Power Series

Definition

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$$

is called a power series.

More generally an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0)^1 + a_2 (x - x_0)^2 + \cdots a_n (x - x_0)^n + \cdots$$

is called a power series ~~is centered at~~

at x_0 where x_0 is a constant.

There exists a constant R for which the power series converges such that

$$|x - x_0| < R$$

or $-R < x - x_0 < R$

$\Rightarrow -R + x_0 < x < x_0 + R$

$\Rightarrow x \in (-R + x_0, R + x_0)$

R is radius of convergence and $(-R + x_0, x_0 + R)$ is interval of convergence.

Example $\sum_{n=0}^{\infty} 3(x-2)^n$

Solution $a_n = 3(x-2)^n$ (i) $|x-2| < 1$

$a_{n+1} = 3(x-2)^{n+1}$ $-1 < x-2 < 1$

$$\frac{a_{n+1}}{a_n} = \frac{3(x-2)^{n+1}}{3(x-2)^n}$$

$1 < x < 3$

$= x - 2$

(ii) $|x-2| > 1$

$$\left|\frac{a_{n+1}}{a_n}\right| = |x-2|$$

then $x-2>1$

or $x-2 < -1$

$\Rightarrow x>3$ or $x<1$

$(-\infty, 1) \cup (3, \infty)$

Apply limit:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}|x-2|$$

$= |x-2| = L$

So the given series converges if $L < 1$ and diverges if $L > 1$

$\Rightarrow$ The given series converges if

$|x-2| < 1$

and diverges if $|x-2| > 1$

So the given series convergent on $(1-3)$ and divergent on $(-\infty, 1) \cup (3, \infty)$

At $x = 1$

$$\sum_{n=0}^{\infty} 3(1-2)^n = 3(-1)^n = 3\sum_{n=0}^{\infty}(-1)^n$$

which is divergent

At $x = 3$

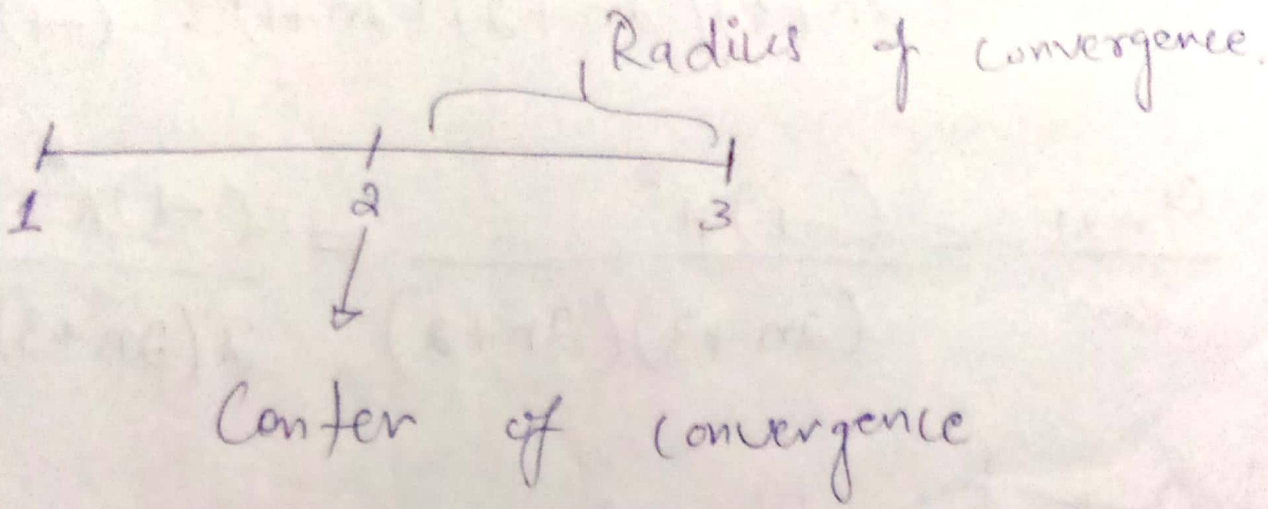
$$\sum_{n=0}^{\infty} 3(3-2)^n = 3(1)^n = 3\sum_{n=0}^{\infty}(1)^n$$

$$= 3(1+1+1+\cdots)$$

which is divergent.

So interval of convergence is $(1, 3)$ where interval of divergence is $(-\infty, 1] \cup [3, \infty)$

Radius of convergence is 1



Example 2 Find radius of convergence and interval of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Sol

$$a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$a_{n+1} = \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}$$

$$a_{n+1} = \frac{(-1)^{n+1}\, x^{2n+3}}{(2n+3)(2n+2)(2n+1)!} \cdot \frac{(2n+1)!}{(-1)^n\, x^{2n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(-1)x^2}{(2n+3)(2n+2)} = \frac{(-1)x^2}{2(2n+3)(n+1)}$$

$\cancel{\left|\frac{a_{n+1}}{a_n}\right| = |x-2|}$ ~~then~~ $\cancel{x-2 > 1}$ ~~or~~ $\cancel{x-2 < -1}$ $\cancel{\Rightarrow}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^2}{2(2n+3)(n+1)}$$

Apply limit:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = x^2 \lim_{n\to\infty} \frac{1}{2(2n+3)(n+1)}$$

$$= x^2 \times 0$$

So the given series is convergent everywhere.

