!! जय श्री राधे कृष्णा जय श्री गिरिराज जी महाराज !! (1)

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER: INTEGRATION CLASS NO: 12

Type

(1) $\int \frac{N^r}{\text{linear}\sqrt{\text{linear}}}\,dx$ put linear $= t^2$

(2) $\int \frac{N^r}{\text{Quadratic}\sqrt{\text{linear}}}\,dx$ put linear $= t^2$

(3) $\int \frac{N^r}{\text{linear}\sqrt{\text{Quadratic}}}\,dx$ put linear $= \frac{1}{t}$

(4) $\int \frac{N^r}{\text{Quad}\sqrt{\text{Quadr..}}}\,dx$ put $x = \frac{1}{t}$

Que 1

$$I = \int \frac{1}{(x+1)\sqrt{x+2}}\,dx$$

put $x+2 = t^2$

$dx = 2t\,dt$

$$I = 2\int \frac{\cancel{t}\,dt}{(t^2-2+1)\cdot\cancel{t}}$$

$$= 2\int \frac{dt}{t^2-1}$$

$$= \cancel{2}\times\frac{1}{\cancel{2}}\log\left|\frac{t-1}{t+1}\right| + C$$

$$I = \log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$$

Qn. 2

$$I = \int \frac{x+1}{(x^2+3)\sqrt{x+2}}\,dx$$

put $x+2 = t^2$

$dx = 2t\,dt$

$$I = 2\int \frac{(t^2-2+1)\,\cancel{t}\,dt}{[(t^2-2)^2+3]\cdot\cancel{t}}$$

$$= 2\int \frac{t^2-1}{t^4+4-4t^2+3}\,dt$$

$$= 2\int \frac{t^2-1}{t^4-4t^2+1}\,dt$$

Divide by t^2

$$I = 2\int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}-4}\,dt$$

$$= 2\int \frac{1-\frac{1}{t^2}}{\underbrace{(t+\tfrac{1}{t})}_{z}^2-2-4}\,dt$$

$$= 2\int \frac{dz}{z^2-(\sqrt{6})^2}$$

$$= \cancel{2}\times\frac{1}{\cancel{2}\sqrt{6}}\log\left|\frac{z-\sqrt{6}}{z+\sqrt{6}}\right|+C$$

$$= \frac{1}{\sqrt{6}}\log\left|\frac{t+\frac{1}{t}-\sqrt{6}}{t+\frac{1}{t}+\sqrt{6}}\right|+C$$

Ques 3 $I = \int \frac{1}{(x+1)\sqrt{(x^2+4)}} dx$

put $x+1 = \frac{1}{t}$

$dx = -\frac{1}{t^2} dt$

$$I = -\int \frac{\frac{1}{t^2} dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2+4}}$$

$$I = -\int \frac{\frac{1}{t^2} dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}+1-\frac{2}{t}+4}}$$

$$= -\int \frac{\cancel{\frac{1}{t^2}} dt}{\cancel{\frac{1}{t}}\sqrt{\frac{1+5t^2-2t}{\cancel{t}}}}$$

$$= -\int \frac{dt}{\sqrt{5t^2-2t+1}}$$ Perfect square proceed

Ques 4 $I = \int \frac{x^2}{(x\sin x + \cos x)^2} dx$

put $x\sin x + \cos x = t$

$(x\cos x + \cancel{\sin x} - \cancel{\sin x})\,dx = dt$

$x\cos x\, dx = dt$

$$I = \int \frac{x\cos x \cdot x\sec x}{(x\sin x + \cos x)^2} dx = \int \underbrace{\frac{x\cos x}{(x\sin x+\cos x)^2}}_{II} \cdot \underbrace{x\sec x}_{I}\, dx$$

