Proof by induction:

Useful for proving statements of the form

$$\forall n \in A, \; P(n)$$

where $A \subseteq \mathbb{Z}+$ the set of positive integers. A is infinite and $P(n)$ is a predicate.

Basis step: Verify that $P(1)$ is true
Inductive step: Assume $P(k)$ is true, and prove that $P(k) \rightarrow P(k+1)$
Conclusion: Therefore $P(n)$ is true $\forall$ positive integers.

Example: Prove by induction

$$1+2+3+\cdots+n = n(n+1)/2$$

This is $P(n)$. Now $P(1) \Rightarrow$

$$1 = 1(1+1)/2$$

which is true.

Assume $P(k)$ is true. Then $P(k+1)$ is

$$1+2+3+\cdots+k+k+1 =$$
$$(1+2+3+\cdots+k)+k+1 =$$
$$P(k)+k+1 =$$
$$\frac{k(k+1)}{2}+k+1 =$$
$$\frac{k(k+1)+2(k+1)}{2} =$$
$$\frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2}$$
$$\frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

hence $P(k+1)$ is true and by induction $P(n)$ is true $\forall n \in \mathbb{Z}+$.

Example: Prove by induction that

$P(n)$: $1+3+5+\cdots+(2n-1)=n^2$

$P(1)$: $1=1^2$ is true.

Assume $P(k)$ is true. Then $P(k+1)$ is

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1)=$$

$$\begin{aligned}
&P(k)+(2(k+1)-1)= \\
&k^2+2(k+1)-1 = \\
&k^2+2k+2-1 = \\
&k^2+2k+1 = \\
&(k+1)^2
\end{aligned}$$

hence $P(k)$ true $\Rightarrow$ $P(k+1)$ is true and the result is true by induction.

Example: Prove by induction that

$P(n)$: $1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2$

$P(1)$: $1^3=1^2$ is true

Assume $P(k)$ is true. Then $P(k+1)$ is

$$\begin{aligned}
&1^3+2^3+\cdots+k^3+(k+1)^3 = \\
&P(k)+(k+1)^3 = \\
&(1+2+\cdots+k)^2+(k+1)^3 = \\
&\left[\frac{k(k+1)}{2}\right]^2+(k+1)^3 =
\end{aligned}$$

$$\frac{k^2(k+1)^2}{4} + (k+1)(k+1)^2 =$$

$$\frac{k^2(k+1)^2}{4} + \frac{4(k+1)(k+1)^2}{4} =$$

$$\frac{(k+1)^2(k^2+4k+4)}{4} =$$

$$\frac{(k+1)^2(k+2)^2}{2^2} =$$

$$(1+\dots+k+1)^2$$