← ULTIMATE MATHEMATICS: BY AJAY MITTAL →

CHAPTER: AOI

CLASS NO: 3

(·) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

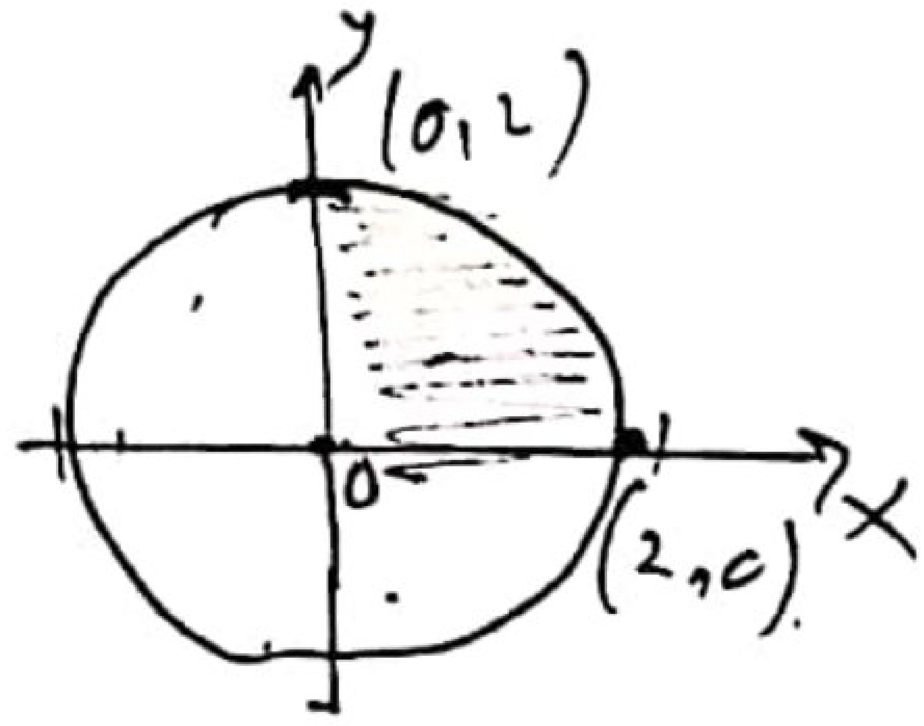
CIRCLE

(·) $(x-h)^2+(y-k)^2=r^2$

Centre (h,k) & rad $= r$

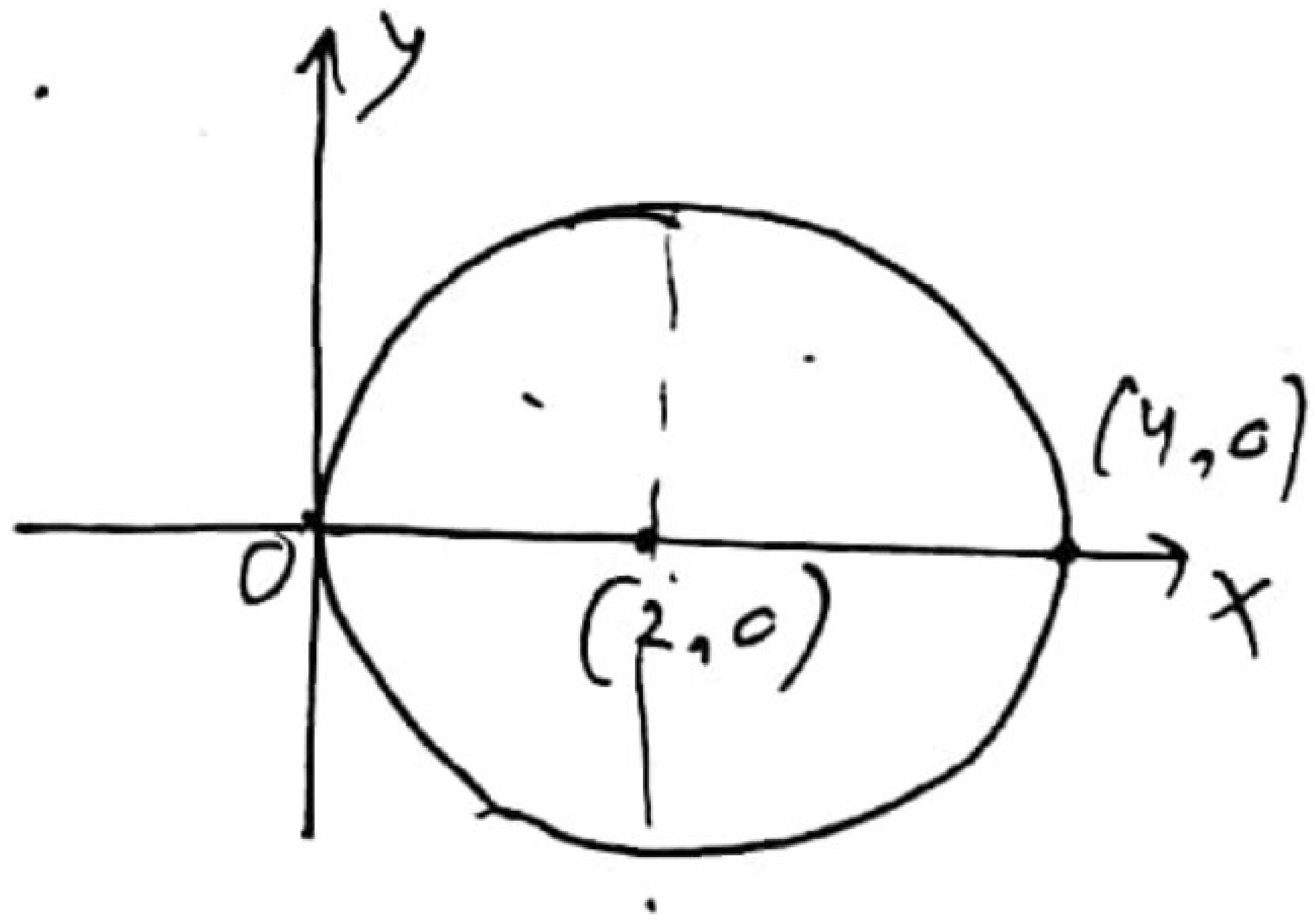
(·) $x^2+y^2=4$

Centre $(0,0)$ Rad $=2$



