MATCHING AND FACTORS

- In this chapter, we focus on the structure of the graph but from the viewpoint of grouping vertices based on a variety of criteria, mainly in terms of making viable pairings.
- We investigate the optimization of pairings through the use of edge-matchings within a graph, more commonly known as a matching.

Independent Vertices & Independent Set:

- Two vertices are said to be independent if they are not adjacent in G; and a set A of vertices in G is called an independent set if every two vertices in A are independent in G.

Vertex Cover & Edge Cover:

- A set *Q* of vertices in *G* is called a vertex cover of *G* if every edge in *G* is incident with a vertex in *Q*.
- A set *W* of edges in *G* is called an edge cover of *G* if every vertex in *G* is incident with an edge in *W*.

Independent Edges & Matching Set:

- Two edges are said to be independent if they are not adjacent, that is, they are not incident with a common vertex.
- A set M of edges in G is called a matching if every two edges in M are independent in G.

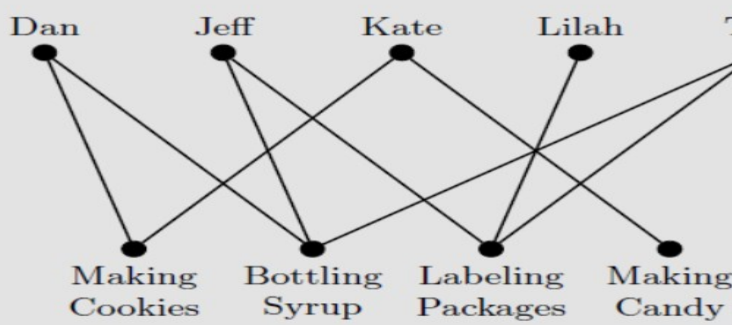
Definition 5.1 Given a graph $G = (V, E)$, a ***matching*** M is a subset of the edges of G so that no two edges share an endpoint. The size of a matching, denoted $|M|$, is the number of edges in the matching.

The most common application of matchings is the pairing of people, usually described in terms of marriages. Other applications of graph matching are task assignment, distinct representatives, and roommate selection.

Example 5.1 The Vermont Maple Factory just received a rush order for 6-dozen boxes of maple cookies, 3-dozen bags of maple candy, and 10-dozen bottles of maple syrup. Some employees have volunteered to stay late tonight to help finish the orders. In the chart below, each employee is shown along with the jobs for which he or she is qualified. Draw a graph to model this situation and find a matching.

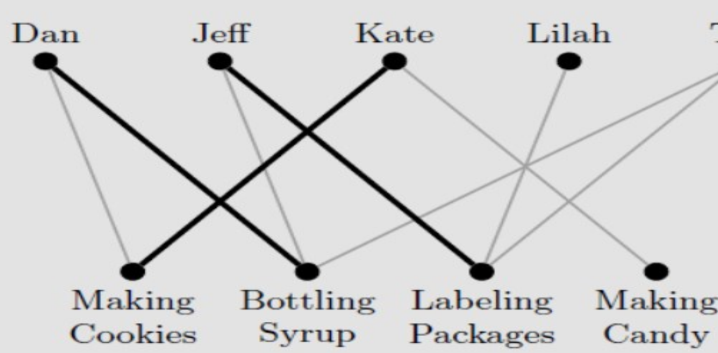
Employee	Task	
Dan	Making Cookies	Bottling Syrup
Jeff	Labeling Packages	Bottling Syrup
Kate	Making Candy	Making Cookies
Lilah	Labeling Packages	
Tori	Labeling Packages	Bottling Syrup

Solution: Model using a bipartite graph where X consists of the employees and Y consists of the tasks. We draw an edge between two vertices a and b if employee a is capable of completing the task b, creating G_1 below.



G_1

A matched edge, which is shown in bold below, represents the assignment of a task to an employee. One possible matching is shown below.



Remark:

- With any matching problem, you should ask yourself what the important criterion for a solution is and how does that translate to a matching.
- In Example 5.1, is it more important for each employee to have a task or for every task to be completed?

We need a way to describe which vertices are the endpoints of a matched edge.

