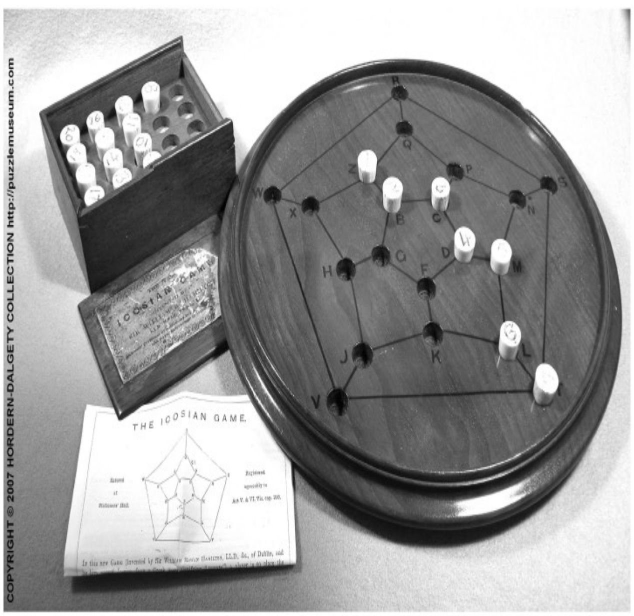
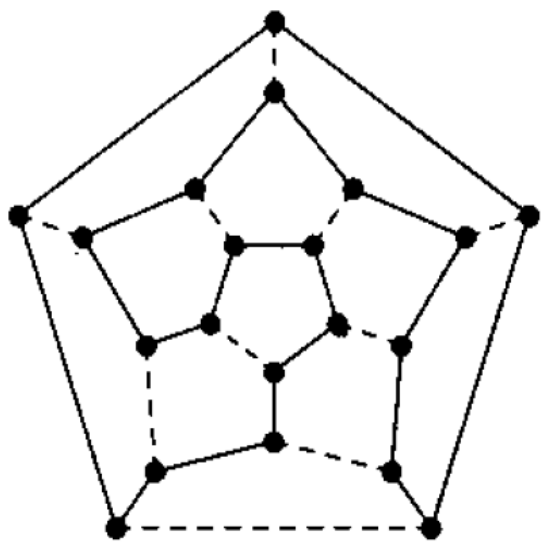
Hamiltonian Cycles:

- In 1857, Sir William Rowan Hamilton (1805{1865), who was then a Royal Astronomer of Ireland, invented a game called the Icosian Game.
- The game involved a regular dodecahedron on which the 20 vertices were labeled using the names of 20 cities in the world.
- The object of the game was to travel `Around the World' by finding a `walk' in the dodecahedron which visited each city once and exactly once, starting and terminating at the same city such that one player specifies a 5-vertex path and the other must extend it to a spanning cycle.
- A regular dodecahedron or pentagonal dodecahedron is a dodecahedron that is regular, which is composed of 12 regular pentagonal faces, three meeting at each vertex.



Copyright © 2014 Hordern-Dalgety Collection. http://puzzlemuseum.com



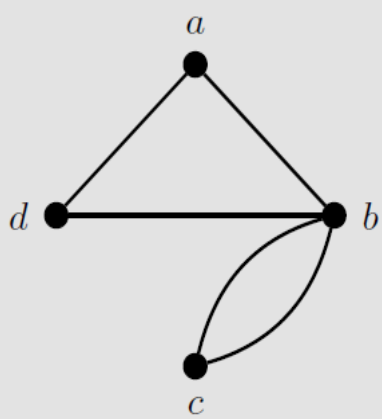
Definition 2.10 A cycle in a graph G that contains every vertex of G is called a ***hamiltonian cycle***. A path that contains every vertex is called a ***hamiltonian path***. A graph that contains a hamiltonian cycle is called ***hamiltonian***.

Remark: When comparing an Eulerian circuit with a Hamiltonian cycle, only one requirement has been lifted: instead of a tour containing every edge and every vertex, we are now only concerned with the vertices.

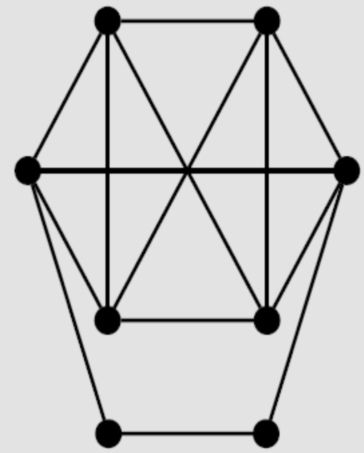
Question: When a graph has a Hamiltonian cycle and how to find an optimal, or near optimal, Hamiltonian cycle?

- For the past one hundred sixty years, numerous mathematicians have Searched for a solution to the Hamiltonian cycle problem; that is, what are the necessary and sufficient conditions for a graph to contain a Hamiltonian cycle?
- Recall that a necessary condition is a property that must be achieved in order for a solution to be possible and a sufficient condition is a property that guarantees the existence of a solution

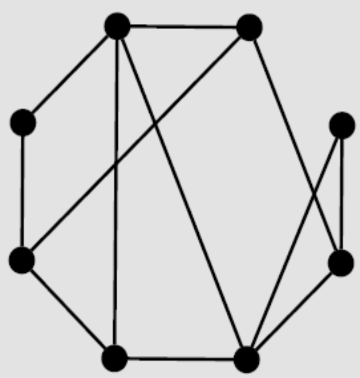
Example 2.8 For each of the graphs below, determine if they have hamiltonian cycles (and paths) and eulerian circuits (and trails).



