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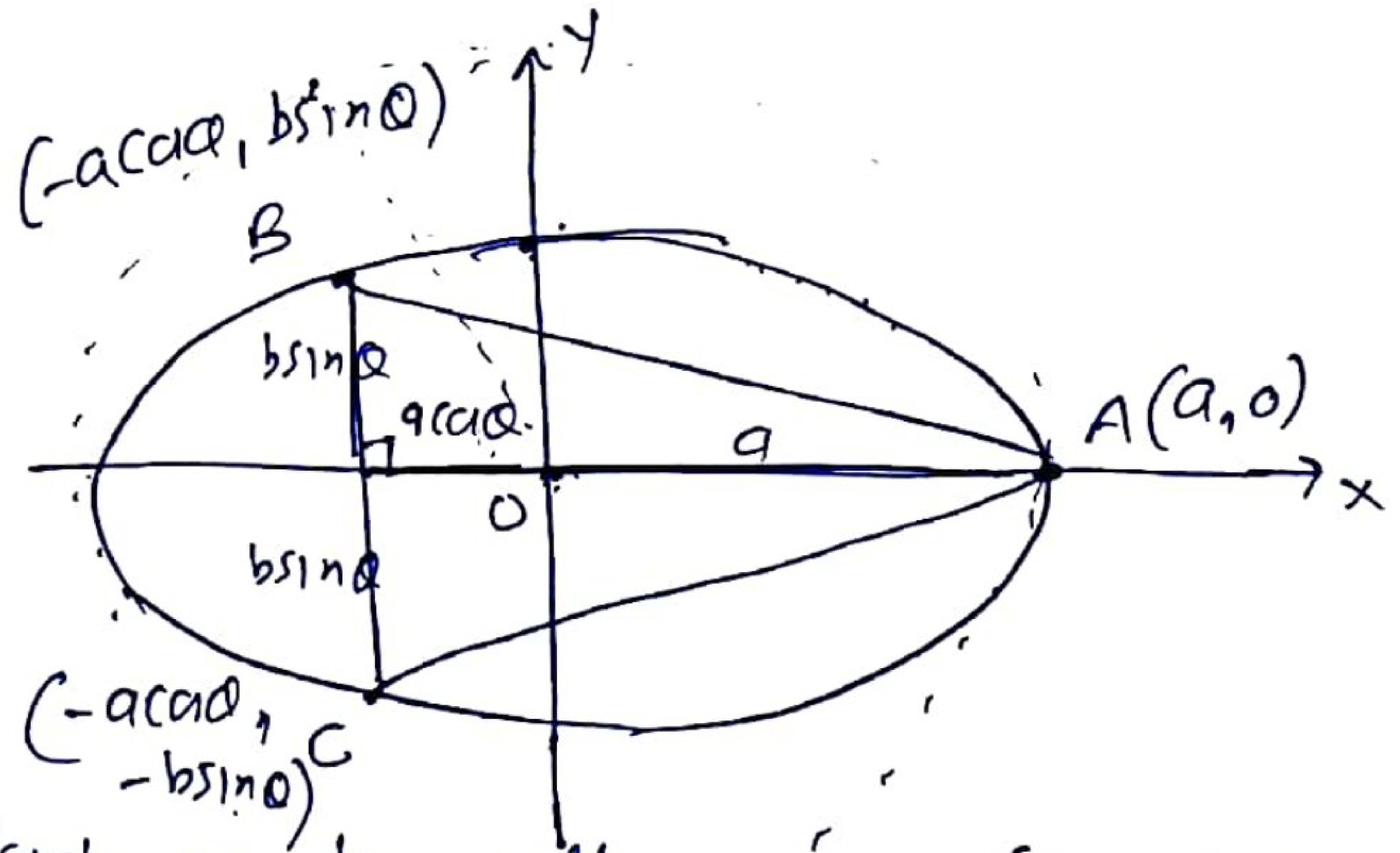
ULTIMATE MATHEMATICS BY: AJAY MITTAL

