CONNECTIVITY AND FLOW

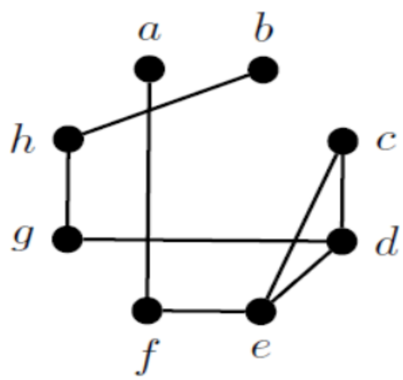
- We have used connectivity in the context of problems. For example, we needed to know if a graph was connected to determine if it has Eulerian circuit, Hamiltonian cycle, and we define trees as minimally connected graphs; since the removal of any edge would disconnect the graph.
- This chapter focuses on connectivity as its own topic, where we now consider how connected a graph is, and not just whether it is connected or not.
- One way to describe the clumping is in a connected graph, how many edges or vertices would need to be removed before the graph is no longer connected, which is one way we measure connectivity.

Connectivity Measures:

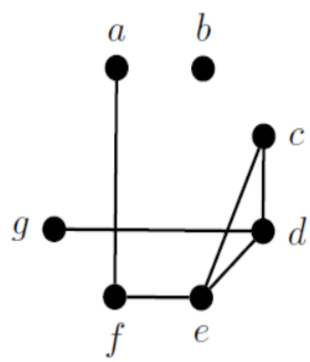
When we define a graph to be connected, we refer to the existence of a way to move between any two vertices in a graph, specifically as the existence of a path between any pair of vertices.

Definition 4.1 A ***cut-vertex*** of a graph G is a vertex v whose removal disconnects the graph, that is, G is connected but $G - v$ is not. A set S of vertices within a graph G is a ***cut-set*** if $G - S$ is disconnected.

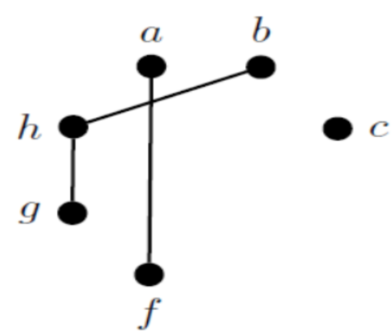
- Note that any connected graph that is not complete has a cut-set, whereas K_n does not have a cut-set.
- Moreover, a graph can have many different cuts-sets of varying sizes.
- For example, two different cut-sets are shown below in graph G_1.



G_1



$G_1 - \{h\}$



$G_1 - \{d, e\}$

k-Connected:

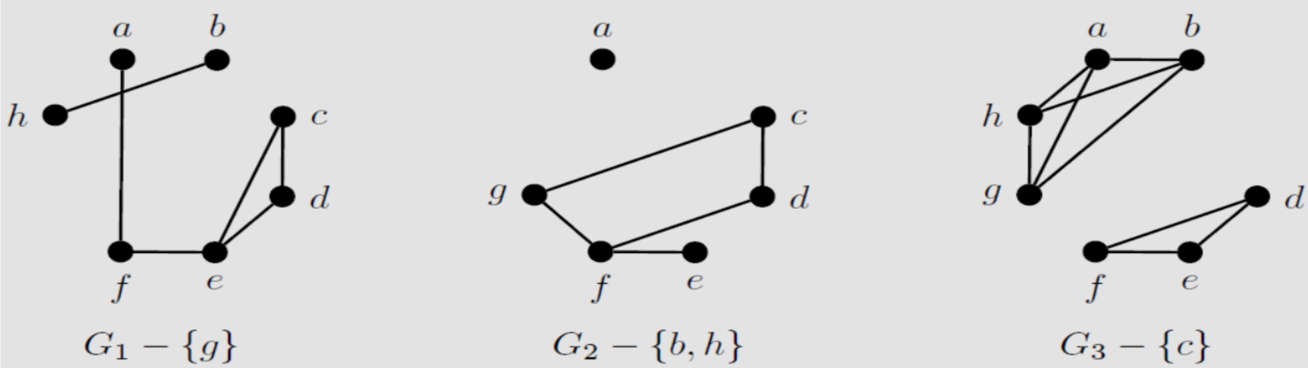
Definition 4.2 For any graph G, we say G is **k-connected** if the smallest cut-set is of size at least k.