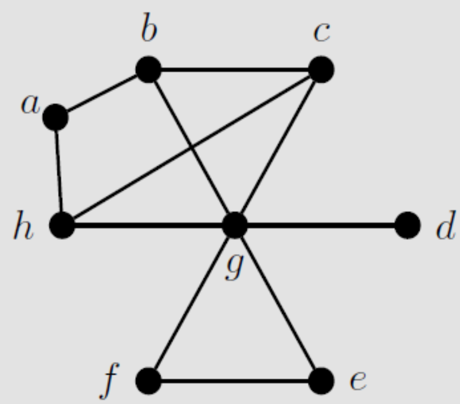
G_1



G_2



G_3



G_4

Solution: Since G_1 is connected and all vertices are even, we know it has an eulerian circuit. There is no hamiltonian cycle since we need to include c in the cycle and by doing so we have already passed through b twice, making it impossible to visit a and d.

Since G_2 is connected and all vertices are even, we know it is eulerian. Hamiltonian cycles and hamiltonian paths also exist. To find one such path, remove any one of edges from a hamiltonian cycle.

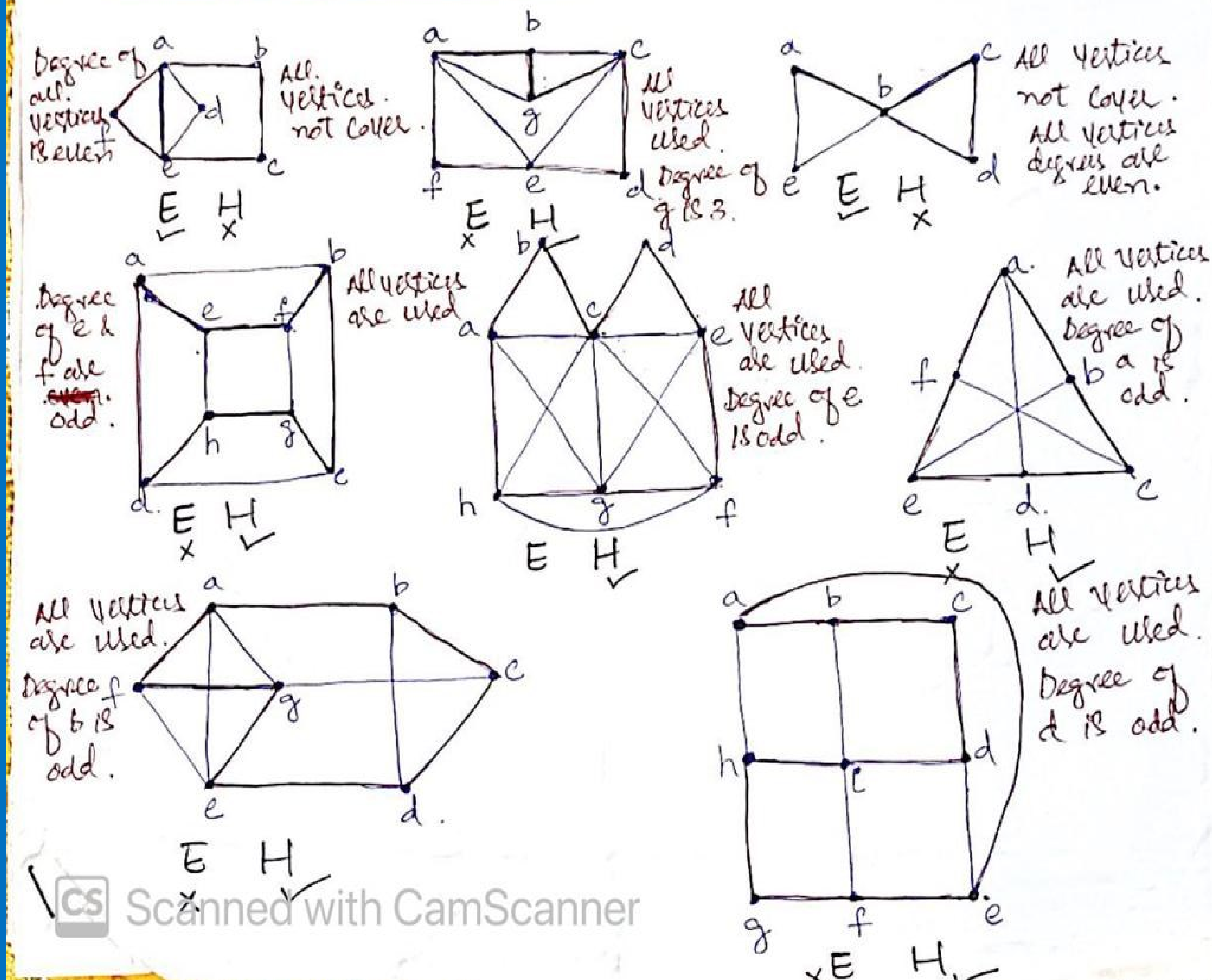
Some vertices of G_3 are odd, so we know it is not eulerian. Moreover, since more than two vertices are odd, the graph is not semi-eulerian. However, this graph does have a hamiltonian cycle (and so also a hamiltonian path). Can you find it?

Four vertices of G_4 are odd, we know it is neither eulerian nor semi-eulerian. This graph does not have a hamiltonian cycle since d cannot be a part of any cycle. Moreover, this graph does not have a hamiltonian path since any traversal of every vertex would need to travel through g multiple times.

Remark: Recall from Section 2.1.3 that if a graph has an eulerian circuit, then it cannot have an eulerian trail, and vice versa. The same is not true for the hamiltonian version.

- If a graph has a hamiltonian cycle, it automatically has a hamiltonian path (just leave of fthe last edge of the cycle to obtain a path).
- If a graph has a hamiltonian path, it may or may not have a hamiltonian cycle.

