← ULTIMATE MATHEMATICS: BY AJAY MITTAL →

Chapter: INTEGRATION : CLASS NO: 6

Type $\int \frac{\text{linear}}{\text{Quadratic}}\,dx$

linear → $ax+b$

Quadratic → cx^2+dx+e

Qn. 1 $I = \int \frac{3x+1}{x^2+x+1}\,dx$ Diff. ऊपर बनाओ

4

$2x+1$

Method 1

$$= 3\int \frac{x+\frac{1}{3}}{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int \frac{2x+2/3+1-1}{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int \frac{(2x+1)-1/3}{x^2+x+1}\,dx$$

Separate

$$= \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$

put $x^2+x+1=t$ in 1st Integral

$(2x+1)\,dx = dt$

$$\therefore I = \frac{3}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1}\,dx$$

$$= \frac{3}{2}\log|t| - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{3}{2}\log|x^2+x+1| - \frac{1}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$ Ans

Method-II $I = \int \frac{3x+1}{x^2+x+1}\,dx$

let $N^r = A \cdot \frac{d}{dx}(D^r) + B$

find A & B value by equating the coefficient of x and constant term

$3x+1 = A(2x+1) + B$

$3 = 2A$; $1 = A + B$

$A = 3/2$; $1 = 3/2 + B$

$B = -1/2$

$\therefore I = \int \frac{\frac{3}{2}(2x+1) - 1/2}{x^2+x+1}\,dx$

$= \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$

proceed

Qn2 $I = \int \frac{x}{x^2+x+1}\,dx$

$\downarrow$

type $\int \frac{\text{linear}}{\text{Quad}}$

$\int \frac{x\,dx}{x^4+x^2+1}$

put $x^2 = t$

Type 2 $\int \frac{\text{linear}\, dx}{\sqrt{\text{Quadratic}}}$ Diff. ऊपर बनाओ

Qn 3 $I = \int \frac{5x-1}{\sqrt{1-x-x^2}}\,dx$ $\qquad -2x-1$

$= 5\int \frac{x - 1/5}{\sqrt{1-x-x^2}}\,dx$

$= -\frac{5}{2}\int \frac{-2x + 2/5 - 1 + 1}{\sqrt{1-x-x^2}}\,dx$

$= -\frac{5}{2}\int \frac{(-2x-1) + 7/5}{\sqrt{1-x-x^2}}\,dx$

Separate $= -\frac{5}{2}\int \frac{-2x-1}{\sqrt{1-x-x^2}}\,dx - \frac{7}{2}\int \frac{1}{\sqrt{1-x-x^2}}\,dx$

put $1-x-x^2 = t$

$(-2x-1)\,dx = dt$

$\therefore I = -\frac{5}{2}\int \frac{dt}{\sqrt{t}} - \frac{7}{2}\int \frac{1}{\sqrt{-(x^2+x-1)}}\,dx$

$= -\frac{5}{\cancel{2}} \times \cancel{2}\sqrt{t} - \frac{7}{2}\int \frac{1}{\sqrt{-\left[\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} - 1\right]}}\,dx$

$= -5\sqrt{1-x-x^2} - \frac{7}{2}\int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2}}\,dx$

$= -5\sqrt{1-x-x^2} - \frac{7}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + C$ Ans

Q. (4) $I = \int \frac{x}{\sqrt{(1-x)(2x+1)}} dx$

open Bracket make quadratic

Type 3

$I = \int \frac{\text{linear}}{\text{Special Integral}} dx \rightarrow$ Seperate

Q.(5) $I = \int \frac{3x+1}{x^2+4} dx$

$= 3\int \frac{x}{x^2+4} dx + \int \frac{1}{x^2+4} dx$

put $x^2+4 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$= \frac{3}{2}\int \frac{dt}{t} + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

$= \frac{3}{2}\log|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$ Ans

Q.6 $I = \int \frac{2x-1}{\sqrt{4-16x^2}} dx$

$= 2\int \frac{x}{\sqrt{4-16x^2}} dx - \int \frac{1}{\sqrt{4-16x^2}} dx$

put $4-16x^2 = t$

$-32x\,dx = dt$

$x\,dx = -\frac{dt}{32}$

$\int \frac{1}{\sqrt{4-(4x)^2}}$

$= \frac{1}{4}\sin^{-1}\left(\frac{4x}{2}\right) + C$

$\therefore I = \frac{-2}{32}\int \frac{dt}{\sqrt{t}} - \frac{1}{4}\int \frac{1}{\sqrt{(\frac{1}{2})^2 - x^2}}\,dt$

$= \frac{-1}{16} \times 2\sqrt{t} - \frac{1}{4}\sin^{-1}\left(\frac{x}{1/2}\right) + C$

