→ ULTIMATE MATHEMATICS →

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WORKSHEET D-7 (with solution)

Que 1 → Show that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

Solution

$\left.\begin{array}{l} C_1 \to C_1 - bC_3 \\ C_2 \to C_2 + aC_3 \end{array}\right\}$ Imp step

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ 2b-b+a^2b+b^3 & -2a+a-a^3-ab^2 & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

taking $(1+a^2+b^2)$ common from C_1 & C_2

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

expanding

$$= (1+a^2+b^2)^2 \left[1(1-a^2-b^2+2a^2) - 2b(0-b) \right]$$

$$= (1+a^2+b^2)^2 (1+a^2-b^2+2b^2)$$

$$= (1+a^2+b^2)^2 (1+a^2+b^2)$$

$$= (1+a^2+b^2)^3$$

Ans

Determinants

Ques 2 → Show
$$\begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$$

Solution

$R_1 \to xR_1$, $R_2 \to yR_2$, $R_3 \to zR_3$.

$$= \frac{1}{xyz}\begin{vmatrix} x(y+z)^2 & x^2y & zx^2 \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

taking x, y, z common from C_1, C_2 & C_3 say

$$= \frac{xyz}{xyz}\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}.$$

$C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} (y+z+x)(y+z-x) & 0 & x^2 \\ 0 & (x+z+y)(x+z-y) & y^2 \\ (z+x+y)(z-x-y) & (z+x+y)(z-x-y) & (x+y)^2 \end{vmatrix}$$

taking $(x+y+z)$ common from C_1 & C_2

$$= (x+y+z)^2\begin{vmatrix} y+z-x & 0 & x^2 \\ 0 & x+z-y & y^2 \\ z-x-y & z-x-y & (x+y)^2 \end{vmatrix}$$

$R_3 \to R_3 - (R_1 + R_2)$.

$$= (x+y+z)^2\begin{vmatrix} y+z-x & 0 & x^2 \\ 0 & x+z-y & y^2 \\ -2y & -2x & 2xy \end{vmatrix}$$

Multiply C_1 & C_2 by x & y resp.

$$= \frac{(x+y+z)^2}{xy} \begin{vmatrix} xy+zx-x^2 & 0 & x^2 \\ 0 & xy+zy-y^2 & y^2 \\ -2xy & -2xy & 2xy \end{vmatrix}$$

$C_1 \to C_1 + C_3$ and $C_2 \to C_2 + C_3$

$$= \frac{(x+y+z)^2}{xy} \begin{vmatrix} xy+zx & x^2 & x^2 \\ y^2 & xy+zy & y^2 \\ 0 & 0 & 2xy \end{vmatrix}$$

taking x & y common from R_1 & R_2 resp

$$= \frac{(x+y+z)^2}{\cancel{xy}} (\cancel{xy}) \begin{vmatrix} y+z & x & x \\ y & x+z & y \\ 0 & 0 & 2xy \end{vmatrix}$$

taking $(2xy)$ common from R_3

$$= 2xy(x+y+z)^2 \begin{vmatrix} y+z & x & x \\ y & x+z & y \\ 0 & 0 & 1 \end{vmatrix}$$

expanding

$$= 2xy(x+y+z)^2 \left[(y+z)(x+z) + x(0)) - x(y) \right.$$

$$= 2xy(x+y+z)^2 (\cancel{xy} + zy + zx + z^2 - \cancel{xy})$$

$$= 2xy(x+y+z)^2 (xz + zy + z^2)$$

$$= 2xyz(x+y+z)^2(x+y+z)$$

$$= 2xyz(x+y+z)^3$$ Ans.

Qn 3: If $\Delta = \begin{vmatrix} 3 & -2 & \sin(3\theta) \\ -7 & 8 & \cos(2\theta) \\ -11 & 14 & 2 \end{vmatrix} = 0$ then

show that $\sin\theta = 0$ (or) $\sin\theta = 1/2$

Solution: we have $\begin{vmatrix} 3 & -2 & \sin(3\theta) \\ -7 & 8 & \cos(2\theta) \\ 11 & 14 & 2 \end{vmatrix} = 0$

expanding

$= 3(16 - 14\cos(2\theta)) + 2(-14 + 11\cos(2\theta)) + \sin(3\theta)(-98 + 88) = 0$

