Algorithm Design & Problem Solving:
Algorithm Analysis

Contents

The challenge

- Q. Will my program be able to solve a large practical input?

Why is my program so slow ?

Why does it run out of memory ?

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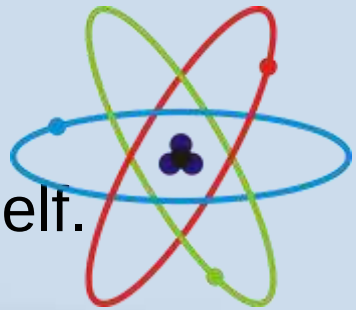
1.4 Analysis of Algorithms

- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- *theory of algorithms*
- *memory*

Scientific method applied to analysis of algorithms

- A framework for predicting performance and comparing algorithms.
- Scientific method.
  - Observe some feature of the natural world.
  - Hypothesize a model that is consistent with the observations.
  - Predict events using the hypothesis.
  - Verify the predictions by making further observations.
  - Validate by repeating until the hypothesis and observations agree.
- Principles.
  - Experiments must be reproducible.
  - Hypotheses must be falsifiable.
- Feature of the natural world. Computer itself.



Example: 3-SUM

- 3-SUM. Given N distinct integers, how many triples sum to exactly zero?
-

```
% more 8ints.txt
8
30 -40 -20 -10 40 0 10 5

% java ThreeSum 8ints.txt
4
```

	a[i]	a[j]	a[k]	sum
1	30	-40	10	0
2	30	-20	-10	0
3	-40	40	0	0
4	-10	0	10	0

Context. Deeply related to problems in computational geometry.

3-SUM: brute-force algorithm

```
int find3NumbersforSum(int A[], int arr_size, int sum)
{
    int i,j,k count=0;
    for ( i = 0; i < arr_size - 2; i++) {
        for (j = i + 1; j < arr_size - 1; j++) {
             for (k = j + 1; k < arr_size; k++)
                if (A[i] + A[j] + A[k] == sum)  count++;
        }
    }
    return count;
}
```

check each triple

for simplicity, ignore integer overflow

Measuring the running time

- Q. How to time a program?
- A. Manual.

```
% java ThreeSum 1Kints.txt
```

```
tick tick tick
70
% java ThreeSum 2Kints.txt
```

```
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
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528
% java ThreeSum 4Kints.txt
```

```
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4039
```

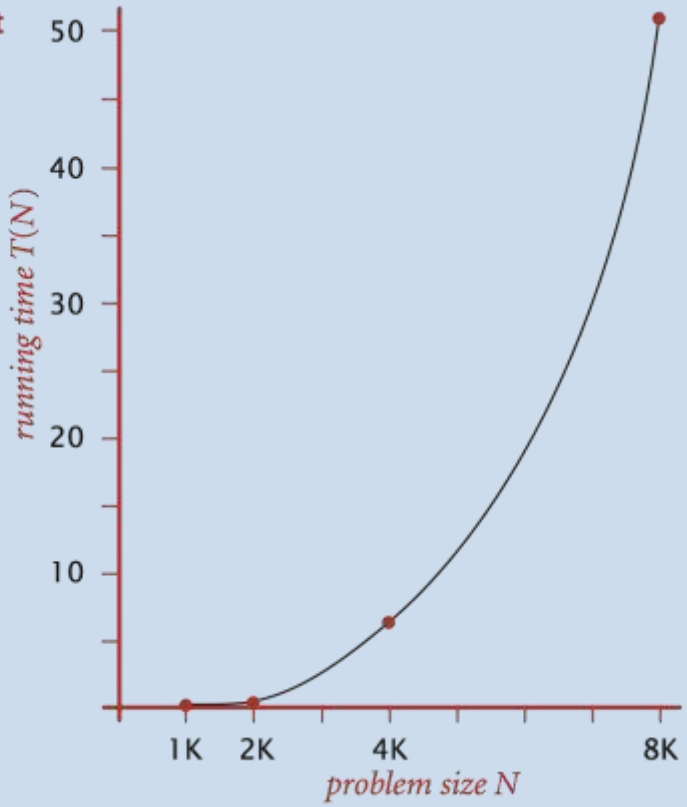
Empirical analysis

- Run the program for various input sizes and measure running time.

N	time (seconds) †
250	0
500	0
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1
16,000	?

