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Linear Algebra (MT-1004)

Lecture # 19

PROPERTIES OF VECTORS:

Theorem 4.1.1

Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:

(a) $0\mathbf{u} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{u} = -\mathbf{u}$

(d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

SUBSPACES:

Definition 1

A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

- Subspace:

  $(V,+,\cdot)$: a vector space

  $\left.\begin{array}{l} W \neq \Phi \\ W \subseteq V \end{array}\right\}$: a nonempty subset of V

  $(W,+,\cdot)$: The nonempty subset W is called a subspace **if** W **is a vector space** under the operations of addition and scalar multiplication defined on V

- Trivial subspace:

  Every vector space V has at least two subspaces

  (1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V (It satisfies the ten axioms)

  (2) V is a subspace of V

* Any subspaces other than these two are called proper (or nontrivial) subspaces

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Axioms that is not inherited for Subspace (Rest are supposed to be inherited as ***W** lies in Vector Space* ***V)***

Axiom 1—Closure of W under addition

Axiom 4—Existence of a zero vector in W

Axiom 5—Existence of a negative in W for every vector in W

Axiom 6—Closure of W under scalar multiplication

NOTE: It is necessary to verify that W is closed under addition and scalar multiplication since it is possible that adding two vectors in W or multiplying a vector in W by a scalar produces a vector in V that is outside of W. If such condition happens, then its not a subspace (Example is in Fig. 4.2.1)

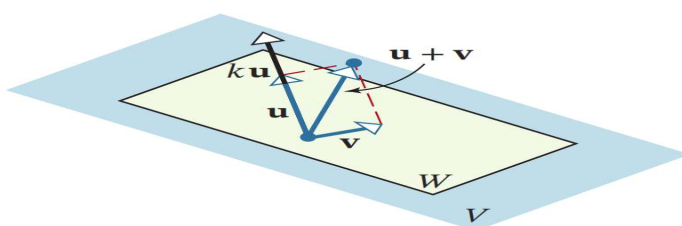

FIGURE 4.2.1 The vectors $\mathbf{u}$ and $\mathbf{v}$ are in W, but the vectors $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ are not.

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Subspace Test

Theorem 4.2.1

Subspace Test

If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.

(b) If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.

Again NOTE:

Note that every vector space has at least two subspaces, itself and its zero subspace.

- Ex : A subspace of $M_{2\times2}$

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times2}$, with the standard operations of matrix addition and scalar multiplication

Sol:

First, we knon that W, the set of all 2×2 symmetric matrices, is an nonempty subset of the vector space $M_{2\times2}$

Second,

$$A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \quad (A_1 + A_2 \in W)$$

$$c \in R,\ A \in W \Rightarrow (cA)^T = cA^T = cA \quad (cA \in W)$$

The definition of a symmetric matrix A is that $A^T = A$

Thus, Th. 2.4 is applied to obtain that W is a subspace of M_{2x2}

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- Ex : The set of singular matrices is not a subspace of $M_{2\times 2}$

Let W be the set of singular (noninvertible) matrices of order 2. Show that W is not a subspace of $M_{2\times 2}$ with the standard matrix operations

Sol:

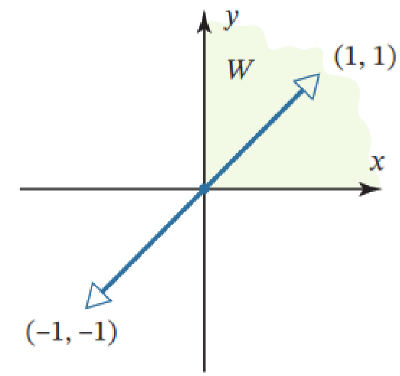
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W, \ B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$$

$$\because A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \notin W$$ (W is not closed under vector addition)

$$\therefore W \text{ is not a subspace of } M_{2\times 2}$$

- Ex : The set of first-quadrant vectors is not a subspace of R^2

  Show that $W = \{(x_1, x_2) : x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2



Sol:

Let $\mathbf{u} = (1,\ 1) \in W$

$\because (-1)\mathbf{u} = (-1)(1, 1) = (-1, -1) \notin W$

(W is not closed under scalar multiplication)

$\therefore W$ is not a subspace of R^2

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Building Subspaces

The following theorem provides a useful way of creating a new subspace from known subspaces.

Theorem 4.2.2

If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

SEE EXAMPLES 1 till 12 FROM TEXT BOOK

Solution Spaces of Homogeneous Systems

The solutions of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns can be viewed as vectors in R^n. The following theorem provides an important insight into the geometric structure of the solution set.

Theorem 4.2.3

The solution set of a homogeneous system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.

EXAMPLE 13 | Solution Spaces of Homogeneous Systems

In each part the solution of the linear system is provided. Give a geometric description of the solution set.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution (*a*) The solutions are

$$x = 2s - 3t, \quad y = s, \quad z = t$$

from which it follows that

$$x = 2y - 3z \quad \text{or} \quad x - 2y + 3z = 0$$

This is the equation of a plane through the origin that has $\mathbf{n} = (1, -2, 3)$ as a normal.

Solution (*b*) The solutions are

$$x = -5t, \quad y = -t, \quad z = t$$

which are parametric equations for the line through the origin that is parallel to the vector $\mathbf{v} = (-5, -1, 1)$.

Solution (*c*) The only solution is $x = 0, y = 0, z = 0$, so the solution space consists of the single point $\{\mathbf{0}\}$.

Solution (*d*) This linear system is satisfied by all real values of x, y, and z, so the solution space is all of R^3.