National University of computer and emerging sciences

Foundation for Advancement of Science and Technology

Linear Algebra (MT-1004)

Lecture # 07

Number of Solutions of Linear System:

{ Theorem 1.6.1 }

"A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities"

Solving Linear Systems by Matrix Inversion:

{ Theorem 1.6.2 }

If A is an invertible $n \times n$ matrix, then for every $n \times 1$ matrix b, the system of equations $A\mathbf{x} = \mathbf{b}$ has **exactly one solution** , namely, $\mathbf{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

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Task for Students
Refer Lecture & Example #1 (Pg # 63)
Question # 1 till 8 (Ex #1.6 ; Pg # 67) is related to above topic. Do for Practice.

Linear Systems with a Common Co-efficient Matrix (A) :

The method below is applicable for both invertible & not invertible co-efficient matrix A

$$[A \mid \boldsymbol{b_1} \mid \boldsymbol{b_2} \mid \cdots \mid \boldsymbol{b_k}]$$

in which the coefficient matrix A is "augmented" by all k of the matrices $\boldsymbol{b_1}$, $\boldsymbol{b_2}$, . . . , $\boldsymbol{b_k}$, and then reduce to reduced row echelon form by Gauss–Jordan elimination

Task for Students

Refer Lecture & Example #2 (Pg # 63)

Question # 9 till 12 (Ex #1.6 ; Pg # 67 & 68) is related to above topic. Do for Practice.

NOTE:

Theorem 1.6.3

Let A be a square matrix.

(a) If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.

(b) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

Theorem 1.6.4

Equivalent Statements

If A is an $n \times n$ matrix, then the following are equivalent.

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

Theorem 1.6.5

Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Determining Consistency by Elimination

If A is an invertible matrix, Theorem 1.6.2 completely solves this problem by asserting that for every m × 1 matrix b, the linear system Ax = b has the unique solution x $= A^{-1}$b.

If A is not square, or if A is square but not invertible, then Theorem 1.6.2 does not apply.

In these cases **b** must usually satisfy certain conditions in order for Ax = b to be consistent.

Task for Students

Refer Lecture & Example #3 & 4 (Pg # 66 & 67)

Question # 13 till 17 (Ex #1.6 ; Pg # 67 & 68) is related to above topic. Do for Practice.

Some Special Types of Matrices

Diagonal Matrices:

A square matrix in which all the entries off the main diagonal are zero is called a diagonal matrix. Here are some examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is **invertible** if and only if all of its **diagonal entries are nonzero**; in this case the inverse of D as mentioned above is:

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

Powers of diagonal matrices:

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Matrix products that involve diagonal matrix:

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

In words, *to multiply a matrix A on the left by a diagonal matrix D, multiply successive rows of A by the successive diagonal entries of D, and to multiply A on the right by D, multiply successive columns of A by the successive diagonal entries of D.*

Task for Students

Refer Lecture & Example #1 (Pg # 69)

Question # 3 till 10 (Ex #1.7 ; Pg # 74) is related to above topic. Do for Practice.