Predicate Logic

Recall logic notation:

and	$\wedge$
or	$\vee$
not	$\neg$
implies	$\longrightarrow$

Recall that a predicate $P(x)$ is a logical statement that evaluates to T or F depending on x. The value for x comes from a specified domain of discourse.

Eg. $P(x)$: $x > x^2$, domain $D = \mathbb{R}$

Then
$P(0)$ is F as $0 \not> 0^2$
$P(1/2)$ is T as $1/2 > (1/2)^2$
$P(2)$ is F as $2 \not> 2^2$

Eg. $Q(x,y): x>y$, domain $D=\mathbb{Z}$

Then $Q(3,1)$ is T

$Q(5,5)$ is F

$Q(6,-6)$ is T

$Q(2^8, 256)$ is F

Universal Quantifier

The universal quantifier is $\forall$ and means 'for all', 'for each' or 'for every'. If $P(x)$ is a predicate then

$$\forall x\, P(x)$$

means $P(x)$ is true for every possible x in the domain.

Eg. $P(x)$: $2x$ is even

If $D=\mathbb{Z}$ then $\forall x\, P(x)$ is true

If $D=\mathbb{Q}$ then $\forall x\, P(x)$ is false since for eg. $(3/7)\times 2$ isn't even.

Eg. $P(x)$: x^2+2x is an odd integer, $D=\mathbb{Z}$. Then $\forall x\, P(x)$ is false since if x is odd, x^2+2x is not even, ie it's not divisible by 2. If $x=1$ then $x^2+2x=3$ and $2\nmid 3$.

Existential Quantifier

The *existential quantifier* is $\exists$ and means 'there exists', or 'there is'. If $P(x)$ is a predicate $\exists x P(x)$ means $P(x)$ is true for at least one x in the domain.

Eg. $Q(x): x+1 > 2x$, $D = \mathbb{Z}$
Then $\exists x\, Q(x)$ is true since $0+1 > 2(0)$ is true.

To show a $\exists x$ is true you only need to find one x which works. To show it's false you need to show there is no x for which the predicate is true.

To show $\forall x P(x)$ is false you only need to find one x for which $P(x)$ is false (a counter-example). To show $\forall x P(x)$ is true you need to show $P(x)$ is true for every possible x.

Eg. $Q(x): x+1 > 2x$, $D = \mathbb{Z}$
$\forall x Q(x)$ is F since $2+1 \not> 2(2)$
$\exists x(\neg Q(x))$ is true since $\neg Q(2)$ is T

What is a Predicate?

A **predicate** (or propositional function) is a statement involving a variable. It usually describes a property about something.

Examples:

- S(x) = "x is a student"
- P(x) = " x is a professor"
- B(x) = " x is blue"
- H(x,y) = " x has y"

A predicate is a **proposition with variables**

Predicates

Suppose we have

$$P(x, y) = [x + 2 = y]$$

If

$x = 1$ and $y = 3$: $P(1,3)$ is true

If

$x = 1$ and $y = 4$: $P(1,4)$ is false

$\neg P(1,4)$ is true

You can think of a propositional function as a function that

- Evaluates to true or false.
- Takes one or more arguments.
- Expresses a predicate involving the argument(s).
- Becomes a proposition when values are assigned to the arguments.

Example

Let $Q(x, y, z)$ denote the statement "$x^2 + y^2 = z^2$". What is the truth value of $Q(3, 4, 5)$? What is the truth value of $Q(2, 2, 3)$? How many values of (x, y, z) make the predicate true?

Example

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Since $3^2 + 4^2 = 25 = 5^2$, $Q(3, 4, 5)$ is true.

Since $2^2 + 2^2 = 8 \neq 3^2 = 9$, $Q(2, 2, 3)$ is false.

There are infinitely many values for (x, y, z) that make this propositional function true—how many right triangles are there?

Consider the previous example. Does it make sense to assign to x the value "blue"?

Intuitively, the *universe of discourse* is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

What would be the universe of discourse for the propositional function $P(x) =$ "The test will be on x the 23rd" be?

Domain of a Predicate

In the predicate

P(x) = “2x is an even integer”,

it is important that the domain D of P(x) is defined, for instance D={Integers}

The **domain, D,** is the set where the x’s come from i.e. the set of possible x values.

Quantifiers

A predicate becomes a proposition when we assign it fixed values. However, another way to make a predicate into a proposition is to *quantify* it. That is, the predicate is true (or false) for *all* possible values in the universe of discourse or for *some* value(s) in the universe of discourse.

Such *quantification* can be done with two *quantifiers*: the *universal* quantifier and the *existential* quantifier.

What is a Quantifier?

Most statements in maths and computer science use terms such as ‘for every’ and ‘for some’.

<u>Example:</u>

- For every triangle T, the sum of the angles of T is 180°.
- For *every* integer n, n is less than p, for *some* prime number p.

A <u>**quantifier**</u> is a logical symbol which makes an assertion about the set of values which make one or more formulas true. We will discuss 2 important quantifiers.

