Explain your steps. The calculations and answers should be written neatly on paper which is attached as a single pdf. Submits the solution on GoogleClassroom with in dua date.

Problem 1

Recall that $\delta(G)$ denotes the minimum degree of G. Prove that if G is a graph with $2n$ vertices and $\delta(G) \geq n$ then G has a perfect matching.

Solution:

Suppose that $G = (V, E)$ is a graph on $|V| = 2n$ vertices and the minimum degree, $\delta = \delta(G) \geq n$. Our goal is to show that G has a perfect matching.

We will construct a perfect matching in n rounds, distinguishing two phases. During the first phase, we apply a trivial, greedy algorithm to construct a maximal matching, that is, a matching that cannot be extended by adding an edge. We start with an empty matching $M_0 = (\emptyset, \emptyset)$. In each round i, we consider the graph $G[V \setminus V(M_{i-1})]$ induced by unsaturated vertices. If it contains edges, then we arbitrarily pick one of them (say, edge $a_i b_i$) and add it to the current matching; that is, $V(M_i) = V(M_{i-1}) \cup \{a_i, b_i\}$ and $E(M_i) = E(M_{i-1}) \cup \{a_i b_i\}$. This phase ends if there are no more edges to pick from. If all the vertices are saturated, then we are done; otherwise, we move on to the second phase.

During the second phase, at the beginning of each round $i \leq n$, set $V \setminus V(M_{i-1})$ contains at least two vertices and it induces an independent set. We pick \emph{any} two vertices, say, p and q from that set. We will show that there is an edge in $E(M_{i-1})$, say, rs such that $pr \in E$ and $qs \in E$. In other words, we will show that there exists a path (p, r, s, q) of length 3 (such paths are often called \textbf{augmenting paths}). The existence of such paths allows us to improve the size of our matching. Indeed, we can simply remove rs from the matching and add edges pr and qs instead. Formally, $V(M_i) = V(M_{i-1}) \cup \{p, q\}$ and $E(M_i) = (E(M_{i-1}) \setminus \{rs\}) \cup \{pr, qs\}$.

To finish the proof, let us note that p has at least n neighbours in $V(M_{i-1})$ (since $\delta \geq n$ and $V \setminus V(M_{i-1})$ induces an independent set). Let $R := N(p) \subseteq V(M_{i-1})$, and let S be the set of vertices matched with vertices from R, that is,
$S = \{s \in V(M_{i-1}) : sr \in E(M_{i-1}) \text{ for some } r \in R\}$. Clearly, $|S| = |R| \geq \delta \geq n$. Moreover, S and R can overlap (and, in fact, they do) but it causes no problem. More importantly, since q has at least n neighbours in $V(M_{i-1})$, $|S| \geq n$, and $|V(M_{i-1})| \leq 2n - 2$, it follows that q has at least one neighbour in S which finishes the argument.

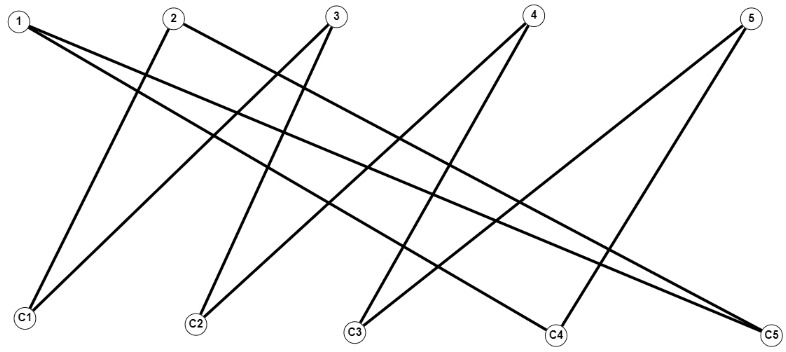
Problem 2

Consider the 3×5 Latin rectangle $L = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix}$. Define a bipartite graph G with bipartition (X, Y) associated with L as follows:

1. $X = \{1, 2, 3, 4, 5\}$,
2. $Y = \{C_1, C_2, C_3, C_4, C_5\}$, where C_i is the ith column of L,
3. i in X adjacent to C_j in Y if and inly if i does not appear in C_j.

