

CHAPTER 6: INNER PRODUCT SPACES

6.1 Inner Products

1.
 - (a) $\langle \mathbf{u}, \mathbf{v} \rangle = 2(1)(3) + 3(1)(2) = 12$
 - (b) $\langle k\mathbf{v}, \mathbf{w} \rangle = 2((3)(3))(0) + 3((3)(2))(-1) = -18$
 - (c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 2(1+3)(0) + 3(1+2)(-1) = -9$
 - (d) $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = [2(3)(3) + 3(2)(2)]^{1/2} = \sqrt{30}$
 - (e) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \langle (-2, -1), (-2, -1) \rangle^{1/2} = [2(-2)(-2) + 3(-1)(-1)]^{1/2} = \sqrt{11}$
 - (f) $\|\mathbf{u} - k\mathbf{v}\| = \langle (-8, -5), (-8, -5) \rangle^{1/2} = [2(-8)(-8) + 3(-5)(-5)]^{1/2} = \sqrt{203}$
2.
 - (a) $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}(1)(3) + 5(1)(2) = \frac{23}{2}$
 - (b) $\langle k\mathbf{v}, \mathbf{w} \rangle = \frac{1}{2}((3)(3))(0) + 5((3)(2))(-1) = -30$
 - (c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \frac{1}{2}(1+3)(0) + 5(1+2)(-1) = -15$
 - (d) $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = \left[\frac{1}{2}(3)(3) + 5(2)(2) \right]^{1/2} = \sqrt{\frac{49}{2}} = \frac{7}{\sqrt{2}}$
 - (e) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \langle (-2, -1), (-2, -1) \rangle^{1/2} = \left[\frac{1}{2}(-2)(-2) + 5(-1)(-1) \right]^{1/2} = \sqrt{7}$
 - (f) $\|\mathbf{u} - k\mathbf{v}\| = \langle (-8, -5), (-8, -5) \rangle^{1/2} = \left[\frac{1}{2}(-8)(-8) + 5(-5)(-5) \right]^{1/2} = \sqrt{157}$
3.
 - (a) $\langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \end{bmatrix} = 34$
 - (b) $\langle k\mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 24 \\ 15 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -39$
 - (c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 11 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -18$
 - (d) $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} 8 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \end{bmatrix} \right)^{1/2} = \sqrt{89}$
 - (e) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} -5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -3 \end{bmatrix} \right)^{1/2} = \sqrt{34}$

$$(f) \quad \|\mathbf{u} - k\mathbf{v}\| = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ -5 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ -5 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} -21 \\ -13 \end{bmatrix} \cdot \begin{bmatrix} -21 \\ -13 \end{bmatrix} \right)^{1/2} = \sqrt{610}$$

$$4. \quad (a) \quad \langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 7$$

$$(b) \quad \langle k\mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 9 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 12$$

$$(c) \quad \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5$$

$$(d) \quad \|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = \left[\left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right)^{1/2} = \sqrt{25} = 5$$

$$(e) \quad d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left[\left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} -2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \end{bmatrix} \right)^{1/2} = \sqrt{13}$$

$$(f) \quad \|\mathbf{u} - k\mathbf{v}\| = \left[\left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -8 \\ -5 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -8 \\ -5 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} -8 \\ -11 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ -11 \end{bmatrix} \right)^{1/2} = \sqrt{185}$$

$$5. \quad \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

$$6. \quad \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$$7. \quad \langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 26 \\ 6 \end{bmatrix} = -24$$

$$8. \quad \langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ 0 \end{bmatrix} = -42$$

$$9. \quad \text{If } \mathbf{u} = U \text{ and } \mathbf{v} = V \text{ then } \langle \mathbf{u}, \mathbf{v} \rangle = \text{tr}(U^T V) = \text{tr} \left(\begin{bmatrix} 1 & 13 \\ 10 & 2 \end{bmatrix} \right) = 3.$$

$$10. \quad \text{If } \mathbf{u} = U \text{ and } \mathbf{v} = V \text{ then } \langle \mathbf{u}, \mathbf{v} \rangle = \text{tr}(U^T V) = \text{tr} \left(\begin{bmatrix} 4 & -18 \\ 8 & 52 \end{bmatrix} \right) = 56.$$

$$11. \quad \langle \mathbf{p}, \mathbf{q} \rangle = (-2)(4) + (1)(0) + (3)(-7) = -29$$

$$12. \quad \langle \mathbf{p}, \mathbf{q} \rangle = (-5)(3) + (2)(2) + (1)(-4) = -15$$

$$13. \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$$14. \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

$$15. \begin{aligned} \langle \mathbf{p}, \mathbf{q} \rangle &= p(-2)q(-2) + p(-1)q(-1) + p(0)q(0) + p(1)q(1) \\ &= (-10)(5) + (-2)(2) + (0)(1) + (2)(2) = -50 \end{aligned}$$

$$16. \begin{aligned} \langle \mathbf{p}, \mathbf{q} \rangle &= p(-1)q(-1) + p(0)q(0) + p(1)q(1) + p(2)q(2) \\ &= (-2)(2) + (0)(1) + (2)(2) + (10)(5) = 50 \end{aligned}$$

$$17. \begin{aligned} \|\mathbf{u}\| &= \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = [2(-3)(-3) + 3(2)(2)]^{1/2} = \sqrt{30} \\ d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| = \langle (-4, -5), (-4, -5) \rangle^{1/2} = [2(-4)(-4) + 3(-5)(-5)]^{1/2} = \sqrt{107} \end{aligned}$$

$$18. \begin{aligned} \|\mathbf{u}\| &= \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = [2(-1)(-1) + 3(2)(2)]^{1/2} = \sqrt{14} \\ d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| = \langle (-3, -3), (-3, -3) \rangle^{1/2} = [2(-3)(-3) + 3(-3)(-3)]^{1/2} = 3\sqrt{5} \end{aligned}$$

$$19. \|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{14}; \quad d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \sqrt{(-6)^2 + 1^2 + 10^2} = \sqrt{137}$$

$$20. \|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \sqrt{(-5)^2 + 2^2 + 1^2} = \sqrt{30}; \quad d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \sqrt{(-8)^2 + 0^2 + 5^2} = \sqrt{89}$$

$$21. \text{ If } \mathbf{u} = U \text{ and } \mathbf{v} = V \text{ then } \|U\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{\text{tr}(U^T U)} = \sqrt{\text{tr}\left(\begin{bmatrix} 25 & 26 \\ 26 & 68 \end{bmatrix}\right)} = \sqrt{93} \text{ and}$$

$$d(U, V) = \|U - V\| = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle^{1/2} = \sqrt{\text{tr}((U - V)^T (U - V))} = \sqrt{\text{tr}\left(\begin{bmatrix} 25 & 1 \\ 1 & 74 \end{bmatrix}\right)} = \sqrt{99} = 3\sqrt{11}.$$

$$22. \text{ If } \mathbf{u} = U \text{ and } \mathbf{v} = V \text{ then } \|U\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{\text{tr}(U^T U)} = \sqrt{\text{tr}\left(\begin{bmatrix} 10 & -13 \\ -13 & 29 \end{bmatrix}\right)} = \sqrt{39} \text{ and}$$

$$d(U, V) = \|U - V\| = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle^{1/2} = \sqrt{\text{tr}((U - V)^T (U - V))} = \sqrt{\text{tr}\left(\begin{bmatrix} 18 & 21 \\ 21 & 25 \end{bmatrix}\right)} = \sqrt{43}.$$

$$23. \begin{aligned} \|\mathbf{p}\| &= \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \sqrt{[p(-2)]^2 + [p(-1)]^2 + [p(0)]^2 + [p(1)]^2} = \sqrt{(-10)^2 + (-2)^2 + 0^2 + 2^2} = 6\sqrt{3} \\ d(\mathbf{p}, \mathbf{q}) &= \|\mathbf{p} - \mathbf{q}\| = \sqrt{[p(-2) - q(-2)]^2 + [p(-1) - q(-1)]^2 + [p(0) - q(0)]^2 + [p(1) - q(1)]^2} \\ &= \sqrt{(-15)^2 + (-4)^2 + (-1)^2 + 0^2} = 11\sqrt{2} \end{aligned}$$

$$24. \quad \|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \sqrt{[p(-1)]^2 + [p(0)]^2 + [p(1)]^2 + [p(2)]^2} = \sqrt{(-2)^2 + 0^2 + 2^2 + 10^2} = 6\sqrt{3}$$

$$d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \sqrt{[p(-1) - q(-1)]^2 + [p(0) - q(0)]^2 + [p(1) - q(1)]^2 + [p(2) - q(2)]^2} \\ = \sqrt{(-4)^2 + (-1)^2 + 0^2 + 5^2} = \sqrt{42}$$

$$25. \quad \|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \left[\left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} -4 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 7 \end{bmatrix} \right)^{1/2} = \sqrt{65}$$

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left[\left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} -12 \\ -24 \end{bmatrix} \cdot \begin{bmatrix} -12 \\ -24 \end{bmatrix} \right)^{1/2} = 12\sqrt{5}$$

$$26. \quad \|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \left[\left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} 3 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right)^{1/2} = \sqrt{58}$$

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left[\left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} -9 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} -9 \\ -6 \end{bmatrix} \right)^{1/2} = 3\sqrt{13}$$

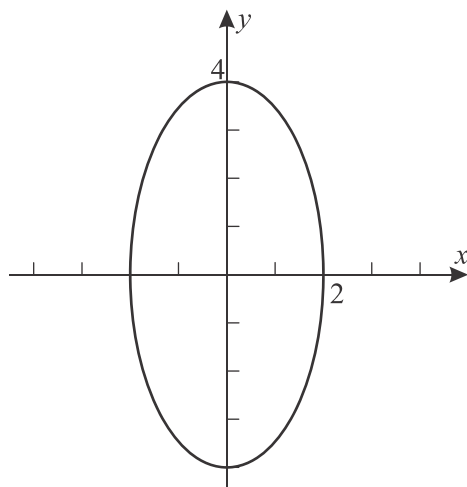
$$27. \quad (a) \quad \langle 2\mathbf{v} - \mathbf{w}, 3\mathbf{u} + 2\mathbf{w} \rangle = \langle 2\mathbf{v}, 3\mathbf{u} + 2\mathbf{w} \rangle - \langle \mathbf{w}, 3\mathbf{u} + 2\mathbf{w} \rangle = \langle 2\mathbf{v}, 3\mathbf{u} \rangle + \langle 2\mathbf{v}, 2\mathbf{w} \rangle - \langle \mathbf{w}, 3\mathbf{u} \rangle - \langle \mathbf{w}, 2\mathbf{w} \rangle \\ = 6\langle \mathbf{v}, \mathbf{u} \rangle + 4\langle \mathbf{v}, \mathbf{w} \rangle - 3\langle \mathbf{w}, \mathbf{u} \rangle - 2\langle \mathbf{w}, \mathbf{w} \rangle = 6\langle \mathbf{u}, \mathbf{v} \rangle + 4\langle \mathbf{v}, \mathbf{w} \rangle - 3\langle \mathbf{u}, \mathbf{w} \rangle - 2\|\mathbf{w}\|^2 \\ = 6(2) + 4(-6) - 3(-3) - 2(49) = -101$$

$$(b) \quad \|\mathbf{u} + \mathbf{v}\| = \sqrt{\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle} = \sqrt{\langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle} = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle} \\ = \sqrt{\|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2} = \sqrt{1 + 2(2) + 4} = 3$$

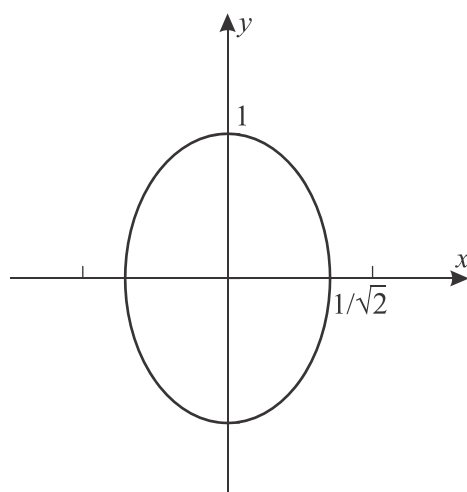
$$28. \quad (a) \quad \langle \mathbf{u} - \mathbf{v} - 2\mathbf{w}, 4\mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, 4\mathbf{u} + \mathbf{v} \rangle - \langle \mathbf{v}, 4\mathbf{u} + \mathbf{v} \rangle - \langle 2\mathbf{w}, 4\mathbf{u} + \mathbf{v} \rangle \\ = \langle \mathbf{u}, 4\mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, 4\mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle - \langle 2\mathbf{w}, 4\mathbf{u} \rangle - \langle 2\mathbf{w}, \mathbf{v} \rangle \\ = 4\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle - 4\langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle - 8\langle \mathbf{w}, \mathbf{u} \rangle - 2\langle \mathbf{w}, \mathbf{v} \rangle \\ = 4\|\mathbf{u}\|^2 - 3\langle \mathbf{u}, \mathbf{v} \rangle - \|\mathbf{v}\|^2 - 8\langle \mathbf{u}, \mathbf{w} \rangle - 2\langle \mathbf{v}, \mathbf{w} \rangle = 4 - 3(2) - 4 - 8(-3) - 2(-6) = 30$$

$$(b) \quad \|2\mathbf{w} - \mathbf{v}\| = \sqrt{\langle 2\mathbf{w} - \mathbf{v}, 2\mathbf{w} - \mathbf{v} \rangle} = \sqrt{\langle 2\mathbf{w}, 2\mathbf{w} - \mathbf{v} \rangle - \langle \mathbf{v}, 2\mathbf{w} - \mathbf{v} \rangle} \\ = \sqrt{\langle 2\mathbf{w}, 2\mathbf{w} \rangle - \langle 2\mathbf{w}, \mathbf{v} \rangle - \langle \mathbf{v}, 2\mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{4\|\mathbf{w}\|^2 - 4\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{v}\|^2} = \sqrt{4(49) - 4(-6) + 4} \\ = \sqrt{224} = 4\sqrt{14}$$

29. If $\mathbf{u} = (x, y)$ then $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{\frac{1}{4}x^2 + \frac{1}{16}y^2}$, so the equation of the unit circle is $\frac{x^2}{4} + \frac{y^2}{16} = 1$.



30. If $\mathbf{u} = (x, y)$ then $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{2x^2 + y^2}$, so the equation of the unit circle is $2x^2 + y^2 = 1$, which can be rewritten as $\frac{x^2}{1/2} + \frac{y^2}{1} = 1$.



31. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{9}u_1v_1 + u_2v_2$ (see Example 3)
32. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{16}{9}u_1v_1 + u_2v_2$ (see Example 3)
33. Axiom 2 does not hold, e.g., with $\mathbf{u} = \mathbf{v} = \mathbf{w} = (1, 0, 0)$ we have $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 4$ but $\langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle = 1 + 1 = 2$; Axiom 3 does not hold either, e.g., with $\mathbf{u} = \mathbf{v} = (1, 0, 0)$ and $k = 2$, $\langle k\mathbf{u}, \mathbf{v} \rangle = 4$ does not equal $k\langle \mathbf{u}, \mathbf{v} \rangle = 2$; This is not an inner product on R^3 .
34. Axiom 4 does not hold, e.g., $\langle (0, 1, 0), (0, 1, 0) \rangle = -1 < 0$; this is not an inner product on R^3 .

35. By Definition 1, Definition 2, and Theorem 6.1.2, we have

$$\begin{aligned}
 \langle 2\mathbf{v} - 4\mathbf{u}, \mathbf{u} - 3\mathbf{v} \rangle &= \langle 2\mathbf{v} - 4\mathbf{u}, \mathbf{u} \rangle - \langle 2\mathbf{v} - 4\mathbf{u}, 3\mathbf{v} \rangle \\
 &= \langle 2\mathbf{v}, \mathbf{u} \rangle - \langle 4\mathbf{u}, \mathbf{u} \rangle - \langle 2\mathbf{v}, 3\mathbf{v} \rangle + \langle 4\mathbf{u}, 3\mathbf{v} \rangle \\
 &= 2\langle \mathbf{v}, \mathbf{u} \rangle - 4\langle \mathbf{u}, \mathbf{u} \rangle - 6\langle \mathbf{v}, \mathbf{v} \rangle + 12\langle \mathbf{u}, \mathbf{v} \rangle \\
 &= 2\langle \mathbf{u}, \mathbf{v} \rangle - 4\langle \mathbf{u}, \mathbf{u} \rangle - 6\langle \mathbf{v}, \mathbf{v} \rangle + 12\langle \mathbf{u}, \mathbf{v} \rangle \\
 &= 14\langle \mathbf{u}, \mathbf{v} \rangle - 4\|\mathbf{u}\|^2 - 6\|\mathbf{v}\|^2.
 \end{aligned}$$

36. By Definition 1, Definition 2, and Theorem 6.1.2, we have

$$\begin{aligned}
 \langle 5\mathbf{u} + 6\mathbf{v}, 4\mathbf{v} - 3\mathbf{u} \rangle &= \langle 5\mathbf{u} + 6\mathbf{v}, 4\mathbf{v} \rangle - \langle 5\mathbf{u} + 6\mathbf{v}, 3\mathbf{u} \rangle \\
 &= \langle 5\mathbf{u}, 4\mathbf{v} \rangle + \langle 6\mathbf{v}, 4\mathbf{v} \rangle - \langle 5\mathbf{u}, 3\mathbf{u} \rangle - \langle 6\mathbf{v}, 3\mathbf{u} \rangle \\
 &= 20\langle \mathbf{u}, \mathbf{v} \rangle + 24\langle \mathbf{v}, \mathbf{v} \rangle - 15\langle \mathbf{u}, \mathbf{u} \rangle - 18\langle \mathbf{v}, \mathbf{u} \rangle \\
 &= 20\langle \mathbf{u}, \mathbf{v} \rangle + 24\langle \mathbf{v}, \mathbf{v} \rangle - 15\langle \mathbf{u}, \mathbf{u} \rangle - 18\langle \mathbf{u}, \mathbf{v} \rangle \\
 &= 2\langle \mathbf{u}, \mathbf{v} \rangle - 15\|\mathbf{u}\|^2 + 24\|\mathbf{v}\|^2.
 \end{aligned}$$

37. (a) $\langle \mathbf{p}, \mathbf{q} \rangle = \left(\int_{-1}^1 x^2 dx \right) = \left(\frac{x^3}{3} \right)_{-1}^1 = \frac{2}{3}$

(b) $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \left(\int_{-1}^1 (1 - x^2)^2 dx \right)^{1/2} = \left(\left(x - \frac{2x^3}{3} + \frac{x^5}{5} \right)_{-1}^1 \right)^{1/2} = \sqrt{\frac{16}{15}} = \frac{4}{\sqrt{15}}$

(c) $\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \left(\int_{-1}^1 1 dx \right)^{1/2} = \left(x \right)_{-1}^1 = \sqrt{2}$

(d) $\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2} = \left(\int_{-1}^1 x^4 dx \right)^{1/2} = \left(\frac{x^5}{5} \right)_{-1}^1 = \sqrt{\frac{2}{5}}$

38. (a) $\langle \mathbf{p}, \mathbf{q} \rangle = \left(\int_{-1}^1 2x^3(1 - x^3) dx \right) = \left(\left(\frac{x^4}{2} - \frac{2x^7}{7} \right)_{-1}^1 \right) = \frac{-4}{7}$

(b) $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \left(\int_{-1}^1 (3x^3 - 1)^2 dx \right)^{1/2} = \left(\left(\frac{9x^7}{7} - \frac{3x^4}{2} + x \right)_{-1}^1 \right)^{1/2} = \sqrt{\frac{32}{7}} = \frac{4\sqrt{2}}{\sqrt{7}}$

(c) $\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \left(\int_{-1}^1 (2x^3)^2 dx \right)^{1/2} = \left(\frac{4x^7}{7} \right)_{-1}^1 = \sqrt{\frac{8}{7}} = \frac{2\sqrt{2}}{\sqrt{7}}$

(d) $\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2} = \left(\int_{-1}^1 (1 - x^3)^2 dx \right)^{1/2} = \left(\left(x - \frac{x^4}{2} + \frac{x^7}{7} \right)_{-1}^1 \right)^{1/2} = \sqrt{\frac{16}{7}} = \frac{4}{\sqrt{7}}$

39. $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 \cos 2\pi x \sin 2\pi x dx = \frac{1}{2\pi} \frac{(\sin 2\pi x)^2}{2} \Big|_0^1 = 0$ (substituted $u = \sin 2\pi x$)

40. $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 x e^x dx = \left(x e^x - e^x \right) \Big|_0^1 = 1$ (used integration by parts with $u = x$ and $dv = e^x dx$)

41. Part (a) follows directly from Definition 2 and Axiom 4 of Definition 1.

To prove part (b), write

$$\|k\mathbf{v}\| \stackrel{\text{Def.2}}{=} \sqrt{(k\mathbf{v}, k\mathbf{v})} \stackrel{\text{Axiom 3}}{=} \sqrt{k(\mathbf{v}, k\mathbf{v})} \stackrel{\text{Axiom 1}}{=} \sqrt{k(k\mathbf{v}, \mathbf{v})} \stackrel{\text{Axiom 3}}{=} \sqrt{k^2(\mathbf{v}, \mathbf{v})} \stackrel{\text{Def.2}}{=} |k| \|\mathbf{v}\|$$

42. Part (c) follows from Definition 2 and part (b) of Theorem 6.1.1, since

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(-1)(\mathbf{v} - \mathbf{u})\| = |-1| \|\mathbf{v} - \mathbf{u}\| = \|\mathbf{v} - \mathbf{u}\| = d(\mathbf{v}, \mathbf{u})$$

Part (d) follows from Definition 2 and Axiom 4 of Definition 1.