Definition 5.2 A vertex is ***saturated*** by a matching M if it is incident to an edge of the matching; otherwise, it is called ***unsaturated***.

- The matching displayed in Example 5.1 has saturated vertices (Dan, Jeff, Kate, Making Cookies, Bottling Syrup, and Labeling Packages) representing the three tasks that will be completed by the three employees.
- The unsaturated vertices (Making Candy, Lilah, and Tori) represent the tasks that are not assigned or the employees without a task assignment.

Question: Is this a good matching?
Answer: No—some items needed for the order are not assigned and so the order will not be fulfilled.

- When searching for matching in a graph, we need to determine what type of matching properly describes the solution.

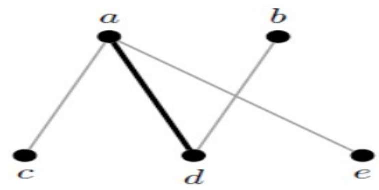
Types of Matching:

Definition 5.3 Given a matching M on a graph G, we say M is

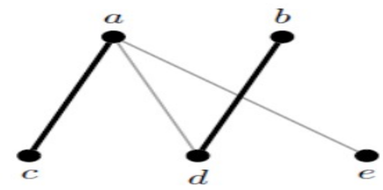
- ***maximal*** if M cannot be enlarged by adding an edge.
- ***maximum*** if M is of the largest size amongst all possible matchings.
- ***perfect*** if M saturates every vertex of G.
- an ***X-matching*** if it saturates every vertex from the collection of vertices X (a similar definition holds for a Y-matching).

- Note that a perfect matching is automatically maximum and a maximum matching is automatically maximal, though the reverse need not be true.

Example: Consider the two matchings shown in bold below of a graph G_2.



Maximal Matching of G_2



Maximum Matching of G_2

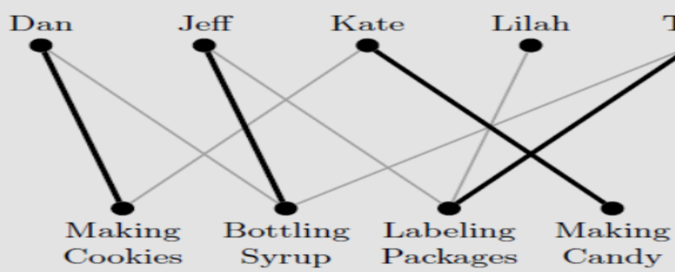
- The matching on the left is maximal as no other edges in the graph can be included in the matching since the remaining edges require the use of a saturated vertex (either a for edges ac and ae, or d for edge bd).
- The matching on the right is maximum since there is no way for a matching to contain three edges (since then vertex a would have two matched edges incident to it).
- In addition, the matching on the right is an X -matching if we define X = {a, b}. Finally, neither matching is perfect since not every vertex is saturated.

Depending on the application, we are often searching for a perfect matching or an X -matching.

- However, when neither of these can be found, we need a good explanation as to the size of a maximum matching.

Example 5.2 Determine and find the proper type of matching for the Vermont Maple Factory graph G_1 from Example 5.1.

Solution: Since we need the tasks to be completed but do not need every employee to be assigned a task, we must find an X-matching where X consists of the vertices representing the tasks. An example of such a matching is shown below.



Note that all tasks are assigned to an employee, but not all employees have a task (Lilah is not matched with a task). In addition, this is not the only matching possible. For example, we could have Lilah labeling packages and Tori bottling syrup with Jeff having no task to complete.

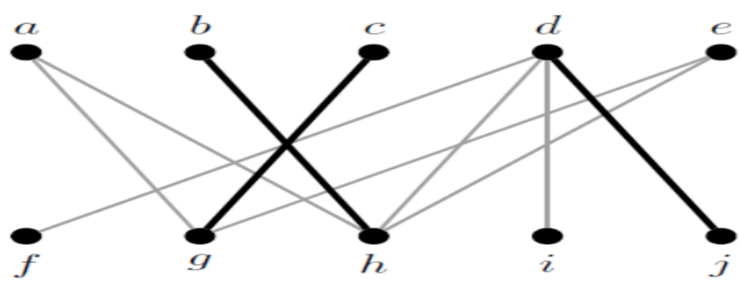
Remark: Above example illustrates that many matchings can have the same size.