Examples of Eulerian & Hamiltonian Graphs:


Degree of all vertices is even
All vertices not cover.
E H
All vertices used.
Degree of g is 3.
E H
All vertices not cover.
All vertices degrees are even.
E H
Degree of e & f are ~~even~~ odd.
All vertices are used
E H
All vertices are used.
Degree of e is odd.
E H
All vertices are used.
Degree of a is odd.
E H
All vertices are used.
Degree of f is odd.
E H
All vertices are used.
Degree of d is odd.
E H

Properties of Hamiltonian Graphs

(1) G must be connected.

(2) No vertex of G can have degree less than 2.

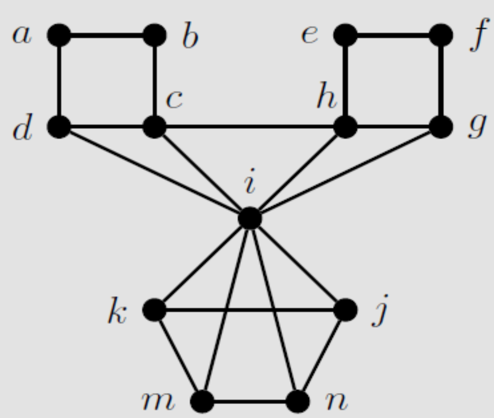
(3) G cannot contain a ***cut-vertex***, that is a vertex whose removal disconnects the graph.

(4) If G contains a vertex x of degree 2 then both edges incident to x must be included in the cycle.

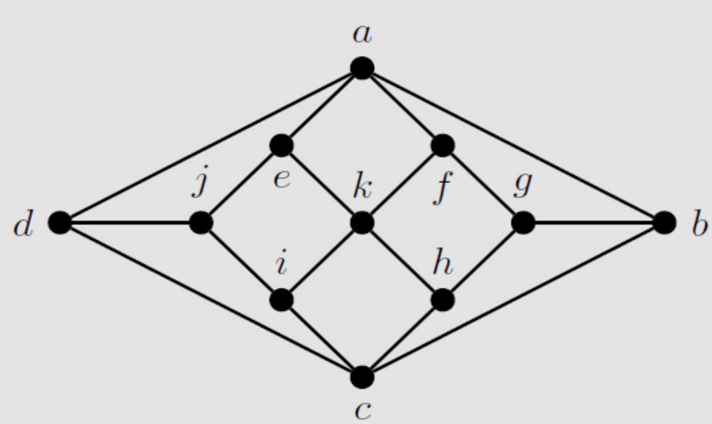
(5) If two edges incident to a vertex x must be included in the cycle, then all other edges incident to x cannot be used in the cycle.

(6) If in the process of attempting to build a hamiltonian cycle, a cycle is formed that does not span G, then G cannot be hamiltonian.

Example 2.9 Use the properties listed above to show that the graphs below are not hamiltonian.

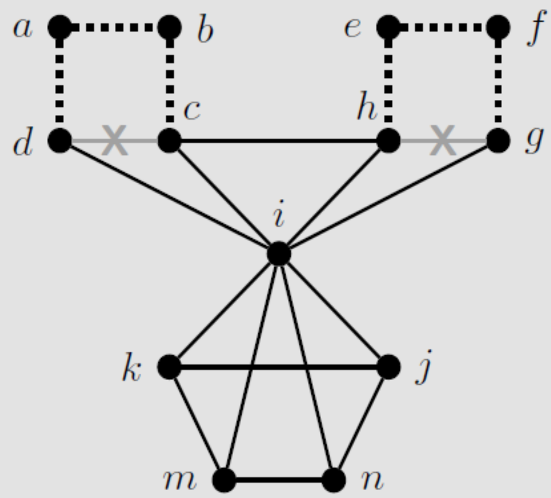


G_5

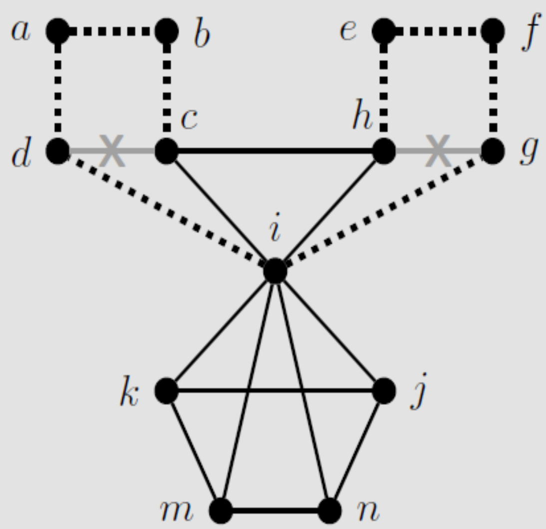


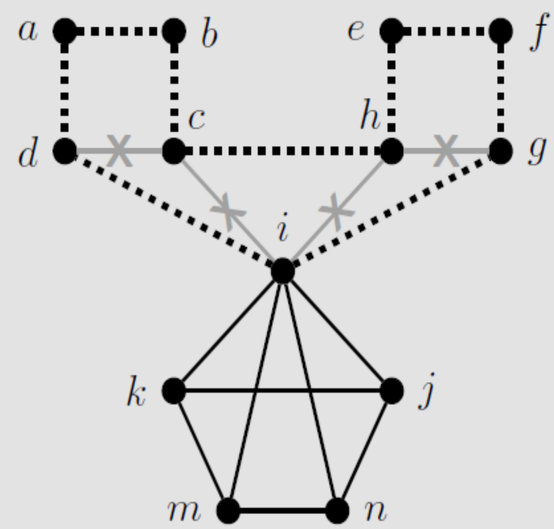
G_6

Solution: For G_5, notice that vertices a, b, e, and f all have degree 2 and so all the edges incident to these vertices must be included in the cycle. But then edges cd and hg cannot be a part of a cycle since they would create smaller cycles that do not include all of the vertices in G_5.