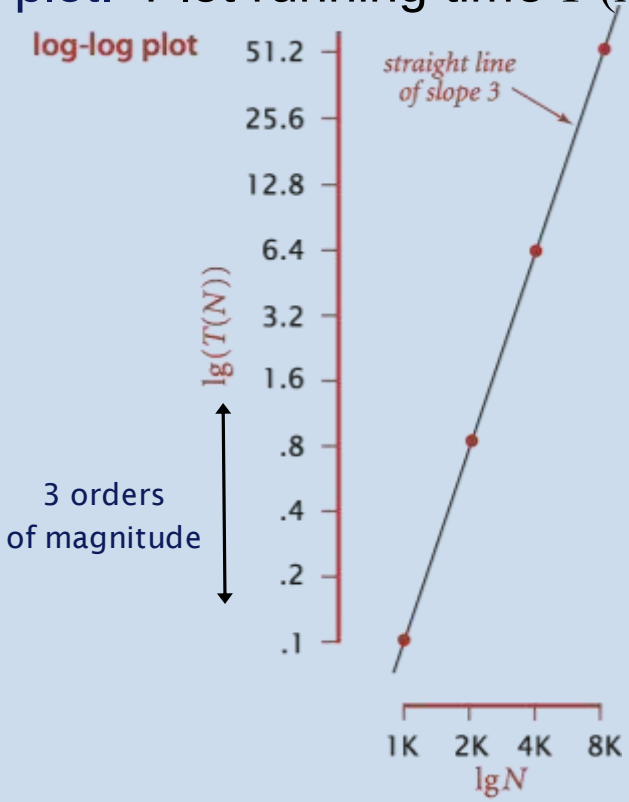
Data analysis

- Standard plot. Plot running time $T(N)$ vs. input size N.



Data analysis

- Log-log plot. Plot running time $T(N)$ vs. input size N using log-log scale.



eqn of straight line :-y=bx+c

$\lg(T(N)) = b \lg N + c$

$b = 2.999$

$c = -33.2103$

$T(N) = a N^{b}$, *where* $a = 2^{c}$

power law

- Regression. Fit straight line through data points: $a N^{b}$.

slope

- Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

Prediction and validation

- Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

"order of growth" of running time is about N^3 [stay tuned]

Predictions.

- 51.0 seconds for $N = 8{,}000$.
- 408.1 seconds for $N = 16{,}000$.

Observations.

N	time (seconds) †
8,000	51.1
8,000	51
8,000	51.1
16,000	410.8

validates hypothesis!

Experimental algorithmics

- System independent effects.
  - Algorithm.
  - Input data.

determines exponent in power law

- System dependent effects.
  - Hardware: CPU, memory, cache, …
  - Software: compiler, interpreter, garbage collector, …
  - System: operating system, network, other apps, …

determines constant in power law

Bad news. Difficult to get precise measurements.

Good news. Much easier and cheaper than other sciences.

e.g., can run huge number of experiments

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1.4 Analysis of Algorithms

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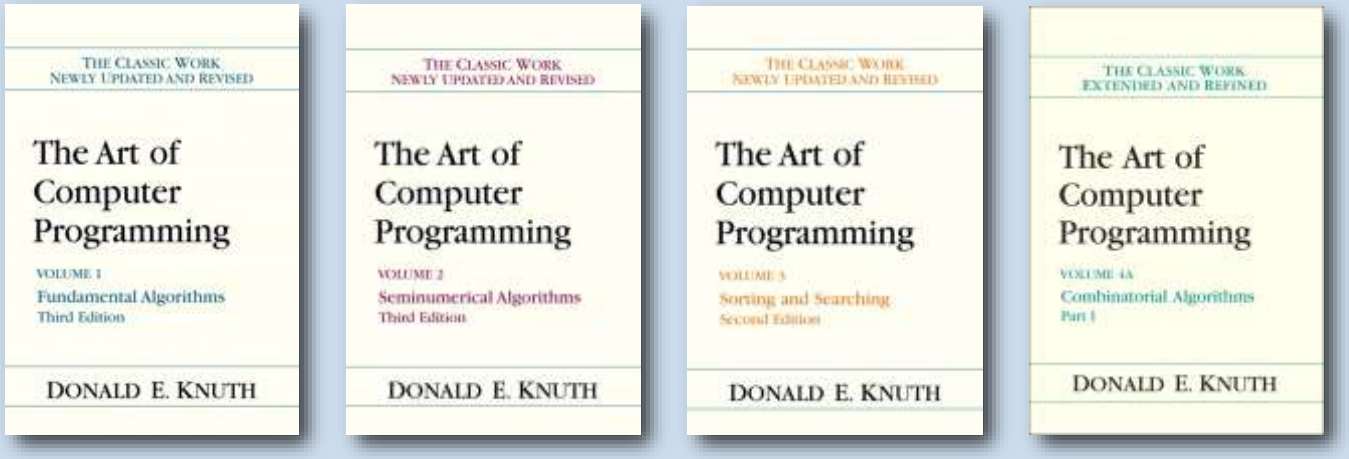
- *mathematical models*
- *order-of-growth classifications*
- *theory of algorithms*
- *memory*