43. (b) k_1 and k_2 must both be positive in order for $\langle \mathbf{u}, \mathbf{v} \rangle$ to satisfy the positivity axiom. (Refer to the discussion following Theorem 6.1.1.)

44. By using Definition 2 and Axioms 1, 2, and 3 of Definition 1, we have

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= \left\langle \frac{1}{2}(\mathbf{u} + \mathbf{v}) + \frac{1}{2}(\mathbf{u} - \mathbf{v}), \frac{1}{2}(\mathbf{u} + \mathbf{v}) - \frac{1}{2}(\mathbf{u} - \mathbf{v}) \right\rangle \\ &= \frac{1}{4} \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle + \frac{1}{4} \langle \mathbf{u} - \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle - \frac{1}{4} \langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle - \frac{1}{4} \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \\ &= \frac{1}{4} \|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4} \|\mathbf{u} - \mathbf{v}\|^2. \end{aligned}$$

45. By using Definition 2 and Axioms 1, 2, and 3 of Definition 1, we have

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= 2\langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{v}, \mathbf{v} \rangle \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2. \end{aligned}$$

48. (b) $T(1, 1, 1) = \langle (1, 1, 1), (1, 0, 2) \rangle = (1)(1) + (1)(0) + (1)(2) = 3$

(c) $T(x + x^2) = \langle x + x^2, 1 + x \rangle = (0)(1) + (1)(1) + (1)(0) = 1$

(d) $T(x + x^2) = \langle x + x^2, 1 + x \rangle = (1 + 1^2)(1 + 1) + (0 + 0^2)(1 + 0) + (-1 + (-1)^2)(1 - 1) = 4$

True-False Exercises

(a) True. The dot product is the special case of the weighted inner product with all the weights equal to 1.

(b) False. For example, if $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} = (1, 1)$, and $\mathbf{v} = (-2, 1)$ then $\langle \mathbf{u}, \mathbf{v} \rangle = -1$.

(c) True. This follows from Axioms 1 and 2 of Definition 1 since $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle$.

(d) True. This follows from Axiom 3 of Definition 1 as well as part (e) of Theorem 6.1.2.

(e) False. For example, if $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} = (1, 1)$, and $\mathbf{v} = (-1, 1)$ then $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ even though both vectors are nonzero.

(f) True. By Definition 2, $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle$ so by Axiom 4 of Definition 1, $\|\mathbf{v}\|^2 = 0$ implies $\mathbf{v} = \mathbf{0}$.

(g) False. A must be invertible; otherwise $A\mathbf{v} = \mathbf{0}$ has nontrivial solutions $\mathbf{v} \neq \mathbf{0}$ even though $\langle \mathbf{v}, \mathbf{v} \rangle = A\mathbf{v} \cdot A\mathbf{v} = \mathbf{0}$ which would violate Axiom 4 of Definition 1.

6.2 Angle and Orthogonality in Inner Product Spaces

1. (a) $\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(1)(2) + (-3)(4)}{\sqrt{1^2 + (-3)^2} \sqrt{2^2 + 4^2}} = -\frac{10}{\sqrt{10} \sqrt{20}} = -\frac{10}{\sqrt{200}} = -\frac{10}{10\sqrt{2}} = -\frac{1}{\sqrt{2}}$
 (b) $\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(-1)(2) + (5)(4) + (2)(-9)}{\sqrt{(-1)^2 + 5^2 + 2^2} \sqrt{2^2 + 4^2 + (-9)^2}} = 0$
 (c) $\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(1)(-3) + (0)(-3) + (1)(-3) + (0)(-3)}{\sqrt{1^2 + 0^2 + 1^2 + 0^2} \sqrt{(-3)^2 + (-3)^2 + (-3)^2 + (-3)^2}} = -\frac{6}{\sqrt{2} \sqrt{36}} = -\frac{1}{\sqrt{2}}$
2. (a) $\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(-1)(3) + (0)(8)}{\sqrt{(-1)^2 + 0^2} \sqrt{3^2 + 8^2}} = -\frac{3}{\sqrt{73}}$
 (b) $\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(4)(1) + (1)(0) + (8)(-3)}{\sqrt{4^2 + 1^2 + 8^2} \sqrt{1^2 + 0^2 + (-3)^2}} = -\frac{20}{\sqrt{81} \sqrt{10}} = -\frac{20}{9\sqrt{10}}$
 (c) $\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(2)(4) + (1)(0) + (7)(0) + (-1)(0)}{\sqrt{2^2 + 1^2 + 7^2 + (-1)^2} \sqrt{4^2 + 0^2 + 0^2 + 0^2}} = \frac{8}{4\sqrt{55}} = \frac{2}{\sqrt{55}}$
3. $\cos \theta = \frac{\langle \mathbf{p}, \mathbf{q} \rangle}{\|\mathbf{p}\| \|\mathbf{q}\|} = \frac{(-1)(2) + (5)(4) + (2)(-9)}{\sqrt{(-1)^2 + 5^2 + 2^2} \sqrt{2^2 + 4^2 + (-9)^2}} = 0$
4. $\cos \theta = \frac{\langle \mathbf{p}, \mathbf{q} \rangle}{\|\mathbf{p}\| \|\mathbf{q}\|} = \frac{(0)(7) + (1)(3) + (-1)(3)}{\sqrt{0^2 + 1^2 + (-1)^2} \sqrt{7^2 + 3^2 + 3^2}} = 0$
5. $\cos \theta = \frac{\langle U, V \rangle}{\|U\| \|V\|} = \frac{\text{tr}(U^T V)}{\sqrt{\text{tr}(U^T U)} \sqrt{\text{tr}(V^T V)}} = \frac{(2)(3) + (6)(2) + (1)(1) + (-3)(0)}{\sqrt{2^2 + 6^2 + 1^2 + (-3)^2} \sqrt{3^2 + 2^2 + 1^2 + 0^2}} = \frac{19}{\sqrt{50} \sqrt{14}} = \frac{19}{10\sqrt{7}}$
6. $\cos \theta = \frac{\langle U, V \rangle}{\|U\| \|V\|} = \frac{\text{tr}(U^T V)}{\sqrt{\text{tr}(U^T U)} \sqrt{\text{tr}(V^T V)}} = \frac{(2)(-3) + (4)(1) + (-1)(4) + (3)(2)}{\sqrt{2^2 + 4^2 + (-1)^2 + 3^2} \sqrt{(-3)^2 + 1^2 + 4^2 + 2^2}} = 0$
7. (a) orthogonal: $\langle \mathbf{u}, \mathbf{v} \rangle = -4 + 6 - 2 = 0$
 (b) not orthogonal: $\langle \mathbf{u}, \mathbf{v} \rangle = -2 - 2 - 2 = -6 \neq 0$
 (c) orthogonal: $\langle \mathbf{u}, \mathbf{v} \rangle = (a)(-b) + (b)(a) = 0$
8. (a) orthogonal: $\langle \mathbf{u}, \mathbf{v} \rangle = 0 + 0 + 0 = 0$
 (b) not orthogonal: $\langle \mathbf{u}, \mathbf{v} \rangle = -8 + 6 + 20 + 9 = 27 \neq 0$
 (c) orthogonal: $\langle \mathbf{u}, \mathbf{v} \rangle = -ac + 0 + ac = 0$
9. $\langle \mathbf{p}, \mathbf{q} \rangle = (-1)(0) + (-1)(2) + (2)(1) = 0$
10. $\langle \mathbf{p}, \mathbf{q} \rangle = (2)(4) + (-3)(2) + (1)(-2) = 0$
11. $\langle U, V \rangle = (2)(-3) + (1)(0) + (-1)(0) + (3)(2) = 0$
12. $\langle U, V \rangle = (5)(1) + (-1)(3) + (2)(-1) + (-2)(0) = 0$

- 13.** The vectors are not orthogonal with respect to the Euclidean inner product since $\langle \mathbf{u}, \mathbf{v} \rangle = (1)(2) + (3)(-1) = -1 \neq 0$. Using the weighted inner product instead yields $\langle \mathbf{u}, \mathbf{v} \rangle = 2(1)(2) + k(3)(-1) = 4 - 3k$, so the vectors are orthogonal with respect to this inner product if $k = \frac{4}{3}$.
- 14.** The vectors are not orthogonal with respect to the Euclidean inner product since $\langle \mathbf{u}, \mathbf{v} \rangle = (2)(0) + (-4)(3) = -12 \neq 0$. Using the weighted inner product instead yields $\langle \mathbf{u}, \mathbf{v} \rangle = 2(2)(0) + k(-4)(3) = -12k$. This would equal 0 if $k = 0$, however, the corresponding formula $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1$ does not represent an inner product since it violates Axiom 4. Consequently, no k values can be found for which the vectors are orthogonal.
- 15.** The orthogonality of the two vectors implies $(w_1)(1)(2) + (w_2)(2)(-4) = 0$. The weights must be positive numbers such that $w_1 = 4w_2$.
- 16.** Begin by forming a matrix A whose rows are the given vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} : $A = \begin{bmatrix} 2 & 1 & -4 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{bmatrix}$. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & \frac{34}{11} \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & \frac{6}{11} \end{bmatrix}$ therefore the general solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is $x_1 = -\frac{34}{11}t$, $x_2 = 4t$, $x_3 = -\frac{6}{11}t$, $x_4 = t$, so that all solution vectors are scalar multiples of $(-34, 44, -6, 11)$. All solution vectors of the homogeneous system are orthogonal to every row vector of the coefficient matrix (see Example 6 in Section 6.2).
The magnitude of the vector $(-34, 44, -6, 11)$ is $\sqrt{(-34)^2 + 44^2 + (-6)^2 + 11^2} = 57$.
We conclude that the two unit vectors that are orthogonal to all three of the vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} are $\frac{1}{57}(-34, 44, -6, 11) = (-\frac{34}{57}, \frac{44}{57}, -\frac{6}{57}, \frac{11}{57})$ and $-\frac{1}{57}(-34, 44, -6, 11) = (\frac{34}{57}, -\frac{44}{57}, \frac{6}{57}, -\frac{11}{57})$.
- 17.** Orthogonality of \mathbf{p}_1 and \mathbf{p}_3 implies $\langle \mathbf{p}_1, \mathbf{p}_3 \rangle = (2)(1) + (k)(2) + (6)(3) = 2k + 20 = 0$ so $k = -10$. Likewise, orthogonality of \mathbf{p}_2 and \mathbf{p}_3 implies $\langle \mathbf{p}_2, \mathbf{p}_3 \rangle = (l)(1) + (5)(2) + (3)(3) = l + 19$ so $l = -19$.
Substituting the values of k and l obtained above yields the polynomials $\mathbf{p}_1 = 2 - 10x + 6x^2$ and $\mathbf{p}_2 = -19 + 5x + 3x^2$ which are not orthogonal since $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = (2)(-19) + (-10)(5) + (6)(3) = -70 \neq 0$. We conclude that no scalars k and l exist that make the three vectors mutually orthogonal.
- 18.** $\langle \mathbf{u}, \mathbf{v} \rangle = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 0$
- 19.** $\langle \mathbf{p}, \mathbf{q} \rangle = p(-2)q(-2) + p(0)q(0) + p(2)q(2) = (-2)(4) + (0)(0) + (2)(4) = 0$
- 20.** For A to be a in the subspace of M_{22} spanned by U and V , there must exist scalars a and b such that

$$a \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} + b \begin{bmatrix} 4 & 0 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Equating corresponding entries in the second column on both sides yields $a = -1$ and $b = 1$. However, for these values, the remaining entries on both sides do not equal (e.g., $(-1)(1) + (1)(4) \neq -1$). We conclude that A is not in the subspace of M_{22} spanned by U and V .

21. $|\langle \mathbf{u}, \mathbf{v} \rangle| = |2(1)(2) + 3(0)(1) + (3)(-1)| = |1| = 1;$

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{2(1)(1) + 3(0)(0) + (3)(3)} = \sqrt{11};$$

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{2(2)(2) + 3(1)(1) + (-1)(-1)} = \sqrt{12};$$

since $\|\mathbf{u}\| \|\mathbf{v}\| = \sqrt{132} \geq 1 = |\langle \mathbf{u}, \mathbf{v} \rangle|$, we conclude that the Cauchy-Schwarz inequality holds.

22. $|\langle U, V \rangle| = |(-1)(1) + (2)(0) + (6)(3) + (1)(3)| = |20| = 20;$

$$\|U\| = \sqrt{\langle U, U \rangle} = \sqrt{(-1)(-1) + (2)(2) + (6)(6) + (1)(1)} = \sqrt{42};$$

$$\|V\| = \sqrt{\langle V, V \rangle} = \sqrt{(1)(1) + (0)(0) + (3)(3) + (3)(3)} = \sqrt{19};$$

since $\|U\| \|V\| = \sqrt{798} \geq \sqrt{400} = 20 = |\langle U, V \rangle|$, we conclude that the Cauchy-Schwarz inequality holds.

23. $|\langle \mathbf{p}, \mathbf{q} \rangle| = |(-1)(2) + (2)(0) + (1)(-4)| = |-6| = 6;$

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{(-1)(-1) + (2)(2) + (1)(1)} = \sqrt{6};$$

$$\|\mathbf{q}\| = \sqrt{\langle \mathbf{q}, \mathbf{q} \rangle} = \sqrt{(2)(2) + (0)(0) + (-4)(-4)} = \sqrt{20};$$

since $\|\mathbf{p}\| \|\mathbf{q}\| = \sqrt{120} \geq \sqrt{36} = 6 = |\langle \mathbf{p}, \mathbf{q} \rangle|$, we conclude that the Cauchy-Schwarz inequality holds.

24. $|\langle \mathbf{u}, \mathbf{v} \rangle| = \left| \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \right| = \left| \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right| = |3| = 3;$

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)} = \sqrt{\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}} = \sqrt{13};$$

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)} = \sqrt{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \sqrt{1} = 1;$$

since $\|\mathbf{u}\| \|\mathbf{v}\| = \sqrt{13} \geq \sqrt{9} = 3 = |\langle \mathbf{u}, \mathbf{v} \rangle|$, we conclude that the Cauchy-Schwarz inequality holds.

25. By inspection, $\langle \mathbf{u}, \mathbf{w}_1 \rangle = -2 \neq 0$. Since \mathbf{u} is not orthogonal to \mathbf{w}_1 , it is not orthogonal to the subspace.

26. We have $\langle \mathbf{p}, \mathbf{w}_1 \rangle = (-1)(2) + (-1)(0) + (2)(-1) + (4)(1) = 0$ and

$$\langle \mathbf{p}, \mathbf{w}_2 \rangle = (-1)(0) + (-1)(4) + (2)(-2) + (4)(2) = 0 \text{ therefore for all scalars } a \text{ and } b$$

$$\langle \mathbf{p}, a\mathbf{w}_1 + b\mathbf{w}_2 \rangle = a\langle \mathbf{p}, \mathbf{w}_1 \rangle + b\langle \mathbf{p}, \mathbf{w}_2 \rangle = 0. \text{ We conclude that } \mathbf{p} \text{ is orthogonal to } \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}.$$

27. Begin by forming a matrix A whose rows are the given vectors:

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \text{ has the reduced row echelon form } \begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ The general solution of the}$$

homogeneous system $A\mathbf{x} = \mathbf{0}$ is $x_1 = -s + \frac{2}{7}t$, $x_2 = -s - \frac{4}{7}t$, $x_3 = s$, $x_4 = t$, therefore

$$\mathbf{x} = s(-1, -1, 1, 0) + t\left(\frac{2}{7}, -\frac{4}{7}, 0, 1\right).$$

A basis for the orthogonal complement is formed by vectors $(-1, -1, 1, 0)$ and $(\frac{2}{7}, -\frac{4}{7}, 0, 1)$.

28. Begin by forming a matrix A whose rows are the given vectors:

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} \text{ has the reduced row echelon form } \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ The general solution of}$$

the homogeneous system $A\mathbf{x} = \mathbf{0}$ is $x_1 = -r - 2s - t$, $x_2 = -r - s - 2t$, $x_3 = r$, $x_4 = s$, $x_5 = t$, therefore

$$\mathbf{x} = r(-1, -1, 1, 0, 0) + s(-2, -1, 0, 1, 0) + t(-1, -2, 0, 0, 1).$$

A basis for the orthogonal complement is formed by vectors $(-1, -1, 1, 0, 0)$, $(-2, -1, 0, 1, 0)$, and $(-1, -2, 0, 0, 1)$.