<u>Interval of convergence</u>

$(-\infty, \infty)$

Radius $= \infty$

Solution about Ordinary Point

Analytic function

A function is said to be analytic at $x = x_0$ if it can be expressed in the form of power series at $x = x_0$

e.g. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin x$ is analytic at $x = 0$

e^x is also analytic ftn

$f(x) = \frac{x^2+1}{(x^2-1)}$ is analytic except at $x = \pm 1$

All polynomials are analytic.

Ordinary point and Singular point

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \longrightarrow (1)$$

A point $x = x_0$, if $P(x)$ and $Q(x)$ both are analytic at $x = x_0$ then $x = x_0$ is ordinary point of DE (1).

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

A point $x = x_0$ is singular point if it is not ordinary point.

Theorem (Existence of power series solution)

If $x = x_0$ is an ordinary point of DE $y'' + P(x)y' + Q(x)y = 0$ then there always exists two linearly independent solutions in the form of power series.

Solve $(x^2+1)y'' + xy' - y = 0 \longrightarrow (1)$

$$\Rightarrow y'' + \frac{x}{(x^2+1)}y' - \frac{1}{(x^2+1)}y = 0$$

$$P(x) = \frac{x}{(x^2+1)}, \quad Q(x) = \frac{-1}{x^2+1}$$

$P(x)$ is undefined if $x^2+1=0$

$$\Rightarrow x = \pm i$$

So $p(x)$ is analytic everywhere on the real line similarly $Q(x)$.

Here realy $x=0$ is an ordinary point.

So let us suppose

$$y = \sum_{n=0}^{\infty} C_n x^n \longrightarrow (2)$$

$$y' = \sum_{n=0}^{\infty} n C_n x^{n-1} = \sum_{n=1}^{\infty} n C_n x^{n-2} \longrightarrow (3)$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) C_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} \longrightarrow (4)$$

Substituting (2) - (4) into (1)

we have

$$(x^2+1)y'' + xy' - y = 0$$

$$\Rightarrow (x^2+1) \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + x \sum_{n=1}^{\infty} n C_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) \underbrace{C_n x^n}_{n=k} + \sum_{n=2}^{\infty} n(n-1) \underbrace{C_n x^{n-2}}_{n-2=k}$$

$$+ \sum_{n=1}^{\infty} n \underbrace{C_n x^n}_{n=k} - \sum_{n=0}^{\infty} \underbrace{C_n x^n}_{n=k} = 0$$

$$\Rightarrow \sum_{k=2}^{\infty} k(k-1) C_k x^k + \sum_{k+\cancel{2}=\cancel{2}}^{\infty} (k+2)(k+1) C_{k+2} x^k$$

$$+ \sum_{k=1}^{\infty} k C_k x^k - \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) C_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n$$

$$+ \sum_{n=1}^{\infty} n C_n x^n - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$- C_0 - C_1 x - \sum_{n=2}^{\infty} C_n x^n + C_1 x + \sum_{n=2}^{\infty} n C_n x^n$$

$$+ 2C_2 + 6C_3 x + \sum_{n=2}^{\infty} n(n+2)(n+1) C_{n+2} x^n$$

$$+\sum_{n=2}^{\infty} n(n-1)C_n x^n = 0$$

$$\Rightarrow (2C_2 - C_0) + 6C_3 x + \sum_{n=2}^{\infty}\left[n(n-1)C_n + (n+2)(n+1)C_{n+2} + nC_n - C_n\right]x^n = 0$$

$$\Rightarrow (2C_2 - C_0) + 6C_3 x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)C_{n+2} + (n^2 - \cancel{n} + \cancel{n} - 1)C_n\right]x^n = 0 \longrightarrow \text{(A)}$$

Now using identity property:

$$\sum_{n=0}^{\infty} C_n (x - x_0)^n = 0 \quad \text{then}$$

$$C_n = 0$$

So we have

$$2C_2 - C_0 = 0 \longrightarrow \text{(i)}$$

$$6C_3 = 0 \longrightarrow \text{(ii)}$$

$$C_{n+2} = -\frac{(n^2-1)C_n}{(n+1)(n+2)}, \quad n \geq 2 \longrightarrow \text{(iii)}$$

eq(i) $\Rightarrow C_2 = \frac{C_0}{2} \longrightarrow (*)$

eq(ii) $\Rightarrow C_3 = 0 \longrightarrow (**)$

for $n=2$ (iii) $\Rightarrow$

$$C_4 = \frac{-(3)C_2}{2(3)(4)} = -\frac{1}{8}C_2 = \frac{-C_0}{16}$$

$$n=3 \Rightarrow C_5 = -\frac{2}{5}C_3 = 0$$

$$n=4 \Rightarrow C_6 = \frac{3}{2^3(2!)}C_0 = \frac{3}{16}C_0$$

$$n=5 \quad C_7 = -\frac{4}{7}C_5 = 0$$

$$n=6, \; C_8 = \frac{-1\cdot 3\cdot 5}{2^3 \; 4!}C_0 = \frac{-15}{8\times(24)}C_0$$

and so on.

