!! जय श्री राधे कृष्णा जय श्री गिरिराज जी महाराज !!

ULTIMATE MATHEMATICS: BY AJAY MITTAL

CHAPTER: D.E

CLASS NO: 2

$F(x, y) = x^2 + y^2$

$F(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2 (x^2 + y^2) = \lambda^2 F(x, y)$

Homogeneous Function of degree zero

$F(x, y) = \sqrt{x} + \sqrt{y}$

$F(\lambda x, \lambda y) = \sqrt{\lambda x} + \sqrt{\lambda y} = \sqrt{\lambda}(\sqrt{x} + \sqrt{y}) = \lambda^{1/2} F(x, y)$

degree $= 1/2$

$F(x, y) = x^2 + y^3$

$F(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^3 y^3 = \lambda^2 (x^2 + \lambda y^3) \neq \lambda^n F(x, y)$

(not a homogeneous function)

Homogeneous D E

$$\frac{dy}{dx} = F(x, y)$$

$$F(\lambda x, \lambda y) = \lambda^0 F(x, y)$$

$$F(x, y) = x^n \cdot f\left(\frac{y}{x}\right) = y^n f\left(\frac{x}{y}\right)$$

$$F(x, y) = x^2 + y^2 = x^2\left(1 + \left(\frac{y}{x}\right)^2\right) = x^2 f\left(\frac{y}{x}\right)$$

$$= y^2\left(\left(\frac{x}{y}\right)^2 + 1\right) = y^2 f(x/y)$$

Type: Homogeneous D·E

(i) $\frac{dy}{dx} = F(x, y)$

degree: The degree of each term in numerator and denominator must be same

(1) $x^2 \to 2$
(2) $x^2 y \to 3$
(3) $\frac{y}{x} \to 0$
(4) $\sqrt{x} \to 1/2$
(5) $\sqrt{x + y^2} \to 1$
(6) $\sqrt{x^2 + y^3} \to \times$
(7) constant $\to 0$

$\frac{dy}{dx}$ = Question must be in fraction

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

$$F(x,y) = \frac{x+y}{x-y}$$

$$F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \lambda^0 \left(\frac{x+y}{x-y}\right)$$

$$= \lambda^0 F(x,y)$$

✓ put $y = vx$

✓ Diff w.r.t x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

✓ $v + x\frac{dv}{dx} = v$ terms

✓ $x\frac{dv}{dx} = v$ terms

variable separate

✓ $f(v)\,dv = \frac{dx}{x}$

✓ $\int f(v)\,dv = \int \frac{dx}{x}$

$g(v) = \log|x| + C$

replace v by $\frac{y}{x}$

✓ $\sin\left(\frac{y}{x}\right)$, $\log\left(\frac{y}{x}\right)$, $e^{y/x}$, $\sin^{-1}\left(\frac{y}{x}\right)$

put $\boxed{y = vx}$

$\sin x$, $\sin y$, $\log x$, e^x, e^y, $\sin^{-1} x$

× (100% Not Homogeneous)

✓ $\sin\left(\frac{x}{y}\right)$, $\log\left(\frac{x}{y}\right)$, $e^{x/y}$, $\sin^{-1}\left(\frac{x}{y}\right)$

put $\boxed{x = vy}$

Diff w.r.t y

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Prepare

$$\frac{dx}{dy} =$$

Q.No-1 Show that the D.E $(x-y)\frac{dy}{dx} = x+2y$ is a homogeneous D.E & solve it when $x=1$ & $y=0$

Soln

$$\frac{dy}{dx} = \frac{x+2y}{x-y} \quad \text{-----} (i)$$

here $F(x,y) = \frac{x+2y}{x-y}$

$$F(\lambda x, \lambda y) = \frac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \frac{\lambda}{\lambda}\left(\frac{x+2y}{x-y}\right) = \lambda^0 F(x,y)$$