29. (a) Every vector in W has form $(x, y) = (x, 2x)$, i.e., $W = \text{span}\{(1, 2)\}$. By inspection, all vectors in R^2 orthogonal to $(1, 2)$ are scalar multiples of the vector $(2, -1)$. Eliminating t from

$$(x, y) = t(2, -1) = (2t, -t) \text{ we obtain } x = 2(-y), \text{ i.e. } W^\perp \text{ can be represented using an equation}$$

$$y = -\frac{1}{2}x.$$

(An alternate method of solving this exercise is to follow the procedure of Example 6: letting

$$A = \begin{bmatrix} 1 & 2 \end{bmatrix}, \text{ the general solution of } A \begin{bmatrix} x \\ y \end{bmatrix} = 0 \text{ is } x = -2t, y = t. \text{ Eliminating } t \text{ yields } y = -\frac{1}{2}x.)$$

- (b) W^\perp will have dimension 1. A normal to the plane is $\mathbf{u} = (1, -2, -3)$, so W^\perp will consist of all scalar multiples of \mathbf{u} or $t\mathbf{u} = (t, -2t, -3t)$ so parametric equations for W^\perp are $x = t$,

$$y = -2t, z = -3t.$$

30. (a) Every vector in W has form $(0, y, 0)$, i.e. $W = \text{span}\{(0, 1, 0)\}$. By inspection, all vectors in R^3 orthogonal to $(0, 1, 0)$ are linear combinations of $(1, 0, 0)$ and $(0, 0, 1)$ i.e. W^\perp is the xz plane.

- (b) Every vector in W has form $(0, y, z)$, i.e. $W = \text{span}\{(0, 1, 0), (0, 0, 1)\}$. By inspection, all vectors in R^3 orthogonal to W have form $(1, 0, 0)$ i.e. W^\perp is the x axis.

31. (a) $\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 x^3 dx = \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4}$

$$(b) \quad \|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{\frac{1}{2}} = \left(\int_0^1 x^2 \, dx \right)^{\frac{1}{2}} = \left(\frac{x^3}{3} \Big|_0^1 \right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2} = \left(\int_0^1 x^4 \, dx \right)^{1/2} = \left(\frac{x^5}{5} \Big|_0^1 \right)^{1/2} = \frac{1}{\sqrt{5}}$$

$$32. (a) \quad \text{Using the results obtained in Exercise 31 we have } \cos \theta = \frac{\langle \mathbf{p}, \mathbf{q} \rangle}{\|\mathbf{p}\| \|\mathbf{q}\|} = \frac{\frac{1}{4}}{\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{5}}} = \frac{\sqrt{15}}{4}.$$

$$(b) \quad d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \left(\int_0^1 (x - x^2)^2 \, dx \right)^{1/2} = \left(\left(\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right) \Big|_0^1 \right)^{1/2} = \sqrt{\frac{1}{30}}$$

$$33. (a) \quad \langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 (x^2 - x)(x + 1) \, dx = \int_{-1}^1 (x^3 - x) \, dx = \left(\frac{x^4}{4} - \frac{x^2}{2} \right) \Big|_{-1}^1 = 0$$

$$(b) \quad \|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{\frac{1}{2}} = \left(\int_{-1}^1 (x^2 - x)^2 \, dx \right)^{\frac{1}{2}} = \left(\left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} \right) \Big|_{-1}^1 \right)^{\frac{1}{2}} = \frac{4}{\sqrt{15}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2} = \left(\int_{-1}^1 (x + 1)^2 \, dx \right)^{1/2} = \left(\left(\frac{x^3}{3} + x^2 + x \right) \Big|_{-1}^1 \right)^{1/2} = \sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}}$$

$$34. (a) \quad \text{Using the results obtained in Exercise 33 we have } \cos \theta = \frac{\langle \mathbf{p}, \mathbf{q} \rangle}{\|\mathbf{p}\| \|\mathbf{q}\|} = 0.$$

$$(b) \quad d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \left(\int_{-1}^1 (x^2 - 2x - 1)^2 \, dx \right)^{1/2} = \left(\left(\frac{x^5}{5} - x^4 + \frac{2x^3}{3} + 2x^2 + x \right) \Big|_{-1}^1 \right)^{1/2} = 2\sqrt{\frac{14}{15}}$$

$$35. (a) \quad \langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 \left(\frac{1}{2} - x \right) \, dx = \left(\frac{1}{2}x - \frac{x^2}{2} \right) \Big|_0^1 = 0$$

$$(b) \quad \|\mathbf{p} + \mathbf{q}\|^2 = \int_0^1 \left(\frac{3}{2} - x \right)^2 \, dx = \left(\frac{9}{4}x - \frac{3x^2}{2} + \frac{x^3}{3} \right) \Big|_0^1 = \frac{13}{12}; \quad \|\mathbf{p}\|^2 = \int_0^1 (1)^2 \, dx = x \Big|_0^1 = 1;$$

$$\mathbf{q}^2 = \int_0^1 \left(\frac{1}{2} - x \right)^2 \, dx = \left(\frac{x}{4} - \frac{x^2}{2} + \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{12}; \text{ we conclude that } \|\mathbf{p} + \mathbf{q}\|^2 = \frac{13}{12} = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2.$$

$$36. (a) \quad \langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 (x^3 - x) \, dx = \left(\frac{x^4}{4} - \frac{x^2}{2} \right) \Big|_{-1}^1 = 0$$

$$(b) \quad \|\mathbf{p} + \mathbf{q}\|^2 = \int_{-1}^1 (x^2 + x - 1)^2 \, dx = \left(\frac{x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} - x^2 + x \right) \Big|_{-1}^1 = \frac{26}{15};$$

$$\|\mathbf{p}\|^2 = \int_{-1}^1 (x)^2 \, dx = \frac{x^3}{3} \Big|_{-1}^1 = \frac{2}{3}; \quad \|\mathbf{q}\|^2 = \int_{-1}^1 (x^2 - 1)^2 \, dx = \left(\frac{x^5}{5} - \frac{2x^3}{3} + x \right) \Big|_{-1}^1 = \frac{16}{15};$$

$$\text{we conclude that } \|\mathbf{p} + \mathbf{q}\|^2 = \frac{26}{15} = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2.$$

$$37. \quad \|\mathbf{u} - \mathbf{v}\|^2 = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} - \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle = 1 - 0 - 0 + 1 = 2 \text{ therefore } \|\mathbf{u} - \mathbf{v}\| = \sqrt{2}.$$

$$38. \quad \text{Assuming } \langle \mathbf{w}, \mathbf{u}_1 \rangle = \langle \mathbf{w}, \mathbf{u}_2 \rangle = 0, \text{ it follows that}$$

$$\langle \mathbf{w}, k_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \rangle = \langle \mathbf{w}, k_1 \mathbf{u}_1 \rangle + \langle \mathbf{w}, k_2 \mathbf{u}_2 \rangle = k_1 \langle \mathbf{w}, \mathbf{u}_1 \rangle + k_2 \langle \mathbf{w}, \mathbf{u}_2 \rangle = (k_1)(0) + (k_2)(0) = 0.$$

When V is R^3 with the Euclidean inner product this means that a vector perpendicular to two vectors is also perpendicular to the entire plane spanned by these vectors (assuming they are noncollinear).

39. Using the trigonometric identity $\cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha - \beta) + \frac{1}{2} \cos(\alpha + \beta)$ we obtain

$$\mathbf{f}_k, \mathbf{f}_l = \frac{1}{2} \int_0^\pi \cos((k-l)x) dx + \frac{1}{2} \int_0^\pi \cos((k+l)x) dx \text{ where both } k-l \text{ and } k+l \text{ are nonzero integers.}$$

Substituting $u = (k-l)x$ in the first integral, and $t = (k+l)x$ in the second integral yields

$$\langle \mathbf{f}_k, \mathbf{f}_l \rangle = \frac{1}{2} \frac{\sin((k-l)x)}{k-l} \Big|_0^\pi + \frac{1}{2} \frac{\sin((k+l)x)}{k+l} \Big|_0^\pi = 0 + 0 = 0 \text{ since } \sin(m\pi) = 0 \text{ for any integer } m.$$

40. We are looking for positive real numbers w_1 and w_2 for which the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2$ satisfies $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = 1$, and $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = 1$ when applied to the given two vectors. These three equations yield a linear system

$$\begin{aligned} -w_1 + 3w_2 &= 0 \\ w_1 + 3w_2 &= 1 \\ w_1 + 3w_2 &= 1 \end{aligned}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 \end{bmatrix}$. Since the system has

only one solution $w_1 = \frac{1}{2}$, $w_2 = \frac{1}{6}$, we conclude that the weighted Euclidean inner product that satisfies the given conditions is $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2} u_1 v_1 + \frac{1}{6} u_2 v_2$.

41. $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$ contains all linear combinations $k_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 + \dots + k_r \mathbf{u}_r$

where k_1, k_2, \dots, k_r are arbitrary scalars. Let $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$.

$$\langle \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{w}, k_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 + \dots + k_r \mathbf{u}_r \rangle = k_1 \langle \mathbf{w}, \mathbf{u}_1 \rangle + k_2 \langle \mathbf{w}, \mathbf{u}_2 \rangle + \dots + k_r \langle \mathbf{w}, \mathbf{u}_r \rangle = 0 + 0 + \dots + 0 = 0$$

Thus if \mathbf{w} is orthogonal to each vector $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r$, then \mathbf{w} must be orthogonal to every vector in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$.

42. Let \mathbf{w} be a vector orthogonal to all of the basis vectors. It must be possible to express \mathbf{w} as a linear combination of the basis vectors: $\mathbf{w} = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_r \mathbf{v}_r$. By the Axioms 1-3 of Definition 1 in Section 6.1, we can write

$$\begin{aligned} \langle \mathbf{w}, \mathbf{w} \rangle &= \langle \mathbf{w}, k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_r \mathbf{v}_r \rangle \\ &= k_1 \langle \mathbf{w}, \mathbf{v}_1 \rangle + k_2 \langle \mathbf{w}, \mathbf{v}_2 \rangle + \dots + k_r \langle \mathbf{w}, \mathbf{v}_r \rangle \\ &= k_1 \cdot 0 + k_2 \cdot 0 + \dots + k_r \cdot 0 \\ &= 0 \end{aligned}$$

By Axiom 4 of Definition 1 in Section 6.1, we must have $\mathbf{w} = \mathbf{0}$.

43. Suppose that \mathbf{v} is orthogonal to every basis vector. Then, as in Exercise 41, \mathbf{v} is orthogonal to the span of the set of basis vectors, which is all of W , hence \mathbf{v} is in W^\perp . If \mathbf{v} is not orthogonal to every basis vector, then \mathbf{v} clearly cannot be in W^\perp . Thus W^\perp consists of all vectors orthogonal to every basis vector.
44. This result can be established by induction.
46. Using the Euclidean inner product, apply the Cauchy-Schwarz inequality to $\mathbf{u} = (a, b)$ and $\mathbf{v} = (\cos \theta, \sin \theta)$.
47. Using the weighted Euclidean inner product of Formula (2) in Section 6.1, the desired inequality follows from the Cauchy-Schwarz inequality.
49. Using the inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)dx$, part (a) follows from the Cauchy-Schwarz inequality and part (b) follows from the triangle inequality (part (a) of Theorem 6.2.2).
50. Squaring both sides of the Cauchy-Schwarz inequality and applying Definition 2 of Section 6.1 yields Formula (4).
51. (a) We are looking for all vectors $\mathbf{v} = (a, b)$ such that $\langle \mathbf{x}, \mathbf{v} \rangle = a + b$ is equal to $\langle T_A(\mathbf{x}), T_A(\mathbf{v}) \rangle = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a+b \\ -a+b \end{bmatrix} = 2a + 2b$. The equation $a + b = 2a + 2b$ yields $a + b = 0$, i.e. $b = -a$. Vectors that satisfy $\langle \mathbf{x}, \mathbf{v} \rangle = \langle T_A(\mathbf{x}), T_A(\mathbf{v}) \rangle$ must have a form $a(1, -1)$ where a is an arbitrary scalar.
- (b) We are looking for all vectors $\mathbf{v} = (a, b)$ such that $\langle \mathbf{x}, \mathbf{v} \rangle = 2a + 3b$ is equal to $\langle T_A(\mathbf{x}), T_A(\mathbf{v}) \rangle = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a+b \\ -a+b \end{bmatrix} = 4a + 4b$. The equation $2a + 3b = 4a + 4b$ yields $2a + b = 0$, i.e. $b = -2a$. Vectors that satisfy $\langle \mathbf{x}, \mathbf{v} \rangle = \langle T_A(\mathbf{x}), T_A(\mathbf{v}) \rangle$ must have a form $a(1, -2)$ where a is an arbitrary scalar.
52. (a) We are looking for all vectors $\mathbf{q} = a + bx + cx^2$ such that $\langle \mathbf{p}, \mathbf{q} \rangle = a + b$ is equal to $\langle T(\mathbf{p}), T(\mathbf{q}) \rangle = \langle 3, 3a - cx^2 \rangle = 9a$. The equation $a + b = 9a$ yields $b = 8a$. Vectors that satisfy $\langle \mathbf{p}, \mathbf{q} \rangle = \langle T(\mathbf{p}), T(\mathbf{q}) \rangle$ must have a form $a + 8ax + cx^2$ where a and c are arbitrary scalars.
- (b) We are looking for all vectors $\mathbf{q} = a + bx + cx^2$ such that $\langle \mathbf{p}, \mathbf{q} \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) = (0)(a - b + c) + (1)(a) + (2)(a + b + c) = 3a + 2b + 2c$ is equal to $\langle T(\mathbf{p}), T(\mathbf{q}) \rangle = \langle 3, 3a - cx^2 \rangle = (3)(3a - c) + (3)(3a) + (3)(3a - c) = 27a - 6c$. The equation $3a + 2b + 2c = 27a - 6c$ yields $b = 12a - 4c$. Vectors that satisfy $\langle \mathbf{p}, \mathbf{q} \rangle = \langle T(\mathbf{p}), T(\mathbf{q}) \rangle$ must have a form $a + (12a - 4c)x + cx^2$ where a and c are arbitrary scalars.

True-False Exercises

- (a) False. If \mathbf{u} is orthogonal to every vector of a subspace W , then \mathbf{u} is in W^\perp .
- (b) True. $W \cap W^\perp = \{\mathbf{0}\}$.

- (c) True. For any vector \mathbf{w} in W , $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle = 0$, so $\mathbf{u} + \mathbf{v}$ is in W^\perp .
- (d) True. For any vector \mathbf{w} in W , $\langle k\mathbf{u}, \mathbf{w} \rangle = k\langle \mathbf{u}, \mathbf{w} \rangle = k(0) = 0$, so $k\mathbf{u}$ is in W^\perp .
- (e) False. If \mathbf{u} and \mathbf{v} are orthogonal $|\langle \mathbf{u}, \mathbf{v} \rangle| = |0| = 0$.
- (f) False. If \mathbf{u} and \mathbf{v} are orthogonal, $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ thus $\mathbf{u} + \mathbf{v} = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2} \neq \|\mathbf{u}\| + \|\mathbf{v}\|$

6.3 Gram-Schmidt Process; QR-Decomposition

1. (a) $\langle (0, 1), (2, 0) \rangle = 0 + 0 = 0$;
 $\|(0, 1)\| = 1$; $\|(2, 0)\| = 2 \neq 1$;
 The set is orthogonal, but is not orthonormal.
- (b) $\left\langle \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle = -\frac{1}{2} + \frac{1}{2} = 0$;
 $\left\| \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$; $\left\| \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$
 The set is orthogonal and orthonormal.
- (c) $\left\langle \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle = -\frac{1}{2} - \frac{1}{2} = -1 \neq 0$;
 The set is not orthogonal (therefore, it is not orthonormal either).
- (d) $\langle (0, 0), (0, 1) \rangle = 0 + 0 = 0$;
 $\|(0, 0)\| = 0 \neq 1$; $\|(0, 1)\| = 1$;
 The set is orthogonal, but is not orthonormal.
2. (a) $\left\langle \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \right\rangle = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0$; $\left\langle \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \right\rangle = -\frac{1}{2} + 0 + \frac{1}{2} = 0$;
 $\left\langle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \right\rangle = -\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} = -\frac{2}{\sqrt{6}} \neq 0$;
 The set is not orthogonal (therefore, it is not orthonormal either).
- (b) $\left\langle \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \right\rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$; $\left\langle \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \right\rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$;
 $\left\langle \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \right\rangle = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$;
 $\left\| \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \right\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$; $\left\| \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \right\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$; $\left\| \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \right\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$;
 The set is orthogonal and orthonormal.
- (c) $\left\langle (1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle = 0 + 0 + 0 = 0$; $\langle (1, 0, 0), (0, 0, 1) \rangle = 0 + 0 + 0 = 0$;
 $\left\langle \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), (0, 0, 1) \right\rangle = 0 + 0 + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \neq 0$;
 The set is not orthogonal (therefore, it is not orthonormal either).

$$(d) \quad \left\langle \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \right\rangle = \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} + 0 = 0;$$

$$\left\| \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right) \right\| = \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}} = 1; \quad \left\| \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \right\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1$$

The set is orthogonal and orthonormal.

$$3. \quad (a) \quad \langle p_1(x), p_2(x) \rangle = \frac{2}{3} \left(\frac{2}{3} \right) - \frac{2}{3} \left(\frac{1}{3} \right) + \frac{1}{3} \left(-\frac{2}{3} \right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0;$$

$$\langle p_1(x), p_3(x) \rangle = \frac{2}{3} \left(\frac{1}{3} \right) - \frac{2}{3} \left(\frac{2}{3} \right) + \frac{1}{3} \left(\frac{2}{3} \right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0;$$

$$\langle p_2(x), p_3(x) \rangle = \frac{2}{3} \left(\frac{1}{3} \right) + \frac{1}{3} \left(\frac{2}{3} \right) - \frac{2}{3} \left(\frac{2}{3} \right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0;$$

The set is orthogonal.

$$(b) \quad \langle p_1(x), p_2(x) \rangle = 1(0) + 0\left(\frac{1}{\sqrt{2}}\right) + 0\left(\frac{1}{\sqrt{2}}\right) = 0; \quad \langle p_1(x), p_3(x) \rangle = 1(0) + 0(0) + 0(1) = 0;$$

$$\langle p_2(x), p_3(x) \rangle = 0(0) + \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}} \neq 0;$$

The set is not orthogonal.

$$4. \quad (a) \quad \text{Denoting the matrices } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}, C = \begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}, \text{ and } D = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix} \text{ we calculate}$$

$$\langle A, B \rangle = (1)(0) + (0)\left(\frac{2}{3}\right) + (0)\left(\frac{1}{3}\right) + (0)\left(-\frac{2}{3}\right) = 0;$$

$$\langle A, C \rangle = (1)(0) + (0)\left(\frac{2}{3}\right) + (0)\left(-\frac{2}{3}\right) + (0)\left(\frac{1}{3}\right) = 0;$$

$$\langle A, D \rangle = (1)(0) + (0)\left(\frac{1}{3}\right) + (0)\left(\frac{2}{3}\right) + (0)\left(\frac{2}{3}\right) = 0;$$

$$\langle B, C \rangle = (0)(0) + \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) = 0;$$

$$\langle B, D \rangle = (0)(0) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{2}{3}\right) = 0;$$

$$\langle C, D \rangle = (0)(0) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 0;$$

The set is orthogonal.

$$(b) \quad \text{Denoting the matrices } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \text{ we calculate}$$

$$\langle A, B \rangle = (1)(0) + (0)(1) + (0)(0) + (0)(0) = 0;$$

$$\langle A, C \rangle = (1)(0) + (0)(0) + (0)(1) + (0)(1) = 0;$$

$$\langle A, D \rangle = (1)(0) + (0)(0) + (0)(1) + (0)(-1) = 0;$$

$$\langle B, C \rangle = (0)(0) + (1)(0) + (0)(1) + (0)(1) = 0;$$

$$\langle B, D \rangle = (0)(0) + (1)(0) + (0)(1) + (0)(-1) = 0;$$

$$\langle C, D \rangle = (0)(0) + (0)(0) + (1)(1) + (1)(-1) = 0;$$

The set is orthogonal.

