GRAPH COLORING

- In the previous chapter we discussed the application of graph matching to a problem where items from two distinct groups must be paired.
- An important aspect of this pairing is that no item could be paired more than once.

Basic problem:

Five student groups are meeting on Saturday, with varying time requirements. The staff at the Campus Center need to determine how to place the groups into rooms while using the fewest rooms possible.

- Although we can think of this problem as pairing groups with rooms, there is no restriction that a room can only be used once.
- In fact, to minimize the number of rooms used, we would hope to use a room as often as possible.

Key topics:

- This chapter explores graph coloring, a strategy often used to model resource restrictions.
- We will explore graph coloring in terms of both vertices and edges, though most of our time will be spent on coloring vertices.
- But before we get into the heart of coloring, we begin with a historically significant problem, known as the **Four-Color Theorem.**

FOUR COLOR THEOREM:

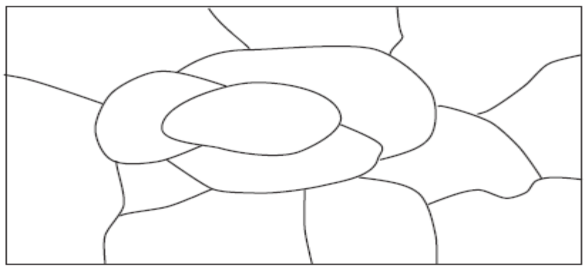
Initial Idea:

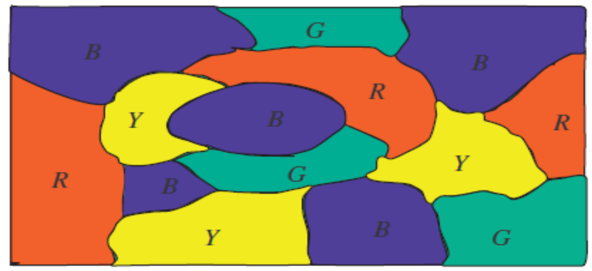
In 1852 Augustus De Morgan sent a letter to his colleague Sir **William Hamilton** (the same mathematician who introduced what we now call hamiltonian cycles) regarding a puzzle presented by one of his students, **Frederick Gutherie** (though Gutherie later clarified that the question originated from his brother, Francis).

- One wishes to color the regions in such a way that adjacent regions (i.e., regions sharing some common boundary) are colored by different colors.
- What is the minimum number of colors needed?

- This question was known for over a century as the Four Color Conjecture, and can be stated as

Any map split into contiguous regions can be colored using at most four colors so that no two bordering regions are given the same color.





- ✓ Can all maps be colored with at most four colors? Many people believed that the answer is in the affirmative, but no one could prove it for a longtime. This is known as the four-color problem.
- ✓ An important aspect of this conjecture is that a region, such as a country or state, cannot be split into two disconnected pieces.

The Four-Color Conjecture started as a map coloring problem yet migrated into a graph coloring problem.

k-coloring:

Definition 6.1 A proper ***k-coloring*** of a graph G is an assignment of colors to the vertices of G so that no two adjacent vertices are given the same color and exactly k colors are used.

Mathematically, a k-coloring of G can be regarded as a mapping

$$\theta : V(G) \to \{1, 2, ..., k\}$$

(not necessarily onto) such that $\theta(u) \neq \theta(v)$ if the vertices u and v are adjacent in G.

Consider the graph G of Fig. 8.2.3.

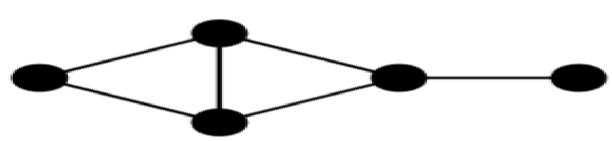
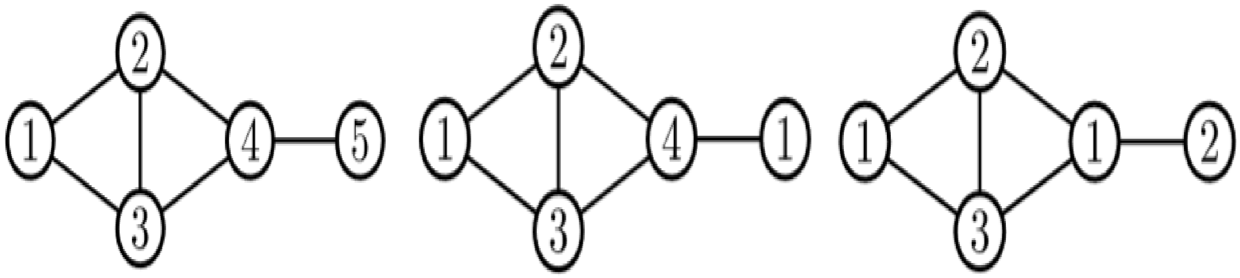


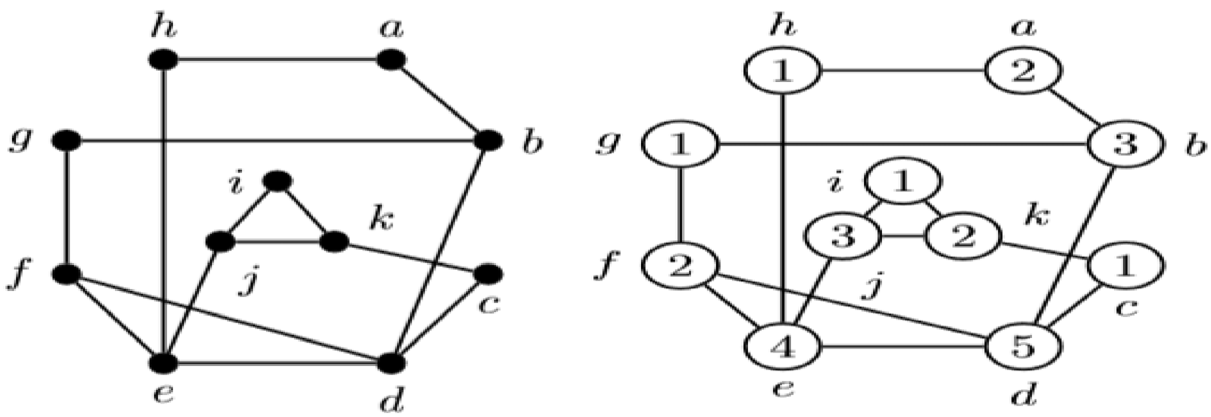
Fig. 8.2.3

A 5-coloring, a 4-coloring and a 3-coloring of G are, respectively, shown in (a), (b) and (c) of Fig. 8.2.4.



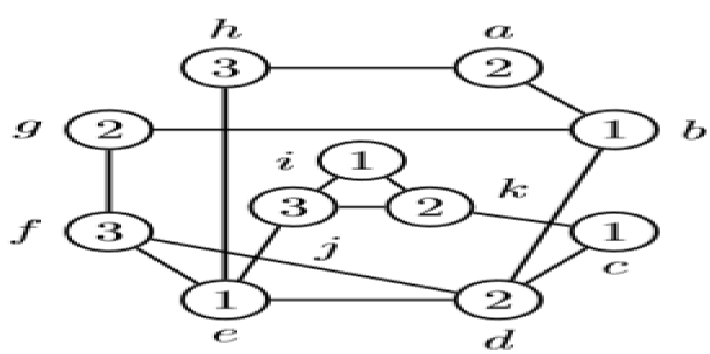
(a) (b) (c)

Remark 8.2.2. By definition, a p-coloring of G is also a q-coloring of G if $p \leq q$. Thus, it is absolutely correct to say that the three colorings in Fig. 8.2.4 are 5-colorings of G.



