DIFFERENTIATION & CONTINUITY.

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SOLUTION of WORKSHEET No: 6

Qn 1 → $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x & ; x \neq 0 \\ 2 & ; x = 0 \end{cases}$

$LHL = \lim\limits_{x \to 0^-} \left(\dfrac{\sin x}{x} + \cos x \right)$ put $x = 0 - h = -h$, $h \to 0$

$LHL = \lim\limits_{h \to 0} \left(\dfrac{-\sin h}{-h} + \cos h \right) = \lim\limits_{h \to 0} \left(\dfrac{\sin h}{h} \right) + \lim\limits_{h \to 0} (\cos h)$

$= 1 + 1 = 2$

$RHL = \lim\limits_{x \to 0^+} \left(\dfrac{\sin x}{x} + \cos x \right)$ put $x = 0 + h = h$, $h \to 0$

$RHL = \lim\limits_{h \to 0} \left(\dfrac{\sin h}{h} + \cos h \right) = \lim\limits_{h \to 0} \left(\dfrac{\sin h}{h} \right) + \lim\limits_{h \to 0} (\cos h)$

$= 1 + 1 = 2$

$f(0) = 2$

Clearly $LHL = RHL = f(0)$

$\therefore f(x)$ is continuous at $x = 2$ **Ans**

Qn.2 → $f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x+1} & ; x \neq -1 \\ \lambda & ; x = -1 \end{cases}$

$LHL = \lim\limits_{x \to -1^-} \left(\dfrac{x^2 - 2x - 3}{x+1} \right)$

$= \lim\limits_{x \to -1^-} \left(\dfrac{\cancel{(x+1)}(x-3)}{\cancel{x+1}} \right)$

put $x = -1 - h$ & $h \to 0$

$\therefore LHL = \lim\limits_{h \to 0} (-1 - h - 3) = -1 - 3$

$\boxed{LHL = -4}$

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Similarly RHL = -4

$f(-1) = \lambda$

Since it is given $f(x)$ is continuous at $x=-1$

$\therefore$ LHL = RHL = $f(-1)$

$\Rightarrow -4 = -4 = \lambda$

$\Rightarrow \boxed{\lambda = -4}$ Ans

Ans 3 → $f(x) = \begin{cases} \dfrac{1-\cos(2x)}{2x^2} & ; x \neq 0 \\ K & ; x = 0 \end{cases}$

Since $f(x)$ is continuous at $x=0$

$\therefore \lim_{x \to 0} (f(x)) = f(0)$

$\Rightarrow \lim_{x \to 0} \left(\dfrac{1-\cos(2x)}{2x^2} \right) = k$

$\Rightarrow \lim_{x \to 0} \left(\dfrac{\cancel{2}\sin^2 x}{\cancel{2}x^2} \right) = k$

$\Rightarrow 1 = k \quad \therefore \boxed{k = 1}$ Ans

Qn. 4 → $f(x) = \begin{cases} 3ax + b & ; x > 1 \\ 11 & ; x = 1 \\ 5ax - 2b & , x < 1 \end{cases}$

Given that $f(x)$ is continuous at $x = 1$

$\Rightarrow$ LHL = RHL = $f(1)$

$\Rightarrow \lim_{x \to 1^-} (5ax - 2b) = \lim_{x \to 1^+} (3ax + b) = 11$

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put $x = 1-h$ & $h \to 0$; put $x = 1+h$ & $h \to 0$

$$\Rightarrow \lim_{h\to 0}\left(5a(1-h)-2b\right) = \lim_{h\to 0}\left(3a(1+h)+b\right) = 11$$

$$\Rightarrow 5a-2b = 3a+b = 11$$

$$\Rightarrow 5a-2b = 11 \quad \times 1$$
$$3a+b = 11 \quad \times 2$$

$$\Rightarrow 5a - \cancel{2b} = 11$$
$$6a + \cancel{2b} = 22$$
$$11a = 33 \qquad \boxed{a = 3 \Rightarrow b = 2} \text{ ANS}$$

