Exercise 5.2.

Date:

Question No 8.

Find a Matrix P that diagonalizes A and check your working by Computing $P^{-1}AP$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Soln:

$$\det(\lambda I - A)$$

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right).$$

$$\det\left(\begin{bmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & 1 \\ 0 & 0 & \lambda-1 \end{bmatrix} \right).$$

$$= (\lambda-1) \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix}.$$

$$= (\lambda-1)\left[(\lambda-1)^2 - 1\right] = 0$$

$$(\lambda-1)(\lambda-2)^2\lambda = 0.$$

$$\lambda = 0, \lambda = 1, \lambda = 2.$$

$\lambda_1 = 0$ Contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ when $x_1 = 0, x_2 = t, x_3 = t$

$$P_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

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Pg No.

Date:

for $\lambda = 1$ echelon form of N.

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$x_1 = t, \ x_2 = 0, \ x_3 = 0$.

$$P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

for $\lambda = 2$. echelon form. is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$x_1 = 0, \ x_2 = t, \ x_3 = t$

$P_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ forms basis for Eigen Space.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

$\therefore P^{-1}AP$.

$$= \begin{bmatrix} 0 & -1/2 & -1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ Ans

Q NO 10

Follow the direction in Exercise 9 for Matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Soln:-

(a) Find eigen values of A.

Soln:-

$\det(\lambda I - A)$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-3 & 0 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & -1 & \lambda-2 \end{bmatrix}.$$

$(\lambda-3)(\lambda-2)^2 = 0$

$\lambda = 3, \lambda = 2.$

(b) for $\lambda = 2$

$$= \begin{bmatrix} 2-3 & 0 & 0 \\ 0 & 2-2 & 0 \\ 0 & -1 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank is 2.

QNO15 :- The Characteristic equation of a matrix A is given. Find the size of the matrix and the possible dimensions of its Eigen values.

(a) $(\lambda-1)(\lambda+3)(\lambda-5)=0$

The degree of the Characteristic polynomial of A is 3 therefore A is 3x3 matrix, All three eigen spaces (for $\lambda=1$, $\lambda=-3$ and $\lambda=5$) must have dimension 1

(b) $\lambda^2(\lambda-1)(\lambda-2)^3=0$

The degree of the characteristic polynomial of A is 6 therefore A is 6x6 matrix. The possible dimensions of the eigenspace corresponding to $\lambda=0$ are 1 or 2. The dimension of the eigen space corresponding to $\lambda=1$ must be 1. The possible dimensions of the Eigenspace corresponding to $\lambda=2$ are 1, 2, or 3.

Q No 20 :-

Soln

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} . \quad \boxed{D = P^{-1} A P}$$

$$P^{-1} A P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) $A^{1000} = P D^{1000} P^{-1}$

$$= \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} (-1)^{1000} & 0 & 0 \\ 0 & (-1)^{1000} & 0 \\ 0 & 0 & (1)^{1000} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans

(b) $A^{-1000} = P D^{-1000} P^{-1}$

$$= \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (-1)^{-1000} & 0 & 0 \\ 0 & (-1)^{-1000} & 0 \\ 0 & 0 & (1)^{-1000} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{Ans.}$$

(c) $A^{2301} = P D^{2301} P^{-1}$

$$\begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} (-1)^{2301} & 0 & 0 \\ 0 & (-1)^{2301} & 0 \\ 0 & 0 & (1)^{2301} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{Ans.}$$

(d) $A^{-2301} = P D^{-2301} P^{-1}$

$$= \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (-1)^{-2301} & 0 & 0 \\ 0 & (-1)^{-2301} & 0 \\ 0 & 0 & (1)^{2301} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{Ans.}$$

Exercise 6.1

Date:

Question #2.

Follow the direction of Exercise 1 using weighted Euclidean inner Product

$\therefore u = (1,1)$, $v = (3,2)$

$\langle u, v \rangle = \frac{1}{2} u_1 v_1 + 5 u_2 v_2$

Sol:-

ⓐ $\langle u, v \rangle = w_1 u_1 v_1 + w_2 u_2 v_2$

$= \frac{1}{2}(1)(3) + 5(1)(2) = \frac{3}{2} + 10 = \frac{3+20}{2}$

$\boxed{= 23/2}$

ⓑ $\langle kv, w \rangle = \frac{1}{2}(ku_1 w_1) + 5(ku_2 w_2)$

$= \frac{1}{2}(3 \times 1 \times 0) + 5(3 \times 2 \times -1)$

$= 0 + 5(-3)(2)$

$\boxed{\langle kv, w \rangle = -30}$

ⓒ $\langle u+v, w \rangle = \frac{1}{2}(u_1 + v_1) w_1 + 5(u_2 + v_2) w_2$

$= \frac{1}{2}(1+3)0 + 5(1+2)-1$

$= 0 + (-15)$

$= -15$ Answer.