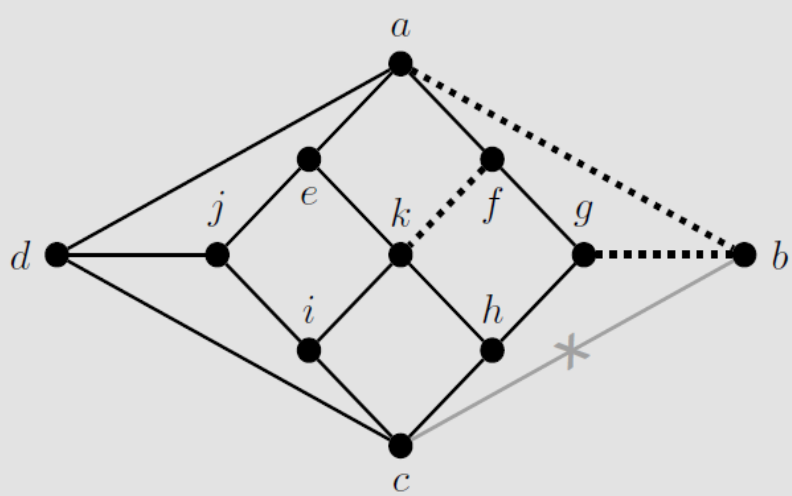


Since only one edge remains incident to d and g, namely di and gi, then these must be a part of the cycle. But in doing so, we would be forced to use ch since the other edges incident to i could not be chosen. This creates a cycle that does not span G_5, and so G_5 is not hamiltonian.

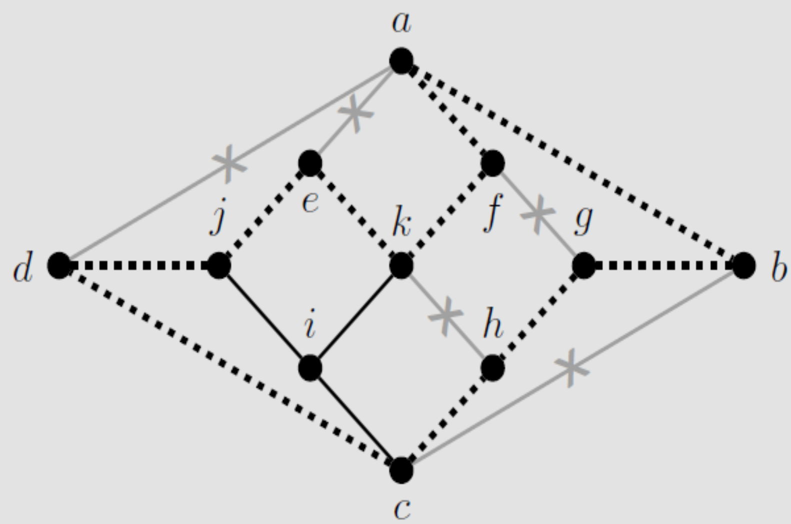




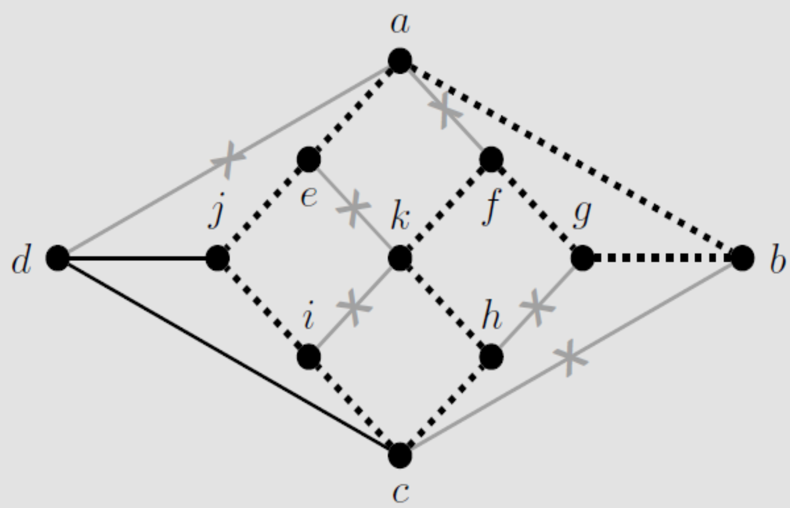
For G_6, we know we cannot use all of the edges ab, ad, bc, and bd, as these four edges together create a cycle that does not span the graph. Since either b or d must have an edge not from this list, by symmetry we will choose bg to be a part of the hamiltonian cycle C we are attempting to build. Then either ab or bc must be the other edge incident to b in the cycle C, and again using symmetry we can choose ab. Now, we cannot use both af and fg, since together with ab and bg we would have a cycle that does not span G_6. Thus we must choose fk to be a part of the cycle C.



At this point we will consider two scenarios based on the last edge incident to f in C. For the first case, we choose to add af to C. Then we cannot include fg and so gh must also be a part of the cycle since it is the only edge left incident to g. We cannot use hk as it would close the cycle before it spans G_6, and so ch must also be a part of the cycle C. Since a already has two incident edges in the cycle, we cannot use either of ae or ad. Thus the other edges incident to d and e must be a part of the cycle, namely dj, dc, ej, and ek. But this closes the cycle without including vertex i.