Qn-5 → $f(x) = 2x - |x|$

$$f(x) = \begin{cases} 2x - x & ; x \geq 0 \\ 2x-(-x) & ; x < 0 \end{cases}$$

$$f(x) = \begin{cases} x & ; x \geq 0 \\ 3x & ; x < 0 \end{cases}$$

$LHL = \lim_{x\to 0^-}(3x)$ put $x = 0-h = -h$ & $h \to 0$

$$LHL = \lim_{h\to 0}\left(3(-h)\right) = 3\times 0 = 0$$

$RHL = \lim_{x\to 0^+}(x)$ put $x = 0+h = h$ & $h \to 0$

$$RHL = \lim_{h\to 0}(h) = 0$$

$f(0) = 0$ since $LHL = RHL = f(0)$

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$\therefore f(x)$ is continuous at $x=0$ Ans

Qn 6 → $f(x) = |x-3| + |x+1|$

$(3) \quad (-1)$

$f(x) = |x+1| + |x-3|$

$(-1) \quad (3)$

$$f(x) = \begin{cases} -(x+1)-(x-3) \; ; & x < -1 \\ (x+1)-(x-3) \; ; & -1 \le x < 3 \\ (x+1)+(x-3) \; ; & 3 \le x < \infty \end{cases}$$

$$f(x) = \begin{cases} -2x+2 \; ; & x < -1 \\ 4 \; ; & -1 \le x < 3 \\ 2x-2 \; ; & x \ge 3 \end{cases}$$

Continuity at $x=-1$

$LHL = \lim_{x \to (-1)^-} (-2x+2)$ put $x = -1-h$ & $h \to 0$

$LHL = \lim_{h \to 0} \left(-2(-1-h)+2\right) = -2(-1)+2 = 4$

$RHL = \lim_{x \to (-1)^+} (4) = 4$

$f(-1) = 4$

Since $LHL = RHL = f(-1)$

$\therefore f(x)$ is continuous at $x=-1$ Ans

Similarly check continuity at $x=3$ (By self)

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Qn. 7 → $f(x) = \begin{cases} \frac{1-\cos(2x)}{2x^2} & ; x<0 \\ k & ; x=0 \\ \frac{x}{|x|} & ; x>0 \end{cases}$

Sol: Redefine the function

$f(x) = \begin{cases} \frac{1-\cos(2x)}{2x^2} & ; x<0 \\ k & ; x=0 \\ \frac{x}{x} = 1 & ; x>0 \end{cases}$ {∵ when $x>0$, $|x| = x$}

$LHL = \lim_{x \to 0^-} \left(\frac{1-\cos(2x)}{2x^2} \right)$

$= \lim_{x \to 0^-} \left(\frac{2\sin^2 x}{2x^2} \right)$ put $x = 0-h = -h$, $h \to 0$

$LHL = \lim_{h \to 0} \left(\frac{\sin^2(-h)}{(-h)^2} \right) = \lim_{h \to 0} \left(\frac{\sin^2 h}{h^2} \right) = 1$

$RHL = \lim_{x \to 0^+} (1) = 1$

$f(0) = k$

Since $f(x)$ is continuous at $x=1$ (given)

∴ $LHL = RHL = f(0)$

$\Rightarrow 1 = 1 = k$

∴ $\boxed{k=1}$ Ans

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Ans 8 → $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & ; \quad x < \pi/2 \\ a & ; \quad x = \pi/2 \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & ; \quad x > \pi/2 \end{cases}$

$LHL = \lim\limits_{x \to \frac{\pi}{2}^-} \left(\dfrac{1-\sin^3 x}{3\cos^2 x} \right)$ put $x = \frac{\pi}{2} - h$ & $h \to 0$

$LHL = \lim\limits_{h \to 0} \left[\dfrac{1-\sin^3(\frac{\pi}{2}-h)}{3\cos^2(\frac{\pi}{2}-h)} \right]$

$LHL = \lim\limits_{h \to 0} \left(\dfrac{1-\cos^3 h}{3\sin^2 h} \right)$

$LHL = \lim\limits_{h \to 0} \left(\dfrac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos^2 h)} \right)$ a^3-b^3 formula used

$LHL = \lim\limits_{h \to 0} \left(\dfrac{\cancel{(1-\cos h)}(1+\cos^2 h+\cos h)}{3\cancel{(1-\cos h)}(1+\cos h)} \right)$

