PLANARITY

- We have not spent much time talking about the way in which we draw a graph, mainly because the drawing itself is only a visual depiction of the information stored within the edge and vertex set.
- This chapter will investigate when the way in which we draw a graph can indicate underlying structural information about the graph.
- In particular, we will be looking at when a graph is planar.

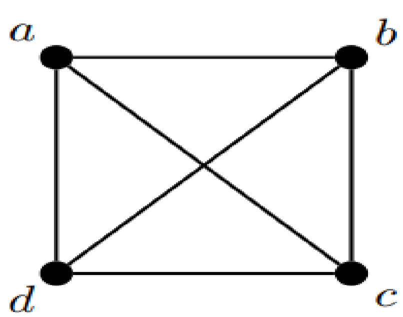
Definition 7.1 A graph G is ***planar*** if and only if the vertices can be arranged on the page so that edges do not cross (or touch) at any point other than at a vertex.

- Note that for a graph to be planar, it is only required that at least one drawing exists without edge crossings.
- It is not required that all possible drawings of the graph be without edge crossings.

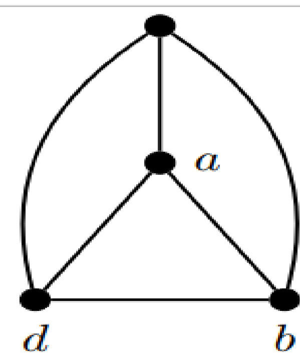
Example:

For example, below are two drawings of the graph K_4 (in the language of Section 1.3, these graphs are isomorphic).

- The drawing on the left is the more standard way of drawing K_4 and contains one edge crossing (ac and bd cross at a location that is not a vertex).
- The drawing on the right is a planar drawing of K_4 so that no edge crossings exist.



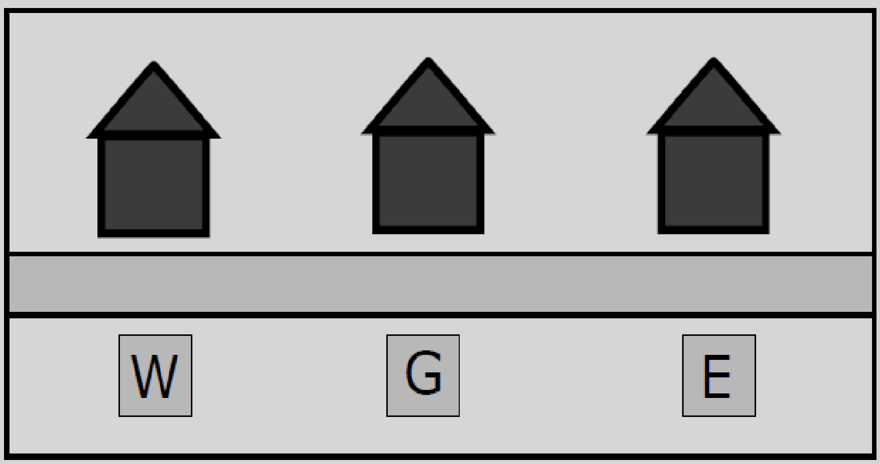
K_4



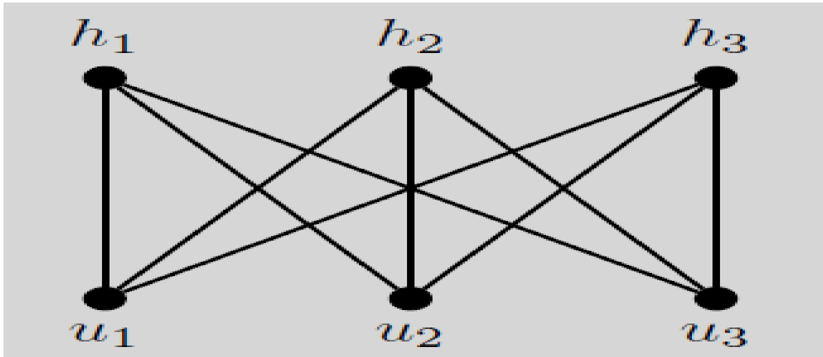
planar drawing of K_4

- Often it is useful to think of taking a graph and moving around the vertices and pulling or stretching the edges so that they can be repositioned without edge crossings.
- Finding a planar drawing of a graph can be very tricky. In fact, simply determining if a graph is planar or not is hardly trivial.

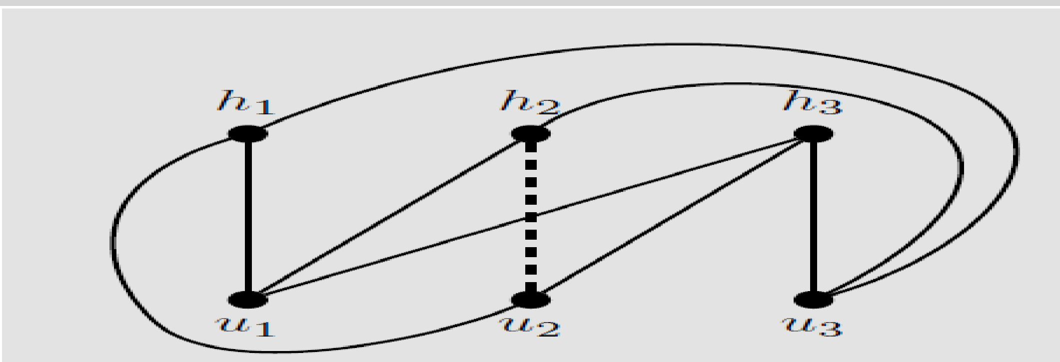
Example 7.1 Three houses are set to be built along a new city block; across the street lie access points to the three main utilities each house needs (water, electricity, and gas). Is it possible to run the lines and pipes underground without any of them crossing?