- Thus, when finding a maximum matching, it is less important which people get paired with a task than it does that we make as many pairings possible.

MATCHING IN BIPARTITE GRAPHS:

- It is often quite clear how to form a matching in a small graph.
- However, as the size of the graph grows or the complexity increases, finding a maximum matching can become difficult.
- Moreover, once you believe a maximum matching has been found, how can you convince someone that a better matching does not exist?

Intuitive Idea: Consider the graph below with a matching shown in bold.



G_3

- If you tried to adjust which edges appear in the matching to find one of larger size (try it!), you would find it impossible.
- We can do this through the use of a neighbor set (A set of vertices S, the neighbor set N (S) consists of all the vertices incident to at least one vertex from S).
- In graph G_3 above, if we consider S = {f, i, j}, then N (S) = {d} and since at most one of the vertices from S can be paired with d, we know the maximum matching can contain at most 3 edges.
- Since we found a matching with 3 edges, we know our matching is in fact maximum.

The Assignment Problem:

There are *m* applicants and *n* jobs, and each applicant is applying for a number of these jobs. Under what conditions is it possible to assign each applicant to a job for which he/she is applying?

HALL'S MATCHING CONDITION:

When we are filling jobs with applicants, there may be many more applicants than jobs; successfully filling the jobs will not use all applicants.

- To model this problem, we consider an X, Y-bipartite graph, and we seek a matching that saturates X.
  - If a matching M saturates X, then for every $S \subset X$ there must be at least $|S|$ vertices that have neighbors in S, because the vertices matched to S must be chosen from that set.
  - We use *N(S)* to denote the set of vertices having a neighbor in S. Thus $N(S) \geq |S|$ is a necessary condition.

The condition, for all $S \subset X$, $N(S) \geq |S|$ is **Hall's Condition.**

- Hall proved that this obvious necessary condition is also sufficient.

Theorem 5.4 (Hall's Marriage Theorem) Given a bipartite graph $G = (X \cup Y, E)$, there exists an X-matching if and only if $|S| \leq |N(S)|$ for any $S \subseteq X$.

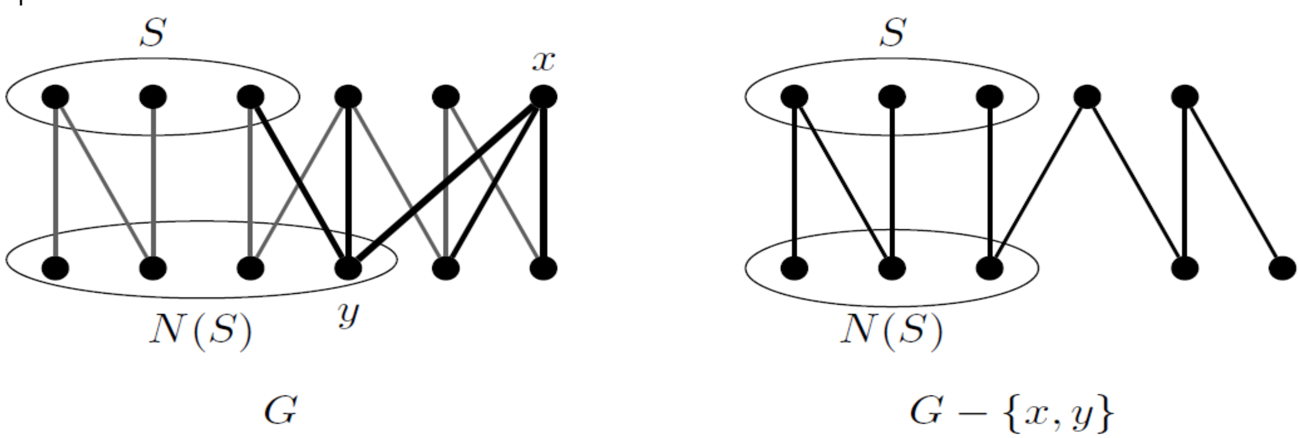
- This result is named for the British mathematician Philip Hall, who published the original proof in terms of a set theory result in 1935 [45], as well as the many iterations of the result that are often described in terms of pairing boys and girls into marriages.

