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Linear Algebra (MT-1004)

Lecture # 21

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Definition 1

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a ***linearly independent set*** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be ***linearly dependent***. If S has only one vector, we will agree that it is linearly independent if and only if that vector is nonzero.

OR

Theorem Based Definition

Linear Independence and Linear Dependence

- Definitions :

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ is a nonempty set of vectors, then the vector equation

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_k\mathbf{v}_k=\mathbf{0}$$

has at least one solution, namely $c_1=0,\ c_2=0,\ldots,c_k=0.$

If this is the only solution, then S is called a linearly independent set. If there are other solutions, then S is called a linearly dependent set.

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- Ex $\boxed{\text{Testing for linear independence}}$

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S=\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}=\{(1,\ 2,\ 3),(0,\ 1,\ 2),(-2,\ 0,\ 1)\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0} \;\Rightarrow\; \begin{aligned} c_1 \qquad\quad -2c_3 &= 0 \\ 2c_1+\ c_2+\qquad\ \ &= 0 \\ 3c_1+2c_2+\ c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1=c_2=c_3=0 \quad (\text{only the trivial solution})$$

$$(\text{or } \det(A)=-1\neq 0, \text{ so there is only the trivial solution})$$

$$\Rightarrow S \text{ is (or } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \text{ are) linearly independent}$$

- Ex : Testing for linear independence **for polynomials**

  Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\{1+x-2x^2, 2+5x-x^2, x+x^2\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$$

i.e., $c_1(1+x-2x^2)+c_2(2+5x-x^2)+c_3(x+x^2)=0+0x+0x^2$

$$\Rightarrow \begin{aligned} c_1+2c_2 \quad &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G. E.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ This system has infinitely many solutions

(i.e., this system has nontrivial solutions, e.g., $c_1=2$, $c_2=-1$, $c_3=3$)

$\Rightarrow$ S is (or $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are) linearly dependent

Theorem 4.4.2

(*a*) A set with finitely many vectors that contains $\mathbf{0}$ is linearly dependent.

(*b*) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

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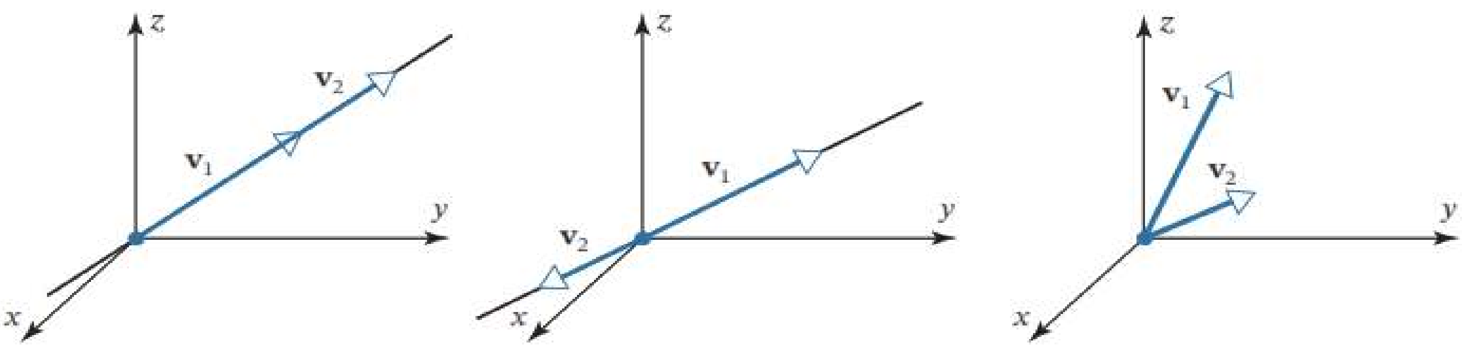
A Geometric Interpretation of Linear Independence

Linear independence has the following useful geometric interpretations in R^2 and R^3:

- Two vectors in R^2 or R^3 are linearly independent if and only if they do not lie on the same line when they have their initial points at the origin. Otherwise one would be a scalar multiple of the other (**Figure 4.4.3**).



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

FIGURE 4.4.3

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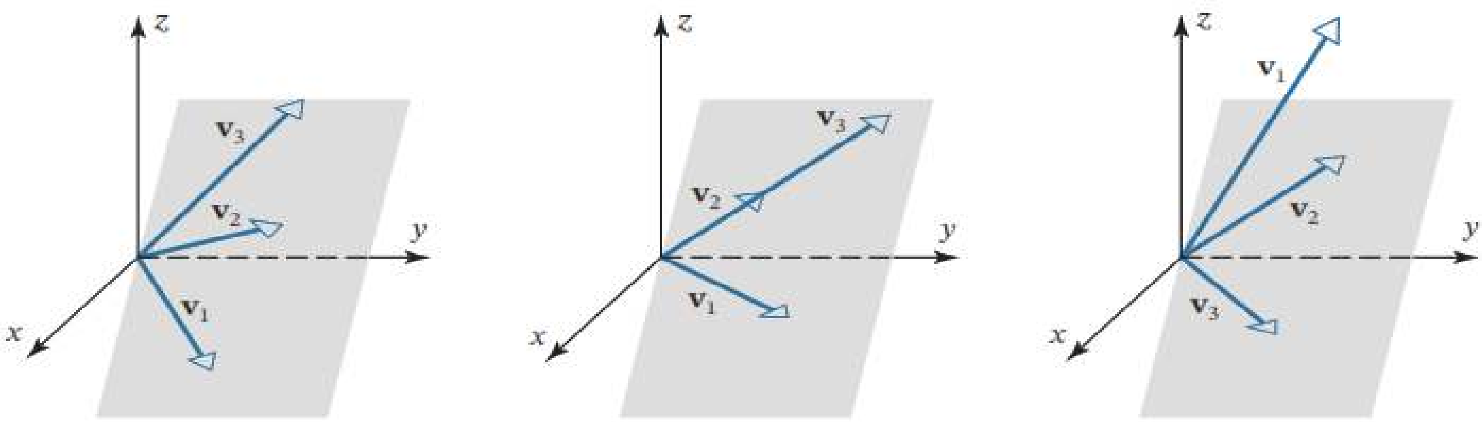
- Three vectors in R^3 are linearly independent if and only if they do not lie in the same plane when they have their initial points at the origin. Otherwise at least one would be a linear combination of the other two (**Figure 4.4.4**).



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

FIGURE 4.4.4

Theorem 4.4.3

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

example

Explain why the following form linearly dependent sets of vectors. (Solve this problem by inspection.)

b. $\mathbf{u}_1 = (3, -1)$, $\mathbf{u}_2 = (4, 5)$, $\mathbf{u}_3 = (-4, 7)$ in R^2

Ans: A set of 3 vectors in R^2 must be linearly dependent by Theorem 4.4.3.

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Do Question # 1-15 from Ex # 4.4