AUGMENTING PATHS & VERTEX COVERS

- We need a way to determine if a matching is in fact maximum. We do this through the use of alternating and augmenting paths.

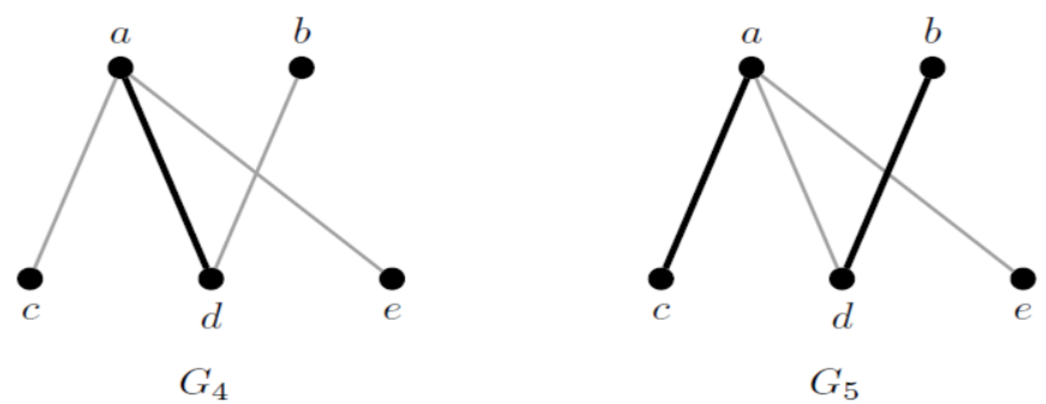
Definition 5.6 Given a matching M of a graph G, a path is called

- *M-alternating* if the edges in the path alternate between edges that are part of M and edges that are not part of M.
- *M-augmenting* if it is an M-alternating path and both endpoints of the path are unsaturated by M, implying both the starting and ending edges of the path are not part of M.

Alternatively:

Let M be a matching in G. A path $v_1v_2\cdots v_p$ in G is said to be M-**alternating** if $v_iv_{i+1} \in M$ whenever $v_{i-1}v_i \notin M$ for each $i = 2, 3, \cdots, p-1$. Likewise, a cycle $u_1u_2\cdots u_pu_1$ in G is said to be M-**alternating** if $u_iu_{i+1} \in M$ whenever $u_{i-1}u_i \notin M$ for each $i = 2, 3, \cdots, p$, where $u_{p+1} = u_1$.

Example:



- Both graphs have alternating paths; for example, the path *c a d b* is alternating in both graphs.

- However, this path is only augmenting in G_4 since both *c* and *b* are unsaturated by the matching.
- If we switch the edges along this path we get a larger matching. This switching procedure removes the matched edges and adds the previously unmatched edges along an augmenting path. Since the path is augmenting, the matching increases in size by one edge.
- Note that switching along the path *c a d b* in G_4 produces the matching shown in G_5.
- The matching shown in G_5 is the maximum matching for that graph.

We could switch along such a path to produce a larger matching.

- This result is stated in the theorem below and was first published by the French mathematician Claude Berge in 1959.
- Note that, unlike Hall's Theorem, **Berge's Theorem** holds for both bipartite graphs and non-bipartite graphs.

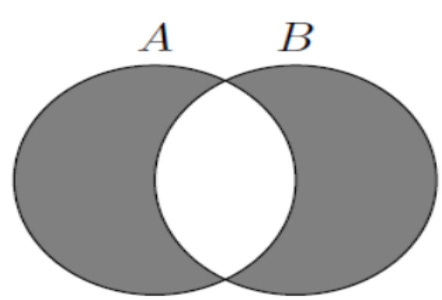
Berge's Theorem:

Theorem 5.7 (Berge's Theorem) A matching M of a graph G is maximum if and only if G does not contain any M-augmenting paths.

- We will need the following set theory definition and result about two different matchings of a graph in our proof of Berge's Theorem.

Definition 5.8 Let A and B be two sets. Then the ***symmetric difference*** $A \triangle B$ is all those elements in exactly one of A and B; that is, $A \triangle B = (A - B) \cup (B - A)$.