(·) $(x-2)^2+y^2=4$

Centre $(2,0)$ Rad $=2$



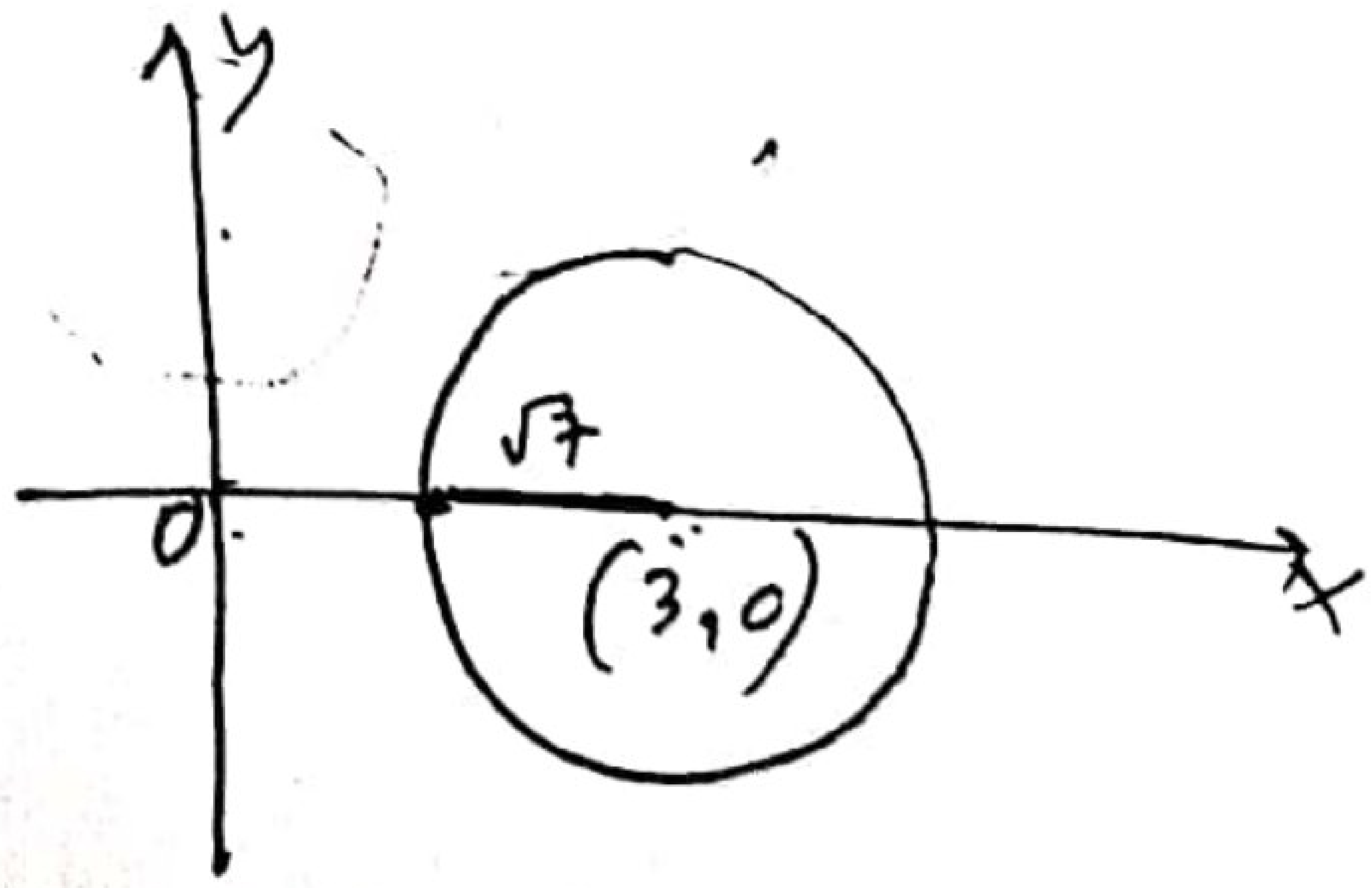
(·) $x^2+y^2-6x+2=0$

$(x^2-6x)+y^2+2=0$

$(x-3)^2-9+y^2+2=0$

$(x-3)^2+y^2=7$

Cen: $(3,0)$ Rad $=\sqrt{7}$



(·) $x^2+y^2=4$

$y=\sqrt{4-x^2}$

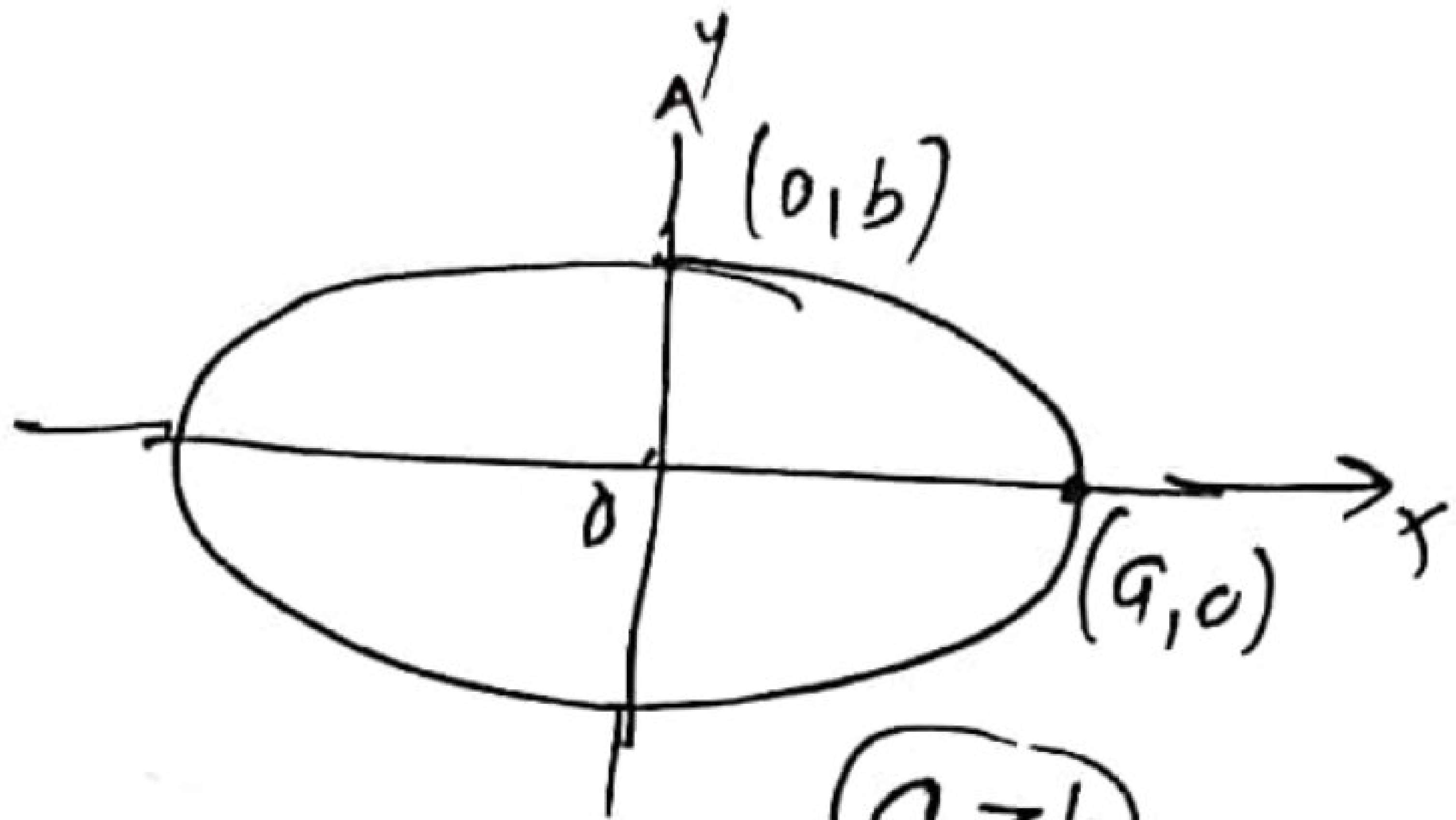
$\int \sqrt{4-x^2}\,dx \rightarrow$ lay formula