$\Rightarrow 48 - 42\cos(2\theta) - 28 + 22\cos(2\theta) - 10\sin(3\theta) = 0$

$\Rightarrow 20 - 20\cos(2\theta) - 10\sin(3\theta) = 0$

divide by 10

$\Rightarrow 2 - 2\cos(2\theta) - \sin(3\theta) = 0$

$\Rightarrow 2 - 2(1 - 2\sin^2\theta) - (3\sin\theta - 4\sin^3\theta) = 0$

$\Rightarrow \cancel{2} - \cancel{2} + 4\sin^2\theta - 3\sin\theta + 4\sin^3\theta = 0$

$\Rightarrow \sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow \sin\theta(4\sin^2\theta + 6\sin\theta - 2\sin\theta - 3) = 0$

$\Rightarrow \sin\theta\left(2\sin\theta(2\sin\theta + 3) - 1(2\sin\theta + 3)\right) = 0$

$\Rightarrow \sin\theta\left[(2\sin\theta - 1)(2\sin\theta + 3)\right] = 0$

$\Rightarrow \sin\theta = 0$ | $\sin\theta = 1/2$ | $\sin\theta = -3/2$ Rejected since $-1 \le \sin\theta \le 1$

$\therefore$ either

$\sin\theta = 0$ or $\sin\theta = 1/2$ Proved

Qns = 4 → In ΔABC if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0$$

prove that ΔABC is an isosceles triangle

Soln we have
$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B-\sin A & \sin C-\sin A \\ \sin A+\sin^2 A & (\sin B-\sin A)+(\sin^2 B-\sin^2 A) & (\sin C-\sin A)+(\sin^2 C-\sin^2 A) \end{vmatrix} = 0$$

taking $(\sin B-\sin A)$ and $(\sin C-\sin A)$ common from C_2 & C_3 respectively

$= \cancel{\sin(B-A}$
$$= (\sin B-\sin A)(\sin C-\sin A)\begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & 1 & 1 \\ \sin A+\sin^2 A & 1+\sin B+\sin A & 1+\sin C+\sin A \end{vmatrix} = 0$$

expanding

$$= (\sin B-\sin A)(\sin C-\sin A)\left[\cancel{1}+\sin C+\cancel{\sin A}-\cancel{1}-\sin B-\cancel{\sin A}\right] = 0$$

$$\Rightarrow (\sin B-\sin A)(\sin C-\sin A)(\sin C-\sin B) = 0$$

$\Rightarrow$ either $\sin B-\sin A = 0$ (or) $\sin C-\sin A = 0$ | $\sin C-\sin B = 0$

$\Rightarrow \sin B = \sin A$ (or) $\sin C = \sin A$ (or) $\sin C = \sin B$

$\Rightarrow B = A$ (or) $C = A$ (or) $C = B$

$\therefore$ Clearly ΔABC is an isosceles triangle.

Qn. 5 → Show that ΔABC is an isosceles triangle

Given that
$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$$

Do yourself (Same as Qn No: 4)

Qn. 6 → Find θ such that
$$\begin{vmatrix} 1 & 1 & \sin(3\theta) \\ -4 & 3 & \cos(2\theta) \\ 7 & -7 & -2 \end{vmatrix} = 0$$

Do yourself (Same as Qn No: 3)

Ans $\theta = n\pi$ (or) $\theta = n\pi + (-1)^n\left(\frac{\pi}{6}\right)$; $n \in Z$

Qn. 7 → evaluate
$$\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{46}+\sqrt{15} & 5 & \sqrt{10} \\ \sqrt{115}+3 & \sqrt{15} & 5 \end{vmatrix}$$

Solution

Taking $\sqrt{5}$ common from C_2 and C_3

$$= (\sqrt{5})(\sqrt{5}) \begin{vmatrix} \sqrt{23}+\sqrt{3} & 1 & 1 \\ \sqrt{46}+\sqrt{15} & \sqrt{5} & \sqrt{2} \\ \sqrt{115}+3 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