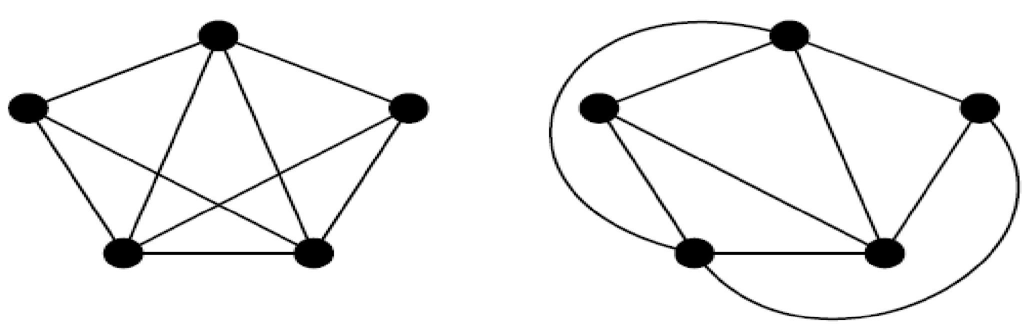
Solution: This scenario can be modeled with three vertices representing the houses (h_1, h_2, and h_3) and three vertices representing the utilities (u_1, u_2, and u_3). First note that if we are not concerned with edge crossings, the proper graph model is $K_{3,3}$, the complete bipartite graph with three vertices in each side of the vertex partition. The standard drawing of $K_{3,3}$, given below, clearly is not a planar drawing as there are many edge crossings; for example, h_1u_2 crosses h_2u_1.



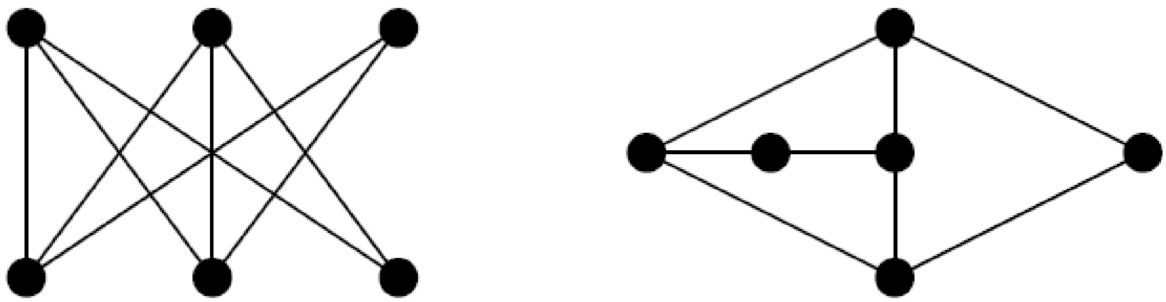
Attempting to find a drawing without edge crossings, we could stretch and move some of the edges as shown in the next graph. However, this drawing is still not planar since edges h_3u_1 and h_2u_2 still cross. In fact, no matter how you try to draw $K_{3,3}$ (try it!), there will always be at least one edge crossing. Thus the utility lines and pipes cannot be placed without any of them crossing.



The complete graph K_5 with an edge deleted is planar and its plane drawing is shown in Fig. 8.7.2.



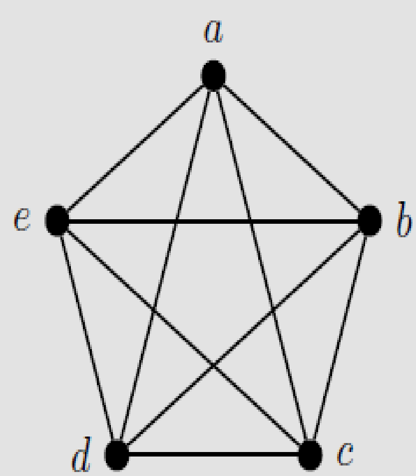
The complete bipartite graph $K(3,3)$ with an edge deleted is planar and its plane drawing is shown in Fig. 8.7.3.



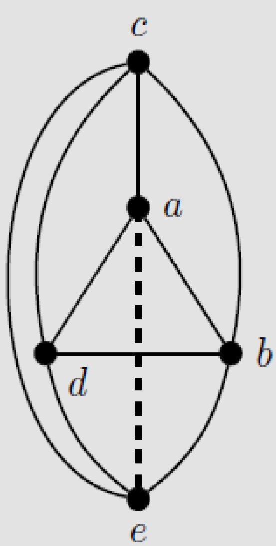
KURATOWSKI'S THEOREM:

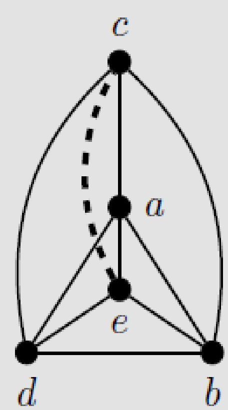
- Given a specific drawing of a graph, it is easy to see if that drawing is planar or not.
- However, just because you cannot find a planar drawing of a graph does not mean a planar drawing does not exist. This is perhaps the most challenging part of planarity.
- Luckily, we have already seen one of the most
- important structures in showing a graph is nonplanar (namely $K_{3,3}$).

