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SOLUTIONS INTEGRATION CLASS NO: 6

→ WORKSHEET NO: 5 →

Ans: 1 → $I = \int \frac{3x-1}{3x^2+4x+2}\,dx$ $\left(\text{type } \int \frac{\text{linear}}{(\text{Quadratic})}\,dx\right)$

$= \frac{1}{2}\int \frac{6x-2+4-4}{3x^2+4x+2}\,dx$

$= \frac{1}{2}\int \frac{(6x+4)-6}{3x^2+4x+2}\,dx$

$= \frac{1}{2}\int \frac{6x+4}{3x^2+4x+2}\,dx - 3\int \frac{1}{3x^2+4x+2}\,dx$

put $3x^2+4x+2 = t$ in I^{st} Integral

$(6x+4)\,dx = dt$

$I = \frac{1}{2}\int \frac{dt}{t} - \frac{3}{3}\int \frac{1}{x^2+\frac{4}{3}x+\frac{2}{3}}\,dx$

$= \frac{1}{2}\log|t| - \int \frac{1}{\left(x+\frac{2}{3}\right)^2 - \frac{4}{9}+\frac{2}{3}}\,dx$

$= \frac{1}{2}\log|t| - \int \frac{1}{\left(x+\frac{2}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}\,dx$

$I = \frac{1}{2}\log|3x^2+4x+2| - \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right) + C$ **Ans**

QN: 2 → $\int \frac{(3\sin x-2)\cos x\,dx}{5-\cos^2 x-4\sin x}$

$= \int \frac{(3\sin x-2)\cos x\,dx}{5-(1-\sin^2 x)-4\sin x}$

$$= \int \frac{(3\sin x - 2)\cos x}{\sin^2 x - 4\sin x + 4}\,dx$$

put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\therefore I = \int \frac{(3t-2)\,dt}{t^2-4t+4}$$

$$= 3\int \frac{t - \frac{2}{3}}{t^2-4t+4}\,dt$$

$$= \frac{3}{2}\int \frac{2t - \frac{4}{3} - 4 + 4}{t^2-4t+4}\,dt$$

$$= \frac{3}{2}\int \frac{(2t-4) + 8/3}{t^2-4t+4}\,dt$$

$$= \frac{3}{2}\int \frac{2t-4}{t^2-4t+4}\,dt + 4\int \frac{1}{t^2-4t+4}\,dt$$

put $t^2-4t+4 = z$

$(2t-4)\,dt = dz$

$$\therefore I = \frac{3}{2}\int \frac{dz}{z} + 4\int \frac{1}{(t-2)^2}\,dt$$

$$= \frac{3}{2}\log|z| + 4\left(-\frac{1}{t-2}\right) + c \quad \cdots \left\{ \int \frac{1}{x^2}\,dx = -\frac{1}{x} \right.$$

Shortcut}

$$= \frac{3}{2}\log|t^2-4t+4| - \frac{4}{t-2} + c$$

$$= \frac{3}{2}\log|(2-t)^2| - \frac{4}{t-2} + c$$

$$= \frac{3}{2}\times 2\log|2-t| - \frac{4}{t-2} + c$$

$$= 3\log|2-\sin x| - \frac{4}{\sin x - 2} + c \qquad \text{Ans}$$

Ans: 3 → $\int \frac{x^3+x}{x^4-9}\,dx$

Separate

$$= \int \frac{x^3}{x^4-9}\,dx + \int \frac{x}{x^4-9}\,dx$$

put $x^4-9=t$ in Ist Integral & put $x^2=z$ in 2nd Integral

$x^3dx = \frac{dt}{4}$ and $x\,dx = \frac{dz}{2}$

$$\therefore I = \frac{1}{4}\int \frac{dt}{t} + \frac{1}{2}\int \frac{dz}{z^2-3^2}$$

$$= \frac{1}{4}\log|t| + \frac{1}{2}\times\frac{1}{2\times 3}\log\left|\frac{z-3}{z+3}\right| + c$$

$$I = \frac{1}{4}\log|x^4-9| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C \quad \text{Ans}$$