$= \dfrac{(1+\cos^2(0)+\cos(0))}{3(1+\cos 0)} = \dfrac{1+1+1}{3(1+1)} = \dfrac{3}{6} = \dfrac{1}{2}$

$RHL = \lim\limits_{x \to \frac{\pi}{2}^+} \left(\dfrac{b(1-\sin x)}{(\pi-2x)^2} \right)$ put $x = \frac{\pi}{2}+h$ & $h \to 0$

$RHL = \lim\limits_{h \to 0} \left(\dfrac{b(1-\sin(\frac{\pi}{2}+h))}{(\pi-2(\frac{\pi}{2}+h))^2} \right)$

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Solution of worksheet No: 6 (7)

$$RHL = \lim_{h\to 0}\left(\frac{b(1-\cos h)}{(\cancel{\pi}-\cancel{\pi}-2h)^2}\right)$$

$$= \lim_{h\to 0}\left(\frac{b \cdot 2\sin^2\left(\frac{h}{2}\right)}{\frac{4h^2}{4}\times 4}\right) = \frac{2b}{16} = \frac{b}{8}$$

$f(\pi/2) = a$

Since $f(x)$ is continuous at $x = \pi/2$

$\Rightarrow LHL = RHL = f(\pi/2)$

$\Rightarrow \frac{1}{2} = \frac{b}{8} = a$

$\Rightarrow \boxed{a = \frac{1}{2} \;;\; b = 4}$ ANS

Qn. 9 → $f(x) = \begin{cases} 1 & : x \le 3 \\ ax+b & : 3 < x < 5 \\ 7 & , x \ge 5 \end{cases}$

Continuity at $x=3$

$LHL = \lim_{x\to 3^-}(1) = 1$

$RHL = \lim_{x\to 3^+}(ax+b)$ put $x = 3+h$ & $h \to 0$

$RHL = \lim_{h\to 0}(a(3+h)+b) = 3a+b$

$f(3) = 1$

$LHL = RHL = f(3)$

$\Rightarrow 1 = 3a+b = 1$

$\Rightarrow 3a+b = 1$ ----(i)

Continuity at $x=5$

$LHL = \lim_{x\to 5^-}(ax+b)$ put $x = 5-h$ & $h\to 0$

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Solution of Worksheet No: 6 (8)

$LHL = \lim_{h \to 0} \left(a(5-h) + b \right) = 5a + b$

$RHL = \lim_{x \to 5^+} (7) = 7$

$f(5) = 7$

$5a + b = 7 = 7$

$\Rightarrow 5a + b = 7$ ---- (2)

Solving equation (1) & (2)

$2a = 6 \Rightarrow \boxed{a = 3 \quad \& \quad b = -8}$ Ans

Qn-10 → $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x} & ; x \neq \pi/2 \\ 3 & ; x = \pi/2 \end{cases}$

Since $f(x)$ is continuous at $x = \pi/2$ (given)

$\therefore \lim_{x \to \frac{\pi}{2}} (f(x)) = f(\pi/2)$

$\Rightarrow \lim_{x \to \frac{\pi}{2}} \left(\dfrac{k\cos x}{\pi - 2x} \right) = 3$

put $x = \frac{\pi}{2} + h$ & $h \to 0$ { you can also put $x = \frac{\pi}{2} - h$; choice hoti hai }

$\Rightarrow \lim_{h \to 0} \left(\dfrac{k\cos(\frac{\pi}{2} + h)}{\pi - 2(\frac{\pi}{2} + h)} \right) = 3$

$\Rightarrow \lim_{h \to 0} \left(\dfrac{-k\sin h}{-2h} \right) = \lim_{h \to 0} \left(\dfrac{k}{2} \dfrac{\sin h}{h} \right) = 3$

$\dfrac{k}{2} \times 1 = \dfrac{k}{2} = 3$

$\Rightarrow \boxed{k = 6}$ Ans

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Solution of worksheet No: 6 (9)

Qn 11 → $f(x) = \begin{cases} \frac{x-4}{|x-4|} + a & ; \quad x < 4 \\ a+b & ; \quad x = 4 \\ \frac{x-4}{|x-4|} + b & ; \quad x > 4 \end{cases}$