CHAPTER: A.O.D (CLASS NO: 8)

Topic: MAXIMA MINIMA (Continued)...
(SPECIAL QUESTIONS)

Ques. 1 → Find the maximum area of an isosceles triangle "inscribed" in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis

Solution



Point → Any point on the Ellipse is $(a\cos\theta, b\sin\theta)$ ←

Base $= 2b\sin\theta$; altitude $= a + a\cos\theta$

$A \rightarrow$ area of ΔABC

$$A = \frac{1}{\cancel{2}}(\cancel{2}b\sin\theta)(a+a\cos\theta)$$

$$A = ab\sin\theta(1+\cos\theta)$$

$$A = ab(\sin\theta + \sin\theta\cos\theta)$$

$$A = ab\left(\sin\theta + \frac{\sin(2\theta)}{2}\right) \quad \ldots \text{(to be Max)}$$

Diff w.r.t. θ

$$\frac{dA}{d\theta} = ab(\cos\theta + \cos(2\theta))$$

for Max/Min put $\frac{dA}{d\theta} = 0$

$\Rightarrow ab(\cos\theta + \cos(2\theta)) = 0$

$\Rightarrow \cos\theta = -\cos(2\theta)$

$\Rightarrow \cos\theta = \cos(\pi - 2\theta)$

$\Rightarrow \theta = \pi - 2\theta$

$\Rightarrow 3\theta = \pi$

$\Rightarrow \boxed{\theta = \pi/3}$

Diff again w.r.t θ

$$\frac{d^2A}{d\theta^2} = ab(-\sin\theta - 2\sin(2\theta))$$

$$\left(\frac{d^2A}{d\theta^2}\right)_{\theta=\pi/3} = ab\left(-\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2}\right) \quad \cdots \left\{\begin{array}{l}\sin(2\pi/3) \\ = \sin(\pi - \pi/3) \\ = \sin(\pi/3) = \frac{\sqrt{3}}{2}\end{array}\right\}$$