G_1 proper 5-coloring

- Above is a graph with two different proper colorings of the vertices.
- The two colorings given above use a different number of colors but are both proper since no two vertices of the same color are adjacent.



proper 3-coloring

Color Classes:

Definition 6.2 Given a proper k-coloring of G, the ***color classes*** are sets $S_1, \ldots S_k$ where S_i consists of all vertices of color i.

The first coloring of G_1 has color classes $S_1 = \{c, g, h, i\}$, $S_2 = \{a, f, k\}$, $S_3 = \{b, j\}, S_4 = \{e\}, S_5 = \{d\}$ and the second coloring has color classes $S_1 = \{b, c, e, i\}, S_2 = \{a, d, g, k\}, S_3 = \{f, h, j\}$. Recall that two vertices are independent if they have no edges between them. Thus a color class must consist of independent vertices. We will see there is a relationship between independent sets and coloring a graph.

Remark:

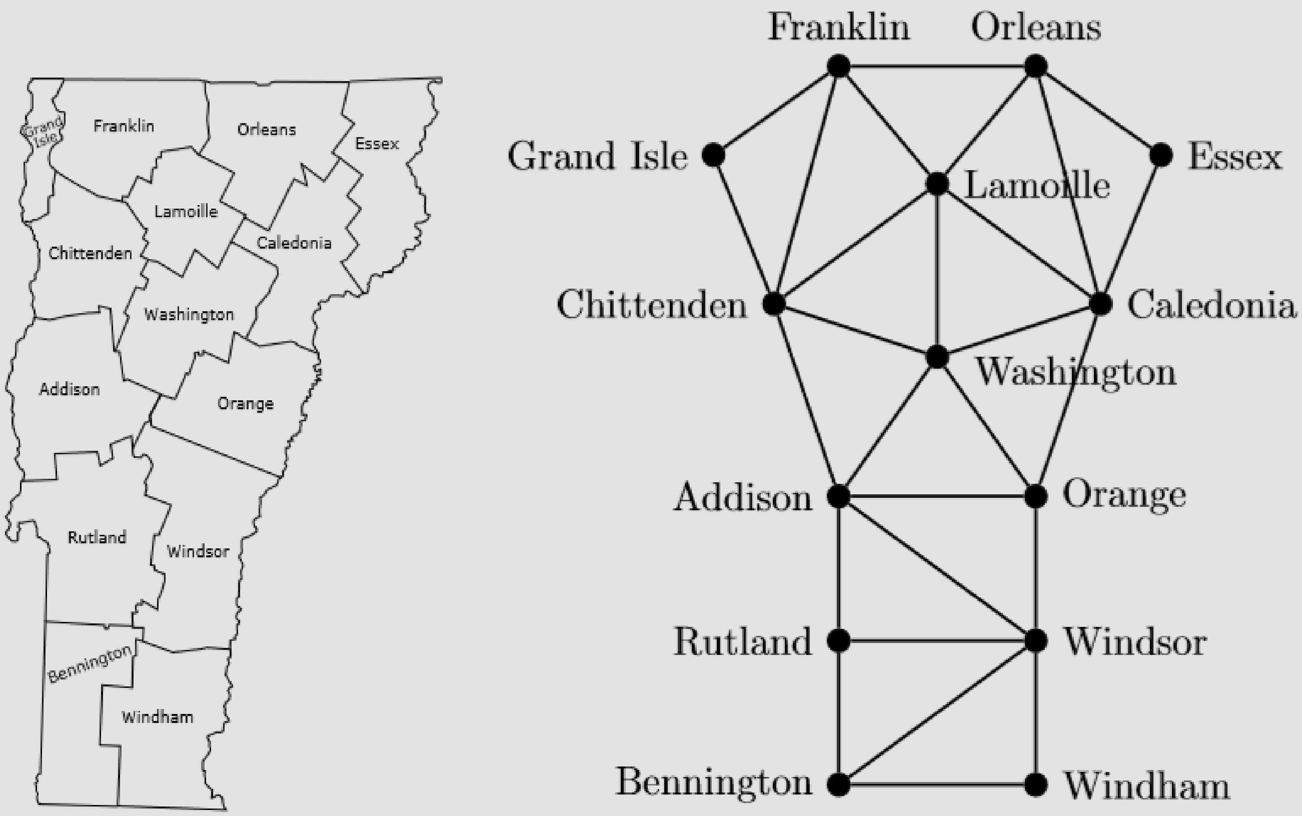
Note that each S_i is non-empty, every two different color classes are disjoint and $V(G) = S_1 \cup S_2 \cup \ldots \cup S_k$.

Independence Number:

Definition 6.3 The ***independence number*** of a graph G is $\alpha(G) = n$ if there exists a set of n vertices with no edges between them but every set of $n + 1$ vertices contains at least one edge.

- Most problems on graph coloring are optimization problems since we want to minimize the number of colors used; that is, find the lowest value of k so that G has a proper k-coloring.
- The example below demonstrates how a map coloring relates to a vertex coloring of a graph.

Example 6.1 Find a coloring of the map of the counties of Vermont and explain why three colors will not suffice.



Note that Lamoille County is surrounded by five other counties. If we try to alternate colors amongst these five counties, for example Orleans – 1, Franklin – 2, Chittenden – 1, Washington – 2, we still need a third color for the fifth county (Caledonia – 3). Since Lamoille touches each of these counties, we know it needs a fourth color.

VERTEX COLORING:

- Any graph we consider can be simple or have multi-edges but cannot have loops, since a vertex with a loop could never be assigned a color.
- In any graph coloring problem, we want to determine the smallest value for k for which a graph has a k-coloring. This value for k is called the **chromatic number** of a graph.

Definition 6.4 The *chromatic number* $\chi(G)$ of a graph is the smallest value k for which G has a proper k-coloring.

- The chromatic number χ(G) of G can well be defined as the minimum number k such that V(G) is partitioned into k pairwise disjoint independent sets in G.

In order to determine the chromatic number of a graph, we often need to complete the following two steps:

(1) Find a vertex coloring of G using k colors.

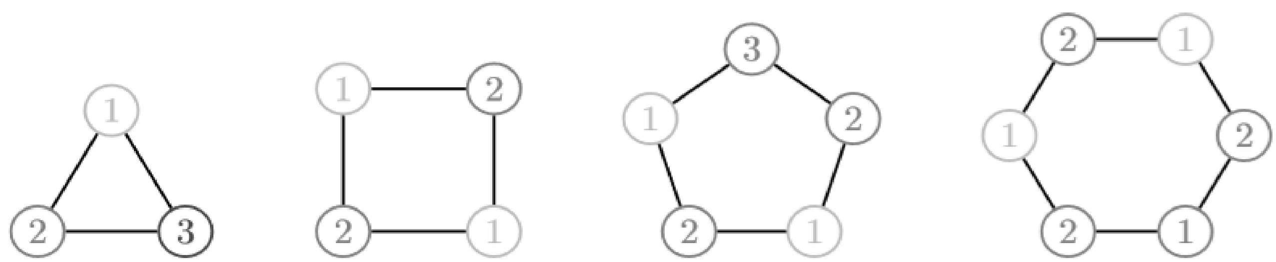
(2) Show why fewer colors will not suffice.