Define the connectivity of G, $\kappa(G) = k$, to be the maximum k such that G is k-connected, that is there is a cut-set S of size k, yet no cut-set exists of size $k-1$ or less. Define $\kappa(K_n) = n - 1$.

- The distinction between *k*-connected and connectivity *k* is subtle yet important.
- For example, if we say a graph is 3-connected, then we know there cannot be a cut-set of size 2 or less in the graph; however, we only know that its connectivity is at least 3 ($\kappa(G) \geq 3$).

Example 4.1 Find $\kappa(G)$ for each of the graphs shown above on page 169.

Solution: The removal of any one of d, e, f, g, or h in G_1 will disconnect the graph, so $\kappa(G_1) = 1$. Similarly, $G_3 - c$ has two components and so $\kappa(G_3) = 1$. However, $\kappa(G_2) = 2$ since the removal of any one vertex will not disconnect the graph, yet $S = \{b, h\}$ is a cut-set. Note this means G_2 is both 1-connected and 2-connected, but not 3-connected.



- The example above demonstrates that more than one minimal cut-set can exist within a graph.
- Moreover, any connected graph is 1-connected.
- We are more interested in how large k can be before G fails to be k-connected.

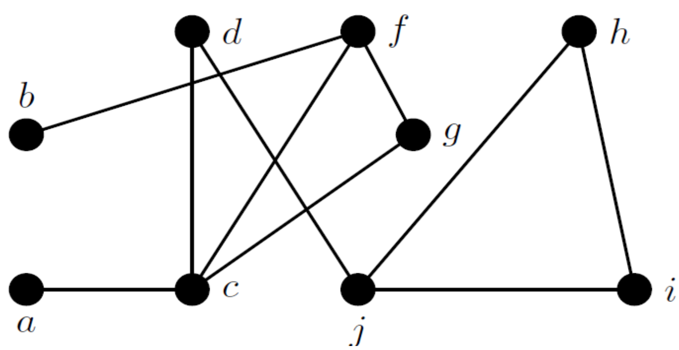
k-Edge-Connected:

- We now look at how many edges need to be removed before the graph is disconnected.
- Recall that when we remove an edge e = xy from a graph, we are not removing the endpoints x and y.

Definition 4.3 A ***bridge*** in a graph $G = (V, E)$ is an edge e whose removal disconnects the graph, that is, G is connected but $G - e$ is not. An ***edge-cut*** is a set $F \subseteq E$ so that $G - F$ is disconnected.

- ✓ Clearly every connected graph has an edge-cut since removing all the edges from a graph will result in just a collection of isolated vertices.
- ✓ As with the vertex version, we are more concerned with the smallest size of an edge-cut.

Find all bridges in the following graph:

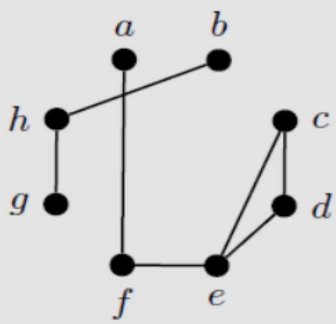


Definition 4.4 We say G is ***k*-edge-connected** if the smallest edge-cut is of size at least k.

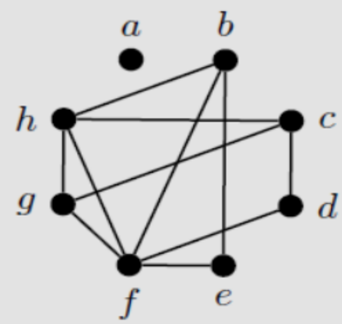
Define $\kappa'(G) = k$ to be the maximum k such that G is k-edge-connected, that is there exists a edge-cut F of size k, yet no edge-cut exists of size $k-1$.

Example 4.2 Find $\kappa'(G)$ for each of the graphs shown on page 169.

