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Cauchy-Euler Equation⇒

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x). \longrightarrow (1)$$

For example

$$3x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = 9$$

$$3x \frac{dy}{dx} + y = 3x$$

$$x^3 \frac{d^3y}{dx^3} + 4x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + y = \sin x.$$

Method of solution ⇒

$$a_2 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = 0. \rightarrow (2)$$

Let $y = x^m$

$$\frac{dy}{dx} = m x^{m-1}, \quad \frac{d^2y}{dx^2} = m(m-1) x^{m-2}$$

$$a_2 x^2 m(m-1)x^{m-2} + a_1 x m x^{m-1} + a_0 x^m = 0$$

$$\Rightarrow a_2 x^m m(m-1) + a_1 m x^m + a_0 x^m = 0$$

$$\Rightarrow (a_2 m(m-1) + a_1 m + a_0) x^m = 0.$$

Since $x^m \neq 0$

$$\rightarrow (a_2 m(m-1) + a_1 m + a_0) = 0$$

This equation is called (3) auxillary equation or charecteristic equation for the DE (2).

Case (1) $m_1 \neq m_2$, (real roots)

then $y = c_1 x^{m_1} + c_2 x^{m_2}$

Case (2), $m = m_1 = m_2$; equal roots

$$y = c_1 x^m + c_2 x^m \ln x$$

Case (3) $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then

$$y = c_1 x^\alpha \cos(\beta \ln x) + c_2 x^\alpha \sin(\beta \ln x).$$

Example: Solve $4x^2\frac{d^2y}{dx^2}+8x\frac{dy}{dx}+y=0$ — (1)

Solution: Suppose $y=x^m$ is a solution

then $\frac{dy}{dx}=mx^{m-1}$, $\frac{d^2y}{dx^2}=m(m-1)x^{m-2}$

$\Rightarrow 4x^2\left(m(m-1)x^{m-2}\right)+8x\,mx^{m-1}+x^m=0$

$\Rightarrow 4m(m-1)x^m+8mx^m+x^m=0$

$\Rightarrow \left(4m(m-1)+8m+1\right)x^m=0$

Thus, the auxillary equation is

$4m^2+4m+1=0$

$m=\frac{-4\pm\sqrt{16-4(4)(1)}}{2(4)}=\frac{-4\pm 0}{8}$

$m=-\frac{1}{2}$

$m_1=-\frac{1}{2},\ m_2=-\frac{1}{2}$

$y=c_1x^{-\frac{1}{2}}+c_2x^{-\frac{1}{2}}\ln x$. Ans.

Q2 Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0 \rightarrow (1)$

Sol: Let $y = x^m$, $y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

putting into (1) we get

$m(m-1)x^{m-2+2} - 2x\,mx^{m-1} + (-4)x^m = 0$

$\Rightarrow (m(m-1) - 2m - 4)x^m = 0$

$\Rightarrow m^2 - 3m - 4 = 0$

$m_1 = -1, \quad m_2 = 4$

So $y = c_1x^{-1} + c_2x^4$. Ans.

Q2. $4x^2y'' + 17y = 0$, $y(1) = -1$, $y'(1) = -\frac{1}{2}$

Let $y = x^m$, then $y' = mx^{m-1}$

& $y'' = m(m-1)x^{m-2}$

$\Rightarrow 4x^2 m(m-1)x^{m-2} + 17x^m = 0$

$\Rightarrow (4m(m-1) + 17)x^m = 0$

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Thus, the auxillary equation is

$$4m^2-4m+17=0$$

$$m=\frac{-(-4)\pm\sqrt{(-4)^2-4(4)(17)}}{2(4)}$$

$$m=\frac{4\pm\sqrt{16-(68)(4)}}{8}$$

$$m_1=\frac{1}{2}+2i, \quad m_2=\frac{1}{2}-2i.$$

As $m=\alpha\pm\beta i$

$$\alpha=\frac{1}{2}, \quad \beta=+2$$

So $y=c_1x^{1/2}\cos(2\ln x)+c_2x^{1/2}\sin(2\ln x) \longrightarrow (i)$

As $y(+1)=-1$

$$(-1)=c_1\cos(0)+c_2(+1)^{1/2}(0)$$

$$c_1=-1$$

$$y=(-1)x^{1/2}\cos(2\ln x)+c_2x^{1/2}\sin(2\ln x).$$

A

$$y'=-\frac{1}{2}x^{-1/2}\cos(2\ln x)+x^{1/2}\sin(2\ln x)\frac{1}{x^2}(2x)$$
$$+c_2\frac{1}{2}x^{-1/2}\cos(2\ln x)\frac{2x}{x^2}$$