(a) Draw the diagram of G.

Solution:



(b) What is the degree of each vertex in X?

Solution:
Degree of each vertex in X is 2

(c) What is the degree of each vertex in Y?

Solution:
Degree of each vertex in Y is 2

(d) Is G 2-regular?

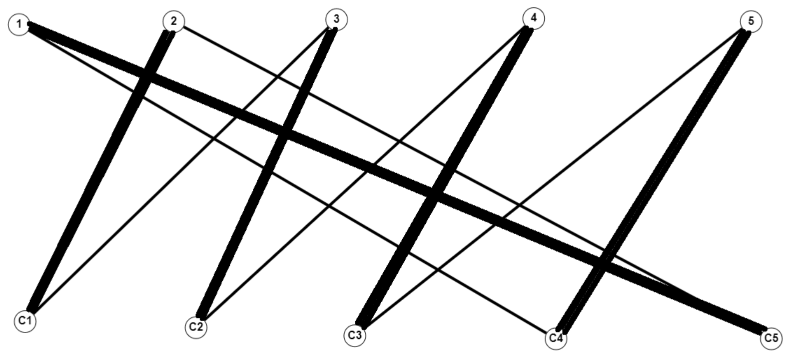
Solution:
Yes.

(e) Does G contain a perfect matching?

Solution:
Yes.

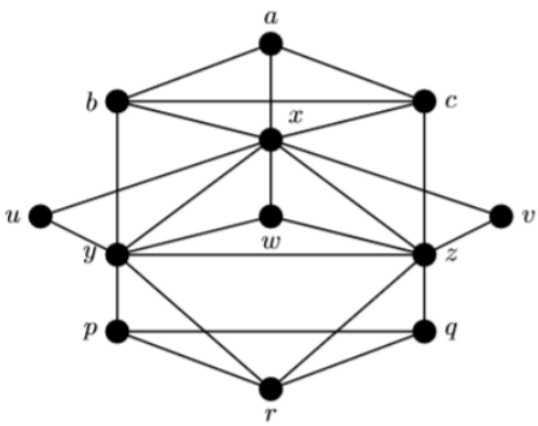
(f) Display a perfect matching in G if your answer to part(e) is yes.

Solution:



Problem 3

Consider the following graph G:

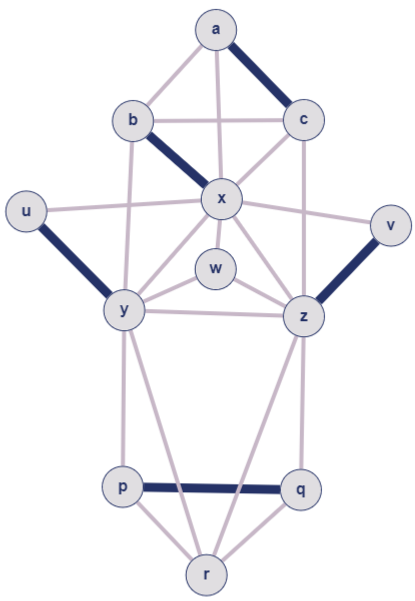


(a) Does G have a perfect matching?

(b) What is the size of a maximum matching in G?

Solution:

We use Edmonds Blossom Algorithm, which gives



There are no free vertices in the graph anymore so we have found a matching of maximum cardinality. Its edges are colored blue. Also some vertices are unsaturated, so given graph does not have perfect matching.

Problem 4

Determine the stable matchings resulting from the Proposal Algorithm run with men proposing and with women proposing, given the preference lists below.