5. Let us denote the column vectors $\mathbf{u}_1 = (1, 0, -1)$, $\mathbf{u}_2 = (2, 0, 2)$, and $\mathbf{u}_3 = (0, 5, 0)$. These vectors are orthogonal since $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 2 + 0 - 2 = 0$, $\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = 0 + 0 + 0 = 0$, and $\langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0 + 0 + 0 = 0$.

It follows from Theorem 6.3.1 that the column vectors are linearly independent, therefore they form an

orthogonal basis for the column space of A . We proceed to normalize each column vector:

$$\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{1+0+1}}(1, 0, -1) = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right); \quad \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{4+0+4}}(2, 0, 2) = \left(\frac{2}{2\sqrt{2}}, 0, \frac{2}{2\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right);$$

$$\frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{1}{\sqrt{0+25+0}}(0, 5, 0) = (0, 1, 0). \text{ A resulting orthonormal basis for the column space is}$$

$$\left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0, 1, 0)\right\}.$$

6. Let us denote the column vectors $\mathbf{u}_1 = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$, $\mathbf{u}_2 = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$, and $\mathbf{u}_3 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.

These vectors are orthogonal since $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = -\frac{1}{10} + \frac{1}{10} + 0 = 0$, $\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \frac{1}{15} + \frac{1}{15} - \frac{2}{15} = 0$, and

$\langle \mathbf{u}_2, \mathbf{u}_3 \rangle = -\frac{1}{6} + \frac{1}{6} + 0 = 0$. It follows from Theorem 6.3.1 that the column vectors are linearly independent, therefore they form an orthogonal basis for the column space of A . We proceed to normalize each column vector:

$$\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{\frac{1}{25} + \frac{1}{25} + \frac{1}{25}}} \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) = \frac{5}{\sqrt{3}} \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right);$$

$$\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4} + 0}} \left(-\frac{1}{2}, \frac{1}{2}, 0\right) = \sqrt{2} \left(-\frac{1}{2}, \frac{1}{2}, 0\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right);$$

$$\frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{1}{\sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}}} \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{3}{\sqrt{6}} \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right).$$

A resulting orthonormal basis for the column space is $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)\right\}$.

7. $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -\frac{12}{25} + \frac{12}{25} + 0 = 0$; $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0 + 0 + 0 = 0$; $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0 + 0 + 0 = 0$;

$$\|\mathbf{v}_1\| = \sqrt{\frac{9}{25} + \frac{16}{25} + 0} = 1; \quad \|\mathbf{v}_2\| = \sqrt{\frac{16}{25} + \frac{9}{25} + 0} = 1; \quad \|\mathbf{v}_3\| = \sqrt{0 + 0 + 1} = 1;$$

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim(R^3) = 3$, this set forms a basis for R^3 by Theorem 4.6.4. This basis is orthonormal, so by Theorem 6.3.2(b),

$$\begin{aligned} \mathbf{u} &= \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3 \\ &= \left(-\frac{3}{5} - \frac{8}{5} + 0\right) \mathbf{v}_1 + \left(\frac{4}{5} - \frac{6}{5} + 0\right) \mathbf{v}_2 + (0 + 0 + 2) \mathbf{v}_3 \\ &= -\frac{11}{5} \mathbf{v}_1 - \frac{2}{5} \mathbf{v}_2 + 2 \mathbf{v}_3. \end{aligned}$$

8. It was shown in Exercise 7 that the vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 form an orthonormal basis for R^3 . By Theorem 6.3.2(b),

$$\begin{aligned} \mathbf{u} &= \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3 \\ &= \left(-\frac{9}{5} - \frac{28}{5} + 0\right) \mathbf{v}_1 + \left(\frac{12}{5} - \frac{21}{5} + 0\right) \mathbf{v}_2 + (0 + 0 + 4) \mathbf{v}_3 \\ &= -\frac{37}{5} \mathbf{v}_1 - \frac{9}{5} \mathbf{v}_2 + 4 \mathbf{v}_3. \end{aligned}$$

9. $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 4 - 2 - 2 = 0$; $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 2 - 4 + 2 = 0$; $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 2 + 2 - 4 = 0$;

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly

independent. Because the number of vectors in the set matches $\dim(R^3) = 3$, this set forms a basis for R^3 by Theorem 4.6.4. By Theorem 6.3.2(a),

$$\begin{aligned}\mathbf{u} &= \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \frac{\langle \mathbf{u}, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2} \mathbf{v}_3 \\ &= \frac{-2+0+2}{4+4+1} \mathbf{v}_1 + \frac{-2+0-4}{4+1+4} \mathbf{v}_2 + 0 \mathbf{v}_3 \\ &= 0 \mathbf{v}_1 - \frac{2}{3} \mathbf{v}_2 + \frac{1}{3} \mathbf{v}_3.\end{aligned}$$

$$\begin{aligned}10. \quad \langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= -2 - 2 + 6 - 2 = 0 & \langle \mathbf{v}_1, \mathbf{v}_3 \rangle &= 1 - 2 + 0 + 1 = 0 & \langle \mathbf{v}_1, \mathbf{v}_4 \rangle &= 1 + 0 + 0 - 1 = 0 \\ \langle \mathbf{v}_2, \mathbf{v}_3 \rangle &= -2 + 4 + 0 - 2 = 0 & \langle \mathbf{v}_2, \mathbf{v}_4 \rangle &= -2 + 0 + 0 + 2 = 0 & \langle \mathbf{v}_3, \mathbf{v}_4 \rangle &= 1 + 0 + 0 - 1 = 0\end{aligned}$$

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim(R^4) = 4$, this set forms a basis for R^4 by Theorem 4.6.4. We use Theorem 6.3.2(a):

$$\begin{aligned}\mathbf{u} &= \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \frac{\langle \mathbf{u}, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2} \mathbf{v}_3 + \frac{\langle \mathbf{u}, \mathbf{v}_4 \rangle}{\|\mathbf{v}_4\|^2} \mathbf{v}_4 \\ &= \frac{1-1+2-1}{1+1+4+1} \mathbf{v}_1 + \frac{-2+2+3+2}{4+4+9+4} \mathbf{v}_2 + \frac{1+2+0-1}{1+4+0+1} \mathbf{v}_3 + \frac{1+0+0+1}{1+0+0+1} \mathbf{v}_4 \\ &= \frac{1}{7} \mathbf{v}_1 + \frac{5}{21} \mathbf{v}_2 + \frac{1}{3} \mathbf{v}_3 + 1 \mathbf{v}_4.\end{aligned}$$

$$11. \quad (\mathbf{u})_S = \left(-\frac{11}{5}, -\frac{2}{5}, 2\right)$$

$$12. \quad (\mathbf{u})_S = \left(-\frac{37}{5}, -\frac{9}{5}, 4\right)$$

$$13. \quad (\mathbf{u})_S = \left(0, -\frac{2}{3}, \frac{1}{3}\right)$$

$$14. \quad (\mathbf{u})_S = \left(\frac{1}{7}, \frac{5}{21}, \frac{1}{3}, 1\right)$$

$$15. \quad (a) \quad \|\mathbf{v}\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1, \text{ so } \mathbf{v} \text{ forms an orthonormal basis for the line } W = \text{span}\{\mathbf{v}\}.$$

$$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{v} = \left(-\frac{3}{5} + \frac{24}{5}\right) \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{21}{5} \left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{63}{25}, \frac{84}{25}\right)$$

$$(b) \quad \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (-1, 6) - \left(\frac{63}{25}, \frac{84}{25}\right) = \left(-\frac{88}{25}, \frac{66}{25}\right);$$

$$\mathbf{w}_2 \text{ is orthogonal to the line since } \langle \mathbf{w}_2, \mathbf{v} \rangle = -\frac{264}{125} + \frac{264}{125} = 0.$$

$$16. \quad (a) \quad \|\mathbf{v}\| = \sqrt{\frac{25}{169} + \frac{144}{169}} = 1, \text{ so } \mathbf{v} \text{ forms an orthonormal basis for the line } W = \text{span}\{\mathbf{v}\}.$$

$$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{v} = \left(\frac{10}{13} + \frac{36}{13}\right) \left(\frac{5}{13}, \frac{12}{13}\right) = \frac{46}{13} \left(\frac{5}{13}, \frac{12}{13}\right) = \left(\frac{230}{169}, \frac{552}{169}\right)$$

$$(b) \quad \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (2, 3) - \left(\frac{230}{169}, \frac{552}{169}\right) = \left(\frac{108}{169}, -\frac{45}{169}\right);$$

$$\mathbf{w}_2 \text{ is orthogonal to the line since } \langle \mathbf{w}_2, \mathbf{v} \rangle = \frac{540}{2197} - \frac{540}{2197} = 0.$$

17. (a) $\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{2+3}{1+1} (1, 1) = \frac{5}{2} (1, 1) = \left(\frac{5}{2}, \frac{5}{2}\right)$
- (b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (2, 3) - \left(\frac{5}{2}, \frac{5}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right);$
 \mathbf{w}_2 is orthogonal to the line since $\langle \mathbf{w}_2, \mathbf{v} \rangle = -\frac{1}{2} + \frac{1}{2} = 0.$
18. (a) $\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{9-4}{9+16} (3, 4) = \frac{5}{25} (3, 4) = \left(\frac{3}{5}, \frac{4}{5}\right)$
- (b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (3, -1) - \left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{12}{5}, -\frac{9}{5}\right);$
 \mathbf{w}_2 is orthogonal to the line since $\langle \mathbf{w}_2, \mathbf{v} \rangle = \frac{36}{5} - \frac{36}{5} = 0.$
19. (a) $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for the plane $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}.$
 $\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 = 2\left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) + 4\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) = \left(\frac{10}{3}, \frac{8}{3}, \frac{4}{3}\right)$
- (b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (4, 2, 1) - \left(\frac{10}{3}, \frac{8}{3}, \frac{4}{3}\right) = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right);$
 \mathbf{w}_2 is orthogonal to the plane since $\langle \mathbf{w}_2, \mathbf{v}_1 \rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$ and $\langle \mathbf{w}_2, \mathbf{v}_2 \rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0.$
20. (a) $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for the plane $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}.$
 $\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 = -\frac{2}{\sqrt{6}} \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) + \frac{4}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
 $= \left(-\frac{2}{6}, -\frac{2}{6}, \frac{4}{6}\right) + \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right) + \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) = (1, 1, 2)$
- (b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (3, -1, 2) - (1, 1, 2) = (2, -2, 0);$
 \mathbf{w}_2 is orthogonal to the plane since $\langle \mathbf{w}_2, \mathbf{v}_1 \rangle = \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} + 0 = 0$ and $\langle \mathbf{w}_2, \mathbf{v}_2 \rangle = \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} = 0.$
21. (a) $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the plane $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}.$
 $\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \frac{4}{6} (1, -2, 1) + \frac{2}{5} (2, 1, 0)$
 $= \left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right) + \left(\frac{4}{5}, \frac{2}{5}, 0\right) = \left(\frac{22}{15}, -\frac{14}{15}, \frac{2}{3}\right)$
- (b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (1, 0, 3) - \left(\frac{22}{15}, -\frac{14}{15}, \frac{2}{3}\right) = \left(-\frac{7}{15}, \frac{14}{15}, \frac{7}{3}\right);$
 \mathbf{w}_2 is orthogonal to the plane since $\langle \mathbf{w}_2, \mathbf{v}_1 \rangle = -\frac{7}{15} - \frac{28}{15} + \frac{7}{3} = \frac{-7-28+35}{15} = 0$ and
 $\langle \mathbf{w}_2, \mathbf{v}_2 \rangle = -\frac{14}{15} + \frac{14}{15} + 0 = 0.$
22. (a) $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the plane $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}.$
 $\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \frac{7}{14} (3, 1, 2) + \frac{1}{3} (-1, 1, 1) = \left(\frac{3}{2}, \frac{1}{2}, 1\right) + \left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
 $= \left(\frac{7}{6}, \frac{5}{6}, \frac{4}{3}\right)$
- (b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (1, 0, 2) - \left(\frac{7}{6}, \frac{5}{6}, \frac{4}{3}\right) = \left(-\frac{1}{6}, -\frac{5}{6}, \frac{2}{3}\right);$
 \mathbf{w}_2 is orthogonal to the plane since $\langle \mathbf{w}_2, \mathbf{v}_1 \rangle = -\frac{3}{6} - \frac{5}{6} + \frac{4}{3} = 0$ and $\langle \mathbf{w}_2, \mathbf{v}_2 \rangle = \frac{1}{6} - \frac{5}{6} + \frac{2}{3} = 0.$

$$\begin{aligned}
 23. \quad \text{proj}_W \mathbf{b} &= \frac{\langle \mathbf{b}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{b}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \frac{1+2+0-2}{1+1+1+1} (1, 1, 1, 1) + \frac{1+2+0+2}{1+1+1+1} (1, 1, -1, -1) \\
 &= \frac{1}{4} (1, 1, 1, 1) + \frac{5}{4} (1, 1, -1, -1) = \left(\frac{3}{2}, \frac{3}{2}, -1, -1 \right)
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \text{proj}_W \mathbf{b} &= \frac{\langle \mathbf{b}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{b}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \frac{0+2+0+2}{0+1+16+1} (0, 1, -4, -1) + \frac{3+10+0-2}{9+25+1+1} (3, 5, 1, 1) \\
 &= \frac{2}{9} (0, 1, -4, -1) + \frac{11}{36} (3, 5, 1, 1) = \left(\frac{11}{12}, \frac{7}{4}, -\frac{7}{12}, \frac{1}{12} \right)
 \end{aligned}$$

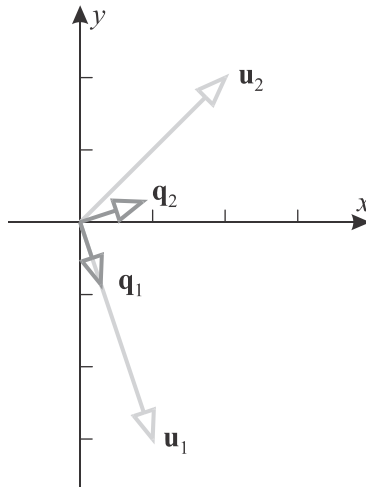
$$\begin{aligned}
 25. \quad \text{proj}_W \mathbf{b} &= \langle \mathbf{b}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{b}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{b}, \mathbf{v}_3 \rangle \mathbf{v}_3 \\
 &= \left(0 + \frac{2}{\sqrt{18}} + 0 + \frac{1}{\sqrt{18}} \right) \mathbf{v}_1 + \left(\frac{1}{2} + \frac{10}{6} + 0 - \frac{1}{6} \right) \mathbf{v}_2 + \left(\frac{1}{\sqrt{18}} + 0 + 0 + \frac{4}{\sqrt{18}} \right) \mathbf{v}_3 = \frac{3}{\sqrt{18}} \mathbf{v}_1 + 2 \mathbf{v}_2 + \frac{5}{\sqrt{18}} \mathbf{v}_3 \\
 &= \left(0, \frac{3}{18}, -\frac{12}{18}, -\frac{3}{18} \right) + \left(1, \frac{5}{3}, \frac{1}{3}, \frac{1}{3} \right) + \left(\frac{5}{18}, 0, \frac{5}{18}, -\frac{20}{18} \right) = \left(\frac{23}{18}, \frac{11}{6}, -\frac{1}{18}, -\frac{17}{18} \right)
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \text{proj}_W \mathbf{b} &= \langle \mathbf{b}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{b}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{b}, \mathbf{v}_3 \rangle \mathbf{v}_3 \\
 &= \left(\frac{1}{2} + 1 + 0 - \frac{1}{2} \right) \mathbf{v}_1 + \left(\frac{1}{2} + 1 + 0 + \frac{1}{2} \right) \mathbf{v}_2 + \left(\frac{1}{2} - 1 + 0 + \frac{1}{2} \right) \mathbf{v}_3 = 1 \mathbf{v}_1 + 2 \mathbf{v}_2 + 0 \mathbf{v}_3 \\
 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) + (1, 1, -1, -1) = \left(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \mathbf{v}_1 &= \mathbf{u}_1 = (1, -3) \\
 \mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (2, 2) - \frac{2-6}{10} (1, -3) = (2, 2) - \left(-\frac{2}{5}, \frac{6}{5} \right) = \left(\frac{12}{5}, \frac{4}{5} \right)
 \end{aligned}$$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{10}} (1, -3) = \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right)$ and

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{\frac{144}{25} + \frac{16}{25}}} \left(\frac{12}{5}, \frac{4}{5} \right) = \frac{1}{\sqrt{\frac{160}{25}}} \left(\frac{12}{5}, \frac{4}{5} \right) = \frac{5}{4\sqrt{10}} \left(\frac{12}{5}, \frac{4}{5} \right) = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right).$$

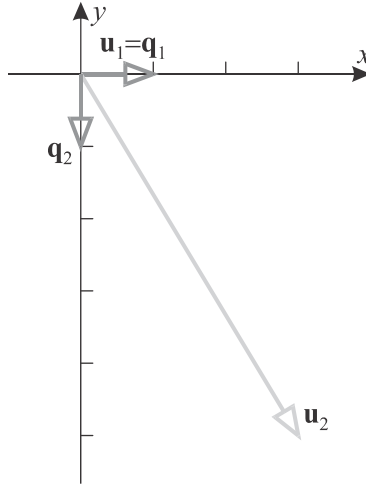


28. $\mathbf{v}_1 = \mathbf{u}_1 = (1, 0)$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (3, -5) - \frac{3+0}{1} (1, 0) = (3, -5) - (3, 0) = (0, -5)$$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{1}} (1, 0) = (1, 0)$ and