Ans 4 → $\int \frac{x^3+x^2+2x+1}{x^2-x+1}\,dx$

Divide

$$\begin{array}{r|l} & x+2 \\ x^2-x+1 & x^3+x^2+2x+1 \\ & -(x^3-x^2+x) \\ \hline & 2x^2+x+1 \\ & -(2x^2-2x+2) \\ \hline & 3x-1 \end{array}$$

$$\therefore I = \int (x+2) + \frac{3x-1}{x^2-x+1}\,dx$$

$$= \frac{x^2}{2} + 2x + \int \frac{3x-1}{x^2-x+1}\,dx$$

$$I = \frac{x^2}{2} + 2x + I_1 + C$$

where $I_1 = \int \frac{3x-2}{x^2-x+1}\,dx$

$$= 3\int \frac{x-2/3}{x^2-x+1}\,dx$$

$$= \frac{3}{2}\int \frac{2x-4/3-1+1}{x^2-x+1}\,dx$$

$$I_1 = \frac{3}{2}\int \frac{(2x-1) - 1/3}{x^2-x+1}\,dx$$

$$I_1 = \frac{3}{2}\int \frac{(2x-1)}{x^2-x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2-x+1}\,dx$$

put $x^2-x+1 = t$ in I^{st} Integral

$(2x-1)\,dx = dt$

$$I_1 = \frac{3}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2 - \frac{1}{4} + 1}\,dx$$

$$= \frac{3}{2}\log|x^2-x+1| - \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\,dx$$

$$I_1 = \frac{3}{2}\log|x^2-x+1| - \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$

$$\therefore I = \frac{x^2}{2} + 2x + \frac{3}{2}\log|x^2-x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C \quad \text{Ans}$$

→ Misprint in worksheet

Ques: 5 → $\displaystyle\int \frac{1-x^2}{x(1-2x)}\,dx$

$$I = \int \frac{x^2-1}{2x^2-x}\,dx$$

Divide

$$2x^2-x \,\big)\, x^2 - 1 \,\big(\, \tfrac{1}{2}$$
$$-(x^2 - \tfrac{x}{2})$$
$$\tfrac{x}{2} - 1$$

$$I = \int \frac{1}{2} + \frac{\frac{x}{2} - 1}{2x^2-x}\,dx$$

$$I = \frac{x}{2} + \frac{1}{2}\int \frac{x-2}{2x^2-x}\,dx$$

$$I = \frac{x}{2} + \frac{1}{2}I_1 + C$$

where $\displaystyle I_1 = \int \frac{x-2}{2x^2-x}\,dx$ --- $\left\{ \text{type } \int \frac{\text{linear}}{\text{quadratic}}\,dx \right\}$

$$I_1 = \frac{1}{4}\int \frac{4x-8-1+1}{2x^2-x}\,dx$$

$$I_1 = \frac{1}{4}\int \frac{(4x-1)-7}{2x^2-x}\,dx$$

$$I_1 = \frac{1}{4}\int \frac{4x-1}{2x^2-x}\,dx - \frac{7}{4}\int \frac{1}{2x^2-x}\,dx$$

put $2x^2-x=t$ in 1^{st} Integral
$(4x-1)dx = dt$

$$I_1 = \frac{1}{4}\int \frac{dt}{t} - \frac{7}{8}\int \frac{1}{x^2-\frac{x}{2}}\,dx$$

$$= \frac{1}{4}\log|t| - \frac{7}{8}\int \frac{1}{(x-\frac{1}{4})^2-(\frac{1}{4})^2}\,dx$$

$$= \frac{1}{4}\log|2x^2-x| - \frac{7}{8}\times\frac{1}{2\times\frac{1}{4}}\log\left|\frac{x-\frac{1}{4}-\frac{1}{4}}{x-\cancel{\frac{1}{4}}+\cancel{\frac{1}{4}}}\right|$$

$$I_1 = \frac{1}{4}\log|2x^2-x| - \frac{7}{4}\log\left|\frac{2x-1}{2x}\right|$$

$$\therefore I = \frac{x}{2} + \frac{1}{2}\left[\frac{1}{4}\log|2x^2-x| - \frac{7}{4}\log\left|\frac{2x-1}{2x}\right|\right] + C$$

$$I = \frac{x}{2} + \frac{1}{8}\log|2x^2-x| - \frac{7}{8}\log\left|\frac{2x-1}{2x}\right| + C \quad \text{Ans}$$

(Remember: Different Ans. are possible in Indefinite Integrals with different methods)

