!! जय श्री राधे कृष्णा जय श्री गिरिराज जी महाराज !!

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ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER: $\boxed{\text{DIFFERENTIAL EQUATION}}$ $\boxed{\text{CLASS No: 1}}$

(i) $y = f(x)$

dependent variable

Independent variable

(i) D.E $\quad x\frac{dy}{dx} + y = 0$

(i) ORDER & DEGREE of D.E

✓ highest derivative in D.E → order

✓ power of the highest order → degree

(Note) D.E is a polynomial in its derivatives

eg $\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 + y = 0$ order: 2 & degree: 1

eg $\left(\frac{d^3y}{dx^3}\right)^3 + \left(\frac{dy}{dx}\right)^4 + \left(\frac{d^4y}{dx^4}\right)^2 + y = 0$ order = 4, degree = 2

eg $\left(\frac{d^2y}{dx^2}\right)^3 + \sin\left(\frac{dy}{dx}\right) = 0$ order: 2 ; degree: Not defined

eg $\left(\frac{d^2y}{dx^2}\right)^3 + \sin y + x^2 = 0$ order = 2 deg = 3

eg $(y'')^{3/2} = (y')^{2/3} \Rightarrow (y'')^3 = (y')^{4/3} \rightarrow (y'')^9 = (y')^4$

eg $\int y\,dx + \frac{d^2y}{dx^2} + xy = 0$

Diff $y + \frac{d^3y}{dx^3} + x\frac{dy}{dx} + y = 0$ order = 3

degree = 1

$\boxed{\text{Solution of D.E}}$

✓ given: D.E

✓ Integral: Solution / equation of curve

✓ general solution & particular solution

↳ in terms of C

↳ Given: Initial conditions

$x=0;\ y=1$ (or) $y(0)=1$

to find: value of C

given $\frac{dy}{dx} = f(x,y)$

$f(y)\,dy = f(x)\,dx$ (variable separable)

$\int f(y)\,dy = \int f(x)\,dx$

$\boxed{g(y) = \phi(x) + C}$ general solution

eg $\log|y| = \log|x| + \log C$

$\log|y| = \log|cx|$

$\Rightarrow |y| = |cx|$

$\Rightarrow y = \pm cx$

$\boxed{y = c_1 x}$

TYPE 1 $\boxed{\text{LINEAR D.E}}$

FORM (A)

$$\boxed{\frac{dy}{dx} + Py = Q}$$ (single - y)

when $P, Q \rightarrow$ Constant term or function of x

I.F = (Integrating factor)

✓ $I.F = e^{\int P dx}$

property $e^{\log x} = x$ eg $e^{-3\log x} = e^{\log(x)^{-3}} = \frac{1}{x^3}$

✓ Solution $y \cdot IF = \int (Q \times IF) dx + c$

100% (X) $\sin y$, y^2, e^y, $\log y$, $\sqrt{y}$, $\sin^{-1} y$, ----

Form (B) :

$$\boxed{\frac{dx}{dy} + Px = Q}$$ (Single - x)

when $P, Q \rightarrow$ Constant term (or) function of y

$I.F = e^{\int P dy}$

Solution $x \cdot IF = \int (Q \times IF) dy + C$

100% (X) x^2, $\sin x$, $\sqrt{x}$, $\sin^{-1} x$, $\log x$, e^x, x^3 -----

Ques 1 Solve $x\log x \cdot \frac{dy}{dx} + y = \frac{2}{x}\log x$

Soln divide by $x\log x$

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2}$$

Comp with $\frac{dy}{dx} + Py = Q$

we have $P = \frac{1}{x\log x}$ & $Q = \frac{2}{x^2}$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{x\log x} dx}$$

put $\log x = t$

$\frac{1}{x} dx = dt$

$$\therefore IF = e^{\int \frac{1}{t} dt}$$

$$= e^{\log t}$$

$$= t$$

$$\boxed{IF = \log x}$$

Solution $y \times IF = \int Q \times IF\, dx + C$

$$\Rightarrow y \cdot \log x = 2\int \underset{\mathrm{II}}{\frac{1}{x^2}} \cdot \underset{\mathrm{I}}{\log x}\, dx + C$$

$$\Rightarrow y\log x = 2\left[\log x \cdot \left(-\frac{1}{x}\right) + \int \frac{1}{x}\cdot\frac{1}{x} dx\right] + C$$

$$\Rightarrow y\log x = 2\left[-\frac{\log x}{x} - \frac{1}{x}\right] + C$$

$$\Rightarrow y\log x = -\frac{2}{x}(\log x + 1) + C \quad \text{Ans}$$

Ques 2 Soln $(\tan^{-1}y - x)\,dy = (1+y^2)\,dx$

given that $y(0)=0$

Soln

$$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

Comp with $\frac{dx}{dy} + Px = Q$

here $P = \frac{1}{1+y^2}$; $Q = \frac{\tan^{-1}y}{1+y^2}$

$$IF = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$$\boxed{IF = e^{\tan^{-1}y}}$$

Solution

$$x \times IF = \int (Q \times IF)\,dy + C$$

$$x \cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} \cdot e^{\tan^{-1}y}\,dy + C$$

put $\tan^{-1}y = t$

$\frac{1}{1+y^2}dy = dt$

$$\Rightarrow x e^{\tan^{-1}y} = \int \underset{I}{t} \cdot \underset{II}{e^t}\,dt + C$$

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$$\Rightarrow x e^{\tan^{-1}y} = t \cdot e^t - \int e^t \, dt + C$$

$$\Rightarrow x e^{\tan^{-1}y} = t e^t - e^t + C$$

$$\Rightarrow x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C$$

put $x = 0$ & $y = 0$

$$0 = e^0 (0 - 1) + C$$

$$\boxed{C = 1}$$

$$\Rightarrow \boxed{x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + 1}$$ is the Required Solution