Solution $x^2+y^2 \le r^2$ (inside the circle)

$x^2+y^2 \ge r^2$ (outside the circle)

ELLIPSE

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

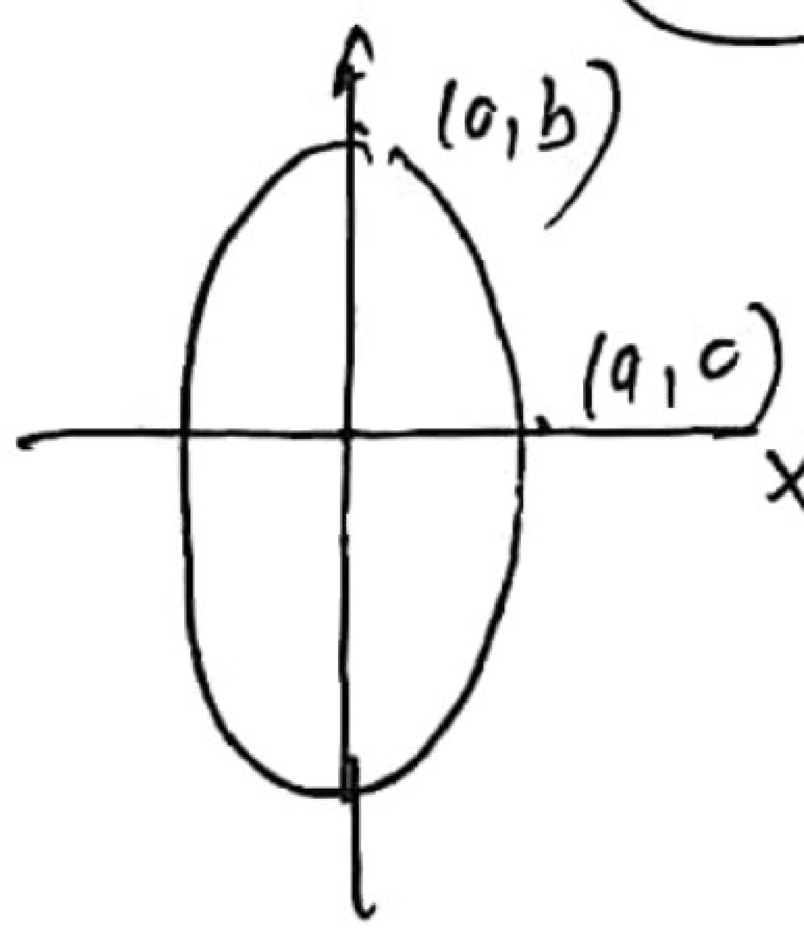


(a > b)

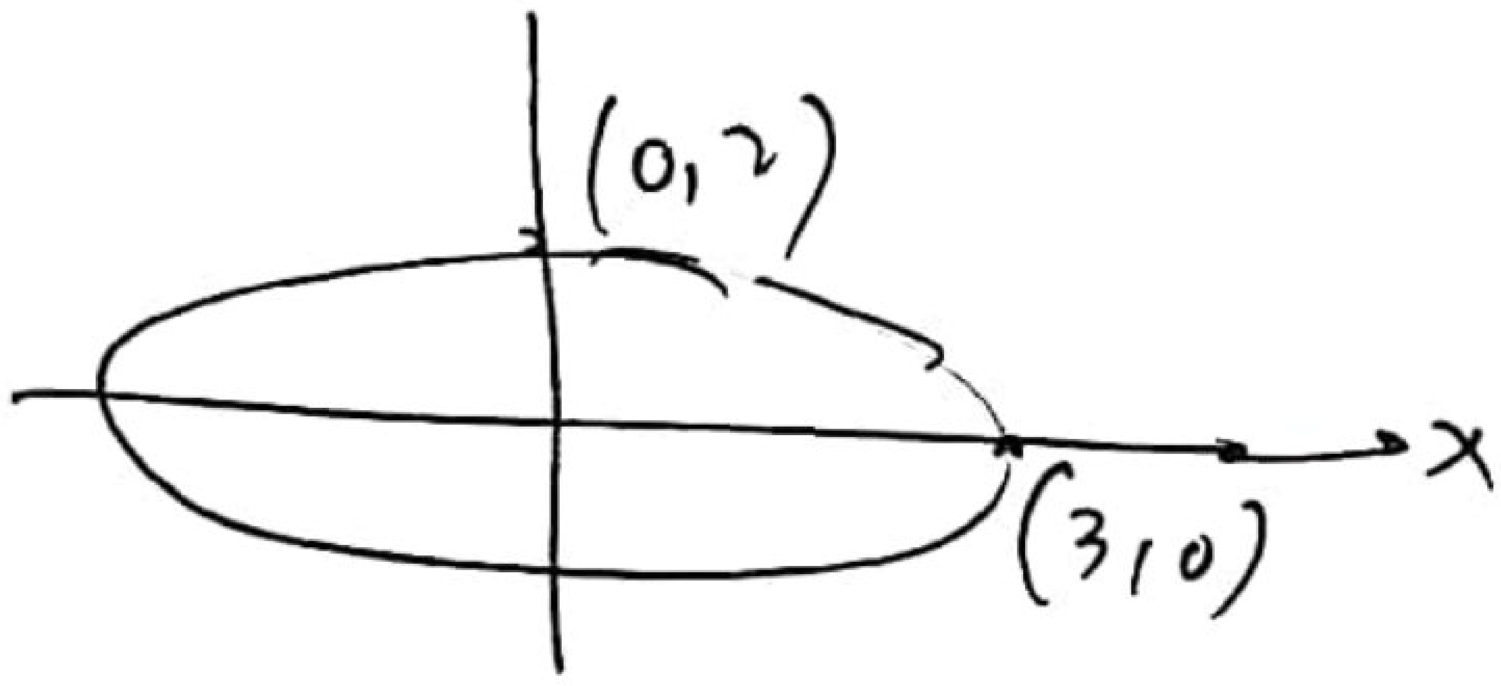
eg $4x^2+9y^2=36$

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

$a=3,\quad b=2$



(b > a)



