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Graph Theory Notes

Graph Coloring:

→ Four Color Theorem: Any map split into contiguous region can be colored using at most 4 colour so that no two bordering regions are given the same colour.

→ K-coloring: A proper K-coloring of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices are given the same color and exactly K colors are used.

→ Color classes: given a proper K-coloring of G, the color classes are sets $S_1 \ldots S_K$ where S_i consists of all vertices of color i.

→ Independence Number:

$\alpha(G) = n$: if there exists a set of n vertices with no edges b/w them but every set of $n+1$ vertices contain at least one edge

→ We want to find the lowest value of K so that G has proper coloring.

⇒ Vertex Coloring:

→ Chromatic Number: Smallest value of K for which a graph has a proper K-coloring.

- Denoted by $\chi(G)$

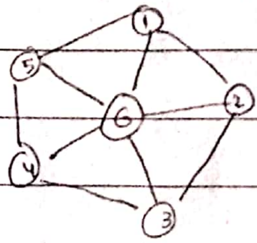
- To find chromatic number of a graph:
  - i) Find a vertex coloring of G using K colors.
  - ii) Show why fewer colors will not suffice.

→ Cycles:

If n (number of vertices) is even, 2 colors needed

If n is odd and $n \geq 3$, 3 colors needed.

→ Wheels: cycle with a central vertex adjacent to each of the vertices in the cycle.



If wheel has odd cycle, 4 ~~vertices~~ colors required

If wheel has even cycle, 3 ~~vertices~~ colors req

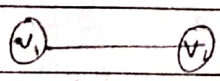
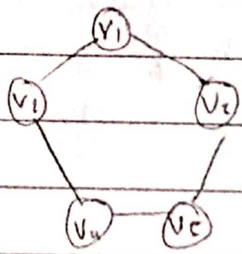
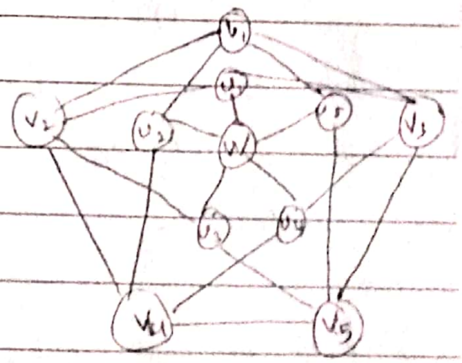
→ Clique: a ~~set~~ subgraph which is a complete graph.

Clique size $= \omega(G)$ is largest value of n for which G contains K_n as a subgraph.

if G contains K_n, then $\chi(G) \geq n$

- Triangle free graphs: $\omega(G) = 2$

→ Mycielski Triangles:

 ⇒  ⇒ 

→ Brooke's Theorem:

$\Delta \Rightarrow$ maximum degree of G.

$\chi(G) \leq \Delta$ as long as G is not a complete graph or an odd cycle. If G is a complete graph or an odd cycle, then ~~$\Delta(G)$~~

$\chi(G) = \Delta + 1$

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1$$

- color higher degree vertices first.
- After assigning a color to an initial vertex, look for cliques within the graph containing that vertex.

→ Equitable Coloring: A minimal proper coloring of G so that the number of vertices of each color differs by at most one.

→ Strategy:
- Begin with vertices of high degree
- Look for locations where colors are forced (cliques, odd cycles, wheels etc) rather than chosen.
- After that, color the remaining vertices while trying to avoid using any additional colors.

→ General Results:
- First we begin with a basic counting argument relating the no. of edges of a graph, with its chromatic number.
- Hence we can create an upper bound based on the no. of edges in a graph rather than maximum degree.

$$\boxed{\chi(G) = \frac{1}{2} + \sqrt{2m + \frac{1}{4}}}$$

→ Let G be a graph and $l(G)$ be the length of the longest path in G. Then $\chi(G) \leq 1 + l(G)$.

→ Given a graph $G = (V, E)$, an induced subgraph is a subgraph $G[V']$ where $V' \subseteq V$ and every available edge from G b/w the vertices in V' is included.

→ Let G be a graph and $\delta(G)$ denote the min. degree of G. The $\chi(G) \leq 1 + \max_H \delta(H)$ for any induced subgraph.

→ Perfect Graphs:

A graph G is perfect iff $\chi(H) = \omega(H)$ for all induced subgraphs H.

→ A graph G is perfect iff $\bar{G}$ is perfect.

→ A graph G is perfect iff no induced subgraph of G or $\bar{G}$ is an odd cycle of length at least 5.

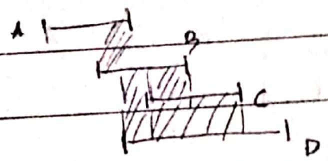
- Trees, bipartite, chordal and Interval graphs are perfect.

→ Cordal Graph:

• If any cycle of length 4 or larger has an edge (called a chord) b/w two consecutive vertices of the cycle.

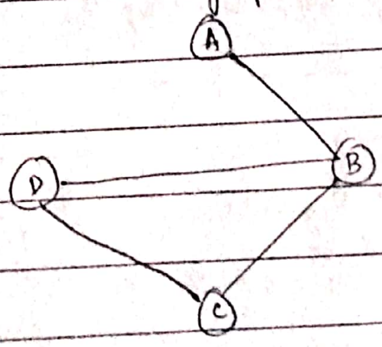
→ Interval Graphs:

• If every vertex can be represented as a finite interval & two vertices are adjacent whenever the corresponding intervals overlap; that is, for every vertex x there exists an interval I_x and xy is an edge in G if $I_x \cap I_y \neq \phi$.



