Linear Algebra (MT-1004)

Lecture # 04

National University of computer and emerging sciences

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Types of Solutions of Augmented Matrices on applying Reduced Echelon Form

a) Unique Solution:

Suppose that the augmented matrix for a linear system in the unknowns x_1, x_2, x_3, and x_4 has been reduced by elementary row operations to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

This matrix is in reduced row echelon form and corresponds to the equations

$$\begin{aligned} x_1 \quad\quad\quad\quad &= 3 \\ x_2 \quad\quad\quad &= -1 \\ x_3 \quad\quad &= 0 \\ x_4 &= 5 \end{aligned}$$

Thus, the system has a unique solution, namely, $x_1 = 3, x_2 = -1, x_3 = 0, x_4 = 5$, which can also be expressed as the 4-tuple $(3, -1, 0, 5)$.

b) No Solution:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The equation that corresponds to the last row of the augmented matrix is

$$0x + 0y + 0z = 1$$

Since this equation is not satisfied by any values of x, y, and z, the system is inconsistent.

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b) Infinitely many solutions: (Concept of leading & free variables)

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation that corresponds to the last row of the augmented matrix is

$$0x + 0y + 0z = 0$$

This equation can be omitted since it imposes no restrictions on x, y, and z; hence, the linear system corresponding to the augmented matrix is

$$\begin{aligned} x \quad + 3z &= -1 \\ y - 4z &= \ \ 2 \end{aligned}$$

In general, the variables in a linear system that correspond to the leading l's in its augmented matrix are called the ***leading variables***, and the remaining variables are called the ***free variables***. In this case the leading variables are x and y, and the variable z is the only free variable. Solving for the leading variables in terms of the free variables gives

$$\begin{aligned} x &= -1 - 3z \\ y &= 2 + 4z \end{aligned}$$

From these equations we see that the free variable z can be treated as a parameter and assigned an arbitrary value t, which then determines values for x and y. Thus, the solution set can be represented by the parametric equations

$$x = -1 - 3t, \quad y = 2 + 4t, \quad z = t$$

By substituting various values for t in these equations we can obtain various solutions of the system. For example, setting $t = 0$ yields the solution

$$x = -1, \quad y = 2, \quad z = 0$$

Gauss Elimination Method: (Based on Echelon Form_Forward pahse)

Following are the steps for *Gauss Elimination Method:*

Step # 01: Change the linear system to form Ax=b

Step # 02: Form the augmented matrix A_b by including the elements of b as an extra column in matrix A

Steps # 03: Convert the augmented matrix in to ***Echelon Form*** by using ***ERO***

Step # 04: Convert the echelon matrix in to ***equation form*** and find unknowns by using backward substitution

Gauss Jordan Method: (Based on Reduced Echelon Form_Forward pahse+Backward Phase)

Following are the steps for *Gauss Jordan Method: (Here we don't need backward substitution)*

Step # 01: Change the linear system to form Ax=b

Step # 02: Form the augmented matrix A_b by including the elements of b as an extra column in matrix A

Steps # 03: Convert the augmented matrix in to ***Reduced Echelon Form*** by using ***ERO***

Step # 04: Convert the reduced echelon matrix in to ***equation form*** and find unknowns directly