Example 7.2 Determine if K_5 is planar. If so, give a planar drawing; if not, explain why not.



Solution: If we attempt the same procedure as we did for K_4 above, then we run into a problem. We start with the planar drawing of K_4 and add another vertex e below the edge bd (shown on the left below). We can easily add the edges from e to b, c, and d without creating crossings, but to reach a from e we would need to cross one of the edges bd, bc, or cd. Placing vertex e anywhere in the interior of any of the triangular sections of K_4 will still create the need for an edge crossing (one of which is shown on the right below).



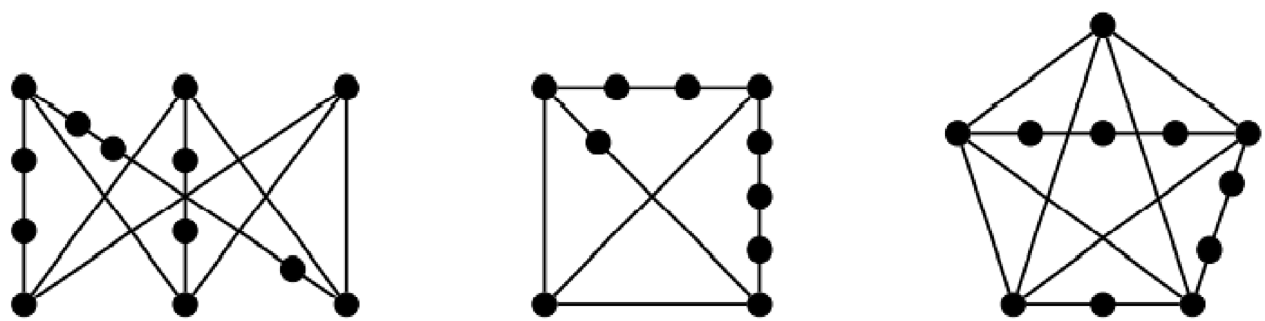


Remarks:

- We now have two graphs that we know to be nonplanar.
- Moreover, having either K_5 or $K_{3,3}$ as a subgraph will guarantee that a graph is nonplanar.
- If a portion of a graph is nonplanar there is no way for the entire graph to be planar.
- What may be surprising is that these two graphs provide the basis for determining the planarity of any graph.
- However, it is not as simple as containing a K_5 or $K_{3,3}$ subgraph, but rather a modified version of these graphs called subdivisions.

Definition 7.2 A *subdivision* of an edge xy consists of inserting vertices so that the edge xy is replaced by a path from x to y. The subdivision of a graph G is obtained by subdividing edges in G.

- Note that a subdivision of a graph can be obtained by subdividing one,
- two, or even all of its edges.
- However, the new vertices placed on the edges from G cannot appear in more than one subdivided edge.
- The graph on the left shows a subdivision of $K_{3,3}$.
- The graph in the middle shows a subdivision of K_4.
- The graph on the right shows a subdivision of K_5.



From the discussion above, we should understand why $K_{3,3}$ and K_5 subgraphs pose a problem for planarity.

- Adding a vertex along any of the edges (thus creating paths between the original vertices) of one of these graphs will not suddenly allow the graph to become planar.
- Thus, containing a subdivision $K_{3,3}$ or K_5 proves a graph is nonplanar.

The Polish mathematician **Kazimierz Kuratowski** proved in 1930 that containing a $K_{3,3}$ or K_5 subdivision was not only enough to prove a graph was nonplanar, but more surprisingly that any nonplanar graph must contain a $K_{3,3}$ or K_5 subdivision.

Theorem 7.3 (Kuratowski's Theorem) A graph G is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5.

Explanation:

- In practice, it is often useful to think of moving vertices and stretching edges at the same time as looking for a $K_{3,3}$ or K_5 subdivision.
- Note that in order to contain a $K_{3,3}$ subdivision, a graph must have at least 6 vertices of degree 3 or greater.
- In order to contain a K_5 subdivision the graph must have at least 5 vertices of degree 4 or greater.
- These conditions are often helpful when searching for a subdivision.

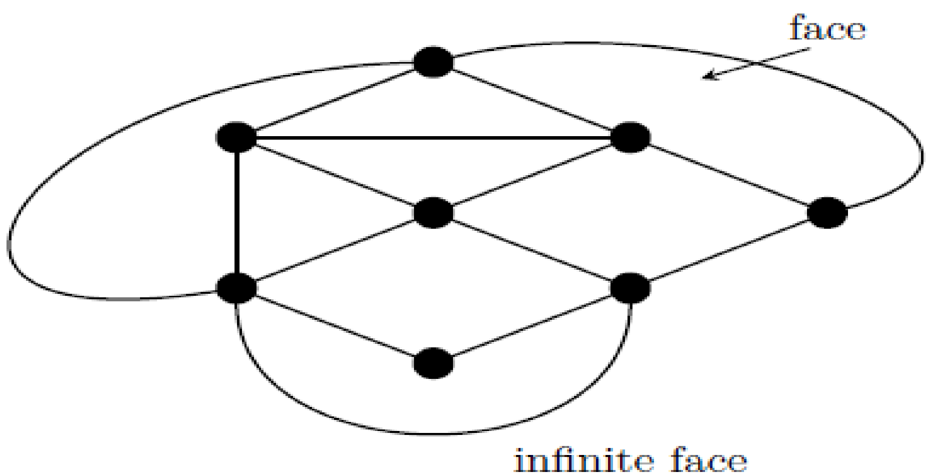
EULER'S FORMULA:

- One major result regarding planarity that is quite useful in gaining some intuition as to the planarity of a graph was proven in 1752 by a mathematician **Leonhard Euler**.
- The result was given in more geometric terms (and planarity is one area of intersection between graph theory and geometry) and uses an additional term relating to the drawing of a graph, namely a region (resp face).

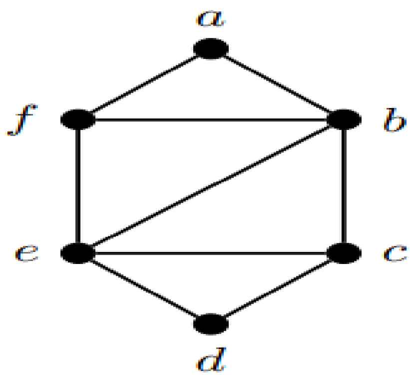
Definition 7.4 Given a planar drawing of a graph G, a ***region*** is a portion of the plane completely bounded by the edges of the graph.

Examples:

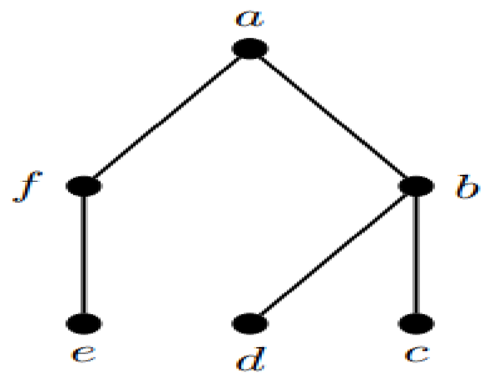
1. The plane graph is of order 8 and size 15; including the infinite face, there are 9 faces in its plane drawing.



2. The following two graphs G_1 and G_2, each have 6 vertices, but G_1 has 9 edges and 5 regions whereas G_2 has 5 edges and only one region, the infinite one.



G_1



G_2

3. Every tree has exactly 1 region since no cycles exist to fully encompass a portion of the plane.

There is a very interesting relationship linking the order, the size and the number of regions in any plane drawing of a planar graph.

Theorem 7.5 (Euler's Formula) If G is a connected planar graph with n vertices, m edges, and r regions then $n - m + r = 2$.

Proof: Argue by induction on m, the number of edges in the graph. If $m = 1$ then since G is connected it is either a tree with one edge and so $n = 2$ and $r = 1$, or G is a graph with a loop, and so $n = 1$ and $r = 2$. In either case, Euler's Formula holds.

Now suppose Euler's Formula holds for all graphs with $m \geq 1$ edges and consider a graph G with $m + 1$ edges, n vertices, and r regions. We will consider the graph that is formed with the removal of an edge from G.

First, if $G' = G - e$ is not connected for any edge e in G then we know e must be a bridge of G. Thus G must be a tree, and so $n = m + 1$ and $r = 1$. Therefore $n - m + r = m + 1 - m + 1 = 2$.

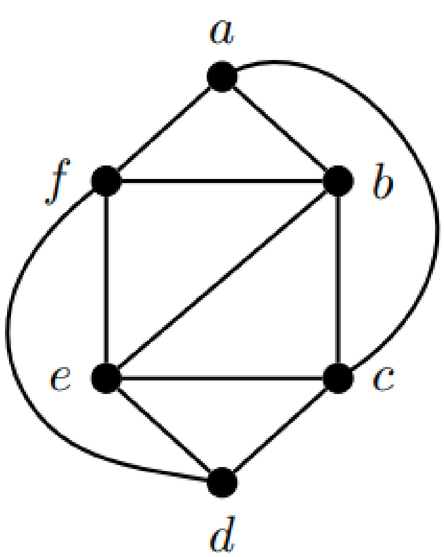
Next, if $G' = G - e$ is connected for some edge e of G, then e must be a part of some cycle in G. Then there must be two different regions on the two sides of e in G, but these regions join into just one with the removal of e. Thus G' has n vertices, $r - 1$ regions, and $m - 1$ edges. By the induction hypothesis applied to G', we know $n - (m - 1) + (r - 1) = 2$, which simplifies to $n - m + r = 2$.

Thus by induction we know Euler's Formula holds for all connected planar graphs.