Thus af cannot be a part of a hamiltonian cycle C (if such a cycle exists). Thus fg must be chosen and we cannot use gh since g already has two incident edges as a part of the cycle. Thus the other edges incident to h must be used, namely hk and hc, and so the other edges incident to k cannot be, namely ke and ki. Based on the edges remaining, all of ci, ij, ej, and ea would be required, which closes a cycle that does not include d.



Since neither option produces a spanning cycle of G_6, we know it is not hamiltonian.

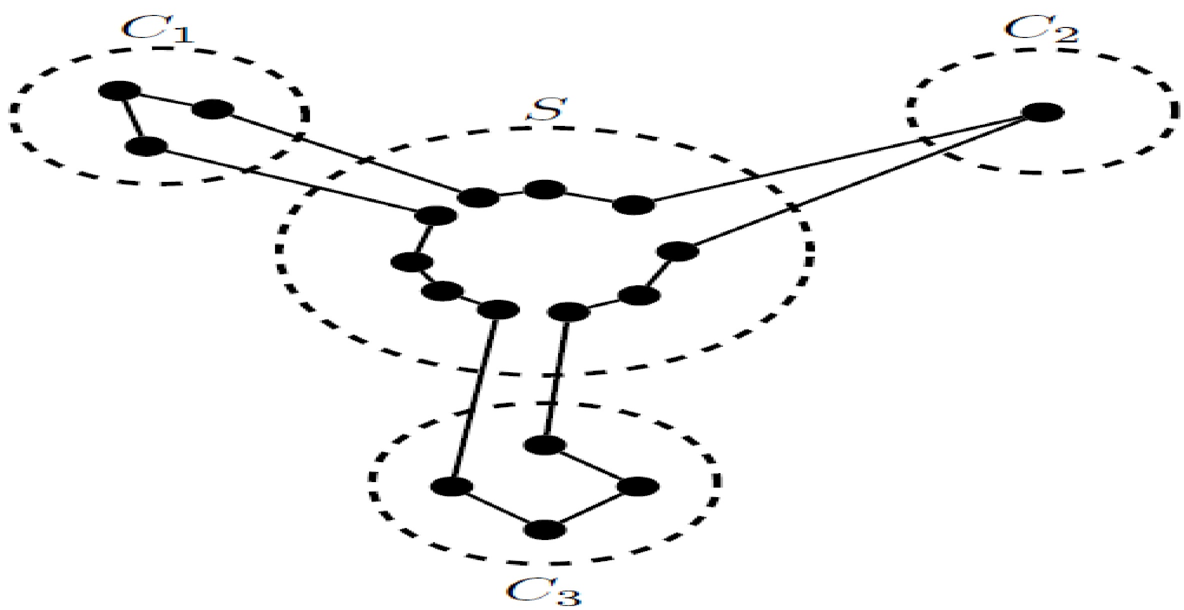
Result: **If G is Hamiltonian then any subgraph G ', $G \subseteq G'$ where G' is obtained by adding new edges between non-adjacent vertices of G then G' is also Hamiltonian.**

Example: **Clearly C_n is Hamiltonian, C_n=G and K_n=G' then K_n is Hamiltonian.**

NECESSARY CONDITION:

Proposition 2.11 If G is a graph with a hamiltonian cycle, then $G - S$ has at most $|S|$ components for every nonempty set $S \subseteq V$.

Proof: Assume G has a hamiltonian cycle C and S is a nonempty subset of vertices. Then as we traverse C we will leave and return to S from distinct vertices in S since no vertex can be used more than once in a cycle. Since C must span $V(G)$, we know S must have at least as many vertices as the number of components of $G - S$.



Proof: When leaving a component of $G - S$, a Hamiltonian cycle can go only to S, and the arrivals in S must use distinct vertices of S. Hence S must have at least as many vertices as $G - S$ has components. ∎

7.2.4. Definition. Let $c(H)$ denote the number of components of a graph H.

Thus the necessary condition is that $c(G - S) \leq |S|$ for all $\emptyset \neq S \subseteq V$. This condition guarantees that G is 2-connected (deleting one vertex leaves at most one component), but it does not guarantee a Hamiltonian cycle.

Example 5.2.4. Consider the graph G of Fig. 5.1.5(a). We have proven in Example 5.1.4 that G is not Hamiltonian. Let us now apply Theorem 5.2.2 to draw the same conclusion.

Let $S = \{a, c, x, z\}$. Then $G - S$ is shown in Fig. 5.2.2.

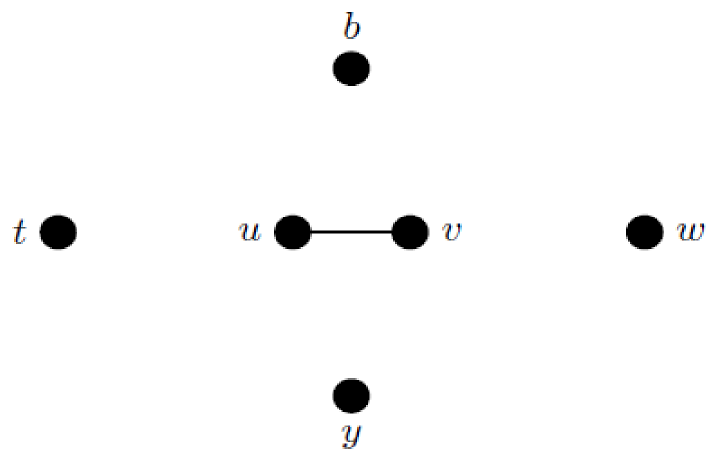
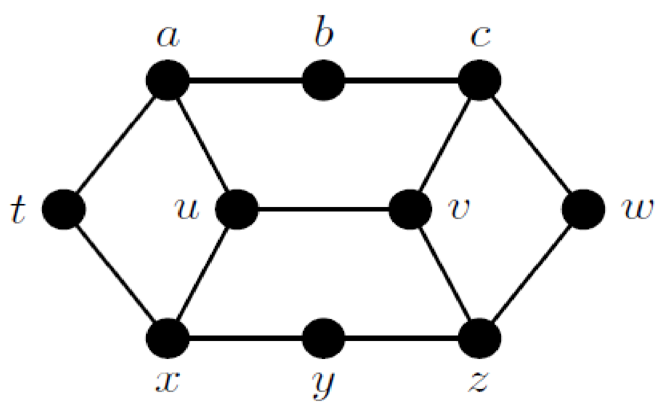