$$y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + \cdots$$

Scanned with CamScanner

(15)

$$y = C_0 + C_1 x + \frac{C_0}{2} x^2 + (0)x^3 - \frac{C_0}{16} x^4 + (0)x^5 + \frac{3}{16} C_0 x^6 + (0)x^7 + \cdots$$

$$y = C_1 x + C_0\left(1 + \frac{x^2}{2} - \frac{x^4}{16} + \frac{3x^6}{16} + \cdots\right)$$

$$y = C_1 y_1(x) + C_2 y_2(x)$$

$$y_1(x) = x,$$

$$y_2(x) = 1 + \frac{x^2}{2} - \frac{x^4}{16} + \frac{3x^6}{16} + \cdots$$

Solve $y'' + xy = 0 \longrightarrow (1)$

Here $x = 0$ is ordinary point

So let $y = \sum_{n=0}^{\infty} C_n x^n \longrightarrow (2)$

(16)

$$y' = \sum_{n=0}^{\infty} n C_n x^{n-1} = \sum_{n=1}^{\infty} n C_n x^{n-1}$$

$$y'' = \sum_{n=1}^{\infty} n(n-1) C_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

Substituting into (1) we have

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + x \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow \underbrace{\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}}_{n-2=k} + \underbrace{\sum_{n=0}^{\infty} C_n x^{n+1}}_{n+1=k} = 0$$

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) C_{k+2} x^k + \sum_{k-1=0}^{\infty} C_{k-1} x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) C_{k+2} x^k + \sum_{k=1}^{\infty} C_{k-1} x^k = 0$$

$$\Rightarrow 2C_2 + \sum_{k=1}^{\infty}(k+2)(k+1)C_{k+2}x^k + \sum_{k=1}^{\infty} C_{k-1}x^k = 0$$

$$\Rightarrow 2C_2 + \sum_{k=1}^{\infty}\left[(k+1)(k+2)C_{k+2} + C_{k-1}\right]x^k = 0 \quad (2)$$

<u>Identity property</u>

if $\sum_{n=0}^{\infty} C_n(x-a)^n = 0$

then for all n we have $C_n = 0$

So invoking identity property we have

$$2C_2 = 0 \Rightarrow \boxed{C_2 = 0}$$

$$(k+1)(k+2)C_{k+2} + C_{k-1} = 0$$

$$\Rightarrow C_{k+2} = -\frac{C_{k-1}}{(k+1)(k+2)}, \quad k = 1, 2, 3, \ldots \quad (3)$$

For $k = 1$ $(3) \Rightarrow$

$$C_3 = -\frac{C_0}{2 \cdot 3}$$

For $k = 2$, $(3) \Rightarrow C_4 = \frac{C_1}{3 \cdot 4}$

For $k = 3$, $(3) \Rightarrow C_5 = -\frac{C_2}{(4 \cdot 5)} = 0$

For $k = 4$

$$C_6 = \frac{-C_3}{5 \cdot 6} = \frac{C_0}{2 \cdot 3 \cdot 5 \cdot 6}$$

For $k = 5$

$$C_7 = \frac{C_4}{7 \cdot 8} = \frac{-C_1}{3 \cdot 4 \cdot 7 \cdot 8}$$

For $k = 6$, $C_8 = 0$

For $k = 7$, $C_9 = \frac{C_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}$

For $k=8$, $C_{10} = \frac{C_1}{3\cdot4\cdot6\cdot7\cdot9\cdot10}$

$$y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3$$

$$+ C_4 x^4 + C_5 x^5 + C_6 x^6 + C_7 x^7$$

$$+ C_8 x^8 + C_9 x^9 + C_{10} x^{10} + \cdots$$

$$y = C_0 + C_1 x - \frac{C_0}{(2\cdot3)} x^3 + \frac{C_1}{3\cdot4} x^4$$

$$+ x^6 \left(\frac{C_0}{2\cdot3\cdot5\cdot6}\right) + \left(-\frac{C_1}{3\cdot4\cdot7\cdot8}\right) x^7$$

$$- \frac{C_0}{2\cdot3\cdot5\cdot6\cdot8\cdot9} x^9 + \frac{C_1}{3\cdot4\cdot6\cdot7\cdot9\cdot10} x^{10}$$

$$+ \cdots$$

$$y = C_0 \Bigg(\underbrace{1 - \frac{x^3}{2\cdot3} + \frac{x^6}{2\cdot3\cdot5\cdot6} - \frac{x^9}{2\cdot3\cdot5\cdot6\cdot9} + \cdots}_{y_1(x)} \Bigg)$$

20

$$+c_1\left(x - \frac{x^7}{3\cdot4\cdot7\cdot8} + \frac{x^4}{3\cdot4} - \frac{x^{10}}{3\cdot4\cdot6\cdot7\cdot9\cdot10} + \cdots\right)$$

$y_2(x)$

$$\boxed{y = c_0\, y_1(x) + c_1\, y_2(x)}$$