(OR) $$\boxed{x = \tan^{-1}y - 1 + e^{-\tan^{-1}y}}$$ Ans

Ans 3 → Solve $\dfrac{dy}{dx} - 2y = \sin x$

Sol: Comp with $\dfrac{dy}{dx} + Py = Q$

here $P = -2$ & $Q = \sin x$

$$IF = e^{-\int 2 dx} = e^{-2x}$$

Solution

$$y \cdot e^{-2x} = \int e^{-2x} \cdot \sin x \, dx + C$$

$$y \cdot e^{-2x} = I_1 + C$$

where $I_1 = \int e^{-2x} \cdot \sin x \, dx$

(process)

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Qn 4 → Find the equation of the curve passing through the point (0,2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5. (7)

Soln Given
$$x+y = \frac{dy}{dx} + 5$$
$$\Rightarrow \frac{dy}{dx} - y = x-5$$
comp with $\frac{dy}{dx} + Py = Q$
$$P = -1 \quad \& \quad Q = x-5$$
$$IF = e^{\int -1\,dx} = e^{-x}$$

Soln
$$y e^{-x} = \int \underset{II}{e^{-x}} \cdot \underset{I}{(x-5)}\, dx + C$$
$$ye^{-x} = \left[-(x-5)\cdot e^{-x} + \int e^{-x}\,dx\right] + C$$
$$ye^{-x} = -(x-5)e^{-x} - e^{-x} + C$$
$$ye^{-x} = e^{-x}(-x+5-1) + C$$
$$\boxed{y = (4-x) + e^{x}C}$$
thus soln pass (0,2) put $x=0$ & $y=2$

$2 = 4 + ce^{0}$

$\Rightarrow \boxed{-2 = c}$

$\therefore \boxed{y = 4 - x - 2e^{x}}$ equation of curve / solution

Qn 5 → Soln $(x+1)\frac{dy}{dx} = 2e^{-y} - 1$; given $y(0) = 0$

$\Rightarrow \frac{dy}{dx} = \frac{2e^{-y}-1}{x+1}$

$\frac{dx}{dy} = \frac{x+1}{2e^{-y}-1}$

$\Rightarrow \frac{dx}{dy} = \frac{x}{2e^{-y}-1} + \frac{1}{2e^{-y}-1}$

$\Rightarrow \frac{dx}{dy} - \frac{x}{2e^{-y}-1} = \frac{1}{2e^{-y}-1}$

$\Rightarrow \frac{dx}{dy} - \frac{e^{y}\cdot x}{2-e^{y}} = \frac{e^{y}}{2-e^{y}}$

here $P = \frac{-e^{y}}{2-e^{y}}$; $Q = \frac{e^{y}}{2-e^{y}}$

$IF = e^{-\int \frac{e^{y}}{2-e^{y}} dy}$

put $2 - e^{y} = t$

$-e^{y} dy = dt$

$IF = e^{\int \frac{dt}{t}} = e^{\log t} = t$

$\boxed{IF = 2 - e^{y}}$

Solution

$x \cdot IF = \int Q \cdot IF\, dy + c$

$x(2-e^{y}) = \int \frac{e^{y}}{\cancel{2-e^{y}}} \times \cancel{(2-e^{y})}\, dy + c$

$\Rightarrow x(2-e^{y}) = e^{y} + c$

put $x=0$ & $y=0$

$0 = 1 + c$

$(c = -1)$

$\therefore x(2-e^{y}) = e^{y} - 1$

$\boxed{x = \frac{e^{y}-1}{2-e^{y}}}$

Ans

WORKSHEET NO:- 1 (D·E)

Q NO:- 1 → Solve $\frac{dy}{dx} - y = \cos x$ ANS $y = \frac{1}{2}(\sin x - \cos x) + ce^{x}$

Q NO 2 → Solve $y\,dx - (x + 2y^2)dy = 0$ ANS $x = 2y^2 + cy$

Q NO 3 → Solve $\frac{dy}{dx} + y\cot x = 2x + x^2 \cot x$; given $y = 0, x = \frac{\pi}{2}$

ANS $y = x^2 - \frac{\pi^2}{4\sin x}$

Q NO 4 → Find the equation of a curve passing through the point $(0,1)$. If the slope of the tangent to the curve at any point (x,y) is equal to the sum of the x coordinate and the product of the x coordinate and y coordinate of that point ANS $y = -1 + 2e^{x^2/2}$

(HINT: $\frac{dy}{dx} = x + xy$)

Q NO 5 → Solve $\frac{dy}{dx} - 3y\cot x = \sin(2x)$; $y(\frac{\pi}{2}) = 2$

ANS $y = 4\sin^3 x - 2\sin^2 x$

Q NO 6 → Solve $x\frac{dy}{dx} + y - x + xy\cot x = 0$

ANS $y = \frac{1}{x} - \cot x + \frac{c}{x\sin x}$

Q NO 7 → $(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$; $y = 0, x = 1$

ANS $y(1+x^2) = \tan^{-1}x - \frac{\pi}{4}$

Q NO 8 → Solve $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1$ ANS $ye^{2\sqrt{x}} = 2\sqrt{x} + c$

Q NO 9 → $(x+y)\frac{dy}{dx} = 1$ ANS $(x+y+1) = ce^{y}$

Q NO 10 → Find only I·F of D·E $(1-y^2)\frac{dx}{dy} + yx = ay$ ANS $\frac{1}{\sqrt{1-y^2}}$

—x—