$\therefore$ D.E is homogeneous of degree 0

put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ eq(i) becomes

$$v + x\frac{dv}{dx} = \frac{x+2vx}{x-vx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+2v-v+v^2}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2+v+1}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{(v^2+v+1)}{v-1}$$

$\Rightarrow \dfrac{v-1}{v^2+v+1}\,dv = -\dfrac{dx}{x}$

Integrating both sides

$\Rightarrow \displaystyle\int \frac{v+1}{v^2+v+1}\,dv = -\int\frac{dx}{x}$

$\Rightarrow \dfrac{1}{2}\displaystyle\int \frac{2v+\cancel{2}+1-1}{v^2+v+1}\,dv = -\log|x|$

$\Rightarrow \dfrac{1}{2}\displaystyle\int \frac{(2v+1)+1}{v^2+v+1}\,dv = -\log|x|$

$\Rightarrow \dfrac{1}{2}\displaystyle\int \frac{2v+1}{v^2+v+1}\,dv + \frac{1}{2}\int\frac{1}{v^2+v+1}\,dv = -\log|x|$

v_t

$\Rightarrow \dfrac{1}{2}\displaystyle\int \frac{dt}{t} + \frac{1}{2}\int \frac{1}{(v+\frac{1}{2})^2-\frac{1}{4}+1}\,dv = -\log|x|$

$\Rightarrow \dfrac{1}{2}\log|v^2+v+1| + \dfrac{1}{2}\displaystyle\int\frac{1}{(v+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\,dv = -\log|x|$

$\Rightarrow \dfrac{1}{2}\log|v^2+v+1| + \dfrac{1}{\cancel{2}}\times\dfrac{\cancel{2}}{\sqrt{3}}\tan^{-1}\left(\dfrac{2v+1}{\sqrt{3}}\right) = -\log|x| + C$

replace v by y/x

$\Rightarrow \dfrac{1}{2}\log\left|\dfrac{y^2}{x^2}+\dfrac{y}{x}+1\right| + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{\frac{2y}{x}+1}{\sqrt{3}}\right) = -\log|x| + C$

$$\Rightarrow \frac{1}{2}\log\left|\frac{y^2+xy+x^2}{x^2}\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}\,x}\right) = -\log|x| + C$$

$$\Rightarrow \log\left|\frac{y^2+xy+x^2}{x^2}\right| + \log|x^2| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}\,x}\right) = 2C$$

$$\Rightarrow \log|y^2+xy+x^2| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}\,x}\right) = C_1$$

put $x=1$ & $y=0$

$$\Rightarrow \log(1) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = C_1$$

$$\Rightarrow \frac{2}{\sqrt{3}}\cdot\frac{\pi}{6} = C_1$$

$$\therefore \log|x^2+y^2+xy| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}\,x}\right) = \frac{\pi}{3\sqrt{3}} \quad \text{Ans}$$

Qn. 2 Show that D.E $2ye^{x/y}dx + (y-2xe^{x/y})dy = 0$ is homogeneous & find its particular solution given that $y(0)=1$

Sol:

$$2ye^{x/y}dx + (y-2xe^{x/y})dy = 0$$

$$\Rightarrow 2ye^{x/y}dx = -(y-2xe^{x/y})dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{2xe^{x/y}-y}{2ye^{x/y}} \quad \text{....(1)}$$

here $F(x,y) = \dfrac{2xe^{x/y} - y}{2ye^{x/y}}$

$F(\lambda x, \lambda y) = \dfrac{2\lambda x e^{\lambda x/\lambda y} - \lambda y}{2\lambda y e^{\lambda x/\lambda y}} = \dfrac{\lambda}{\lambda}\left(\dfrac{2xe^{x/y} - y}{2ye^{x/y}}\right)$