At times it can be quite complex to show a graph cannot be colored with fewer colors. There are a few properties of graphs and the existence of certain subgraphs that can immediately provide a basis for these arguments.

Explanation & Example:

1. **Cycles**

Look back at Example 6.1 about coloring the counties in Vermont and the discussion of alternating colors around a central vertex. In doing so, we were using one of the most basic properties in graph coloring: the number of colors needed to color a cycle. Recall that a cycle on n vertices is denoted C_n. The examples below show optimal colorings of C_3, C_4, C_5, and C_6.



C_3 C_4 C_5 C_6

Remarks:

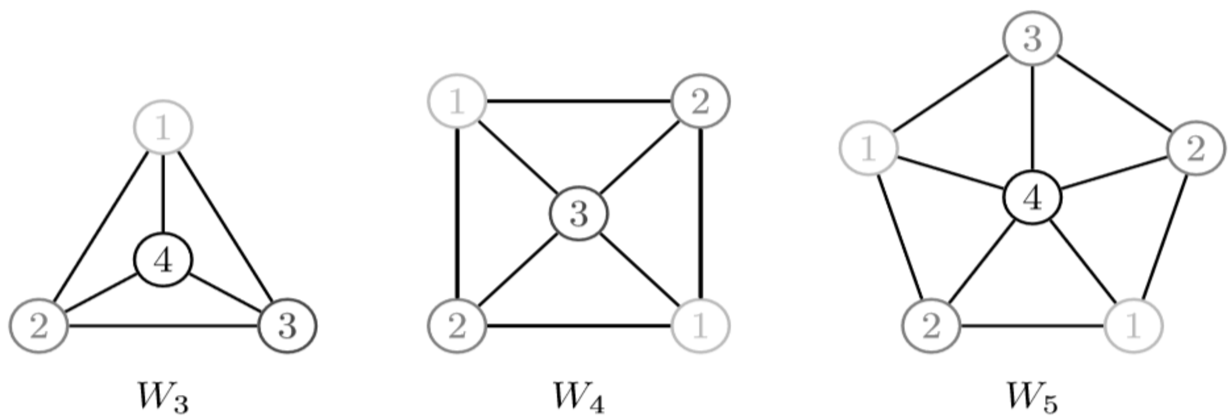
- Notice that in all the graphs we try to alternate colors around the cycle. When n is even, we can color C_n in two colors since this alternating pattern can be completed around the cycle.

- When n is odd, we need three colors for C_n since the final vertex visited when traveling around the cycle will be adjacent to a vertex of color 1 and of color 2.
- This was demonstrated in the coloring of the five counties surrounding Lamoille County in Example 6.1.

2. Wheels.

The next structure that provides additional reasoning for the lower bound of the chromatic number is based upon an odd cycle.

Definition 6.5 A *wheel* W_n is a graph in which n vertices form a cycle around a central vertex that is adjacent to each of the vertices in the cycle.

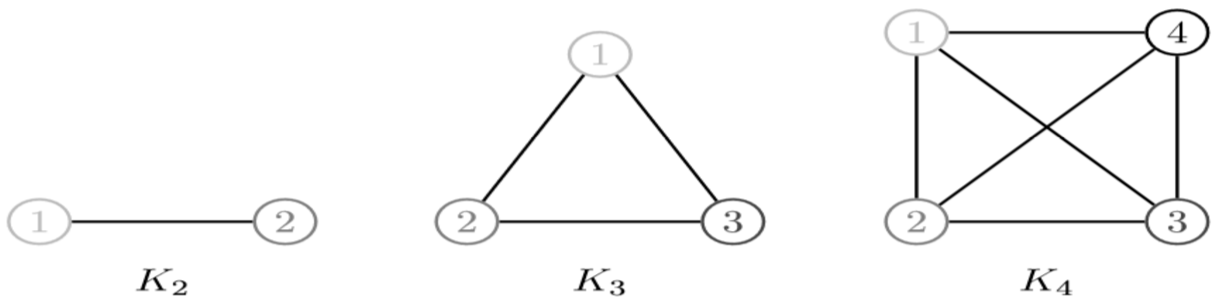


- In general, we can use odd wheels to explain why 3 colors will not suffice.

3. Complete Graph:

The final structure we search for within a graph is based on the notion of a complete graph.

- Recall that in a complete graph each vertex is adjacent to every other vertex in the graph.
- Thus, if we assign colors to the vertices, we cannot use a color more than once.
- When a complete graph appears as a subgraph within a larger graph, we call it a **clique**.
- Possible colorings of a few complete graphs are shown below.



Definition 6.6 A ***clique*** in a graph is a subgraph that is itself a complete graph. The ***clique size*** of a graph G, denoted $\omega(G)$, is the largest value of n for which G contains K_n as a subgraph.

- The clique size of a graph automatically provides a nice lower bound for the chromatic number.
- For example, if G contains K_5 as a subgraph, then we know this portion of the graph needs at least 5 colors. Thus $\chi(G) \geq 5$.

Remarks:

- Clique size of a graph provide a lower bound for chromatic number.
- Thus, when trying to argue that fewer colors will not suffice, we look for odd cycles (which require 3 colors), odd wheels (which require 4 colors), and cliques (which require as many colors as the number of vertices in the clique).

Below is a summary of our discussion so far regarding lower bounds for the chromatic number of a graph.

Special Classes of Graphs with known $\chi(G)$

- $\chi(C_n) = 2$ if n is even $(n \geq 2)$
- $\chi(C_n) = 3$ if n is odd $(n \geq 3)$
- $\chi(K_n) = n$
- $\chi(W_n) = 4$ if n is odd $(n \geq 3)$

Computation of chromatic Number:

Question:

Which graphs G are such that χ(G) = 1? Which graphs G have their χ(G) equal to its order n?

- Let G be a graph of order n. Then χ(G) = 1 if and only if G ≅ O_n (empty graph).
- Let G be a graph of order n ≥ 2. Then χ(G) = n if and only if G ≅ K_n.
- Let G be a graph which contains an odd cycle as a subgraph. Then χ(G) ≥ 3.
- Let G be a graph and let p be any positive integer such that G contains a K_p as a subgraph. Then χ(G) ≥ p if and only if G is bipartite.
- If H is a subgraph of a graph G, then χ(H) ≤ χ(G).
- Let G be a graph which contains an odd cycle as a subgraph. Then χ(G) ≥ 3.

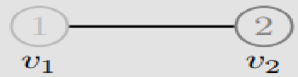
A graph can have a chromatic number that is much larger than its clique size.

In fact, **Jan Mycielski** showed that there exist graphs with an arbitrarily large chromatic number yet have a clique size of 2.