Solution

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\ge 1 \text{ (outside the ellipse)}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1 \text{ (inside the ellipse)}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \Rightarrow \frac{y^2}{b^2}=\frac{a^2-x^2}{a^2}$$

$$\Rightarrow y^2=\frac{b^2}{a^2}(a-x^2)$$

$$\Rightarrow \boxed{y=\frac{b}{a}\sqrt{a^2-x^2}}$$

direct learn

Ques 1 Find the area of the region
$\{(x,y): y^2 \le 4x ; 4x^2+4y^2 \le 9\}$

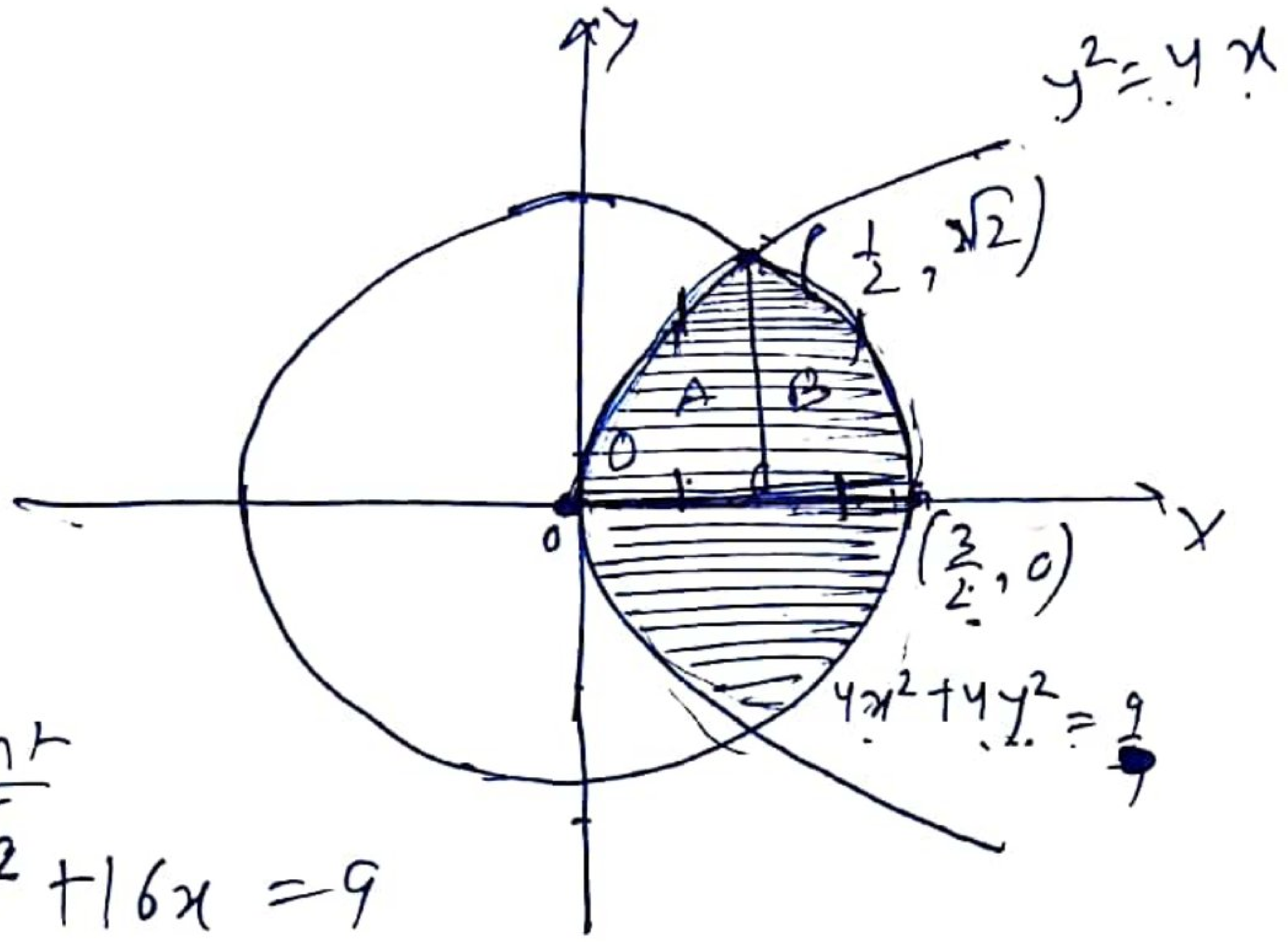
Sol. (1) $y^2 \le 4x$

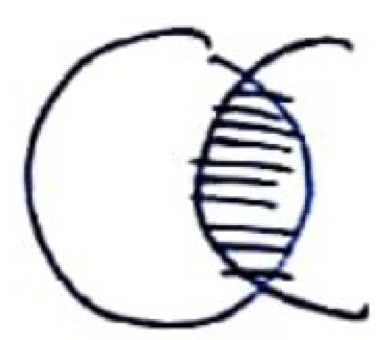
Parabola, vertex $(0,0)$, solution is inside the parabola

(2) $4x^2+4y^2 \le 9$

$\Rightarrow x^2+y^2 \le \frac{9}{4}$

Circle, Centre $(0,0)$ & Radius $= 3/2$; solution : Inside the circle





Int point

$$4x^2+16x=9$$

$$4x^2+16x-9=0$$

$$4x^2+18x-2x-9=0$$

$$2x(2x+9)-1(2x+9)=0$$

$$x = 1/2 ; x = \frac{-9}{2}$$

Required area $= 2\int_0^{1/2} 2\sqrt{x}\, dx + 2\int_{1/2}^{3/2} \sqrt{\frac{9}{4}-x^2}\, dx$

(4)

$$= 2\times2\times\frac{2}{3}\left(x^{3/2}\right)_0^{1/2} + 2\left[\frac{x}{2}\sqrt{\frac{9}{4}-x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{1/2}^{3/2}$$

$$= \frac{8}{3}\left(\frac{1}{2}\times\frac{1}{\sqrt{2}}\right) + 2\left[\left(0+\frac{9}{8}\times\frac{\pi}{2}\right) - \left(\frac{1}{4}\cdot\sqrt{2} + \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right)\right]$$

$$= \frac{4}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{\sqrt{2}}{2} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{4}{3\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