Men $\{u, v, w, x, y, z\}$	**Women** $\{a, b, c, d, e, f\}$
$u : a > b > d > c > f > e$	$a : z > x > y > u > v > w$
$v : a > b > c > f > e > d$	$b : y > z > w > x > v > u$
$w : c > b > d > a > f > e$	$c : v > x > w > y > u > z$
$x : c > a > d > b > e > f$	$d : w > y > u > x > z > v$
$y : c > d > a > b > f > e$	$e : u > v > x > w > y > z$
$z : d > e > f > c > b > a$	$f : u > w > x > v > z > y$

Solution:

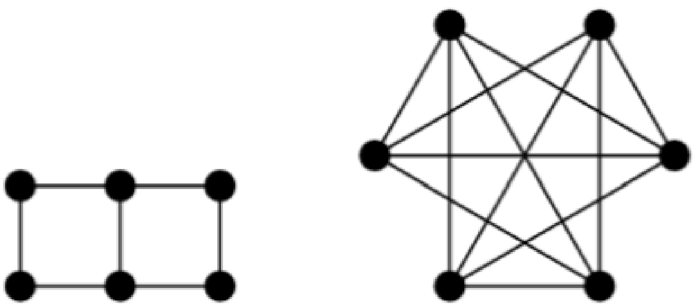
When men propose, the steps of the algorithm are as below. For each round, we list the proposals by u, v, w, x, y, z in order, followed by the resulting rejections. Round 1: a, a, c, c, c, d; $a \times v$, $c \times w$, $c \times y$. Round 2: a, b, b, c, d, d; $b \times v$, $d \times z$. Round 3: a, c, b, c, d, e; $c \times x$. Round 4: a, c, b, a, d, e; $a \times u$. Round 5: b, c, b, a, d, e; $b \times u$. Round 6: d, c, b, a, d, e; $d \times u$. Round 7: c, c, b, a, d, e; $c \times u$. Round 8: f, c, b, a, d, e; stable matching.

When women propose, the steps of the algorithm are as below. For each round, we list the proposals by a, b, c, d, e, f in order, followed by the resulting rejections. Round 1: z, y, v, w, u, u; $u \times e$. Round 2: z, y, v, w, v, u; $v \times e$. Round 3: z, y, v, w, x, u; stable matching.

Note that the pairs uf and vc occur in both results, and in all other cases the women are happier when the women propose and the men are happier when the men propose.

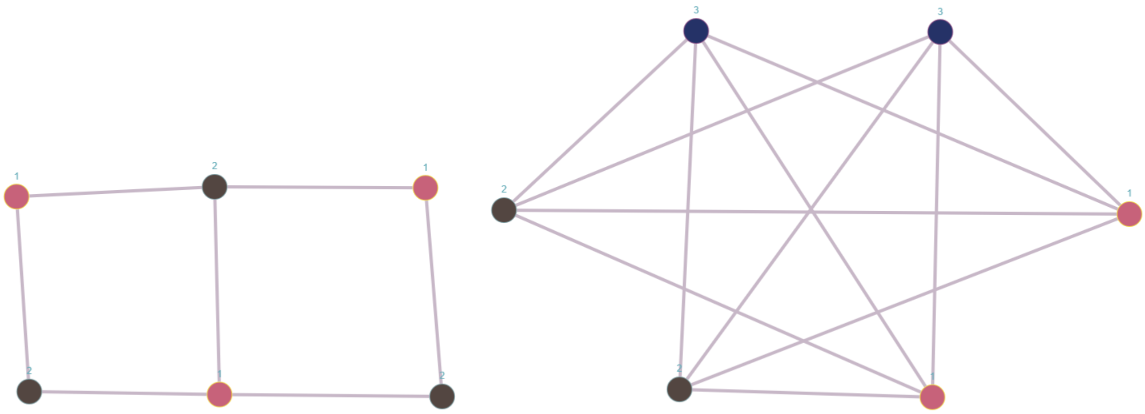
Problem 5

For each of the following graphs, find its chromatic number.