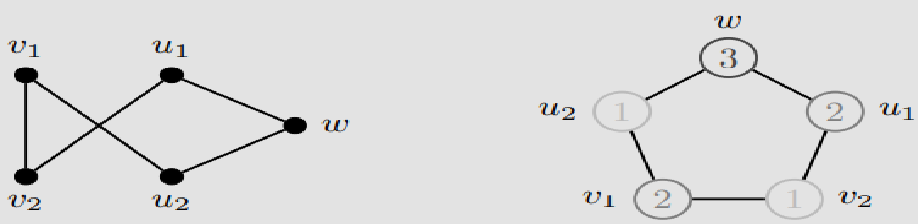
- We often refer to graphs with **ω(G) = 2** as triangle-free.
- Mycielski's proof provided a method for finding a triangle-free graph that requires the desired number of colors.

Example 6.2 Mycielski's Construction is a well-known procedure in graph theory that produces triangle-free graphs with increasing chromatic numbers. The idea is to begin with a triangle-free graph G where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and add new vertices $U = \{u_1, u_2, \ldots, u_n\}$ so that $N(u_i) = N(v_i)$ for every i; that is, add an edge from u_i to v_j whenever v_i is adjacent to v_j. In addition, we add a new vertex w so that $N(w) = U$; that is, add an edge from w to every vertex in U. The resulting graph will remain triangle-free but need one more color than G. If you perform enough iterations of this procedure, you can obtain a graph with $\omega(G) = 2$ and $\chi(G) = k$ for any desired value of k.

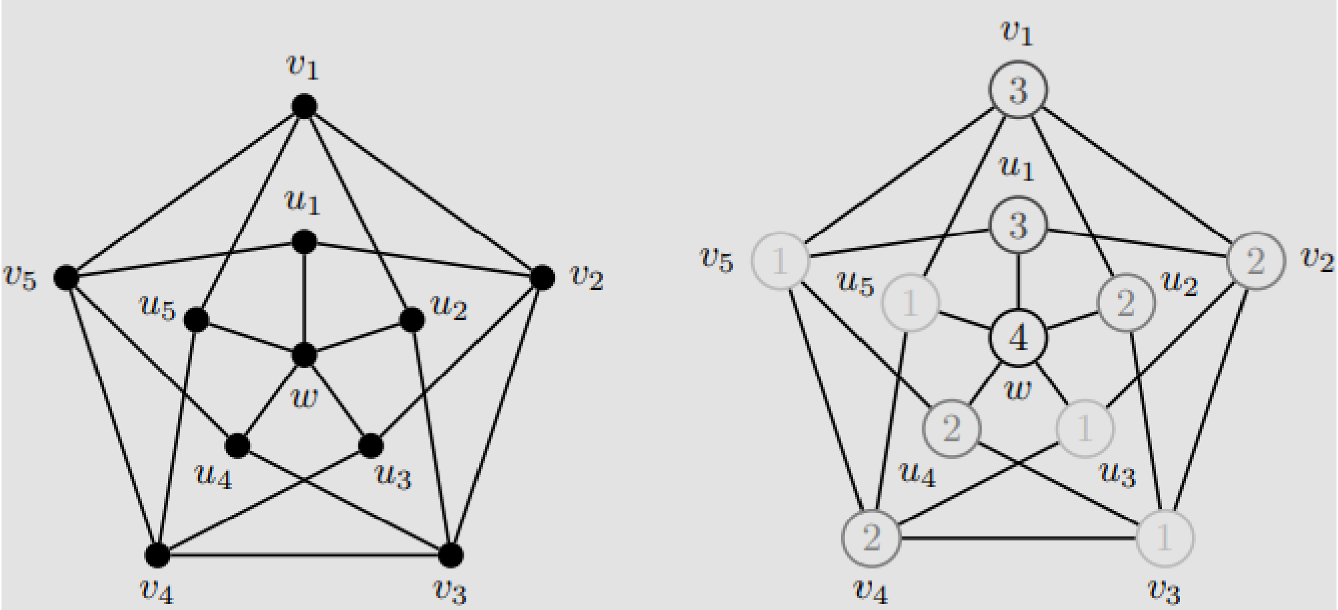
Consider G to be the complete graph on two vertices, K_2, which is clearly triangle free and has chromatic number 2, as shown in the following graph.



After the first iteration of Mycielski's Construction, we get the graph shown below on the left. Notice that u_1 has an edge to v_2 since v_1 is adjacent to v_2. Similarly, u_2 has an edge to v_1. In addition, w is adjacent to both u_1 and u_2. The graph on the right below is an unraveling of the graph on the left. Thus we have obtained C_5, which we know needs 3 colors.



After the second iteration, we obtain the graph shown below. The outer cycle on 5 vertices represents the graph obtained above in the first iteration. The inner vertices are the new additions to the graph, with u_1 adjacent to v_2 and v_5 since v_1 is adjacent to v_2 and v_5. Similar arguments hold for the remaining u-vertices and the center vertex w is adjacent to all of the u-vertices. A coloring of the graph is shown below on the right. Note that the outer cycle needs 3 colors, as does the group of u-vertices. This forces w to use a fourth color. In addition, no matter which three vertices you choose, you cannot find a triangle among them, and so the graph remains triangle-free.



If we continue this procedure through one more step, we obtain a graph needing 5 colors with a clique size of 2.

Remark:
Although Mycielski's Construction should warn you not to rely too heavily on the clique size of a graph, most real-world applications have a chromatic number close to their clique size.

Brooks' Theorem:

Theorem 6.7 (Brooks' Theorem) Let G be a connected graph and Δ denote the maximum degree among all vertices in G. Then $\chi(G) \leq \Delta$ as long as G is not a complete graph or an odd cycle. If G is a complete graph or an odd cycle then $\chi(G) = \Delta + 1$.

- The reasoning behind Brooks' Theorem is that if all the neighbors of x have been given different colors, then one additional color is needed for x.
- If x has the maximum degree over all vertices in G, then we have used Δ+ 1 colors for x and its neighbors.
- Perhaps more surprising is that unless a graph equals K_n or C_m (for an odd m), the neighbors of the vertex of maximum degree cannot all be given different colors and so the bound tightens to Δ.
- For example, the third graph from Mycielski's construction in Example 6.2 has a maximum degree of 5. Since this graph is neither a complete graph nor an odd cycle (although it does contain an odd cycle), we know the chromatic number is at most 5.

To summarize our discussion so far, we have:

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1$$

COLORING STRATEGIES:

- The bounds above provide starting points for determining the range in which to search for a proper k-coloring of a graph.

- The process for finding a minimum coloring is not trivial, though we will discuss some strategies for determining the chromatic number of a graph.

