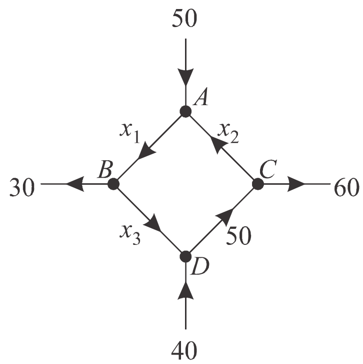
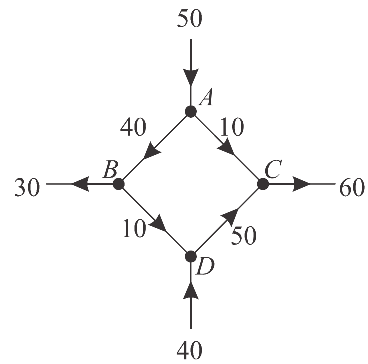
1. There are four nodes, which we denote by A, B, C, and D (see the figure on the left).
We determine the unknown flow rates x_1, x_2, and x_3 assuming the counterclockwise direction (if any of these quantities are found to be negative then the flow direction along the corresponding branch will be reversed).





Network node	Flow In		Flow Out
A	$x_2 + 50$	$=$	x_1
B	x_1	$=$	$x_3 + 30$
C	50	$=$	$x_2 + 60$
D	$x_3 + 40$	$=$	50

This system can be rearranged as follows

$$\begin{array}{rcrcrcr} -x_1 & + & x_2 & & & = & -50 \\ x_1 & & & - & x_3 & = & 30 \\ & - & x_2 & & & = & 10 \\ & & & & x_3 & = & 10 \end{array}$$

By inspection, this system has a unique solution $x_1 = 40$, $x_2 = -10$, $x_3 = 10$. This yields the flow rates and directions shown in the figure on the right.

2. **(a)** There are five nodes – each of them corresponds to an equation.

Network node	Flow In		Flow Out
top left	200	$=$	$x_1 + x_3$
top right	$x_3 + 150$	$=$	$x_4 + x_5$
bottom left	$x_1 + 25$	$=$	x_2
bottom middle	$x_2 + x_4$	$=$	$x_6 + 175$
bottom right	$x_5 + x_6$	$=$	200

This system can be rearranged as follows

$$\begin{array}{rcrcrcrcrcrcr} x_1 & & & + & x_3 & & & & & & & = & 200 \\ & & & - & x_3 & + & x_4 & + & x_5 & & & = & 150 \\ -x_1 & + & x_2 & & & & & & & & & = & 25 \\ & & x_2 & & & + & x_4 & & & - & x_6 & = & 175 \\ & & & & & & & & x_5 & + & x_6 & = & 200 \end{array}$$

(b) The augmented matrix of the linear system obtained in part (a) has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 150 \\ 0 & 1 & 0 & 1 & 0 & -1 & 175 \\ 0 & 0 & 1 & -1 & 0 & 1 & 50 \\ 0 & 0 & 0 & 0 & 1 & 1 & 200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. If we assign x_4 and x_6 the arbitrary values s and t, respectively, the general solution is given by the formulas

$x_1 = 150 - s + t$, $x_2 = 175 - s + t$, $x_3 = 50 + s - t$, $x_4 = s$, $x_5 = 200 - t$, $x_6 = t$

(c) When $x_4 = 50$ and $x_6 = 0$, the remaining flow rates become $x_1 = 100$, $x_2 = 125$, $x_3 = 100$, and $x_5 = 200$.

The directions of the flow agree with the arrow orientations in the diagram.

3. **(a)** There are four nodes – each of them corresponds to an equation.