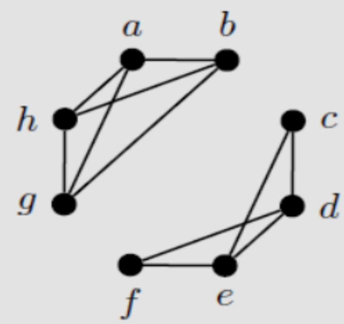
Solution: There are many options for a single edge whose removal will disconnect G_1 (for example af or dg). Thus $\kappa'(G_1) = 1$. For G_2, no one edge can disconnect the graph with its removal, yet removing both ab and ah will isolate a and so $\kappa(G_2) = 2$. Similarly $\kappa'(G_3) = 2$, since the removal of bc and cg will create two components, one with vertices a, b, g, h and the other with c, d, e, f.



$G_1 - dg$



$G_2 - \{ab, ah\}$



$G_3 - \{bc, cg\}$

Whitney's Theorem:

- What do you think, is there any relationship between the vertex and edge connectivity measures?
- The examples above should demonstrate that these measures need not be equal, though they can be.
- How does the minimum degree of a graph play a role in these?
- Notice how in both G_2 and G_3 above we found an edge-cut by removing both edges incident to a specific vertex.

Theorem 4.5 (Whitney's Theorem) For any graph G, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Remark:

- ✓ Whitney's Theorem provides an indication that high connectivity (or edge-connectivity) requires a large minimum degree. But is the converse true?
- ✓ Can a graph have a high minimum degree but low connectivity?

CONNECTIVITY AND PATHS:

- Now we have some familiarity with connectivity, we turn to its relationship to paths within a graph.
- We will assume the graphs are connected, as otherwise, the results are trivial.
- We begin by relating cut-vertices and bridges to paths.

Theorem 4.6 A vertex v is a cut-vertex of a graph G if and only if there exist vertices x and y such that v is on every $x - y$ path.

Proof: First suppose v is a cut-vertex in a graph G. Then $G-v$ must have at least two components. Let x and y be vertices in different components of $G - v$. Since G is connected, we know there must exist an $x - y$ path in G that does not exist in $G - v$. Thus v must lie on this path.

Conversely, let v be a vertex and suppose there exist vertices x and y such that v is on every $x - y$ path. Then none of these paths exist in $G - v$, and so x and y cannot be in the same component of $G - v$. Thus G must have at least two components and so v is a cut-vertex.

Theorem 4.7 An edge e is a bridge of G if and only if there exist vertices x and y such that e is on every $x - y$ path.

- It should be obvious that any edge along a cycle cannot be a bridge since its removal will only break the cycle, not disconnect the graph.
- More surprising is that all edges not on a cycle are in fact bridges.

Theorem 4.8 Every nontrivial connected graph contains at least two vertices that are not cut-vertices.

Theorem 4.9 An edge e is a bridge of G if and only if e lies on no cycle of G.

Definition 4.10 Let P_1 and P_2 be two paths within the same graph G. We say these paths are

- ***disjoint*** if they have no vertices or edges in common.
- ***internally disjoint*** if the only vertices in common are the starting and ending vertices of the paths.
- ***edge-disjoint*** if they have no edges in common.

✓ Two disjoint paths are automatically internally disjoint and edge-disjoint, but two edge-disjoint paths may or may not be internally disjoint.

Definition 4.11 Let x and y be two vertices in a graph G. A set S (of either vertices or edges) ***separates*** x and y if x and y are in different components of $G - S$. When this happens, we say S is a separating set for x and y.

- Let G be a connected graph, and u, v be two non-adjacent vertices in G. A subset S of $V - \{u, v\}$ is said to separate u and v if $G - S$ is disconnected and u and v are in different components of $G-S$.
- A set $I = \{Q_1, Q_2, \ldots, Q_k\}$ of u–v paths (each Q_i is a u–v path) in G is said to be internally-disjoint if $V(Q_i) \cap V(Q_j) = \{u, v\}$ for all i, j with $1 \le i < j \le k$.

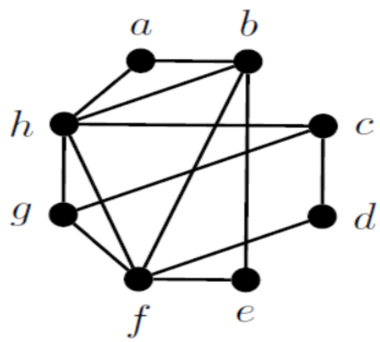
Remarks:

✓ Note that a cut-set may or may not be a separating set for a specific pair of vertices.