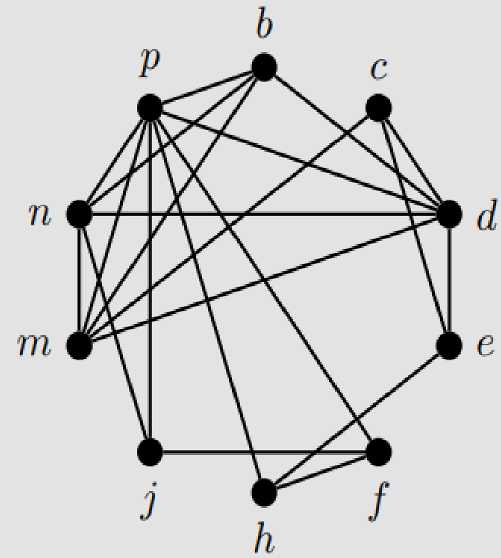
Intuitive Idea:

- In our discussion of Brooks' Theorem, we noted that if every neighbor of a vertex has a different color, then one additional color would be needed for that vertex.
- This implies that large degree vertices are more likely to increase the value for the chromatic number of a graph and thus should be assigned a color earlier rather than later in the process.
- In addition, it is better to look for locations in which colors are forced rather than chosen; that is, once an initial vertex is given color 1, look for cliques within the graph containing that vertex as these have very clear restrictions on assigning future colors.

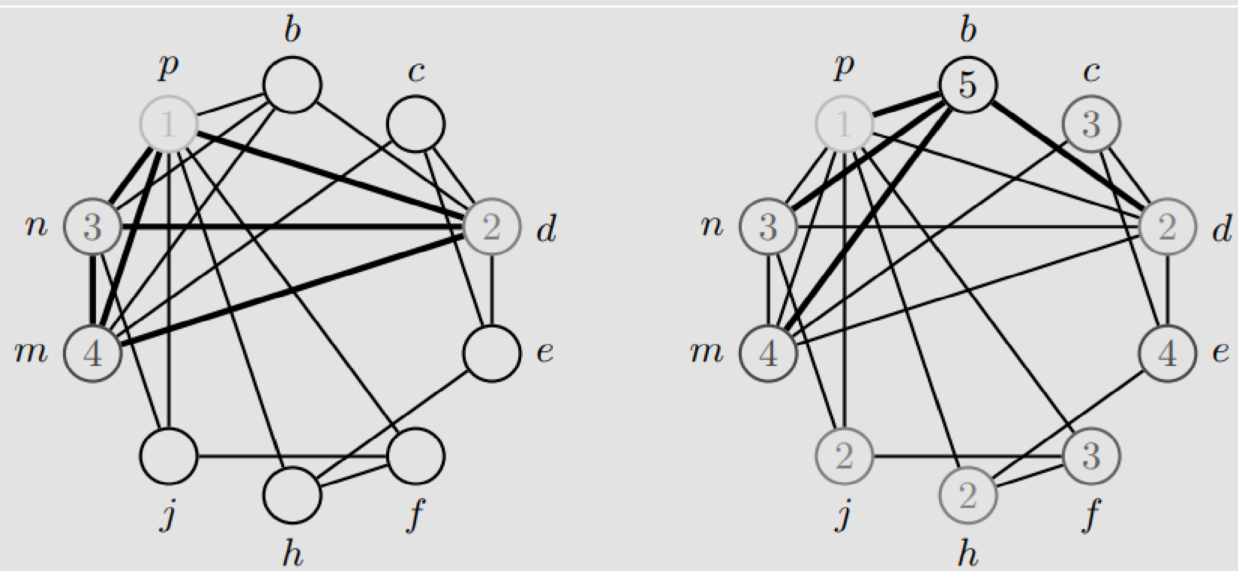
Example 6.3 Every year on Christmas Eve, the Petrie family compete in a friendly game of Trivial Pursuit. Unfortunately, due to longstanding disagreements and the outcome of previous years' games, some family members are not allowed on the same team. The table below lists the ten family members competing in this year's Trivial Pursuit game, where an entry of N in the table indicates people who are incompatible. Model the information as a graph and find the minimum number of teams needed to keep the peace this Christmas.

	Betty	Carl	Dan	Edith	Frank	Henry	Judy	Marie	Nell	Pete
Betty	·	·	N	·	·	·	·	N	N	N
Carl	·	·	N	N	·	·	·	N	·	·
Dan	N	N	·	N	·	·	·	N	N	N
Edith	·	N	N	·	·	·	·	N	·	·
Frank	·	·	·	·	·	N	N	·	·	N
Henry	·	·	·	·	N	·	·	·	·	N
Judy	·	·	·	·	·	N	·	·	N	N
Marie	N	N	N	N	·	·	·	·	N	N
Nell	N	·	N	·	·	·	N	N	·	N
Pete	N	·	N	·	N	N	N	N	N	·

Solution: Each person will be represented by a vertex in the graph and an edge indicates two people who are incompatible. Colors will be assigned to the vertices, where each color represents a Trivial Pursuit team.



At our initial step, we want to find a vertex of highest degree (p) and give it color 1. Once p has been assigned a color, we look at its neighbors with high degree as well, namely d (degree 6), m and n (both of degree 5). These four vertices are also all adjacent to each other (forming a K_4 shown in bold below on the left) and so must use three additional colors.



Finally, b has the next highest degree (4) and is also adjacent to all the previously colored vertices (forming a K_5) and so a fifth color is needed. The remaining vertices all have degree 3 and can be colored without introducing any new colors. One possible solution is shown above on the right. This solution translates into the following teams:

Team	Members		
1	Pete		
2	Dan	Henry	Judy
3	Carl	Frank	Nell
4	Edith	Marie	
5	Betty		

Remark:

The coloring obtained in Example 6.3 was not unique.

- ✓ There are many ways to find a proper coloring for the graph; however, every proper coloring would need at least five colors.

Question:

In terms of the graph model (forming teams) does the solution above seem fair?

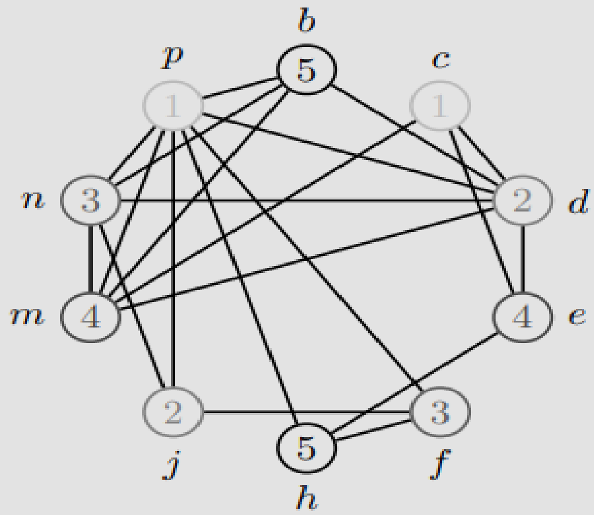
- ✓ Often, we are not only looking for the minimal k-coloring, but also one that adds in a notion of fairness.

Definition 6.8 An *equitable coloring* is a minimal proper coloring of G so that the number of vertices of each color differs by at most one.

- By this definition, the final coloring from Example 6.3 is not equitable.
- Note that not all graphs have equitable coloring using exactly $\chi(G)$ colors.

Example 6.4 Find an equitable coloring for the graph from Example 6.3.