$= \lambda^0 F(x,y)$

$\therefore$ D.E is homogen.. of degree 0

put $x = vy$

$\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

$\therefore$ eq (i) becomes

$\Rightarrow v + y\dfrac{dv}{dy} = \dfrac{2vye^{vy/y} - y}{2ye^{x/y}}$

$\Rightarrow v + y\dfrac{dv}{dy} = \dfrac{2ve^{v} - 1}{2e^{v}}$

$\Rightarrow y\dfrac{dv}{dy} = \dfrac{2ve^{v} - 1}{2e^{v}} - v$

$\Rightarrow y\dfrac{dv}{dy} = \dfrac{\cancel{2ve^{v}} - 1 - \cancel{2ve^{v}}}{2e^{v}}$

$\Rightarrow y\dfrac{dv}{dy} = -\dfrac{1}{2e^{v}}$

$\Rightarrow e^{v}dv = -\dfrac{1}{2}\dfrac{dy}{y}$

$\Rightarrow \int e^{v}dv = -\dfrac{1}{2}\int\dfrac{dy}{y}$

$\Rightarrow e^{v} = -\dfrac{1}{2}\ln|y| + c$ replace v by x/y

$\Rightarrow e^{x/y} = -\frac{1}{2}\log|y| + c$

put $x=0$ & $y=1$

$\Rightarrow 1 = 0 + c$

$\Rightarrow e^{x/y} = -\frac{1}{2}\log|y| + 1$

$\Rightarrow \boxed{2e^{x/y} + \log|y| = 2}$ Ans

Ques:- 3 Solve

$(xdy - ydx)\, y\sin\left(\frac{y}{x}\right) = (ydx + xdy)\, x\cos\left(\frac{y}{x}\right)$

Soln

$\Rightarrow dy\left(xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)\right) = dx\left(xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)\right)$

$\Rightarrow \frac{dy}{dx} = \frac{xy\cos\left(\frac{y}{x}\right) + y^2\sin(y/x)}{xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)}$ --- (i)

the degree of each term in N & D is same

put $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ eq (1) becomes

$v + x\frac{dv}{dx} = \frac{vx^2\cos v + v^2x^2\sin v}{vx^2\sin v - x^2\cos v}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v}$

$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + \cancel{v^2\sin v} - \cancel{v^2\sin v} + v\cos v}{v\sin v - \cos v}$$

$$x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \frac{v\sin v - \cos v}{v\cos v}\,dv = 2\frac{dx}{x}$$

$$\Rightarrow \int \frac{v\sin v - \cos v}{v\cos v}\,dv = 2\int \frac{dx}{x}$$

$$\Rightarrow \int \tan v - \frac{1}{v}\,dv = 2\log|x|$$

$$\Rightarrow \log|\sec v| - \log|v| = 2\log|x| + \log C$$

$$\Rightarrow \log\left|\frac{\sec v}{v x^2}\right| = \log C$$

$$\Rightarrow \log\left|\frac{\sec(y/x)}{\frac{y}{x}\cdot x^2}\right| = \log C$$

$$\Rightarrow \left|\frac{\sec(y/x)}{xy}\right| = C$$

$$\Rightarrow \boxed{|\sec(y/x)| = C|xy|} \quad \underline{\text{Ans}}$$

$$\Rightarrow \sec(y/x) = \pm C(xy)$$

$$\Rightarrow \boxed{\sec(y/x) = C_1 xy} \quad \underline{\text{Ans}}$$

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Qn 4 → Solve $ye^{x/y}dx = (xe^{x/y} + y^2)dy$

Soln

$$\frac{dx}{dy} = \frac{xe^{x/y} + y^2}{ye^{x/y}}$$ (Not homogeneous D.E)

put $x = vy$

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\therefore \quad v + y\frac{dv}{dy} = \frac{vye^{vy/y} + y^2}{ye^{vy/y}}$$

$$\Rightarrow v + y\frac{dv}{dy} = \frac{ve^v + y}{e^v}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{ve^v + y}{e^v} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{\cancel{ve^v} + y - \cancel{ve^v}}{e^v}$$

$$\Rightarrow \cancel{y}\frac{dv}{dy} = \frac{\cancel{y}}{e^v}$$

$$\Rightarrow e^v dv = dy$$

$$\Rightarrow \int e^v dv = \int dy$$

$$\Rightarrow e^v = y + c$$

$$\Rightarrow \boxed{e^{x/y} = y + c} \quad \text{Ans}$$

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Type = 3 Variable Separate

$$\frac{dy}{dx} = F(x, y)$$

Variable separate always by "Cross-Multiply"