$$= -ab\frac{3\sqrt{3}}{2} < 0$$

$\therefore$ Area of ΔABC is Max at $\theta = \pi/3$

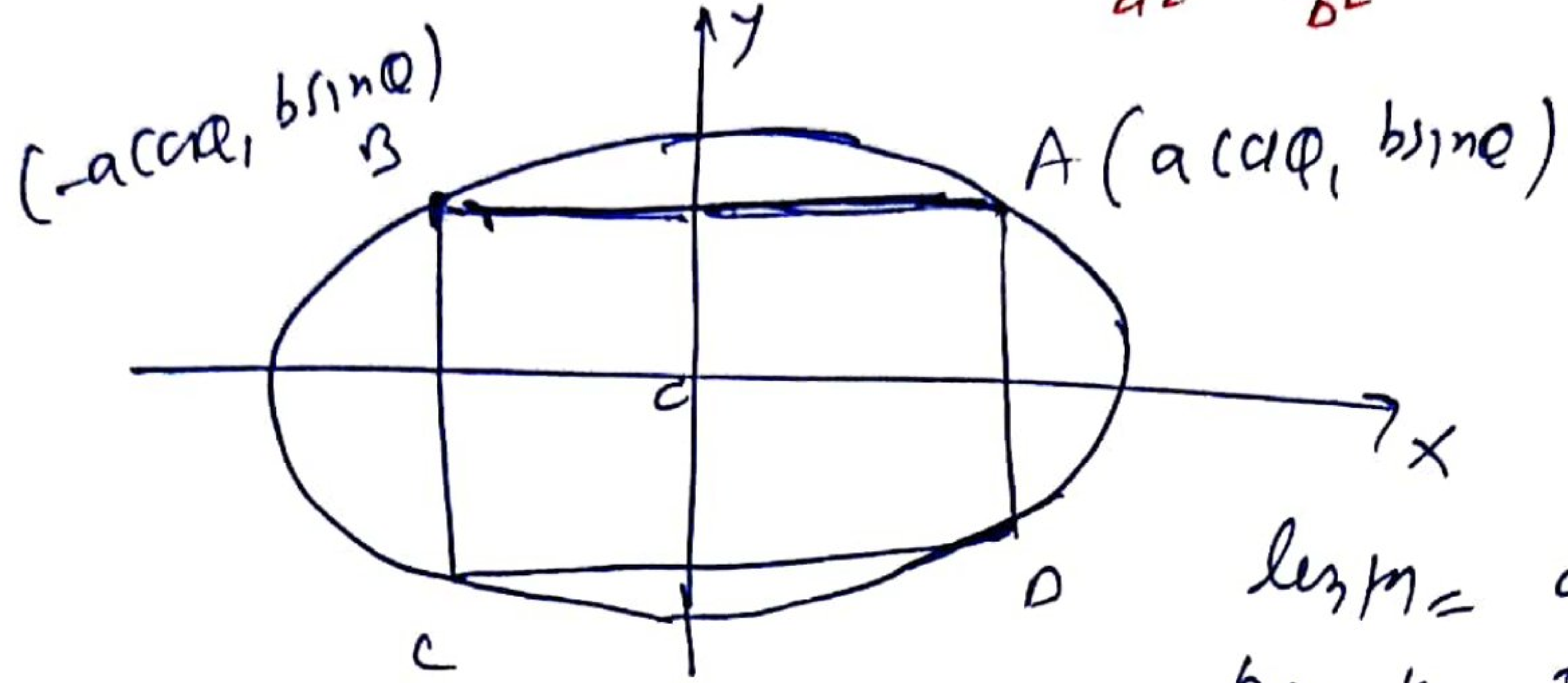
Now $A_{max} = ab\left(\sin\theta + \frac{\sin(2\theta)}{2}\right)$

$= ab\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right)$

$= ab\left(\frac{2\sqrt{3} + \sqrt{3}}{4}\right)$

$A_{max} = \frac{3\sqrt{3}ab}{4}$ square units Ans

Ques 2 → Find the area of greatest "rectangle" that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



