← ULTIMATE MATHEMATICS →

Solutions of M-6

Q1 → given $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

let $P(n) : A^n = \begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{bmatrix}$

$P(1) : A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ which is equal to A

$\therefore P(1)$ is true

let $P(k)$ be true

$P(k) : A^k = \begin{bmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{bmatrix}$

To prove $P(k+1)$ is true

$P(k+1) : A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$

LHS A^{k+1}

$= A^k \cdot A$

$= \begin{bmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta & \cos(k\theta)\sin\theta + \sin(k\theta)\cos\theta \\ -\sin(k\theta)\cos\theta - \cos(k\theta)\sin\theta & -\sin(k\theta)\sin\theta + \cos(k\theta)\cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$ = RHS

(M-6) solution

$\therefore P(k+1)$ is true

$\therefore$ By PMI, $P(n)$ is true for all values of $n \in N$

Ans-2 given $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

let $P(n): (aI + bA)^n = a^n I + n a^{n-1} bA$

$P(1)$: $aI + bA = aI + bA$

Clearly $P(1)$ is true

let $P(k)$ be true

$P(k)$: $(aI + bA)^k = a^k I + k a^{k-1} bA$

To prove $P(k+1)$ is true

$P(k+1)$: $(aI + bA)^{k+1} = a^{k+1} I + (k+1) a^k bA$

LHS $(aI + bA)^{k+1}$

$= (aI + bA)^k \cdot (aI + bA)$

$= (a^k I + k a^{k-1} bA) \cdot (aI + bA)$ --- {from $P(k)$}

$= a^{k+1} I + a^k bA + k a^k bA + k a^{k-1} b^2 A^2$

$= a^{k+1} I + a^k bA (1+k) + k a^{k-1} b^2 A^2$

$= a^{k+1} I + a^k bA (k+1) + 0$

--- $\left\{ \because A^2 = AA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \right\}$

$=$ RHS

$\therefore P(k+1)$ is true

$\therefore P(n)$ is true for all $n \in N$.

(M-6) Solutions

Ans. 3 → given $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

let $P(n)$:

$$A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$$

$$P(1): A = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Clearly $P(1)$ is true

let $P(k)$ be true

$$P(k) = A^k = \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$$

To prove $P(k+1)$ is true

$$P(k+1): A^{k+1} = \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix}$$

Taking LHS

$$A^{k+1}$$

$$= A^k \cdot A$$

$$= \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{k-1}+3^{k-1}+3^{k-1} & 3^{k-1}+3^{k-1}+3^{k-1} & 3^{k-1}+3^{k-1}+3^{k-1} \\ — & — & — \\ — & — & — \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 3^{k-1} & 3\cdot 3^{k-1} & 3\cdot 3^{k-1} \\ 3\cdot 3^{k-1} & 3\cdot 3^{k-1} & 3\cdot 3^{k-1} \\ 3\cdot 3^{k-1} & 3\cdot 3^{k-1} & 3\cdot 3^{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix} = \underline{RHS}$$

$\therefore$ $P(k+1)$ is true

$\therefore$ By PMI, $P(n)$ is true for all $n \in N$.

Qn 4 → given $AB = BA$

(i) let $P(n)$: $AB^n = B^n A$

$P(1)$: $AB = BA$

$P(1)$ is true $\{\because$ given $AB = BA\}$

let $P(k)$ be true

$P(k)$: $AB^k = B^k A$

To prove $P(k+1)$ is true

$P(k+1)$: $AB^{k+1} = B^{k+1} A$

LHS AB^{k+1}

$= AB^k \cdot B$

$= B^k A \cdot B$ --- $\{$from $P(k)\}$

$= B^k BA$ --- $\{$given $AB = BA\}$

(M-6) Solutions

$= B^{k+1} A$

$= Rhs$

$\therefore P(k+1)$ is true

By PMI, $P(n)$ is true for all $n \in N$

(ii) let $P(n)$: $(AB)^n = A^n B^n$

$P(1)$: $AB = AB$

Clearly $P(1)$ is true

let $P(k)$ be true

$P(k)$: $(AB)^k = A^k B^k$

To prove $P(k+1)$ is true

$P(k+1)$: $(AB)^{k+1} = A^{k+1} B^{k+1}$

Lhs $(AB)^{k+1}$

$= (AB)^k (AB)$

$= A^k B^k AB$ --- {from $P(k)$}

$= A^k B^k BA$ -- {given $AB = BA$}

$= A^k B^{k+1} A$

$= A^k . A B^{k+1}$ --- {from result of part (i)}

$= A^{k+1} B^{k+1}$

$= Rhs$

$\therefore P(k+1)$ is true

$\therefore P(n)$ is true for all $n \in N$ Ans

Qn 5 → Do yourself

Qn 6 → Do yourself

Qn 7 → given $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

let $P(n)$: $A^n = \begin{bmatrix} a^n & \frac{b(a^n - 1)}{a-1} \\ 0 & 1 \end{bmatrix}$

$P(1)$: $A = \begin{bmatrix} a & \frac{b\cancel{(a-1)}}{\cancel{a-1}} \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ which is true

let $P(k)$ be true

$P(k)$: $A^k = \begin{bmatrix} a^k & \frac{b(a^k - 1)}{a-1} \\ 0 & 1 \end{bmatrix}$

To prove $P(k+1)$ is true

$P(k+1)$: $A^{k+1} = \begin{bmatrix} a^{k+1} & \frac{b(a^{k+1} - 1)}{a-1} \\ 0 & 1 \end{bmatrix}$

LHS A^{k+1}

$= A^k \cdot A$

$= \begin{bmatrix} a^k & \frac{b(a^k - 1)}{a-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

(M-6) Solutions

$$= \begin{bmatrix} a^{k+1}+0 & a^k b + \frac{b(a^k-1)}{a-1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{k+1} & \frac{a^{k+1}b - \cancel{a^k b} + \cancel{ba^k} - b}{a-1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{k+1} & \frac{b(a^{k+1}-1)}{a-1} \\ 0 & 1 \end{bmatrix} = \underline{\underline{RHS}}$$

$\therefore$ $P(k+1)$ is true

By PMI, $P(n)$ is true for all $n \in N$.