Proof: First, suppose G has an X-matching, call it M. Then every vertex in X is saturated by an edge from M. Thus for any $S \subseteq X$ we know there is an M-edge from a vertex $v \in S$ to a vertex $y \in N(S)$ and so $|N(S)| \geq |S|$.

Conversely, call the property "$|S| \leq |N(S)|$ for any $S \subseteq X$" the marriage condition and suppose G satisfies the marriage condition. We will argue by induction on $|X|$ that G has an X-matching. If $|X| = 1$, then since G satisfies the marriage condition we know $|N(X)| \geq 1$ and so there must be an edge to some vertex in Y. Thus G has an X-matching.

Suppose $|X| \geq 2$ and that whenever the marriage condition is met for all X' of size smaller than X that there is a matching that saturates X'. We will consider two cases based on how much larger $N(S)$ is compared to the size of S.

First, suppose $|N(S)| \geq |S| + 1$ for all non-empty proper subsets S of X. Let $x \in X$. Then there exists some $y \in Y$ such that $e = xy$ is an edge of G. Let $G' = G - \{x, y\}$, that is the graph obtained by removing the vertices x and y. Consider any $S \subseteq X - \{x\}$.

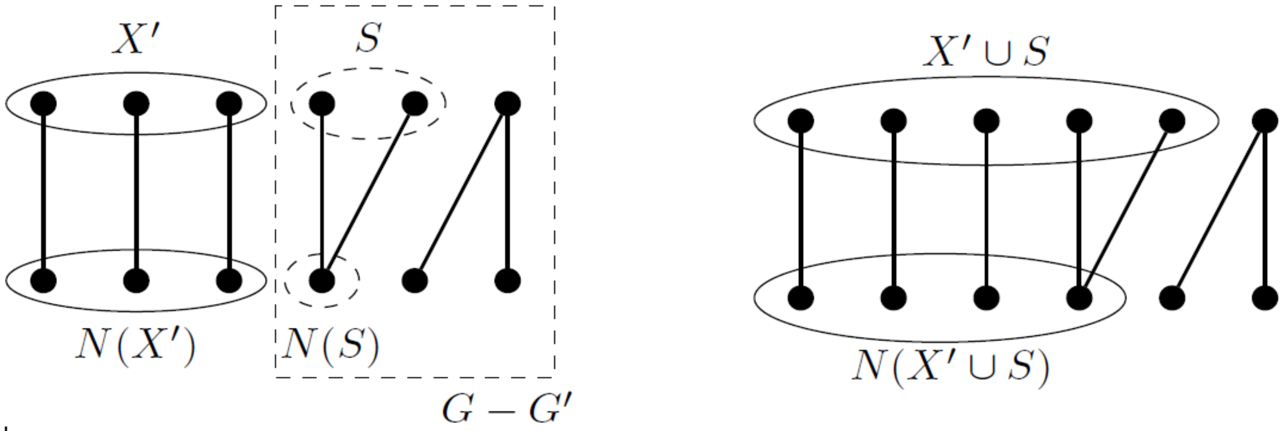


Then the neighbors of S in G' contains at most 1 fewer than the neighbors of S in G, since the only adjacent vertex possibly missing is y. This gives

$$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S|$$

and so G' satisfies the marriage condition. Thus by the induction hypothesis we know G' has a matching M' that saturates $X - \{x\}$. We can obtain an X-matching of G from M' by adding in the edge e.

Finally, suppose $|N(X')| = |X'|$ for some non-empty proper subset X' of X. Let G' be the graph consisting of the vertices in $X' \cup N(X')$ and all edges from X' to $|N(X')|$. Then G' satisfies the marriage condition and since $|X'| < |X|$, we know G' has an X'-matching by the induction hypothesis. It remains to show that $G - G'$ also satisfies the marriage condition.



Let S be any subset of $X - X'$. If $|S| > |N_{G-G'}(S)|$ then $S \cup X'$ would have size $|S| + |X'|$ and the neighbors of $S \cup X'$ in G could have size at most $|S| + |N(X')|$. Thus $|S \cup X'| > |N_G(S \cup X')|$, which contradicts that

G satisfies the marriage condition. Thus $G - G'$ also satisfies the marriage condition and so by the induction hypothesis has a matching that saturates $X - X'$. Together with the X'-matching, we have a matching that saturates all of X.