Mathematical models for running time

- Total running time: sum of cost $\times$ frequency for all operations.
  - Need to analyze program to determine set of operations.
  - Cost depends on machine, compiler.
  - Frequency depends on algorithm, input data.



1974 Turing Award

- In principle, accurate mathematical models are available.

Cost of basic operations

- Challenge. How to estimate constants.

operation	example	nanoseconds †
integer add	a + b	2.1
integer multiply	a * b	2.4
integer divide	a / b	5.4
floating-point add	a + b	4.6
floating-point multiply	a * b	4.2
floating-point divide	a / b	13.5
sine	Math.sin(theta)	91.3
arctangent	Math.atan2(y, x)	129
...	...	...

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

Example: 1-Sum

- Q. How many instructions as a function of input size N ?

```
int i,count = 0;
for ( i = 0; i < N; i++) {
  if (a[i] == 0)
    count++;
```

N array accesses

operation	frequency
variable declaration	2
assignment statement	2
less than compare	$N + 1$
equal to compare	N
array access	N
Increment (i++ plus count++)	N to $2\,N$

Example: 2-Sum

- Q. How many instructions as a function of input size N ?

```
int i,j,count = 0;
for (i = 0; i < N; i++) {
   for ( j = i+1; j < N; j++) {
      if (a[i] + a[j] == 0)
         count++;
```

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$\frac{1}{2}(N+1)(N+2)$
equal to compare	$\frac{1}{2}N(N-1)$
array access	$N(N-1)$
increment	$\frac{1}{2}N(N-1)$ to $N(N-1)$

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2}N(N-1) = \binom{N}{2}$$

tedious to count exactly

Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care

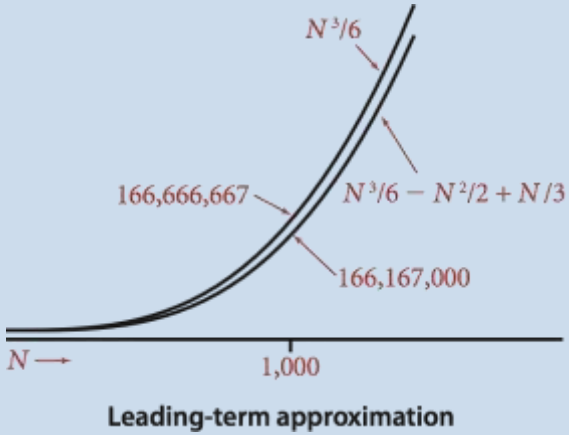
Ex 1. $\frac{1}{6} N^3 + 20 N + 16 \sim \frac{1}{6} N^3$

Ex 2. $\frac{1}{6} N^3 + 100 N^{4/3} + 56 \sim \frac{1}{6} N^3$

Ex 3. $\frac{1}{6} N^3 - \frac{1}{2} N^2 + \frac{1}{3} N \sim \frac{1}{6} N^3$

discard lower-order terms

(e.g., N = 1000: 166.67 million vs. 166.17 million)



Leading-term approximation

Technical definition. $f(N) \sim g(N)$ means

$$\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$$

Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care

operation	frequency	tilde notation
variable declaration	$N + 2$	$\sim N$
assignment statement	$N + 2$	$\sim N$
less than compare	$\frac{1}{2}(N + 1)(N + 2)$	$\sim \frac{1}{2} N^2$
equal to compare	$\frac{1}{2} N (N - 1)$	$\sim \frac{1}{2} N^2$
array access	$N (N - 1)$	$\sim N^2$
increment	$\frac{1}{2} N (N - 1)$ to $N (N - 1)$	$\sim \frac{1}{2} N^2$ to $\sim N^2$

Example: 2-Sum

- Q. Approximately how many array accesses as a function of input size N?

```
int i,j,count = 0;
for (i = 0; i < N; i++)
  for (j = i+1; j < N; j++)
    if (a[i] + a[j] == 0)
      count++;
```

"inner loop"

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1)$$
$$= \binom{N}{2}$$

- A. $\sim N^2$ array accesses.
- Bottom line. Use cost model and tilde notation to simplify counts.

