ULTIMATE MATHEMATICS

(Solution of I-3)

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Qns. 1 → $\tan^{-1}\left(\dfrac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)$

= divide by $\sqrt{1+\cos x}$

$= \tan^{-1}\left(\dfrac{1+\sqrt{\dfrac{1-\cos x}{1+\cos x}}}{1-\sqrt{\dfrac{1-\cos x}{1+\cos x}}}\right)$

$= \tan^{-1}(1) + \tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right)$

$= \dfrac{\pi}{4} + \tan^{-1}\sqrt{\dfrac{2\sin^2(x/2)}{2\cos^2(x/2)}}$ ---- $\{\tan^{-1}x + \tan^{-1}y\}$

$= \dfrac{\pi}{4} + \tan^{-1}\left(\tan\dfrac{x}{2}\right)$

$= \dfrac{\pi}{4} + \dfrac{x}{2}$ ANS.

Qns. 2 → $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$\tan^{-1}\left(\dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$ ---- $\left\{\cot^{-1}x = \tan^{-1}\dfrac{1}{x}\right\}$

divide by $\sqrt{1+\sin x}$

$= \tan^{-1}\left(\dfrac{1-\sqrt{\dfrac{1-\sin x}{1+\sin x}}}{1+\sqrt{\dfrac{1-\sin x}{1+\sin x}}}\right)$

$= \tan^{-1}(1) - \tan^{-1}\left(\sqrt{\dfrac{1-\sin x}{1+\sin x}}\right)$ ---- $\{\tan^{-1}x - \tan^{-1}y\}$

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$$= \frac{\pi}{4} - \tan^{-1}\sqrt{\frac{1-\cos(\frac{\pi}{2}-x)}{1+\cos(\frac{\pi}{2}-x)}}$$

$$= \frac{\pi}{4} - \tan^{-1}\sqrt{\frac{\cancel{2}\sin^2(\frac{\pi}{4}-\frac{x}{2})}{\cancel{2}\cos^2(\frac{\pi}{4}-\frac{x}{2})}}$$

$$= \frac{\pi}{4} - \tan^{-1}\sqrt{\tan^2(\frac{\pi}{4}-\frac{x}{2})}$$

$$= \frac{\pi}{4} - \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)$$

$$= \cancel{\frac{\pi}{4}} - \cancel{\frac{\pi}{4}} + \frac{x}{2} = \frac{x}{2} \quad \text{Ans}$$

Ans 3 → already solved in class

Ans 4 → $\tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right)$

divide by $\sqrt{1+x^2}$

$$= \tan^{-1}\left(\frac{1 + \sqrt{\frac{1-x^2}{1+x^2}}}{1 - \sqrt{\frac{1-x^2}{1+x^2}}}\right)$$

$$= \tan^{-1}(1) + \tan^{-1}\sqrt{\frac{1-x^2}{1+x^2}} \quad \cdots \{\tan^{-1}x - \tan^{-1}y\}$$

$=$ put $x^2 = \cos(2\theta)$

$$= \frac{\pi}{4} + \tan^{-1}\sqrt{\frac{1-\cos(2\theta)}{1+\cos(2\theta)}}$$

$$= \frac{\pi}{4} + \tan^{-1}\sqrt{\frac{\cancel{2}\sin^2\theta}{\cancel{2}\cos^2\theta}}$$

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$= \frac{\pi}{4} + \tan^{-1}\sqrt{\tan^2\theta}$

$= \frac{\pi}{4} + \tan^{-1}(\tan\theta)$

$= \frac{\pi}{4} + \theta$

replace θ ...... $\left\{ \begin{array}{l} x^2 = \cos(2\theta \\ \cos^{-1}(x^2) = 2\theta \\ \theta = \frac{1}{2}\cos^{-1}x^2 \end{array} \right\}$

$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$ Ans

5→ Given $\tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right) = \alpha$

divide by $\sqrt{1+x^2}$

$\Rightarrow \tan^{-1}\left(\frac{1 - \sqrt{\frac{1-x^2}{1+x^2}}}{1 + \sqrt{\frac{1-x^2}{1+x^2}}}\right) = \alpha$

