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Linear Algebra (MT-1004)

Lecture # 09 & 10

Reflection Operators:

Some of the most basic matrix operators on R^2 and R^3 are those that map each point into its symmetric image about a fixed line or a fixed plane that contains the origin; these are called reflection operators

TABLE 1

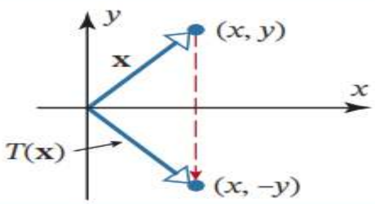
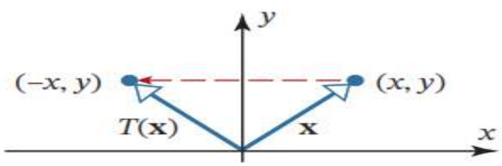
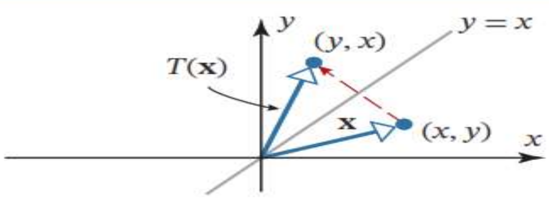
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the x-axis $T(x,y)=(x,-y)$		$T(\mathbf{e}_1)=T(1,0)=(1,0)$ $T(\mathbf{e}_2)=T(0,1)=(0,-1)$	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$
Reflection about the y-axis $T(x,y)=(-x,y)$		$T(\mathbf{e}_1)=T(1,0)=(-1,0)$ $T(\mathbf{e}_2)=T(0,1)=(0,1)$	$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$
Reflection about the line $y=x$ $T(x,y)=(y,x)$		$T(\mathbf{e}_1)=T(1,0)=(0,1)$ $T(\mathbf{e}_2)=T(0,1)=(1,0)$	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

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TABLE 2

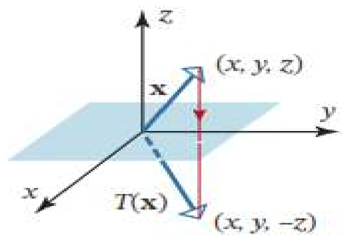
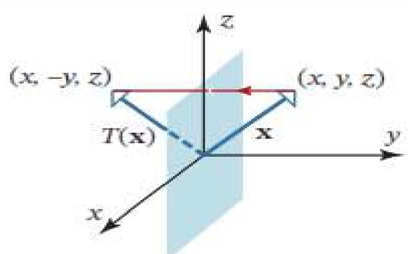
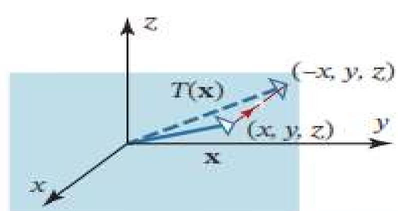
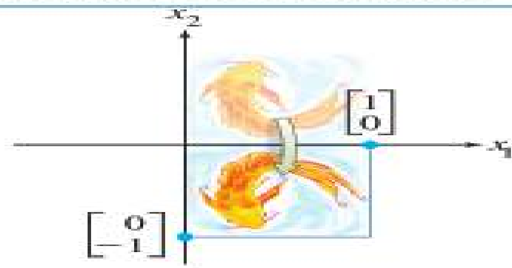
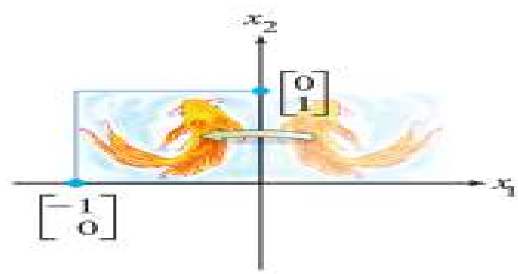
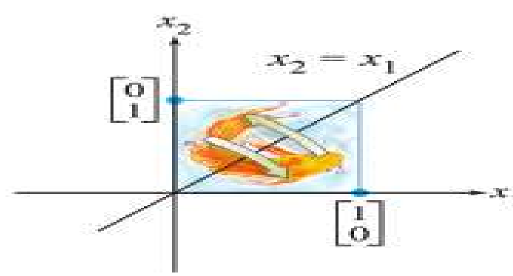
Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$T(\mathbf{e}_1) = T(1,0,0) = (1,0,0)$ $T(\mathbf{e}_2) = T(0,1,0) = (0,1,0)$ $T(\mathbf{e}_3) = T(0,0,1) = (0,0,-1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$T(\mathbf{e}_1) = T(1,0,0) = (1,0,0)$ $T(\mathbf{e}_2) = T(0,1,0) = (0,-1,0)$ $T(\mathbf{e}_3) = T(0,0,1) = (0,0,1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$T(\mathbf{e}_1) = T(1,0,0) = (-1,0,0)$ $T(\mathbf{e}_2) = T(0,1,0) = (0,1,0)$ $T(\mathbf{e}_3) = T(0,0,1) = (0,0,1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