Example: 3-Sum

- Q. Approximately how many array accesses as a function of input size N?

```
int i,j,k,count = 0;
for ( i = 0; i < N; i++) {
  for ( j = i+1; j < N; j++) {
    for ( k = j+1; k < N; k++) {
      if (a[i] + a[j] + a[k] == 0)
        count++;
```

"inner loop"

$$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!} \sim \frac{1}{6}N^3$$

- A. $\sim \frac{1}{2} N^3$ array accesses.
- Bottom line. Use cost model and tilde notation to simplify counts.

3-Sum Approximate Running Time

- Q. How long will it take to run?

A	1	`int i,j,k,count = 0;`
		`for ( i = 0; i < N; i++) {`
B	N	`for ( j = i+1; j < N; j++) {`
C	$\sim N^2/2$	`for ( k = j+1; k < N; k++) {`
D	$\sim n^3/6$	`if (a[i] + a[j] + a[k] == 0)` "inner loop"
E	input dependent	`count++;`

- Only count instructions that execute most frequently – the inner loop (D)
- So, including all the variable init and add ops
  - Total Time T(N) := $(T_1 * A) + (T_2 * B) + (T_3 * C) + (T_4 * D) + (x * E)$
  - Note that E is execution dependent – how many of N's add up to 0

Order of Growth

- T(n) of 3sum function is ~$N^3/6$
- **Order of Growth** of function is N^3

- Note that this reflects what we saw with the power law aN^3 where *a* is some machine dependent factor

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- *mathematical models*
- *order-of-growth classifications*
- *theory of algorithms*
- *memory*

Common order-of-growth classifications

- Definition. If $f(N) \sim c\, g(N)$ for some constant $c > 0$, then the order of growth of $f(N)$ is $g(N)$.
  - Ignores leading coefficient.
  - Ignores lower-order terms.

