!! जय श्री राधे कृष्णा, जय श्री गिरिराज जी महराज !! ①

ULTIMATE MATHEMATICS: BY AJAY MITTAL

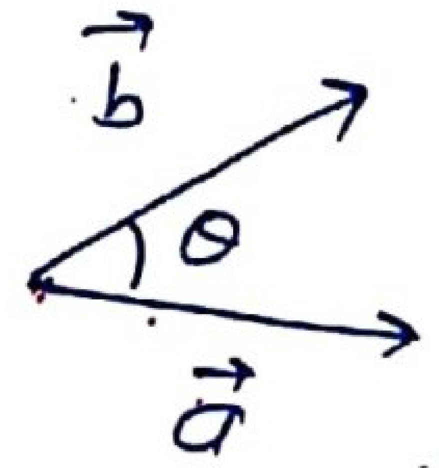
VECTORS: CLASS NO: 3

Topic Dot Product (OR) Scalar product of two vectors

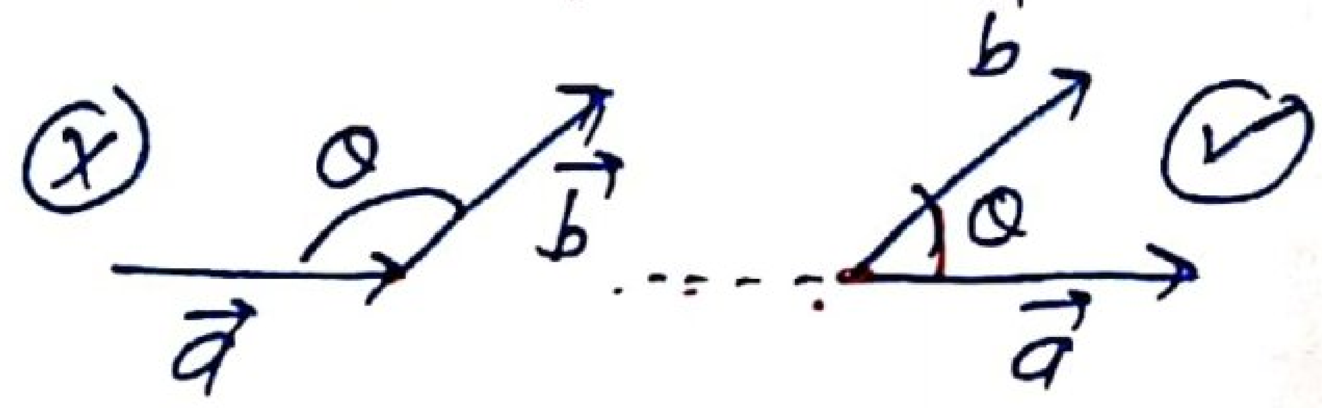
(1) $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

(2) $\vec{a}\cdot\vec{b}$ is always a scalar quantity

(3) angle b/w two vectors



$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$



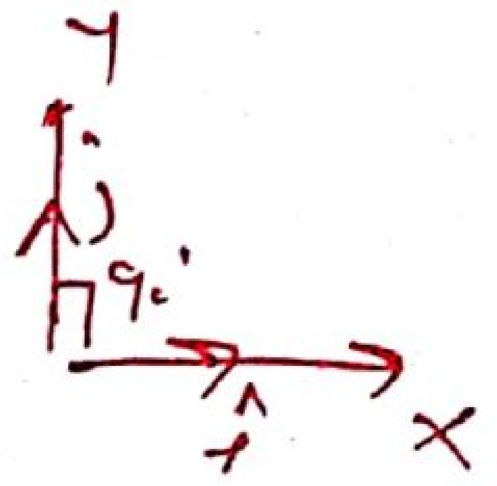
(vectors must be coinitial)

(4) If $\vec{a}\perp\vec{b}$ (or) orthogonal then $\vec{a}\cdot\vec{b} = 0$

(5) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Reason $\hat{i}\cdot\hat{i} = |\hat{i}|\,|\hat{i}|\cos 0^\circ$

$= (1)(1)(1) = 1$



(6) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Reason: angle b/w $\hat{i}$ & $\hat{j}$ is 90° and $\cos(90^\circ) = 0$