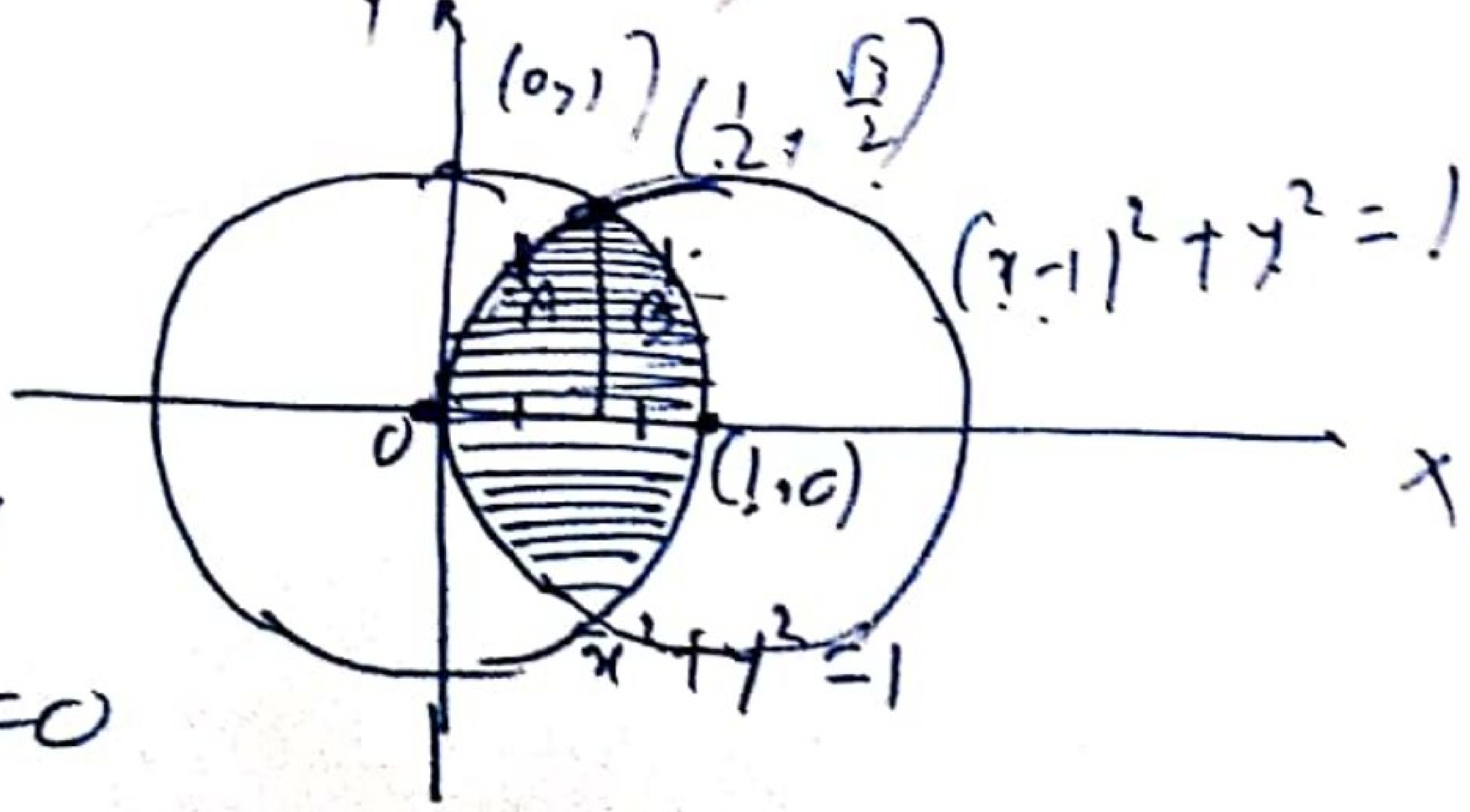
Area $= \frac{1}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}(1/3)$ square units <u>Ans</u>

<u>Que. 2</u> Find the area bounded by the curves

$(x-1)^2+y^2=1$ & $x^2+y^2=1$

(1) $(x-1)^2+y^2=1$ Centre $(1,0)$ Rad $=1$

(2) $x^2+y^2=1$ Centre $(0,0)$ Rad $=1$

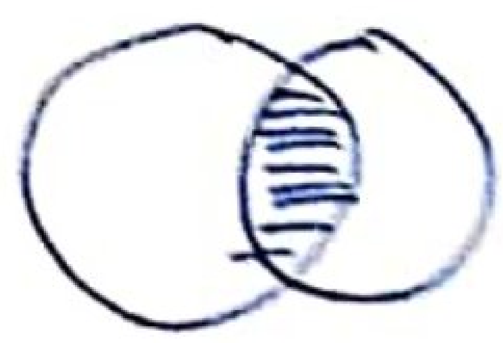


<u>Int point</u>

$(x-1)^2 + \cancel{1} - x^2 = \cancel{1}$

$\cancel{x^2} - 2x + 1 - \cancel{x^2} = 0$

$x = 1/2$



Req area $= 2\int_0^{1/2}\sqrt{1-(x-1)^2}\,dx + 2\int_{1/2}^{1}\sqrt{1-x^2}\,dx$

$$= 2\left[\frac{(x-1)}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{1/2} +$$

$$2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_{1/2}^{1}$$

$$= 2\left[\left(-\frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right)\right) - \left(0 + \frac{1}{2}\sin^{-1}(-1)\right)\right]$$

$$+2\left[\left(0 + \frac{1}{2}\sin^{-1}(1)\right) - \left(\frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right)\right]$$

$$= 2\left[-\frac{\sqrt{3}}{8} - \frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{2}\frac{\pi}{2}\right] + 2\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{\sqrt{3}}{8} - \frac{1}{2}\cdot\frac{\pi}{6}\right]$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6}$$

$$= \pi - \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \text{ sq. units}$$

Qn 13 Find the area of the smaller region by the curve $4x^2 + 9y^2 = 36$ & $2x+3y = 6$

(OR) $\left\{(x,y) : \frac{x^2}{9} + \frac{y^2}{4} \le 1 \ ; \ \frac{x}{3} + \frac{y}{2} \ge 1\right\}$

