EDGE CROSSING & THICKNESS

- Once a graph is known to be nonplanar, one could ask how close the graph is to being planar.
- You could quantify this in many different ways, such as how many vertices or edges would need to be removed to create a planar subgraph.
- Here we consider instead how many times the edges of a graph cross, called the crossing number.

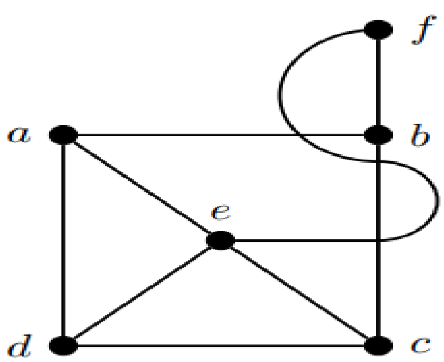
Definition 7.18 For any simple graph G the ***crossing number*** of G, denoted $cr(G)$, is the minimum number of edge crossings in any drawing of G satisfying the conditions below:

(i) no edge crosses another more than once, and

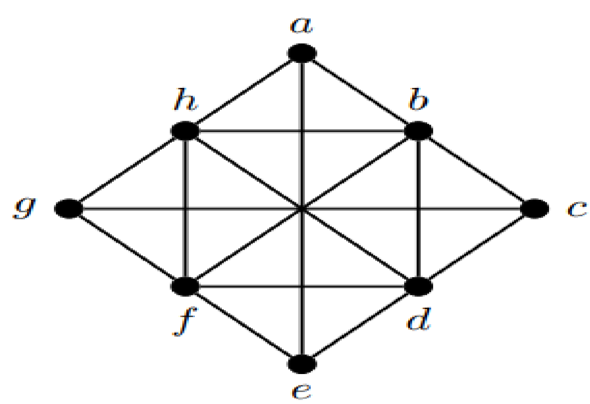
(ii) at most two edges cross at a given point.

Explanation:

- Note that we need to be a bit precise when defining the crossing number, since we could theoretically draw edges as complex curves requiring a large number of crossings.
- For example the graph G_7 on the left below violates (i) since the edge ef crosses bc more than once.
- whereas the graph G_8 on the right below violates (ii) since edges ae, bf, cg, and dh all cross at a point (not a vertex) in the center of the graph.



G_7



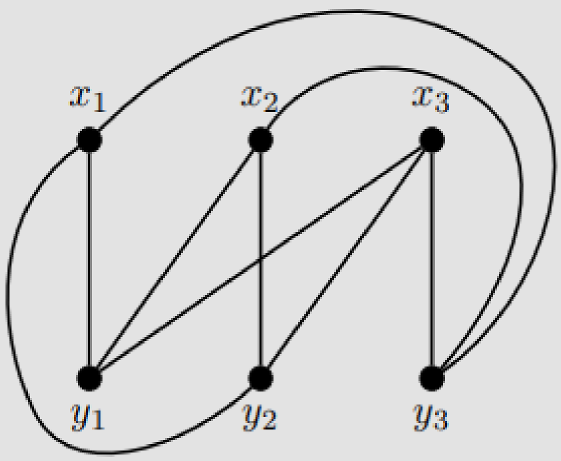
G_8

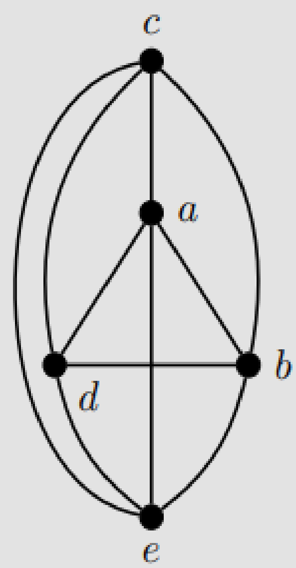
Remark: It should be clear that cr(G) = 0 if and only if G is planar.

Question: But what about K_5 and $K_{3,3}$, the two graphs that are instrumental in determining planarity?

Example 7.4 Determine the crossing numbers for K_5 and $K_{3,3}$.

Solution: Since we know K_5 and $K_{3,3}$ are not planar, $cr(K_5), cr(K_{3,3}) \geq 1$. A drawing of each of these adhering to the criteria above is shown below, proving they each have crossing number 1.





Result for Crossing Number of a Graph:

Theorem 7.19 Let G be a simple graph with m edges and n vertices. Then $cr(G) \geq m - 3n + 6$. Moreover, if G is bipartite then $cr(G) \geq m - 2n + 4$.