- Ex. The order of growth of the running time of this code is N^3.

```
int i,j,k,count = 0;
for (i = 0; i < N; i++) {
   for (j = i+1; j < N; j++) {
      for (k = j+1; k < N; k++) {
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

Typical usage. With running times.

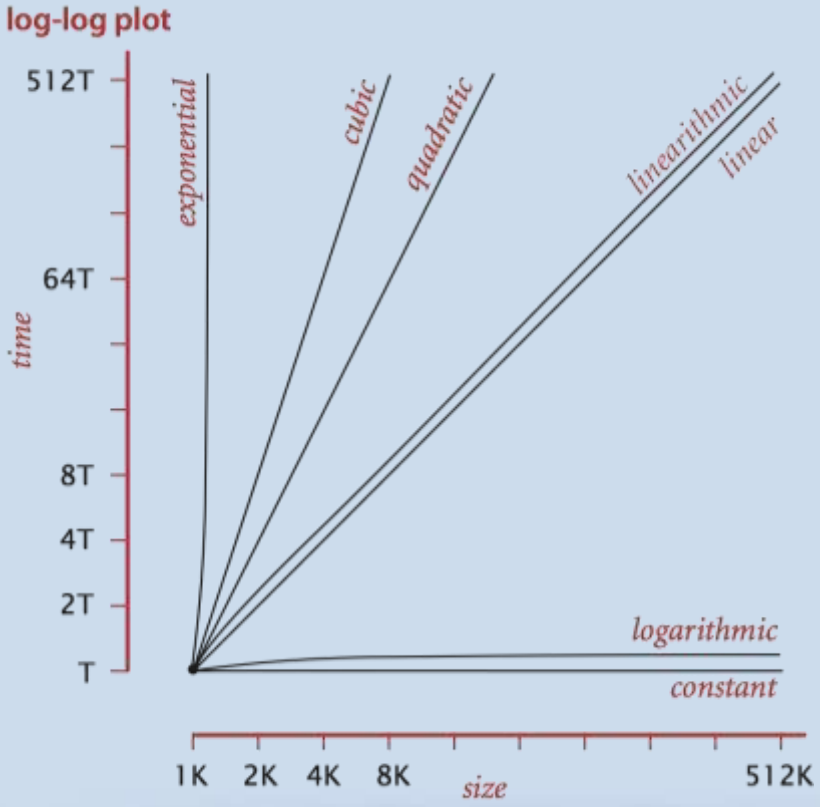
where leading coefficient depends on machine, compiler, JVM, ...

Common order-of-growth classifications

- Good news. The set of functions

  $1,\ \log N,\ N,\ N \log N,\ N^2,\ N^3$, and 2^N

- suffices to describe the order of growth of most common algorithms.



Typical orders of growth

Common order-of-growth classifications

order of growth	name	typical code framework	description	example	$T(2N)$ / $T(N)$
1	**constant**	a = b + c;	statement	add two numbers	1
$\log N$	**logarithmic**	while (N > 1) { N = N / 2; ... }	divide in half	binary search	~ 1
N	**linear**	for (int i = 0; i < N; i++) { ... }	loop	find the maximum	2
$N \log N$	**linearithmic**	[see mergesort lecture]	divide and conquer	mergesort	~ 2

N^2	**quadratic**	for (int i = 0; i < N; i++)
		for (int j = 0; j < N; j++)
		{ ... }
			double loop	check all pairs	4

N^3	**cubic**	for (int i = 0; i < N; i++)
		for (int j = 0; j < N; j++)
		for (int k = 0; k < N; k++)
		{ ... }
			triple loop	check all triples	8

2^N	**exponential**	[see combinatorial search lecture]	exhaustive search	check all subsets	$T(N)$

Practical implications of order-of-growth

growth rate	problem size solvable in minutes			
	1970s	1980s	1990s	2000s
1	any	any	any	any
log N	any	any	any	any
N	millions	tens of millions	hundreds of millions	billions
N log N	hundreds of thousands	millions	millions	hundreds of millions
N^2	hundreds	thousand	thousands	tens of thousands
N^3	hundred	hundreds	thousand	thousands
2^N	20	20s	20s	30

Practical implications of order-of-growth

growth rate	problem size solvable in minutes				time to process millions of inputs			
	1970s	1980s	1990s	2000s	1970s	1980s	1990s	2000s
1	any	any	any	any	instant	instant	instant	instant
log N	any	any	any	any	instant	instant	instant	instant
N	millions	tens of millions	hundreds of millions	billions	minutes	seconds	second	instant
N log N	hundreds of thousands	millions	millions	hundreds of millions	hour	minutes	tens of seconds	seconds
N^2	hundreds	thousand	thousands	tens of thousands	decades	years	months	weeks
N^3	hundred	hundreds	thousand	thousands	never	never	never	millennia

Practical implications of order-of-growth

growth rate	name	description	effect on a program that runs for a few seconds	
			time for 100x more data	size for 100x faster computer
1	constant	independent of input size	–	–
log N	logarithmic	nearly independent of input size	–	–
N	linear	optimal for N inputs	a few minutes	100x
N log N	linearithmic	nearly optimal for N inputs	a few minutes	100x
N^2	quadratic	not practical for large problems	several hours	10x
N^3	cubic	not practical for medium problems	several weeks	4–5x
2^N	exponential	useful only for tiny problems	forever	1x

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- *memory*

Types of analyses

- Best case. Lower bound on cost.
  - Determined by "easiest" input.
  - Provides a goal for all inputs.
- Worst case. Upper bound on cost.
  - Determined by "most difficult" input.
  - Provides a guarantee for all inputs.
- Average case. Expected cost for random input.
  - Need a model for "random" input.
  - Provides a way to predict performance.

this course

Ex 1. Array accesses for brute-force 3-SUM.

Best: $\sim \frac{1}{2} N^3$

Average: $\sim \frac{1}{2} N^3$

Worst: $\sim \frac{1}{2} N^3$

Ex 2. Compares for binary search.

Best: ~ 1

Average: $\sim \lg N$

Worst: $\sim \lg N$

Theory of algorithms

- Goals.
  - Establish “difficulty” of a problem.
  - Develop “optimal” algorithms.

- Approach.