$\Rightarrow \tan^{-1}(1) - \tan^{-1}\sqrt{\frac{1-x^2}{1+x^2}} = \alpha$

put $x^2 = \cos(2\theta)$

$\Rightarrow \frac{\pi}{4} - \tan^{-1}\sqrt{\frac{1-\cos(2\theta)}{1+\cos(2\theta)}} = \alpha$

$\Rightarrow \frac{\pi}{4} - \tan^{-1}\sqrt{\frac{\cancel{2}\sin^2\theta}{\cancel{2}\cos^2\theta}} = \alpha$

$\Rightarrow \frac{\pi}{4} - \tan^{-1}(\tan\theta) = \alpha$

$\Rightarrow \frac{\pi}{4} - \theta = \alpha$

replace θ

$\Rightarrow \frac{\pi}{4} - \frac{1}{2}\cos^{-1}(x^2) = \alpha$

$\Rightarrow \frac{1}{2}\left(\frac{\pi}{2} - \cos^{-1}(x^2)\right) = \alpha$

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(By property $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} - \cos^{-1}x = \sin^{-1}x$)

$\Rightarrow \frac{1}{2}\sin^{-1}(x^2) = \alpha$

$\rightarrow \sin^{-1}(x^2) = 2\alpha$

$\Rightarrow x^2 = \sin(2\alpha)$ Proved

Qn 6 → already solved in class

Qn 7 → we have

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{(x-1)(x+2) + (x+1)(x-2)}{(x-2)(x+2) - (x-1)(x+1)}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x^2-4}{-3}\right) = \frac{\pi}{4}$$

$$\rightarrow \frac{2x^2-4}{-3} = \tan\left(\frac{\pi}{4}\right)$$

$$\rightarrow \frac{2x^2-4}{-3} = 1$$

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$$2x^2-4=-3$$

$$\Rightarrow 2x^2=1$$

$$\Rightarrow x^2=1/2$$

$$\Rightarrow \boxed{x=\pm 1/\sqrt{2}}$$

<u>Check</u> we have to check $xy<1$ or not

$$\left(\frac{x-1}{x-2}\right)\times\left(\frac{x+1}{x+2}\right)=\frac{x^2-1}{x^2-4}$$

put $x=\pm\frac{1}{\sqrt{2}} \Rightarrow \frac{\frac{1}{2}-1}{\frac{1}{2}-4}=\frac{-1/2}{-7/2}=\frac{1}{7}<1$

Clearly both values $x=\pm\frac{1}{\sqrt{2}}$ satisfy the condition

$\therefore \boxed{x=\pm\frac{1}{\sqrt{2}}}$ Ans.

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Ans 8 → $\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1-\left(\frac{x-1}{x+1}\right)\left(\frac{2x-1}{2x+1}\right)}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{4x^2-2}{6x}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$

$$\Rightarrow \frac{4x^2-2}{6x} = \frac{23}{36}$$

$$\Rightarrow \frac{2x^2-1}{3x} = \frac{23}{36}$$

$$\Rightarrow 72x^2 - 36 = 69x$$

$$\Rightarrow 72x^2 - 69x - 36 = 0$$

$$\Rightarrow 24x^2 - 23x - 12 = 0$$

$$\Rightarrow 2\cancel{4x^2} \quad (3x-4)(8x+3) = 0$$

$$\Rightarrow \boxed{x = 4/3} \quad \text{or} \quad \boxed{x = -3/8}$$

verify

$$\left(\frac{x-1}{x+1}\right)\left(\frac{2x-1}{2x+1}\right)$$

when $x = \frac{4}{3}$

$$\left(\frac{\frac{4}{3}-1}{\frac{7}{3}+1}\right)\left(\frac{\frac{8}{3}-1}{\frac{8}{3}+1}\right) = \left(\frac{1}{7}\right)\left(\frac{5}{11}\right) = \frac{5}{72} < 1$$

when $x = -3/8$

$$\left(\frac{-\frac{3}{8}-1}{-\frac{3}{8}+1}\right)\left(\frac{-\frac{6}{8}-1}{-\frac{6}{8}+1}\right) = \left(\frac{-11}{5}\right)\left(\frac{-14}{2}\right) = \frac{154}{10} > 1$$

rejected

$\boxed{\text{Ans} = 4/3}$ (Misprint in worksheet

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Q. 9 → given

$$2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right) = \tan^{-1}(2\sec x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\sin x}{\cos^2 x}\right) = \tan^{-1}(2\sec x)$$