- Given is a Venn diagram representing the symmetric difference.
- Note that we can also think of the symmetric difference as what remains in the union when we

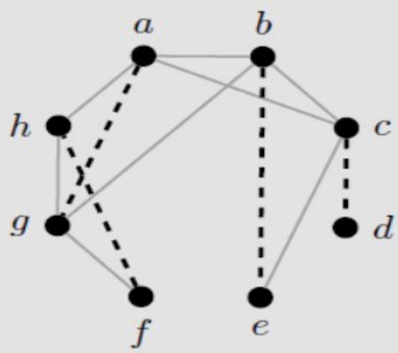


remove the intersection of two sets, that is A Δ B = A ∪ B − A ∩ B.

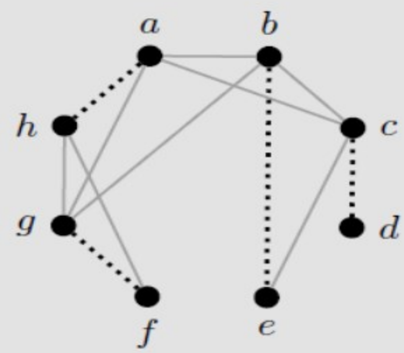
Remark: In graph theoretic terms, we can look at the symmetric difference of various subsets of vertices or edges of a larger graph G.

- Our use here will be in considering the symmetric difference of two matchings, which would consist of the edges in a graph that are in one but not both of two different matchings.

Example 5.4 Below are two different matchings of a graph G. Find $M_1 \triangle M_2$.

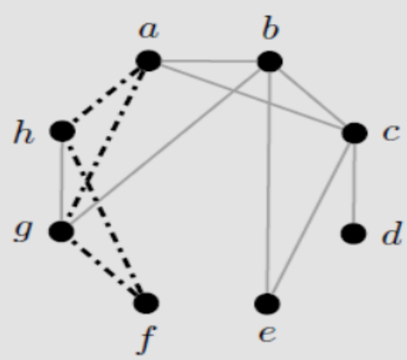


M_1



M_2

Solution: Note that the symmetric difference will only show those edges involved in exactly one of the two matchings. Thus if an edge is in both matchings or left unmatched in both matchings, it does not appear in $M_1 \triangle M_2$.



$M_1 \triangle M_2$

Remark: The symmetric difference of two matchings may not be itself a matching. As seen in the example above with the cycle created by the vertices *a, g, f, h.*

- However, as the lemma below shows, the edges in $M_1 \Delta M_2$ must result in only one of two specific types of structures.

Lemma 5.9 Let M_1 and M_2 be two matchings in a graph G. Then every component of $M_1 \triangle M_2$ is either a path or an even cycle.

Proof: Let $H = M_1 \triangle M_2$. Since M_1 and M_2 are both matchings, we know any vertex has at most one edge incident to it from either matching. Thus each vertex has degree at most 2 in H. Thus every component of H consists of paths and cycles. Moreover, any such path or cycle must alternate between edges in M_1 and M_2, and so every cycle must be of even length.

Remark:

- We can use the symmetric difference of two matchings to gain an understanding of how they relate to each other.
- In particular, we will use this concept to gain an understanding about their relative sizes, allowing us to prove Berge's Theorem.

Theorem 5.7 (Berge's Theorem, restated) A matching M of a graph G is not maximum if and only if G contains some M-augmenting path.

Proof: First suppose M is a matching of G and G contains an M-augmenting path P. Then we can switch edges along P to produce a larger matching M' and so M is not a maximum matching of G.

Conversely, suppose M is a matching of G and M is not maximum. Then there must exist a different matching M' that is larger than M. We will produce an M-augmenting path by looking at $M \triangle M'$. By Lemma 5.9 above we know that every component of $M \triangle M'$ is either a path or even cycle. Since $|M'| > |M|$ we know not all components can be even cycles, since even cycles will contain the same number of edges from M and M'. Thus there must be some component that is a path with more edges from M' than M. This path must start and end with an M' edge, and so is an M-augmenting path.

- ✓ Now we understand how to determine if a matching is maximum (search for augmenting paths), we need a procedure or algorithm to actually find one.
- ✓ The algorithm described below is closely related to the Hungarian Algorithm proposed by Harold Kuhn in 1955 [61].
- ✓ Also, even though Berge's Theorem holds for graphs that are not bipartite, the algorithm below requires the input of a bipartite graph.

Augmenting Path Algorithm

Input: Bipartite graph $G = (X \cup Y, E)$.