$= -\frac{\sqrt{t}}{8} - \frac{1}{4}\sin^{-1}(2x) + C$ Ans

Type-4 → $\boxed{\text{Divide}}$

when degree of $N^r \geq$ degree of D^r then divide

and write $\int \frac{N}{D}\,dx = \int Q + \frac{R}{D}\,dx$

Q-1

$I = \int \frac{x^2+5x+3}{x^2+3x+2}\,dx$

$$x^2+3x+2 \overline{\big)\, x^2+5x+3 \,}\; 1$$
$$-(x^2+3x+2)$$
$$2x+1$$

$\therefore I = \int 1 + \frac{2x+1}{x^2+3x+2}\,dx$

$= x + \int \frac{2x+1}{x^2+3x+2}\,dx$

$\int \frac{\text{linear}}{\text{Quad}}\,dx$

Proceed

Qn 8 → $I = \int \frac{x^3+x+1}{x^2-1}\,dx$

$$\begin{array}{r|l} & x \\ x^2-1 & x^3+x+1 \\ & -(x^3-x) \\ \hline & 2x+1 \end{array}$$

$\therefore I = \int x + \frac{2x+1}{x^2-1}\,dx$

$= \frac{x^2}{2} + \int \frac{2x+1}{x^2-1}\,dx$ → Special

$= \frac{x^2}{2} + 2\int \frac{x}{x^2-1}\,dx + \int \frac{1}{x^2-1}\,dx$

put $x^2-1=t$

$2x\,dx = dt$

$= \frac{x^2}{2} + \int \frac{dt}{t} + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$ process

— x —

Qn 9 $I = \int \sqrt{\frac{1+x}{x}}\,dx$

$I = \int \sqrt{\frac{1+x}{x} \times \frac{1+x}{1+x}}\,dx$

$= \int \frac{x+1}{\sqrt{x^2+x}}\,dx$

$= \int \frac{\text{linear}}{\sqrt{\text{Quad}}}\,dx$ process

Qn 10 → $I = \int \frac{2\sin(2x) - \cos x}{6 - \cos^2 x - 4\sin x}\,dx$

$= \int \frac{4\sin x\cos x - \cos x}{6 - \cos^2 x - 4\sin x}\,dx$

$$I = \int \frac{(4\sin x - 1)\cos x \, dx}{6 - \cos^2 x - 4\sin x}$$

$$= \int \frac{(4\sin x - 1)\cos x \, dx}{6 - (1 - \sin^2 x) - 4\sin x}$$

put $\sin x = t$ $\therefore$ $\cos x \, dx = dt$

$$I = \int \frac{(4t - 1)\, dt}{6 - (1 - t^2) - 4t}$$

$$= \int \frac{4t - 1}{t^2 - 4t + 5} dt$$

$2t - 4$

$$= 2\int \frac{2t - 1/2 - 4 + 4}{t^2 - 4t + 5} dt$$

$$= 2\int \frac{(2t - 4) + 7/2}{t^2 - 4t + 5} dt$$

$$= 2\int \frac{2t - 4}{t^2 - 4t + 5} dt + 7\int \frac{1}{t^2 - 4t + 5} dt$$

put $t^2 - 4t + 5 = z$

$(2t - 4)\, dt = dz$

$$I = 2\int \frac{dz}{z} + 7\int \frac{1}{\text{perfect square}}$$

Qn 11) $I = \int \frac{1}{2e^{2x}+3e^{x}+1} dx$

$= \int \frac{1}{\frac{2}{e^{-2x}} + \frac{3}{e^{-x}} + 1} dx$

$= \int \frac{e^{-2x}}{2+3e^{-x}+e^{-2x}} dx$

Good step

$= \int \frac{e^{-x} \cdot e^{-x} dx}{e^{-2x}+3e^{-x}+2}$

put $e^{-x} = t$

$-e^{-x} dx = dt \Rightarrow e^{-x} dx = -dt$

$I = -\int \frac{t\, dt}{t^2+3t+2}$

$\downarrow$

$\int \frac{\text{linear}}{\text{Quadratic}} dx$ process

Qn 12 → $I = \int \frac{ax^3+bx}{x^4+c^2} dx$

$I = a\int \frac{x^3}{x^4+c^2} dx + b\int \frac{x\, dx}{x^4+c^2}$

put $x^4+c^2 = t$

$4x^3 dx = dt$

$x^3 dx = \frac{dt}{4}$

$x^2 = z$

$x dx = \frac{dz}{2}$

$\therefore I = \frac{a}{4}\int \frac{dt}{t} + \frac{b}{2}\int \frac{dz}{z^2+c^2}$