length $= 2a\cos\theta$

Breadth $= 2b\sin\theta$

$$A = 4ab\sin\theta\cos\theta$$

$$A = 4ab\left(\frac{\sin(2\theta)}{2}\right)$$

$$A = 2ab\sin(2\theta)$$

Diff w.r.t θ

$$\frac{dA}{d\theta} = 2ab\cdot\cos(2\theta)\cdot 2 = 0$$

$$\Rightarrow \cos(2\theta) = 0$$

$$\Rightarrow 2\theta = \pi/2$$

$$\Rightarrow \theta = \pi/4$$

Diff $\frac{d^2A}{d\theta^2} = -4ab\cdot\sin(2\theta)\cdot 2$

$$\left(\frac{d^2A}{d\theta^2}\right)_{\theta=\pi/4} = -8ab\cdot 1 = -8ab < 0$$

$\therefore$ Max

$$A_{max} = 2ab\sin(2\theta) = 2ab\sin(\pi/2)$$

$$= 2ab \text{ sq. units}$$

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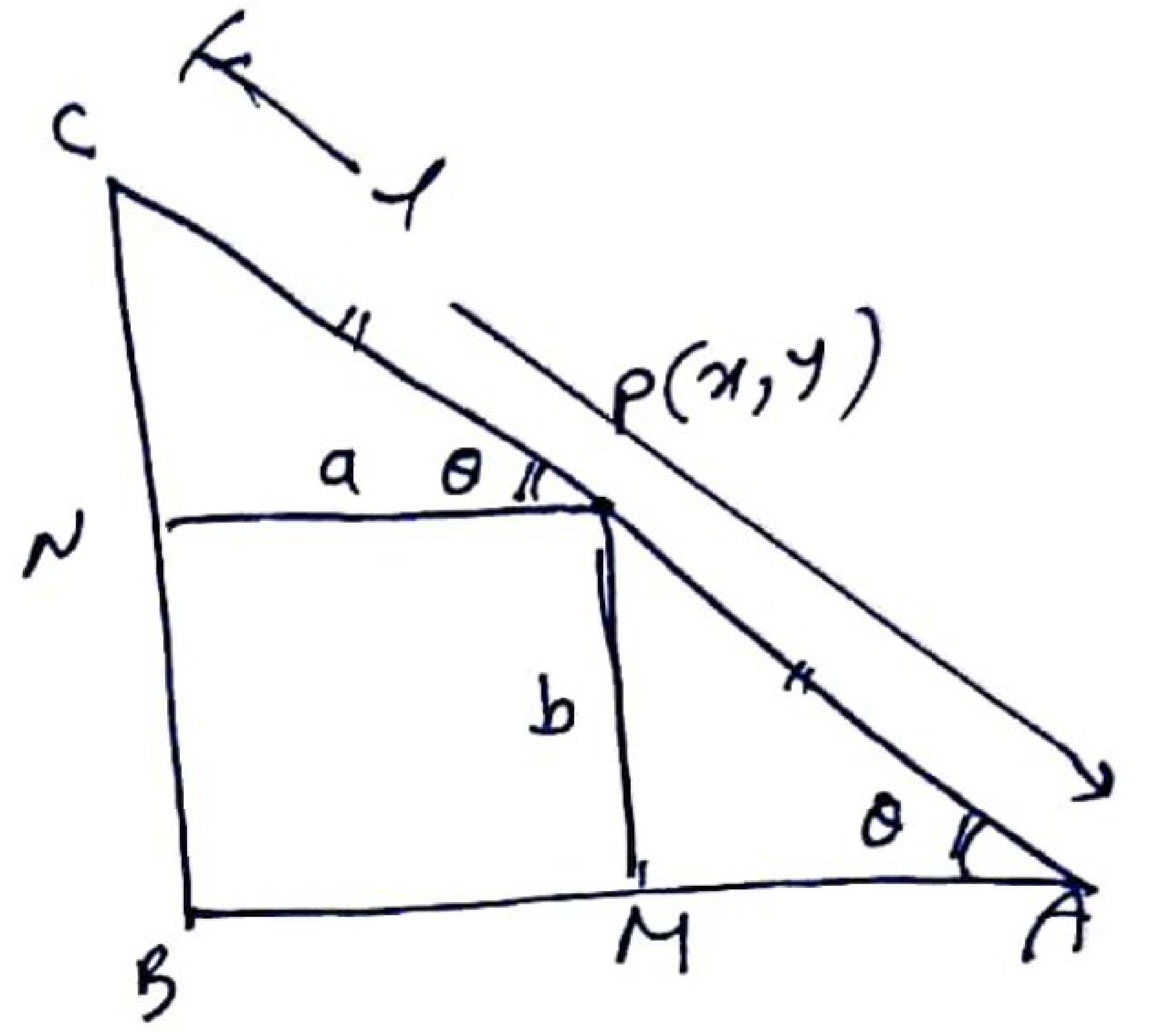
A.CD (Class No: 8) (4)

Qn. 3 → A point on the hypotenuse of a triangle is at a distance of 'a' & 'b' from the sides of the triangle.

Show that the minimize length of the hypotenuse is $(a^{2/3}+b^{2/3})^{3/2}$



Soln

$l = AP + PC$

$\Delta PNC \quad \cos\theta = \frac{a}{PC}$

$\Rightarrow PC = a\sec\theta$

$\Delta PMA \quad \sin\theta = \frac{b}{AP}$

$\Rightarrow AP = b\,\mathrm{cosec}\,\theta$

$\therefore \quad l = a\sec\theta + b\,\mathrm{cosec}\,\theta \quad \ldots$ (to be Min.)