Quantifiers

Definition

The *universal quantification* of a predicate $P(x)$ is the proposition "$P(x)$ is true for all values of x in the universe of discourse" We use the notation

$$\forall x P(x)$$

which can be read "for all x"

Definition

The *existential quantification* of a predicate $P(x)$ is the proposition "There exists an x in the universe of discourse such that $P(x)$ is true." We use the notation

$$\exists x P(x)$$

which can be read "there exists an x"

Example 1:

Suppose

$P(x)$ = "x has a mobile phone"

and D = {set of students}

Then

1. $\forall x, P(x)$ means "Every student has a mobile phone".
2. $\exists x, P(x)$ means "There exists some student(s) who has a mobile phone".

Example 2:

- Let $P(x)$ be the predicate "x must take a discrete mathematics course" and let $Q(x)$ be the predicate "x is a computer science student".
- The universe of discourse for both $P(x)$ and $Q(x)$ is all DIT students.
- Express the statement "Every computer science student must take a discrete mathematics course".
- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

Example 3:

Express the statement "for every x and for every y, $x + y > 10$"

- Express the statement "Every computer science student must take a discrete mathematics course".

$$\forall x(Q(x) \rightarrow P(x))$$

- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

$$\forall x(Q(x) \vee P(x))$$

Express the statement "for every x and for every y, $x + y > 10$"

Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

Answer:

$$\forall x \forall y P(x, y)$$

Note that we can also use the shorthand

$$\forall x, y P(x, y)$$

Example 4:

Let $P(x, y)$ denote the statement, "$x + y = 5$".

What does the expression,

$$\exists x \exists y P(x)$$

mean?

Example 5:

Express the statement "there exists a real solution to $ax^2 + bx - c = 0$"

Example 4:

$\exists x \exists y$ P(x) means "there exists some x and some y such that x+y=5".

Example 5:

Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where the universe of discourse for x is the set of reals. Note here that a, b, c are all fixed constants.

The statement can thus be expressed as

$$\exists x P(x)$$

Quantifiers
Truth Values

In general, when are quantified statements true/false?

Statement	True When	False When
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Table: Truth Values of Quantifiers

Mixing Quantifiers

Existential and universal quantifiers can be used together to quantify a predicate statement; for example,

$$\forall x \exists y P(x, y)$$

is perfectly valid. However, you must be careful—it must be read left to right.

For example, $\forall x \exists y P(x, y)$ is not equivalent to $\exists y \forall x P(x, y)$. Thus, ordering is important.

For example:

- $\forall x \exists y Loves(x, y)$: everybody loves somebody
- $\exists y \forall x Loves(x, y)$: There is someone loved by everyone

Those expressions do not mean the same thing!

Note that $\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$, but the converse does not hold

However, you *can* commute *similar* quantifiers; $\exists x \exists y P(x, y)$ is equivalent to $\exists y \exists x P(x, y)$ (which is why our shorthand was valid).

Statement	True When	False When
$\forall x \forall y P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is at least one pair, x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x, there is a y for which $P(x, y)$ is true.	There is an x for which $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x, there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$	There is at least one pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Table: Truth Values of 2-variate Quantifiers

Example 1:

Express, in predicate logic, the statement that there are an infinite number of integers.

Solution:

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Then the statement can be expressed by the following.

$$\forall x \exists y P(x, y)$$

Example 2:

Express the *commutative law of addition* for $\mathbb{R}$.

Solution: We want to express that for every pair of reals, x, y the following identity holds:

$$x + y = y + x$$

Then we have the following:

$$\forall x \forall y (x + y = y + x)$$

Rules of inference for quantified statements

1. Universal instantiation

- $\forall x \in D, P(x)$
- $d \in D$
- Therefore $P(d)$

2. Universal generalization

- $P(d)$ for any $d \in D$
- Therefore $\forall x, P(x)$

3. Existential instantiation

- $\exists x \in D, P(x)$
- Therefore $P(d)$ for some $d \in D$

4. Existential generalization

- $P(d)$ for some $d \in D$
- Therefore $\exists x, P(x)$

Example 1: Write the following as a quantified statement

English:

Everyone likes either Microsoft or Apple.
Jill does not like Microsoft.
Therefore Jill likes Apple.

Maths:

M(x) = "x likes Microsoft"
A(x) = "x likes Apple"

$$\forall x\ M(x) \vee A(x)$$

$M(\text{Jill}) \vee A(\text{Jill})$.
$\neg M(\text{Jill})$.
$\Rightarrow A(\text{Jill})$.

Example 2: Write the following as a quantified statement

English: Every student is intelligent

Maths: Let S(x) = "x is a student"

Let I(x) = "x is intelligent"

$\forall x\, S(x) \rightarrow I(x)$

Example 3: Write the following as a quantified statement

English: Some student(s) is intelligent

Maths: Let S(x) = "x is a student"

Let I(x) = "x is intelligent"

$\exists x\, S(x) \wedge I(x)$

Negation

Just as we can use negation with propositions, we can use them with quantified expressions.

Lemma

Let $P(x)$ be a predicate. Then the following hold.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

This is essentially a quantified version of De Morgan's Law (in fact if the universe of discourse is finite, it is *exactly* De Morgan's law).