Network node	Flow In		Flow Out
top left	$x_2 + 300$	=	$x_3 + 400$
top right (A)	$x_3 + 750$	=	$x_4 + 250$
bottom left	$x_1 + 100$	=	$x_2 + 400$
bottom right (B)	$x_4 + 200$	=	$x_1 + 300$

This system can be rearranged as follows

$$\begin{array}{rcrcrcrcr} & & x_2 & - & x_3 & & & = & 100 \\ & & & & x_3 & - & x_4 & = & -500 \\ x_1 & - & x_2 & & & & & = & 300 \\ -x_1 & & & & & + & x_4 & = & 100 \end{array}$$

(b) The augmented matrix of the linear system obtained in part (a) $\begin{bmatrix} 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 1 & -1 & 0 & 0 & 300 \\ -1 & 0 & 0 & 1 & 100 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & -1 & -400 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. If we assign x_4 the arbitrary value s, the general solution is given by the formulas

$x_1 = -100 + s$, $x_2 = -400 + s$, $x_3 = -500 + s$, $x_4 = s$

(c) In order for all x_i values to remain positive, we must have $s > 500$. Therefore, to keep the traffic flowing on all roads, the flow from *A* to *B* must exceed 500 vehicles per hour.

4. **(a)** There are six intersections – each of them corresponds to an equation.

Intersection	Flow In		Flow Out
top left	$500+300$	$=$	x_1+x_3
top middle	x_1+x_4	$=$	x_2+200
top right	x_2+100	$=$	x_5+600
bottom left	x_3+x_6	$=$	$400+350$
bottom middle	x_7+600	$=$	x_4+x_6
bottom right	x_5+450	$=$	x_7+400

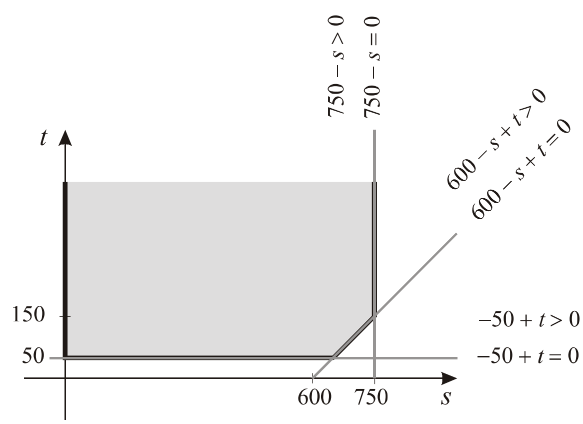
We rewrite the system as follows

$$\begin{array}{rrrrrrrcr}
x_1 & & +\ x_3 & & & & & = & 800 \\
-x_1 & +\ x_2 & & -\ x_4 & & & & = & -200 \\
& -\ x_2 & & & +\ x_5 & & & = & -500 \\
& & x_3 & & & +\ x_6 & & = & 750 \\
& & & x_4 & & +\ x_6 & -\ x_7 & = & 600 \\
& & & & x_5 & & +\ x_7 & = & 50
\end{array}$$

(b) The augmented matrix of the linear system obtained in part (a) has the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 450 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 750 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

. If we assign x_6 and x_7 the arbitrary values s and t, respectively, the general solution is given by the formulas $x_1 = 50+s$, $x_2 = 450+t$, $x_3 = 750-s$, $x_4 = 600-s+t$, $x_5 = -50+t$, $x_6 = s$, $x_7 = t$ subject to the restriction that all seven values must be nonnegative. Obviously, we need both $s = x_6 \ge 0$ and $t = x_7 \ge 0$, which in turn imply $x_1 \ge 0$ and $x_2 \ge 0$. Additionally imposing the three inequalities $x_3 = 750-s \ge 0$, $x_4 = 600-s+t \ge 0$, and $x_5 = -50+t \ge 0$ results in the set of allowable s and t values depicted in the grey region on the graph.



(c) Setting $x_1 = 0$ in the general solution obtained in part (b) would result in the negative value $s = x_6 = -50$ which is not allowed (the traffic would flow in a wrong way along the street marked as x_6.)

5. From Kirchhoff's current law at each node, we have $I_1 + I_2 - I_3 = 0$. Kirchhoff's voltage law yields