when $k = -1/2$

then $k\vec{a} + \frac{1}{2}\vec{a} =$ becomes $\vec{0}$

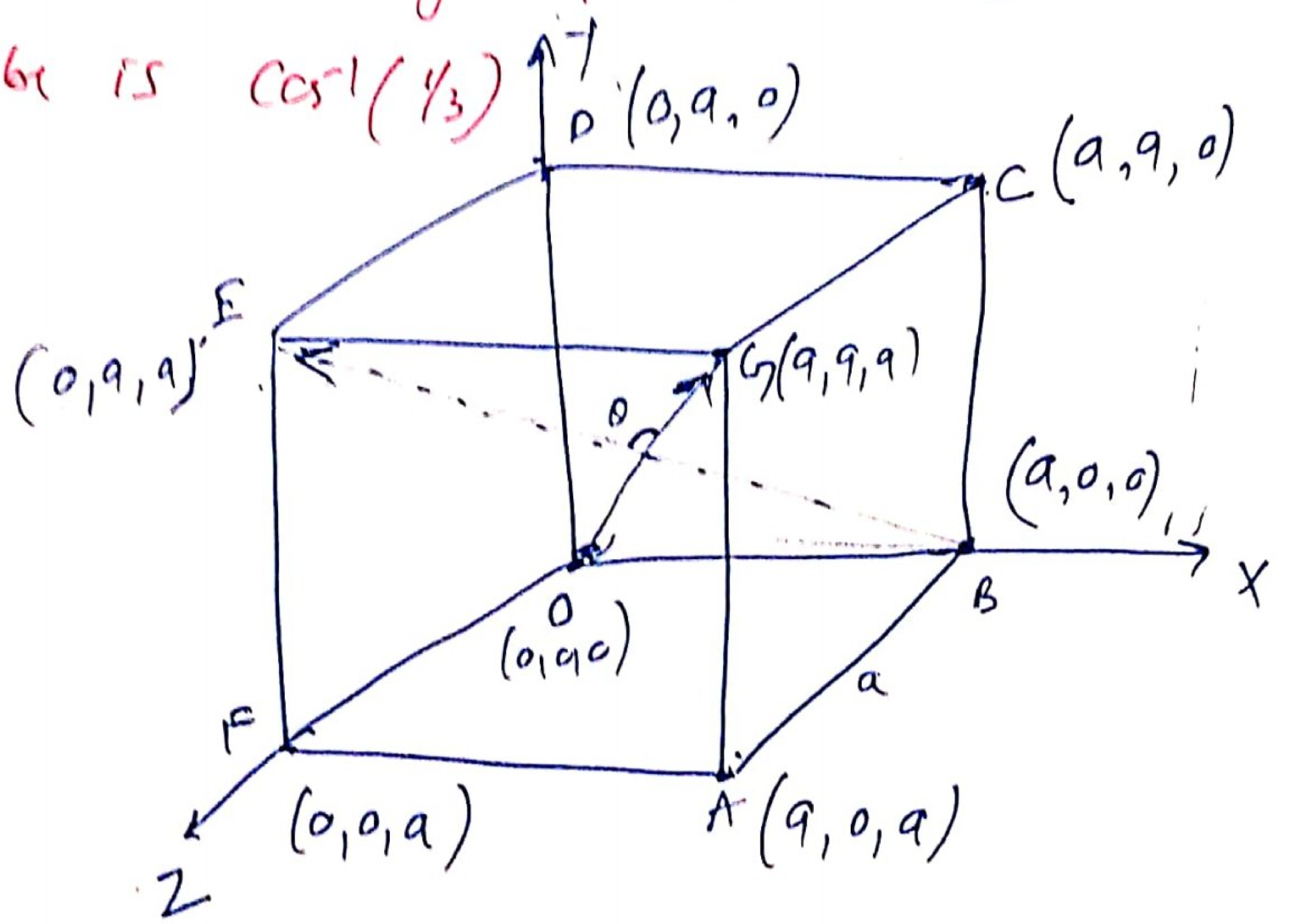
but $\vec{a}$ can't be $\parallel$ to $\vec{0}$

$\therefore k \neq -1/2$

$\therefore$ Required values of k $(-1, 1) - \{-1/2\}$

$k \in (-1, -1/2) \cup (-1/2, 1)$ Ans

Qn 10 → Show that the angle b/w two diagonals of a cube is $\cos^{-1}(1/3)$



$\overrightarrow{BE} = -a\hat{i} + a\hat{j} + a\hat{k}$

$\overrightarrow{OG} = a\hat{i} + a\hat{j} + a\hat{k}$

let θ be the angle b/w them

$$\cos\theta = \frac{\overrightarrow{BE} \cdot \overrightarrow{OG}}{|\overrightarrow{BE}|\,|\overrightarrow{OG}|}$$

$$\Rightarrow \cos\theta = \frac{(-a\hat{i} + a\hat{j} + a\hat{k}) \cdot (a\hat{i} + a\hat{j} + a\hat{k})}{\sqrt{a^2 + a^2 + a^2}\ \sqrt{a^2 + a^2 + a^2}}$$

$$= \frac{\cancel{-a^2} + \cancel{a^2} + a^2}{(\sqrt{3}\,a)\ (\sqrt{3}\,a)}$$

$$= \frac{a^2}{3a^2}$$

$$\cos\theta = \frac{1}{3}$$

$$\theta = \cos^{-1}(1/3)$$ Ans

WORKSHEET NO: 5 (VECTORS)

