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# Linear Algebra (MT-1004)

Lecture # 16 & 17

# Vectors in $R^n$

- An ordered $n$-tuple :

  a sequence of $n$ real numbers $(x_1, x_2, \ldots, x_n)$

- $R^n$-space :

  the set of all ordered $n$-tuples

$n = 1$ $\quad$ $R^1$-space = set of all real numbers
($R^1$-space can be represented geometrically by the $x$-axis)

$n = 2$ $\quad$ $R^2$-space = set of all ordered pair of real numbers $(x_1, x_2)$
($R^2$-space can be represented geometrically by the $xy$-plane)

$n = 3$ $\quad$ $R^3$-space = set of all ordered triple of real numbers $(x_1, x_2, x_3)$
($R^3$-space can be represented geometrically by the $xyz$-space)

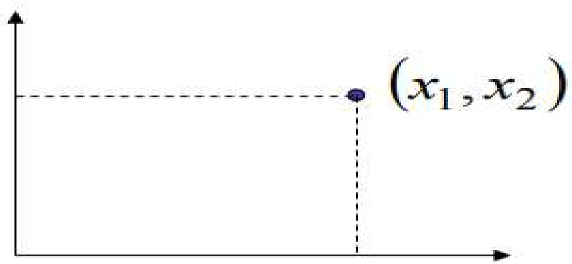
$n = 4$ $\quad$ $R^4$-space = set of all ordered quadruple of real numbers $(x_1, x_2, x_3, x_4)$

- Notes:

  (1) An $n$-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as a point in $R^n$ with the $x_i$'s as its coordinates

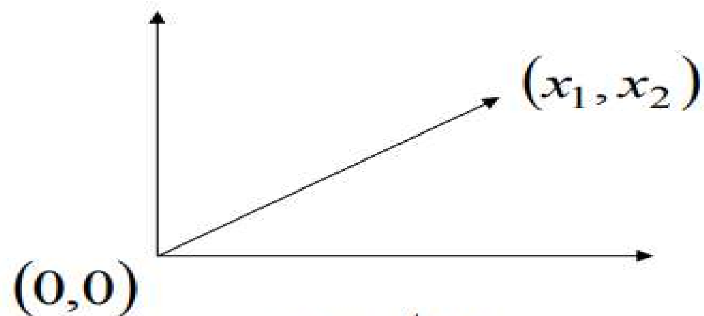
  (2) An $n$-tuple $(x_1, x_2, \cdots, x_n)$ also can be viewed as a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in $R^n$ with the $x_i$'s as its components

- Ex:1




a point

or




a vector

※ A vector on the plane is expressed geometrically by a directed line segment whose initial point is the origin and whose terminal point is the point $(x_1, x_2)$

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$$\mathbf{u}=(u_1,u_2,\cdots,u_n),\ \ \mathbf{v}=(v_1,v_2,\cdots,v_n) \qquad \text{(two vectors in } R^n\text{)}$$

- Equality:

  $\mathbf{u}=\mathbf{v}$ if and only if $u_1= v_1$, $u_2= v_2$, . . . . . . , $u_n= v_n$

- Vector addition (the sum of **u** and **v**):

  $$\mathbf{u}+\mathbf{v}=(u_1+v_1, u_2+v_2,\cdots,u_n+v_n)$$

- Scalar multiplication (the scalar multiple of **u** by $c$):

  $$c\mathbf{u}=(cu_1,cu_2,\cdots,cu_n)$$

- Notes:

  The sum of two vectors and the scalar multiple of a vector in $R^n$ are called the standard operations in $R^n$

- Difference between **u** and **v**:

$$\mathbf{u}-\mathbf{v} \equiv \mathbf{u}+(-1)\mathbf{v}=(u_1-v_1,\ u_2-v_2,\ u_3-v_3,\ldots,\ u_n-v_n)$$

- Zero vector :

$$\mathbf{0}=(0,0,\ldots,0)$$

# Notations for Vectors

Up to now we have been writing vectors in $R^n$ using the notation

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) \tag{15}$$

We call this the ***comma-delimited*** form. However, since a vector in $R^n$ is just a list of its $n$ components in a specific order, any notation that displays those components in the

correct order is a valid way of representing the vector. For example, the vector in (15) can be written as

$$\mathbf{v} = [v_1 \quad v_2 \quad \cdots \quad v_n] \tag{16}$$

which is called ***row-vector*** form, or as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{17}$$

which is called ***column-vector*** form. The choice of notation is often a matter of taste or convenience, but sometimes the nature of a problem will suggest a preferred notation. Notations (15), (16), and (17) will all be used at various places in this text.

The following theorem summarizes the most important properties of vector operations.

**Theorem 3.1.1**

If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in $R^n$, and if $k$ and $m$ are scalars, then:

(*a*) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(*b*) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(*c*) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(*d*) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(*e*) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(*f*) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(*g*) $k(m\mathbf{u}) = (km)\mathbf{u}$

(*h*) $1\mathbf{u} = \mathbf{u}$

## Linear Combinations

Addition, subtraction, and scalar multiplication are frequently used in combination to form new vectors. For example, if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in $R^n$, then the vectors

$$\mathbf{u} = 2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3 \quad \text{and} \quad \mathbf{w} = 7\mathbf{v}_1 - 6\mathbf{v}_2 + 8\mathbf{v}_3$$

are formed in this way. In general, we make the following definition.