$\rightarrow \frac{a}{4} \log|t| + \frac{b}{2} \times \frac{1}{c} \tan^{-1}\left(\frac{z}{c}\right) + C$

process

13 Ans.

$$I = \int \sqrt{\frac{a-x}{a+x}}\, dx$$

$$= \int \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}}\, dx$$

$$= \int \frac{a-x}{\sqrt{a^2-x^2}}\, dx$$

$$= a\int \frac{1}{\sqrt{a^2-x^2}}\, dx - \int \frac{x}{\sqrt{a^2-x^2}}\, dx$$

put $a^2-x^2=t$

Proceed

— x —

Type $\boxed{\text{Single } \sin x,\ \cos x}$

$$I = \int \frac{1}{\sin x,\ \cos x,\ \text{constant}}\, dx$$

- Write $\sin x = \dfrac{2\tan(x/2)}{1+\tan^2(x/2)}$ & $\cos x = \dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$
- L.C.M
- N^r we get $1+\tan^2(x/2)\, dx = \sec^2(x/2)\, dx$
- put $\tan(x/2) = t$

$\frac{1}{2}\sec^2(x/2)\, dx = dt \Rightarrow \sec^2(x/2)\, dx = 2dt$ (always)

Q.14 $I = \displaystyle\int \frac{1}{2\sin x + 3\cos x + 4}\, dx$

$$I = \int \frac{1}{2\times\frac{2\tan x/2}{1+\tan^2(x/2)} + 3\frac{(1-\tan^2(x/2))}{1+\tan^2(x/2)} + 4}\, dx$$

$$I = \int \frac{1+\tan^2(x/2)\,dx}{4\tan(x/2) + 3 - 3\tan^2(x/2) + 4 + 4\tan^2(x/2)}$$

$$= \int \frac{\sec^2(x/2)\,dx}{\tan^2(x/2) + 4\tan(x/2) + 7}$$

put $\tan(x/2) = t$

$\sec^2(x/2)\,dx = 2dt$

$$I = 2\int \frac{dt}{t^2 + 4t + 7}$$

$\downarrow$

$$\int \frac{1}{\text{Quadrat}}\,d.$$

$$= 2\int \frac{1}{(t+2)^2 - 4 + 7}\,dt$$

$$= 2\int \frac{1}{(t+2)^2 + (\sqrt{3})^2}\,dt$$

$$= 2 \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t+2}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan(x/2) + 2}{\sqrt{3}}\right) + C \quad \underline{\text{Ans}}$$

– x –

$$I = \int \frac{1}{1+\sin x}\,dx$$

Rationalize

$$\int \frac{1}{2+\sin x}\,dx$$

$\hookrightarrow$ single $\sin x$, $\cos x$ type

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WORKSHEET No: 5 (Class No: 6)

INTEGRATION

Qn. 1 $\int \frac{3x-1}{3x^2+4x+2}\,dx$ Ans $\frac{1}{2}\log|3x^2+4x+2| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right)+C$

Qn. 2 → $\int \frac{(3\sin x-2)\cos x\,dx}{5-\cos^2 x-4\sin x}$ Ans $3\log|2-\sin x| + \frac{4}{2-\sin x}+C$

Qn. 3 → $\int \frac{x^3+x}{x^4-9}\,dx$ Ans $\frac{1}{4}\log|x^4-9| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right|+C$

Qn 4 → $\int \frac{x^3+x^2+2x+1}{x^2-x+1}\,dx$ Ans $\frac{x^2}{2}+2x+\frac{3}{2}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$

Qn 5 → $\int \frac{1-x^2}{x(1-2x)}\,dx$ Ans $\frac{x}{2}+\log|x| - \frac{3}{4}\log|1-2x|+C$

Qn 6 → $\int \frac{3x+1}{\sqrt{5-2x-x^2}}\,dx$ Ans $-3\sqrt{5-2x-x^2} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$

Qn 7 → $\int \sqrt{\frac{1-x}{1+x}}\,dx$ Ans $\sin^{-1}x + \sqrt{1-x^2}+C$

Qn 8 → $\int \frac{6x-5}{\sqrt{3x^2-5x+1}}\,dx$ Ans $2\sqrt{3x^2-5x+1}\,dx$

Qn. 9 → $\int \frac{x}{\sqrt{x^2+x+1}}\,dx$ Ans $\sqrt{x^2+x+1} - \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+C$

Qn 10 → $\int \frac{x+1}{\sqrt{x^2+1}}\,dx$ Ans $\sqrt{x^2+1} + \log\left|x+\sqrt{x^2+1}\right|+C$

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Qn 11 → $\int \frac{1}{3+2\sin x+\cos x}\,dx$ Ans $\tan^{-1}\left(1+\tan\frac{x}{2}\right)+C$

Qn 12 → $\int \frac{1}{5+7\cos x+\sin x}\,dx$ Ans $\frac{1}{5}\log\left|\frac{\tan\frac{x}{2}+2}{\tan\frac{x}{2}-3}\right|+C$

Qn 13 → $\int \frac{1}{1-2\sin x}\,dx$ Ans $\frac{1}{\sqrt{3}}\log\left|\frac{\tan(x/2)-2-\sqrt{3}}{\tan(\frac{x}{2})-2+\sqrt{3}}\right|+C$

—x—