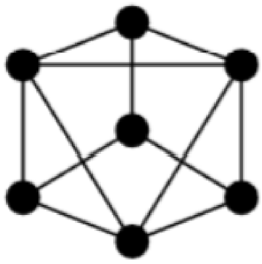
Solution:

The choromatic number of graph on left is 2, while choromatic number of graph on right is 3



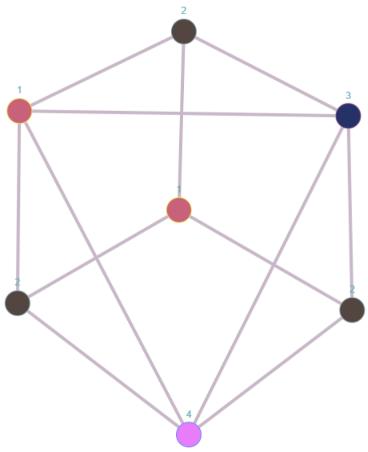
Problem 6

Let G be the graph



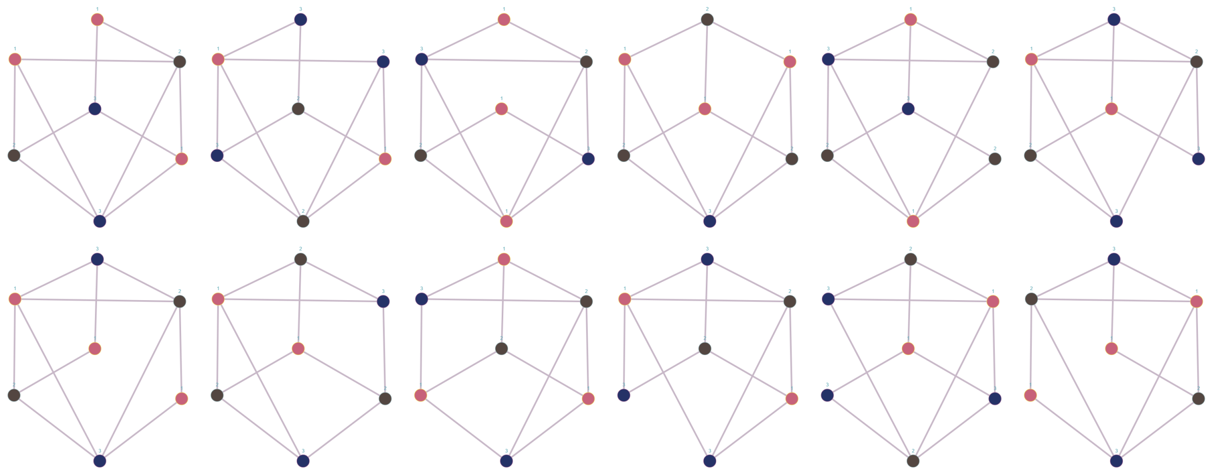
(a) Find $\chi(G)$.

Solution:
The choromatic number is 4.



(b) Vrify that $\chi(G - e) = \chi(G) - 1$ for each edge e in G.

Solution: In each subgraph, given condition is true.



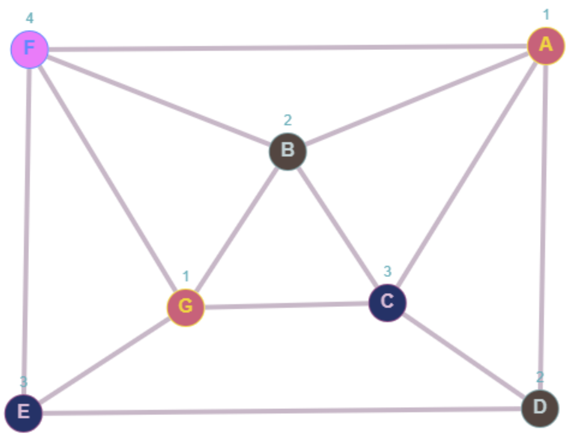
Problem 7

A school is preparing a timetable for exams in 7 different subjects, labeled A to G. It is understood that if there is a pupil taking two of these subjects, their exams must be held in different time slots. The table below shows (by crosses) the pairs of subjects which are taken by at least one pupil in common. The school wants to find the minimum number of time slots necessary and also to allocate subjects to the time slots accordingly. Interpreting this problem as a vertex-coloring problem, find the minimum number of time slots needed and a suitable time allocation of the subjects.