(7) e.g. $\vec{a} = 2\hat{i} + 3\hat{j} - 2\hat{k}$ & $\vec{b} = \hat{i} + 4\hat{j} + 3\hat{k}$

then $\vec{a}\cdot\vec{b} = (2\hat{i} + 3\hat{j} - 2\hat{k})\cdot(\hat{i} + 4\hat{j} + 3\hat{k})$

$= 2 + 12 - 6$

$\vec{a}\cdot\vec{b} = 8$ (clearly a scalar quantity)

(8) If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then $\vec{a}\cdot\vec{b} = 0$

Note (but converse need not be true: Reason they can be $\perp$)

(9) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)

(10) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = \vec{a}\cdot(\vec{b}+\vec{c})$ (distributive)

(11) $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

(12) $|\vec{a}|^2 = \vec{a}\cdot\vec{a}$

(Mainly used in $|\vec{a}+\vec{b}|$ or $|\vec{a}-\vec{b}|$)

IMP $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$

$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$

(13) (i) $m(\vec{a}\cdot\vec{b}) = (m\vec{a})\cdot\vec{b} = \vec{a}\cdot(m\vec{b})$

(ii) $m\vec{a}\cdot n\vec{b} = mn(\vec{a}\cdot\vec{b})$

(iii) $|m\vec{a}| = |m|\,|\vec{a}|$ Magnitude, Modulus

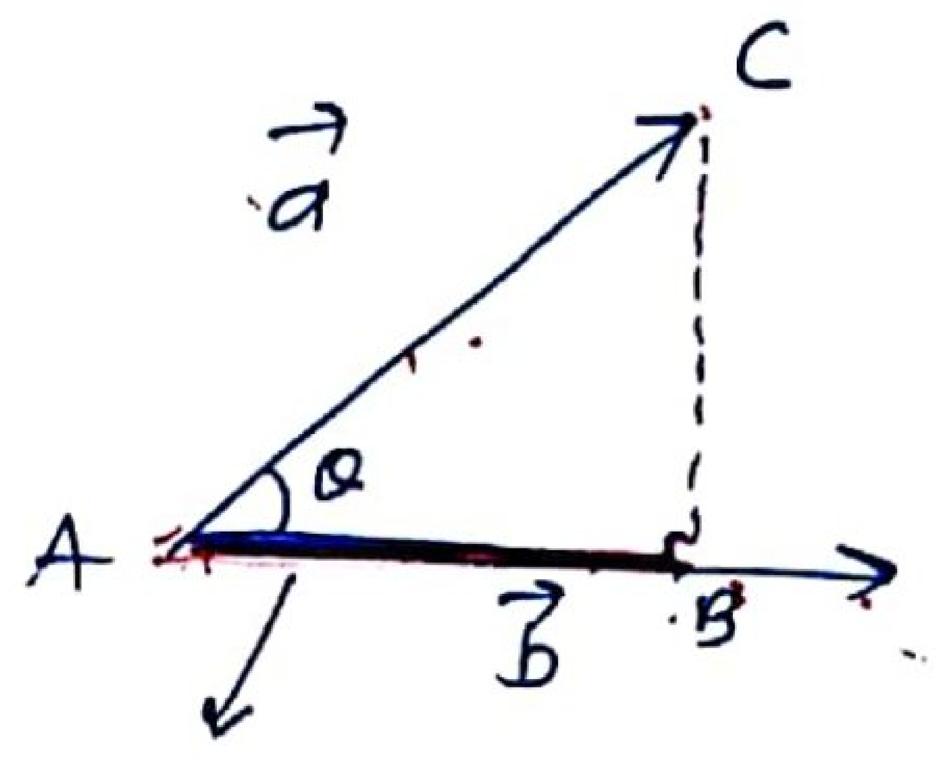
(14) If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

and $\vec{a}$ & $\vec{b}$ are perpendicular

then $\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$ Imp

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$$

(15) Projection of $\vec{a}$ on $\vec{b}$



projection of $\vec{a}$ on $\vec{b}$

Projection of $\vec{a}$ on $\vec{b}$ (AB) $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ ✓

Projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Projection of $\vec{a}$ on $\vec{b}$ $= \vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$ ✓

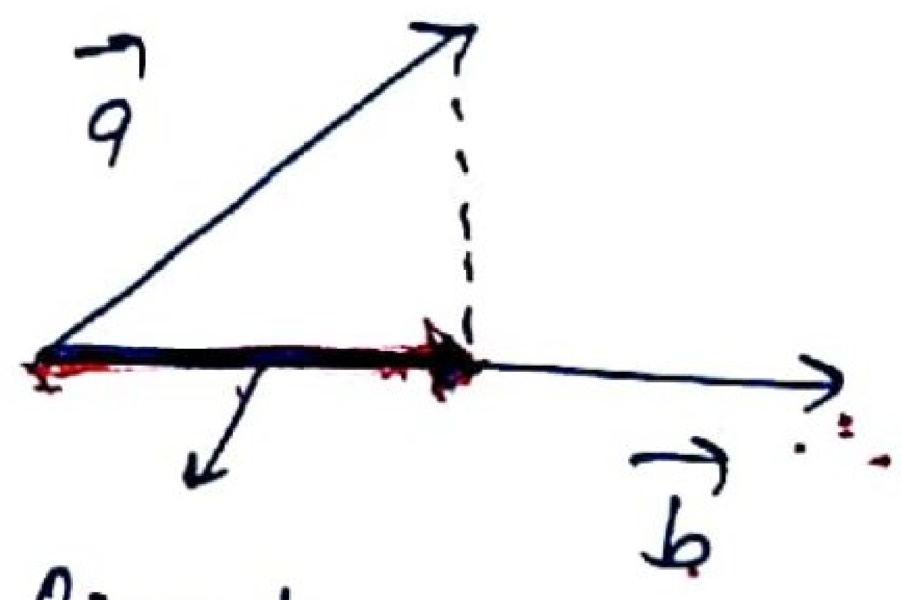
→ unit vector

$\cos\theta = \dfrac{AB}{AC} = \dfrac{\text{projection}}{|\vec{a}|}$

→ projection of $\vec{a}$ on $\vec{b}$ $= |\vec{a}|\cos\theta$ ✓

IMP Projection vector

Projection vector $=$ (projection) $\hat{b}$



Projection vector

(16) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

Reason

$\vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{b}$

$|\vec{a}|^2 - \cancel{\vec{a}\cdot\vec{b}} + \cancel{\vec{a}\cdot\vec{b}} - |\vec{b}|^2$

$= |\vec{a}|^2 - |\vec{b}|^2$

$m\cdot\vec{a}$ ✗

$\vec{a}\cdot\vec{b}$ ✓

$m\cdot n$ ✗

mn ✓

$\vec{a} - \vec{a}$ (ɑ) $\vec{0}$

–x–

QUESTIONS

Qus 1 Find the angle b/w the two vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ & $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

Soln

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\vec{a}\cdot\vec{b} = (\hat{i} - 2\hat{j} + 3\hat{k})\cdot(3\hat{i} - 2\hat{j} + \hat{k})$$

$$= 3 + 4 + 3 = 10$$

$$|\vec{a}| = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{9+4+1} = \sqrt{14}$$

Now $\cos\theta = \frac{10}{\sqrt{14}\sqrt{14}} = \frac{10}{14} = \frac{5}{7}$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{5}{7}\right)} \text{ Ans}$$

Qn. 2 Find the value of p so that the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} + p\hat{j} + 3\hat{k}$ are

(i) orthogonal (ii) collinear / parallel

Soln let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ & $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$

(i) orthogonal $\Rightarrow \vec{a}\cdot\vec{b} = 0$

$$\Rightarrow 3 + 2p + 27 = 0$$

$$\Rightarrow p = -15$$

(ii) parallel: $\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \Rightarrow 3 = \frac{2}{p} \Rightarrow p = 2/3$ Ans

Qn 3 → let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$

$\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$

$\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$

find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and satisfying $\vec{d}\cdot\vec{c} = 21$

Sol let required vector $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

given $\vec{d} \perp \vec{a}$ and $\vec{d} \cdot \vec{b}$ and $\vec{d}\cdot\vec{c} = 21$