$$\Rightarrow \frac{\cancel{2}\sin x}{\cos^2 x} = \cancel{2}\sec x$$

$$\Rightarrow \sin x = \cos^2 x \times \sec x$$

$$\Rightarrow \sin x = \cos x$$

$$\Rightarrow \tan x = 1$$

$$\boxed{x = \pi/4} \quad \text{Ans}$$

Q. 10 → already solved in class

Q. 11 → given $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}(3x)$

$$\Rightarrow \tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}(3x) - \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{x-1+x+1}{1-(x-1)(x+1)}\right) = \tan^{-1}\left(\frac{3x-x}{1+3x\times x}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2+1}\right) = \tan^{-1}\left(\frac{2x}{1+3x^2}\right)$$

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don't cancel x (Remember)

$$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2}$$

$$\Rightarrow 2x + 6x^3 = 4x - 2x^3$$

$$\Rightarrow 8x^3 - 2x = 0$$

$$\Rightarrow 2x(4x^2-1) = 0$$

$$\Rightarrow 2x(2x+1)(2x-1) = 0$$

$$\boxed{x = 0, \quad x = -1/2, \quad x = 1/2}$$ Ans

Check: we have to check two conditions

$(x-1)(x+1) < 1$ and $(3x)(x) > -1$

$x^2 - 1 < 1$ $\qquad 3x^2 > -1$

when $x=0$: $0-1 = -1 < 1$ $\qquad 0 > -1$

when $x = 1/2$: $\frac{1}{4} - 1 = -\frac{3}{4} < 1$ $\qquad 3\left(\frac{1}{2}\right)^2 = \frac{3}{4} > -1$

when $x = -1/2$: $\frac{1}{4} - 1 = \frac{-3}{4} < 1$ $\qquad 3\left(\frac{-1}{2}\right)^2 = \frac{3}{4} > -1$

$\therefore x = 0, \; x = 1/2, \; x = -1/2$ Ans

Qn. 12 Given $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$

$$\Rightarrow \tan^{-1}\left(\frac{x+1+x-1}{1-(x+1)(x-1)}\right) = \tan^{-1}\left(\frac{8}{31}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2+1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$$

$\Rightarrow \dfrac{2x}{2-x^2} = \dfrac{8}{31}$

$\Rightarrow 62x = 16 - 8x^2$

$\Rightarrow 8x^2 + 62x - 16 = 0$

$\Rightarrow 4x^2 + 31x - 8 = 0$

$\Rightarrow 4x^2 + 32x - x - 8 = 0$

$\Rightarrow 4x(x+8) - 1(x+8) = 0$

$\Rightarrow \boxed{x = 1/4, \quad x = -8}$

Check We have to check

$(x+1)(x-1) < 1$

$x^2 - 1 < 1$

When $x = 1/4$

$\frac{1}{16} - 1 = -\frac{15}{16} < 1$

When $x = -8$ $\quad 64 - 1 = 63 > 1$ (Rejected)

$\therefore \boxed{x = 1/4}$ Ans

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3 → $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

$= \tan^{-1}\left(\dfrac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right)$

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$$= \tan^{-1}\left(\frac{x^2 + \cancel{xy} - \cancel{xy} + y^2}{\cancel{xy} + y^2 + x^2 - \cancel{xy}}\right)$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)$$

$$= \tan^{-1}(1)$$

$$= \pi/4 \quad \underline{\text{Ans}}$$

Qn <u>14</u> → already done in class

Qn <u>15</u> → already done in class

Qn <u>16</u> → $\sin\left[\cot^{-1}\left(\cos\left(\tan^{-1}x\right)\right)\right]$

use <u>conversions</u> here $P = x$, $B = 1$ $\therefore H = \sqrt{x^2+1}$

$$= \sin\left[\cot^{-1}\left(\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)\right]$$

$$= \sin\left[\cot^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right]$$

here $B = 1$, $P = \sqrt{x^2+1}$ $\therefore H = \sqrt{x^2+1+1}$
$= \sqrt{x^2+2}$

$$= \sin\left[\sin^{-1}\left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}\right)\right]$$

$$= \sqrt{\frac{x^2+1}{x^2+2}} \quad \underline{\text{Ans}}$$

Qn <u>17</u> → Same as Qn 16