Diff. w.r.t. θ

$\frac{dl}{d\theta} = a\sec\theta\tan\theta - b\,\mathrm{cosec}\,\theta\cot\theta$

for Max/Min put $\frac{dl}{d\theta} = 0$

$\Rightarrow a\sec\theta\tan\theta = b\,\mathrm{cosec}\,\theta\cot\theta$

$\Rightarrow a \cdot \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} = b \cdot \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta}$

$\Rightarrow a\sin^3\theta = b\cos^3\theta$

$\Rightarrow \tan^3\theta = \frac{b}{a}$

$\Rightarrow \boxed{\tan\theta = \left(\frac{b}{a}\right)^{1/3}}$

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Diff again w.r.t θ

$$\frac{d^2 l}{d\theta^2} = a\left[\sec^3\theta + \tan^2\theta \cdot \sec\theta\right] - b\left[-\operatorname{cosec}^3\theta \cdot + \cot^2\theta \cdot \operatorname{cosec}\theta\right]$$

$$\frac{d^2 l}{d\theta^2} = a\left[\sec^3\theta + \sec\theta\tan^2\theta\right] + b\left[\operatorname{cosec}^3\theta + \cot^2\theta\operatorname{cosec}\theta\right] > 0$$

$\therefore$ lgth of hypotenuse is Minimum

Now $l = a\sec\theta + b\operatorname{cosec}\theta$

$$l = a\sqrt{1+\tan^2\theta} + b\sqrt{1+\cot^2\theta}$$

$$l = a\left(1+\left(\frac{b}{a}\right)^{2/3}\right)^{1/2} + b\left(1+\left(\frac{a}{b}\right)^{2/3}\right)^{1/2}$$

$$l = a\left[1+\frac{b^{2/3}}{a^{2/3}}\right]^{1/2} + b\left[1+\frac{a^{2/3}}{b^{2/3}}\right]^{1/2}$$

$$l = \frac{a}{a^{1/3}}\left[a^{2/3}+b^{2/3}\right]^{1/2} + \frac{b}{b^{1/3}}\left[b^{2/3}+a^{2/3}\right]^{1/2}$$

$$l = a^{2/3}\left(a^{2/3}+b^{2/3}\right)^{1/2} + b^{2/3}\left(b^{2/3}+a^{2/3}\right)^{1/2}$$