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{25}} (0, -5) = (0, -1).$$



29. $\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (-1, 1, 0) - \frac{-1+1+0}{1+1+1} (1, 1, 1) = (-1, 1, 0) - 0(1, 1, 1) = (-1, 1, 0)$$

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = (1, 2, 1) - \frac{1+2+1}{1+1+1} (1, 1, 1) - \frac{-1+2+0}{1+1+0} (-1, 1, 0) \\ &= (1, 2, 1) - \frac{4}{3} (1, 1, 1) - \frac{1}{2} (-1, 1, 0) = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right) \end{aligned}$$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{3}} (1, 1, 1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$,

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{2}} (-1, 1, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \text{ and } \mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{1/\sqrt{6}} \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right).$$

30. $\mathbf{v}_1 = \mathbf{u}_1 = (1, 0, 0)$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (3, 7, -2) - \frac{3+0+0}{1+0+0} (1, 0, 0) = (3, 7, -2) - 3(1, 0, 0) = (0, 7, -2)$$

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = (0, 4, 1) - \frac{0+0+0}{1+0+0} (1, 0, 0) - \frac{0+28-2}{0+49+4} (0, 7, -2) \\ &= (0, 4, 1) - 0(1, 0, 0) - \frac{26}{53} (0, 7, -2) = \left(0, \frac{30}{53}, \frac{105}{53}\right) \end{aligned}$$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{1} (1, 0, 0) = (1, 0, 0)$,

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{53}} (0, 7, -2) = \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right), \text{ and } \mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{15/\sqrt{53}} \left(0, \frac{30}{53}, \frac{105}{53}\right) = \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right).$$

31. First, transform the given basis into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

$$\mathbf{v}_1 = \mathbf{u}_1 = (0, 2, 1, 0)$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (1, -1, 0, 0) - \frac{0-2+0+0}{5} (0, 2, 1, 0) = (1, -1, 0, 0) + \left(0, \frac{4}{5}, \frac{2}{5}, 0\right)$$

$$= \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = (1, 2, 0, -1) - \frac{0+4+0+0}{5} (0, 2, 1, 0) - \frac{1-\frac{2}{5}+0+0}{\frac{6}{5}} \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \\ &= (1, 2, 0, -1) - (0, \frac{8}{5}, \frac{4}{5}, 0) - \left(\frac{1}{2}, -\frac{1}{10}, \frac{1}{5}, 0\right) = \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)\end{aligned}$$

$$\mathbf{v}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2} \mathbf{v}_3$$

$$\begin{aligned}&= (1, 0, 0, 1) - \frac{0+0+0+0}{5} \mathbf{v}_1 - \frac{1+0+0+0}{\frac{6}{5}} \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) - \frac{\frac{1}{2}+0+0-1}{\frac{5}{2}} \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \\ &= (1, 0, 0, 1) - \frac{0+0+0+0}{5} \mathbf{v}_1 - \frac{1+0+0+0}{\frac{6}{5}} \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) - \frac{\frac{1}{2}+0+0-1}{\frac{5}{2}} \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) = \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)\end{aligned}$$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(0, 2, 1, 0)}{\sqrt{5}} = \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$,

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(1, -\frac{1}{5}, \frac{2}{5}, 0)}{\frac{\sqrt{30}}{5}} = \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right), \quad \mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{(\frac{1}{2}, \frac{1}{2}, -1, -1)}{\frac{\sqrt{10}}{2}} = \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right),$$

$$\mathbf{q}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5})}{\frac{4}{\sqrt{15}}} = \left(\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}}\right).$$

32. We begin by forming a matrix whose columns are the three given vectors: $\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$. The reduced row

echelon form of this matrix is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. By Theorem 4.8.5(b), vectors $\mathbf{u}_1 = (0, 1, 2)$ and

$\mathbf{u}_2 = (-1, 0, 1)$ form a basis for the span of the three given vectors (the third vector can be expressed as a linear combination of the first two).

We proceed to use the Gram-Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ to an orthonormal basis.

$$\mathbf{v}_1 = \mathbf{u}_1 = (0, 1, 2)$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (-1, 0, 1) - \frac{0+0+2}{0+1+4} (0, 1, 2) = (-1, 0, 1) - \frac{2}{5} (0, 1, 2) = \left(-1, -\frac{2}{5}, \frac{1}{5}\right)$$

An orthonormal basis is formed by the vectors

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{5}} (0, 1, 2) = \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \text{ and } \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6/5}} \left(-1, -\frac{2}{5}, \frac{1}{5}\right) = \left(-\frac{5}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right).$$

33. From Exercise 23, $\mathbf{w}_1 = \text{proj}_W \mathbf{b} = \left(\frac{3}{2}, \frac{3}{2}, -1, -1\right)$, so $\mathbf{w}_2 = \mathbf{b} - \text{proj}_W \mathbf{b} = \left(-\frac{1}{2}, \frac{1}{2}, 1, -1\right)$.

34. From Exercise 25, $\mathbf{w}_1 = \text{proj}_W \mathbf{b} = \left(\frac{23}{18}, \frac{11}{6}, -\frac{1}{18}, -\frac{17}{18}\right)$, so $\mathbf{w}_2 = \mathbf{b} - \mathbf{w}_1 = \left(-\frac{5}{18}, \frac{1}{6}, \frac{1}{18}, -\frac{1}{18}\right)$.

35. Let W be the plane spanned by the vectors \mathbf{u}_1 and \mathbf{u}_2 .

$$\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (2, 0, -1) - \frac{2+0-1}{1+1+1} (1, 1, 1) = (2, 0, -1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$$

$$\mathbf{w}_1 = \text{proj}_W \mathbf{w} = \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \frac{1+2+3}{3} \mathbf{v}_1 + \frac{\frac{5}{3}-\frac{2}{3}-\frac{12}{3}}{\frac{42}{9}} \mathbf{v}_2 = 2(1, 1, 1) - \frac{9}{14} \left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$$

$$= (2, 2, 2) - \left(\frac{15}{14}, -\frac{3}{14}, -\frac{6}{7}\right) = \left(\frac{13}{14}, \frac{31}{14}, \frac{20}{7}\right)$$

$$\mathbf{w}_2 = \mathbf{w} - \mathbf{w}_1 = (1, 2, 3) - \left(\frac{13}{14}, \frac{31}{14}, \frac{20}{7}\right) = \left(\frac{1}{14}, -\frac{3}{14}, \frac{1}{7}\right)$$

36. The vectors \mathbf{u}_1 and \mathbf{u}_2 are not orthogonal, therefore we begin by using the Gram-Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ to an orthonormal basis.

$$\mathbf{v}_1 = \mathbf{u}_1 = (-1, 0, 1, 2)$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (0, 1, 0, 1) - \frac{0+0+0+2}{1+0+1+4} (-1, 0, 1, 2) = (0, 1, 0, 1) - \frac{1}{3} (-1, 0, 1, 2) = \left(\frac{1}{3}, 1, -\frac{1}{3}, \frac{1}{3}\right)$$

An orthonormal basis is formed by the vectors

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{6}} (-1, 0, 1, 2) = \left(-\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) \text{ and } \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{4/3}} \left(\frac{1}{3}, 1, -\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right).$$

$$\mathbf{w}_1 = \text{proj}_{\mathbf{w}} \mathbf{w} = \langle \mathbf{w}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{w}, \mathbf{q}_2 \rangle \mathbf{q}_2 = \frac{1+0+6+0}{\sqrt{6}} \left(-\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) + \frac{-1+6-6+0}{2\sqrt{3}} \left(\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)$$

$$= \left(-\frac{7}{6}, 0, \frac{7}{6}, \frac{7}{3}\right) + \left(-\frac{1}{12}, -\frac{1}{4}, \frac{1}{12}, -\frac{1}{12}\right) = \left(-\frac{5}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{9}{4}\right)$$

$$\mathbf{w}_2 = \mathbf{w} - \mathbf{w}_1 = (-1, 2, 6, 0) - \left(-\frac{5}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{9}{4}\right) = \left(\frac{1}{4}, \frac{9}{4}, \frac{19}{4}, -\frac{9}{4}\right).$$

37. First, transform the given basis into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$$

$$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} = \sqrt{1^2 + 2(1)^2 + 3(1)^2} = \sqrt{1+2+3} = \sqrt{6}$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (1, 1, 0) - \frac{1(1)+2(1)(1)+3(0)(1)}{6} (1, 1, 1) = (1, 1, 0) - \frac{1}{2} (1, 1, 1) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$\|\mathbf{v}_2\| = \sqrt{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} = \sqrt{\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 3\left(-\frac{1}{2}\right)^2} = \sqrt{6\left(\frac{1}{4}\right)} = \frac{\sqrt{6}}{2}$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = (1, 0, 0) - \frac{1+0+0}{6} (1, 1, 1) - \frac{\frac{1}{2}+0+0}{\frac{6}{4}} \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$= (1, 0, 0) - \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) - \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}\right) = \left(\frac{2}{3}, -\frac{1}{3}, 0\right)$$

$$\|\mathbf{v}_3\| = \sqrt{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle} = \sqrt{\left(\frac{2}{3}\right)^2 + 2\left(-\frac{1}{3}\right)^2 + 3(0)^2} = \sqrt{\frac{4}{9} + \frac{2}{9}} = \frac{\sqrt{6}}{3}$$

The orthonormal basis is $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 1, 1)}{\sqrt{6}} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})}{\frac{\sqrt{6}}{2}} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$, and

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{(\frac{2}{3}, -\frac{1}{3}, 0)}{\frac{\sqrt{6}}{3}} = \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right).$$

38. Let us denote $\mathbf{w}_1 = (1, 0)$ and $\mathbf{w}_2 = (0, 1)$. The set is orthogonal because $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 4(1)(0) + (0)(1) = 0$.

$$\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{\mathbf{w}_1}{\sqrt{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}} = \frac{1}{\sqrt{4(1)(1) + (0)(0)}} (1, 0) = \left(\frac{1}{2}, 0\right); \quad \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{\mathbf{w}_2}{\sqrt{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}} = \frac{1}{\sqrt{4(0)(0) + (1)(1)}} (0, 1) = (0, 1)$$

By normalizing the given orthogonal set, we obtained an orthonormal set $\left\{\left(\frac{1}{2}, 0\right), (0, 1)\right\}$.

39. For example, $\mathbf{x} = \left(\frac{1}{\sqrt{3}}, 0\right)$ and $\mathbf{y} = \left(0, \frac{1}{\sqrt{2}}\right)$.

40. The line through the origin making $\frac{\pi}{6}$ angle with the x -axis is a subspace of R^2 spanned by the vector

$\mathbf{u} = (\cos \frac{\pi}{6}, \sin \frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$. Since this vector has norm 1, the orthogonal projection of $\mathbf{x} = (1, 5)$ onto

$W = \text{span}\{\mathbf{u}\}$ is $\text{proj}_W \mathbf{x} = \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u} = \left(\frac{\sqrt{3}}{2} + \frac{5}{2}\right) \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(\frac{3+5\sqrt{3}}{4}, \frac{5+\sqrt{3}}{4}\right)$.

41. (a) By inspection, $\mathbf{v}'_1 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}'_2 = \mathbf{v}_1 - 2\mathbf{v}_2$ so \mathbf{v}'_1 and \mathbf{v}'_2 are in W . The dimension of W is 2 since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W . By Theorem 6.3.1, $\{\mathbf{v}'_1, \mathbf{v}'_2\}$ is linearly independent, so by Theorem 4.6.4 it is a basis for W , hence it spans W .

(b) Calculating $\text{proj}_W \mathbf{u}$ using $\{\mathbf{v}_1, \mathbf{v}_2\}$ we obtain

$$\frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \frac{-3+0+7}{1+0+1} (1, 0, 1) + \frac{0+1+0}{0+1+0} (0, 1, 0) = (2, 0, 2) + (0, 1, 0) = (2, 1, 2).$$

Calculating $\text{proj}_W \mathbf{u}$ using $\{\mathbf{v}'_1, \mathbf{v}'_2\}$ instead yields the same vector:

$$\frac{\langle \mathbf{u}, \mathbf{v}'_1 \rangle}{\|\mathbf{v}'_1\|^2} \mathbf{v}'_1 + \frac{\langle \mathbf{u}, \mathbf{v}'_2 \rangle}{\|\mathbf{v}'_2\|^2} \mathbf{v}'_2 = \frac{-3+1+7}{1+1+1} (1, 1, 1) + \frac{-3-2+7}{1+4+1} (1, -2, 1) = \left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right) + \left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right) = (2, 1, 2).$$

42. The first three Legendre polynomials are $q_1(x) = 1$, $q_2(x) = x$, and $q_3(x) = \frac{1}{2}(3x^2 - 1) = -\frac{1}{2} + \frac{3}{2}x^2$.

By Example 8 and the Remark that follows it, these polynomials form an orthogonal set in the vector space

P_2 with the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x)dx$.

We have $\|\mathbf{q}_1\|^2 = \int_{-1}^1 (1)^2 dx = x \Big|_{-1}^1 = 2$, $\|\mathbf{q}_2\|^2 = \int_{-1}^1 (x)^2 dx = \frac{x^3}{3} \Big|_{-1}^1 = \frac{2}{3}$, and

$$\|\mathbf{q}_3\|^2 = \int_{-1}^1 \left(-\frac{1}{2} + \frac{3}{2}x^2\right)^2 dx = \left(\frac{x}{4} - \frac{x^3}{2} + \frac{9x^5}{20}\right) \Big|_{-1}^1 = \frac{2}{5}.$$

- (a) For $p(x) = 1 + x + 4x^2$, Theorem 6.3.2(a) yields

$$\begin{aligned} \mathbf{p} &= \frac{\langle \mathbf{p}, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2} \mathbf{q}_1 + \frac{\langle \mathbf{p}, \mathbf{q}_2 \rangle}{\|\mathbf{q}_2\|^2} \mathbf{q}_2 + \frac{\langle \mathbf{p}, \mathbf{q}_3 \rangle}{\|\mathbf{q}_3\|^2} \mathbf{q}_3 \\ &= \frac{\int_{-1}^1 (1+x+4x^2)dx}{2} \mathbf{q}_1 + \frac{\int_{-1}^1 (x+x^2+4x^3)dx}{\frac{2}{3}} \mathbf{q}_2 + \frac{\int_{-1}^1 \left(-\frac{1}{2} - \frac{x}{2} - \frac{x^2}{2} + \frac{3x^3}{2} + 6x^4\right)dx}{\frac{2}{5}} \mathbf{q}_3 \\ &= \frac{1}{2} \left(x + \frac{x^2}{2} + \frac{4x^3}{3}\right) \Big|_{-1}^1 \mathbf{q}_1 + \frac{3}{2} \left(\frac{x^2}{2} + \frac{x^3}{3} + x^4\right) \Big|_{-1}^1 \mathbf{q}_2 + \frac{5}{2} \left(-\frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{6} + \frac{3x^4}{8} + \frac{6x^5}{5}\right) \Big|_{-1}^1 \mathbf{q}_3 \\ &= \left(\frac{1}{2}\right) \left(\frac{14}{3}\right) \mathbf{q}_1 + \left(\frac{3}{2}\right) \left(\frac{2}{3}\right) \mathbf{q}_2 + \left(\frac{5}{2}\right) \left(\frac{16}{15}\right) \mathbf{q}_3 \\ &= \frac{7}{3} \mathbf{q}_1 + 1 \mathbf{q}_2 + \frac{8}{3} \mathbf{q}_3 \end{aligned}$$

- (b) For $p(x) = 2 - 7x^2$, Theorem 6.3.2(a) yields

$$\mathbf{p} = \frac{\langle \mathbf{p}, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2} \mathbf{q}_1 + \frac{\langle \mathbf{p}, \mathbf{q}_2 \rangle}{\|\mathbf{q}_2\|^2} \mathbf{q}_2 + \frac{\langle \mathbf{p}, \mathbf{q}_3 \rangle}{\|\mathbf{q}_3\|^2} \mathbf{q}_3$$

$$\begin{aligned}
&= \frac{\int_{-1}^1 (2-7x^2) dx}{2} \mathbf{q}_1 + \frac{\int_{-1}^1 (2x-7x^3) dx}{\frac{2}{3}} \mathbf{q}_2 + \frac{\int_{-1}^1 \left(-1 + \frac{13x^2}{2} - \frac{21x^4}{2}\right) dx}{\frac{2}{5}} \mathbf{q}_3 \\
&= \frac{1}{2} \left(2x - \frac{7x^3}{3}\right) \Big|_{-1}^1 \mathbf{q}_1 + \frac{3}{2} \left(x^2 - \frac{7x^4}{4}\right) \Big|_{-1}^1 \mathbf{q}_2 + \frac{5}{2} \left(-x + \frac{13x^3}{6} - \frac{21x^5}{10}\right) \Big|_{-1}^1 \mathbf{q}_3 \\
&= \left(\frac{1}{2}\right) \left(-\frac{2}{3}\right) \mathbf{q}_1 + \left(\frac{3}{2}\right) (0) \mathbf{q}_2 + \left(\frac{5}{2}\right) \left(-\frac{28}{15}\right) \mathbf{q}_3 \\
&= -\frac{1}{3} \mathbf{q}_1 + 0 \mathbf{q}_2 - \frac{14}{3} \mathbf{q}_3
\end{aligned}$$

(c) For $p(x) = 4 + 3x$, Theorem 6.3.2(a) yields

$$\begin{aligned}
\mathbf{p} &= \frac{\langle \mathbf{p}, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2} \mathbf{q}_1 + \frac{\langle \mathbf{p}, \mathbf{q}_2 \rangle}{\|\mathbf{q}_2\|^2} \mathbf{q}_2 + \frac{\langle \mathbf{p}, \mathbf{q}_3 \rangle}{\|\mathbf{q}_3\|^2} \mathbf{q}_3 \\
&= \frac{\int_{-1}^1 (4+3x) dx}{2} \mathbf{q}_1 + \frac{\int_{-1}^1 (4x+3x^2) dx}{\frac{2}{3}} \mathbf{q}_2 + \frac{\int_{-1}^1 \left(-2 - \frac{3x}{2} + 6x^2 + \frac{9x^3}{2}\right) dx}{\frac{2}{5}} \mathbf{q}_3 \\
&= \left(\frac{1}{2}\right) \left(4x + \frac{3x^2}{2}\right) \Big|_{-1}^1 \mathbf{q}_1 + \frac{3}{2} \left(2x^2 + x^3\right) \Big|_{-1}^1 \mathbf{q}_2 + \frac{5}{2} \left(-2x - \frac{3x^2}{4} + 2x^3 + \frac{9x^4}{8}\right) \Big|_{-1}^1 \mathbf{q}_3 \\
&= \left(\frac{1}{2}\right) (8) \mathbf{q}_1 + \left(\frac{3}{2}\right) (2) \mathbf{q}_2 + \left(\frac{5}{2}\right) (0) \mathbf{q}_3 \\
&= 4\mathbf{q}_1 + 3\mathbf{q}_2 + 0\mathbf{q}_3
\end{aligned}$$