Steps:

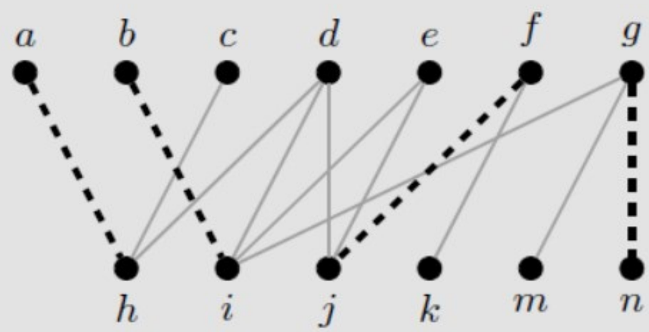
1. Find an arbitrary matching M.
2. Let U denote the set of unsaturated vertices in X.
3. If U is empty, then M is a maximum matching; otherwise, select a vertex x from U.
4. Consider y in $N(x)$.
5. If y is also unsaturated by M, then add the edge xy to M to obtain a larger matching M'. Return to Step (2) and recompute U. Otherwise, go to Step (6).
6. If y is saturated by M, then find a maximal M-alternating path from x using xy as the first edge.
   - (a) If this path is M-augmenting, then switch edges along that path to obtain a larger matching M'; that is, remove from M the matched edges along the path and add the unmatched edges to create M'. Return to Step (2) and recompute U.
   - (b) If the path is not M-augmenting, return to Step (4), choosing a new vertex from $N(x)$.
7. Stop repeating Steps (2)–(4) when all vertices from U have been considered.

Output: Maximum matching for G.

Remarks:

- The arbitrary matching in Step (1) could be the empty matching (no edges are initially included), though in practice starting with a quick simple matching allows for fewer iterations of the algorithm.
- You should not spend much time trying to create an initial maximum matching, but rather choose obvious edges to include.

Example 5.5 Apply the Augmenting Path Algorithm to the bipartite graph G_6 below, where $X = \{a, b, c, d, e, f, g\}$ and $Y = \{h, i, j, k, m, n\}$, with an initial matching shown as dashed lines.

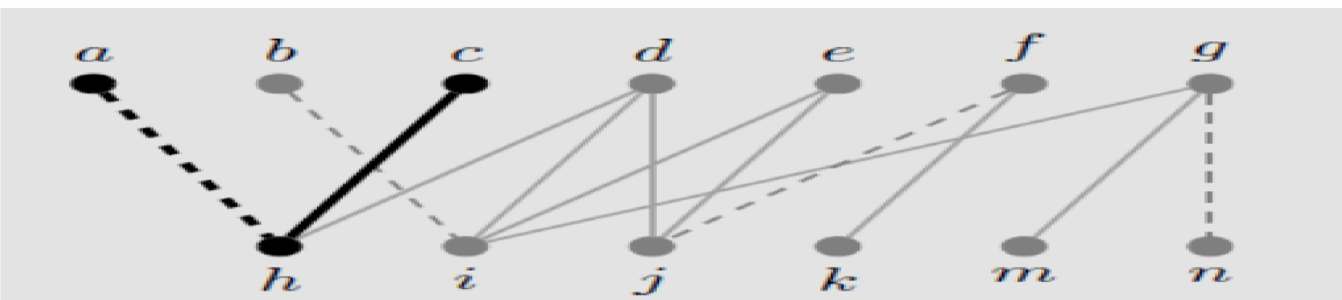


G_6

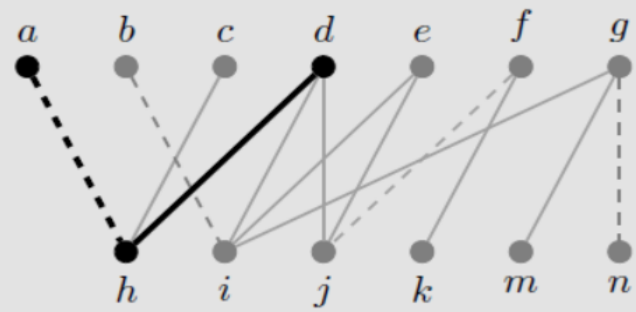
Solution: During each step shown below, the path under consideration will be in bold, with the matching shown as dashed lines throughout.