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Now using $y''(1) = -\frac{1}{2}$

$C_2 = 0$

Thus, the solution is

$y = -x^{1/2} C_5 (2 \ln x)$. Ans//

Q. Solve

$$x^2y'' - 3xy' + 3y = 2x^4e^x. \longrightarrow (1)$$

The associated homogenous equation is

$$x^2y'' - 3xy' + 3y = 0, \longrightarrow (2)$$

Substitution of $y = x^m$, $y' = mx^{m-1}$ & $y'' = m(m-1)x^{m-2}$; in (2) yields

$$m(m-1)x^2x^{m-2} - 3xmx^{m-1} + 3x^m = 0$$

$$\Rightarrow \left(m(m-1) - 3m + 3\right)x^m = 0$$

$$\Rightarrow m^2 - 4m + 3 = 0, \longrightarrow (3)$$

is the auxillary quation.

$$(3) \Rightarrow m^2 - 3m - m + 3 = 0$$

$$\Rightarrow m(m-3) - 1(m-3) = 0$$

$$\Rightarrow (m-1)(m-3) = 0$$

$$\Rightarrow m_1 = 1, \; m_2 = 3$$

So $y_c = c_1x + c_2x^3$. $\longrightarrow (4)$

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Now transforming equation 1 into standard form we have:

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$$

So $f(x) = 2x^2e^x$.

Now $y_p = U_1(x)y_1(x) + U_2(x)y_2(x)$

$\Rightarrow$ $y_p = U_1(x)x + U_2(x)x^3 \longrightarrow$ 5

where

$U_1(x) = \int \frac{W_1}{W}dx, \longrightarrow$ 6

$U_2(x) = \int \frac{W_2}{W}dx. \longrightarrow$ 7

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3$$

$\Rightarrow$ $W = 2x^3$.

$$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = 0 - 2x^5e^x$$

$$W_1 = -2x^5e^x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x$$

So $$U_1(x) = \int \frac{(-2x^5e^x)}{2x^3}\,dx$$

$$= -\int x^2e^x\,dx = x^2e^x - \int (2x)e^x\,dx$$

$$= x^2e^x - 2\{xe^x - e^x\}$$

$$= x^2e^x - 2xe^x + 2e^x.$$

& $$U_2(x) = \int \frac{2x^3e^x}{2x^3}\,dx = \int e^x\,dx = e^x$$

Then $$y_p = (x^2e^x - 2xe^x + 2e^x)x + e^x x^3 = -2x^2e^x + 2xe^x$$

And so general solution is

$$y = y_c + y_p = c_1x + c_2x^3 + 2xe^x - 2x^2e^x.$$ Ans.

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Solve 2

$x^2y''-3xy'+3y=2x^4e^x \cdots \rightarrow$ (A)

The associated homogenous equation is

$x^2y''-3xy'+3y=0$

$y=x^m, \quad y'=mx^{m-1}, \quad y''=m(m-1)x^{m-2}$

$m_1=1, \quad m_2=3$

$y_c=c_1x+c_2x^3. \longrightarrow (i)$

Let $y_p=U_1(x)\,x+U_2(x)\,x^3. \rightarrow (ii)$

$U_1=\int\frac{W_1}{W}dx, \quad U_2=\int\frac{W_2}{W}dx$

Transforming (A) into standard form

$\Rightarrow \quad y''-\frac{3}{x}y'+\frac{3}{x^2}y=2x^2e^x$

$f(x)=2x^2e^x.$

$W=\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}, \quad W_2=\begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix}$

$W_1=\begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix},$

$$u_1 = -\int \underset{I}{x^2} \underset{II}{e^{x}}\, dx = -\left\{x^2 e^x - \int (2x) e^x dx\right\}$$

$$= -x^2 e^x + 2\left\{x e^x - e^x\right\}$$

$$= -x^2 e^x + 2x e^x - 2e^x$$

$$u_2 = \int e^x dx = e^x$$

$$\boxed{y_p = \left(-x^2 e^x + 2x e^x - 2e^x\right)x + e^x x^3}$$

$$y = y_c + y_p$$ Ans//