43. First transform the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{1, x, x^2\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$\mathbf{v}_1 = \mathbf{p}_1 = 1$$

$$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} = \sqrt{\int_0^1 1^2 dx} = \sqrt{x} \Big|_0^1 = 1$$

$$\langle \mathbf{p}_2, \mathbf{v}_1 \rangle = \int_0^1 x \cdot 1 dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2}$$

$$\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = x - \frac{\frac{1}{2}}{1} (1) = x - \frac{1}{2} (1) = -\frac{1}{2} + x$$

$$\|\mathbf{v}_2\|^2 = \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \int_0^1 \left(-\frac{1}{2} + x\right)^2 dx = \int_0^1 \left(\frac{1}{4} - x + x^2\right) dx = \left(\frac{1}{4}x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right) \Big|_0^1 = \frac{1}{12}$$

$$\|\mathbf{v}_2\| = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}}$$

$$\langle \mathbf{p}_3, \mathbf{v}_1 \rangle = \int_0^1 x^2 \cdot 1 dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$$

$$\langle \mathbf{p}_3, \mathbf{v}_2 \rangle = \int_0^1 x^2 \left(-\frac{1}{2} + x\right) dx = \int_0^1 \left(-\frac{1}{2}x^2 + x^3\right) dx = \left(-\frac{1}{6}x^3 + \frac{1}{4}x^4\right) \Big|_0^1 = \frac{1}{12}$$

$$\mathbf{v}_3 = \mathbf{p}_3 - \frac{\langle \mathbf{p}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{p}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = x^2 - \frac{\frac{1}{3}}{1} (1) - \frac{\frac{1}{12}}{\frac{1}{12}} \left(-\frac{1}{2} + x\right) = x^2 - \frac{1}{3} + \frac{1}{2} - x = \frac{1}{6} - x + x^2$$

$$\|\mathbf{v}_3\|^2 = \langle \mathbf{v}_3, \mathbf{v}_3 \rangle = \int_0^1 \left(\frac{1}{6} - x + x^2 \right)^2 dx = \left(\frac{1}{36}x - \frac{1}{6}x^2 + \frac{4}{9}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right) \Big|_0^1 = \frac{1}{180}$$

$$\|\mathbf{v}_3\| = \sqrt{\frac{1}{180}} = \frac{1}{6\sqrt{5}}$$

The orthonormal basis is

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{1} = 1;$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{-\frac{1}{2} + x}{\frac{1}{2\sqrt{3}}} = 2\sqrt{3}\left(-\frac{1}{2} + x\right) = \sqrt{3}(-1 + 2x);$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\frac{1}{6} - x + x^2}{\frac{1}{6\sqrt{5}}} = 6\sqrt{5}\left(\frac{1}{6} - x + x^2\right) = \sqrt{5}(1 - 6x + 6x^2).$$

44. The reduced row echelon form of the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the three column vectors of the

original matrix, $\mathbf{u}_1 = (6, 2, -2, 6)$, $\mathbf{u}_2 = (1, 1, -2, 8)$, and $\mathbf{u}_3 = (-5, 1, 5, -7)$ form a basis for the column space. Applying the Gram-Schmidt process yields an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

$$\mathbf{v}_1 = \mathbf{u}_1 = (6, 2, -2, 6)$$

$$\begin{aligned} \mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (1, 1, -2, 8) - \frac{6+2+4+48}{36+4+4+36} (6, 2, -2, 6) = (1, 1, -2, 8) - \frac{3}{4} (6, 2, -2, 6) \\ &= \left(-\frac{7}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{7}{2}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = (-5, 1, 5, -7) - \frac{-30+2-10-42}{36+4+4+36} (6, 2, -2, 6) - \frac{\frac{35}{2}-\frac{1}{2}-\frac{5}{2}-\frac{49}{2}}{\frac{49}{4}+\frac{1}{4}+\frac{1}{4}+\frac{49}{4}} \left(-\frac{7}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{7}{2}\right) \\ &= (-5, 1, 5, -7) + (6, 2, -2, 6) + \frac{2}{5} \left(-\frac{7}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{7}{2}\right) = \left(-\frac{2}{5}, \frac{14}{5}, \frac{14}{5}, \frac{2}{5}\right) \end{aligned}$$

45. Let $\mathbf{u}_1 = (1, 2)$, $\mathbf{u}_2 = (-1, 3)$, $\mathbf{q}_1 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$, and $\mathbf{q}_2 = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$. A QR -decomposition of the matrix A is formed by the given matrix Q and the matrix

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} & -\frac{1}{\sqrt{5}} + \frac{6}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}.$$

46. Let $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (2, 1, 4)$, $\mathbf{q}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, and $\mathbf{q}_2 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. A QR -decomposition of the matrix A is formed by the given matrix Q and the matrix

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{2}} + 0 + \frac{4}{\sqrt{2}} \\ 0 & -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

47. Let $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (0, 1, 2)$, $\mathbf{u}_3 = (2, 1, 0)$, $\mathbf{q}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\mathbf{q}_2 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and

$\mathbf{q}_3 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$. A QR -decomposition of the matrix A is formed by the given matrix Q and the matrix

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} & 0 + 0 + \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} + 0 + 0 \\ 0 & 0 + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 \\ 0 & 0 & \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} + 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{4}{\sqrt{6}} \end{bmatrix}.$$

48. Let $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (2, 1, 3)$, $\mathbf{u}_3 = (1, 1, 1)$, $\mathbf{q}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\mathbf{q}_2 = \left(\frac{\sqrt{2}}{2\sqrt{19}}, -\frac{\sqrt{2}}{2\sqrt{19}}, \frac{3\sqrt{2}}{\sqrt{19}}\right)$, and

$\mathbf{q}_3 = \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}\right)$. A QR -decomposition of the matrix A is formed by the given matrix Q and the matrix

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 & \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 & \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 \\ 0 & \frac{2\sqrt{2}}{2\sqrt{19}} - \frac{\sqrt{2}}{2\sqrt{19}} + \frac{9\sqrt{2}}{\sqrt{19}} & \frac{\sqrt{2}}{2\sqrt{19}} - \frac{\sqrt{2}}{2\sqrt{19}} + \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & -\frac{3}{\sqrt{19}} + \frac{3}{\sqrt{19}} + \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}.$$

49. In partitioned form, $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$. By inspection, $\mathbf{u}_3 = \mathbf{u}_1 + 2\mathbf{u}_2$, so the column vectors

of A are not linearly independent and A does not have a QR -decomposition.

51. The proof of part (a) mirrors the proof of part (b) in the book. By Theorem 6.3.1, an orthogonal set of nonzero vectors in W is linearly independent. It follows from part (b) of Theorem 4.6.5 that this set can be enlarged to form a basis for W . Applying the Gram-Schmidt process (without the normalization step) will yield an enlarged orthogonal set (the original orthogonal set will not be affected).
52. If $\mathbf{v}_3 = \mathbf{0}$ then

$$\begin{aligned} \mathbf{u}_3 &= \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 \\ &= \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{u}_1 + \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{u}_2 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle \langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_2\|^2 \|\mathbf{v}_1\|^2} \mathbf{u}_1 \\ &= \left(\frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle \langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_2\|^2 \|\mathbf{v}_1\|^2} \right) \mathbf{u}_1 + \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{u}_2 \end{aligned}$$

making \mathbf{u}_3 a linear combination of \mathbf{u}_1 and \mathbf{u}_2 , which contradicts the assumption of the linear independence of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

53. The diagonal entries of R are $\langle \mathbf{u}_i, \mathbf{q}_i \rangle$ for $i = 1, 2, \dots, n$, where $\mathbf{q}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$ is the normalization of a vector \mathbf{v}_i that is the result of applying the Gram-Schmidt process to $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$. Thus, \mathbf{v}_i is \mathbf{u}_i minus a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{i-1}$, so $\mathbf{u}_i = \mathbf{v}_i + k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_{i-1} \mathbf{v}_{i-1}$. Thus, $\langle \mathbf{u}_i, \mathbf{v}_i \rangle = \langle \mathbf{v}_i, \mathbf{v}_i \rangle$ and $\langle \mathbf{u}_i, \mathbf{q}_i \rangle = \left\langle \mathbf{u}_i, \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|} \right\rangle = \frac{1}{\|\mathbf{v}_i\|} \langle \mathbf{v}_i, \mathbf{v}_i \rangle = \|\mathbf{v}_i\|$. Since each vector \mathbf{v}_i is nonzero, each diagonal entry of R is nonzero.

55. (b) The range of T is W ; the kernel of T is W^\perp .

True-False Exercises

- (a) False. For example, the vectors $(1, 0)$ and $(1, 1)$ in \mathbb{R}^2 are linearly independent but not orthogonal.
- (b) False. The vectors must be nonzero for this to be true.
- (c) True. A nontrivial subspace of \mathbb{R}^3 will have a basis, which can be transformed into an orthonormal basis with respect to the Euclidean inner product.
- (d) True. A nonzero finite-dimensional inner product space will have finite basis which can be transformed into an orthonormal basis with respect to the inner product via the Gram-Schmidt process with normalization.
- (e) False. $\text{proj}_W \mathbf{x}$ is a vector in W .
- (f) True. Every invertible $n \times n$ matrix has a QR -decomposition.

6.4 Best Approximation; Least Squares

$$1. \quad A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}; \quad A^T A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}; \quad A^T \mathbf{b} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix};$$

$$\text{The associated normal equation is } \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}.$$

$$2. \quad A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}; \quad A^T A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 4 & 2 \\ 0 & 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 15 & -1 & 5 \\ -1 & 22 & 30 \\ 5 & 30 & 45 \end{bmatrix};$$

$$A^T \mathbf{b} = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 4 & 2 \\ 0 & 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 13 \end{bmatrix};$$

The associated normal equation is $\begin{bmatrix} 15 & -1 & 5 \\ -1 & 22 & 30 \\ 5 & 30 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 13 \end{bmatrix}.$

$$3. \quad A^T A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}; \quad A^T \mathbf{b} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix};$$

The normal system $A^T A \mathbf{x} = A^T \mathbf{b}$ is $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}.$

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{20}{11} \\ 0 & 1 & -\frac{8}{11} \end{bmatrix}.$

The solution of this system $x_1 = \frac{20}{11}$, $x_2 = -\frac{8}{11}$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

$$4. \quad A^T A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix}; \quad A^T \mathbf{b} = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix};$$

The normal system $A^T A \mathbf{x} = A^T \mathbf{b}$ is $\begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}.$

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{3}{7} \\ 0 & 1 & -\frac{2}{3} \end{bmatrix}.$

The solution of this system $x_1 = \frac{3}{7}$, $x_2 = -\frac{2}{3}$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

$$5. \quad A^T A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix};$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix};$$

The normal system $A^T A \mathbf{x} = A^T \mathbf{b}$ is
$$\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}.$$

The reduced row echelon form of the augmented matrix of the normal system is
$$\begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{bmatrix}.$$

The solution of this system $x_1 = 12$, $x_2 = -3$, $x_3 = 9$ is the unique least squares solution of $A \mathbf{x} = \mathbf{b}$.

6.
$$A^T A = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & -2 & -1 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 9 & -4 & 0 \\ -4 & 6 & -5 \\ 0 & -5 & 6 \end{bmatrix};$$

$$A^T \mathbf{b} = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & -2 & -1 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix};$$

The normal system $A^T A \mathbf{x} = A^T \mathbf{b}$ is
$$\begin{bmatrix} 9 & -4 & 0 \\ -4 & 6 & -5 \\ 0 & -5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix}.$$

The reduced row echelon form of the augmented matrix of the normal system is
$$\begin{bmatrix} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 26 \end{bmatrix}.$$

The solution of this system $x_1 = 14$, $x_2 = 30$, $x_3 = 26$ is the unique least squares solution of $A \mathbf{x} = \mathbf{b}$.

7. Least squares error vector: $\mathbf{e} = \mathbf{b} - A \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} \frac{20}{11} \\ -\frac{8}{11} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} \frac{28}{11} \\ \frac{16}{11} \\ \frac{40}{11} \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} \\ -\frac{27}{11} \\ \frac{15}{11} \end{bmatrix};$

$$A^T \mathbf{e} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} -\frac{6}{11} \\ -\frac{27}{11} \\ \frac{15}{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$
 therefore the least squares error vector is orthogonal to every vector in the column space of A .

Least squares error: $\|\mathbf{b} - A \mathbf{x}\| = \sqrt{\left(-\frac{6}{11}\right)^2 + \left(-\frac{27}{11}\right)^2 + \left(\frac{15}{11}\right)^2} = \frac{3}{11} \sqrt{110} \approx 2.86.$

8. Least squares error vector: $\mathbf{e} = \mathbf{b} - A \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{7} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{46}{21} \\ -\frac{5}{21} \\ \frac{13}{21} \end{bmatrix} = \begin{bmatrix} -\frac{4}{21} \\ -\frac{16}{21} \\ \frac{8}{21} \end{bmatrix};$

$A^T \mathbf{e} = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{4}{21} \\ -\frac{16}{21} \\ \frac{8}{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; therefore the least squares error vector is orthogonal to every vector in the column space of A .

Least squares error: $\|\mathbf{b} - A\mathbf{x}\| = \sqrt{\left(-\frac{4}{21}\right)^2 + \left(-\frac{16}{21}\right)^2 + \left(\frac{8}{21}\right)^2} = \frac{4}{\sqrt{21}} \approx 0.873$.

9. Least squares error vector: $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 3 \end{bmatrix};$

$A^T \mathbf{e} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; therefore the least squares error vector is orthogonal to every vector

in the column space of A .

Least squares error: $\|\mathbf{b} - A\mathbf{x}\| = \sqrt{3^2 + (-3)^2 + 0^2 + 3^2} = 3\sqrt{3} \approx 5.196$.

10. Least squares error vector: $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 14 \\ 30 \\ 26 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \end{bmatrix};$

$A^T \mathbf{e} = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & -2 & -1 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ therefore the least squares error vector is orthogonal to every vector

in the column space of A .

Least squares error: $\|\mathbf{b} - A\mathbf{x}\| = \sqrt{(-2)^2 + 0^2 + 2^2 + 2^2} = 2\sqrt{3} \approx 3.464$.

11. $A^T A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 24 & 12 \\ 12 & 6 \end{bmatrix}; \quad A^T \mathbf{b} = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix};$

The normal system $A^T A \mathbf{x} = A^T \mathbf{b}$ is $\begin{bmatrix} 24 & 12 \\ 12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$.

The general solution of the normal system is $x_1 = \frac{1}{2} - \frac{1}{2}t$, $x_2 = t$. All of these are least squares solutions of $A\mathbf{x} = \mathbf{b}$. The error vector is the same for all solutions:

$$\mathbf{e} = \mathbf{b} - \mathbf{Ax} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} - \frac{1}{2}t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

$$12. \quad A^T A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & -6 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 42 \\ 42 & 126 \end{bmatrix}; \quad A^T \mathbf{b} = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix};$$

$$\text{The normal system } A^T \mathbf{Ax} = A^T \mathbf{b} \text{ is } \begin{bmatrix} 14 & 42 \\ 42 & 126 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}.$$

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 3 & \frac{2}{7} \\ 0 & 0 & 0 \end{bmatrix}$.

The general solution of the normal system is $x_1 = \frac{2}{7} - 3t$, $x_2 = t$. All of these are least squares solutions of $\mathbf{Ax} = \mathbf{b}$. The error vector is the same for all solutions:

$$\mathbf{e} = \mathbf{b} - \mathbf{Ax} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -2 & -6 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} \frac{2}{7} - 3t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ \frac{6}{7} \end{bmatrix} = \begin{bmatrix} \frac{5}{7} \\ \frac{4}{7} \\ \frac{1}{7} \end{bmatrix}.$$

$$13. \quad A^T A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix}; \quad A^T \mathbf{b} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \\ 7 \end{bmatrix};$$

$$\text{The normal system } A^T \mathbf{Ax} = A^T \mathbf{b} \text{ is } \begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \\ 7 \end{bmatrix}.$$

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & 1 & -\frac{7}{6} \\ 0 & 1 & 1 & \frac{7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution of the normal system is $x_1 = -\frac{7}{6} - t$, $x_2 = \frac{7}{6} - t$, $x_3 = t$. All of these are least squares solutions of $\mathbf{Ax} = \mathbf{b}$. The error vector is the same for all solutions:

$$\mathbf{e} = \mathbf{b} - \mathbf{Ax} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{7}{6} - t \\ \frac{7}{6} - t \\ t \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \begin{bmatrix} \frac{14}{3} \\ -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{7}{6} \\ -\frac{49}{6} \end{bmatrix}.$$

$$14. \quad A^T A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -4 & 10 \\ -1 & 3 & -7 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix} = \begin{bmatrix} 11 & 12 & -7 \\ 12 & 120 & -84 \\ -7 & -84 & 59 \end{bmatrix}; \quad A^T \mathbf{b} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -4 & 10 \\ -1 & 3 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 22 \\ -15 \end{bmatrix};$$

The normal system $A^T A \mathbf{x} = A^T \mathbf{b}$ is $\begin{bmatrix} 11 & 12 & -7 \\ 12 & 120 & -84 \\ -7 & -84 & 59 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 22 \\ -15 \end{bmatrix}.$

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & -\frac{5}{7} & \frac{13}{84} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

The general solution of the normal system is $x_1 = \frac{2}{7} - \frac{1}{7}t$, $x_2 = \frac{13}{84} + \frac{5}{7}t$, $x_3 = t$. All of these are least squares solutions of $A\mathbf{x} = \mathbf{b}$. The error vector is the same for all solutions:

$$\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix} \begin{bmatrix} \frac{2}{7} - \frac{1}{7}t \\ \frac{13}{84} + \frac{5}{7}t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{7}{6} \\ -\frac{1}{3} \\ \frac{11}{6} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ -\frac{5}{3} \\ -\frac{5}{6} \end{bmatrix}.$$

$$15. \quad A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}; \quad A^T \mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix};$$

The normal system $A^T A \mathbf{x} = A^T \mathbf{b}$ is $\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}.$

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{17}{95} \\ 0 & 1 & \frac{143}{285} \end{bmatrix}.$

The solution of this system $x_1 = \frac{17}{95}$, $x_2 = \frac{143}{285}$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

By Theorem 6.4.2, $\text{proj}_W \mathbf{b} = A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{17}{95} \\ \frac{143}{285} \end{bmatrix} = \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}.$

This matches the result obtained using Theorem 6.4.4:

$$\begin{aligned} \text{proj}_W \mathbf{b} &= A(A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \left(\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \left(\frac{1}{(14)(21) - (-3)(-3)} \begin{bmatrix} 21 & 3 \\ 3 & 14 \end{bmatrix} \right) \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \\ &= \frac{1}{285} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 21 & 3 \\ 3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \end{aligned}$$

$$= \frac{1}{285} \begin{bmatrix} -92 \\ 439 \\ 470 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}$$

$$16. \quad A^T A = \begin{bmatrix} 5 & 1 & 4 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix}; \quad A^T \mathbf{b} = \begin{bmatrix} 5 & 1 & 4 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \end{bmatrix};$$

The normal system $A^T A \mathbf{x} = A^T \mathbf{b}$ is $\begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & -\frac{1}{7} \\ 0 & 1 & -\frac{2}{7} \end{bmatrix}$.