Sum property in C_1

$$= 5\begin{vmatrix} \sqrt{23} & 1 & 1 \\ \sqrt{46} & \sqrt{5} & \sqrt{2} \\ \sqrt{115} & \sqrt{3} & \sqrt{5} \end{vmatrix} + 5\begin{vmatrix} \sqrt{3} & 1 & 1 \\ \sqrt{15} & \sqrt{5} & \sqrt{2} \\ 3 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

taking $\sqrt{23}$ common from C_1 & $\sqrt{3}$ common from C_1

$$= \sqrt{23} \times 5 \begin{vmatrix} 1 & 1 & 1 \\ \sqrt{2} & \sqrt{5} & \sqrt{2} \\ \sqrt{5} & \sqrt{3} & \sqrt{5} \end{vmatrix} + 5\sqrt{3}\begin{vmatrix} 1 & 1 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

identical   Identical

$$= 5\sqrt{23} \times 0 + 5\sqrt{3} \times 0$$

$$= 0 \text{ Ans}$$

Ans 8 → Without expanding show that

SELF

$$\begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0$$

Hint $C_2 \to C_2 + C_3$

Qn. 9 → without expanding show that

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$$

Solution

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Sum property in C_3

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta \end{vmatrix} - \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\sin\delta \end{vmatrix}$$

$\cos\delta$ common C_3 of I^{st} determinant
and $\sin\delta$ common from C_3 of 2^{nd} determinant

$$= \cos\delta \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha \\ \sin\beta & \cos\beta & \cos\beta \\ \sin\gamma & \cos\gamma & \cos\gamma \end{vmatrix} - \sin\delta \cdot \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta & \sin\beta \\ \sin\gamma & \cos\gamma & \sin\gamma \end{vmatrix}$$

Identical   Identical

$$= \cos\delta \times 0 - \sin\delta \times 0$$
$$= 0 \quad \underline{\text{Ans}}$$

Que 10 → find $\begin{vmatrix} 1 & 1 & 1 \\ {}^nC_1 & {}^{n+2}C_1 & {}^{n+4}C_1 \\ {}^nC_2 & {}^{n+2}C_2 & {}^{n+4}C_2 \end{vmatrix}$

Soln we know that ${}^nC_1 = n$ and ${}^nC_2 = \frac{n(n-1)}{2}$
{from XI class}

$$= \begin{vmatrix} 1 & 1 & 1 \\ n & n+2 & n+4 \\ \frac{n(n-1)}{2} & \frac{(n+2)(n+1)}{2} & \frac{(n+4)(n+3)}{2} \end{vmatrix}$$

$\frac{1}{2}$ common from R_3

$$= \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ n & (n+2) & (n+4) \\ n^2-n & n^2+3n+2 & n^2+7n+12 \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ n & 2 & 4 \\ n^2-n & 4n+2 & 8n+12 \end{vmatrix}$$

expanding

$$= \frac{1}{2}\left[1(16n+24-16n-8)\right)$$

$$= \frac{1}{2}\left[16\right)$$

$$= 8 \quad \underline{\underline{Ans}}$$

QN 11 → $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ & $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

Show that $\Delta + \Delta_1 = 0$

Soluh.

Consider $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

$C_1 \to xC_1$, $C_2 \to yC_2$, $C_3 \to zC_3$

$$= \frac{1}{xyz}\begin{vmatrix} x & y & z \\ xyz & xyz & xyz \\ x^2 & y^2 & z^2 \end{vmatrix}$$

xyz common from R_2

$$= \frac{\cancel{xyz}}{\cancel{xyz}}\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$\downarrow$ Transpose

$$\Delta_1 = \begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} \quad \text{--- \{Since } |A'| = |A| \text{\}}$$

$C_1 \leftrightarrow C_2$

$$\Delta_1 = - \underbrace{\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}}_{\text{given equal to } \Delta}$$

$\Delta_1 = -\Delta$

$\boxed{\Delta + \Delta_1 = 0}$ Ans

QN 12 → Find matrix X such that

$$\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} X \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$$

Soln let $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$; $C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$

$\Rightarrow A X B = C$

(Pre-multiply by A^{-1} and post multiply by B^{-1})

$$A^{-1} A X B B^{-1} = A^{-1} C B^{-1}$$

$\Rightarrow \quad I X I = A^{-1} C B^{-1}$

$\Rightarrow \quad X = A^{-1} C B^{-1}$

$|A| = 15 - 14 = 1$

$\text{Adj } A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$|B| = -1 + 2 = 1$

$\text{Adj } B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

$B^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

$$X = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$X = \begin{bmatrix} -16 & 3 \\ 24 & -5 \end{bmatrix}$$ Ans

Note: Don't let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(bht lengthy ho jayega)

Qn 13 Find matrix X such that

Soln $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} X \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Do yourself

Ans $X = \begin{bmatrix} 9 & -14 \\ -16 & 25 \end{bmatrix}$

— x —