Qn 8 → given $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$

let $P(n)$: $A^n = \begin{bmatrix} \cos(n\theta) & i\sin(n\theta) \\ i\sin(n\theta) & \cos(n\theta) \end{bmatrix}$

$P(1)$: $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$ clearly $P(1)$ is true

let $P(k)$ be true

$P(k)$: $A^k = \begin{bmatrix} \cos(k\theta) & i\sin(k\theta) \\ i\sin(k\theta) & \cos(k\theta) \end{bmatrix}$

To prove $P(k+1)$ is true

(M-6) Solutions

$P(k+1)$: $A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & i\sin(k+1)\theta \\ i\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$

LHS A^{k+1}

$= A^k \cdot A$

$= \begin{bmatrix} \cos(k\theta) & i\sin(k\theta) \\ i\sin(k\theta) & \cos(k\theta) \end{bmatrix} \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos(k\theta)\cos\theta + i^2\sin(k\theta)\sin\theta & i\cos(k\theta)\sin\theta + i\sin(k\theta)\cos\theta \\ i\sin(k\theta)\cos\theta + i\cos(k\theta)\sin\theta & i^2\sin(k\theta)\sin\theta + \cos(k\theta)\cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta & i(\sin\theta\cdot\cos(k\theta) + \cos\theta\sin(k\theta)) \\ i(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta) & -\sin(k\theta)\sin\theta + \cos(k\theta)\cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos(k+1)\theta & i\sin(k+1)\theta \\ i\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$

= RHS

$\therefore$ By PMI, $P(n)$ is true for all $n \in \mathbb{N}$

Ans 9 → $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

(i) $f(x) = x^2 - 4x + 7$

$f(A) = A^2 - 4A + 7I$

By self show $= 0$

ii) we have $A^2 - 4A + 7I = 0$

$$\boxed{A^2 = 4A - 7I}$$

multiply by A

$A^3 = 4A^2 - 7A$

$A^3 = 4(4A - 7I) - 7A$

$A^3 = 9A - 28I$

multiply by A

$A^4 = 9A^2 - 28A$

$A^4 = 9(4A - 7I) - 28A$

$A^4 = 8A - 63I$

multiply by A

$A^5 = 8A^2 - 63A$

$A^5 = 8(4A - 7I) - 63A$

$A^5 = -31A - 56I$

$$A^5 = -31\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 56\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -62 & -93 \\ 31 & -62 \end{bmatrix} - \begin{bmatrix} 56 & 0 \\ 0 & 56 \end{bmatrix}$$

$$= \begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$$ Ans

---

Ans 10 → given $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

(i) $A^2 - 5A + 7I$

By yourself show $= 0$

(ii) we have

$$A^2 - 5A + 7I = 0$$

$$\Rightarrow \boxed{A^2 = 5A - 7I}$$

multiply by A

$$A^3 = 5A^2 - 7A$$

$$A^3 = 5(5A - 7I) - 7A$$

$$A^3 = 18A - 35I$$

multiply by A

$$A^4 = 18A^2 - 35A$$

$A^4 = 18(5A - 7I) - 35A$

$A^4 = 55A - 126I$

$= 55\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 126\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{bmatrix} 165 & 55 \\ -55 & 110 \end{bmatrix} - \begin{bmatrix} 126 & 0 \\ 0 & 126 \end{bmatrix}$

$A^4 = \begin{bmatrix} 39 & 55 \\ -55 & -16 \end{bmatrix}$ Ans

---

11 → given $A = \text{diag}(a, b, c)$

i.e $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$P(n)$. $A^n = \text{diag}(a^n, b^n, c^n)$

i.e $A^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}$

$P(1)$: $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ which is true

$P(k)$: $A^k = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix}$

Proceed and do yourself

Qn 12 → (i)

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0$

(ii) let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Ans

Qn 13 → given $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$A^4 = A^2 A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$

$= 8 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 8A$

$A = 8$ Ans

Qn (14) $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$A^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}$

$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^4 = A^2 A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$(A^4)^n = (I)^n$

$A^{4n} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .. {Since $I I I \ldots = I$}

Ans

Qn 15 given $A \to$ Skew Symm Matrix

$\Rightarrow \boxed{A' = -A}$

let $P = A^n$

$A = AAA \ldots$ ntimes

$P' = (AAA \ldots A)'$

$P' = A'A'A' \ldots A'$

$P' = (-A)(-A)(-A) \ldots (-A)$ ... {given $A' = -A$}

$P' = (-1)^n A$

Case I when $n \to$ even

$$P^n = (-1)^n A^n$$

$$P^n = A^n \quad \text{---} \left\{ \begin{array}{l} (-1)^n = 1 \\ n \to \text{even} \end{array} \right\}$$

$$P^n = P$$

$\therefore$ P is a symm Matrix

Case II when $n \to$ odd

$$P^n = (-1)^n A^n$$

$$P^n = -A^n \quad \text{---} \left\{ \begin{array}{l} (-1)^n = -1 \\ n \to \text{odd} \end{array} \right\}$$

$$P^n = -P$$

$\therefore$ P $\to$ Skew symm Matrix

— ^ —