$$l = \left(a^{2/3}+b^{2/3}\right)^{1/2} \cdot \left(a^{2/3}+b^{2/3}\right)$$

$$l = \left(a^{2/3}+b^{2/3}\right)^{3/2} \quad \text{Ans}$$

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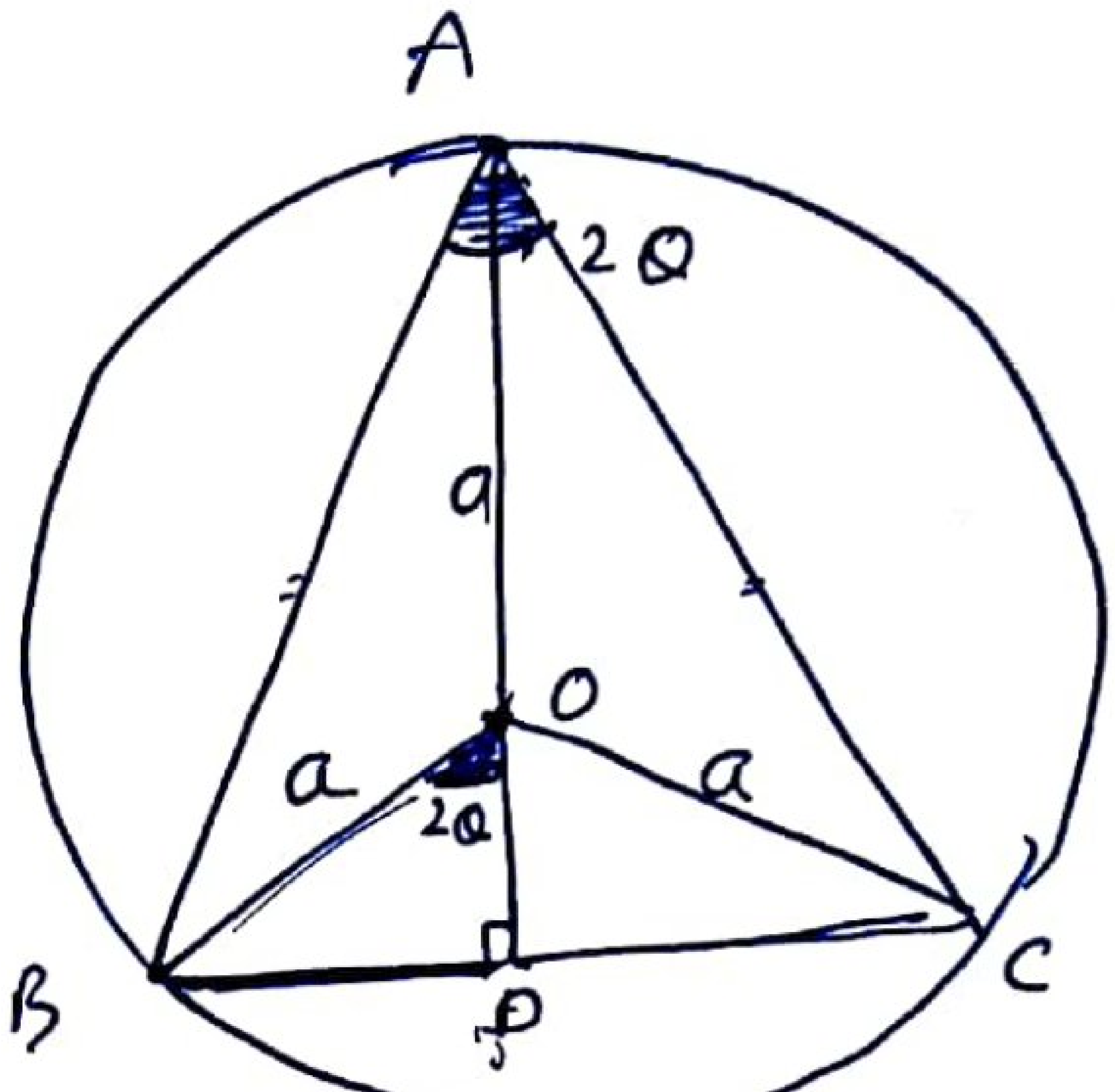
Q.No.4 — An isosceles triangle of vertical angle 2θ is "inscribed" in a circle of radius 'a'. Show that the area of triangle is maximum at $\theta = \pi/6$

Soln:



ΔOBD

$$\sin(2\theta) = \frac{BD}{a}$$

$$\Rightarrow BD = a\sin(2\theta)$$

$\therefore$ Base $= 2a\sin(2\theta)$

$$\cos(2\theta) = \frac{OD}{a}$$

$$OD = a\cos(2\theta)$$

Altitude: $a + a\cos(2\theta) = a(1+\cos(2\theta))$

$A \to$ area of ΔABC

$$A = \frac{1}{2}\left(2a\sin(2\theta)\right)\cdot a\left(1+\cos(2\theta)\right)$$

$$A = a^2\left(\sin(2\theta) + \sin(2\theta)\cos(2\theta)\right)$$

$$A = a^2\left(\sin(2\theta) + \frac{\sin(4\theta)}{2}\right)$$

Diff w.r.t 'θ'

$$\frac{dA}{d\theta} = a^2\left[2\cos(2\theta) + 2\cos(4\theta)\right] = 0$$

$$\Rightarrow \cos(2\theta) + \cos(4\theta) = 0$$

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$\Rightarrow \cos(2\theta) = -\cos(4\theta)$