$\Rightarrow \vec{d}\cdot\vec{a} = 0$ and $\vec{d}\cdot\vec{b} = 0$

$4x + 5y - z = 0$

$x - 4y + 5z = 0$

$3x + y - z = 21$

solving we get $x = 7$; $y = -7$, $z = -7$

$\therefore \boxed{\vec{d} = 7\hat{i} - 7\hat{j} - 7\hat{k}}$ Ans

Qn 4 → given $|\vec{a}| = 3$; $|\vec{b}| = 2$ & $\vec{a}\cdot\vec{b} = 6$

Find $|\vec{a} - \vec{b}|$

Sol we have $|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b})$

$= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$

$= |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$

$= 9 - 12 + 4$

$|\vec{a} - \vec{b}|^2 = 1$

$\Rightarrow |\vec{a} - \vec{b}| = 1$ Ans

Scanned with CamScanner

Que 5 → If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$; $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$
find the ~~any~~ value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$

Sol we have $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\boxed{\text{If } \vec{a}=\vec{b} \text{ then } |\vec{a}|=|\vec{b}|}$

If $|\vec{a}|=|\vec{b}|$ then $\vec{a}\neq\vec{b}$

$$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|=|\vec{0}|$$

$$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|=0$$

$$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2=0$$

$$\Rightarrow (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=0$$

$$\Rightarrow |\vec{a}|^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+|\vec{b}|^2+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+|\vec{c}|^2=0$$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$$

$$\Rightarrow 9+25+49+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$$

$$\Rightarrow \boxed{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\frac{-83}{2}} \text{ Ans}$$

OR

$$\vec{a}+\vec{b}+\vec{c}=\vec{0}$$

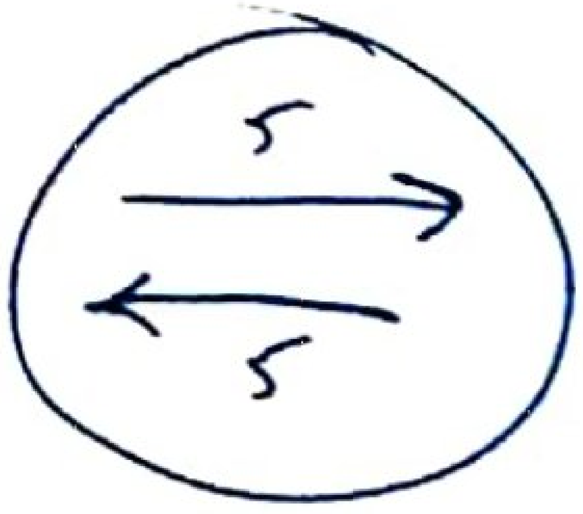
$$\Rightarrow (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{0}\cdot\vec{0}$$
$$=0$$

$\boxed{\vec{a}=\vec{b} \quad \vec{a}\cdot\vec{a}=\vec{b}\cdot\vec{b} \quad \text{Prop}}$

Que 6 If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$; $|\vec{a}|=3$, $|\vec{b}|=5$, $|\vec{c}|=7$
find the angle between $\vec{b}$ & $\vec{c}$

Sol Given $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$$\Rightarrow \vec{b}+\vec{c}=-\vec{a}$$



$\Rightarrow |\vec{b}+\vec{c}| = |-\vec{a}|$

$\Rightarrow |\vec{b}+\vec{c}| = |\vec{a}|$

$\Rightarrow |\vec{b}+\vec{c}|^2 = |\vec{a}|^2$

$\Rightarrow (\vec{b}+\vec{c})\cdot(\vec{b}+\vec{c}) = |\vec{a}|^2$

$\Rightarrow |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c} = |\vec{a}|^2$

$\Rightarrow 25 + 49 + 2\vec{b}\cdot\vec{c} = 9$

$\Rightarrow 2\vec{b}\cdot\vec{c} = -65$

$\Rightarrow 2|\vec{b}||\vec{c}|\cos\theta = -65$

$\Rightarrow 2(5)(7)\cos\theta = -65$

$\Rightarrow \cos\theta = \frac{-\cancel{65}^{13}}{2\times\cancel{5}\times 7}$

$\Rightarrow \cos\theta = -\frac{13}{14}$

$\Rightarrow \boxed{\theta = \cos^{-1}\left(-\frac{13}{14}\right)}$ Ans

Q.No. 7* If $\hat{a}$ & $\hat{b}$ are unit vectors inclined at an angle θ, then show that