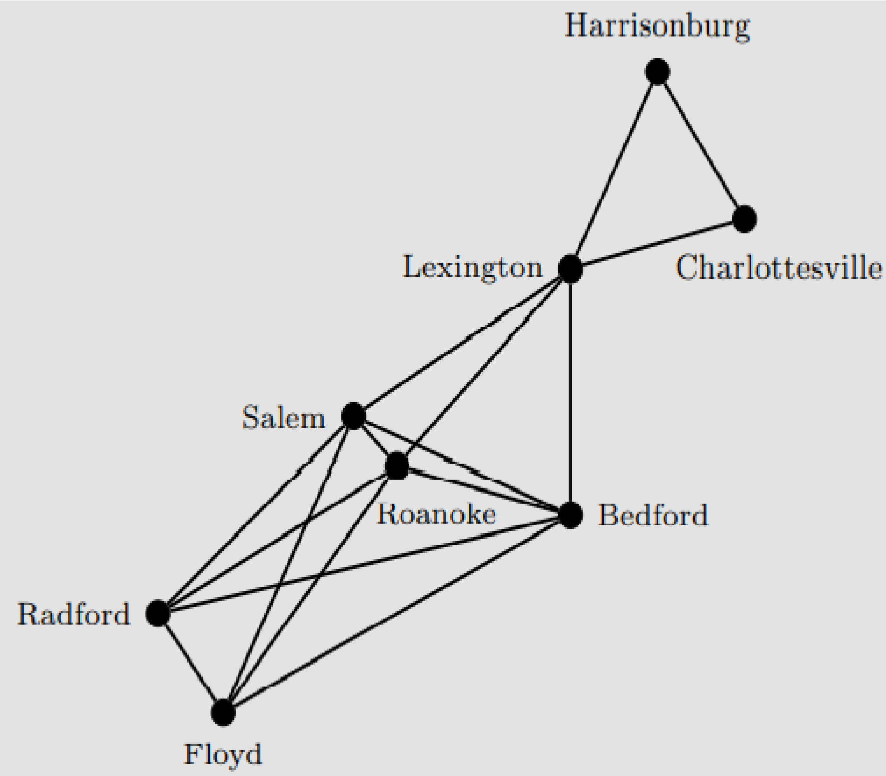
Solution: We begin with the 5-coloring obtained in Example 6.3. Note that colors 2 and 3 are each used three times, color 4 twice, and colors 1 and 5 each once. This implies we should try to move one vertex each from color 2 and color 3 and assign either color 1 or color 5. One possible solution is shown below.



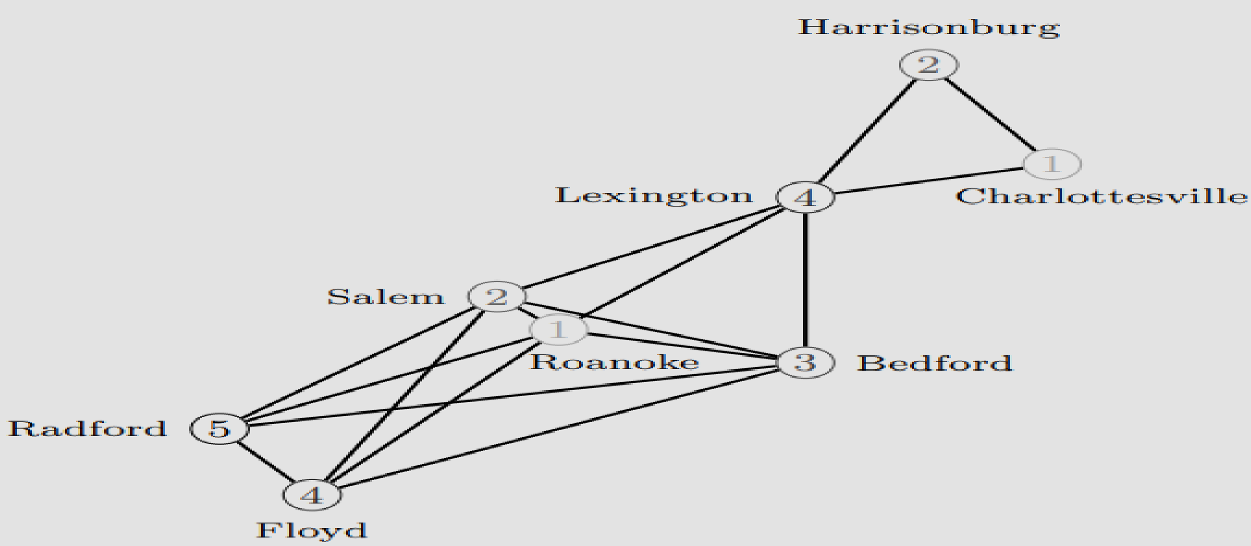
Basic Coloring Strategies

- Begin with vertices of high degree.
- Look for locations where colors are forced (cliques, wheels, odd cycles) rather than chosen.
- When these strategies have been exhausted, color the remaining vertices while trying to avoid using any additional colors.

Example 6.5 Due to the nature of radio signals, two stations can use the same frequency if they are at least 70 miles apart. An edge in the graph below indicates two cities that are at most 70 miles apart, necessitating different radio stations. Determine the fewest number of frequencies need for each city shown below (not drawn to scale) to have its own municipal radio station.



Solution: Each vertex will be assigned a color that corresponds to a radio frequency. This graph has $\chi = 5$ since we have a 5-coloring, as shown below, and fewer than 5 colors will not suffice as there is a K_5 among the vertices representing the cities of Roanoke, Salem, Bedford, Floyd, and Radford.



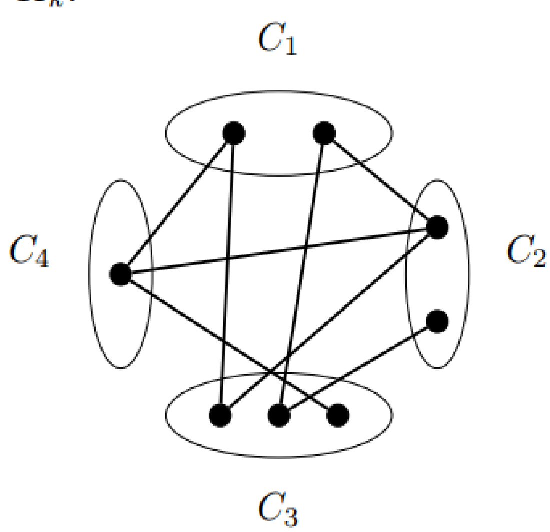
General Results:

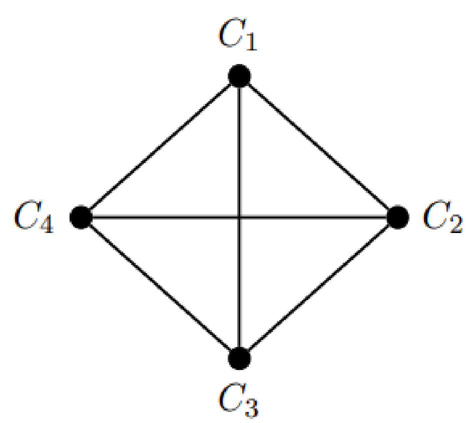
- First, we begin with a basic counting argument relating the number of edges of a graph with its chromatic number.
- In essence we can create an upper bound based on the number of edges in a graph rather than the maximum degree.