$\Rightarrow \cos(2\theta) = \cos(\pi - 4\theta)$

$\Rightarrow 2\theta = \pi - 4\theta$

$\Rightarrow 6\theta = \pi$

$\Rightarrow \theta = \pi/6$

Diff. again w.r.t θ

$$\frac{d^2A}{d\theta^2} = a^2\left[-4\sin(2\theta) - 8\sin(4\theta)\right]$$

$$\left(\frac{d^2A}{d\theta^2}\right)_{\theta=\pi/6} = a^2\left(-4\sin(60^\circ) - 8\sin(180-60)\right)$$

$$= a^2\left[-4\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= a^2(-6\sqrt{3}) < 0$$

$\therefore$ Area of ΔABC is max at $\theta = \pi/6$

$$A_{max} = a^2\left(\sin(60^\circ) + \frac{\sin(120)}{2}\right)$$

$$= a^2\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right)$$

$$= a^2\left(\frac{2\sqrt{3}+\sqrt{3}}{4}\right)$$

$$= \frac{3\sqrt{3}a^2}{4}$$ sq. unit Ans

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Qu. 5 → The sum of the surface areas of a rectangular parallelopiped with sides $x, 2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of the sphere. Also find the minimum value of the sum of their volumes.

Soln

$r \to$ Radius of sphere

$l = x, \quad b = 2x, \quad h = x/3$

$S \to$ sum of ~~to~~ their surface areas

$$S = 2(x)(2x) + 2(2x)\left(\frac{x}{3}\right) + 2(x)\left(\frac{x}{3}\right) + \cancel{\tfrac{4}{3}} 4\pi r^2$$

$$\boxed{S = 6x^2 + \cancel{\tfrac{4}{3}} 4\pi r^2 \text{ (given)}} \quad \cdots (1)$$

$V \to$ sum of their volumes

$$\boxed{V = \frac{2x^3}{3} + \frac{4}{3}\pi r^3} \text{ - (to be Min)}$$

$$V = \frac{2}{3}\left(\frac{S - 4\pi r^2}{6}\right)^{3/2} + \frac{4}{3}\pi r^3 \quad \cdots \text{(from eq (1))}$$

$$V = \frac{2}{3 \cdot 6\sqrt{6}}\left(S - 4\pi r^2\right)^{3/2} + \frac{4}{3}\pi r^3$$

Diff

$$\frac{dV}{dr} = \frac{1}{9\sqrt{6}} \cdot \frac{3}{2}\left(S - 4\pi r^2\right)^{1/2} \cdot (-8\pi r) + 4\pi r^2$$

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$$\frac{dv}{dr} = -\frac{12\pi}{9\sqrt{6}} r\cdot(5-4\pi r^2)^{1/2} + 4\pi r^2$$

put $\frac{dv}{dr} = 0$

$$\Rightarrow \frac{4}{3\sqrt{6}}\pi r\cdot\sqrt{5-4\pi r^2} = 4\pi r^2$$

Squaring

$$\frac{\cancel{16}}{\cancel{54}}(5-4\pi r^2) = r^2 \cancel{4} r^2$$

$$\Rightarrow 5-4\pi r^2 = 54 r^2$$

$$\Rightarrow \boxed{5 = 54 r^2 + 4\pi r^2}$$

or $r = \sqrt{\frac{5}{4\pi+54}}$

Diff. again

$$= \frac{d^2v}{dr^2} = -\frac{12\pi}{9\sqrt{6}}\left[r\cdot\frac{1}{2\sqrt{5-4\pi r^2}}\cdot(-8\pi r) + \sqrt{5-4\pi r^2}\right] + 8\pi r$$

$$\Rightarrow \frac{d^2v}{dr^2} = -\frac{4\pi}{3\sqrt{6}}\left[\frac{r\cdot(-8\pi r)}{2\cdot\sqrt{54 r^2}} + \sqrt{54 r^2}\right] + 8\pi r$$