	A	B	C	D	E	F	G
A		x	x	x		x	
B	x		x			x	x
C	x	x		x			x
D	x		x		x		
E				x		x	x
F	x	x			x		x
G		x	x		x	x	

Solution:

The vertex coloring with mimimum colors are

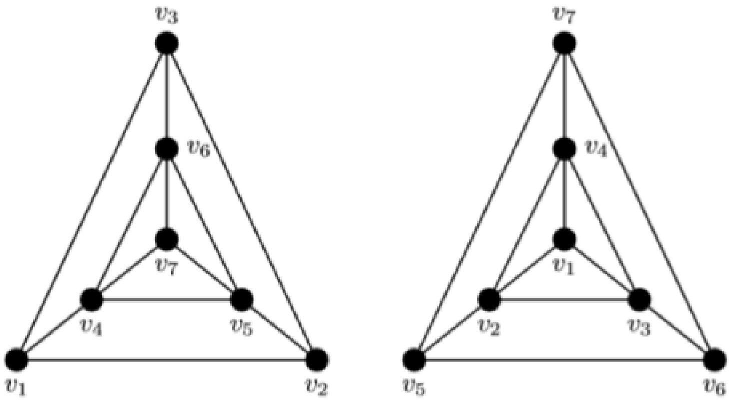


So we conclude that papers can be arrange in the according to the following table

Time slot	Paper
1	A,G
2	C,F
3	B,D
4	E

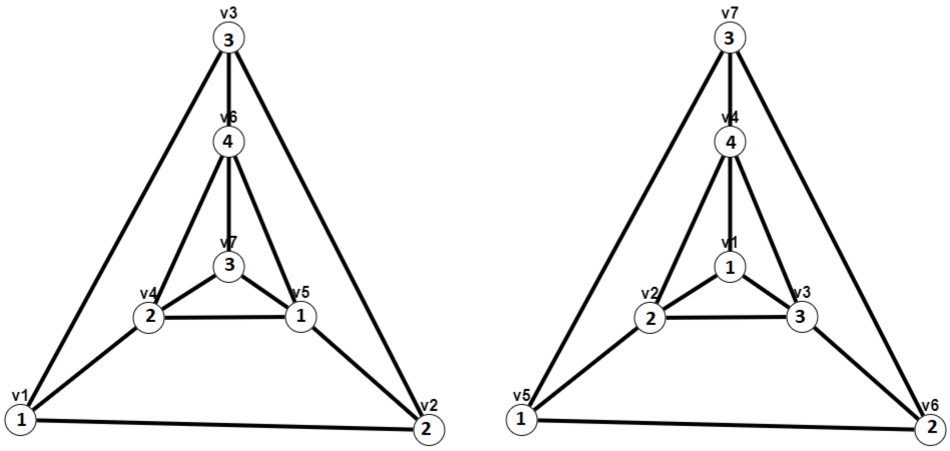
Problem 8

Let G be the graph shown in Figure below with two different ways of ordering its vertices. Apply First-Fit Algorithm (greedy coloring algorithm) to color G and find the number of colors produced in each case.


v_3
v_6
v_7
v_4
v_5
v_1
v_2
v_7
v_4
v_1
v_2
v_3
v_5
v_6

(a) (b)

Solution:


v3 3
v6 4
v7 3
v4 2
v5 1
v1 1
v2 2
v7 3
v4 4
v1 1
v2 2
v3 3
v5 1
v6 2