Proposition 6.9 Let G be a graph with m edges. Then

$$\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$$

Proof: Assume $\chi(G) = k$. First note that there must be at least one edge between color classes since otherwise two color classes without any edges between them could have been given the same color. Now, if we viewed each color class as a vertex and represented any edge between color classes as a singular edge in this new graph, we would obtain the complete graph K_k.





Thus G must have at least as many edges as the complete graph K_k, that is $m \geq \dfrac{k(k-1)}{2}$. We can rewrite this as $2m \geq k(k-1) = k^2 - k$. Completing the square gives us $2m + \frac{1}{4} \geq k^2 - k - \frac{1}{4} = (k - \frac{1}{2})^2$. Thus $k \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$.

Proposition 6.10 Let G be a graph and $l(G)$ be the length of the longest path in G. Then $\chi(G) \leq 1 + l(G)$.

Definition 6.11 Given a graph $G = (V, E)$, an ***induced subgraph*** is a subgraph $G[V']$ where $V' \subseteq V$ and every available edge from G between the vertices in V' is included.

The main reason we need induced subgraphs for coloring problems is that if we took any subgraph and colored it, we may be missing edges that would indicate two vertices need different colors in the larger graph.

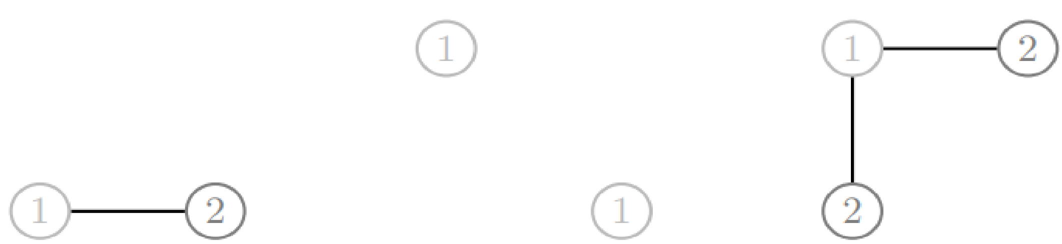
Proposition 6.12 Let G be a graph and $\delta(G)$ denote the minimum degree of G. Then $\chi(G) \leq 1 + \max_H \delta(H)$ for any induced subgraph H.

Perfect Graphs:

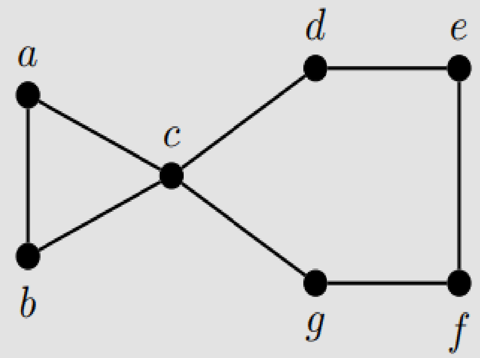
Definition 6.13 A graph G is ***perfect*** if and only if $\chi(H) = \omega(H)$ for all induced subgraphs H.

First note that $\omega(C_n) = 2$ for all values of $n \geq 4$. However, since G is an induced subgraph of itself, and whenever n is odd that $\chi(C_n) = 3$, we know that any odd cycle of length at least 5 cannot be perfect. For the even cycles, whenever we consider a proper induced subgraph, we will either have an edge or a set of independent vertices. In both cases, the induced subgraph will satisfy $\chi(H) = \omega(H)$.

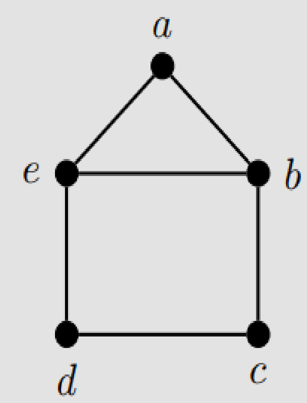
Some sample induced subgraphs are shown for C_4 below.



Example 6.6 Determine if either of the two graphs below are perfect.



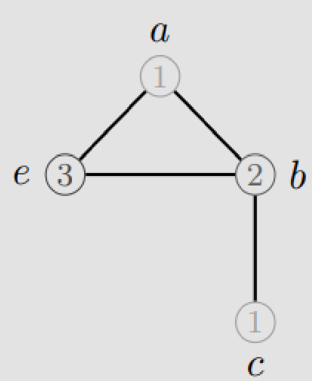
G_2



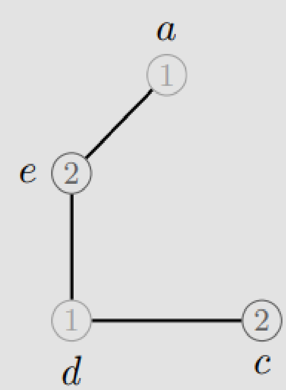
G_3

Solution: Without much work we can see that both graphs above satisfy $\chi(G) = \omega(G)$. However, if we look at the subgraph H induced by $\{c, d, e, f, g\}$ in G_2 we see that H is just C_5 and so $\chi(H) = 3$ even though $\omega(H) = 2$. Thus G_2 is not perfect.

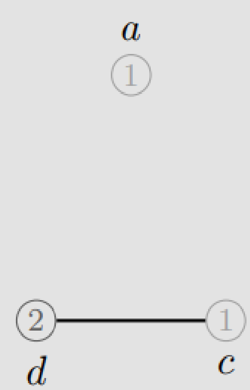
However, G_3 is in fact perfect. If an induced subgraph H contains $\{a, b, e\}$, then $\omega(H) = 3 = \chi(H)$; otherwise one of $\{a, b, e\}$ will not be in H and so $\omega(H) \leq 2$ and without much difficulty we can show $\omega(H) = \chi(H)$. A few illustrative induced subgraphs are shown below.



$G_3[a, b, c, e]$



$G_3[a, c, d, e]$



$G_3[a, c, d]$

Theorem 6.14 A graph G is perfect if and only if $\overline{G}$ is perfect.

As we have already seen, having an induced graph isomorphic to C_5 would disqualify a graph from being perfect.

Theorem 6.15 A graph G is perfect if and only if no induced subgraph of G or $\overline{G}$ is an odd cycle of length at least 5.