Ques 6 $\rightarrow$ $I = \int \frac{3x+1}{\sqrt{5-2x-x^2}}\,dx$ ... { type $= \int \frac{\text{linear}}{\sqrt{\text{Quadratic}}}\,dx$ }

$$I = 3\int \frac{x+1/3}{\sqrt{5-2x-x^2}}\,dx$$

$$I = -\frac{3}{2}\int \frac{-2x - \frac{2}{3} - 2 + 2}{\sqrt{5-2x-x^2}}\,dx$$

$$= -\frac{3}{2}\int \frac{(-2x-2) + 4/3}{\sqrt{5-2x-x^2}}\,dx$$

$$= -\frac{3}{2}\int \frac{-2x-2}{\sqrt{5-2x-x^2}}\,dx - 2\int \frac{1}{\sqrt{5-2x-x^2}}\,dx$$

put $5-2x-x^2 = t$

$(-2x-2)\,dx = dt$

$$I = -\frac{3}{2}\int \frac{dt}{\sqrt{t}} - 2\int \frac{1}{\sqrt{-(x^2+2x-5)}}\,dx$$

$$= -\frac{3}{\cancel{2}} \times \cancel{2}\sqrt{t} - 2\int \frac{1}{\sqrt{-[(x+1)^2-1-5]}}\,dx$$

$$= -3\sqrt{t} - 2\int \frac{1}{\sqrt{(\sqrt{6})^2-(x+1)^2}}\,dx$$

$$I = -3\sqrt{5-2x-x^2} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + C$$ Ans

Ans 7 → $I = \int \sqrt{\frac{1-x}{1+x}}\,dx$

$$I = \int \sqrt{\frac{1-x}{1+x} \times \frac{1-x}{1+x}}\,dx$$

$$I = \int \frac{1-x}{\sqrt{1-x^2}}\,dx \quad \text{--- \{ type } \int \frac{\text{linear}}{\text{Special Integral}}\,dx \text{ \}}$$

Separate

$$I = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

put $1-x^2 = t$ in 2^{nd} Integral

$x\,dx = -dt/2$

$$I = \sin^{-1}x + \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$I = \sin^{-1}x + \frac{1}{\cancel{2}} \times \cancel{2}\sqrt{t} + C$$

$$I = \sin^{-1}x + \sqrt{1-x^2} + C \quad \underline{\text{ANS}}$$

Ans 8 → $I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}}\,dx$ --- { type $\int \frac{\text{linear}}{\sqrt{\text{Quadratic}}}$

{ But here directly we can put $3x^2-5x+1 = t$
$(6x-5)\,dx = dt$ }

$$\therefore I = \int \frac{dt}{\sqrt{t}}$$

$$= 2\sqrt{t} + c$$

$$I = 2\sqrt{3x^2-5x+1} + C \quad \underline{\text{Ans}}$$

Ans 9 → $I = \int \frac{x}{\sqrt{x^2+x+1}}\,dx$ --- { type $\int \frac{\text{linear}}{\sqrt{\text{Quadratic}}}\,dx$ }

$$I = \frac{1}{2}\int \frac{2x+1-1}{\sqrt{x^2+x+1}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x+1)}{\sqrt{x^2+x+1}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2+x+1}}\,dx$$

put $x^2+x+1 = t$ in I^{st} Integral $\Rightarrow (2x+1)\,dx = dt$

$$\therefore I = \frac{1}{2}\int \frac{dt}{\sqrt{t}} - \frac{1}{2}\int \frac{1}{\sqrt{(x+\frac{1}{2})^2 - \frac{1}{4} + 1}}\,dx$$

$$= \frac{1}{\cancel{2}} \times \cancel{2}\sqrt{t} - \frac{1}{2}\int \frac{1}{\sqrt{(x+\frac{1}{2})^2 + (\sqrt{3}/2)^2}}\,dx$$

$$I = \sqrt{t} - \frac{1}{2} \log\left|(x+\tfrac{1}{2}) + \sqrt{x^2+x+1}\right| + C$$

$$I = \sqrt{x^2+x+1} - \frac{1}{2} \log\left|(x+\tfrac{1}{2}) + \sqrt{x^2+x+1}\right| + C$$

Qn 10 → $I = \int \frac{x+1}{\sqrt{x^2+1}} dx$ -- { type $\int \frac{\text{linear}}{\text{special Integral}} dx$ }

Seperate

$$I = \int \frac{x}{\sqrt{x^2+1}} dx + \int \frac{1}{\sqrt{x^2+1}} dx$$

put $x^2+1 = t$

$x\,dx = dt/2$

$$I = \frac{1}{2}\int \frac{dt}{\sqrt{t}} + \log\left|x + \sqrt{x^2+1}\right|$$

$$= \frac{1}{\cancel{2}} \times \cancel{2}\sqrt{t} + \log\left|x + \sqrt{x^2+1}\right| + C$$

$$\therefore I = \sqrt{x^2+1} + \log\left|x + \sqrt{x^2+1}\right| + C$$ ANS

Qn 11 → $I = \int \frac{1}{3 + 2\sin x + \cos x} dx$ --- { type single $\sin x$, $\cos x$ }

$$I = \int \frac{1}{3 + 2 \cdot \frac{2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}} dx$$