The solution of this system $x_1 = -\frac{1}{7}$, $x_2 = -\frac{2}{7}$ is the unique least squares solution of $A \mathbf{x} = \mathbf{b}$.

By Theorem 6.4.2, $\text{proj}_W \mathbf{b} = A \mathbf{x} = \begin{bmatrix} 5 & 1 \\ 1 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$.

This matches the result obtained using Theorem 6.4.4:

$$\text{proj}_W \mathbf{b} = A(A^T A)^{-1} A^T \mathbf{b}$$

$$= \begin{bmatrix} 5 & 1 \\ 1 & 3 \\ 4 & -2 \end{bmatrix} \left(\begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix} \right)^{-1} \begin{bmatrix} 5 & 1 & 4 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ 1 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1/42 & 0 \\ 0 & 1/14 \end{bmatrix} \begin{bmatrix} 5 & 1 & 4 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

17. We follow the procedure of Example 2. For $A = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix}$, we have

$$A^T A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 24 \end{bmatrix} \text{ and } A^T \mathbf{u} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \end{bmatrix};$$

The normal system $A^T A \mathbf{x} = A^T \mathbf{u}$ is $\begin{bmatrix} 6 & 6 \\ 6 & 24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \end{bmatrix}$. The reduced row echelon form of the

augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & -\frac{7}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$ so that the least squares solution of $A \mathbf{x} = \mathbf{u}$ is

$$\mathbf{x} = \begin{bmatrix} -\frac{7}{3} \\ \frac{1}{3} \end{bmatrix}. \text{ Denoting } W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \text{ we obtain } \text{proj}_W \mathbf{u} = A \mathbf{x} = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -\frac{7}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}.$$

18. Let $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$. $A^T A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix};$

$$A^T \mathbf{u} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ -21 \end{bmatrix}.$$

The normal system $A^T A \mathbf{x} = A^T \mathbf{u}$ is $\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ -21 \end{bmatrix}$. The reduced row echelon form of the

augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ so that the least squares solution of $A \mathbf{x} = \mathbf{u}$ is

$$\mathbf{x} = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}. \text{ Denoting } W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ we obtain } \text{proj}_W \mathbf{u} = A \mathbf{x} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 9 \\ 5 \end{bmatrix}.$$

19. Letting $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have $P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ([1])^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} =$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \text{ This matches the matrix in Table 3 of Section 1.8.}$$

20. Letting $A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have $P = A(A^T A)^{-1} A^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ([1])^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1] \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. This matches the matrix in Table 3 of Section 1.8.

21. Letting $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have $A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matches the matrix in Table 4 of Section 1.8.

22. Letting $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have $A^T A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matches the matrix in Table 4 of Section 1.8.

23. We use Theorem 6.4.6: $\mathbf{x} = R^{-1} Q^T \mathbf{b} = \frac{1}{(5)(\frac{7}{5})} \begin{bmatrix} \frac{7}{5} & \frac{1}{5} \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{35} \\ 0 & \frac{5}{7} \end{bmatrix} \begin{bmatrix} \frac{1}{5} \\ \frac{18}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{7} \\ \frac{18}{7} \end{bmatrix}.$

24. We use Theorem 6.4.6: $\mathbf{x} = R^{-1} Q^T \mathbf{b} = \frac{1}{(5)(1)} \begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$

25. (a) If $x=s$ and $y=t$, then a point on the plane is $(s, t, -5s+3t) = s(1, 0, -5) + t(0, 1, 3)$.

$\mathbf{w}_1 = (1, 0, -5)$ and $\mathbf{w}_2 = (0, 1, 3)$ form a basis for W (they are linearly independent since neither of them is a scalar multiple of the other).

(b) Letting $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}$, Formula (11) yields

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix} \left(\begin{bmatrix} 26 & -15 \\ -15 & 10 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix} \left(\frac{1}{(26)(10) - (-15)(-15)} \begin{bmatrix} 10 & 15 \\ 15 & 26 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} \\
&= \frac{1}{35} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 10 & 15 \\ 15 & 26 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 10 & 15 & -5 \\ 15 & 26 & 3 \\ -5 & 3 & 34 \end{bmatrix}.
\end{aligned}$$

26. (a) $W = \text{span}\{(2, -1, 4)\}$ so that the vector $(2, -1, 4)$ forms a basis for W (its linear independence follows from Theorem 4.4.2(b))

(b) Letting $A = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, Formula (11) yields

$$\begin{aligned}
P &= A(A^T A)^{-1} A^T = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} [21]^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}.
\end{aligned}$$

27. The reduced row echelon form of the augmented matrix of the given homogeneous system is

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \text{ so that the general solution is } x_1 = -\frac{1}{2}s + \frac{1}{2}t, \quad x_2 = -\frac{1}{2}s - \frac{1}{2}t, \quad x_3 = s, \quad x_4 = t.$$

The solution space W is spanned by vectors $(-1, -1, 2, 0)$ and $(1, -1, 0, 2)$.

We construct the matrix with these vectors as its columns $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$ then follow the procedure of

Example 2 in Section 6.4.

$$A^T A = \begin{bmatrix} -1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & -1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}; \quad A^T \mathbf{u} = \begin{bmatrix} -1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix};$$

The normal system $A^T A \mathbf{x} = A^T \mathbf{u}$ is $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. The reduced row echelon form of the augmented

matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$ so that the least squares solution of $A \mathbf{x} = \mathbf{u}$ is $\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. We

$$\text{obtained } \text{proj}_W \mathbf{u} = A \mathbf{x} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

28. Letting $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, Formula (11) in Section 6.4 yields

$$\begin{aligned} P &= A(A^T A)^{-1} A^T \\ &= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \left(\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right)^{-1} \begin{bmatrix} a & b & c \end{bmatrix} \\ &= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \left[a^2 + b^2 + c^2 \right]^{-1} \begin{bmatrix} a & b & c \end{bmatrix} \\ &= \frac{1}{a^2 + b^2 + c^2} \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}. \end{aligned}$$

29. Let W be the row space of A . Since W is also the column space of A^T , by Formula (11) we have

$$P = A^T \left((A^T)^T A^T \right)^{-1} (A^T)^T = A^T (A A^T)^{-1} A.$$

30. Multiplying both sides of $A \mathbf{x} = \mathbf{b}$ on the left by A^T yields $A^T A \mathbf{x} = A^T \mathbf{b}$. By Theorem 6.4.4, the least squares solution is $(A^T A)^{-1} A^T \mathbf{b} = (A^T A)^{-1} A^T A \mathbf{x} = \mathbf{x}$.

31. Since \mathbf{b} is orthogonal to the column space of A , it follows that $A^T \mathbf{b} = \mathbf{0}$. By Theorem 6.4.4, the least squares solution is $\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = (A^T A)^{-1} \mathbf{0} = \mathbf{0}$.

32. Partitioning A into columns $A = [\mathbf{u}_1 | \cdots | \mathbf{u}_n]$, we have $A^T A = [A^T \mathbf{u}_1 | \cdots | A^T \mathbf{u}_n]$.

Assume the columns of A are linearly dependent, i.e., there exist scalars k_1, \dots, k_n , not all equal 0, such that

$$k_1 \mathbf{u}_1 + \cdots + k_n \mathbf{u}_n = \mathbf{0}.$$

Multiplying both sides on the left by A^T we obtain

$$k_1 A^T \mathbf{u}_1 + \cdots + k_n A^T \mathbf{u}_n = \mathbf{0}.$$

Since scalars k_1, \dots, k_n are not all equal 0, that means the columns of $A^T A$ are linearly dependent. However, this contradicts the assumption of the invertibility of $A^T A$.

Consequently, the columns of A must be linearly independent.

True-False Exercises

- (a) True. $A^T A$ is an $n \times n$ matrix.
- (b) False. Only square matrices have inverses, but $A^T A$ can be invertible when A is not a square matrix.
- (c) True. If A is invertible, so is A^T , so the product $A^T A$ is also invertible.
- (d) True. Multiplying both sides of $A\mathbf{x} = \mathbf{b}$ on the left by A^T yields $A^T A\mathbf{x} = A^T \mathbf{b}$.
- (e) False. By Theorem 6.4.2, the normal system $A^T A\mathbf{x} = A^T \mathbf{b}$ is always consistent.
- (f) True. This follows from Theorem 6.4.2.
- (g) False. There may be more than one least squares solution as shown in Example 2.
- (h) True. This follows from Theorem 6.4.4.

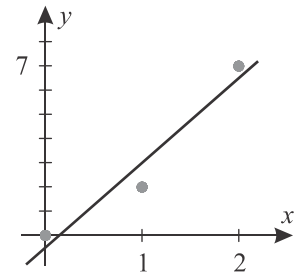
6.5 Mathematical Modeling Using Least Squares

1. We have $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, $M^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $M^T M = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$,

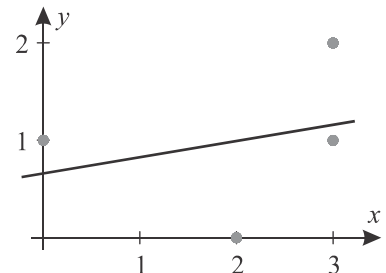
$$(M^T M)^{-1} = \frac{1}{15-9} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}, \text{ and}$$

$$\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{7}{2} \end{bmatrix}$$

so the least squares straight line fit to the given data points is $y = -\frac{1}{2} + \frac{7}{2}x$.



2. We have $M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}$, $M^T M = \begin{bmatrix} 4 & 8 \\ 8 & 22 \end{bmatrix}$, $(M^T M)^{-1} = \frac{1}{24} \begin{bmatrix} 22 & -8 \\ -8 & 4 \end{bmatrix}$,



and $\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} = \frac{1}{24} \begin{bmatrix} 22 & -8 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{6} \end{bmatrix}$ so the least squares straight line fit to the

given data points is

$$y = \frac{2}{3} + \frac{1}{6}x.$$

3. We have $M = \begin{bmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 5 & 5^2 \\ 1 & 6 & 6^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix},$

$$M^T M = \begin{bmatrix} 4 & 16 & 74 \\ 16 & 74 & 376 \\ 74 & 376 & 2018 \end{bmatrix},$$

$$(M^T M)^{-1} = \frac{1}{90} \begin{bmatrix} 1989 & -1116 & 135 \\ -1116 & 649 & -80 \\ 135 & -80 & 10 \end{bmatrix}, \text{ and}$$

$$\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} = \frac{1}{90} \begin{bmatrix} 1989 & -1116 & 135 \\ -1116 & 649 & -80 \\ 135 & -80 & 10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{bmatrix} \begin{bmatrix} 0 \\ -10 \\ -48 \\ -76 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

so the least squares quadratic fit to the given data points is $y = 2 + 5x - 3x^2$.

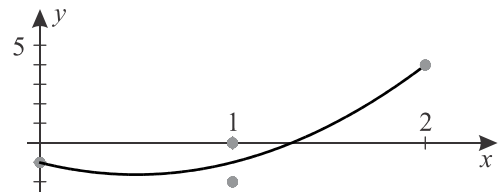
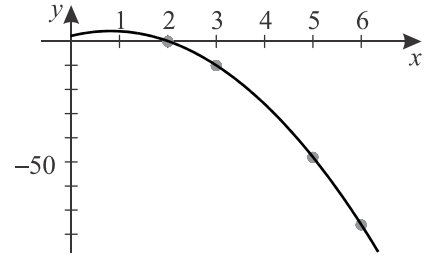
4. We have $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, M^T M = \begin{bmatrix} 4 & 4 & 6 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{bmatrix},$

$$(M^T M)^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{9}{2} & -2 \\ \frac{1}{2} & -2 & 1 \end{bmatrix}, \text{ and}$$

$$\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{9}{2} & -2 \\ \frac{1}{2} & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{5}{2} \\ \frac{5}{2} \end{bmatrix} \text{ so the least squares quadratic fit to the}$$

given data points is $y = -1 - \frac{5}{2}x + \frac{5}{2}x^2$.

5. With the substitution $X = \frac{1}{x}$, the problem becomes to find a line of the form $y = a + b \cdot X$ that best fits the data points $(1, 7), (\frac{1}{3}, 3), (\frac{1}{6}, 1)$.



We have $M = \begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{6} \end{bmatrix}$, $M^T M = \begin{bmatrix} 3 & \frac{3}{2} \\ \frac{3}{2} & \frac{41}{36} \end{bmatrix}$, $(M^T M)^{-1} = \frac{1}{42} \begin{bmatrix} 41 & -54 \\ -54 & 108 \end{bmatrix}$, and

$$\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} = \frac{1}{42} \begin{bmatrix} 41 & -54 \\ -54 & 108 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{21} \\ \frac{48}{7} \end{bmatrix}. \text{ The line in terms of } X \text{ is } y = \frac{5}{21} + \frac{48}{7}X, \text{ so the}$$

required curve is $y = \frac{5}{21} + \frac{48}{7x}$.

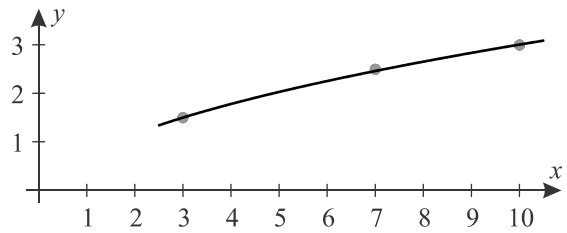
6. With the substitution $X = \sqrt{x}$, the problem becomes to find a line of the form $y = a + b \cdot X$ that best fits the data points $(\sqrt{3}, \frac{3}{2})$, $(\sqrt{7}, \frac{5}{2})$, $(\sqrt{10}, 3)$.

We have $M = \begin{bmatrix} 1 & \sqrt{3} \\ 1 & \sqrt{7} \\ 1 & \sqrt{10} \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \\ 3 \end{bmatrix}$, and

$M^T M = \begin{bmatrix} 3 & \sqrt{3} + \sqrt{7} + \sqrt{10} \\ \sqrt{3} + \sqrt{7} + \sqrt{10} & 20 \end{bmatrix}$. We conduct the remaining computations approximately:

$\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} \approx \begin{bmatrix} -0.316 \\ 1.054 \end{bmatrix}$. The line in terms of X is approximately $y = -0.316 + 1.054X$, so the

required curve is approximately $y = -0.316 + 1.054\sqrt{x}$.



7. The two column vectors of M are linearly independent if and only if neither is a multiple of the other. Since all the entries in the first column are equal, the columns are linearly independent if and only if the second column has at least two different entries, i.e., if and only if at least two of the numbers x_1, x_2, \dots, x_n are distinct.

True-False Exercises

- (a) False. There is only a unique least squares straight line fit if the data points do not all lie on a vertical line.
- (b) True. If the points are not collinear, there is no solution to the system.
- (c) True.
- (d) False. The line minimizes the sum of the *squares* of the data errors.

6.6 Function Approximation; Fourier Series

1. $a_0 = \frac{1}{\pi} \int_0^{2\pi} (1+x) dx = \frac{1}{\pi} \left(x + \frac{x^2}{2} \right) \Big|_0^{2\pi} = 2 + 2\pi$

Using integration by parts to integrate both $x \cos(kx)$ and $x \sin(kx)$ we obtain

$$a_k = \frac{1}{\pi} \int_0^{2\pi} (1+x) \cos(kx) dx = \left(\frac{1+x}{k\pi} \sin(kx) + \frac{1}{k^2\pi} \cos(kx) \right) \Big|_0^{2\pi} = 0 \text{ and}$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} (1+x) \sin(kx) dx = \left(-\frac{1+x}{k\pi} \cos(kx) + \frac{1}{k^2\pi} \sin(kx) \right) \Big|_0^{2\pi} = -\frac{2}{k}$$

(a) $1+x \approx \frac{a_0}{2} + a_1 \cos x + a_2 \cos(2x) + a_3 \cos(3x) + b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x)$ yields

$$1+x \approx 1+\pi + 0 \cos x + 0 \cos(2x) - \frac{2}{1} \sin x - \frac{2}{2} \sin(2x) = 1+\pi - 2 \sin x - \sin 2x$$

(b) $1+x \approx \frac{a_0}{2} + a_1 \cos x + a_2 \cos(2x) + \cdots + a_n \cos(nx) + b_1 \sin x + b_2 \sin(2x) + \cdots + b_n \sin(nx)$ yields

$$1+x \approx 1+\pi - \frac{2}{1} \sin x - \frac{2}{2} \sin(2x) - \cdots - \frac{2}{n} \sin(nx)$$

2. $a_0 = \frac{1}{\pi} \int_0^{2\pi} x^2 dx = \frac{x^3}{3\pi} \Big|_0^{2\pi} = \frac{8\pi^2}{3}$

Using integration by parts twice to integrate both $x^2 \cos(kx)$ and $x^2 \sin(kx)$ we obtain

$$a_k = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos(kx) dx = \left(\frac{x^2}{k\pi} \sin(kx) + \frac{2x}{k^2\pi} \cos(kx) - \frac{2}{k^3\pi} \sin(kx) \right) \Big|_0^{2\pi} = \frac{4}{k^2} \text{ and}$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin(kx) dx = \left(-\frac{x^2}{k\pi} \cos(kx) + \frac{2x}{k^2\pi} \sin(kx) + \frac{2}{k^3\pi} \cos(kx) \right) \Big|_0^{2\pi} = -\frac{4\pi}{k}$$

(a) $x^2 \approx \frac{a_0}{2} + a_1 \cos x + a_2 \cos(2x) + a_3 \cos(3x) + b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x)$ yields

$$x^2 \approx \frac{4\pi^2}{3} + 4 \cos x + \cos(2x) + \frac{4}{9} \cos(3x) - 4\pi \sin x - 2\pi \sin(2x) - \frac{4\pi}{3} \sin(3x)$$

(b) $x^2 \approx \frac{a_0}{2} + a_1 \cos x + a_2 \cos(2x) + \cdots + a_n \cos(nx) + b_1 \sin x + b_2 \sin(2x) + \cdots + b_n \sin(nx)$ yields

$$x^2 \approx \frac{4\pi^2}{3} + \frac{4}{1^2} \cos x + \frac{4}{2^2} \cos(2x) + \cdots + \frac{4}{n^2} \cos(nx) - \frac{4\pi}{1} \sin x - \frac{4\pi}{2} \sin(2x) - \cdots - \frac{4\pi}{n} \sin(nx)$$

3. (a) Let us denote $W = \text{span}\{1, e^x\}$. Applying the Gram-Schmidt process to the basis $\mathbf{u}_1 = 1$ and $\mathbf{u}_2 = e^x$ we obtain an orthogonal basis

$$\mathbf{v}_1 = 1, \quad \mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = e^x - \frac{\int_0^1 e^x dx}{\int_0^1 1 dx} 1 = e^x - \frac{e^x \Big|_0^1}{x \Big|_0^1} 1 = e^x - (e-1)1 = e^x - e + 1.$$

$$\begin{aligned} \text{Since } \int_0^1 (e^x - e + 1)^2 dx &= \int_0^1 (1 - 2e + e^2 + 2e^x - 2ee^x + e^{2x}) dx \\ &= \left(x - 2ex + e^2x + 2e^x - 2e^{x+1} + \frac{1}{2}e^{2x} \right) \Big|_0^1 = -\frac{3}{2} + 2e - \frac{1}{2}e^2 = \frac{1}{2}(e-1)(3-e), \text{ an orthonormal basis is} \end{aligned}$$

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{\int_0^1 1 dx}} = \frac{1}{\sqrt{x \Big|_0^1}} = 1, \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{e^x - e + 1}{\sqrt{\frac{1}{2}(e-1)(3-e)}}.$$

The least squares approximation to $f(x) = x$ from W is

$$\begin{aligned} \text{proj}_W \mathbf{f} &= \langle \mathbf{f}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{f}, \mathbf{q}_2 \rangle \mathbf{q}_2 = \int_0^1 x dx + \frac{2}{(e-1)(3-e)} \left(\int_0^1 x(e^x - e + 1) dx \right) (e^x - e + 1) \\ &= \frac{1}{2} + \frac{2}{(e-1)(3-e)} \left(xe^x - e^x - \frac{x^2e}{2} + \frac{x^2}{2} \right) \Big|_0^1 (e^x - e + 1) = \frac{1}{2} + \frac{2}{(e-1)(3-e)} \left(-\frac{e}{2} + \frac{3}{2} \right) (e^x - e + 1) \\ &= \frac{1}{2} + \frac{e^x - e + 1}{e-1} = \frac{1}{2} + \frac{e^x}{e-1} - 1 = \frac{e^x}{e-1} - \frac{1}{2}. \end{aligned}$$

(b) The mean square error is $\int_0^1 \left(x - \left(\frac{e^x}{e-1} - \frac{1}{2} \right) \right)^2 dx = \frac{7e-19}{12e-12} \approx 0.00136$.