(i) $\cancel{4x^2+9y^2}$ $4x^2 + 9y^2 = 36$

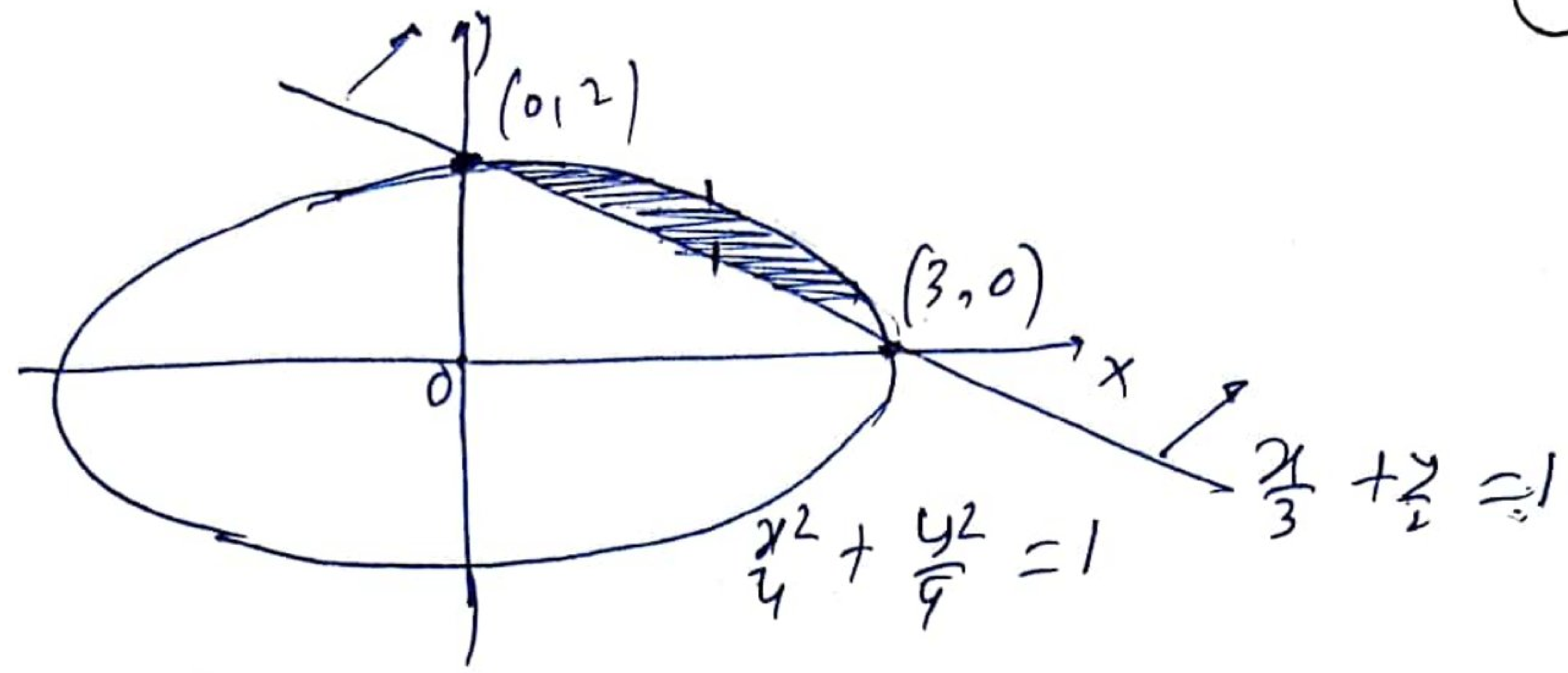
$\frac{x^2}{9} + \frac{y^2}{4} = 1$ (Inside the ellipse)

$a = 3, \ b = 2$ $(a > b)$ I^{11}

$2x + 3y = 6$

points $(0, 2)$ $(3, 0)$

$0 \ge 1$ (away from origin)



Req. area $= \int_0^3 \frac{2}{3}\sqrt{9-x^2} - \frac{2}{3}(3-x)\,dx$

$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - 3x + \frac{x^2}{2}\right]_0^3$

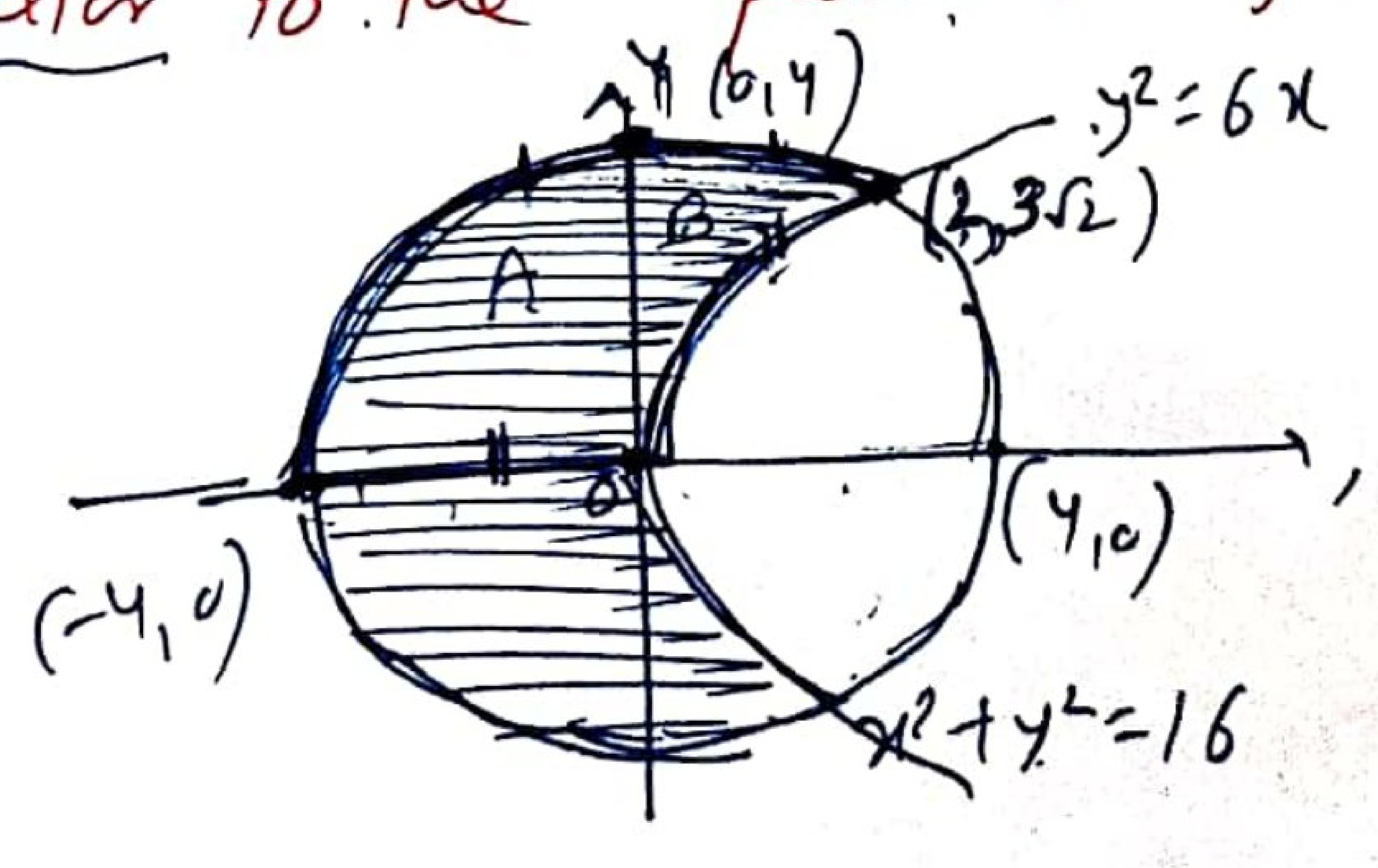
$= \frac{2}{3}\left[\left(0 + \frac{9}{2}\cdot\frac{\pi}{2} - 9 + \frac{9}{2}\right) - (0+0)\right]$