(i) $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$ (2) $\cos\frac{\theta}{2} = \frac{1}{2}|\hat{a}+\hat{b}|$ (3) $\tan\frac{\theta}{2} = \frac{|\hat{a}-\hat{b}|}{|\hat{a}+\hat{b}|}$

Sol: given $|\hat{a}|=1$ & $|\hat{b}|=1$

we have $|\hat{a}-\hat{b}|^2 = (\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$

$\Rightarrow |\hat{a}-\hat{b}|^2 = |\hat{a}|^2 - 2\hat{a}\cdot\hat{b} + |\hat{b}|^2$

$= 1 - 2|\hat{a}||\hat{b}|\cos\theta + 1$

$= 1 - 2\cos\theta + 1$

$= 2 - 2\cos\theta$

$= 2(1-\cos\theta)$

$|\hat{a}-\hat{b}|^2 = 2\times 2\sin^2\frac{\theta}{2}$

$\Rightarrow |\hat{a}-\hat{b}| = 2\sin\frac{\theta}{2}$

$\Rightarrow \sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$ find

(ii) we have $|\hat{a}+\hat{b}|^2 = (\hat{a}+\hat{b})\cdot(\hat{a}+\hat{b})$

$\vdots$

get $\cos\frac{\theta}{2} = \frac{1}{2}|\hat{a}+\hat{b}|$

(iii) $\tan\frac{\theta}{2} = \frac{\sin\theta/2}{\cos\theta/2} = \frac{|\hat{a}-\hat{b}|}{|\hat{a}+\hat{b}|}$ Ans

Qn. 8 → The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of the vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1. Find the value of λ.

Sol. let $\vec{a} = \hat{i}+\hat{j}+\hat{k}$

$\vec{b} = (2\hat{i}+4\hat{j}-5\hat{k}) + (\lambda\hat{i}+2\hat{j}+3\hat{k}) = (2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

Now $\hat{b} = \dfrac{\vec{b}}{|\vec{b}|}$

$$\hat{b} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 36 + 4}}$$

$$\hat{b} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$$

given $\vec{a} \cdot \hat{b} = 1$

$$\Rightarrow (\hat{i}+\hat{j}+\hat{k}) \cdot \frac{\left((2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}\right)}{\sqrt{\lambda^2+4\lambda+44}} = 1$$

$$\Rightarrow (\cancel{2}+\lambda) + 6 - \cancel{2} = \sqrt{\lambda^2+4\lambda+44}$$

$$\Rightarrow \lambda + 6 = \sqrt{\lambda^2+4\lambda+44}$$

Squaring

$$\cancel{\lambda^2} + 36 + 12\lambda = \cancel{\lambda^2} + 4\lambda + 44$$

$$\Rightarrow 8\lambda = 8$$

$$\Rightarrow \boxed{\lambda = 1} \quad \text{Ans}$$

Qn. 9 → given $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$; $\vec{\alpha} = 3\hat{i} - \hat{j}$

Express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta_1} + \vec{\beta_2}$, where $\vec{\beta_1}$ is parallel to $\vec{\alpha}$ and $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$.

Soln

given $\vec{\beta_1}$ is parallel to $\vec{\alpha}$

$\Rightarrow \vec{\beta_1} = \lambda \vec{\alpha}$

$\Rightarrow \vec{\beta_1} = \lambda(3\hat{i} - \hat{j})$

$\Rightarrow \boxed{\vec{\beta_1} = 3\lambda\hat{i} - \lambda\hat{j}}$

given $\vec{\beta} = \vec{\beta_1} + \vec{\beta_2}$

$\Rightarrow \vec{\beta_2} = \vec{\beta} - \vec{\beta_1}$

$= (2\hat{i} + \hat{j} - 3\hat{k}) - (3\lambda\hat{i} - \lambda\hat{j})$

$\boxed{\vec{\beta_2} = \hat{i}(2 - 3\lambda) + \hat{j}(1 + \lambda) - 3\hat{k}}$

given $\vec{\beta_2} \perp \vec{\alpha}$

$\Rightarrow \vec{\beta_2} \cdot \vec{\alpha} = 0$

$\Rightarrow (\hat{i}(2 - 3\lambda) + \hat{j}(1 + \lambda) - 3\hat{k}) \cdot (3\hat{i} - \hat{j}) = 0$

