Section A

Q.1: (Marks=04)

(a) $\mathbf{u}+\mathbf{v}=(-1+3,2+4)=(2,6)$; $\quad k\mathbf{u}=(0,3\cdot 2)=(0,6)$

(b) For any $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ in V, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2)$ is an ordered pair of real numbers, therefore $\mathbf{u}+\mathbf{v}$ is in V. Consequently, V is closed under addition.

For any $\mathbf{u}=(u_1,u_2)$ in V and for any scalar k, $k\mathbf{u}=(0,ku_2)$. is an ordered pair of real numbers, therefore $k\mathbf{u}$ is in V. Consequently, V is closed under scalar multiplication.

(c) Axioms 1-5 hold for V because they are known to hold for R^2.

(d) Axiom 7: $k((u_1,u_2)+(v_1,v_2))=k(u_1+v_1,u_2+v_2)=(0,k(u_2+v_2))=(0,ku_2)+(0,kv_2)$
$=k(u_1,u_2)+k(v_1,v_2)$ for all real k, u_1, u_2, v_1, and v_2;

Axiom 8: $(k+m)(u_1,u_2)=(0,(k+m)u_2)=(0,ku_2+mu_2)=(0,ku_2)+(0,mu_2)$
$=k(u_1,u_2)+m(u_1,u_2)$ for all real k, m, u_1, and u_2;

Axiom 9: $k(m(u_1,u_2))=k(0,mu_2)=(0,kmu_2)=(km)(u_1,u_2)$ for all real k, m, u_1, and u_2;

(e) Axiom 10 fails to hold: $1(u_1,u_2)=(0,u_2)$ does not generally equal (u_1,u_2).
Consequently, V is not a vector space.

The above are not all 10 axioms. Just check whether student have mentioned it's a vector space or not

Q.2: (Marks=04)

Solution (a) If A and B are matrices in U, then they can be expressed in the form

$$A = \begin{bmatrix} a & 0 \\ 2a & b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & 0 \\ 2c & d \end{bmatrix}$$

for some real numbers a, b, c, and d. But

$$A + B = \begin{bmatrix} a+c & 0 \\ 2(a+c) & b+d \end{bmatrix}$$

is also a matrix in U since it is of form (2) with $x = a + c$ and $y = b + d$. Thus, U is closed under addition. Similarly, U is closed under scalar multiplication since

$$kA = \begin{bmatrix} ka & 0 \\ 2ka & kb \end{bmatrix}$$

is of form (2) with $x = ka$ and $y = kb$. These two results establish that $\boxed{U \text{ is a subspace of } M_{22}.}$

Ans

Q.3: **(Marks=04)**

Solution In order for $\mathbf{w}$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

Solving this system using Gaussian elimination yields $k_1 = -3$, $k_2 = 2$, so

$$\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$$

Section B

Q1: (Marks=04)

Determine whether the vectors

$$\mathbf{v}_1 = (1, -2, 3), \quad \mathbf{v}_2 = (5, 6, -1), \quad \mathbf{v}_3 = (3, 2, 1)$$

are linearly independent or linearly dependent in R^3.

Solution The linear independence or dependence of these vectors is determined by the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$

$$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$$

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}$$

Thus, our problem reduces to determining whether this system has nontrivial solutions. There are various ways to do this; one possibility is to simply solve the system, which yields

$$k_1 = -\tfrac{1}{2}t, \quad k_2 = -\tfrac{1}{2}t, \quad k_3 = t$$

(we omit the details). This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent. A second method for establishing the linear dependence is to take advantage of the fact that the coefficient matrix

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

is square and compute its determinant. We leave it for you to show that $\det(A) = 0$ from which it follows that (4) has nontrivial solutions by parts (b) and (g) of Theorem 2.3.8.

Because we have established that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in (2) are linearly dependent, we know that at least one of them is a linear combination of the others. We leave it for you to confirm, for example, that

$$\mathbf{v}_3 = \tfrac{1}{2}\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2$$

Q.2: (Marks=04)

Let W be the set of all matrices of form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$. This set contains at least one matrix, e.g. the zero matrix. Adding two matrices in W results in another matrix in W:

$$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} a' & 0 \\ b' & 0 \end{bmatrix} = \begin{bmatrix} a+a' & 0 \\ b+b' & 0 \end{bmatrix}.$$

Likewise, a scalar multiple of a matrix in W is also in W:

$k\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} ka & 0 \\ kb & 0 \end{bmatrix}$. According to Theorem 4.2.1, $\boxed{W \text{ is a subspace of } M_{22}}$.

Ans

Q.3: (Marks=04)

The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros

appended to this matrix. The general solution of the system $x_1 = 16t$, $x_2 = 19t$, $x_3 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ therefore the vector $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ forms a basis for the null space of A.

A basis for the row space is formed by the nonzero rows of the reduced row echelon form of A: $[1 \quad 0 \quad -16]$ and $[0 \quad 1 \quad -19]$.

Section C

Q.1: (Marks=04)

Determine whether the vectors

$$\mathbf{v}_1 = (1,2,2,-1), \quad \mathbf{v}_2 = (4,9,9,-4), \quad \mathbf{v}_3 = (5,8,9,-5)$$

in R^4 are linearly dependent or linearly independent.

Solution The linear independence or linear dependence of these vectors is determined by whether there exist nontrivial solutions of the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$

or, equivalently, of

$$k_1(1,2,2,-1) + k_2(4,9,9,-4) + k_3(5,8,9,-5) = (0,0,0,0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned}$$

From Echelon Form, We get the values of Constants, If you have any query while you are checking, feel free to discuss

We leave it for you to show that this system has only the trivial solution

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$$

from which you can conclude that $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

Q.2: (Marks=04)

Let W be the set of all matrices of form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$. This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of M_{22}.

Ans

Q.3: (Marks=04)

The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = \frac{1}{2}t$, $x_2 = s$, $x_3 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

A basis for the row space is formed by the nonzero row of the reduced row echelon form of A:
$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \end{bmatrix}$.