$$= -\frac{4\pi}{3\sqrt{6}}\left[-\frac{8\pi r \cancel{r}}{6\sqrt{6}\,\cancel{r}} + 3\sqrt{6}\cdot r\right] + 8\pi r$$

$$= \frac{32\pi^2\cdot r}{108} - 4\pi r + 8\pi r$$

$$\frac{d^2v}{dr^2} = \frac{32\pi^2 r}{108} + 4\pi r > 0$$ $\therefore$ sum of their volumes is Minimum

from eq(1)

$$S = 6x^2 + 4\pi r^2$$

$$\Rightarrow S - 4\pi r^2 = 6x^2$$

$$\Rightarrow 54r^2 = 6x^2$$

$$\Rightarrow 9r^2 = x^2$$

$$\Rightarrow 3r = x \qquad r = x/3$$

$$V_{min} = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\left(\frac{x^3}{27}\right)$$

$$= \frac{54x^3 + 4\pi x^3}{81}$$

$$= x^3\left(\frac{54 + 4\pi}{81}\right) \text{ cubic unit}$$

Qu.6 → If the sum of the surface area of cube and a sphere is constant, what is the ~~~~ ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum

Sol

Sol^n $x \to$ side

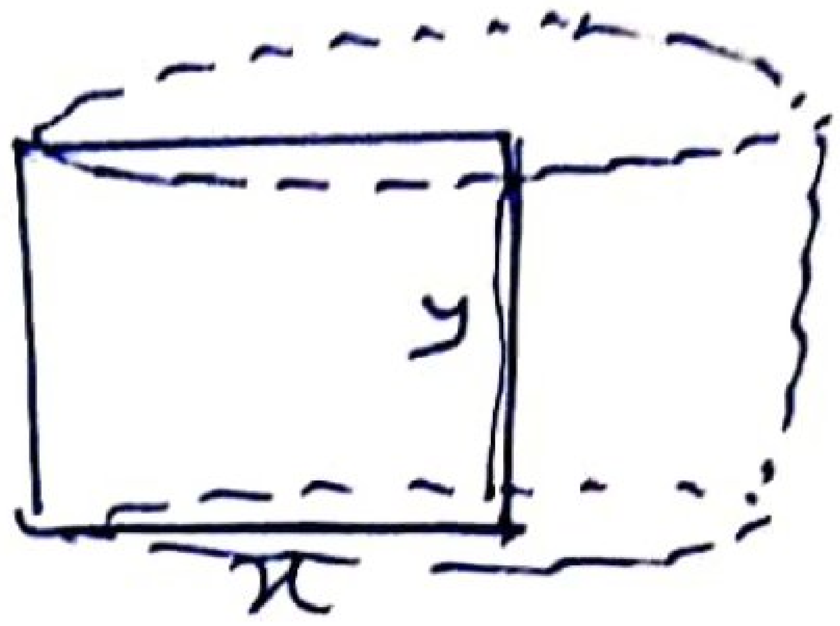
$$S = 4\pi r^2 + 6x^2 \quad \text{--- (1)}$$

$$V = \frac{4}{3}\pi r^3 + x^3 \text{ -- (to Min)}$$

$\boxed{x : 2r = ?}$

Q.N. 7 → Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.

Sol.



Sol:

$$36 = 2x + 2y$$

$$x + y = 18 \quad \cdots (1)$$

$$V = \pi x^2 y \quad \cdots (\text{Max})$$

Q.N. 8 → A telephone Company in a town has 500 subscribers on its list and collects fixed charges of Rs 300/- per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of Re 1/- one subscriber will discontinue the service. Find what increase will bring maximum ~~profit~~ revenue?

Sol: Let Rs x be increased in annual subscription

new quantity $= 500 - x$

new price $= 300 + x$

$$R = (300 + x)(500 - x) \quad \cdots (\text{to be Max})$$

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