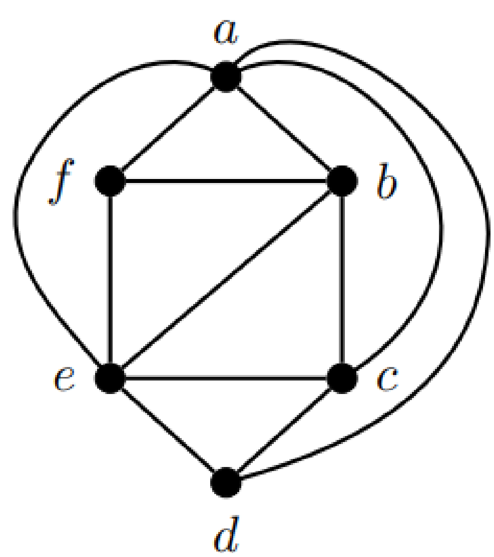
Question: If additional edges could be added to the graph while maintaining planarity (such as adding additional utility lines without creating crossings).

Answer: Clearly additional edges can be added to a tree without incurring edge crossings, but what about G_1 above? We cannot add edges along the interior of the graph, but we could cut into the infinite region with some additional edges.

Note: we are not considering adding multi-edges since we could theoretically add an infinite number of edges between two vertices.



$G_1 + \{ac, df\}$



$G_1 + \{ac, ae, ad\}$

For the graph on the right above, the border for the infinite region is created by the edges between vertices a, d, and e, and so it also is triangular. This graph has 6 vertices and 12 edges.

- Note that if we try to add any additional edges to the graph on the right that an edge-crossing would be required. This graph is called **maximally planar**.

Definition 7.6 A graph G is ***maximally planar*** if $G + e$ is nonplanar for any edge $e = xy$ for any two nonadjacent vertices $x, y \in V(G)$.

Remark: For a graph the be maximally planar, we basically need every region to be bounded by a triangle, including the infinite region.

Theorem 7.7 If G is a maximally planar simple graph with $n \geq 3$ vertices and m edges, then $m = 3n - 6$.

Proof: Assume G is maximally planar. Then every region must be bounded by a triangle, as otherwise we could add a chord to a region bounded by a longer cycle. Since every edge separates two regions, and every region is bounded by three edges, we know $r = \frac{2m}{3}$. Thus by Euler's Formula, we have $n - m + \frac{2m}{3} = 2$, and so $3n - 3m + 2m = 6$, giving $m = 3n - 6$.

- One of the implications of this theorem is that a planar graph cannot be too dense, that is there cannot be too many edges in relation to the number of vertices.

The two theorems below are more useful in practice though, as they allow us to use simple counting arguments when investigating graph planarity.

Theorem 7.8 If $G = (V, E)$ is a simple planar graph with m edges and $n \geq 3$ vertices, then $m \leq 3n - 6$.

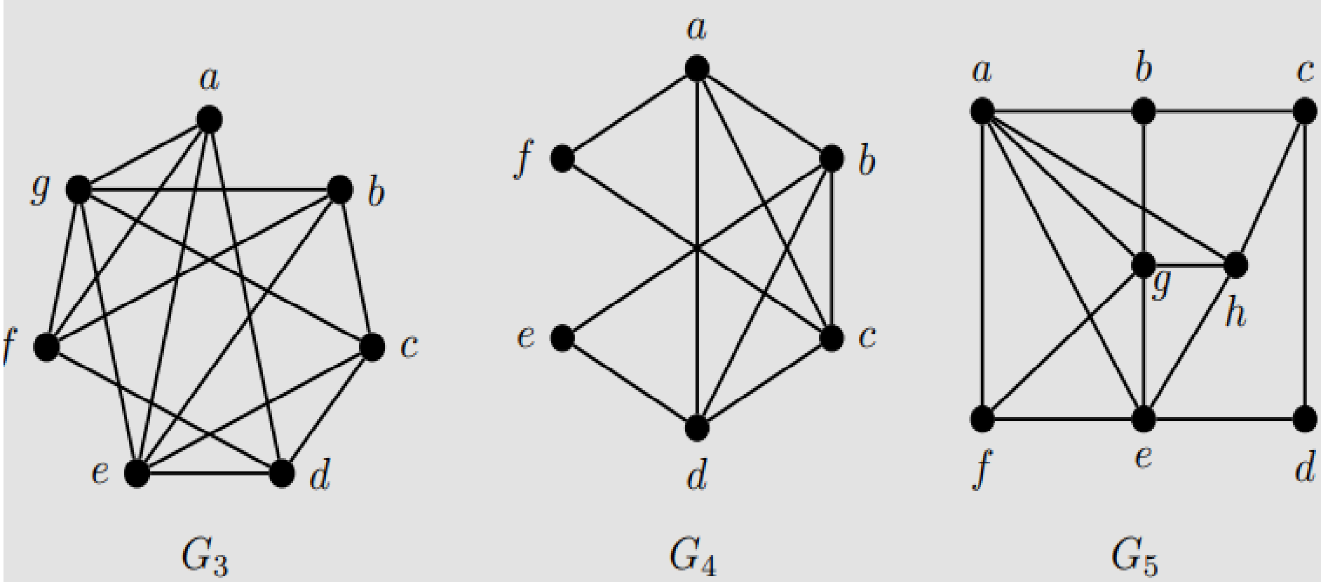
Theorem 7.9 If $G = (V, E)$ is a simple planar graph with m edges and $n \geq 3$ and no cycles of length 3, then $m \leq 2n - 4$.

Remarks:

- These theorems are less about verifying the relationship between edges and vertices when a graph is known to be planar, but rather in determining if a graph satisfies this inequality.
- If a graph does not satisfy the inequality, then it must be nonplanar.