Redefine the function

$f(x) = \begin{cases} \frac{x-4}{-(x-4)} + a & ; \quad x < 4 \\ a+b & ; \quad x = 4 \\ \frac{x-4}{x-4} + b & ; \quad x > 4 \end{cases}$

$f(x) = \begin{cases} -1 + a & ; \quad x < 4 \\ a+b & ; \quad x = 4 \\ 1+b & , \quad x > 4 \end{cases}$

Given that $f(x)$ is continuous at $x = 4$

$\Rightarrow$ LHL = RHL = $f(4)$

$\Rightarrow \lim_{x \to 4^-} (-1+a) = \lim_{x \to 4^+} (1+b) = a+b$

$\Rightarrow -1+a = 1+b = a+b$

$\Rightarrow -1+\cancel{a} = \cancel{a}+b$ and $1+\cancel{b} = a+\cancel{b}$

$\Rightarrow b = -1$ and $a = 1$ Ans

Qns 12 → $f(x) = \begin{cases} ax+1 & ; \quad x \leq 3 \\ bx+3 & ; \quad x > 3 \end{cases}$

Given $f(x)$ is continuous at $x = 3$

$\Rightarrow$ LHL = RHL = $f(3)$

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$\Rightarrow \lim_{x \to 3^-} (ax+1) = \lim_{x \to 3^+} (bx+3) = f(3)$

put $x = 3-h$, $h \to 0$ put $x = 3+h$, $h \to 0$

$\Rightarrow \lim_{h \to 0} (a(3-h)+1) = \lim_{h \to 0} (b(3+h)+3) = a(3)+1$

$\Rightarrow 3a+1 = 3b+3 = 3a+1$

$\Rightarrow 3a+1 = 3b+3$

$\Rightarrow 3a-3b = 2$ is the required Relation Ans

Ans 13 $\rightarrow$ $f(x) = |x| + |x-1|$
$(0) \quad (1)$

~~Ref~~ Redefine the function

$$f(x) = \begin{cases} -(x)-(x-1) & ; \ x < 0 \\ (x)-(x-1) & ; \ 0 \le x < 1 \\ (x)+(x-1) & ; \ x \ge 1 \end{cases}$$

$$f(x) = \begin{cases} -2x+1 & ; \ x < 0 \\ 1 & ; \ 0 \le x < 1 \\ 2x-1 & ; \ x \ge 1 \end{cases}$$

Continuity at $x=0$

$LHL = \lim_{x \to 0^-} (-2x+1)$ put $x = 0-h = -h$ & $h \to 0$

$LHL = \lim_{h \to 0} (-2(-h)+1) = 0+1 = 1$

$RHL = \lim_{x \to 0^+} (1) = 1$

$f(0) = 1$

$\Rightarrow$ RHL $= 3/2$

$f(0) = 3/2$

Since LHL $=$ RHL $= f(0)$

$\therefore f(x)$ is continuous at $x=0$ **An**

Qn. 15 $\rightarrow$ $f(x) = \begin{cases} x + a\sqrt{2}\sin x & ; 0 \le x < \pi/4 \\ 2x\cot x + b & ; \pi/4 \le x < \pi/2 \\ a\cos(2x) - b\sin x & ; \frac{\pi}{2} \le x \le \pi \end{cases}$

Continuity at $x = \pi/4$

$f(x)$ is continuous at $x = \pi/4$ (given)

$\Rightarrow$ LHL $=$ RHL $= f(\pi/4)$

$\Rightarrow \lim_{x \to \frac{\pi}{4}^-} \left(x + a\sqrt{2}\sin x\right) = \lim_{x \to \frac{\pi}{4}^+} \left(2x\cot x + b\right) = f\left(\frac{\pi}{4}\right)$

put $x = \frac{\pi}{4} - h$, $h \to 0$; put $x = \frac{\pi}{4} + h$, $h \to 0$

$\Rightarrow \lim_{h \to 0} \left[\frac{\pi}{4} - h + a\sqrt{2}\sin\left(\frac{\pi}{4} - h\right)\right] = \lim_{h \to 0}\left(2\left(\frac{\pi}{4}+h\right)\cot\left(\frac{\pi}{4}+h\right) + b\right)$