L.C.M

$$= \int \frac{\sec^2(x/2)\,dx}{3 + 3\tan^2(x/2) + 4\tan(x/2) + 1 - \tan^2(x/2)}$$

$$I = \int \frac{\sec^2(x/2)\,dx}{2\tan^2(x/2) + 4\tan(x/2) + 4}$$

put $\tan(x/2) = t$

$\frac{1}{2}\sec^2(x/2)\,dx = dt \Rightarrow \sec^2(x/2)\,dx = 2dt$

$$\therefore I = 2\int \frac{dt}{2t^2 + 4t + 4}$$

$$= \frac{\cancel{2}}{\cancel{2}}\int \frac{dt}{t^2 + 2t + 2}$$

$$= \int \frac{1}{(t+1)^2 - 1 + 2}\,dt$$

$$= \int \frac{1}{(t+1)^2 + 1^2}\,dt$$

$$= \tan^{-1}(t+1) + C$$

$$I = \tan^{-1}\left(\tan\frac{x}{2} + 1\right) + C \quad \underline{\text{Ans}}$$

Ques 12 → $I = \int \frac{1}{5 + 7\cos x + \sin x}\,dx$

$$I = \int \frac{1}{5 + 7\frac{(1 - \tan^2 x/2)}{(1 + \tan^2 x/2)} + \frac{2\tan x/2}{1 + \tan^2 x/2}}\,dx$$

L.C.M

$$I = \int \frac{1 + \tan^2(x/2)\,dx}{5 + 5\tan^2(x/2) + 7 - 7\tan^2(x/2) + 2\tan x/2}$$

$$= \int \frac{\sec^2(x/2)\,dx}{-2\tan^2(x/2) + 2\tan x/2 + 12}\,dx$$

put $\tan x/2 = t$

$\sec^2(x/2)\,dx = 2\,dt$

$$I = 2\int \frac{dt}{-2t^2 + 2t + 12}$$

$$= \frac{2}{-2}\int \frac{dt}{t^2 - t - 6}$$

$$= -\int \frac{1}{\left(t-\frac{1}{2}\right)^2 - \frac{1}{4} - 6}\,dt$$

$$= -\int \frac{1}{\left(t-\frac{1}{2}\right)^2 - \left(\frac{5}{2}\right)^2}\,dt$$

$$= \int \frac{1}{\left(\frac{5}{2}\right)^2 - \left(t-\frac{1}{2}\right)^2}\,dt$$

$$= \frac{1}{2\times\frac{5}{2}}\log\left|\frac{\frac{5}{2} + t - \frac{1}{2}}{\frac{5}{2} - t + \frac{1}{2}}\right| + C$$

$$= \frac{1}{5}\log\left|\frac{5 + 2t - 1}{5 - 2t + 1}\right| + C$$

$$= \frac{1}{5}\log\left|\frac{2t + 4}{6 - 2t}\right| + C$$

$$= \frac{1}{5}\log\left|\frac{t + 2}{3 - t}\right| + C$$

$$= \frac{1}{5}\log\left|\frac{t + 2}{t - 3}\right| + C \quad \text{---} \left\{\because |a-b| = |b-a|\right\}$$

$$I = \frac{1}{5}\log\left|\frac{\tan x/2 + 2}{\tan\frac{x}{2} - 3}\right| + C$$

Q No 13 → $I = \int \frac{1}{1 - 2\sin x}\,dx$ --- {type single $\sin x$, $\cos x$}

$$I = \int \frac{1}{1 - 2\times\frac{2\tan x/2}{1+\tan^2(x/2)}}\,dx$$

L.C.M

$$= \int \frac{1+\tan^2(x/2)}{1+\tan^2(x/2) - 4\tan x/2} dx$$

$$I = \int \frac{\sec^2(x/2)\, dx}{\tan^2(x/2) - 4\tan(x/2) + 1}$$

put $\tan(x/2) = t$

$\sec^2(x/2)\, dx = 2dt$

$$\therefore I = 2\int \frac{dt}{t^2 - 4t + 1}$$

$$= 2\int \frac{1}{(t-2)^2 - 4 + 1} dt$$

$$= 2\int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt$$

$$= 2 \times \frac{1}{2\sqrt{3}} \log\left|\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right| + C$$

$$I = \frac{1}{\sqrt{3}} \log\left|\frac{\tan(x/2) - 2 - \sqrt{3}}{\tan(x/2) - 2 + \sqrt{3}}\right| + C$$ Ans

—x—