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Linear Algebra (MT-1004)

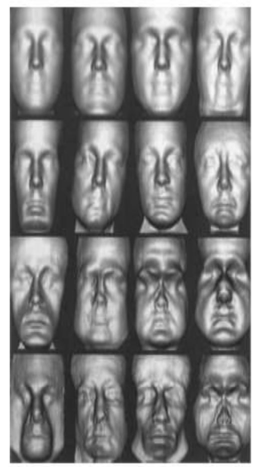
Lecture # 28 & 29

Eigen Values & Eigen Vectors:

Some Applications of Eigen Values & Eigen Vectors:

- **Eigen Values are used to reduce noise in data.**
- **Eigen Values & Eigen Vectors lives in the heart of data science field**
- **It is must-know topic for anyone who wants to understand machine learning in depth**

Historical Note



Methods of linear algebra are used in the emerging field of computerized face recognition. Researchers are working with the idea that every human face in a racial group is a combination of a few dozen primary shapes. For example, by analyzing three-dimensional scans of many faces, researchers at Rockefeller University have produced both an average head shape in the Caucasian group—dubbed the ***meanhead*** (top row left in the figure to the left)—and a set of standardized variations from that shape, called ***eigenheads*** (15 of which are shown in the picture). These are so named because they are eigenvectors of a certain matrix that stores digitized facial information. Face shapes are represented mathematically as linear combinations of the eigenheads.

[Image: © Dr. Joseph J. Atick, adapted from *Scientific American*]

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Definition 1

If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in R^n is called an ***eigenvector*** of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an ***eigenvalue*** of A (or of T_A), and $\mathbf{x}$ is said to be an ***eigenvector corresponding to*** λ.

The requirement that an eigenvector be nonzero is imposed to avoid the unimportant case $A\mathbf{0} = \lambda\mathbf{0}$, which holds for every A and λ.

Theorem 5.1.1

If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\det(\lambda I - A) = 0 \tag{1}$$

This is called the ***characteristic equation*** of A.

When the determinant $\det(\lambda I - A)$ in (1) is expanded, the characteristic equation of A takes the form

$$\boxed{\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0} \tag{3}$$

where the left side of this equation is a polynomial of degree n in which the coefficient of λ^n is 1 (Exercise 37). The polynomial

$$\boxed{p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n} \tag{4}$$

is called the ***characteristic polynomial*** of A. For example, it follows from (2) that the characteristic polynomial of the 2×2 matrix in Example 2 is

$$p(\lambda) = (\lambda - 3)(\lambda + 1) = \lambda^2 - 2\lambda - 3$$

which is a polynomial of degree 2.

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Theorem 5.1.2

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

Theorem 5.1.3

If A is an $n \times n$ matrix, the following statements are equivalent.

(*a*) λ is an eigenvalue of A.

(*b*) λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.

(*c*) The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

(*d*) There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

Finding Eigenvectors and Bases for Eigenspaces

Now that we know how to find the eigenvalues of a matrix, we will consider the problem of finding the corresponding eigenvectors. By definition, the eigenvectors of A corresponding to an eigenvalue λ are the nonzero vectors that satisfy

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

Thus, we can find the eigenvectors of A corresponding to λ by finding the nonzero vectors in the solution space of this linear system. This solution space, which is called the ***eigenspace*** of A corresponding to λ, can also be viewed as:

1. the null space of the matrix $\lambda I - A$
2. the kernel of the matrix operator $T_{\lambda I - A}: R^n \rightarrow R^n$
3. the set of vectors for which $A\mathbf{x} = \lambda\mathbf{x}$

Eigenvalues and Invertibility

The next theorem establishes a relationship between the eigenvalues and the invertibility of a matrix.

Theorem 5.1.4

A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

More on the Equivalence Theorem

As our final result in this section, we will use Theorem 5.1.4 to add one additional part to Theorem 4.9.8.

Theorem 5.1.5

Equivalent Statements

If A is an $n \times n$ matrix in which there are no duplicate rows and no duplicate columns, then the following statements are equivalent.

(*a*) A is invertible.
(*b*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(*c*) The reduced row echelon form of A is I_n.
(*d*) A is expressible as a product of elementary matrices.
(*e*) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
(*f*) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
(*g*) $\det(A) \neq 0$.
(*h*) The column vectors of A are linearly independent.
(*i*) The row vectors of A are linearly independent.
(*j*) The column vectors of A span R^n.
(*k*) The row vectors of A span R^n.
(*l*) The column vectors of A form a basis for R^n.
(*m*) The row vectors of A form a basis for R^n.
(*n*) A has rank n.
(*o*) A has nullity 0.
(*p*) The orthogonal complement of the null space of A is R^n.
(*q*) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.
(*r*) $\lambda = 0$ is not an eigenvalue of A.

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Do Question # 1-15 from Ex # 5.1