TABLE 1 Reflections

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

Projection Operators:

Matrix operators on R^2 and R^3that map each point into its orthogonal projection onto a fixed line or plane through the origin are called projection operators (or more precisely, orthogonal projection operators)

TABLE 3

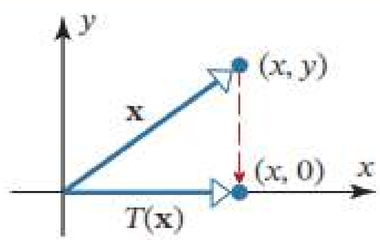
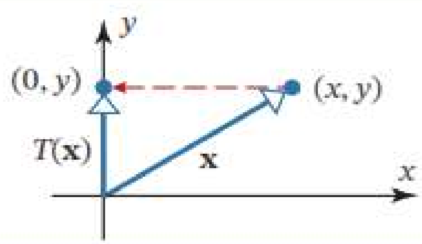
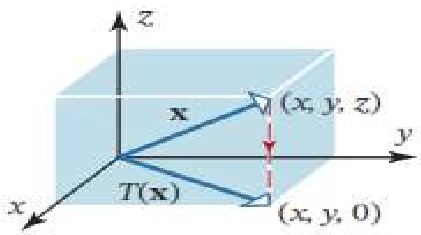
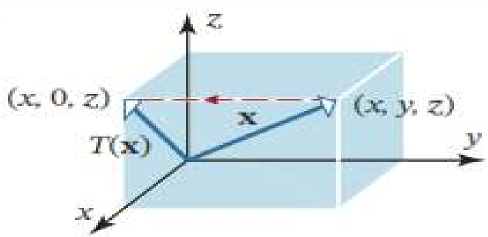
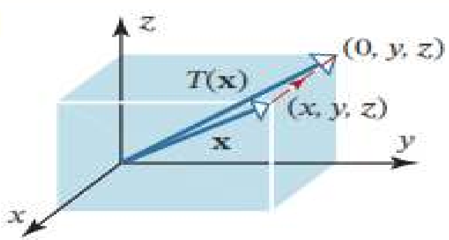
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

TABLE 4

Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Orthogonal projection onto the xy-plane $T(x,y,z) = (x,y,0)$		$T(\mathbf{e}_1) = T(1,0,0) = (1,0,0)$ $T(\mathbf{e}_2) = T(0,1,0) = (0,1,0)$ $T(\mathbf{e}_3) = T(0,0,1) = (0,0,0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection onto the xz-plane $T(x,y,z) = (x,0,z)$		$T(\mathbf{e}_1) = T(1,0,0) = (1,0,0)$ $T(\mathbf{e}_2) = T(0,1,0) = (0,0,0)$ $T(\mathbf{e}_3) = T(0,0,1) = (0,0,1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection onto the yz-plane $T(x,y,z) = (0,y,z)$		$T(\mathbf{e}_1) = T(1,0,0) = (0,0,0)$ $T(\mathbf{e}_2) = T(0,1,0) = (0,1,0)$ $T(\mathbf{e}_3) = T(0,0,1) = (0,0,1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotation Operators:

Matrix operators on R^2that move points along arcs of circles centered at the origin are called rotation operators

