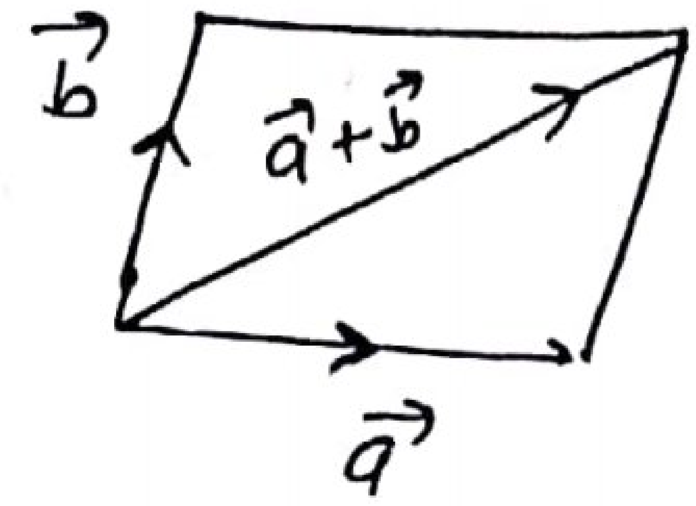
Solutions VECTORS WORKSHEET No: 4 ①

Ans: 1 → let $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ & $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$

let $\vec{d}$ be its diagonal

then $\vec{d} = \vec{a} + \vec{b}$



$\Rightarrow \vec{d} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

$|\vec{d}| = \sqrt{9 + 36 + 4} = 7$

Now $\hat{d} = \dfrac{\vec{d}}{|\vec{d}|} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$

Area of parallelogram $= |\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix} = \hat{i}(+12+10) - \hat{j}(-6-5) + \hat{k}(-4+4)$$

$\Rightarrow \vec{a} \times \vec{b} = 22\hat{i} + 11\hat{j} + 0\hat{k}$

Area $= |\vec{a} \times \vec{b}| = \sqrt{(22)^2 + (11)^2} = \sqrt{(11)^2(2)^2 + (11)^2}$

$= (11)\sqrt{4+1} = 11\sqrt{5}$ sq units

Ans

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Ans 2 → given $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$; $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\vec{a} + \vec{b} = 4\hat{i} + 4\hat{j}$

$\vec{a} - \vec{b} = 2\hat{i} + 4\hat{k}$

$$(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = 16\hat{i} - 16\hat{j} - 8\hat{k}$$

let $\vec{c} = 16\hat{i} + 16\hat{j} - 8\hat{k}$

$$|\vec{c}| = \sqrt{(16)^2 + (16)^2 + (8)^2} = \sqrt{(8)^2(4+4+1)}$$

$$= 8\sqrt{9} = 8 \times 3 = 24$$

Now unit vector $\perp^r$ to both $(\vec{a}+\vec{b})$ & $(\vec{a}-\vec{b})$ is given by $= \pm \hat{c}$

$$= \pm \frac{\vec{c}}{|\vec{c}|}$$

$$= \pm \left(\frac{16\hat{i} + 16\hat{j} - 8\hat{k}}{24} \right)$$

$$= \pm \left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k} \right)$$ Ans

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Ans 3 → Given $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$

and $\vec{a} \times \vec{b}$ is a unit vector

$\Rightarrow$ $|\vec{a} \times \vec{b}| = 1$

let θ be the angle b/w $\vec{a}$ & $\vec{b}$

Now $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

$\Rightarrow$ $1 = (\cancel{3})\left(\frac{\sqrt{2}}{\cancel{3}}\right) \sin\theta$

$\Rightarrow$ $\sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\boxed{\theta = \pi/4}$ Ans

Ans 4 → given $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\vec{b}+\vec{c} = (b_1+c_1)\hat{i} + (b_2+c_2)\hat{j} + (b_3+c_3)\hat{k}$$

$$\vec{a}\times(\vec{b}+\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$$

$$= \hat{i}(a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2) - \hat{j}(a_1b_3 + a_1c_3 - b_1a_3 - c_1a_3)$$
$$+ \hat{k}(a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1)$$

Now $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}(a_2b_3 - b_2a_3) - \hat{j}(a_1b_3 - b_1a_3) + \hat{k}(a_1b_2 - a_2b_1)$

$$\vec{a}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{i}(a_2c_3 - a_3c_2) - \hat{j}(a_1c_3 - c_1a_3) + \hat{k}(a_1c_2 - c_1a_2)$$

$$(\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}) = \hat{i}(a_2b_3 + a_2c_3 - b_2a_3 - a_3c_2) - \hat{j}(a_1b_3 + a_1c_3 - b_1a_3 - c_1a_3) + \hat{k}(a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1)$$