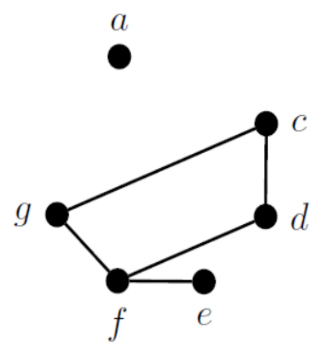
✓ Consider graph G_2 from page 169. We have already shown that {b, h} is a cut-set and {ab, ah} is an edge-cut. If we want to separate b and c then we cannot use b in the separating set and using the edges ab and ah will only isolate a, leaving b and c in the same component.

✓ We can separate b and c using the vertices {f, h} and the edges {cd, cg, ch}.

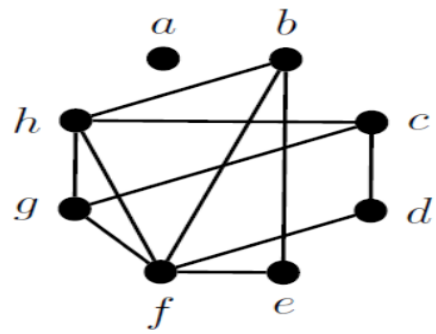
✓ Note that you cannot separate b and c with fewer or edges.



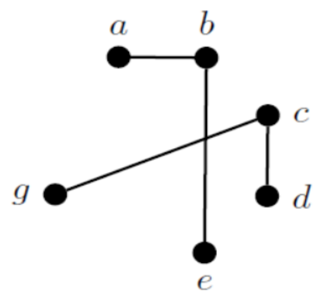
G_2



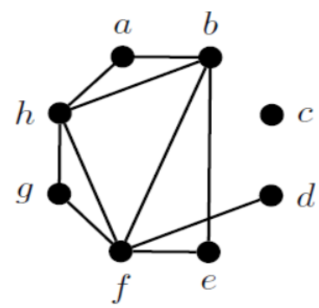
$G_2 - \{b, h\}$



$G_2 - \{ab, ah\}$



$G_2 - \{f, h\}$



$G_2 - \{cd, cg, ch\}$

Menger's Theorem:

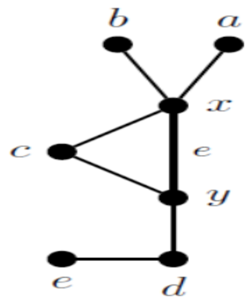
- The following theorems generalize the results relating a cut-vertex or bridge to paths in a graph.
- Menger's Theorem, and the resulting theorems, show the number of internally disjoint (or edge-disjoint) paths directly corresponds to the connectivity (or edge-connectivity) of a graph.

Intuitive Idea:

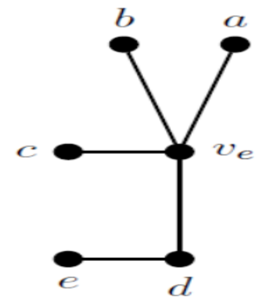
For example, in G_2 above we could separate *b* and *c* using two vertices and it should be easy to see that *b h c* and *b e f d c* are internally disjoint *b* – *c* paths.

- ✓ However, if we try to find more than two *b* – *c* paths then one of them cannot be internally disjoint from the others.

Definition 4.12 Let $e = xy$ be an edge of a graph G. The ***contraction*** of e, denoted G/e, replaces the edge e with a vertex v_e so that any vertices adjacent to either x or y are now adjacent to v_e.



G_4



G_4/e

Remarks:

- Contracting an edge creates a smaller graph, both in terms of the number of vertices and edges but keeps much of the structure of a graph intact.
- In particular, contracting an edge cannot disconnect a graph.

Menger's Theorem Statement:

Theorem 4.13 (Menger's Theorem) Let x and y be nonadjacent vertices in G. Then the minimum number of vertices that separate x and y equals the maximum number of internally disjoint $x - y$ paths in G.

Example:

Consider the graph *G* of Fig. 6.4.2(a) with two non-adjacent vertices *u* and *v*. It can be checked that the minimum number of vertices separating *u* and *v* is 4, which is equal to the maximum number of internally-disjoint *u* − *v* paths in *G*, as shown in Fig. 6.4.2(b).

