National University of computer and emerging sciences

Foundation for Advancement of Science and Technology

Linear Algebra (MT-1004)

Lecture # 20

Linear Combination in a Vector Space

- Linear combination:

A vector $\mathbf{u}$ in a vector space V is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ in V if $\mathbf{u}$ can be written in the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k,$$

where $c_1, c_2, \ldots, c_k$ are real-number scalars

Spanning Set

- This section introduces the spanning set, this notion is associated with the representation of any vector in a vector space as a **linear combination** of a selected set of vectors in that vector space.

- Spanning Set:
  If $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots\ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space W of V consisting of all linear combinations of the vectors in S is called space spanned by $\mathbf{v}_1$, $\mathbf{v}_2,\ldots\ldots,\mathbf{v}_k$ and we say that the vectors $\mathbf{v}_1$, $\mathbf{v}_2,\ldots\ldots\ldots,\mathbf{v}_k$ span W. It is denoted by

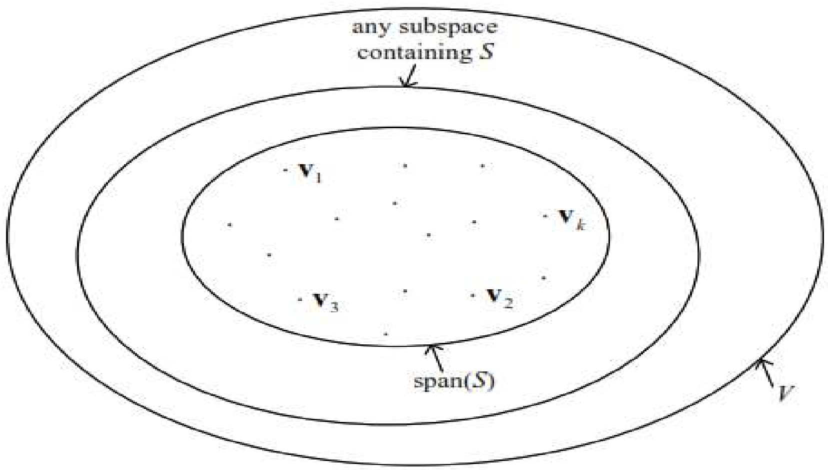
$$W = \text{Span}(S) \text{ or } W = \text{Span}\{\mathbf{v}_1,\mathbf{v}_2,\ldots\ldots\ldots,\mathbf{v}_k\}$$

- Theorem $\boxed{\text{Span}(S) \text{ is a subspace of } V}$

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then

(a) span(S) is a subspace of V

(b) span(S) is the smallest subspace of V that contains S, i.e., every other subspace of V containing S must contain span(S)



Problem Let $\mathbf{v}_1 = (2,5)$ and $\mathbf{v}_2 = (1,3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

Take any vector $\mathbf{w} = (a, b) \in \mathbb{R}^2$. We have to check that there exist $r_1, r_2 \in \mathbb{R}$ such that

$$\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = a \\ 5r_1 + 3r_2 = b \end{cases}$$

Coefficient matrix: $C = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$. $\det C = 1 \neq 0$.

Since the matrix C is invertible, the system has a unique solution for any a and b.
Thus $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2$.

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A Procedure for Identifying Spanning Sets

Step 1. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ be a given set of vectors in V, and let $\mathbf{x}$ be an arbitrary vector in V.

Step 2. Set up the augmented matrix for the linear system that results by equating corresponding components on the two sides of the vector equation

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r = \mathbf{x} \tag{2}$$

Step 3. Use the techniques developed in Chapters 1 and 2 to investigate the consistency or inconsistency of that system. If it is consistent for *all* choices of $\mathbf{x}$, the vectors in S span V, and if it is inconsistent for *some* vector $\mathbf{x}$, they do not.

Go through all examples

Theorem 4.3.2

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.