- However, if the graph satisfies the inequality, then it may or may not be planar and further investigations are needed to determine planarity.

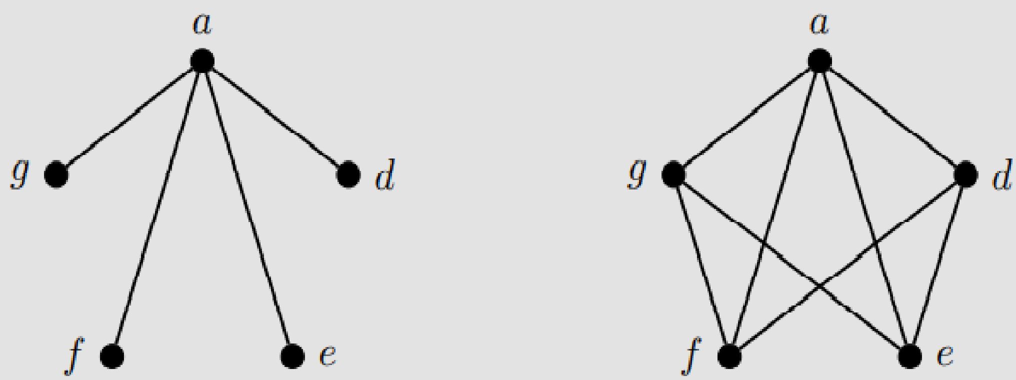
Example 7.3 Determine which of the following graphs are planar. If planar, give a drawing with no edge crossings. If nonplanar, find a $K_{3,3}$ or K_5 subdivision.



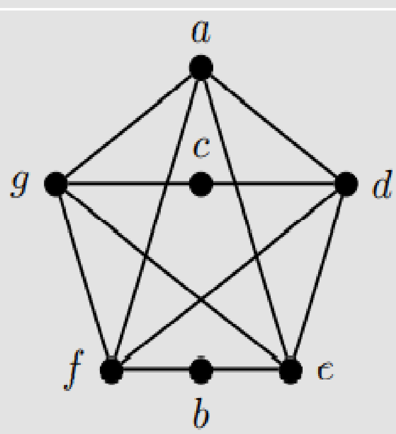
Solution: First note that all of the graphs above are drawn with edge crossings, but this does not indicate their planarity. Also, we can begin by using Euler's Theorem above to give us an indication of how likely the graph is to be planar.

For G_3, we have $|E| = 15$ and $|V| = 7$, giving us the inequality from Theorem 7.8 of $15 \leq 3 \cdot 7 - 6 = 15$. Thus the number of edges is as high as possible for a graph with 7 vertices. Although this does not guarantee the graph is nonplanar, it provides good evidence that we should search for a K_5 or $K_{3,3}$ subdivision. Also notice that every vertex has degree at least 4, so we will begin by looking for a K_5 subdivision.

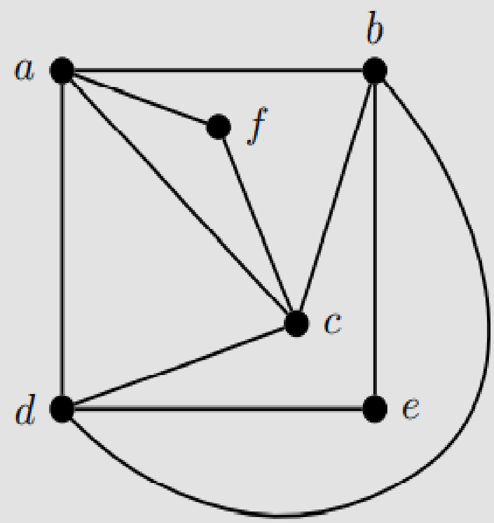
To do this, we start by picking a vertex and looking at its neighbors, hoping to find as many as possible that form a complete subgraph. Beginning with a as a main vertex in K_5, we see the other main vertices would have to be either its neighbors or vertices reachable by a short path. We will start by choosing the other main vertices of the K_5 to be the neighbors of a, namely d, e, f, and g. A starting graph is shown below on the left. Next we fill in the edges between these four vertices, as shown on the right below.



At this point, we are only missing two edges in forming K_5, namely dg and ef. We have two vertices available to use for paths between these nonadjacent vertices, and using them we find a K_5 subdivision. Thus G_3 is nonplanar.



For G_4, we have $|E| = 11$ and $|V| = 6$, giving us the inequality from Theorem 7.8 of $11 \leq 3 \cdot 6 - 6 = 12$. We cannot deduce from this result that the graph is nonplanar. However, notice that two vertices have degree 2 and the remaining 4 vertices have degree 4. There cannot be a K_5 subdivision since there are not enough vertices of degree at least 4 and there cannot be a $K_{3,3}$ subdivision since there are not enough vertices of degree at least 3. Thus we can conclude that G_4 is in fact planar. A planar drawing is shown below.