Clearly $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ **PROVED**

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Ans. 5 → T.P $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = 2|\vec{a}|^2$

let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

LHS $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2$

$$= |(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i}|^2 + |(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j}|^2 + |(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{k}|^2$$

$$= |-y\hat{k}+z\hat{j}|^2 + |x\hat{k}-z\hat{i}|^2 + |-x\hat{j}+y\hat{i}|^2$$

$$= \left(\sqrt{y^2+z^2}\right)^2 + \left(\sqrt{x^2+z^2}\right)^2 + \left(\sqrt{x^2+y^2}\right)^2$$

$$= y^2+z^2 + x^2+z^2 + x^2+y^2$$

$$= 2(x^2+y^2+z^2)$$

$$= 2\left(\sqrt{x^2+y^2+z^2}\right)^2$$

$$= 2|\vec{a}|^2$$

$= \underline{\text{RHS}}$ PROVED

Q.N. 6 → let $\vec{a} = 4\hat{i}-\hat{j}+3\hat{k}$ & $\vec{b} = -2\hat{i}+\hat{j}-2\hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = -\hat{i}+2\hat{j}+2\hat{k}$$

let $\vec{c} = -\hat{i}+2\hat{j}+2\hat{k}$

unit vector $\perp^r$ to both $\vec{a}$ & $\vec{b}$ is $= \pm\hat{c}$

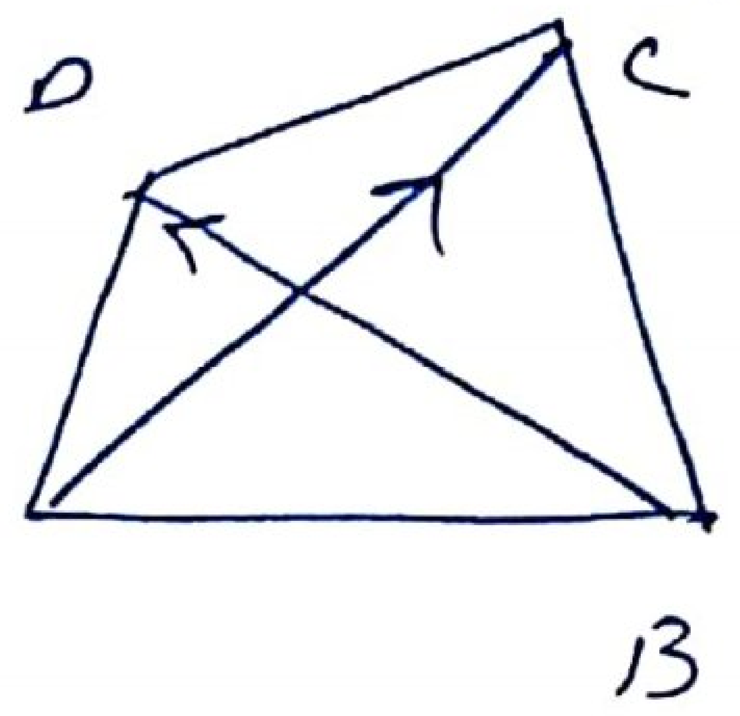
$$= \pm\frac{\vec{c}}{|\vec{c}|}$$

$$= \pm\frac{(-\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{1+4+4}} = \pm\left(-\frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$$

vector $\perp^r$ to $\vec{a}$ & $\vec{b}$ having magnitude 9 units $= \pm 9\hat{c}$

$$= \pm(-3\hat{i}+6\hat{j}+6\hat{k})$$ Ans

QN. 7 → given $A(0,1,1)$ $B(2,3,-2)$ $C(22,19,-5)$ $D(1,-2,1)$



$\vec{AC} = 22\hat{i} + 18\hat{j} - 6\hat{k}$

$\vec{BD} = -\hat{i} - 5\hat{j} + 3\hat{k}$

$$\vec{AC} \times \vec{BD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 22 & 18 & -6 \\ -1 & -5 & 3 \end{vmatrix} = \hat{i}(54-30) - \hat{j}(66-6) + \hat{k}(-110+18)$$

$$= 24\hat{i} - 60\hat{j} - 92\hat{k}$$

$$|\vec{AC} \times \vec{BD}| = \sqrt{(24)^2 + (60)^2 + (92)^2}$$

$$= \sqrt{12640}$$

$$= \sqrt{4 \times 3160}$$

$$= 2\sqrt{3160}$$

Now area of quadrilateral ABCD $= \frac{1}{2}|\vec{AC} \times \vec{BD}|$

$$= \frac{1}{2}(2\sqrt{3160})$$

$$= \sqrt{3160} \text{ square units}$$ Ans

QN 8 → given $|\vec{a}| = 10$; $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 12$

Now $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$12 = (10)(2)\cos\theta$$

$$\Rightarrow \cos\theta = \frac{12}{20} = \frac{3}{5}$$

Now $\sin\theta = \sqrt{1-\cos^2\theta}$

$= \sqrt{1-\frac{9}{25}}$

$\sin\theta = \frac{4}{5}$

we have $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$= (10)(2)\left(\frac{4}{5}\right)$