4. (a) Let us denote $W = \text{span}\{1, x\}$.

Applying the Gram-Schmidt process to the basis $\mathbf{u}_1 = 1$ and $\mathbf{u}_2 = x$ we obtain an orthogonal basis

$$\mathbf{v}_1 = 1, \quad \mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = x - \frac{\int_0^1 x dx}{\int_0^1 1 dx} 1 = x - \frac{\frac{x^2}{2} \Big|_0^1}{x \Big|_0^1} = x - \frac{1}{2} \text{ and an orthonormal basis}$$

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{\int_0^1 1 dx}} = \frac{1}{\sqrt{x \Big|_0^1}} = 1, \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 \left(x - \frac{1}{2}\right)^2 dx}} = \frac{x - \frac{1}{2}}{\sqrt{\left(\frac{x - \frac{1}{2}}{3}\right)^3 \Big|_0^1}} = 2\sqrt{3} \left(x - \frac{1}{2}\right).$$

The least squares approximation to $f(x) = e^x$ from W is

$$\begin{aligned} \text{proj}_W \mathbf{f} &= \langle \mathbf{f}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{f}, \mathbf{q}_2 \rangle \mathbf{q}_2 = \int_0^1 e^x dx + \left(\int_0^1 2\sqrt{3} \left(x - \frac{1}{2}\right) e^x dx \right) 2\sqrt{3} \left(x - \frac{1}{2}\right) \\ &= e - 1 + \left(2\sqrt{3} \left(x e^x - \frac{3}{2} e^x \right) \Big|_0^1 \right) 2\sqrt{3} \left(x - \frac{1}{2}\right) = 4e - 10 + 6(3 - e)x. \end{aligned}$$

(b) The mean square error is $\int_0^1 \left(e^x - (4e - 10 + 6(3 - e)x) \right)^2 dx = -\frac{57}{2} + 20e - \frac{7e^2}{2} \approx 0.00394$.

5. (a) Let us denote $W = \text{span}\{1, x, x^2\}$.

Applying the Gram-Schmidt process to the basis $\mathbf{u}_1 = 1$, $\mathbf{u}_2 = x$, and $\mathbf{u}_3 = x^2$ we obtain an

$$\text{orthogonal basis } \mathbf{v}_1 = 1, \quad \mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = x - \frac{\int_{-1}^1 x dx}{\int_{-1}^1 1 dx} 1 = x - \frac{\frac{x^2}{2} \Big|_{-1}^1}{x \Big|_{-1}^1} = x - 0 = x,$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = x^2 - \frac{\int_{-1}^1 x^2 dx}{\int_{-1}^1 1 dx} 1 - \frac{\int_{-1}^1 x^3 dx}{\int_{-1}^1 x^2 dx} x = x^2 - \frac{\frac{x^3}{3} \Big|_{-1}^1}{x \Big|_{-1}^1} - \frac{\frac{x^4}{4} \Big|_{-1}^1}{\frac{x^3}{3} \Big|_{-1}^1} x = x^2 - \frac{1}{3},$$

$$\text{and an orthonormal basis } \mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{\int_{-1}^1 1 dx}} = \frac{1}{\sqrt{x \Big|_{-1}^1}} = \frac{1}{\sqrt{2}},$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{x}{\sqrt{\int_{-1}^1 x^2 dx}} = \frac{x}{\sqrt{\frac{x^3}{3} \Big|_{-1}^1}} = \sqrt{\frac{3}{2}} x, \quad \mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^1 \left(x^2 - \frac{1}{3}\right)^2 dx}} = \frac{x^2 - \frac{1}{3}}{\frac{2}{3}\sqrt{\frac{2}{5}}}.$$

The least squares approximation to $f(x) = \sin \pi x$ from W is

$$\begin{aligned} \text{proj}_W \mathbf{f} &= \langle \mathbf{f}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{f}, \mathbf{q}_2 \rangle \mathbf{q}_2 + \langle \mathbf{f}, \mathbf{q}_3 \rangle \mathbf{q}_3 \\ &= \frac{1}{4} \int_{-1}^1 \sin \pi x dx + \frac{3}{2} \left(\int_{-1}^1 x \sin \pi x dx \right) x + \frac{45}{8} \left(\int_{-1}^1 \left(x^2 - \frac{1}{3}\right) \sin \pi x dx \right) \left(x^2 - \frac{1}{3}\right) \\ &= 0 + \frac{3}{2} \left(-\frac{x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} \right) \Big|_{-1}^1 x + 0 = \frac{3}{2} \cdot \frac{2}{\pi} x = \frac{3x}{\pi}. \end{aligned}$$

(b) The mean square error is $\int_{-1}^1 \left(\sin \pi x - \frac{3x}{\pi} \right)^2 dx = 1 - \frac{6}{\pi^2} \approx 0.392$.

8. We use Formulas (8) in Section 6.6

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} (\pi - x) dx = -\frac{(\pi - x)^2}{2\pi} \Big|_0^{2\pi} = 0.$$

Using integration by parts to integrate both $(\pi - x)\cos(kx)$ and $(\pi - x)\sin(kx)$ we obtain

$$a_k = \frac{1}{\pi} \int_0^{2\pi} (\pi - x)\cos(kx) dx = \left(\frac{\pi - x}{k\pi} \sin(kx) - \frac{1}{k^2\pi} \cos(kx) \right) \Big|_0^{2\pi} = 0 \text{ and}$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} (\pi - x)\sin(kx) dx = \left(-\frac{\pi - x}{k\pi} \cos(kx) - \frac{1}{k^2\pi} \sin(kx) \right) \Big|_0^{2\pi} = \frac{2}{k}.$$

The Fourier series for $\pi - x$ over the interval $[0, 2\pi]$ is $\sum_{k=1}^{\infty} \frac{2}{k} \sin(kx)$.

9. Let $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi \leq x \leq 2\pi \end{cases}$.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} dx = 1$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx = \frac{1}{\pi} \int_0^{\pi} \cos kx dx = 0$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx = \frac{1}{\pi} \int_0^{\pi} \sin kx dx = \frac{1}{k\pi} (1 - (-1)^k)$$

So the Fourier series is $\frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi} (1 - (-1)^k) \sin kx$.

10. All the coefficients of the series are zero except for $b_3 = 1$, i.e., $\sin(3x)$ is its own Fourier series.

True-False Exercises

- (a) False. The area between the graphs is the error, not the mean square error.
 (b) True.
 (c) True.
 (d) False. $\|1\| = \langle 1, 1 \rangle = \int_0^{2\pi} 1^2 dx = 2\pi \neq 1$.
 (e) True.

Chapter 6 Supplementary Exercises

1. (a) Let $\mathbf{v} = (v_1, v_2, v_3, v_4)$.

$$\langle \mathbf{v}, \mathbf{u}_1 \rangle = v_1, \langle \mathbf{v}, \mathbf{u}_2 \rangle = v_2, \langle \mathbf{v}, \mathbf{u}_3 \rangle = v_3, \langle \mathbf{v}, \mathbf{u}_4 \rangle = v_4$$

If $\langle \mathbf{v}, \mathbf{u}_1 \rangle = \langle \mathbf{v}, \mathbf{u}_4 \rangle = 0$, then $v_1 = v_4 = 0$ and $\mathbf{v} = (0, v_2, v_3, 0)$. Since the angle θ between \mathbf{u} and \mathbf{v} satisfies $\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}$, \mathbf{v} making equal angles with \mathbf{u}_2 and \mathbf{u}_3 means that $v_2 = v_3$. In order for the angle between \mathbf{v} and \mathbf{u}_3 to be defined $\|\mathbf{v}\| \neq 0$. Thus, $\mathbf{v} = (0, a, a, 0)$ with $a \neq 0$.

- (b) As in part (a), since $\langle \mathbf{x}, \mathbf{u}_1 \rangle = \langle \mathbf{x}, \mathbf{u}_4 \rangle = 0$, $x_1 = x_4 = 0$.

Since $\|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1$ and we want $\|\mathbf{x}\| = 1$, the cosine of the angle between \mathbf{x} and \mathbf{u}_2 is $\cos \theta_2 = \langle \mathbf{x}, \mathbf{u}_2 \rangle = x_2$ and, similarly, $\cos \theta_3 = \langle \mathbf{x}, \mathbf{u}_3 \rangle = x_3$, so we want $x_2 = 2x_3$, and

$$\mathbf{x} = \langle 0, x_2, 2x_2, 0 \rangle.$$

$$\|\mathbf{x}\| = \sqrt{x_2^2 + 4x_2^2} = \sqrt{5x_2^2} = |x_2|\sqrt{5}.$$

If $\|\mathbf{x}\| = 1$, then $x_2 = \pm \frac{1}{\sqrt{5}}$, so $\mathbf{x} = \pm \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$.

3. Recall that if $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$, then $\langle U, V \rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$.

(a) If U is a diagonal matrix, then $u_2 = u_3 = 0$ and $\langle U, V \rangle = u_1v_1 + u_4v_4$.

For V to be in the orthogonal complement of the subspace of all diagonal matrices, then it must be the case that $v_1 = v_4 = 0$ and V must have zeros on the main diagonal.

(b) If U is a symmetric matrix, then $u_2 = u_3$ and $\langle U, V \rangle = u_1v_1 + u_2(v_2 + v_3) + u_4v_4$.

Since u_1 and u_4 can take on any values, for V to be in the orthogonal complement of the subspace of all symmetric matrices, it must be the case that $v_1 = v_4 = 0$ and $v_2 = -v_3$, thus V must be skew-symmetric.

5. Let $\mathbf{u} = (\sqrt{a_1}, \dots, \sqrt{a_n})$ and $\mathbf{v} = (\frac{1}{\sqrt{a_1}}, \dots, \frac{1}{\sqrt{a_n}})$. By the Cauchy-Schwarz inequality,

$$\langle \mathbf{u}, \mathbf{v} \rangle^2 = \left(\underbrace{1 + \dots + 1}_{n \text{ terms}} \right)^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \text{ or } n^2 \leq (a_1 + \dots + a_n) \left(\frac{1}{a_1} + \dots + \frac{1}{a_n} \right).$$

7. Let $\mathbf{x} = (x_1, x_2, x_3)$.

$$\langle \mathbf{x}, \mathbf{u}_1 \rangle = x_1 + x_2 - x_3$$

$$\langle \mathbf{x}, \mathbf{u}_2 \rangle = -2x_1 - x_2 + 2x_3$$

$$\langle \mathbf{x}, \mathbf{u}_3 \rangle = -x_1 + x_3$$

$\langle \mathbf{x}, \mathbf{u}_3 \rangle = 0 \Rightarrow -x_1 + x_3 = 0$, so $x_1 = x_3$. Then $\langle \mathbf{x}, \mathbf{u}_1 \rangle = x_2$ and $\langle \mathbf{x}, \mathbf{u}_2 \rangle = -x_2$, so $x_2 = 0$ and $\mathbf{x} = (x_1, 0, x_1)$.

Then $\|\mathbf{x}\| = \sqrt{x_1^2 + x_1^2} = \sqrt{2x_1^2} = |x_1|\sqrt{2}$.

If $\|\mathbf{x}\| = 1$ then $x_1 = \pm \frac{1}{\sqrt{2}}$ and the vectors are $\pm \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$.

8. By inspection, the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + \frac{1}{2}u_2v_2 + \frac{1}{3}u_3v_3 + \dots + \frac{1}{n}u_nv_n$ satisfies $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \langle \mathbf{v}_3, \mathbf{v}_3 \rangle = \dots = \langle \mathbf{v}_n, \mathbf{v}_n \rangle = 1$ and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for all $i \neq j$.

9. For $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ in \mathbb{R}^2 , let $\langle \mathbf{u}, \mathbf{v} \rangle = au_1v_1 + bu_2v_2$ be a weighted inner product.

If $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, -1)$ form an orthonormal set, then $\|\mathbf{u}\|^2 = a(1)^2 + b(2)^2 = a + 4b = 1$,

$\|\mathbf{v}\|^2 = a(3)^2 + b(-1)^2 = 9a + b = 1$, and $\langle \mathbf{u}, \mathbf{v} \rangle = a(1)(3) + b(2)(-1) = 3a - 2b = 0$.

This leads to the system
$$\begin{bmatrix} 1 & 4 \\ 9 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 4 & 1 \\ 9 & 1 & 1 \\ 3 & -2 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the system is inconsistent and there is no such weighted inner product.

11. (a) Let $\mathbf{u}_1 = (k, 0, 0, \dots, 0)$, $\mathbf{u}_2 = (0, k, 0, \dots, 0)$, ..., $\mathbf{u}_n = (0, 0, 0, \dots, k)$ be the edges of the ‘cube’ in R^n and $\mathbf{u} = (k, k, k, \dots, k)$ be the diagonal.

Then $\|\mathbf{u}_i\| = k$, $\|\mathbf{u}\| = k\sqrt{n}$, and $\langle \mathbf{u}_i, \mathbf{u} \rangle = k^2$, so $\cos \theta = \frac{\langle \mathbf{u}_i, \mathbf{u} \rangle}{\|\mathbf{u}_i\| \|\mathbf{u}\|} = \frac{k^2}{k(k\sqrt{n})} = \frac{1}{\sqrt{n}}$.

(b) As n approaches ∞ , $\frac{1}{\sqrt{n}}$ approaches 0, so θ approaches $\frac{\pi}{2}$.

13. Recall that \mathbf{u} can be expressed as the linear combination $\mathbf{u} = a_1 \mathbf{v}_1 + \dots + a_n \mathbf{v}_n$ where $a_i = \langle \mathbf{u}, \mathbf{v}_i \rangle$ for $i = 1, \dots, n$. Since $\|\mathbf{v}_i\| = 1$, we have $\cos^2 \alpha_i = \left(\frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{u}\| \|\mathbf{v}_i\|} \right)^2 = \left(\frac{a_i}{\|\mathbf{u}\|} \right)^2 = \frac{a_i^2}{a_1^2 + a_2^2 + \dots + a_n^2}$.

Therefore $\cos^2 \alpha_1 + \dots + \cos^2 \alpha_n = \frac{a_1^2 + a_2^2 + \dots + a_n^2}{a_1^2 + a_2^2 + \dots + a_n^2} = 1$.

15. To show that $(W^\perp)^\perp = W$, we first show that $W \subseteq (W^\perp)^\perp$. If \mathbf{w} is in W , then \mathbf{w} is orthogonal to every vector in W^\perp , so that \mathbf{w} is in $(W^\perp)^\perp$. Thus $W \subseteq (W^\perp)^\perp$.

To show that $(W^\perp)^\perp \subseteq W$, let \mathbf{v} be in $(W^\perp)^\perp$. Since \mathbf{v} is in V , we have, by the Projection Theorem, that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ where \mathbf{w}_1 is in W and \mathbf{w}_2 is in W^\perp . By definition, $\langle \mathbf{v}, \mathbf{w}_2 \rangle = \langle \mathbf{w}_1 + \mathbf{w}_2, \mathbf{w}_2 \rangle = 0$. But

$$\langle \mathbf{v}, \mathbf{w}_2 \rangle = \langle \mathbf{w}_1 + \mathbf{w}_2, \mathbf{w}_2 \rangle = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle + \langle \mathbf{w}_2, \mathbf{w}_2 \rangle = \langle \mathbf{w}_2, \mathbf{w}_2 \rangle$$

so that $\langle \mathbf{w}_2, \mathbf{w}_2 \rangle = 0$. Hence $\mathbf{w}_2 = \mathbf{0}$ and therefore $\mathbf{v} = \mathbf{w}_1$, so that \mathbf{v} is in W . Thus $(W^\perp)^\perp \subseteq W$.

$$17. \quad A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}, \quad A^T \mathbf{b} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} 4s+3 \\ 5s+2 \end{bmatrix}$$

The associated normal system is $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4s+3 \\ 5s+2 \end{bmatrix}$.

If the least squares solution is $x_1 = 1$ and $x_2 = 2$, then $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 71 \\ 95 \end{bmatrix} = \begin{bmatrix} 4s+3 \\ 5s+2 \end{bmatrix}$.

The resulting equations have solutions $s = 17$ and $s = 18.6$, respectively, so no such value of s exists.

18. Using the trigonometric identity $\sin \alpha \sin \beta = \frac{1}{2} \cos(\alpha - \beta) - \frac{1}{2} \cos(\alpha + \beta)$ we obtain

$$\langle \mathbf{f}, \mathbf{g} \rangle = \frac{1}{2} \int_0^{2\pi} \cos((p-q)x) dx - \frac{1}{2} \int_0^{2\pi} \cos((p+q)x) dx \quad \text{where both } p+q \text{ and } p-q \text{ are nonzero integers.}$$

Substituting $u = (p-q)x$ in the first integral, and $t = (p+q)x$ in the second integral yields

$$\langle \mathbf{f}, \mathbf{g} \rangle = \frac{1}{2} \frac{\sin((p-q)x)}{p-q} \Big|_0^{2\pi} - \frac{1}{2} \frac{\sin((p+q)x)}{p+q} \Big|_0^{2\pi} = 0 - 0 = 0 \quad \text{since } \sin(m\pi) = 0 \text{ for any integer } m.$$