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Linear Algebra (MT-1004)

Lecture # 06

Elementary Matrix:

"A matrix (E) is called Elementary Matrix if it is obtained from an identity matrix by performing a single elementary row operation"

Examples:

Listed below are four elementary matrices and the operations that produce them.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the second row of I_2 by -3.

Interchange the second and fourth rows of I_4.

Add 3 times the third row of I_3 to the first row.

Multiply the first row of I_3 by 1.

Elementary Matrix:

{ Theorem 1.5.1 }

Row Operations by Matrix Multiplication: If the elementary matrix E results from performing a certain row operation on I_m and if A is an m × n matrix, then the product EA is the matrix that results when this same row operation is performed on A.

Example:

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which results from adding 3 times the first row of I_3 to the third row. The product EA is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

which is precisely the matrix that results when we add 3 times the first row of A to the third row.

Elementary Matrix:

{ Theorem 1.5.2 }

"Every elementary matrix is invertible, and the inverse is also an elementary matrix"

i.e. if E is an elementary matrix and E_0 is its inverse then It always follows:

$$\boldsymbol{E_0 E = I} \textbf{ and } \boldsymbol{E E_0 = I}$$

Method for Inverting Matrices:

Assume that the reduced row echelon form of A is I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations. By Theorem 1.5.1, each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n \tag{3}$$

assume for the moment, that A is an invertible $n \times n$ matrix. In Equation (3), the elementary matrices execute a sequence of row operations that reduce A to I_n. If we multiply both sides of this equation on the right by A^{-1} and simplify, we obtain

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

But this equation tells us that *the same sequence of row operations that reduces A to I_n will transform I_n to A^{-1}*. Thus, we have established the following result.

Method for Inverting Matrices:

Inversion Algorithm To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.

Solved examples/pattern of inverse of a 3x3 & 4x4 matrices are from next slides

Inverse of a Matrix by ERO :-

$$[A \mid I] \xrightarrow[\text{ERO}]{} [I \mid A^{-1}]$$

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} \boxed{2} & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$R_{12}\quad \left[\begin{array}{ccc|ccc} \boxed{1} & -2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2-2R_1 \\ R_3-2R_1 \end{array}\quad \left[\begin{array}{cccccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & \boxed{7} & 2 & 1 & -2 & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$R_2-2R_3\quad \left[\begin{array}{cccccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 2 & -2 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$(-1)R_2\quad \left[\begin{array}{cccccc} \boxed{1} & -2 & -1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & -1 & -2 & 2 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right]$$

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$$R_3 - 4R_2 \begin{bmatrix} \boxed{1} & -2 & -1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & -1 & -2 & 2 \\ 0 & 0 & \boxed{1} & 4 & 6 & -7 \end{bmatrix}$$

$$R_1 + 2R_2 \begin{bmatrix} 1 & 0 & -1 & -2 & -3 & 4 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{bmatrix}$$

$$R_1 + R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right]$$

I $\quad A^{-1}$

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$$A^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$$

Ex#1.5 Q.16

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - R_1$
$R_3 - R_1$
$R_4 - R_1$

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - R_2$
$R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$R_4 - R_3$

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \boxed{3} & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & \boxed{5} & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{7} & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$
\begin{array}{l} R_2/3 \\ R_3/5 \\ R_4/7 \end{array}
\quad
\left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1/3 & 1/3 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1/5 & 1/5 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1/7 & 1/7
\end{array}\right]
$$

$$
A^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1/3 & 1/3 & 0 & 0 \\
0 & -1/5 & 1/5 & 0 \\
0 & 0 & -1/7 & 1/7
\end{bmatrix}
$$