National University of computer and emerging sciences

Foundation for Advancement of Science and Technology

Linear Algebra (MT-1004)

Lecture # 23 & 24

Dimension

- Dimension:

  The dimension of a vector space V is defined to be the number of vectors in a basis for V

  V: a vector space $\qquad$ S: a basis for V

  $$\Rightarrow \dim(V) = \#(S) \quad \text{(the number of vectors in a basis } S\text{)}$$

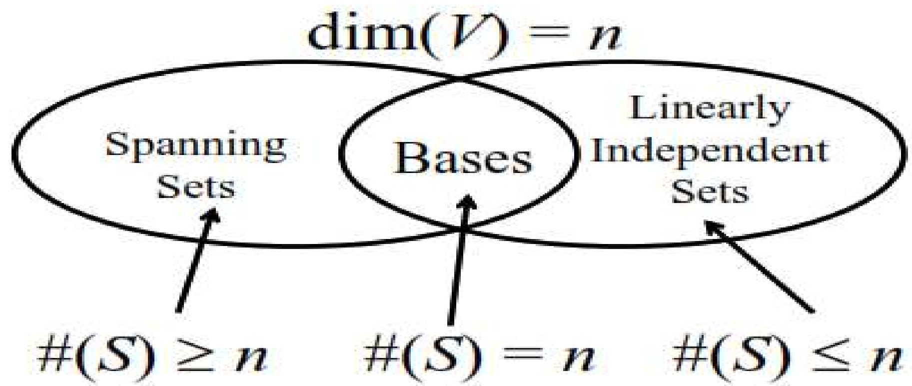
- Finite dimensional:

  A vector space V is finite dimensional if it has a basis consisting of a finite number of elements

- Infinite dimensional:

  If a vector space V is not finite dimensional, then it is called infinite dimensional

- Notes:



(1) $\dim(\{\mathbf{0}\}) = 0$

(If V consists of the zero vector alone, the dimension of V is defined as zero)

(2) Given $\dim(V) = n$, for $S \subseteq V$

S: a spanning set $\Rightarrow \#(S) \geq n$

S: a L.I. set $\Rightarrow \#(S) \leq n$ (from Theorem 4.2)

S: a basis $\Rightarrow \#(S) = n$ (Since a basis is defined to be a set of L.I. vectors that can spans V, $\#(S) = \dim(V) = n$) (see the above figure)

(3) Given $\dim(V) = n$, if W is a subspace of $V \Rightarrow \dim(W) \leq n$

※ For example, if $V = R^3$, you can infer the $\dim(V)$ is 3, which is the number of vectors in the standard basis

Theorem 4.6.2

Let V be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be any basis for V.

(a) If a set in V has more than n vectors, then it is linearly dependent.

(b) If a set in V has fewer than n vectors, then it does not span V.

- Ex4.2: Find the dimension of a vector space according to the standard basis

※ The simplest way to find the dimension of a vector space is to count the number of vectors in the standard basis for that vector space

(1) Vector space R^n $\Rightarrow$ standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$$\Rightarrow \dim(R^n) = n$$

(2) Vector space $M_{m\times n}$ $\Rightarrow$ standard basis $\{E_{ij} \mid 1 \le i \le m,\ 1 \le j \le n\}$

$$\text{and in } E_{ij} \begin{cases} a_{ij} = 1 \\ \text{other entries are zero} \end{cases}$$

$$\Rightarrow \dim(M_{m\times n}) = mn$$

(3) Vector space $P_n(x)$ $\Rightarrow$ standard basis $\{1, x, x^2, \ldots, x^n\}$

$$\Rightarrow \dim(P_n(x)) = n+1$$

(4) Vector space $P(x)$ $\Rightarrow$ standard basis $\{1, x, x^2, \ldots\}$

- Ex 4.3: Determining the dimension of a subspace of R^3

  (a) $W = \{(d, c-d, c): c$ and d are real numbers$\}$

  (b) $W = \{(2b, b, 0): b$ is a real number$\}$

Sol: (Hint: find a set of L.I. vectors that spans the subspace, i.e., find a basis for subspace.)

(a) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S spans W)

$\Rightarrow S$ is a basis for W

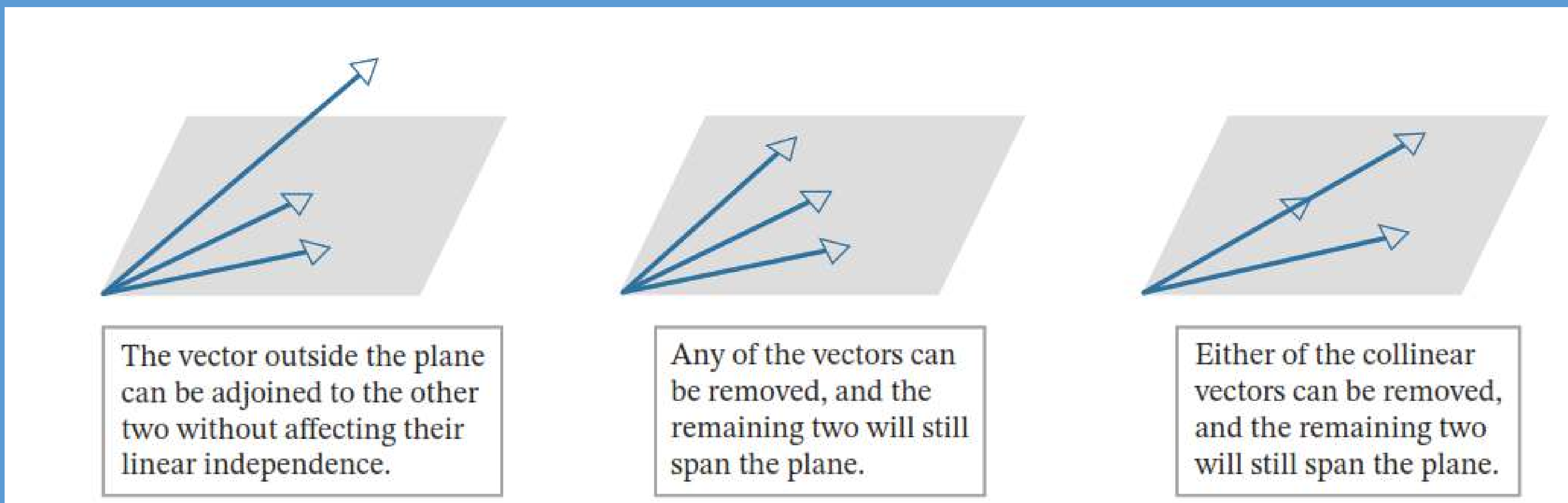
$\Rightarrow \dim(W) = \#(S) = 2$

(b) $\because (2b, b, 0) = b(2, 1, 0)$

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1$


The vector outside the plane can be adjoined to the other two without affecting their linear independence.
Any of the vectors can be removed, and the remaining two will still span the plane.
Either of the collinear vectors can be removed, and the remaining two will still span the plane.

Theorem 4.6.3

Plus/Minus Theorem

Let S be a nonempty set of vectors in a vector space V.

(*a*) If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.

(*b*) If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is,

$$\operatorname{span}(S) = \operatorname{span}(S - \{\mathbf{v}\})$$

Theorem 4.6.4

Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.

1. Every spanning set for a subspace is either a basis for that subspace or has a basis as a subset.
2. Every linearly independent set in a subspace is either a basis for that subspace or can be extended to a basis for it.

Theorem 4.6.5

Let S be a finite set of vectors in a finite-dimensional vector space V.

(a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

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Do Question # 1-20 from Ex # 4.6