- This result cannot determine the crossing number of a given graph, but it does provide a starting point.
- Similar to determining the chromatic number of a graph, stating cr(G) = k often requires some explanation that cr(G) ≥ k and then exhibiting a drawing of G with exactly k edge-crossings.

Example:

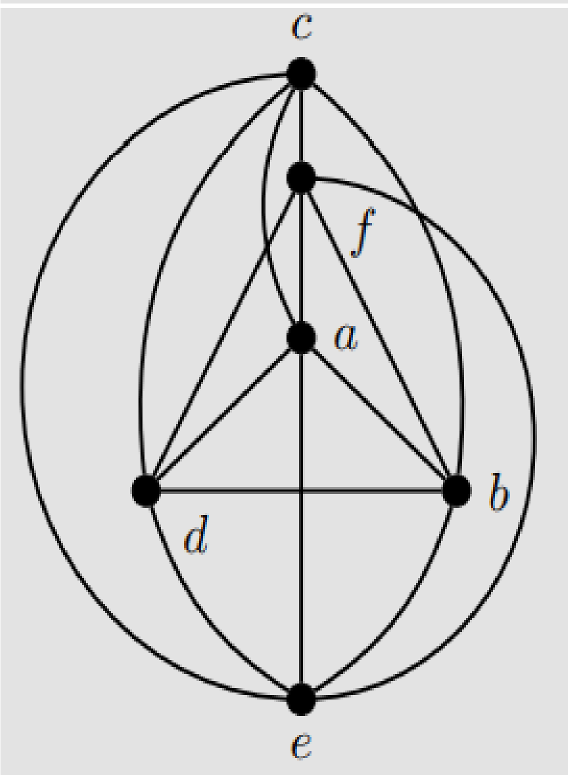
K5 has 10 edges and 5 vertices and so $cr(K_5) \geq 10 - 3 * 5 + 6 = 1$.

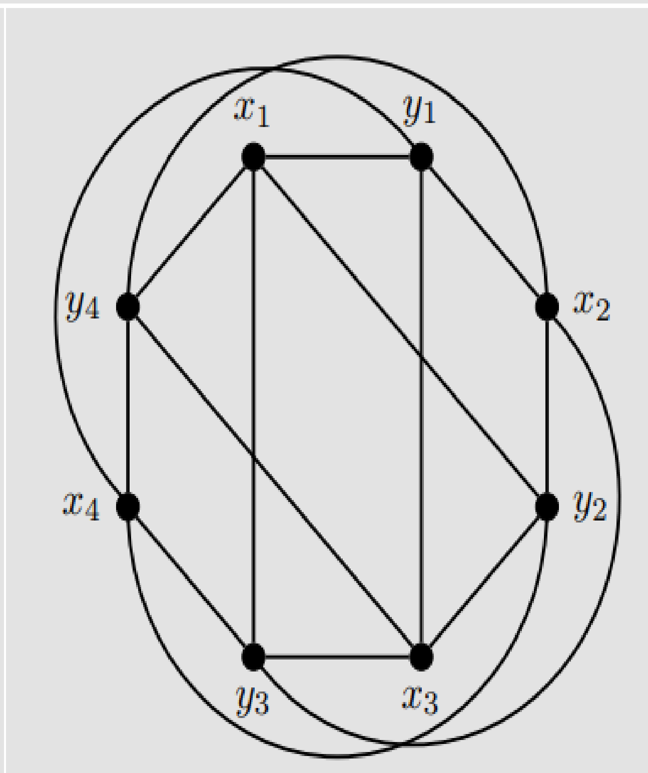
If we didn't already know that K_5 was nonplanar, this would confirm it for us.

- However, we will mainly use Theorem 7.19 to provide a lower bound for the crossing number and then find a drawing of the graph with the given number of crossings.

Example 7.5 Find the crossing number for K_6 and $K_{4,4}$.

Solution: First note that K_6 has 15 edges. Then by Theorem 7.7, we know $cr(K_6) \geq 15 - 3 * 6 + 6 = 3$. Below is a drawing of K_6 with 3 edge crossings, and so we know $cr(K_6) = 3$.





Next, we see that $K_{4,4}$ has 16 edges and so by Theorem 7.7, we know $cr(K_{4,4}) \geq 16 - 2 * 8 + 4 = 4$. Since the drawing below of $K_{4,4}$ has 4 edge crossings, we know $cr(K_{4,4}) = 4$.

There is no clear formula for the crossing number of a graph, even for complete graphs!

- The two results below give upper bounds for K_n and $K_{m,n}$ in terms of the floor function.

