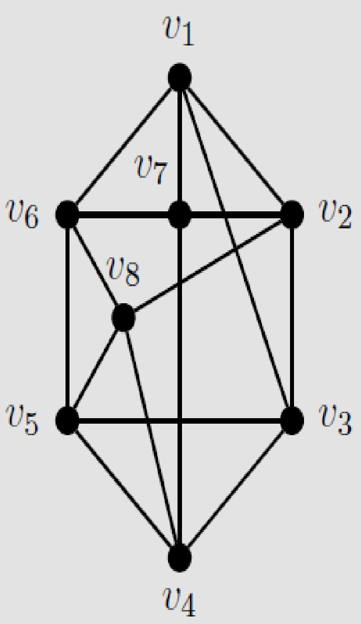
FACTORS

- In the search for a prefect matching in a graph G, we are in essence looking for a spanning subgraph H that consists of independent edges.
- But what if we do not want to restrict ourselves to just edges?
- To that end, we finish this chapter on the generalization of matchings, called factors.

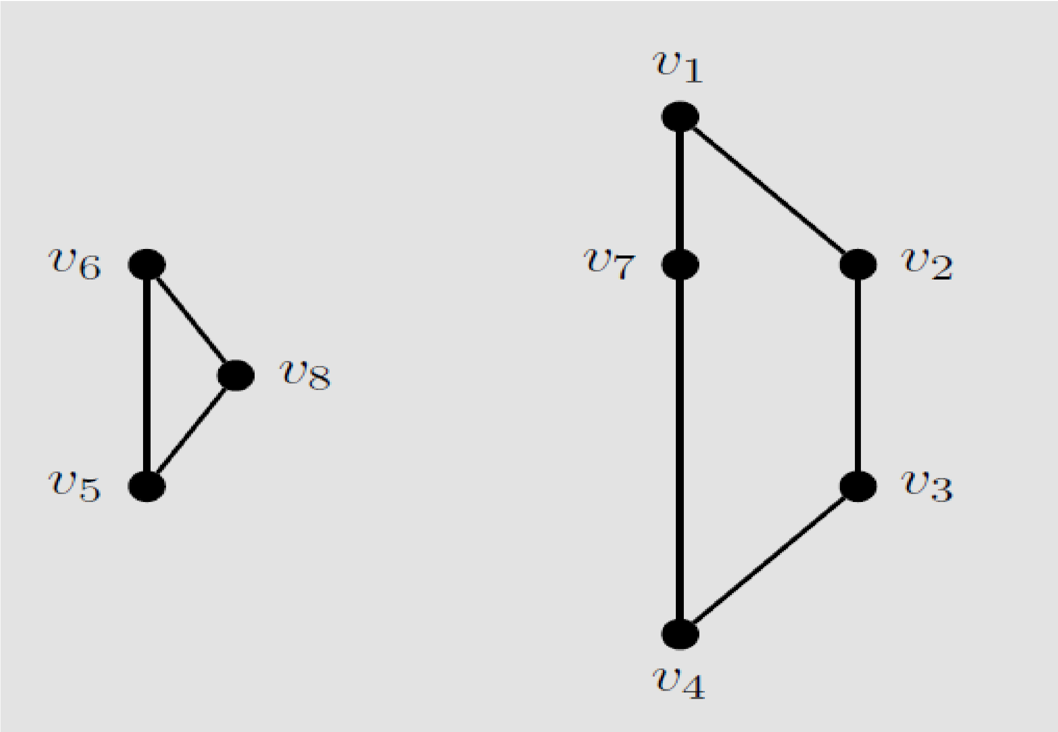
Definition 5.19 Let G be a graph with spanning subgraph H and let k be a positive integer. Then H is a ***k-factor*** of G if H is a k-regular.

- Note that a perfect matching is a 1-factor since the requirement that the edges in a matching are independent results in each vertex to have degree 1.
- So, what would a 2-factor entail? Each vertex must have degree 2, so the spanning subgraph H would consist of some collection of cycles.

Example 5.19 Find a 2-factor for the graph shown below.



Solution: More than one 2-factor exists for the graph above, as we simply need to find a spanning subgraph that is 2-regular. One such solution is given below. Note that this solution consists of two components, one of which is isomorphic to C_3 and the other to C_5.



Remarks:

- Recall that Tutte's Theorem determined exactly when a graph contains a 1-factor but what if we want a 2-factor?
- The theorem below is due to the Danish mathematician **Julius Petersen** and is one of his main contributions to graph theory.

Theorem 5.20 If G is a $2k$-regular graph, then G has a 2-factor.

Proof: First, we will assume G is connected as otherwise we could apply the argument below to each component of G. Next, since every vertex of G has even degree we know G must contain an eulerian circuit C by Theorem 2.7. We will create a bipartite graph G' based upon this eulerian circuit and use a matching on G' to produce our 2-factor of G.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and define the vertices of G' as $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots y_n$ so that $x_i y_j$ is an edge of G' when v_i immediately precedes v_j on the eulerian circuit C. Since G is $2k$-regular, we know C enters and exits each vertex of G exactly k times, and so G' is a k-regular bipartite graph. Thus by Corollary 5.5, G' contains a perfect matching M.