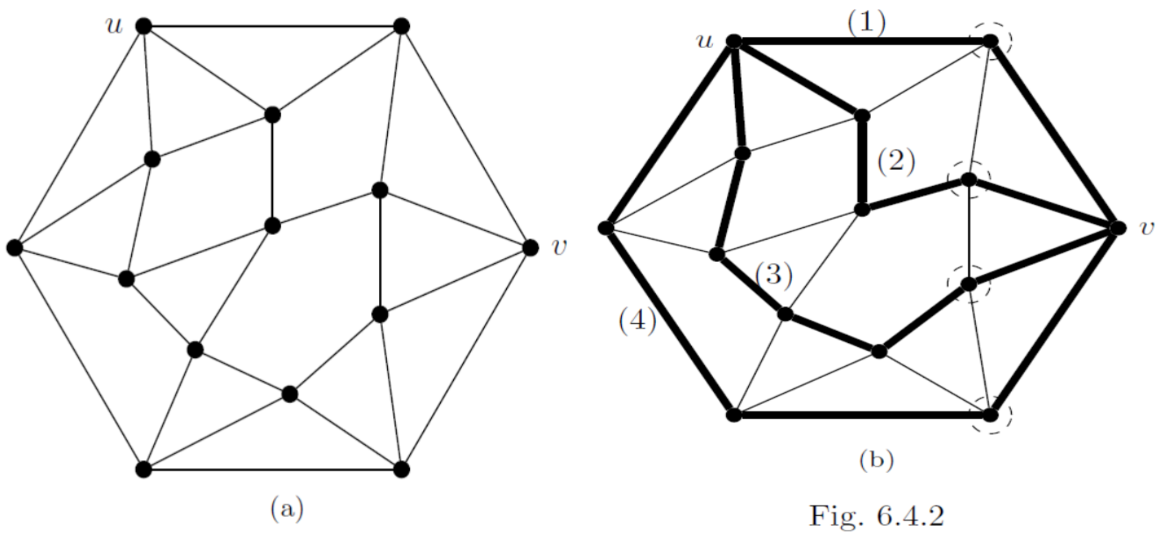


Fig. 6.4.2

✓ An immediate result from Menger's Theorem refers to the global condition of connectivity as opposed to the separation of two specific vertices.

Theorem 4.14 A nontrivial graph G is k-connected if and only if for each pair of distinct vertices x and y there are at least k internally disjoint $x - y$ paths.

✓ Now an edge version exists for the two previous theorems.

Theorem 4.15 Let x and y be distinct vertices in G. Then the minimum number of edges that separate x and y equals the maximum number of edge-disjoint $x - y$ paths in G.

Theorem 4.16 A nontrivial graph G is k-edge-connected if and only if for each pair of distinct vertices x and y there are at least k edge disjoint $x - y$ paths.

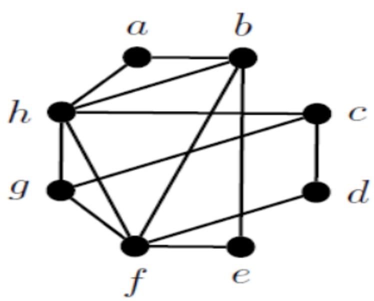
Remark:

- When we are investigating graphs that are not trees, Menger's Theroem (and the resulting theorems) allow us to conduct similar analyses (about connectivity & paths).

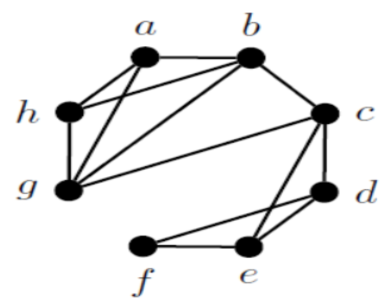
- Where the level to which a graph is connected is equal to the number of paths that would need to be broken in order to separate two vertices.

2-Connected Graphs:

- As we have already seen, 1-connected graphs are simply those graphs that we more commonly call connected.
- *k*-connected graphs can be described in terms of *k* number of paths between two vertices.
- 2-connected graphs hold a special area in the study of connectivity—they are known to be connected and as we will see cannot contain any cut-vertices.
- The class of 2-connected graphs provides both some easy results and some more technical and complex areas of study.



G_2



G_3

✓ Recall that we showed $\kappa(G_2) = 2 = \kappa'(G_2)$ but that $\kappa(G_3) = 1$ and $\kappa'(G_3) = 2$. So, what is the structural difference between G_2 and G_3 that provides the difference in the connectivity measures?