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

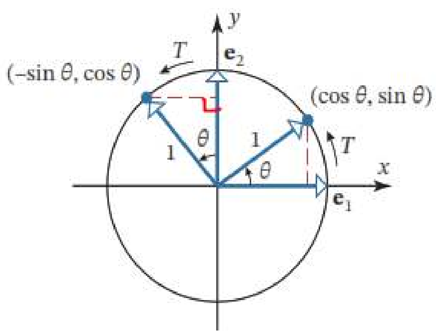
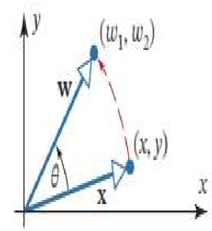


TABLE 5

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Counterclockwise rotation about the origin through an angle θ		$T(\mathbf{e}_1) = T(1,0) = (\cos\theta, \sin\theta)$ $T(\mathbf{e}_2) = T(0,1) = (-\sin\theta, \cos\theta)$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

In the plane, counterclockwise angles are positive and clockwise angles are negative. The rotation matrix for a *clockwise* rotation of $-\theta$ radians can be obtained by replacing θ by $-\theta$ in (19). After simplification this yields

$$R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Some Extra Example of Transformation (not included in Ex # 1.8)

TABLE 2 Contractions and Expansions

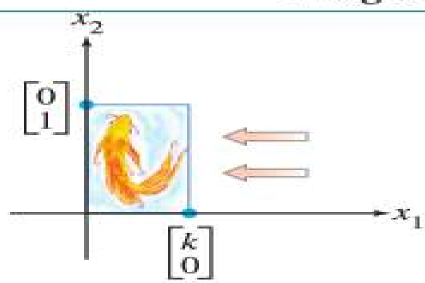
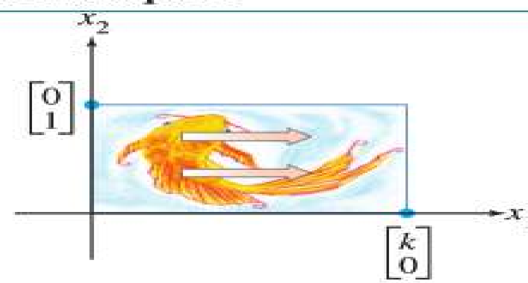
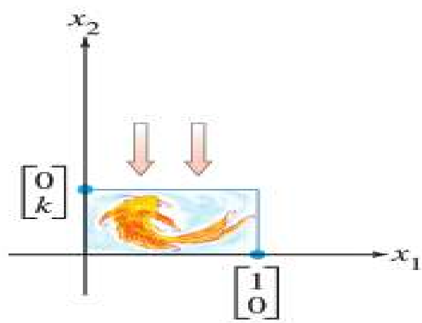
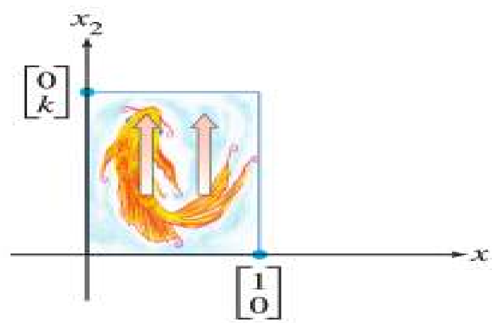
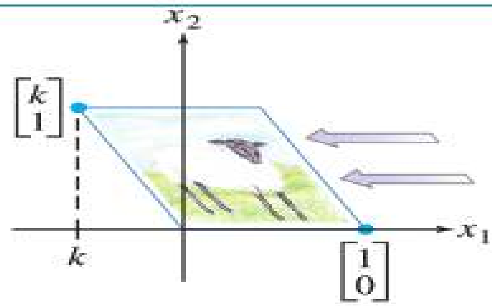
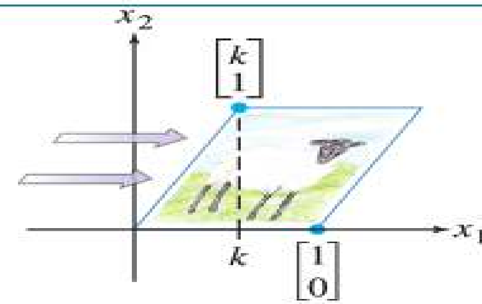
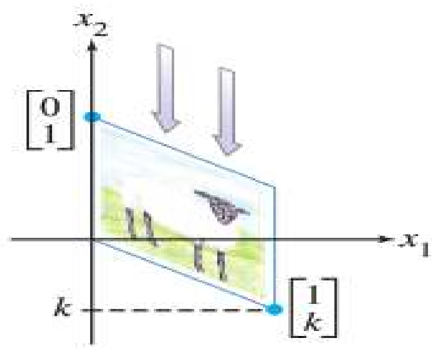
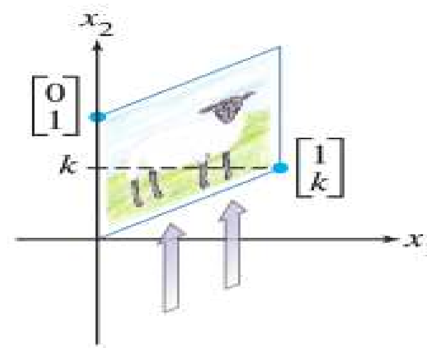
Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$