$= 2\left(\frac{\pi}{4}\right)\cot\left(\frac{\pi}{4}\right) + b$

$\Rightarrow \frac{\pi}{4} + a\cancel{\sqrt{2}} \times \frac{1}{\cancel{\sqrt{2}}} = 2\left(\frac{\pi}{4}\right) \times 1 + b = 2\left(\frac{\pi}{4}\right) \times 1 + b$

$\Rightarrow a + \frac{\pi}{4} = \frac{\pi}{2} + b = \frac{\pi}{2} + b$

$\Rightarrow a - b = \frac{\pi}{2} - \frac{\pi}{4}$

$\Rightarrow a - b = \frac{\pi}{4}$ ------ (1)

Continuity at $x = \pi/2$

Solution of worksheet No: 6 (12)

Since LHL = RHL = f(0)

∴ f(x) is continuous at $x=0$ **Ans**

Similarly check yourself continuity at $x=1$

Q.N. 14 → $f(x) = \begin{cases} \dfrac{\sin(3x)}{\tan(2x)} & ; \; x<0 \\ 3/2 & ; \; x=0 \\ \dfrac{\log(1+3x)}{e^{2x}-1} & ; \; x>0 \end{cases}$

$LHL = \lim_{x\to 0^-}\left(\dfrac{\sin(3x)}{\tan(2x)}\right)$ put $x = 0-h = -h$, $h\to 0$

$LHL = \lim_{h\to 0}\left(\dfrac{\sin(-3h)}{\tan(-2h)}\right) = \lim_{h\to 0}\left(\dfrac{\sin(3h)}{\tan(2h)}\right)$

$= \lim_{h\to 0}\left(\dfrac{\dfrac{\sin(3h)}{3h}\times 3\cancel{h}}{\dfrac{\tan(2h)}{2h}\times 2\cancel{h}}\right) = \dfrac{1\times 3}{1\times 2} = 3/2$

$RHL = \lim_{x\to 0^+}\left(\dfrac{\log(1+3x)}{e^{2x}-1}\right)$ put $x = 0+h = h$, $h\to 0$

$RHL = \lim_{h\to 0}\left(\dfrac{\log(1+3h)}{e^{2h}-1}\right)$

$= \lim_{h\to 0}\left(\dfrac{\dfrac{\log(1+3h)}{3h}\times 3\cancel{h}}{\dfrac{e^{2h}-1}{2h}\times 2\cancel{h}}\right) = \dfrac{1\times 3}{1\times 2}$ ... $\left\{\because \lim_{x\to 0}\left(\dfrac{\log(1+x)}{x}\right) = 1 \text{ and } \lim_{x\to 0}\left(\dfrac{e^x-1}{x}\right) = 1\right\}$

Solution of Worksheet No 8 (13)

$f(x)$ is continuous at $x = \pi/2$ (given)

$\Rightarrow$ LHL $=$ RHL $= f(\pi/2)$

$\Rightarrow \lim_{x\to\frac{\pi}{2}^-}(2x\cot x + b) = \lim_{x\to\frac{\pi}{2}^+}(a\cos(2x) - b\sin x) = f(\frac{\pi}{2})$

put $x = \frac{\pi}{2} - h$; put $x = \frac{\pi}{2} + h$

$h \to 0$ $\quad$ $h \to 0$

$\Rightarrow \lim_{h\to 0}\left[2(\frac{\pi}{2}-h)\cot(\frac{\pi}{2}-h)+b\right] = \lim_{h\to 0}\left[a\cos\left(2(\frac{\pi}{2}+h)\right) - b\sin(\frac{\pi}{2}+h)\right]$

$= a\cos(2\times\frac{\pi}{2}) - b\sin\frac{\pi}{2}$

$\Rightarrow 2(\frac{\pi}{2})(0) + b = a\cos(\pi) - b\sin\frac{\pi}{2} = a\cos(\pi) - b\sin\frac{\pi}{2}$

$\Rightarrow 0 + b = -a - b = -a - b$

$\Rightarrow b = -a - b$

$\Rightarrow a = -2b$ --- (2)

from (1) & (2)

put $a = -2b$ in eq (1)

$-2b - b = \pi/4$

$-3b = \pi/4$

$\boxed{b = -\frac{\pi}{12}}$; $a = -2(-\frac{\pi}{12}) = \frac{\pi}{6}$ Ans.

—x—