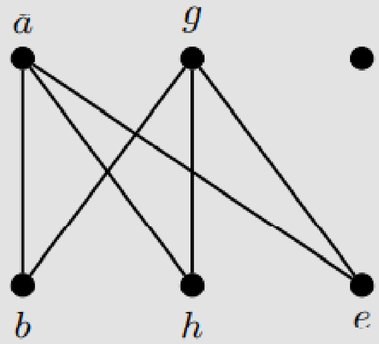


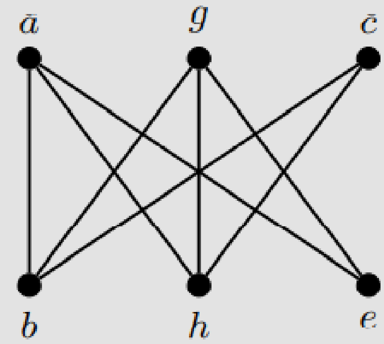
For G_5, we have $|E| = 15$ and $|V| = 8$, giving us the inequality from Theorem 7.8 of $15 \leq 3 \cdot 8 - 6 = 18$. As in the previous examples, we cannot deduce that the graph is nonplanar from Euler's Theorem but the high number of edges relative to the number of vertices should give us some suspicion that the graph may not be planar.

When inspecting the vertex degrees, we see only four vertices of degree at least 4 (namely a, e, g, h), indicating the graph cannot contain a K_5 subdivision. However, there are another three vertices of degree 3, allowing for a possibility of a $K_{3,3}$ subdivision.

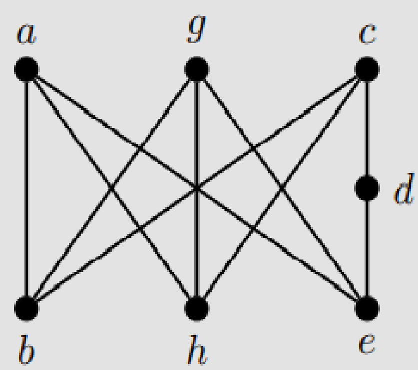
Again, we start by selecting vertex a to be one of the main vertices of a possible $K_{3,3}$ subdivision. At the same time, we will look for another vertex that is adjacent to three of the neighbors of a. We see that a and g are both adjacent to b, h, e, and f. Let us begin with b, h, and e for the vertices on the other side of the $K_{3,3}$, as shown below on the left.

We now search among the remaining vertices (f, d, c) for one that is adjacent to as many of b, h, and e as possible and find c is adjacent to both b and h, as shown below on the right.





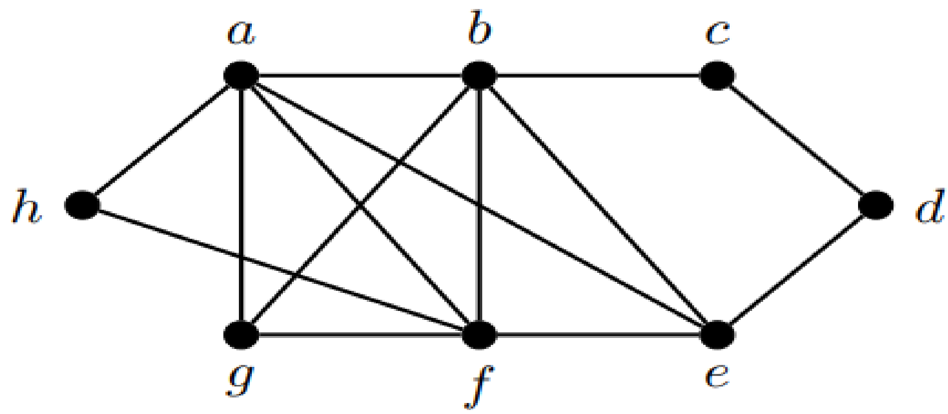
At this point we are only missing one edge to form a $K_{3,3}$ subdivision, namely ce. Luckily we can form a path from c to e using the available vertex d. Thus we have found a $K_{3,3}$ subdivision and proven that G_5 is nonplanar.



One final note of caution: subdivisions are not necessarily unique. In fact, G_3 and G_5 from the example above both contain more than one subdivision.

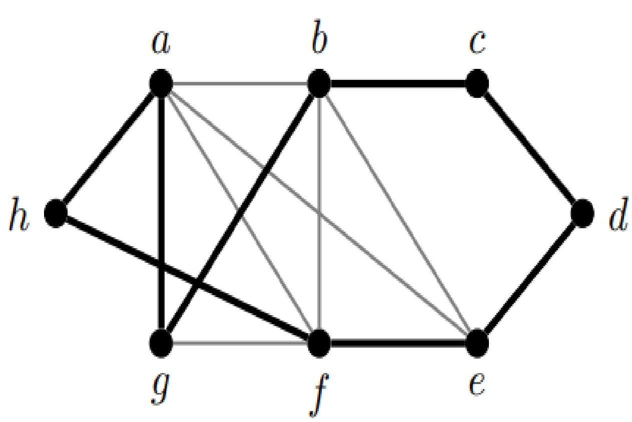
Cycle-Chord Method:

- When a graph is drawn so the vertices are roughly arranged around a circle, it can often be easier to think about shifting their positions on the page or stretching the edges to obtain a planar drawing.
- But when the graph is drawn to highlight some other attribute, such as it being bipartite or showing some clumping of vertices, it can be challenging to find a planar drawing.
- Here we define a detail of one method for finding a planar drawing, called the Cycle-Chord Method.
- The graph G_6 below will serve as an example of how to use this method.