(d) $||v|| = \sqrt{w_1 v_1 v_1 + v_2 v_2}$

$$= \sqrt{\frac{1}{2}(3)^2 + 5(2)^2}$$

$$= \sqrt{\frac{1}{2}(9) + 5 \times 4}$$

$$= \sqrt{\frac{9}{2} + 20} = \sqrt{\frac{9+40}{2}} = \frac{\sqrt{49}}{\sqrt{2}} = \frac{7}{\sqrt{2}} \text{ Ans}$$

(e) $d(u,v) = ||u - v||$

Soln.

$$\langle (-2,-1), (-2,-1) \rangle^{1/2}$$

$$= \sqrt{\frac{1}{2}(-2)(-2) + 5(-1)(-1)}$$

$$\sqrt{\frac{4}{2} + 5} = \sqrt{\frac{4+10}{2}} = \sqrt{\frac{14}{\cancel{2}}} = \sqrt{7} \text{ Ans.}$$

(f) $||u - kv||$

$$\langle (-8,-5), (-8,-5) \rangle^{1/2}$$

$$= \sqrt{\frac{1}{2}(-8)(-8) + 5(-5)(-5)}$$

$$= \sqrt{\frac{64}{2} + 125}$$

$$= \sqrt{32 + 125}$$

$$= \sqrt{157} \text{ Ans}$$

Q#13

Date:

A weighted Euclidean inner Product on R^2 is given for a vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ find Matrix that generates

$$\langle u, v \rangle = 3u_1v_1 + 5u_2v_2$$

Sol:-

$$\langle u, v \rangle = w_1u_1v_1 + w_2u_2v_2.$$

$$\begin{bmatrix} \sqrt{w_1} & 0 \\ 0 & \sqrt{w_2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$ Answer.

Q NO 15.

A Sequence of Sample points. use evaluation inner Product on P_3 at those Sample points to find $\langle p, q \rangle$ for Polynomials.

$$p = x + x^3$$
$$q = 1 + x^2$$

15. $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$

Sol:-

$$\langle p, q \rangle = p(x_0) \cdot q(x_0) + p(x_1)\, q(x_1) + p(x_2)\, q(x_2) + p(x_3)\, q(x_3).$$

$$p(-2)\, q(-2) + p(-1)\, q(-1) + p(0)\, q(0) + p(1)\, q(1)$$

$$= (-2)(2) + (0)(1) + (2)(2) + (10)(5)$$

$$= 50$$ Ans.

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Q No 22.

Date:

Find $\|U\|$ and $\|V\|$ and $d(U,V)$ relative to standard inner Product on M_{22}.

$$U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}, \quad V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}.$$

Sol.

$$\|U\| = \langle U, U\rangle^{1/2} = \sqrt{tr(U^T U)}.$$

$$= \sqrt{tr\left(\begin{bmatrix} 10 & -13 \\ -13 & 29 \end{bmatrix}\right)} = \sqrt{39}$$

$$d(U,V) = \|U-V\| = \langle U-V, U-V\rangle^{1/2}.$$

$$= \sqrt{tr\,(U-V)^t\,(U-V)}$$

$$= \sqrt{tr\left(\left(\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}\right)^t \left(\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}\right)\right)}$$

$$= \sqrt{tr\left(\begin{bmatrix} -3 & -4 \\ -3 & -3 \end{bmatrix}^t * \begin{bmatrix} -3 & -4 \\ -3 & -3 \end{bmatrix}\right)}.$$

$$= \sqrt{tr\left(\begin{bmatrix} -3 & -3 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ -3 & -3 \end{bmatrix}\right)}$$

$$= \sqrt{tr\left(\begin{bmatrix} 9+9 & 12+9 \\ 12+9 & 16+9 \end{bmatrix}\right)} = \sqrt{tr\begin{bmatrix} 18 & 21 \\ 21 & 25 \end{bmatrix}}.$$

$\sqrt{43}$ Ans.

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Q No 26.

Find $||u||$ and $d(u,v)$ for vectors $u = (-1, 2)$, $v = (2, 5)$ relative to inner Product of on R^2 generated by matrix A.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

Sol:

$$||u|| = \langle u, u \rangle^{1/2}.$$

$$= \sqrt{\left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)}$$

$$\left(\begin{bmatrix} 3 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \end{bmatrix}\right)^{1/2} = \sqrt{58}.$$

$$d(u,v) = ||u - v||$$

$$= \sqrt{\left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 3 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right)}.$$

$$= \sqrt{\begin{bmatrix} -9 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} -9 \\ -6 \end{bmatrix}} = 3\sqrt{3} \text{ Ans}$$

Date:

Ques 28.
Suppose that u, v and w are vectors in an inner Product space such that

$\langle u,v\rangle = 2$, $\langle v,w\rangle = -6$, $\langle u,w\rangle = -3$

$\|u\| = 1$, $\|v\| = 2$ $\|w\| = 7$.

Evaluate the Expression.