  - Suppress details in analysis: analyze “to within a constant factor.”
  - Eliminate variability in input model: focus on the worst case.

- Upper bound. Performance guarantee of algorithm for any input.
- Lower bound. Proof that no algorithm can do better.
- Optimal algorithm. Lower bound = upper bound (to within a constant factor).

Commonly-used notations in the theory of algorithms

notation	provides	example	shorthand for	used to
Big Theta	asymptotic order of growth	$\Theta(N^2)$	$\frac{1}{2} N^2$ $10 N^2$ $5 N^2 + 22 N$ $\log N + 3N$ ⋮	classify algorithms
Big Oh	$\Theta(N^2)$ and smaller	$O(N^2)$	$10 N^2$ $100 N$ $22 N \log N + 3$ N ⋮	develop upper bounds
Big Omega	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$\frac{1}{2} N^2$ N^5 $N^3 + 22 N \log$ $N + 3 N$ ⋮	develop lower bounds

Theory of algorithms: example 1

- Goals.
  - Establish "difficulty" of a problem and develop "optimal" algorithms.
  - Ex. 1-SUM = "*Is there a* 0 *in the array?* "
- Upper bound. A specific algorithm.
  - Ex. Brute-force algorithm for 1-SUM: Look at every array entry.
  - Running time of the optimal algorithm for 1-SUM is $O(N)$.
- Lower bound. Proof that no algorithm can do better.
  - Ex. Have to examine all *N* entries (any unexamined one might be 0).
  - Running time of the optimal algorithm for 1-SUM is $\Omega(N)$.
- Optimal algorithm.
  - Lower bound equals upper bound (to within a constant factor).
  - Ex. Brute-force algorithm for 1-SUM is optimal: its running time is $\Theta(N)$.

Theory of algorithms: example 2

- Goals.
  - Establish “difficulty” of a problem and develop “optimal” algorithms.
  - Ex. 3-SUM.

- Upper bound. A specific algorithm.
  - Ex. Brute-force algorithm for 3-SUM.
  - Running time of the optimal algorithm for 3-SUM is $O(N^3)$.

Theory of algorithms: example 2

- Goals.
  - Establish “difficulty” of a problem and develop “optimal” algorithms.
  - Ex. 3-SUM.
- Upper bound. A specific algorithm.
  - Ex. Improved algorithm for 3-SUM.
  - Running time of the optimal algorithm for 3-SUM is $O(N^2 \log N)$.
- Lower bound. Proof that no algorithm can do better.
  - Ex. Have to examine all N entries to solve 3-SUM.
  - Running time of the optimal algorithm for solving 3-SUM is $\Omega(N)$.
- Open problems.
  - Optimal algorithm for 3-SUM?
  - Subquadratic algorithm for 3-SUM?
  - Quadratic lower bound for 3-SUM?

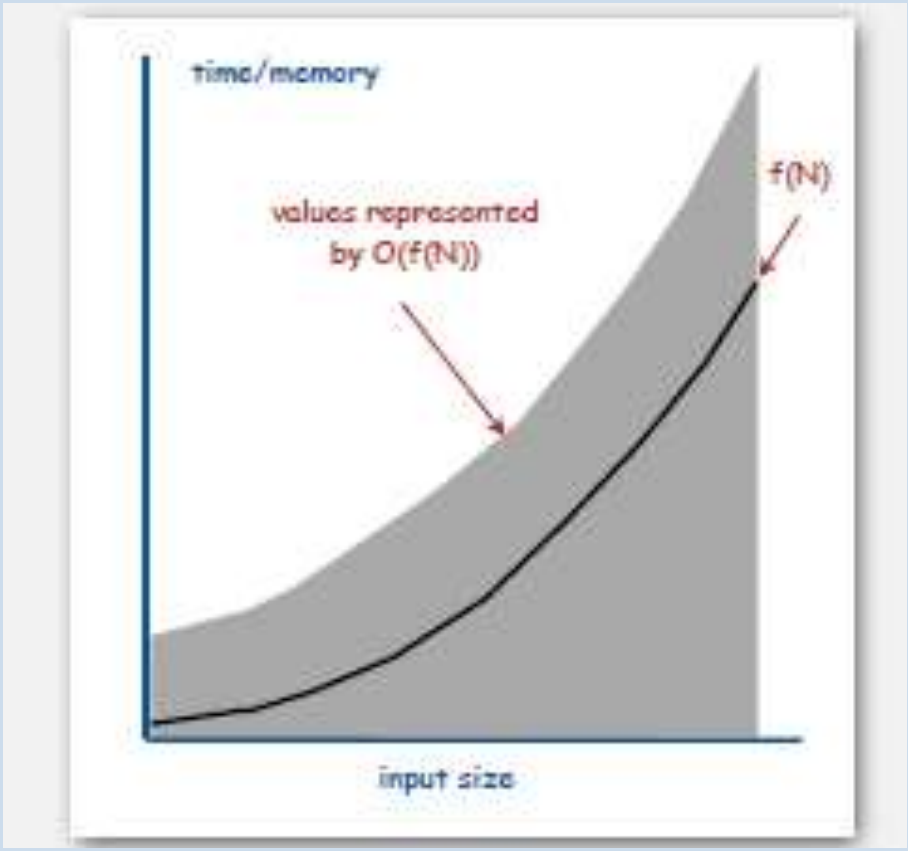
Algorithm design approach

- Start.
  - Develop an algorithm.
  - Prove a lower bound.
- Gap?
  - Lower the upper bound (discover a new algorithm).
  - Raise the lower bound (more difficult).
- Golden Age of Algorithm Design.
  - 1970s-.
  - Steadily decreasing upper bounds for many important problems.
  - Many known optimal algorithms.
- Caveats.
  - Overly pessimistic to focus on worst case?
  - Need better than “to within a constant factor” to predict performance.

Commonly-used notations in the theory of algorithms

notation	provides	example	shorthand for	used to
Tilde	leading term	$\sim 10\,N^2$	$10\,N^2$ $10\,N^2 + 22\,N$ $\log N$ $10\,N^2 + 2\,N + 37$	provide approximate model
Big Theta	Asymptotic order of growth	$\Theta(N^2)$	$\frac{1}{2}\,N^2$ $10\,N^2$ $5\,N^2 + 22\,N$ $\log N + 3N$	classify algorithms
Big Oh	$\Theta(N^2)$ and smaller	$O(N^2)$	$10\,N^2$ $100\,N$ $22\,N \log N + 3\,N$	develop upper bounds
Big Omega	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$\frac{1}{2}\,N^2$ N^5 $N^3 + 22\,N \log N + 3\,N$	develop lower bounds

big-Oh notation



- Big Oh/Big O
  - Worst Case performance
  - Big Oh is not going to show exact timing of algorithm
  - Just how long it takes to run with its worst case data set in terms of the number of items (N) being processed

Big-O Notation

Test yourself!

What is the worst-case complexity of the each of the following code fragments?

1