> Using the matching M as our guide, we can construct the 2-factor H of G. Note that the edge incident to x_i in M will represent an edge exiting v_i, whereas an edge incident to y_i will represent an edge entering v_i. Thus to find H, we start at v_1 and take the edge to v_i that arises from the matched edge x_1y_i. The next edge in the 2-factor will be from v_i to v_j arising from the matched edge x_iy_j. Thus will continue until all vertices are listed, creating a 2-regular spanning subgraph of G.

- There is different way to look at factors.
- We are not just concerned with obtaining a k-factor, but now want to find a collection of k-factors that encompass the entire graph.
- These are called factorizations.

Definition 5.21 A *k-factorization* of G is a partition of the edges into disjoint k-factors.

- As to be expected, having a k-factorization is more difficult than simply containing a k-factor.
- For example, as we have previously noted a 1-factor is equivalent to having a perfect matching in a graph.

Question: But what would a 1-factorization be?

Answer: It would mean that a graph can be partitioned into distinct perfect matchings $M_1, M_2, \ldots M_k$ so that each matching spans the graph, no two matchings have an edge in common, and all edges appear exactly once in one of these matchings.

- Finding a perfect matching was difficult enough!

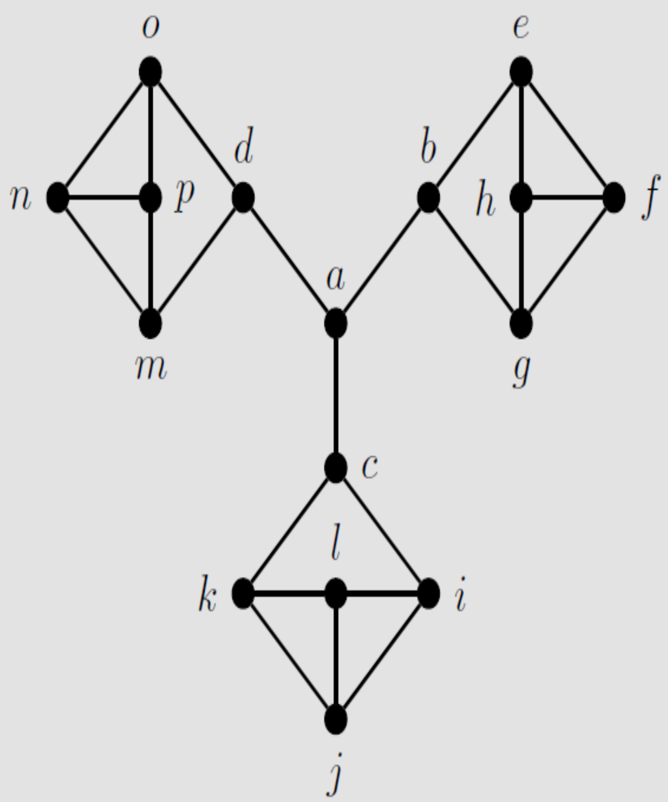
Explanation: However, it should be clear that some conditions can be placed on a graph to have a 1-factorization.

- In particular, the graph must have an even number of vertices and must be regular!
- Once a perfect matching has been found, we could remove all those edges and examine the graph that remains.
- If we are to find another **1-factor (perfect matching),** then no vertex can be isolated, and so as we continue this process until all edges are

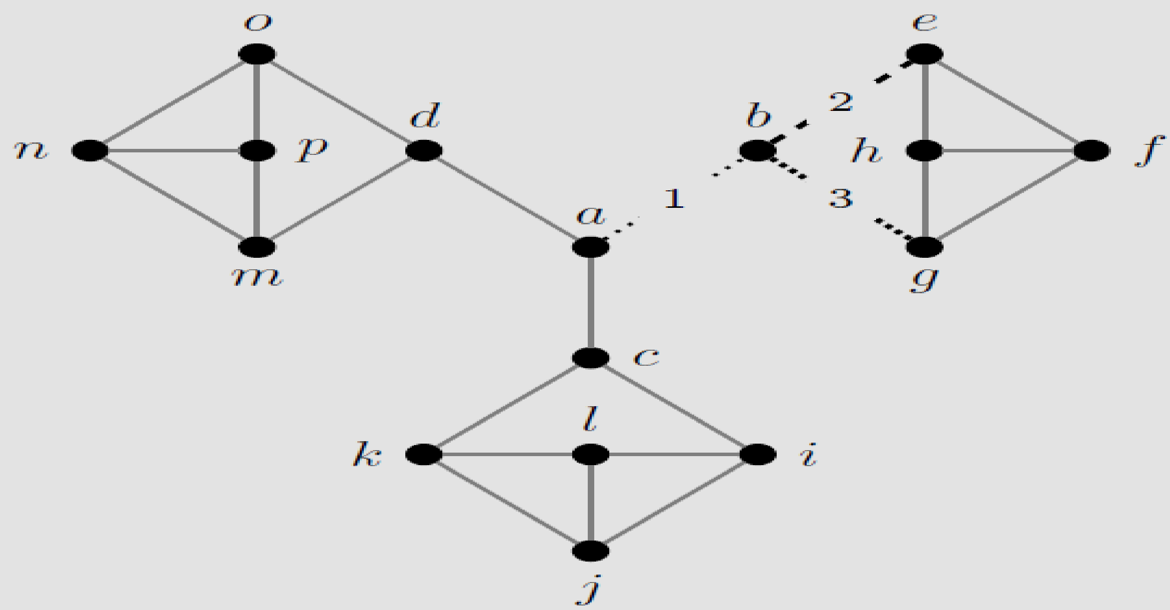
saturated by exactly one perfect matching, we must have all vertices to be of the same degree.