Theorem 7.20

$$cr(K_n) \leq \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

- In 1972 Richard Guy proved this bound and in fact conjectured that equality holds for all n .
- He proved equality for $n \leq 10$ and in 2007 Shengjun Pan and Bruce Richter showed equality holds for $n \leq 12$.
- Note that for K_6, the inequality above gives $cr \leq 3$ and from Theorem 7.19 we have $cr(K_6) \geq 3$.

Remark:

Thus, we know the crossing number without exhibiting a specific drawing of K6. Unfortunately, these bounds quickly diverge, as even moving up just one size to K_7 we get $6 \leq cr(K_7) \leq 9$.

Theorem 7.21

$$cr(K_{m,n}) \leq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

- Similar to K_6, using this equation in tandem with Theorem 7.19, we can determine $cr(K_{4,4}) = 4$.

THICKNESS:

- When designing electrical circuits, such as the circuit board within a computer, it is important that wires do not cross.

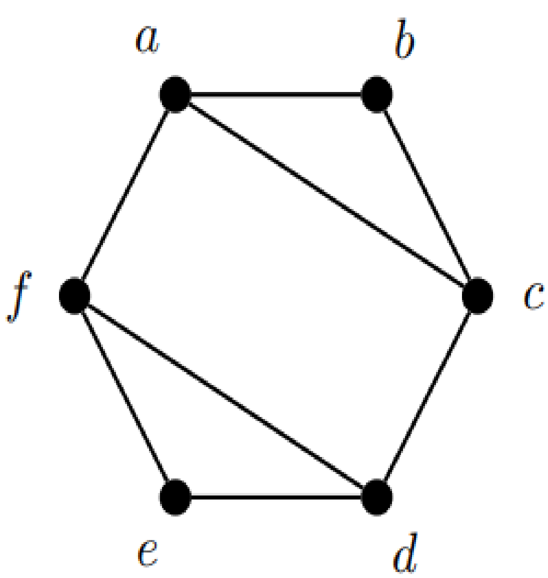
- If there are too many connections between the points then crossings are necessary on a plane. To combat this, we can break up the wires into different layers, each one of which contains no crossings.
- Since each new layer would incur additional cost, we want to minimize the number of layers necessary.
- In graph theoretic terms, we want to decompose the graph into spanning subgraphs, each of which are planar, using the smallest number of subgraphs possible.
- This minimum value is called the **thickness of a graph.**

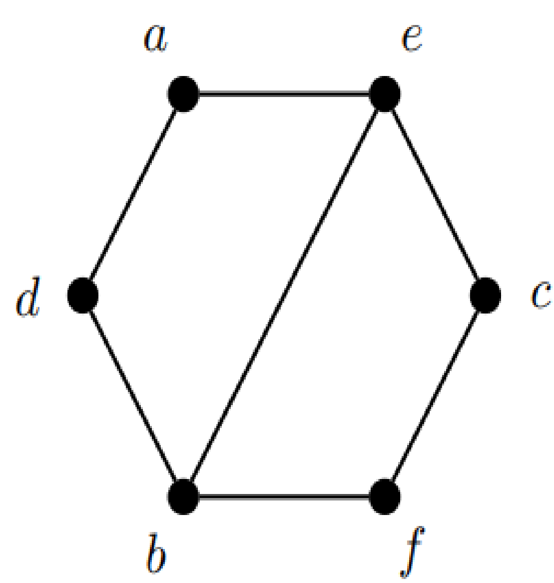
Definition 7.22 Let $T = \{H_1, H_2, \ldots, H_t\}$ be a set of spanning subgraphs of G so that each H_i is planar and every edge of G appears in exactly one graph from T. The ***thickness*** of a graph G, denoted $\theta(G)$, is the minimum size of T among all possible such collections.

Clearly $\theta(G) = 1$ if and only if G is planar, since T would contain only G itself.

Example:

Below is a decomposition of K_6 into two planar spanning subgraphs and since we know that K_6 is not planar we have shown $\theta(K_6) = 2$.





Result for Thickness of Graph:

Corollary 7.23 Let G be a connected simple graph with n vertices and m edges. Then

$$\theta(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil$$

Corollary 7.24 Let G be a connected simple bipartite graph with n vertices and m edges. Then

$$\theta(G) \geq \left\lceil \frac{m}{2n-4} \right\rceil$$

- While a general formula is not known for the thickness of a graph, the theorem below does establish the thickness for a complete graph (and so could serve as an upper bound for any graph on n vertices).

Theorem 7.25

$$\theta(K_n) = \begin{cases} \left\lfloor \frac{n+7}{6} \right\rfloor & n \neq 9, 10 \\ 3 & n = 9, 10 \end{cases}$$