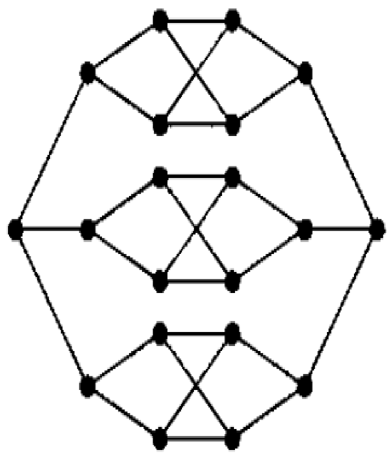
Fig. 5.2.2

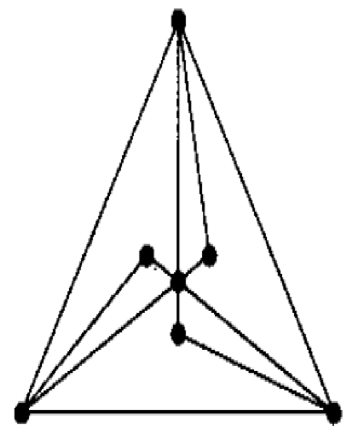
Check that $|S| = 4$ and $c(G - S) = 5$, and we have $c(G - S) > |S|$. We thus conclude by Theorem 5.2.2 that G is not Hamiltonian.



(a)

7.2.5. Example. The graph on the left below is bipartite with partite sets of equal size. However, it fails the necessary condition of Proposition 7.2.3. Hence it is not Hamiltonian.





The graph on the right shows that the necessary condition is not sufficient. This graph satisfies the condition but has no spanning cycle. All edges incident to vertices of degree 2 must be used, but in this graph that requires three edges incident to the central vertex.

SUFFICIENT CONDITIONS:

Theorem 2.12 (Dirac's Theorem) Let G be a graph with $n \geq 3$ vertices. If every vertex of G satisfies $\deg(v) \geq \frac{n}{2}$, then G has a hamiltonian cycle.

Remark: Dirac's Theorem relies on the high degree at every vertex of a graph in order to guarantee a Hamiltonian cycle, yet this isn't quite necessary.

Theorem 2.13 (Ore's Theorem) Let G be a graph with $n \geq 3$ vertices. If $\deg(x) + \deg(y) \geq n$ for all distinct nonadjacent vertices, then G has a hamiltonian cycle.

Remarks:

- Notice that Ore's Theorem is only concerned with the sum of the degrees of nonadjacent pairs of vertices: if this sum is large enough then we can guarantee a Hamiltonian cycle, but if the sum is too small, we cannot make any conclusion about the graph.
- If G is known to be Hamiltonian, then adding edges to G cannot destroy the existence of a Hamiltonian cycle, it can only make it easier to find one.
- So, if G is Hamiltonian, then so must be G + e for some edge e.
- However, the converse is not true: removing an edge from a Hamiltonian graph can create a graph that is not Hamiltonian

What we will see below is that if we are careful with which edges, we choose to add to G, and if the resulting graph is Hamiltonian, then G itself will be Hamiltonian.

QUESTIONS:

1. Determine whether the following graphs are Hamiltonian.

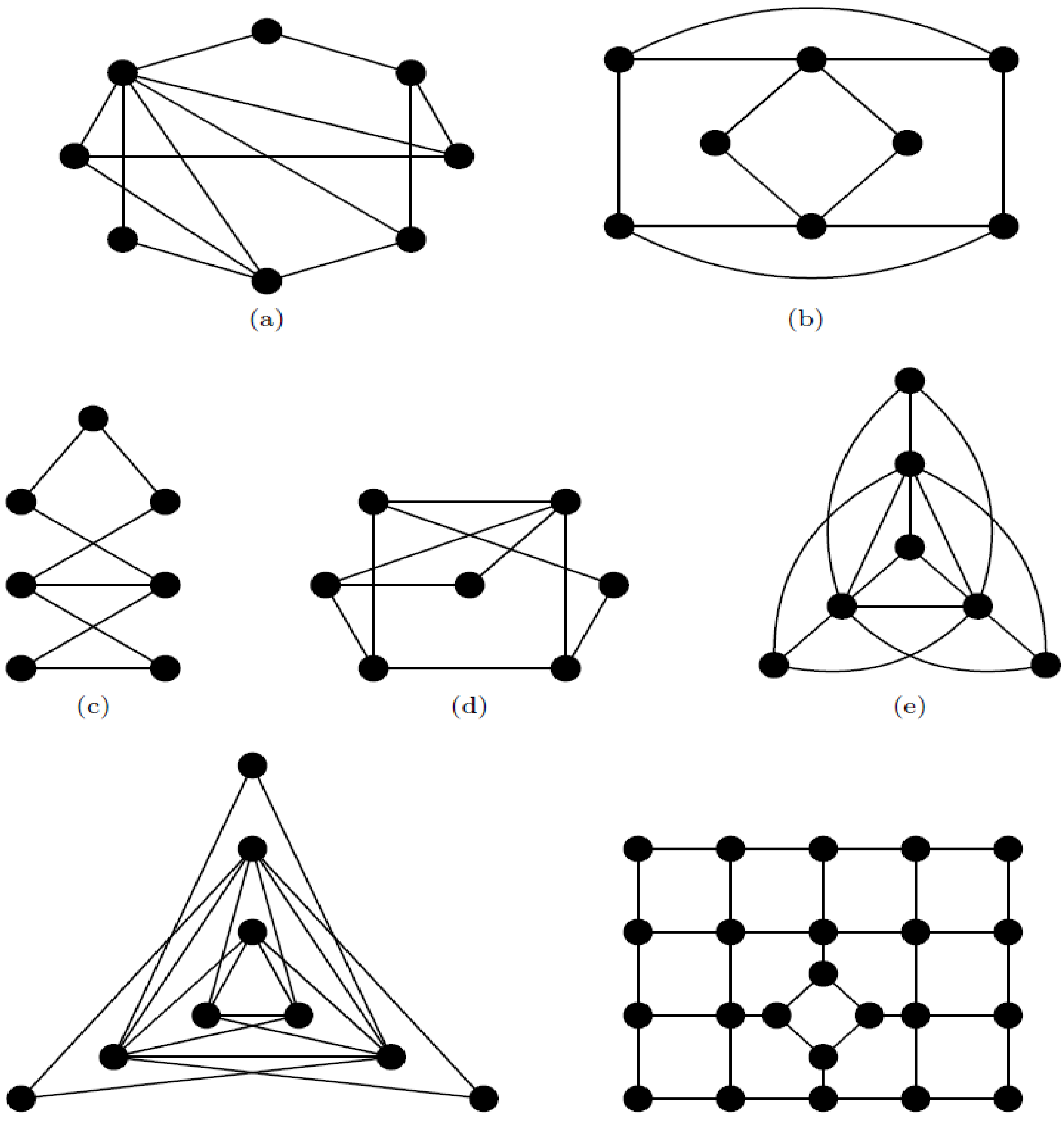
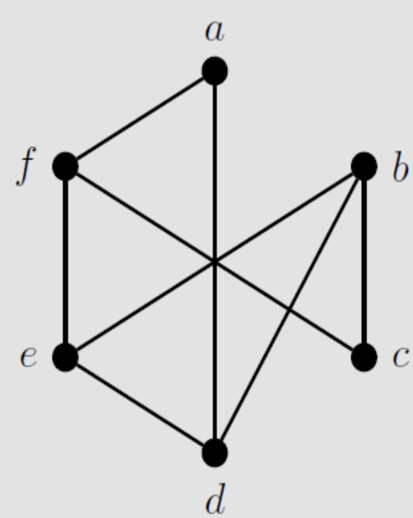