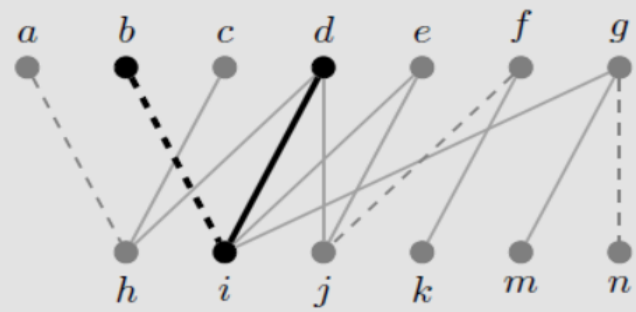
Step 1: The unsaturated vertices from X are $U = \{c, d, e\}$.

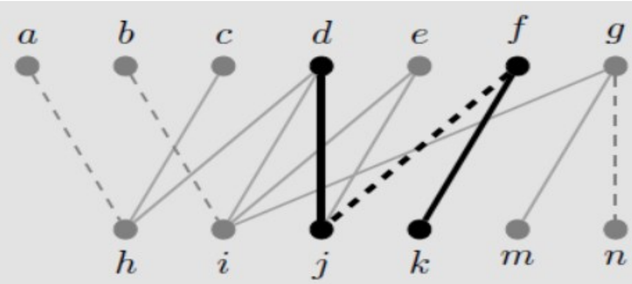
Step 2: Choose c. The only neighbor of c is h, which is saturated by M. Form an M-alternating path starting with the edge ch. This produces the path $c\,h\,a$, as shown on the next page, which is not augmenting.



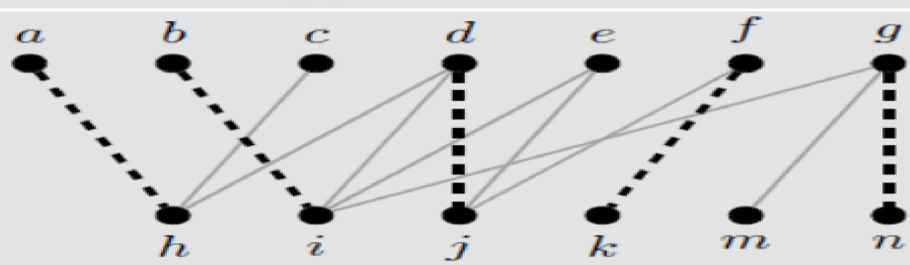
Step 3: Choose a new vertex from U, say d. Then $N(d) = \{h, i, j\}$. Below are the alternating paths originating from d.





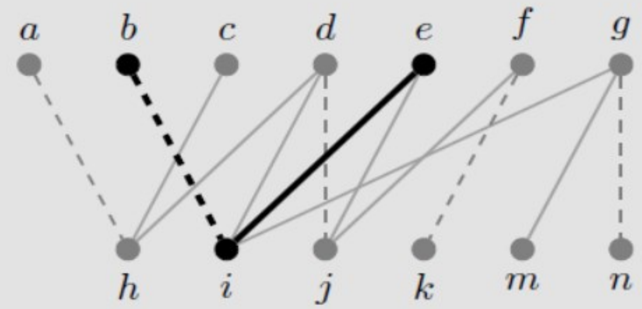


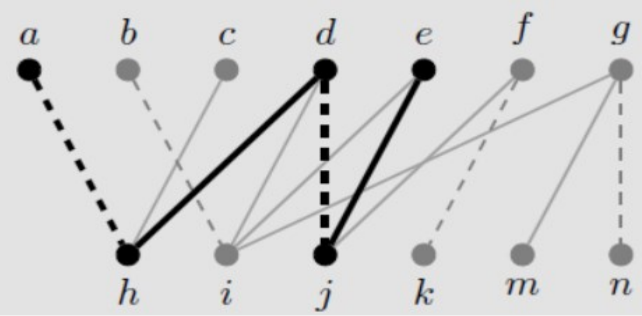
Note that the last path $(d\,j\,f\,k)$ is M-augmenting. Form a new matching M' by removing edge fj from M and adding edges dj and fk, as shown in the following graph.

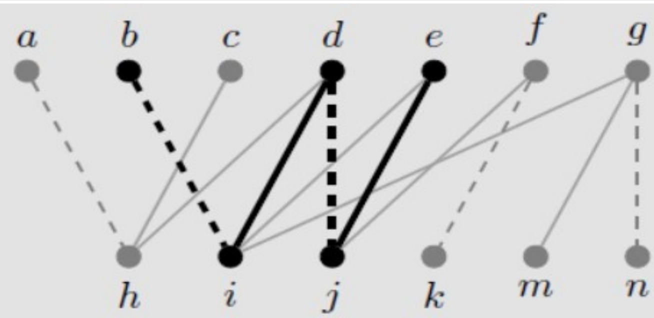


Step 4: Recalculate $U = \{c, e\}$. We must still check c since it is possible for the change in matching to modify possible alternating paths from a previously reviewed vertex; however, the path obtained is $c\,h\,a$, the same as from Step 2.