(a) $\langle u-v-2w, 4u+v\rangle$

Sol:

$= \langle u, 4u+v\rangle - \langle v, 4u+v\rangle - \langle -2w, 4u+v\rangle$

$= \langle u, 4u\rangle + \langle u+v\rangle - \langle v, 4u\rangle - \langle v,v\rangle - \langle 2w, 4u\rangle - \langle 2w, v\rangle$

$= 4\langle u,u\rangle + \langle u,v\rangle - 4\langle v,u\rangle - \langle v,v\rangle - 8\langle w,u\rangle - 2\langle w,v\rangle$

$4\|u\|^2 - 3\langle u,v\rangle - \|v\|^2 - 8\langle u,w\rangle - 2\langle v,w\rangle$

$= 4 - 3(2) - 4 - 8(-3) - 2(-6) = 30$.

(b) $\|2w-v\|$

Sol:

$= \sqrt{\langle 2w-v, 2w-v\rangle}$

$= \sqrt{\langle 2w, 2w-v\rangle - \langle v, 2w-v\rangle}$

$= \sqrt{\langle 2w, 2w\rangle - \langle 2w, v\rangle - \langle v, 2w\rangle + \langle v,v\rangle}$

$= \sqrt{4\|w\|^2 - 4\langle v,w\rangle + \|v\|^2}$

$= \sqrt{4(49) - 4(-6) + 4}$

$\sqrt{224}$

$= 4\sqrt{14}$

Exercise 6.2.

Date:

Q No 2.

Find Cosine angle between vectors.

(a) $u = (-1, 0)$, $v = (3, 8)$

Sol:-

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{(u_1v_1 + u_2v_2)}{\sqrt{u_1^2+u_1^2} * \sqrt{v_1^2+v_1^2}}$$

$$= \frac{(-1\times 3 + 0\times 8)}{\sqrt{(-1)^2+(0)^2} + \sqrt{3^2+8^2}} = \frac{-3}{\sqrt{1}\times\sqrt{9+64}}$$

$$= \frac{-3}{\sqrt{73}}$$ Ans.

(b) $u = (4, 1, 8)$, $v = (1, 0, -3)$

Soln:-

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{4(1) + (1)(0) + (8)(-3)}{\sqrt{4^2+1^2+8^2} * \sqrt{(-3)^2+(1)^2+(0)^2}}$$

$$= \frac{-20}{\sqrt{81}\,\sqrt{20}} = \frac{-20}{9\sqrt{10}}$$

(c) $u = (2, 1, 7, -1)$, $v = (4, 0, 0, 0)$.

Soln

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{(2)(4) + (1)(0) + (7)(0) + (-1)(0)}{\sqrt{2^2+1^2+7^2+(-1)^2} * \sqrt{4^2+0^2+0^2+0^2}}$$

$$= \frac{8}{4\sqrt{55}} = \frac{2}{\sqrt{55}}$$

Date:

Q No 4
Find Cosine of angle between vectors with respect to standard linear product on P_2 -

$P = x - x^2$; $q = 7 + 3x + 3x^2$

Sol:-

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\| \, \|v\|} = \frac{(0)(7) + (1)(3) + (-1)(3)}{\sqrt{0^2 + 1^2 + (-1)^2} * \sqrt{7^2 + 3^2 + 3^2}}$$

$$= \frac{0 + 3 - 3}{\sqrt{1+1}\,\sqrt{49+9+9}} = 0$$

Question #6.

Find cosine of angle Between A and B with respect to Standard inner product on M_{22}.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$$

Solve

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\| \, \|v\|} = \frac{tr(U^T V)}{\sqrt{tr(U^T U)}\,\sqrt{tr(V V^T)}}$$

$$= U^T V = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}.$$

$$= \begin{bmatrix} -6-4 & 2-2 \\ -1+12 & 4+6 \end{bmatrix}$$

$$tr\ U^T V = \begin{bmatrix} -10 & 0 \\ 0 & 10 \end{bmatrix} = 100$$

Date:

$$\cos\theta = \frac{19}{\sqrt{50}\sqrt{14}} = \boxed{\frac{19}{10\sqrt{7}}}$$ Ans

Q No 8:-

Determine whether vectors are orthogonal with respect to Euclidean inner product.

(a) $u = (u_1, u_2, u_3)$, $v = (0, 0, 0)$

Solve.

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3$$
$$= u_1 \times 0 + u_2 \times 0 + u_3 \times 0$$
$$= 0$$

Its orthogonal.

(b) $u = (a, b, c)$, $v = (-c, 0, a)$.

Soln:-

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3.$$
$$= -ac + 0 + ac.$$
$$= 0.$$

Q No 12:- Show the Matrices are orthogonal with respect to Standard inner product on M_{22}.

$$u = \begin{bmatrix} 5 & -1 \\ 2 & -2 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}.$$

Multiply corresponding elements and Add them

$$= 5(1) + (-1)(3) + 2(-1) + (-2)(0).$$
$$= 5 - 3 - 2 + 0$$
$$= 0$$

$\Rightarrow$ It's orthogonal.

Q NO 18:-

Date:

Show Vectors $u = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $V = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$.

are orthogonal with respect to inner product on R^2 that is generated by the Matrix.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Soln:-

$$\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right).$$

$$= \begin{bmatrix} 9 \\ 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 18 - 18$$

$$= 0$$

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