$\Rightarrow 6 - 9\lambda - 1 - \lambda = 0$

$\Rightarrow 5 = 10\lambda \Rightarrow \lambda = 1/2$

$\therefore \vec{\beta_1} = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}$

$\vec{\beta_2} = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$

Now $\vec{\beta_1} + \vec{\beta_2} = 2\hat{i} + \hat{j} - 3\hat{k} = \vec{\beta}$ Hence verified

WORKSHEET NO: 2 VECTORS

QNS 1 Find the projection of the vector $2\hat{i}+3\hat{j}+2\hat{k}$ on the vector $\hat{i}+2\hat{j}+\hat{k}$

ANS $\frac{5}{3}\sqrt{6}$

QNS 2 → If $\vec{a} = 5\hat{i}-\hat{j}-3\hat{k}$ & $\vec{b} = \hat{i}+3\hat{j}+\lambda\hat{k}$. If $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are orthogonal. Find the value of λ

ANS $\lambda = -5$

QNS 3 → Find $|\vec{a}-\vec{b}|$, if two vectors $\vec{a}$ & $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 4$ **ANS** $\sqrt{5}$

QN 4 → If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$ Find $|\vec{x}|$

ANS 3

QN 5 → Find the angle b/w two vectors $\vec{a}$ & $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a}\cdot\vec{b} = \sqrt{6}$

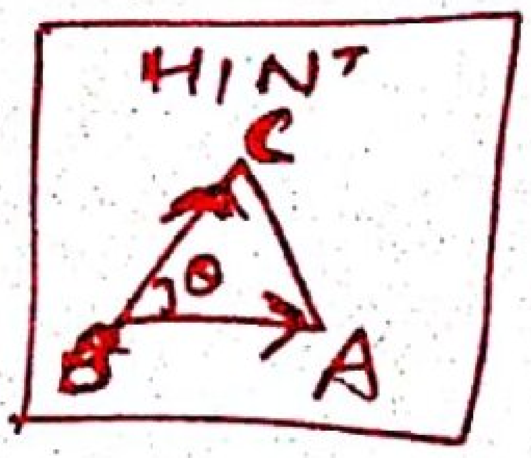
ANS $\pi/4$

QN 6 → Show that the vectors

$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$; $\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$ & $\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$

are unit vectors and also show that they are mutually perpendiculars to each other.

QN 7 → If the vertices A, B, C of a triangle ABC are (1,2,3), (−1,0,0), (0,1,2) respectively then find $\angle ABC$ **ANS** $\angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$



prepare $\vec{BA}$ & $\vec{BC}$ then $\cos C = \frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}|\,|\vec{BC}|}$

Qn 8 → If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$. Find the value of λ Ans = 10

Qn 9 → If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Find the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$

Ans = $-3/2$

Qn 10 → Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is $60°$ and their scalar product is $1/2$ Ans $|\vec{a}| = |\vec{b}| = 1$

Qn 11 → Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 5$ and each one of them being perpendicular to the sum of the other two. Find $|\vec{a} + \vec{b} + \vec{c}|$ Ans $5\sqrt{2}$

Qn 12 → Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ & $\vec{b}$ and $\vec{c}\cdot\vec{d} = 15$

Ans $\frac{1}{3}(160\hat{i} - 5\hat{j} + 70\hat{k})$

Qn 13 → Decompose (break) the vector $6\hat{i} - 3\hat{j} - 6\hat{k}$ into two vectors which ~~were~~ are parallel and perpendicular to the vector $\hat{i} + \hat{j} + \hat{k}$

Ans $-\hat{i} - \hat{j} - \hat{k}$ & $7\hat{i} - 2\hat{j} - 5\hat{k}$

—x—