
Q. 1.
Using proof by induction prove that, for all $n \in \mathbb{Z}^+$:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Q. 2.
Using proof by induction prove that, for all $n \in \mathbb{Z}^+$:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

Q. 3.
Using proof by induction prove that, for all $n \in \mathbb{Z}^+$:

$$1 + 8 + 16 + \cdots + 8n = (2n + 1)^2$$

Q. 4.
Using proof by induction prove that, for all $n \in \mathbb{Z}^+$:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Q. 5.
Using proof by induction prove that, for all $n \in \mathbb{Z}^+$:

$$1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Answers:

In the following solutions I have omitted the step where you show the results is true for the case $n = 1$. You must include that part of the solution in your answers otherwise you will lose marks. I've omitted it here because it's generally very simple and it would just add to the length of the solutions. For the same reasons I've omitted the final step where you conclude that the result is true for all $n \in \mathbb{Z}^+$. By way of an example, I've given the full details for Q. 1.

Q. 1.
If $n = 1$, the LHS is just 1 and the RHS is 1^2 and since

$$1 = 1^2$$

the result is true when $n = 1$.

Assume the result is true when $n = k$, for some positive integer k, i.e.

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

is true.

Then, if $n = k + 1$, we have

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) &= k^2 + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2. \end{aligned}$$

That is

$$1 + 3 + 5 + \cdots + (2(k + 1) - 1) = (k + 1)^2$$

is also true. But this is just the statement we want to prove with n set to $k + 1$. Thus, if the statement is true for $n = k$, then it is also true for $n = k + 1$ and since the statement is true when $n = 1$, by induction it is true for all $n \in \mathbb{Z}^+$. In other words, we've shown that if the statement if true for any value of n then it is also true for the next value. Since we shown the statement is true for the value $n = 1$, we can conclude that it is true for $n = 2, n = 3$ and so on and therefore it is true for every positive integer n.

Q. 2.

Assume true for $n = k$.

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= (1 + 2 + 3 + \cdots + k)^2 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)(k+1)^2}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{2^2} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= (1 + 2 + 3 + \cdots + (k+1))^2
\end{aligned}$$

Q. 3.

Assume true for $n = k$.

$$\begin{aligned}
1 + 8 + 16 + \cdots + 8k + 8(k+1) &= (2k+1)^2 + 8(k+1) \\
&= 4k^2 + 4k + 1 + 8k + 8 \\
&= 4k^2 + 12k + 9.
\end{aligned}$$

Now

$$\begin{aligned}
(2(k+1)+1)^2 &= 2^2(k+1)^2 + 2(2)(k+1) + 1^2 \\
&= 4(k^2 + 2k + 1) + 4(k+1) + 1 \\
&= 4k^2 + 8k + 4 + 4k + 4 + 1 \\
&= 4k^2 + 12k + 9.
\end{aligned}$$

Other variations are possible. For example, having found

$$1 + 8 + 16 + \cdots + 8k + 8(k+1) = 4k^2 + 12k + 9,$$

we could continue as follows.

$$\begin{aligned}
4k^2 + 12k + 9 &= (2k)^2 + 2(2k)(3) + 3^2 \\
&= (2k+3)^2 \\
&= (2k+2+1)^2 \\
&= (2(k+1)+1)^2
\end{aligned}$$

as required.

Q. 4.
Assume true for $n = k$.

$$\begin{aligned}1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6}.\end{aligned}$$

Q. 5.
Assume true for $n = k$.

$$\begin{aligned}1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 + (2(k+1)+1)^2 &= \frac{(k+1)(2k+1)(2k+3)}{3} + (2(k+1)+1)^2 \\ &= \frac{(k+1)(2k+1)(2k+3) + 3(2k+3)^2}{3} \\ &= \frac{(2k+3)[(k+1)(2k+1) + 3(2k+3)]}{3} \\ &= \frac{(2k+3)[2k^2 + k + 2k + 1 + 6k + 9]}{3} \\ &= \frac{(2k+3)(2k^2 + 9k + 10)}{3} \\ &= \frac{(2k+3)(k+2)(2k+5)}{3} \\ &= \frac{(k+2)(2k+3)(2k+5)}{3} \\ &= \frac{((k+1)+1)(2(k+1)+1)(2(k+1)+3)}{3}\end{aligned}$$