Theorem 4.17 A graph G with at least 3 vertices is 2-connected if and only if G is connected and does not have any cut-vertices.

Proof: Assume G is 2-connected. Then any cut-set of G must be of size at least 2. Therefore G must be connected and cannot have a cut-vertex, as in either of these situations we would have a cut-set of size less than 2.

Conversely, suppose G is not 2-connected. Then either G is disconnected or by Menger's Theorem there exist two non-adjacent vertices x and y for which there is exactly one path between them. Removing any vertex along this path will disconnect x and y, and so that vertex serves as a cut-vertex of G.

Remark:

We proved that every non-leaf of a tree is a cut-vertex. Therefore, no tree is 2-connected and so every 2-connected graph must contain a cycle, or more specifically every vertex lies on a cycle.

Corollary 4.18 A graph G with at least 3 vertices is 2-connected if and only if for every pair of vertices x and y there exists a cycle through x and y.

COROLLARY:

Let G be a graph with v(G) ≥ 3. The following statements are equivalent:

(i) G is 2-connected.

(ii) Every two vertices in G are joined by at least two internally-disjoint paths.

(iii) Every two vertices in G are contained in a common cycle. (iv) Every two edges in G are contained in a common cycle.

(v) Any vertex together with any edge are both contained in a common

cycle.

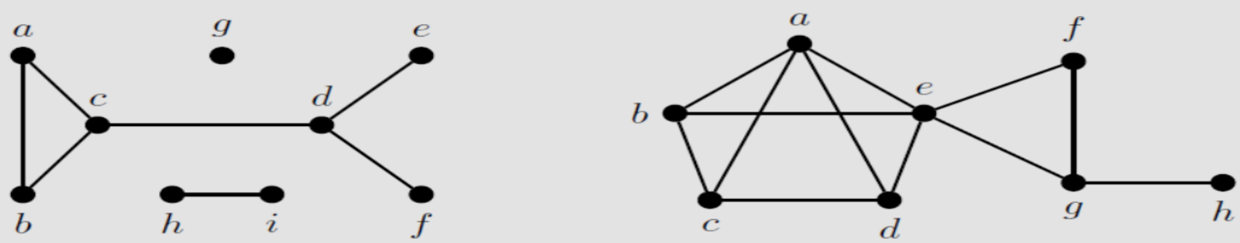
BLOCKS & SUBDIVISION:

Definition 4.19 A *block* of a graph G is a maximal 2-connected subgraph of G, that is, a subgraph with as many edges as possible without a cut-vertex.

- A connected graph of order n ≥ 2 is called a block if it is K_2 when n = 2 or it is 2-connected when n ≥ 3.
- Let G be a connected graph of order n ≥ 3. A subgraph H of G is called a block of G if H is itself a block and H is not properly contained in any subgraph which is a block.
- Every block of G must be an induced subgraph of G.
- A block with more than 3 vertices must contain a cycle and all vertices on that cycle must be in the block.
- It is noted that any disconnected graph is a disjoint union of components.

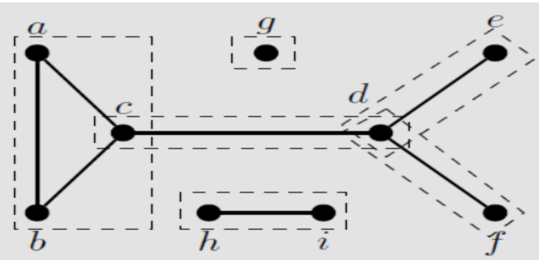
- Any connected graph with cut-vertices is formed by blocks which meet at cut-vertices.
- Let G be a connected graph with cut vertices. A block of G is called an end-block if it contains only one cut-vertex of G.

Example 4.3 Determine all blocks for the two graphs below.

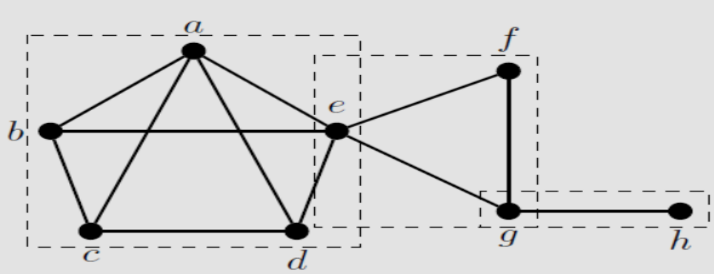


G H

Solution: First note that any isolated vertex will create its own block. Likewise, any component isomorphic to K_2 will also be its own block. The other blocks for graph G are either singular edges (cd, de, and df) or the 3-cycle $a\,b\,c\,a$, as shown below.