Step 5: Check the paths from e. The alternating paths are shown below.







None of these paths are augmenting. Thus no M'-augmenting paths exist in G and so M' must be maximum by Berge's Theorem.

Output: The maximum matching $M' = \{ah, bi, dj, fk, gn\}$ from Step 3.

- ✓ The Augmenting Path Algorithm provides a method for not only finding a maximum matching, but also a reasoning why a larger matching does not exist since no augmenting paths exist at the completion of the algorithm.

- ✓ However, there are other ways to determine if a matching is maximum without the need to work through this algorithm.
- ✓ The simplest, and most elegant, is through the use of a specific set of vertices known as a vertex cover.

VERTEX COVER:

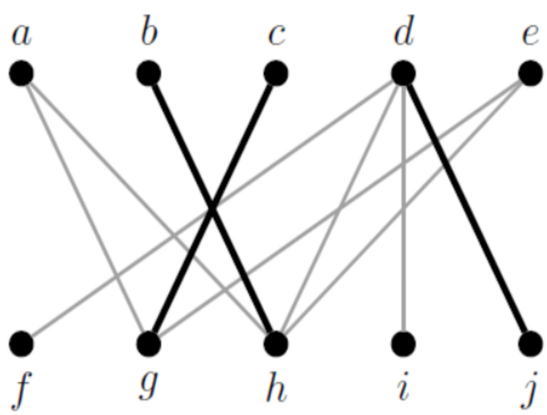
Definition 5.10 A *vertex cover* Q for a graph G is a subset of vertices so that every edge of G has at least one endpoint in Q.

- Every graph has a vertex cover (for example *Q* could contain all the vertices in the graph).
- We want to optimize the vertex cover; that is, find a minimum vertex cover.
- If every edge has an endpoint to one of the vertices in a vertex cover, then at most one matched edge can be incident to any single vertex in the cover.
- This result, stated in the theorem below, was first published in 1931 by the Hungarian mathematicians D´enes K¨onig and (independently) Jen¨o Egerv´ary, and as noted above was the inspiration behind the Augmenting Path Algorithm.

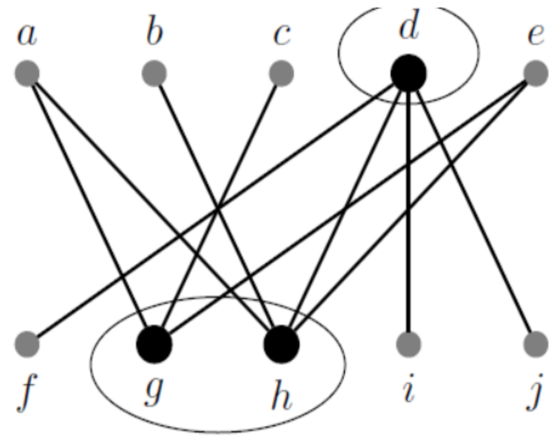
Theorem 5.11 (König-Egerváry Theorem) For a bipartite graph G, the size of a maximum matching of G equals the size of a minimum vertex cover for G.

Example:

Consider the graph G_3. Based on the K¨onig-Egerv´ary Theorem, to show the matching in bold is maximum we need to find a vertex cover of size 3. One such cover is shown; you should check that every edge has an endpoint that is either *d*, *g* or *h*.



G_3 maximum matching



G_3 vertex cover

- Note that the proof of the K¨onig-Egerv´ary Theorem provides a method for finding a vertex cover, which we describe below.
- In most cases, a minimum vertex cover for a bipartite graph will require some vertices from both pieces of the vertex partition.
- It should not come to much surprise that the Augmenting Path Algorithm can be used to find a vertex cover.