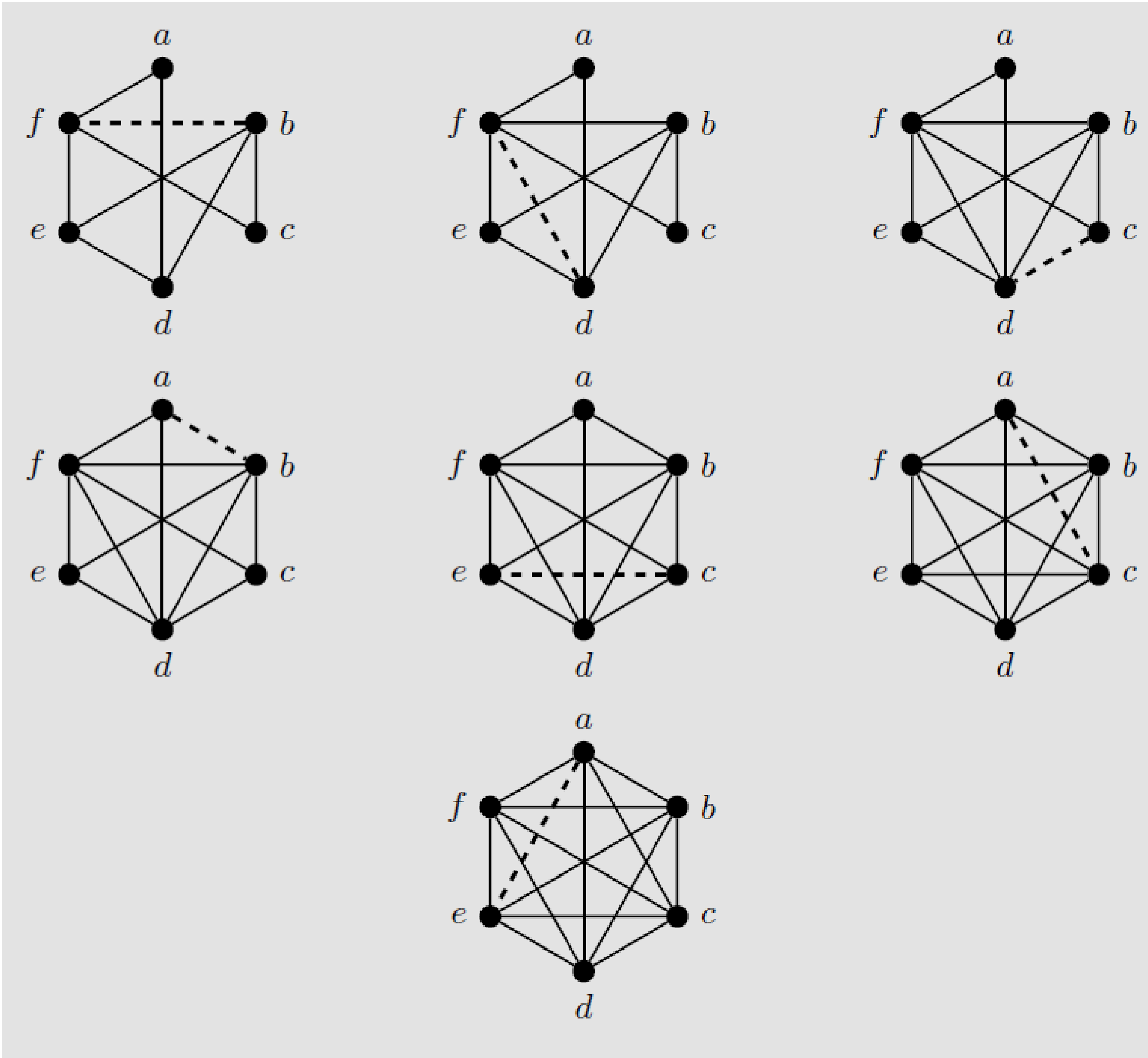
Fig. 5.2.3

Definition 2.14 The ***hamiltonian closure*** of a graph G, denoted $cl(G)$, is the graph obtained from G by iteratively adding edges between pairs of nonadjacent vertices whose degree sum is at least n, that is we add an edge between x and y if $\deg(x) + \deg(y) \geq n$, until no such pair remains.

Example 2.10 Find a hamiltonian closure for the graph below.



Solution: First note that the degrees of the vertices are $2, 3, 2, 3, 3, 3$ (in alphabetic order). Since the closure of G is formed by putting an edge between nonadjacent vertices whose degree sum is at least 6, we must begin with edge bf or df. We chose to start with bf. In the sequence of graphs shown on the next page, the edge added at each stage is a thick dashed line.



- ✓ Note that in the example above we get cl(G) = K6. While the process of finding the Hamiltonian closure can end in a complete graph.

Theorem 2.15 The closure of G is well-defined.

Theorem 2.16 A graph G is hamiltonian if and only if its closure $cl(G)$ is hamiltonian.

Lemma 2.17 If G is a graph with at least 3 vertices such that its closure $cl(G)$ is complete, then G is hamiltonian.

Theorem 2.18 Let G be a simple graph where the vertices have degree $d_1, d_2, \ldots, d_n$ such that $n \geq 3$ and the degrees are listed in increasing order. If for any $i < \frac{n}{2}$ either $d_i > i$ or $d_{n-i} \geq n-i$, then G is hamiltonian.

Conclusion:

- As should not be obvious, the question of whether a graph is Hamiltonian is non-trivial and can be quite difficult to answer for even moderate-sized graphs.
- We now turn to a class of graphs known to contain (many!) Hamiltonian cycles, where instead of questioning if the graph is Hamiltonian, we focus on finding an optimal Hamiltonian cycle.

ALGORITHMS TO FIND HAMILTONIAN CYCLES:

1. Brute Force Algorithm:

To find the Hamiltonian cycle of least total weight, one obvious method is to find all possible Hamiltonian cycles and pick the cycle with the smallest total. The method of trying every possibility to find an optimal solution is referred to as an exhaustive search, or use of the Brute Force Algorithm. This method can be used for any number of problems.

Brute Force Algorithm

Input: Weighted complete graph K_n.

Steps:

1. Choose a starting vertex, call it v.
2. Find all hamiltonian cycles starting at v. Calculate the total weight of each cycle.
3. Compare all $(n-1)!$ cycles. Pick one with the least total weight. (Note: there should be at least two options).

Output: Minimum hamiltonian cycle.

- ✓ Although both K3 and K4 contain Hamiltonian cycles, the first graph with some complexity is K5.

Example 2.11 (pg. # 82): (H.W)

Nearest Neighbor Algorithm:

If you do not have time to run Brute Force, how would you find a good Hamiltonian cycle? You could begin by taking the edge to the closest "vertex from your starting location, that is the edge of the least weight. And then? May be move to the closest vertex from your new location?

- ✓ This strategy is called the Nearest Neighbor Algorithm.

Nearest Neighbor Algorithm

Input: Weighted complete graph K_n.

Steps:

1. Choose a starting vertex, call it v. Highlight v.
2. Among all edges incident to v, pick the one with the smallest weight. If two possible choices have the same weight, you may randomly pick one.
3. Highlight the edge and move to its other endpoint u. Highlight u.
4. Repeat Steps (2) and (3), where only edges to unhighlighted vertices are considered.
5. Close the cycle by adding the edge to v from the last vertex highlighted. Calculate the total weight.

Output: hamiltonian cycle.

Example 2.12 (pg. # 87): (H.W)

Repetitive Nearest Neighbor:

By using a different starting vertex, the Nearest Neighbor Algorithm may identify a new Hamiltonian cycle, which may be better or worse than the initial cycle. Instead of only considering the circuit starting at the chosen vertex, we will run Nearest Neighbor with each of the vertices as a starting point. This is called Repetitive Nearest Neighbor.

Repetitive Nearest Neighbor Algorithm

Input: Weighted complete graph K_n.

Steps:

1. Choose a starting vertex, call it v.
2. Apply the Nearest Neighbor Algorithm.
3. Repeat Steps (1) and (2) so each vertex of K_n serves as the starting vertex.
4. Choose the cycle of least total weight. Rewrite it with the desired reference point.

Output: hamiltonian cycle.

Example 2.13 (pg. # 89): (H.W)