$= \frac{2}{3}\left[\frac{9\pi}{4} - \frac{9}{2}\right]$

$= \frac{2}{3}\times\frac{3}{2}\left[\frac{3\pi}{2} - 3\right] = \left(\frac{3\pi}{2} - 3\right)$ squ. unit Ans

Q.N. 4 Find the area of the circle $x^2+y^2=16$ exterior to the parabola $y^2=6x$

Soln



Int point

$x^2+6x-16=0$

$(x+8)(x-2)=0$

$x=-8$

$x=2$

Req'd area

$$= 2\int_{-4}^{0} \sqrt{16-x^2}\,dx + 2\int_{0}^{2} \sqrt{16-x^2} - \sqrt{6}\sqrt{x}\,dx$$

$$= 2\int_{-4}^{2} \sqrt{16-x^2}\,dx - 2\sqrt{6}\int_{0}^{2} \sqrt{x}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{16-x^2} + 8\sin^{-1}\left(\frac{x}{4}\right)\right]_{-4}^{2} - 2\sqrt{6}\times\frac{2}{3}\left[(x)^{3/2}\right]_{0}^{2}$$

$$= 2\left[\left(2\sqrt{3} + 8\cdot\frac{\pi}{6}\right) - \left(0 - 8\cdot\frac{\pi}{2}\right)\right] - \frac{4\sqrt{6}}{3}\left(2\sqrt{2}\right)$$

$$= 4\sqrt{3} + \frac{8\pi}{3} + 8\pi - \frac{8\times 2\sqrt{3}}{3}$$

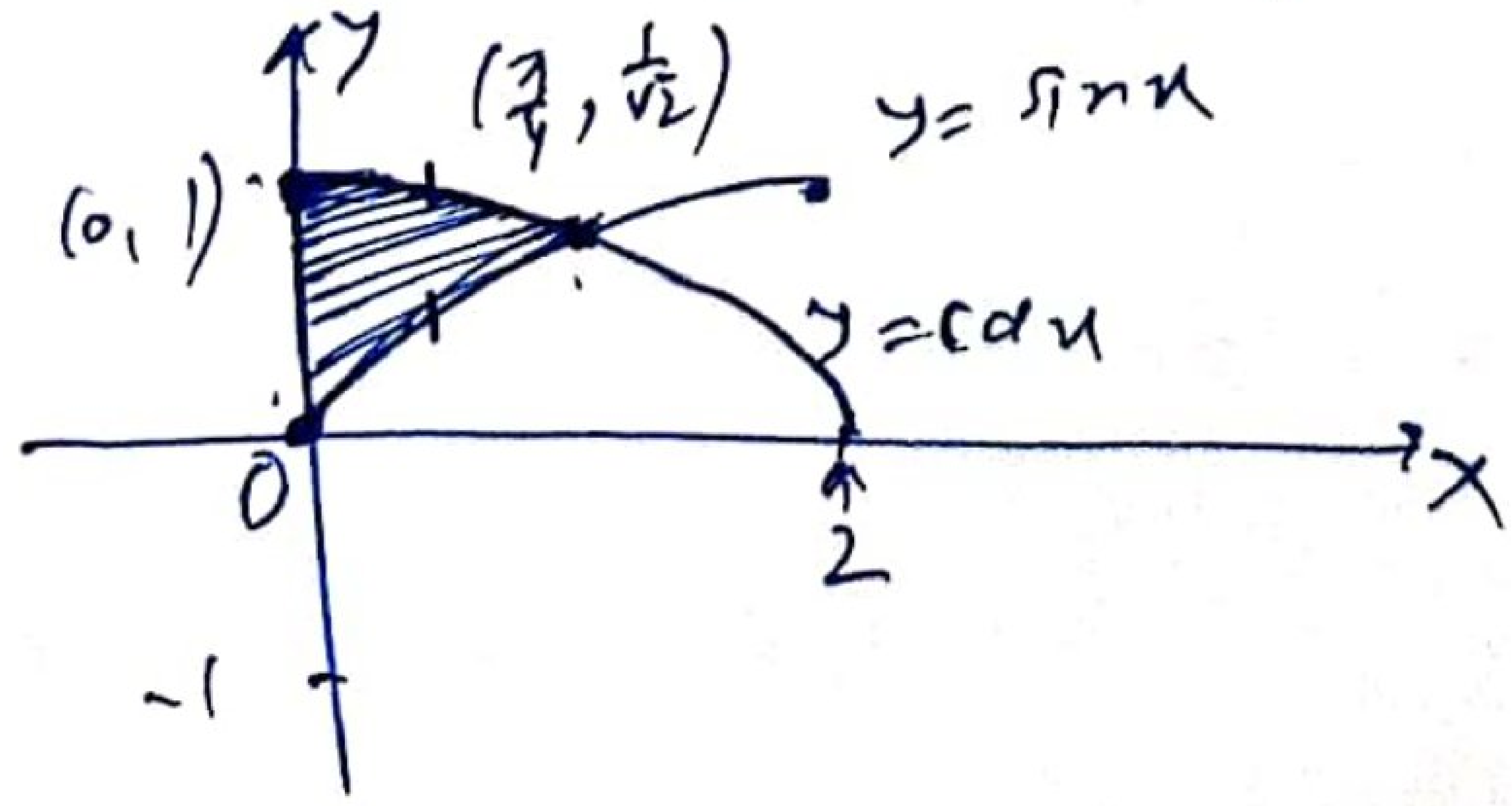
$$= 4\sqrt{3} + \frac{32\pi}{3} - \frac{16\sqrt{3}}{3}$$

$$= \frac{32\pi}{3} - \frac{4\sqrt{3}}{3}$$

$$= \frac{4}{3}\left(8\pi - \sqrt{3}\right) \text{ sq. unit}$$

Qn. 5 Find the area bounded by y-axis, $y = \cos x$ & $y = \sin x$; $0 \le x \le \frac{\pi}{2}$