For H, we have one block consisting of the subgraph induced by vertices a, b, c, d, e, the 3-cycle $e\,f\,g\,h$, and the edge gh, as shown below.

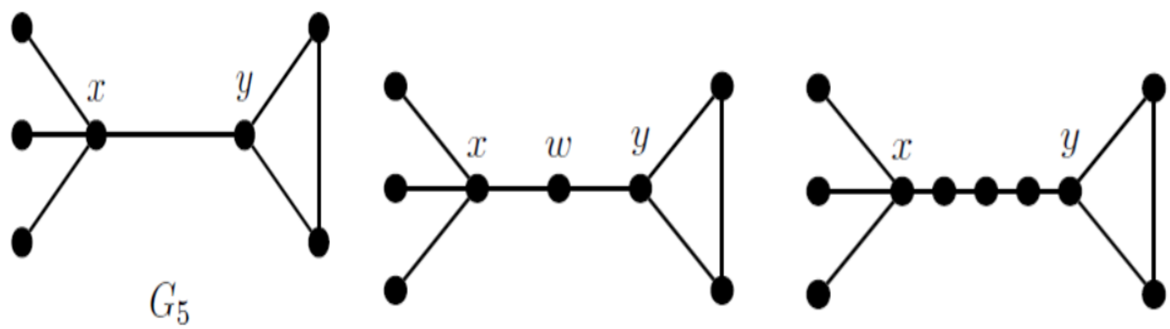


- Notice that some blocks within the same component of a graph *G* will necessarily overlap, and this occurs exactly at any cut-vertex of the larger graph (such as vertex *e* from graph *H in* Example 4.3).
- Moreover, a vertex may be a cut-vertex of *G* but cannot be a cut-vertex of the blocks to which it belongs.

- Essentially, every graph can be viewed at being built from its blocks, which are pasted together at the cut-vertices of the graph.

Recall that in the proof for Menger's Theorem we use a contraction, where an edge e = xy is replaced by the vertex v_e. A reverse process called a subdivision is described below and is useful when discussing 2-connected graphs.

Definition 4.20 Let $e = xy$ be an edge in a graph G. The ***subdivision*** of e adds a vertex v in the edge so as to replace it with the path $x\,v\,y$.



How does subdividing an edge effect connectivity, in particular with respect to cut-vertices?

Theorem 4.21 Let G be a graph and G' the graph obtained by subdividing any edge of G. Then G is 2-connected if and only if G' is 2-connected.

Proof: Let $e = xy$ be an edge of G and let G' be formed by subdividing e into the path $x\,z\,y$.

First suppose G is 2-connected. Then by Corollary 4.18 we know there exists a cycle between any two vertices of G. If a cycle does not use e then it still exists in G'. If a cycle uses e, then by replacing the edge with the path $x\,z\,y$ will maintain the cycle. Thus G' is 2-connected.

Conversely, suppose G is not 2-connected. Clearly if G is disconnected or has fewer than three vertices than G' cannot be 2-connected either. So we may assume G is connected with at least 3 vertices. Thus G must contain a cut-vertex, call it v and consider $G - v$. If e remains in $G - v$ then the subdivision of e would simply occur in one of the components of $G - v$. Thus v is still a cut-vertex in G' and so G' is also not 2-connected.

G' $\qquad$ $G' - v$

If e does not exist in $G - v$ then either $x = v$ or $y = v$. In either case, z would remain adjacent to the other vertex when v is removed, and so would exist in the same component of $G' - v$. Once again, this implies v is a cut-vertex for G' and so G' is also not 2-connected.

G' $\qquad$ $G' - v$

Therefore, G is 2-connected if and only if G' is 2-connected.

Every two vertices of a 2-connected graph are contained in a common cycle.

Theorem:

Let G be a k-connected graph, where $k \geq 2$. Then every set of k vertices is contained in a common cycle in G.