Ques. 5 → Solve

$$(1+e^{2x})dy + (1+y^2)e^x dx = 0 \quad \text{given } y(0)=1$$

Soln:

$$(1+e^{2x})dy = -(1+y^2)e^x dx$$

$$\frac{dy}{dx} = -\frac{(1+y^2)e^x}{1+e^{2x}}$$

$$\Rightarrow \frac{dy}{1+y^2} = -\frac{e^x}{1+e^{2x}}dx$$

$$\Rightarrow \int \frac{1}{1+y^2}dy = -\int \frac{e^x}{1+e^{2x}}dx$$

put $e^x = t$
$e^x dx = dt$

$$\Rightarrow \tan^{-1}y = -\int \frac{dt}{1+t^2}$$

$$\Rightarrow \tan^{-1}y = -\tan^{-1}(e^x) + C$$

$$\Rightarrow \tan^{-1}(y) + \tan^{-1}(e^x) = C$$

put $x=0$ & $y=1$

$$\Rightarrow \frac{\pi}{4} + \frac{\pi}{4} = C \quad \rightarrow C = \pi/2$$

$\Rightarrow \tan^{-1}y + \tan^{-1}(e^{x}) = \frac{\pi}{2}$

$\Rightarrow \tan^{-1}\left(\frac{y+e^x}{1-ye^x}\right) = \frac{\pi}{2}$

$\Rightarrow \frac{y+e^x}{1-ye^x} = \tan\left(\frac{\pi}{2}\right) = \frac{1}{0}$

$\Rightarrow 0 = 1-ye^x$

$\Rightarrow \boxed{y = \frac{1}{e^x}}$ Ans

WORKSHEET No: 2 (D.E)

Qns: 1 → Show D.E is homogeneous & solve it

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$ **ANS** $\sin\left(\frac{y}{x}\right) = \log|cx|$

Qns 2 → Show that the family of curves / solution for which the slope of the tangent at any point (x, y) on it is $\cancel{x^2}$ $\frac{x^2+y^2}{2xy}$, is given by $x^2 - y^2 = cx$

Qns 3 → Solve $(x+y)dy + (x-y)dx = 0$; $y=1$, $x=1$

ANS $\log|x^2+y^2| + 2\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \log 2$

Qns 4 → Solve $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$

ANS $cy = \log\left|\frac{y}{x}\right| - 1$

Qns 5 → Solve $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$; $y = 2$, $x = 1$

ANS $y = \frac{2x}{1 - \log|x|}$

Qn 6 → Solve $\frac{dy}{dx} + y\cot x = 4x\,\mathrm{cosec}\,x$; $y(\pi/2) = 0$

ANS $y\sin x = 2x^2 - \frac{\pi^2}{2}$

Qn. 7 → Solve $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$ **ANS** $\sin^{-1}y + \sin^{-1}x = C$

Qns 8 → Find the equation of the curve passing through the point $(0, \pi/4)$ whose D.E is

$\sin x \cos y\, dx + \cos x \cdot \sin y\, dy = 0$

Ans $\cos y = \frac{\sec x}{\sqrt{2}}$

Qns 9 → Solve $\log\left(\frac{dy}{dx}\right) = 3x + 4y$; given $x = 0$, $y = 0$

Ans $4e^{3x} + 3e^{-4y} = 7$

[Hint If $\log x = y \Rightarrow x = e^{y}$]

Qn 10 → Solve $(x^2 + xy)dy = (x^2 + y^2)dx$

Ans $(x-y)^2 = cxe^{-y/x}$

Qn 11 → Solve $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Ans $\tan x \tan y = c$

—x—