- Hall's Marriage Theorem allows us to determine when a graph has a perfect matching or X -matching for bipartite graphs.
- When the answer to this question is negative, we can still ask for the size of a maximum matching.
- Hall's Marriage Theorem does not give a definitive answer about the size of a maximum matching but rather gives us the tools to reason why an X -matching does not exist.
- The next section uses a specific type of path and a collection of vertices as a way to determine not only the size but also produce a maximum matching within a bipartite graph.

As an immediate consequence of Hall's Theorem, we have:

Corollary 7.3.4. *Let G be a bipartite graph with bipartition (X, Y) such that $|X| = |Y|$. Then G has a perfect matching if and only if $|S| \leq |N(S)|$ for every subset S of X.*

Question 7.3.1. *Let G be the bipartite graph with bipartition (X, Y) as shown in Fig. 7.3.1.*

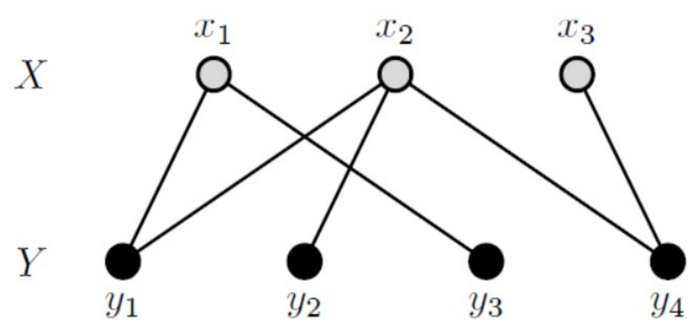


Fig. 7.3.1

(i) Complete the following table:

S	$N(S)$
$\{x_1\}$	$\{y_1, y_3\}$
$\{x_2\}$	
$\{x_3\}$	
$\{x_1, x_2\}$	
$\{x_1, x_3\}$	$\{y_1, y_3, y_4\}$
$\{x_2, x_3\}$	
X	

(ii) Does there exist a complete matching from X to Y?

Solution:

The bipartite graph G with bipartition (X, Y) has a complete matching from X to Y by Hall's Theorem since, as shown in the table therein, $|S| \leq |N(S)|$ for every subset S of X.

Question 7.3.2. *Consider the bipartite graph G with bipartition (X, Y) as shown in Fig. 7.3.2.*

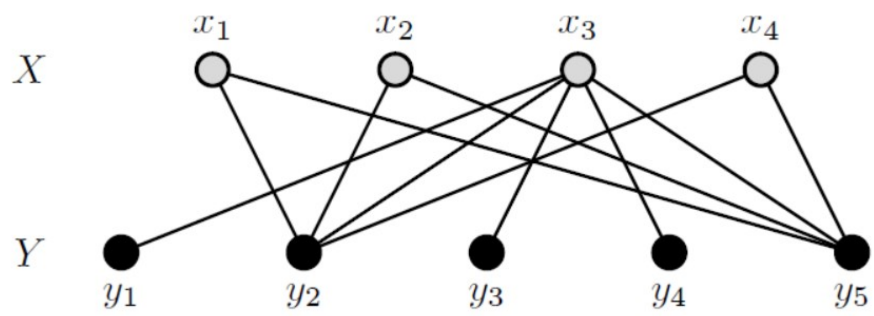


Fig. 7.3.2

(i) Let $S = \{x_1, x_2, x_4\}$. Find $N(S)$.

(ii) Is there a complete matching from X to Y in G? Why?

Solution:

The bipartite graph *G* with bipartition (*X, Y*) has no complete matching from *X* to *Y* by Hall's Theorem as there exists a subset *S* of *X* , namely, *S* = {*x*1 , *x*2 , *x*4 }, such that |*S*| > |*N* (*S*)| .

- An immediate result from Hall's Theorem that will be useful in Factors.

Corollary 5.5 Every k-regular bipartite graph has a perfect matching for all $k > 0$.