Ques 1 → Find a vector $\vec{r}$ of magnitude $3\sqrt{2}$ units which makes an angle of $\frac{\pi}{4}$ and $\frac{\pi}{2}$ with Y and Z axes respectively

Ans $\vec{r} = \pm 3\hat{i} + 3\hat{j}$

Hint First find α (or) $\cos\alpha$ (or) l

then $\vec{r} = |\vec{r}|\,\hat{r}$ and $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$

Ques 2 → Find all vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane containing the vectors $\hat{i} + 2\hat{j} + \hat{k}$ and $-\hat{i} + 3\hat{j} + 4\hat{k}$

Ans $\pm 10(\hat{i} - \hat{j} + \hat{k})$

Hint required vector $= (10\sqrt{3})\hat{n}$ where $\vec{n} = \vec{a} \times \vec{b}$

Ques 3 → If $\vec{a}$ & $\vec{b}$ are unit vectors that what is the angle between $\vec{a}$ & $\vec{b}$ for $\sqrt{3}\vec{a} - \vec{b}$ to be a unit vector?

$\boxed{\text{Ans } \theta = \pi/6}$

Ques 4 → If $|\vec{a}| = 3$ and $-1 \leq k \leq 2$ then find the interval in which $|k\vec{a}|$ lies.

Ans $[0, 6]$

Ques 5 → If $\vec{a}$ & $\vec{b}$ are the position vectors of A & B respectively. Find the position vector of a point C in BA produced such that $BC = 1.5\,BA$

Ans $\vec{c} = \dfrac{3\vec{b} - \vec{a}}{2}$

Q.No. 6 → Using vectors, find the value of 'k' such that the points $(k, -10, 3)$, $(1, -1, 3)$ and $(3, 5, 3)$ are collinear. **Ans** $k = -2$

Q.No. 7 → A vector $\vec{r}$ is inclined at equal angles to the three axes. If the magnitude of $\vec{r}$ is $2\sqrt{3}$ units. Find $\vec{r}$ **Ans** $\pm 2(\hat{i} + \hat{j} + \hat{k})$

Q.No. 8 → In ΔABC, using vectors show that

$$\cos C = \frac{b^2 + a^2 - c^2}{2ab},$$

where a, b, c are the magnitudes of the sides opposite to the vertices A, B, C respectively.

Q.No. 9 → State True or False

(i) If $|\vec{a}| = |\vec{b}|$, then necessarily it implies $\vec{a} = \pm \vec{b}$

(ii) If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then the vectors $\vec{a}$ & $\vec{b}$ are orthogonal

Q.No 10 If $\vec{a}$ is a non-zero vector then show that $(\vec{a} \cdot \hat{i})\hat{i} + (\vec{a} \cdot \hat{j})\hat{j} + (\vec{a} \cdot \hat{k})\hat{k} = \vec{a}$

Hint let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

Q.No 11 → The vectors $3\hat{i} - 2\hat{j} + 2\hat{k}$ & $-\hat{i} - 2\hat{k}$ are the adjacent sides of a parallelogram. Then find the acute angle b/w its diagonals **Ans** $\pi/4$

Qu: 12 → Fill in the blanks

(i) The vector $\vec{a}+\vec{b}$ bisects the angle b/w the non-collinear vectors $\vec{a}$ & $\vec{b}$ if ______

(2) value of expression $|\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2$ is ______

(3) If $|\vec{a}\times\vec{b}|^2 + |\vec{a}\cdot\vec{b}|^2 = 144$ & $|\vec{a}| = 4$ then $|\vec{b}| =$ ______

Qu 13 → Show that the points $A(6,-7,0)$, $B(16,-19,-4)$, $C(0,3,-6)$ and $D(2,-5,10)$ are such that AB and CD intersect at the point $P(1,-1,2)$

Special

Qu 14 → If angle b/w $\vec{a}$ & $\vec{b}$ is 120°. If $|\vec{a}|=1$ & $|\vec{b}|=2$ then $|(\vec{a}+3\vec{b})\times(3\vec{a}-\vec{b})|^2$ is equal to

(A) 300 (B) 325 (C) 275 (D) 225 Ans (A)

Qu 15 → If $\vec{p}$ & $\vec{q}$ are unit vectors forming an angle of 30°. Find the area of the parallelogram having $\vec{a} = \vec{p}+2\vec{q}$ and $\vec{b} = 2\vec{p}+\vec{q}$ as its diagonals

Ans $\frac{3}{4}$ square units

Qu 16 → Show that the line segments joining the mid points of the adjacent sides of a quadrilateral taken in order form a parallelogram (using vectors)

– x –