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$$I = \int \underset{\text{II}}{\frac{x\cos x}{(x\sin x+\cos x)^2}} \cdot \underset{\text{I}}{x\sec x}\, dx$$

$$I = x\sec x \cdot \int \frac{x\cos x}{(x\sin x+\cos x)^2}dx - \int (x\sec x\tan x + \sec x) \cdot \int \frac{x\cos x}{(x\sin x+\cos x)^2} dx \cdot dx$$

put $x\sin x + \cos x = t$

$(x\cos x)\,dx = dt$

$$I = x\sec x \cdot \int \frac{dt}{t^2} - \int \left((x\sec x\tan x + \sec x) \cdot \int \frac{dt}{t^2}\right) dx$$

$$= \frac{-x\sec x}{x\sin x+\cos x} + \int \frac{\sec x\,(x\tan x+1)}{x\sin x+\cos x} \cdot dx$$

$$= \frac{-x\sec x}{x\sin x+\cos x} + \int \frac{\sec x\,(\cancel{x\sin x+\cos x})}{\cos x\,(\cancel{x\sin x+\cos x})} dx$$

$$= \frac{-x\sec x}{x\sin x+\cos x} + \int \sec^2 x\, dx$$

$$I = \frac{-x\sec x}{x\sin x+\cos x} + \tan x + C \quad \text{Ans}$$

Q.5 $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

$$\left\{\begin{array}{ll} \sqrt{1+x^2} & x=\tan\theta \\ \sqrt{a^2+x^2} & x=a\tan\theta \\ \sqrt{1+x} & \text{put } x=\tan^2\theta \\ \sqrt{a+x} & x=a\tan^2\theta \end{array}\right.$$

put $x = a\tan^2\theta$

$dx = 2a\tan\theta \cdot \sec^2\theta\, d\theta$

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$$\therefore I = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a + a\tan^2\theta}} \cdot 2a\tan\theta\sec^2\theta\, d\theta$$

$$= 2a\int \sin^{-1}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}} \cdot \tan\theta\sec^2\theta\, d\theta$$

$$= 2a\int \sin^{-1}(\sin\theta) \cdot \tan\theta\sec^2\theta\, d\theta$$

$$= 2a\int \underset{\mathrm{I}}{\theta} \cdot \underbrace{\tan\theta\sec^2\theta}_{\mathrm{II}}\, d\theta$$

$$\boxed{\begin{aligned}&\int \tan\theta\cdot\sec^2\theta\, d\theta\\ &= \int t\, dt\\ &= \frac{t^2}{2} = \frac{\tan^2\theta}{2}\end{aligned}}$$

$$= 2a\left[\theta\cdot\frac{\tan^2\theta}{2} - \frac{1}{2}\int \tan^2\theta\, d\theta\right]$$

$$= 2a\left[\frac{\theta\tan^2\theta}{2} - \frac{1}{2}\int \sec^2\theta - 1\, d\theta\right]$$

$$= 2a\left[\frac{\theta\cdot\tan^2\theta}{2} - \frac{1}{2}(\tan\theta - \theta)\right] + C$$

$$= a\left[\tan^{-1}\sqrt{\frac{x}{a}}\cdot\frac{x}{a} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C$$

$$I = a\left[\tan^{-1}\sqrt{\frac{x}{a}}\left(\frac{x}{a}+1\right) - \sqrt{\frac{x}{a}}\right] + C \quad \text{Ans}$$

Q1.6 $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$

put $x = t^2$

$dx = 2t\,dt$

$$I = 2\int \sqrt{\frac{1-t}{1+t}} \cdot t\, dt$$

$$I = 2\int \frac{(1-t)\cdot t}{\sqrt{1-t^2}}\,dt$$

$$= 2\int \frac{t-t^2}{\sqrt{1-t^2}}\,dt$$

$$= 2\int \frac{t}{\sqrt{1-t^2}}\,dt - 2\int \frac{t^2}{\sqrt{1-t^2}}\,dt$$

put $1-t^2 = z$ in I^r Integral

$$-2t\,dt = dz$$

$$t\,dt = \frac{-dz}{2}$$

$$I = -\frac{2}{2}\int \frac{dz}{\sqrt{z}} + 2\int \frac{-t^2}{\sqrt{1-t^2}}\,dt$$

$$= -\int \frac{dz}{\sqrt{z}} + 2\int \frac{1-t^2-1}{\sqrt{1-t^2}}\,dt$$

$$= -2\sqrt{z} + 2\int \sqrt{1-t^2} - \frac{1}{\sqrt{1-t^2}}\,dt$$

$$= -2\sqrt{1-t^2} + 2\left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}t - \sin^{-1}t\right] + C$$

$$= -2\sqrt{1-t^2} + t\sqrt{1-t^2} - \sin^{-1}t + C$$

$$= -2\sqrt{1-x} + \sqrt{x}\sqrt{1-x} - \sin^{-1}\sqrt{x} + C$$

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Qn. 7 $I = \int \frac{1}{x^{1/2}+x^{1/3}} dx$