TABLE 3 Shears

Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Task for Students

As per Course outline : Do (Q.1 till 24 & 27 till 41 from Ex # 1.8)

Compositions of Matrix Transformations

Simply stated, the "composition" of matrix transformations is the process of first applying a matrix transformation to a vector and then applying another matrix transformation to the image vector. For example, suppose that T_A is a matrix transformation from R^n to R^k and T_B is a matrix transformation from R^k to R^m. If $\mathbf{x}$ is a vector in R^n, then T_A maps this vector into a vector $T_A(\mathbf{x})$ in R^k, and T_B, in turn, maps that vector into the vector $T_B(T_A(\mathbf{x}))$ in R^m. This process creates a transformation directly from R^n to R^m that we call the ***composition of T_B with T_A*** and which we denote by the symbol

$$T_B \circ T_A$$

which is read "T_B circle T_A." As illustrated in **Figure 1.9.1**, the transformation T_A in the formula is performed first; that is,

$$\boxed{(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))} \qquad (1)$$

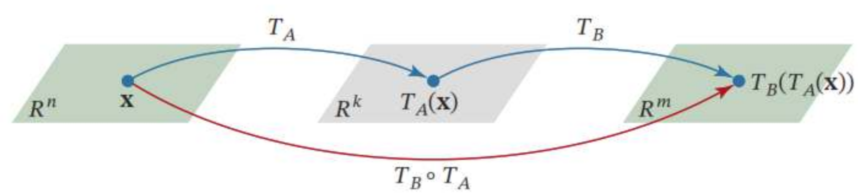


FIGURE 1.9.1

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Theorem 1.9.1

If $T_A: R^n \to R^k$ and $T_B: R^k \to R^m$ are matrix transformations, then $T_B \circ T_A$ is also a matrix transformation and

$$T_B \circ T_A = T_{BA} \tag{2}$$

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EXAMPLE 1 | The Standard Matrix for a Composition

Let $T_1 : R^3 \to R^2$ and $T_2 : R^2 \to R^3$ be the linear transformations given by

$$T_1(x, y, z) = (x + 2y, x + 2z - y)$$

and

$$T_2(x, y) = (3x + y, x, x - 2y)$$

Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

Solution The standard basis vectors for R^3 are $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$. From which it follows that

$$T_1(\mathbf{e}_1) = (1,\ 1), \quad T_1(\mathbf{e}_2) = (2, -1) \quad \text{and} \quad T_1(\mathbf{e}_3) = (0, 2)$$

Thus

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

is the standard matrix for T_1. Similarly, the standard basis vectors for R^2 are $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$, so

$$T_2(\mathbf{e}_1) = (3, 1, 1) \quad \text{and} \quad T_2(\mathbf{e}_2) = (1, 0, 2)$$

Thus

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix}$$

is the standard matrix for T_2. Applying equation (3), the standard matrix for $T_2 \circ T_1$ is

$$BA = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 2 \\ 1 & 2 & 0 \\ -1 & 4 & -4 \end{bmatrix}$$

and the standard matrix for $T_1 \circ T_2$ is

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix}$$