Vertex Cover Method

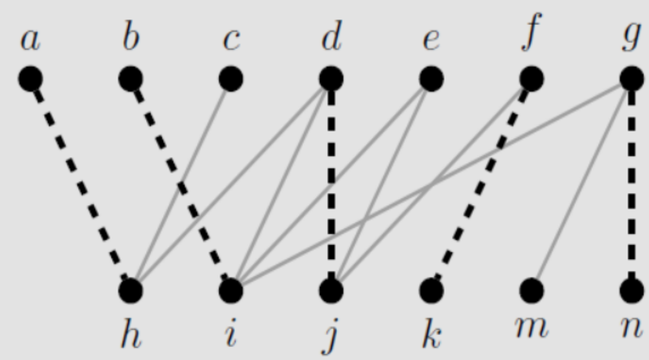
1. Let $G = (X \cup Y, E)$ be a bipartite graph.
2. Apply the Augmenting Path Algorithm and mark the vertices considered throughout its final implementation.
3. Define a vertex cover Q as the unmarked vertices from X and the marked vertices from Y.
4. Q is a minimum vertex cover for G.

Remarks:

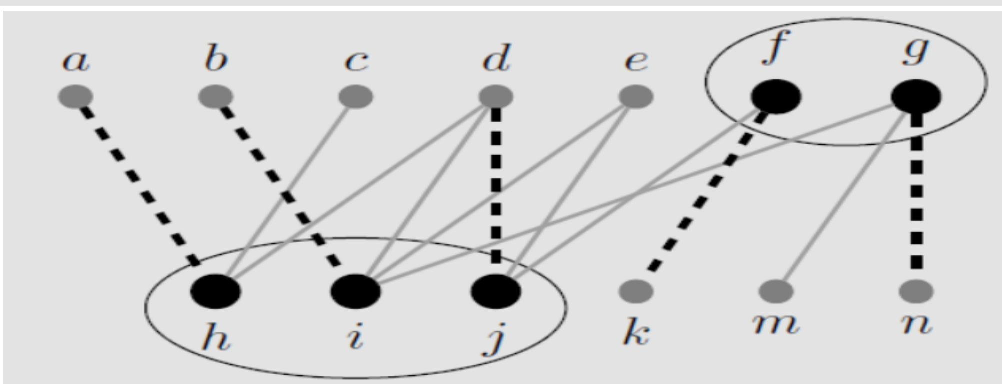
- In Step 2, a vertex is marked if it was considered during the final step in the implementation of the Augmenting Path Algorithm
- Note that this is not just the vertices in U, the unsaturated vertices from X, but also any vertex that was reached through an alternating path that originated at a vertex from U.

- Thus, the unmarked vertices will be those that are never mentioned during the final step of the implementation of the Augmenting Path Algorithm.

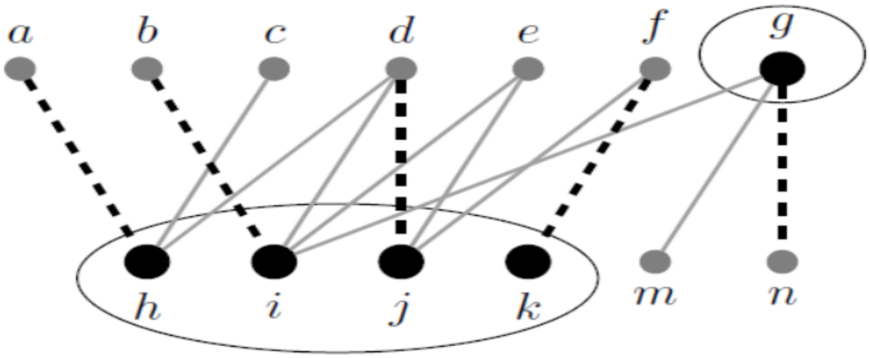
Example 5.6 Apply the Vertex Cover Method to the output graph from Example 5.5.



Solution: Recording the vertices considered throughout the last step of the Augmenting Path Algorithm, the marked vertices from X are a, b, c, d, and e, and the marked vertices from Y are h, i, and j. This produces the vertex cover $Q = \{f, g, h, i, j\}$ of size 5, shown below. Recall that the maximum matching contained 5 edges.



Note that more than one minimum cover may exist for the same graph, just as more than one maximum matching may exist. In the example above, we found one such vertex cover through the matching found using the Augmenting Path Algorithm, though the set Q' = {g, h, i, j, k} is also a valid minimum vertex cover (as shown below).



✓ By finding a minimum vertex cover, we are able to definitively answer how large of a matching can be formed. Vertex covers provide another avenue for proving a maximum matching has been found.