$= \frac{80}{5}$

$\Rightarrow \boxed{|\vec{a}\times\vec{b}| = 16}$ Ans

Ans. 9 → given $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$

$\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$

To prove $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$,
it is sufficient to show that their cross product equals to zero

Now $(\vec{a}-\vec{d})\times(\vec{b}-\vec{c})$

$= \vec{a}\times\vec{b} - \vec{a}\times\vec{c} - \vec{d}\times\vec{b} + \vec{d}\times\vec{c}$

$= \vec{c}\times\vec{d} - \vec{b}\times\vec{d} - \vec{d}\times\vec{b} + \vec{d}\times\vec{c}$ ... {from given equations

$= \vec{c}\times\vec{d} - \vec{b}\times\vec{d} + \vec{b}\times\vec{d} - \vec{c}\times\vec{d}$ ... $\{\because \vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})\}$

$= \vec{0}$

$\therefore (\vec{a}-\vec{d})$ is parallel to $(\vec{b}-\vec{c})$ Proved

Qn. 10 →

given

$\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$	$\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$	$\vec{a} \neq \vec{0}$
$\Rightarrow \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = 0$	$\Rightarrow \vec{a}\times\vec{b} - \vec{a}\times\vec{c} = \vec{0}$	
$\Rightarrow \vec{a}\cdot(\vec{b}-\vec{c}) = 0$	$\Rightarrow \vec{a}\times(\vec{b}-\vec{c}) = \vec{0}$	
$\Rightarrow$ either $\vec{a} = \vec{0}$	$\Rightarrow$ either $\vec{a} = \vec{0}$	
(or) $\vec{b}-\vec{c} = \vec{0}$	(or) $\vec{b}-\vec{c} = \vec{0}$	
(or) $\vec{a} \perp (\vec{b}-\vec{c})$	(or) $\vec{a} \parallel^{l} (\vec{b}-\vec{c})$	
but $\vec{a} \neq \vec{0}$ (given)	but $\vec{a} \neq \vec{0}$ (given)	

But $\vec{a}$ cannot $\perp^{r}$ & $\parallel^{l}$ to $(\vec{b}-\vec{c})$ simultaneously

$\Rightarrow \therefore \vec{b}-\vec{c} = \vec{0}$

$\Rightarrow \boxed{\vec{b} = \vec{c}}$ Proved

Qn. 11 → given $\vec{a} = \hat{i}+4\hat{j}+2\hat{k}$; $\vec{b} = 3\hat{i}-2\hat{j}+7\hat{k}$

$\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$

given $\vec{d}$ is $\perp^{r}$ to both $\vec{a}$ & $\vec{b}$

$\Rightarrow \vec{d} = \lambda(\vec{a}\times\vec{b})$

$$\Rightarrow \vec{d} = \lambda\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix} = \lambda(32\hat{i}-\hat{j}-14\hat{k})$$

also given $\vec{c}\cdot\vec{d} = 15$

$\Rightarrow (2\hat{i}-\hat{j}+4\hat{k})\cdot(32\lambda\hat{i}-\lambda\hat{j}-14\lambda\hat{k}) = 15$

$\Rightarrow 64\lambda + \lambda - 56\lambda = 15$

$\Rightarrow 9\lambda = 15$

$\Rightarrow \boxed{\lambda = \frac{15}{9}}$ $\boxed{\lambda = \frac{5}{3}}$ ✓

$\therefore \boxed{\vec{d} = \frac{5}{3}(32\hat{i}-\hat{j}+14\hat{k})}$ Ans

Q^n 12 → given $\vec{a} = \hat{i}+\hat{j}+\hat{k}$

$\vec{c} = 0\hat{i}+\hat{j}-\hat{k}$

let $\vec{b} = x\hat{i}+y\hat{j}+z\hat{k}$

given $\vec{c} = \vec{a}\times\vec{b}$

$$\Rightarrow 0\hat{i}+\hat{j}-\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$$

$\Rightarrow 0\hat{i}+\hat{j}-\hat{k} = \hat{i}(z-y) - \hat{j}(z-x) + \hat{k}(y-x)$

$\Rightarrow z-y=0$ | $-z+x=1$ | $y-x=-1$

$\Rightarrow z=y$ | $-y+x=1$ | $x-y=1$

$\Rightarrow x-y=1$

~~also~~ also given that

$\vec{a}\cdot\vec{b} = 3$

$\Rightarrow (\hat{i}+\hat{j}+\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k}) = 3$

$\Rightarrow \quad x + y + z = 3$

formal 3 equations

$z = y \quad \text{--} (1)$

$x - y = 1 \quad \text{-} (2)$

$x + y + z = 3 \quad \text{--} (3)$

Solving these equations we get

$x = 5/3 \; ; \quad y = 2/3 \; ; \quad z = 2/3$

$\therefore$ required vector $\boxed{\vec{b} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}}$ Ans

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