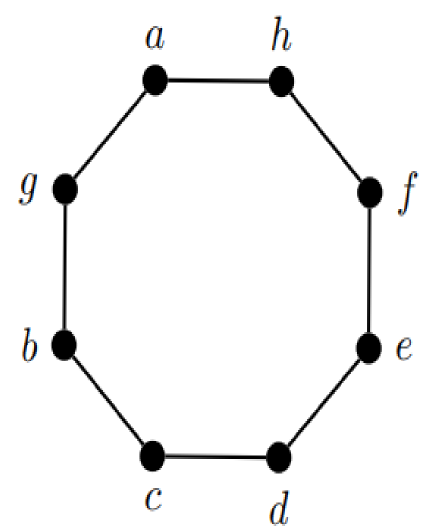


G_6

- To begin, put the vertices in a circular pattern, but with some care in their arrangement.
- We want to find a spanning cycle (also called a Hamiltonian cycle) or something approximating a spanning cycle, when placing the vertices.
- The edges in **bold** on the left represent those that are currently being placed in the planar drawing; the **gray** edges are ones not yet placed.

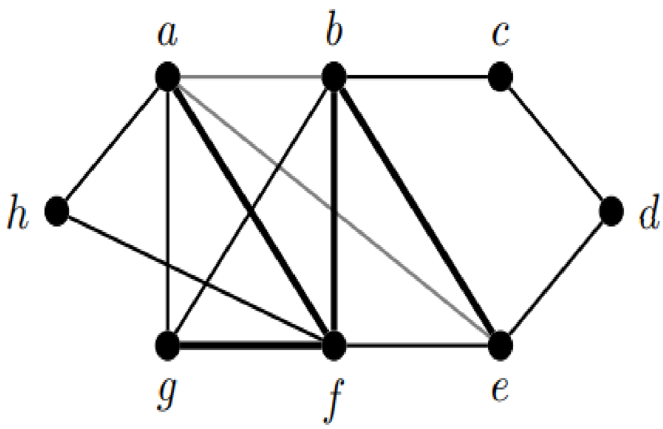


highlighted portion of G_6

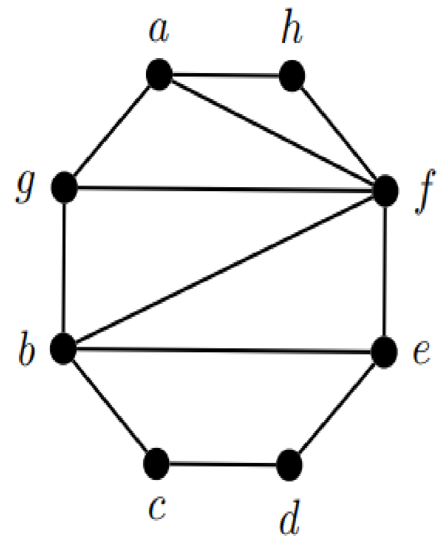


planar drawing

- Once we have created the spanning cycle, we attempt to place as many
- edges in the interior of the cycle as possible so that they do not cross.
- This should resemble **chords of a circle**.
- In the example G_6 below, we are making most of the interior regions of the cycle into **triangulations**.

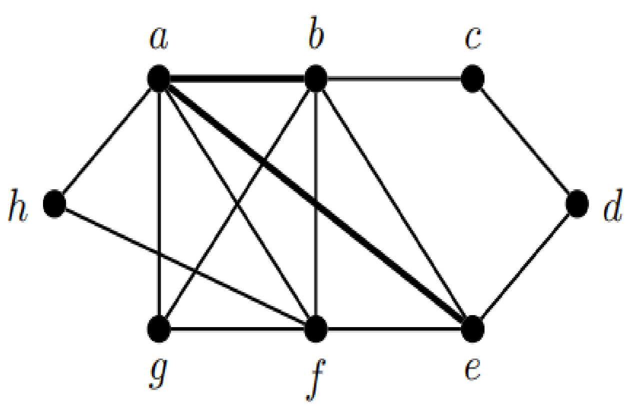


highlighted portion of G_6

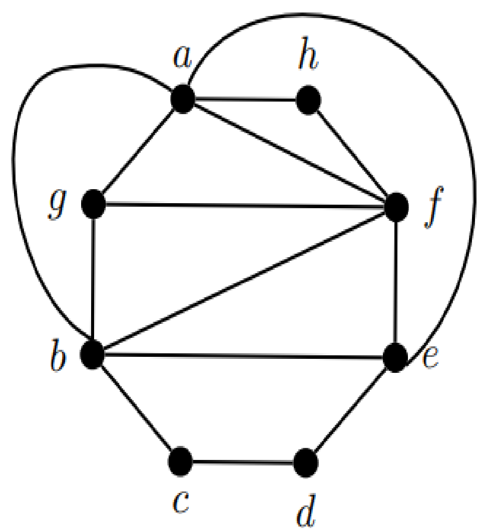


planar drawing

- When placing these chords, we should take care to notice any vertices that have incident edges remaining, as those will need to be placed as curves along the outside of the cycle.
- For example, in G_6 only two edges remain to be placed outside the spanning cycle, namely ab and ae.
- Having multiple edges from the same vertex left to place is often a benefit since they can be drawn in the same or opposite directions of the cycle without creating edge crossings.



highlighted portion of G_6



planar drawing