



National University
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Foundation for Advancement
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Linear Algebra (MT-1004)

Lecture # 37



Orthogonal Matrices

Definition 1

A square matrix A is said to be **orthogonal** if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^T A = I \quad (1)$$

A matrix transformation $T_A: R^n \rightarrow R^n$ is said to be an **orthogonal transformation** or an **orthogonal operator** if A is an orthogonal matrix.



EXAMPLE 1 | A 3×3 Orthogonal Matrix

The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal since

$$A^T A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



EXAMPLE 2 | Rotation and Reflection Matrices Are Orthogonal

Recall from Table 5 of Section 1.8 that the standard matrix for the counterclockwise rotation about the origin of R^2 through an angle θ is

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

This matrix is orthogonal for all choices of θ since

$$A^T A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We leave it for you to verify that the reflection matrices in Tables 1 and 2 of Section 1.8 are all orthogonal.



Theorem 7.1.1

The following are equivalent for an $n \times n$ matrix A .

- (a) A is orthogonal.
- (b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.
- (c) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Warning Note that an orthogonal matrix has *orthonormal* rows and columns—not simply orthogonal rows and columns.

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \mathbf{c}_1^T & \mathbf{r}_1 \mathbf{c}_2^T & \cdots & \mathbf{r}_1 \mathbf{c}_n^T \\ \mathbf{r}_2 \mathbf{c}_1^T & \mathbf{r}_2 \mathbf{c}_2^T & \cdots & \mathbf{r}_2 \mathbf{c}_n^T \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \mathbf{c}_1^T & \mathbf{r}_n \mathbf{c}_2^T & \cdots & \mathbf{r}_n \mathbf{c}_n^T \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{r}_1 & \mathbf{r}_1 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{r}_n \\ \mathbf{r}_2 \cdot \mathbf{r}_1 & \mathbf{r}_2 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{r}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \cdot \mathbf{r}_1 & \mathbf{r}_n \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_n \cdot \mathbf{r}_n \end{bmatrix}$$

It is evident from this formula that $AA^T = I$ if and only if

$$\mathbf{r}_1 \cdot \mathbf{r}_1 = \mathbf{r}_2 \cdot \mathbf{r}_2 = \cdots = \mathbf{r}_n \cdot \mathbf{r}_n = 1$$

and

$$\mathbf{r}_i \cdot \mathbf{r}_j = 0 \quad \text{when } i \neq j$$

which are true if and only if $\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n\}$ is an orthonormal set in R^n . ■



Theorem 7.1.2

- (a) The transpose of an orthogonal matrix is orthogonal.
- (b) The inverse of an orthogonal matrix is orthogonal.
- (c) A product of orthogonal matrices is orthogonal.
- (d) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

Do Q.1 till 6 from Ex # 7.1



Orthogonal Diagonalization:

Definition 1

If A and B are square matrices, then we say that B is **orthogonally similar** to A if there is an orthogonal matrix P such that $B = P^T A P$.

If A is orthogonally similar to some diagonal matrix, say

$$P^T A P = D$$

then we say A is **orthogonally diagonalizable** and P **orthogonally diagonalizes** A .



Theorem 7.2.1

If A is an $n \times n$ matrix with real entries, then the following are equivalent.

- (a) A is orthogonally diagonalizable.
- (b) A has an orthonormal set of n eigenvectors.
- (c) A is symmetric.

Properties of Symmetric Matrices

Our next goal is to devise a procedure for orthogonally diagonalizing a symmetric matrix, but before we can do so, we need the following critical theorem about eigenvalues and eigenvectors of symmetric matrices.

Theorem 7.2.2

If A is a symmetric matrix with real entries, then:

- (a) The eigenvalues of A are all real numbers.
- (b) Eigenvectors from different eigenspaces are orthogonal.



Orthogonally Diagonalizing an $n \times n$ Symmetric Matrix

Step 1. Find a basis for each eigenspace of A .

Step 2. Apply the Gram–Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.

Step 3. Form the matrix P whose columns are the vectors constructed in Step 2. This matrix will orthogonally diagonalize A , and the eigenvalues on the diagonal of $D = P^TAP$ will be in the same order as their corresponding eigenvectors in P .



EXAMPLE 1 | Orthogonally Diagonalizing a Symmetric Matrix

Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Solution We leave it for you to verify that the characteristic equation of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 4 & -2 & -2 \\ -2 & \lambda - 4 & -2 \\ -2 & -2 & \lambda - 4 \end{bmatrix} = (\lambda - 2)^2(\lambda - 8) = 0$$

Thus, the distinct eigenvalues of A are $\lambda = 2$ and $\lambda = 8$. By the method used in Example 7 of Section 5.1, it can be shown that

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad (5)$$

form a basis for the eigenspace corresponding to $\lambda = 2$. Applying the Gram–Schmidt process to $\{\mathbf{u}_1, \mathbf{u}_2\}$ yields the following orthonormal eigenvectors (verify):

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \quad (6)$$

The eigenspace corresponding to $\lambda = 8$ has

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

as a basis. Applying the Gram–Schmidt process to $\{\mathbf{u}_3\}$ (i.e., normalizing \mathbf{u}_3) yields

$$\mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Finally, using \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 as column vectors, we obtain

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

which orthogonally diagonalizes A . As a check, we leave it for you to confirm that

$$P^T A P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$