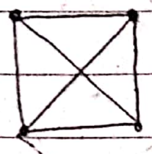
PLANARITY

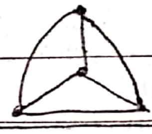
Date:

→ Planar:

vertices can be arranged so that edges donot cross at any point other than at a vertex

K_4: Planar Drawing:



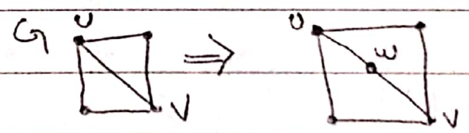


- $K_{3,3}$ and K_5 are not planar. — These two graphs provide a basis for determining the planarity of graph
  - Deleting an edge can make them planar.
- If a portion of graph is non-planar, then entire graph is not planar.

→ Subdivisions:

Subdivision of an edge xy consists of inserting vertices so that the edge xy is replaced by a path from x to y. The subdivision of G is obtained by subdividing edges in G.

* Any graph which contains a subdivision of K_5 or $K_{3,3}$ is non-planar.



⇒ subdivision of original graph.

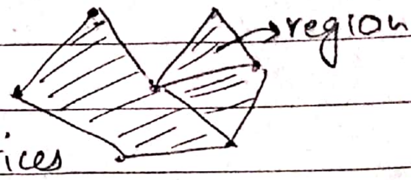
⇒ Kuratowski's Theorems

A graph is planar if and only if it doesnot contain a subdivision of $K_{3,3}$ or K_5.

★

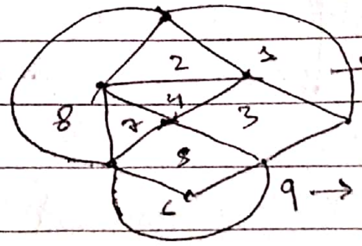
- In order for a graph to have a subdivision of $K_{3,3}$, the graph must have at least 6 vertices of degree 3 or greater
- In order for a graph to have a subdivision of K_5, the graph must have at least 5 vertices of degree 4 or greater.

→ Region : portion of a plane completely bounded by the edges of a graph.



- size of planar graphs: no. of edges
- order " " " : no. of vertices
- faces: no. of different regions
  ↳ If no cycles, then only 1 region (Trees)

9 faces →



⇒ Euler's Formula:

n = vertices, m = edges, r = regions

$$\boxed{n - m + r = 2}$$

→ Proof by Induction:

If $m = 1$, then G is connected, it is either a tree with one edge, hence $n = 2$ an $r = 1$
then $2 - 1 + 1 = 2$ ⇒ Euler's formula hold.

Now consider a graph with $m+1$ edges, n vertices and r regions. We will consider graph formed if we remove e from G.

First, if $G' = G - e$ is not connected for any edge e in G, then e must be a bridge. Then G must be a tree hence $n = m+1$ and $r = 1$ ⇒ $n - m + r$
⇒ $\cancel{m} + 1 - \cancel{m} + 1 = 2$ ⇒ Euler's formula hold.

Next, if $G' = G - e$ is connected, then e must be a part of some cycle in G. Then there must be two different regions on two sides of e, but these regions join into one with the removal of e. Thus G' has n vertices, r-1 regions and m-1 edges. Putting it together,

$n - m + r = n - (m - 1) + (r - 1)$
$= n - m + \cancel{1} + r - \cancel{1}$ ⇒ $n - m + r = 2$ ⇒ We assumed this to be true. Hence it's true for all connected planar graphs.

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⇒ Maximally Planar:

- A graph G is maximally planar if $G+e$ is non planar for any edge $e = xy$ for any two non-adjacent vertices $x, y \in V(G)$.
  - ↳ If every region is bounded by a triangle, it is maximally planar.

⇒ If G is maximally planar simple graph with $n \geq 3$ vertices and m edges, then $m = 3n - 6$.

<u>Proof</u>:

Assume G is maximally planar. Then ~~for~~ every region ~~b~~ must be bounded by a triangle.
Since every edge seperates two regions and every region is bounded by 3 edges we know $r = \frac{2m}{3}$. Thus by Euler's formula:

$$n - m + \frac{2m}{3} = 2 \Rightarrow 3n - 3m + 2m = 6 \Rightarrow 3n - m = 6$$

$$\Rightarrow m = 3n - 6.$$

★ Two important theorems:

⇒ If $G = (V, E)$ is a simple planar graph with m edges and $n \geq 3$ vertices, then $m \leq 3n - 6$

⇒ If $G = (V, E)$ is a simple planar graph with m edges and $n \geq 3$ and no cycles of length 3, then $m \leq 2n - 4$.

★ If a graph doesnot satisfy these conditions, it must be non-planar.

⇒ Cycle-Chord Method:

- Finding a planar drawing
  - Draw a cycle with all ~~as many~~ vertices. ~~as possible.~~
  - Try to add as many edges in the interior of cycle as possible.
  - Put the remaining vertices outside as curves.

⇒ Edge - Crossing And Thickness:

⇒ Edge - Crossing:

→ Crossing Number:

- denoted by $cr(G)$
- minimum number of edges crossings in any drawing of G satisfying the conditions below:
  - i) No edge crosses other more than once.
  - ii) At most two edges cross at a given point.

→ $cr(G) = 0$ if and only if graph is planar.

⇒ Graph G with n vertices and m edges:

$$cr(G) \geq m - 3n + 6$$

If G is bipartite:

$$cr(G) \geq m - 2n + 4$$

→ Upper bounds for Complete Graphs:

$$cr(K_n) \leq \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

→ Upper bound for k-partite graphs:

$$cr(K_{m,n}) \leq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

⇒ Thickness

include all vertices and some edges

- decompose the graph into spanning subgraphs, each of which are planar, using the smallest no. of subgraphs possible.
- Minimum value is called thickness of graph.
- denoted by $\theta(G)$

- $\theta(G) = 1$ if and only if G is planar, since T would contain only G itself.

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⇒ Let G be a connected simple graph with n vertices and m edges. Then

$$\theta(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil$$

⇒ Let G be a connected simple bipartite graph, with n vertices and m edges, Then:

$$\theta(G) \geq \left\lceil \frac{m}{2n-4} \right\rceil$$

⇒ For complete graphs:

$$\theta(K_n) = \begin{cases} \left\lfloor \frac{n+7}{6} \right\rfloor & n \neq 9, 10 \\ 3 & n = 9, 10 \end{cases}$$

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