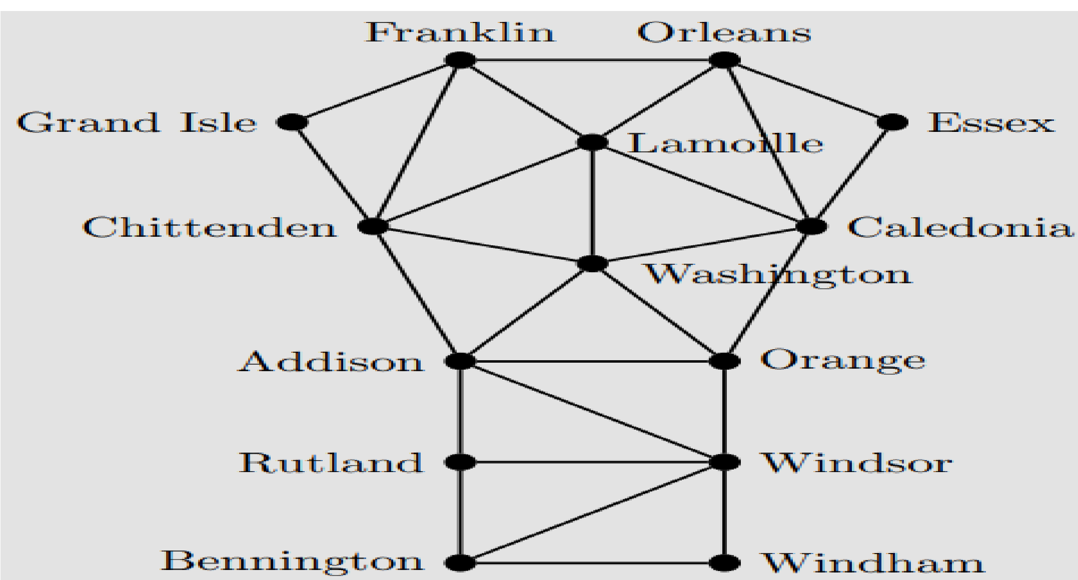
Perfect Graphs

The following classes of graphs are known to be perfect:

- Trees
- Bipartite graphs
- Chordal graphs
- Interval graphs

Definition 6.16 A graph G is *chordal* if any cycle of length four or larger has an edge (called a chord) between two nonconsecutive vertices of the cycle.

Example of Chordal Graph:

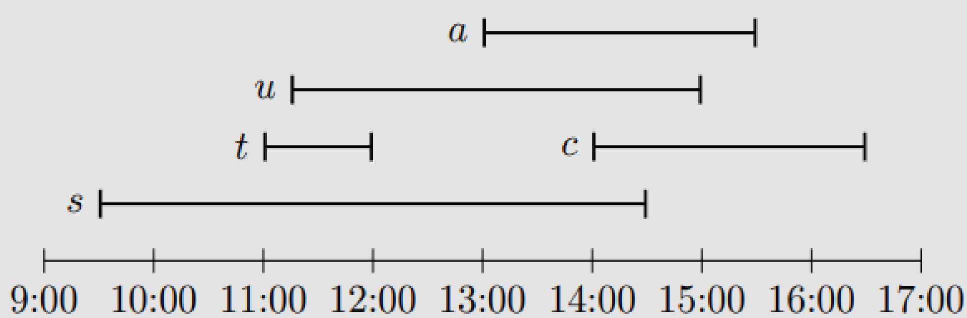


Definition 6.17 A graph G is an ***interval graph*** if every vertex can be represented as a finite interval and two vertices are adjacent whenever the corresponding intervals overlap; that is, for every vertex x there exists an interval I_x and xy is an edge in G if $I_x \cap I_y \neq \emptyset$.

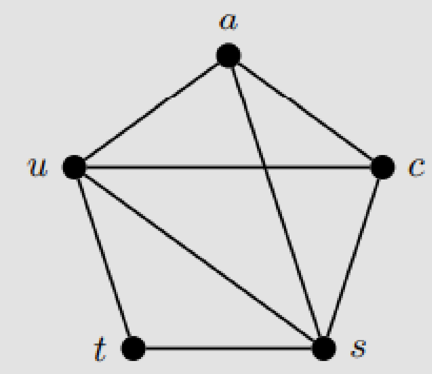
Example 6.7 Five student groups are meeting on Saturday, with varying time requirements. The staff at the Campus Center need to determine how to place the groups into rooms while using the fewest rooms possible. The times required for these groups is shown in the table below. Model this as a graph and determine the minimum number of rooms needed.

Student Group	Meeting Time
Agora	13:00–15:30
Counterpoint	14:00–16:30
Spectrum	9:30–14:30
Tupelos	11:00–12:00
Upstage	11:15–15:00

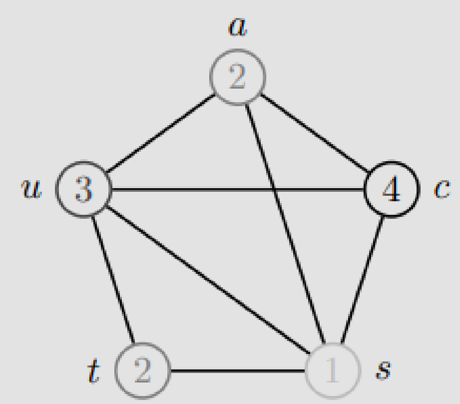
Solution: First we display the information in terms of the intervals. Although this step is not necessary, sometimes the visual aids in determining which vertices are adjacent.



Below is the graph where each vertex represents a student group and two vertices are adjacent if their corresponding intervals overlap.



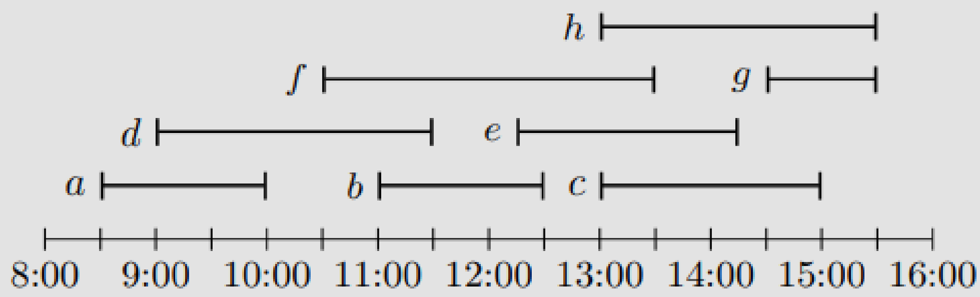
A proper coloring of this graph is shown below. Note that four colors are required since there is a K_4 subgraph with a, c, s, and u.



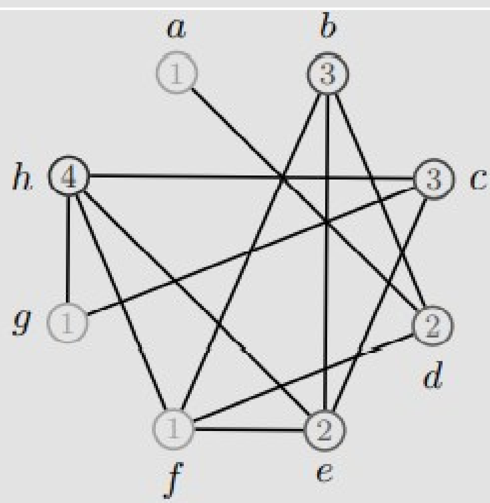
Remark: It should be noted that in most applications of interval graphs, you are given the intervals and must form the graph. A much harder problem is determining if an interval representation of a graph exists and then finding one.

- But once an interval representation is known, coloring interval graphs is quite easy by simply coloring the vertices based on when the correspon-ding interval is first seen as we sweep from left to right.

Example 6.8 Eight meetings must occur during a conference this upcoming weekend, as noted below. Determine the minimum number of rooms that must be reserved.



Solution: Each meeting is represented by a vertex, with an edge between meetings that overlap and colors indicating the room in which a meeting will occur. If we color the vertices according to their start time (so in the order a, d, f, b, e, c, h, g), we get the coloring below.



Note that four meeting rooms are needed since there is a point at which four meetings are all in session, which is demonstrated by the K_4 among the vertices c, e, f, and h.