**Definition 4**

If $\mathbf{w}$ is a vector in $R^n$, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ in $R^n$ if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{14}$$

where $k_1, k_2, \dots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination. In the case where $r = 1$, Formula (14) becomes $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just a scalar multiple of that vector.

Note that this definition of a linear combination is consistent with that given in the context of matrices (see Definition 6 in Section 1.3).

# JUST FOR UNDERSTANDING

## Application of Linear Combinations to Color Models

Colors on computer monitors are commonly based on what is called the **RGB *color model***. Colors in this system are created by adding together percentages of the primary colors red (R), green (G), and blue (B). One way to do this is to identify the primary colors with the vectors

$$\begin{aligned} \mathbf{r} &= (1,0,0) \quad \text{(pure red)}, \\ \mathbf{g} &= (0,1,0) \quad \text{(pure green)}, \\ \mathbf{b} &= (0,0,1) \quad \text{(pure blue)} \end{aligned}$$

in $R^3$ and to create all other colors by forming linear combinations of $\mathbf{r}$, $\mathbf{g}$, and $\mathbf{b}$ using coefficients between 0 and 1, inclusive; these coefficients represent the percentage of each pure color in the mix. The set of all such color vectors is called **RGB *space*** or the **RGB *color cube*** (Figure 3.1.14). Thus, each color vector $\mathbf{c}$ in this cube is expressible as a linear combination of the form

$$\begin{aligned} \mathbf{c} &= k_1\mathbf{r} + k_2\mathbf{g} + k_3\mathbf{b} \\ &= k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) \\ &= (k_1, k_2, k_3) \end{aligned}$$

where $0 \leq k_i \leq 1$. As indicated in the figure, the corners of the cube represent the pure primary colors together with the colors black, white, magenta, cyan, and yellow. The vectors along the diagonal running from black to white correspond to shades of gray.

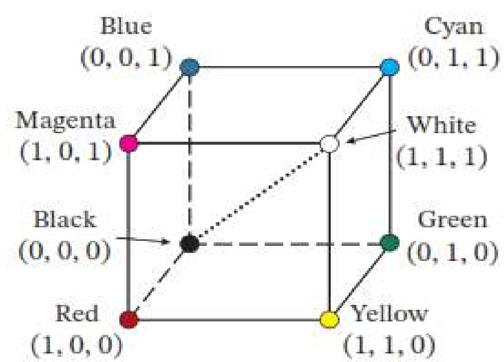


FIGURE 3.1.14

# Vector Spaces

- Vector spaces:

  Let $V$ be a set on which two operations (addition and scalar multiplication) are defined. **If the following ten axioms are satisfied** for every element **u**, **v**, and **w** in $V$ and every scalar (real number) $c$ and $d$, then ***V* is called a vector space**, and the **elements** in $V$ are called **vectors**

  Addition:

  (1) $\mathbf{u}+\mathbf{v}$ is in $V$

  (2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

  (3) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$

  (4) $V$ has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in $V$, $\mathbf{u}+\mathbf{0} = \mathbf{u}$

  (5) For every $\mathbf{u}$ in $V$, there is a vector in $V$ denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

Scalar multiplication:

(6) $c\mathbf{u}$ is in $V$

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

※ This type of definition is called an **abstraction** because you abstract a collection of properties from $R^n$ to form the axioms for defining a more general space $V$

※ Thus, we can conclude that $R^n$ is of course a vector space

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OR

WE CAN DEFINE VECTOR SPACE AS;

**Definition 1**

Let $V$ be an arbitrary nonempty set of objects for which two operations are defined: addition and multiplication by numbers called ***scalars***. By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in $V$ an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar $k$ and each object $\mathbf{u}$ in $V$ an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by $k$. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $V$ and all scalars $k$ and $m$, then we call $V$ a ***vector space*** and we call the objects in $V$ ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in $V$, then $\mathbf{u}+\mathbf{v}$ is in $V$.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There exists an object in $V$, called the ***zero vector***, that is denoted by $\mathbf{0}$ and has the property that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in $V$.
5. For each $\mathbf{u}$ in $V$, there is an object $-\mathbf{u}$ in $V$, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If $k$ is any scalar and $\mathbf{u}$ is any object in $V$, then $k\mathbf{u}$ is in $V$.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

In this text scalars will be either real numbers or complex numbers. Vector spaces with real scalars will be called ***real vector spaces*** and those with complex scalars will be called ***complex vector spaces***. For now we will consider only real vector spaces.

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## Steps to Show That a Set with Two Operations Is a Vector Space

**Step 1.** Identify the set $V$ of objects that will become vectors.

**Step 2.** Identify the addition and scalar multiplication operations on $V$.

**Step 3.** Verify Axioms 1 and 6; that is, adding two vectors in $V$ produces a vector in $V$, and multiplying a vector in $V$ by a scalar also produces a vector in $V$.

Axiom 1 is called ***closure under addition***, and Axiom 6 is called ***closure under scalar multiplication***.

**Step 4.** Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.