But is it enough for a graph to be regular to have a 1-factorization?

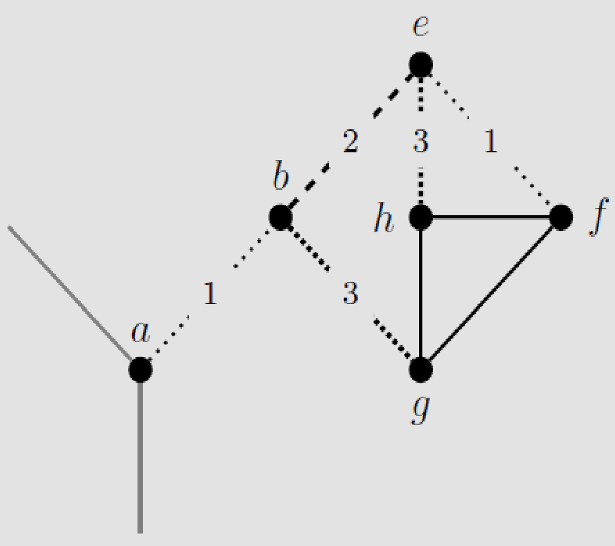
Example 5.20 Determine if the graph below has a 1-factorization.



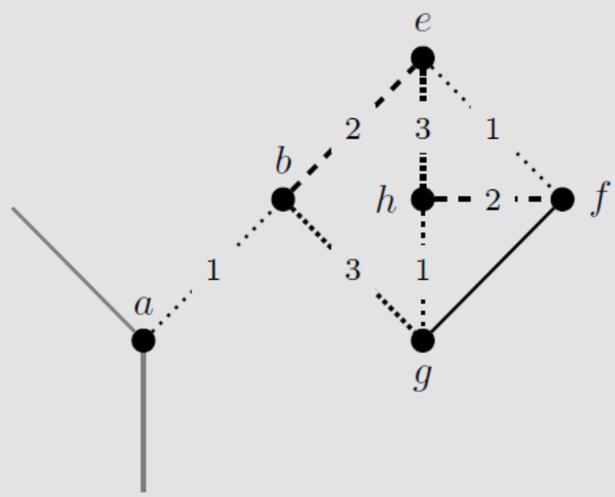
Solution: This graph is 3-regular, implying any 1-factorization would be made from three disjoint 1-factors (or perfect matchings). Suppose we have such a 1-factorization, say M_1, M_2, and M_3, which will be indicated by various line styles. Then each edge out of a would be from a different 1-factor. Without loss of generality, we will assume ab is from M_1. Thus the other two edges out of b would be from M_2 and M_3, and by symmetry we can assume be is from M_2 and bg is from M_3.



Since the other edges out of e cannot also be from M_2, and each vertex must be incident to an edge from each 1-factor, by symmetry we can assume ef is from M_1 and eh is from M_3.



This forces fh to be from M_2 and gh from M_1.



But then fg cannot be in any of M_1, M_2, or M_3 since either f or g is incident to a edge from each of these 1-factors. Thus the graph above does not have a 1-factorization.

Although a graph being regular is necessary, it is not a sufficient condition.

Determining if a graph has a 1-factorization is not an easy question to answer in the general case.

- However, there is one special case where we can get a definitive answer without a complicated proof.

Proposition 5.22 Every k-regular bipartite graph has a 1-factorization for all $k \geq 1$.

- we will show that K_{2n} also has a 1-factorization and discuss other results on graphs with 1-factorizations.
- Surprisingly, 2-factorizations are not much more difficult to prove than 2-factors.

Theorem 5.23 A graph G has a k-factorization if and only if G is $2k$-regular.

- We have already shown that a 2k-regular graph has a 2-factor, what remains to show is when we remove the 2-factor the resulting graph also contains a 2-factor, and that this process can continue until all edges in the graph have been included in a 2-factor.