$\boxed{\text{L.C.M of } 2 \& 3 = 6}$

put $x = t^6$

$dx = 6t^5 dt$

$I = 6\int \frac{t^5 dt}{t^3+t^2}$

$= 6\int \frac{t^3}{t+1} dt$

$$\begin{array}{r} t^2-t+1 \\ t+1 \,\big)\, t^3 \\ -(t^3+t^2) \\ \hline -t^2 \\ -(-t^2-t) \\ \hline t \\ -(t+1) \\ \hline -1 \end{array}$$

$= 6\int t^2-t+1 - \frac{1}{t+1} dt$

$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1|\right] + C$

$I = 6\left[\frac{x^{1/2}}{3} - \frac{x^{1/3}}{2} + x^{1/6} - \log|x^{1/6}+1|\right] + C$ Ans.

Qn 8 * $I = \int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$

$I = \int e^x \left(\frac{1+x^2-2x}{(1+x^2)^2}\right) dx$

$= \int e^x \left(\underbrace{\frac{1}{1+x^2}}_{f(x)} - \underbrace{\frac{2x}{(1+x^2)^2}}_{f'(x)}\right) dx$

$= \int e^x . \frac{1}{1+x^2} dx - 2\int e^x . \frac{x}{(1+x^2)^2} dx$

II 2

Proceed

Ans $\frac{e^x}{1+x^2} + C$

Q No. 9 → $I = \int \frac{x^9}{(4x^2+1)^6} dx$

$$= \int \frac{x^9}{x^{12}\left(4+\frac{1}{x^2}\right)^6} dx$$

$$= \int \frac{1}{x^3\left(4+\frac{1}{x^2}\right)^6} dx$$

put $4+\frac{1}{x^2} = t$ $\Rightarrow$ $-\frac{2}{x^3} dx = dt$

$\frac{1}{x^3} dx = -\frac{dt}{2}$

$$I = -\frac{1}{2}\int \frac{dt}{t^6}$$

$$= -\frac{1}{2}\int t^{-6} dt$$

$$= -\frac{1}{2}\frac{(t)^{-5}}{-5} + c$$

$$I = \frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5} + c$$ Ans

Q No. 10 → $I = \int e^{-3x} \cdot \cos^3 x \, dx$

$$I = \int e^{-3x}\left(\frac{3\cos x + \cos(3x)}{4}\right) dx$$

$$I = \frac{3}{4}\int e^{-3x}\cos x \, dx + \frac{1}{4}\int e^{-3x} \cdot \cos(3x) dx$$

$$I = \frac{3}{4} I_1 + \frac{1}{4} I_2 + c$$

Process

$I_1 = \int e^{-3x} \cos x \, dx$ & $I_2 = \int e^{-3x} \cdot \cos(3x) dx$

(class No 12)

Qn. 11 → $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x}\, dx$

$$= \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cdot \cos^2 x}\, dx$$

$$\frac{\cancel{\sin^4 x}}{\cancel{\cos^2 x}} \quad \tan^2 x \cdot \sin^2 x$$

$$= \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)}{\sin^2 x \cdot \cos^2 x}\, dx$$

Separate

$$= \int \tan^2 x + \cot^2 x - 1\, dx$$

$$= \int \sec^2 x - 1 + \operatorname{cosec}^2 x - 1 - 1\, dx$$

$$I = \tan x - \cot x - 3x + C \quad \text{Ans}$$

Qn. 12 →

$$I = \int \frac{\sqrt{1+x^2}}{x^4}\, dx$$

$$= \int \frac{x\sqrt{\frac{1}{x^2}+1}}{x^4}\, dx$$

$$= \int \frac{\sqrt{\frac{1}{x^2}+1}}{x^3}\, dx$$

put $\frac{1}{x^2} + 1 = t$

$$-\frac{2}{x^3}\, dx = dt \Rightarrow \frac{dx}{x^3} = \frac{-dt}{2}$$

$$I = -\frac{1}{2}\int \sqrt{t}\, dt = -\frac{1}{2} \times \frac{2}{3}\, t^{3/2} + C \quad \text{Ans}$$