* create vertices for each interval.
* Create an edge if two intervals overlap

Interval graph:



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→ Edge Coloring:

*can be used in questions where we need to schedule.

- assignment of colors to ~~ver~~ edges such that two edges have different colors if they share a vertex
- denoted by $\chi'(G) \Rightarrow$ chromatic index

$\Downarrow$ minimum no. of colors required

- $\chi'(K_n) = n-1$ if n is even
- $\chi'(K_n) = n$ if n is odd

→ Vizing Theorem:

$$\underbrace{\Delta(G) \leq \chi'(G)}_{\text{Class 1 graphs}} \overbrace{\leq \Delta(G)+1}^{\text{Class 2 graphs. — regular graphs with odd vertices}}$$ for all simple graphs G.

bipartite

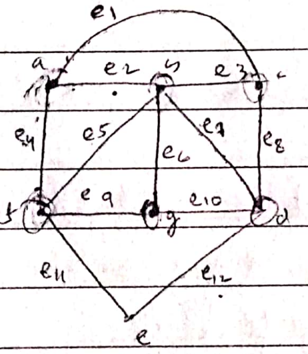
- greedy algorithm will produce an edge coloring with at most $2\Delta(G) - 1$ colors

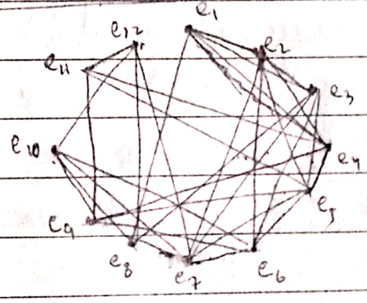
→ Relation between vertex & edge coloring:

→ Line Graph:

$G = (V, E)$, the line graph $L(G) = (V', E')$ is the graph formed from G where each vertex x' in $L(G)$ represents the edge x' from G and $x'y'$ is an edge of $L(G)$ if the edges x' and y' share an end point in G.

$$\boxed{\chi'(G) = \chi(L(G))}$$





* Technique: Pick a vertex from G. Note the adjacent ~~vertices~~ edges of that vertex. Draw a ~~com~~ clique on the corresponding edges in line graph.

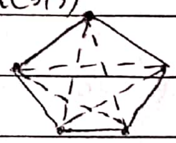
→ Ramsey Number:

- given positive integers m and n, the Ramsey number (m, n) is the minimum number of vertices r so that all simple graphs on r vertices contain either a clique of size m or independent set of size n.
  - often described as guests at a party.

R(3,3)

Prove $R(m,n) = r$ requires two steps
- first, we find an edge coloring of K_{r-1} without a red m-clique and without a blue n-clique.
- second, we must show that any edge coloring of K_r will either have a red m-clique, or a blue n-clique.

Determine R(3,3) Not possible for K_5.



It is possible for K_6

$R(3,3) = 6$.

- $R(n,m) = R(m,n)$
- $R(2,n) = n$

$R(4,4) = 18$

⇒ Color Variation:

→ ~~Online~~ On-line Coloring:
- vertices examined one at a time, in a linear manner.
- relies on "induced subgraph"

- Consider a graph G with the vertices ordered as $x_1 < x_2 < x_3 \ldots < x_n$. An on-line algorithm colors the vertices one at a time where the color for x_i depends on the induced subgraph $[x_1, x_2, \ldots x_i]$, which consists of the vertices up to and including x_i. The maximum number of colors a specific algorithm A uses on any possible ordering of the vertices is denoted $\chi_A(G)$.

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→ First-Fit Algorithm:

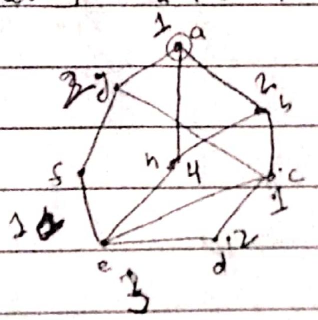
Input: Graph G with vertices ordered as $x_1 < x_2 < x_3 \dots < x_n$.

- Steps:
  - Assign x_1 color 1
  - Assign x_2 color 1 if x_1 and x_2 are not adjacent; otherwise assign x_2 color 2.
  - For all future vertices, assign x_i, the least number color available to x_i in $G[x_1, \dots, x_i]$ that is, give x_i the first color not used by any neighbor of x_i that has already been covered.

Example:

Apply First-Fit:

vertex ordering: $a < b < c < d < e < f < g < h$



⇒ Weighted Coloring:

- assigns each vertex a set of colors so that
  - i) the set consists of consecutive colors (or numbers)
  - ii) the number of colors assigned to vertex equals its weight.
  - iii) If two vertices are adjacent, their set of colors must be disjoint

* → List Coloring.

- Assign colors with added constraint that colors must come from a predefined list.

→ K-Choosability:

- If for every collection of lists, each of size k, a proper list coloring exists, then G is K-choosable. The minimum value for k for which G is K-choosable is called the choosability of G, denoted by ch(G).

→ $ch(G) \geq \chi(G)$

- Proof:

Let G $\chi(G) = k$ and give each vertex of G the list $\{1, 2, \ldots, k\}$ there is a proper coloring for G from these lists, namely one exhibited by the fact that $\chi(G) = k$. However, if we remove the same one element from each of the lists, then G cannot be colored since otherwise $\chi(G) < k$.

→ For any simple graph G,

$$ch \leq \Delta(G) + 1$$