```
for (i = 0; i < N; i++)
   …sequence of statements

for (j = 0; j < M; j++) {
   …sequence of statements }
```

How would the complexity change if the second loop went to N instead of M?

2

```
for (i = 0; i < N; i++) {
   for (j = 0; j < N; j++) {
      …sequence of statements
   }
}

for (k = 0; k < N; k++) {
   …sequence of statements
}
```

Give the order of growth (as a function of N) of the running times of each of the following code fragments:

```
int sum = 0;
for (int k = N; k > 0; k = k/2)
   for (int i = 0; i < k; i++)
      sum++;
```

```
int sum = 0;
for (int i = 1; i < N; i = i* 2)
   for(int j = 0; j < i; j++)
      sum++;
```

```
int sum = 0;
for (int i = 1; i < N; i *= 2)
   for (int j = 0; j < N; j++)
      sum++;
```

Give the order of growth (as a function of N) of the running times of each of the following code fragments:

1

```
void function(int n)
{
 if (n==1)
   return;
 for (int i=1; i<=n; i++) {
   for (int j=1; j<=n; j++){
     putchar('*');
     break;
   }
 }
}
```

2

```
void function(int n)
{
    int count = 0;
    for (int i=n/2; i<=n; i++)
        for (int j=1; j<=n; j=2 * j)
            for (int k=1;k<=n;k=k * 2)
                count++;
}
```

3

```
void function(int n)
{
  int count = 0;
  for (int i=n/2; i<=n; i++)
    for (int j=1;j+n/2<=n; j=j++)
      for (int k=1;k<=n;k=k*2)
                count++;
}
```

4

```
void function(int n)
{
  int count = 0;
  for (int i=0; i<n; i++)
    for (int j=i; j< i*i; j++)
      if (j%i == 0){
        for (int k=0; k<j; k++)
          printf("*");
      }
}
```

Thank You !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