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(class No: 12)

Qn 13 → $I = \int \frac{x^3\,dx}{x^4+3x^2+2}$

$= \int \frac{x^2 \cdot x\,dx}{x^4+3x^2+2}$

put $x^2 = t$

$x\,dx = \frac{dt}{2}$

$I = \frac{1}{2}\int \frac{t\,dt}{t^2+3t+2}$

$= \frac{1}{2}\int \frac{t}{(t+1)(t+2)}\,dt$

let $\frac{t}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$ Proceed

Qn 14 → If $\int \frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}}\,dx = ax + b\log|4e^x + 5e^{-x}| + C$

then find value of a & b

Sol. let $3e^x - 5e^{-x} = A \cdot \frac{d}{dx}(4e^x + 5e^{-x}) + B(4e^x + 5e^{-x})$

$\Rightarrow 3e^x - 5e^{-x} = A(4e^x - 5e^{-x}) + B(4e^x + 5e^{-x})$

equaty the cofficients of e^x & e^{-x}

$3 = 4A + 4B \quad \times 5 \quad \Rightarrow 15 = 20A + 20B$

$-5 = -5A + 5B \quad \times 4 \quad\quad -20 = -20A + 20B$

$-5 = 40B$

$B = -\frac{1}{8}$ $\quad A = \frac{7}{8}$

$\therefore$ given equation becomes

$$\int \frac{\frac{7}{8}(4e^x - 5e^{-x}) - \frac{1}{8}(4e^x + 5e^{-x})\,dx}{4e^x + 5e^{-x}}$$

$$= \frac{7}{8}\int \frac{4e^x - 5e^{-x}}{\underbrace{4e^x + 5e^{-x}}_{t}}\,dx + \int \frac{1}{8}\,dx$$

$$= \frac{7}{8}\int \frac{dt}{t} - \frac{1}{8}x + c$$

$$\Rightarrow \frac{7}{8}\log|4e^x + 5e^{-x}| - \frac{1}{8}x + c = ax + b\log|4e^x + 5e^{-x}| + c$$

Comp $b = 7/8$ $a = -1/8$ Ans

Ans 15 $\rightarrow$ $I = \int \frac{x^2\,dx}{x^4 + x^2 - 2}$

$$I = \int \frac{x^2\,dx}{(x^2+2)(x^2-1)}$$

partial fraction type IV

let $x^2 = y$

$$\frac{x^2}{(x^2+2)(x^2-1)} = \frac{y}{(y+2)(y-1)}$$

let $\frac{y}{(y+2)(y-1)} = \frac{A}{y+2} + \frac{B}{y-1}$. (10 case)

(class no 12)

12

Qn $\underline{16}$ → $I = \int \frac{x^{1/2}}{1+x^{3/4}} dx$

L.C.M of 2 & 4 = 4

put $x = t^4$

$dx = 4t^3 dt$

$I = 4\int \frac{t^2 \cdot t^3 dt}{1+t^3}$

$$t^3+1 \overline{\big) \cancel{t^5}} \; t^2$$
$$-(\cancel{t^5}+t^2)$$
$$\underline{-t^2}$$

$= 4\int \frac{t^5}{t^3+1} dt$

$= 4\int t^2 - \frac{t^2}{t^3+1} dt$

$= 4\left[\frac{t^3}{3}\right] - 4\int \frac{t^2}{t^3+1} dt$

put $t^3+1 = z$

$t^2 dt = \frac{dz}{3}$

$I = \frac{4t^3}{3} - \frac{4}{3}\int \frac{dz}{z}$

$= \frac{4}{3}t^3 - \frac{4}{3}\log|t^3+1| + C$

$I = \frac{4}{3}(x)^{3/4} - \frac{4}{3}\log|x^{3/4}+1| + C$

—x—