Soln



Req. area

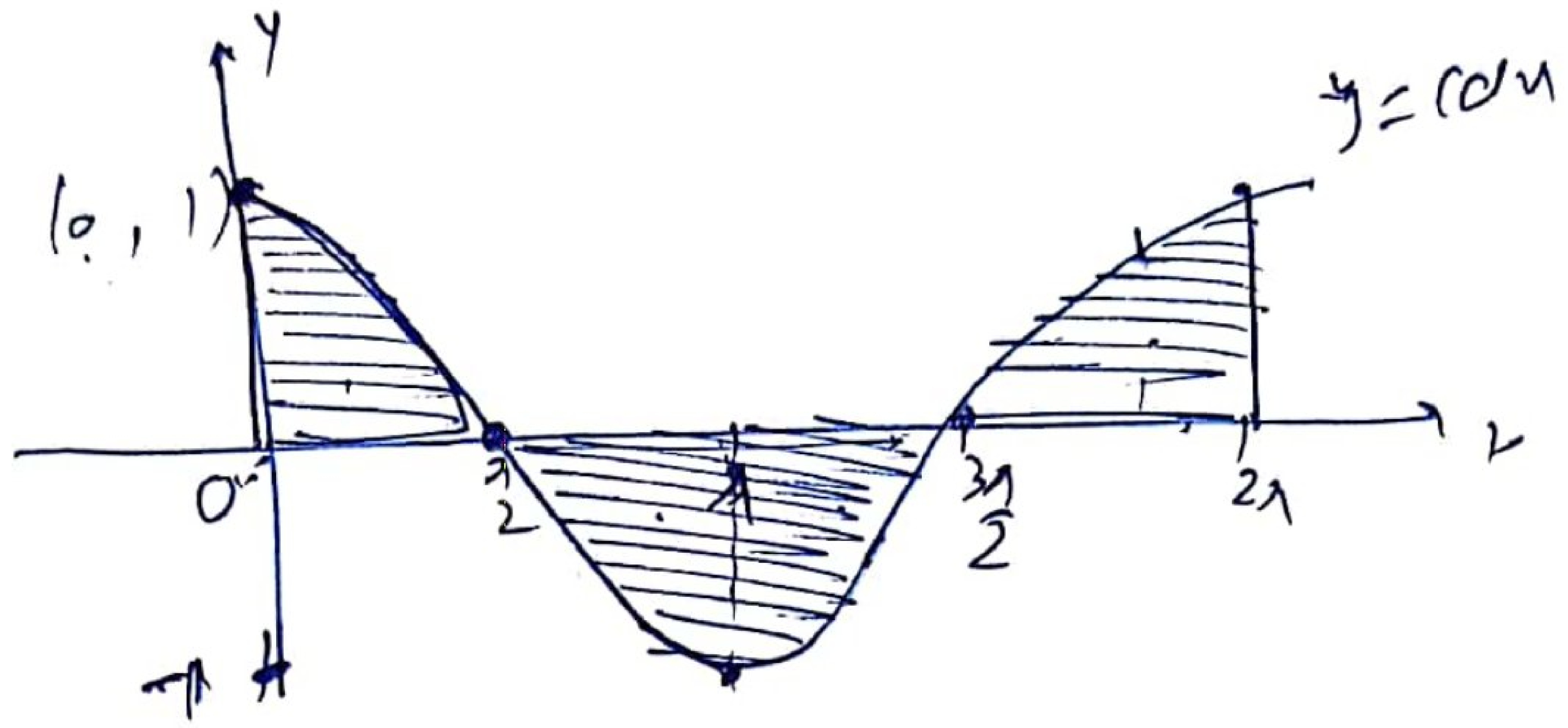
$$= \int_{0}^{\pi/4} \cos x - \sin x\,dx$$

$$= \left(\sin x + \cos x\right)_{0}^{\pi/4}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (1) = \sqrt{2} - 1$$

Qn. 6 Find the area bounded by the curve $y = \cos x$ between $x = 0$ & 2π

Sol



Req. area $= \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{3\pi/2} (0 - \cos x) dx + \int_{3\pi/2}^{2\pi} \cos x \, dx$

$= (\sin x)_0^{\pi/2} + (-\sin x)_{\pi/2}^{3\pi/2} + (\sin x)_{3\pi/2}^{2\pi}$

$= (1 - 0) + (1 + 1) + (0 - (-1))$

$= 4$ square units

AOI WORKSHEET No: 2 (Class No: 3)

Ques 1 Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$

Ans $\frac{ab}{4}(\pi - 2)$ sq. units

Ques 2 → Find the area bounded by the curve $y = \sin x$ b/w $x = 0$ & $x = 2\pi$

Ans 4 square units.

Ques 3 → Find the area of the region $\{(x, y) : x^2 + y^2 \leq 4$ and $x + y \geq 2\}$

Ans $(\pi - 2)$ sq. units

Ques 4 → Find the area bounded by the two curves $(x-2)^2 + y^2 = 4$ and $x^2 + y^2 = 4$

Ans $\left(\frac{8\pi}{3} - 2\sqrt{3}\right)$

Ques 5 → Find the area lying above X-axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$

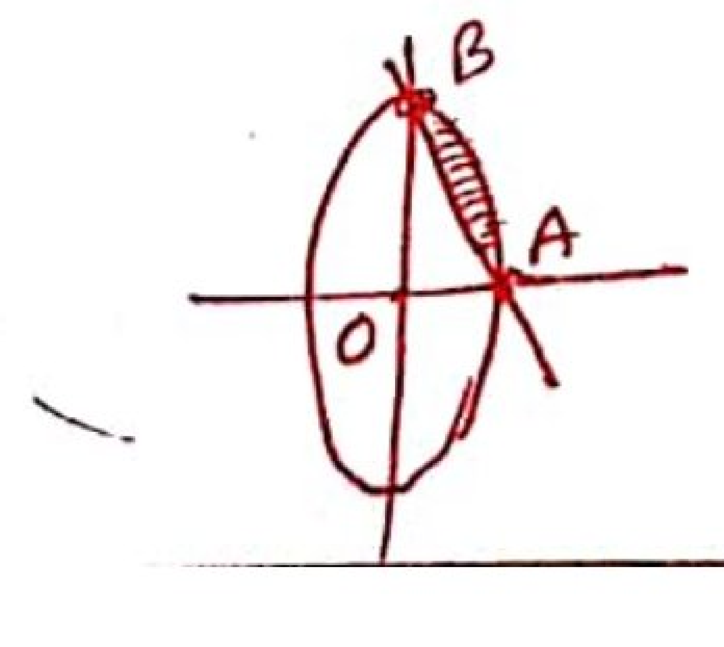
Ans $\frac{4}{3}(8 + 3\pi)$ sq. units.

Ques 6 → Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$

Ans $\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Ques 7 → AOBA is the part of the ellipse $9x^2 + y^2 = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB

Hint



Ans $(3\pi - 6)$ square units

Ans