First Order LINEAR EQUATION

The general form of 1st order equation is

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \longrightarrow (1)$$

Transforming (1) into standard form:

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{f(x)}{a_1(x)}$$

$$\Rightarrow \frac{dy}{dx} + P(x)y = Q(x), \longrightarrow (2)$$

where $P(x) = \frac{a_0(x)}{a_1(x)}$, $Q(x) = \frac{f(x)}{a_1(x)}$. $\longrightarrow (3)$

Integrating factor $= \mu(x) = e^{\int P(x)dx}$

Multiplying Eq 2 both sides by integrating factor we have

$$e^{\int P(x)dx} \frac{dy}{dx} + P(x) e^{\int P(x)dy} y = e^{\int P(x)dx} Q(x)$$

$$\Rightarrow \frac{d}{dx}\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} Q(x), \rightarrow \text{Eq } 4$$

b/c $\frac{d}{dx} e^{\int P(x)dx} = P(x) e^{\int P(x)dx}$.

Integrating Eq. 4 both sides w.r.t x

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$$e^{\int p(x)dx} y = \int Q(x) e^{\int p(x)dx} dx + C$$

$$\Rightarrow y = e^{-\int p(x)dx} \int Q(x) e^{\int p(x)dx} + C e^{-\int p(x)dx} \longrightarrow (5)$$

Example: $\frac{dy}{dx} - 3y = 6 \longrightarrow (1)$

Comparing with standard form we get

$P(x) = -3$, so integrating factor is

$$\mu(x) = e^{\int p(x)dx} = e^{\int (-3)dx} = e^{-3x} \longrightarrow (2)$$

Multiplying Eq 1 both sides by 2 we get

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}$$

$$\Rightarrow \frac{d}{dx}\left[e^{-3x}y\right] = 6e^{-3x}$$

Integrating we have

$$e^{-3x}y = \int 6e^{-3x}\,dx$$

$$e^{-3x}y = \frac{6e^{-3x}}{(-3)} + C$$

$$\Rightarrow y = -2 + Ce^{3x}$$ Ans.

Solve $x\dfrac{dy}{dx} - 4y = x^6 e^x \longrightarrow (1)$

Transforming into standard form we have

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x, \longrightarrow (2)$$

Integrating factor is given by

$$\mu(x) = e^{\int P(x)dx} = e^{\int(-\frac{4}{x})dx} = e^{-4\ln x} = x^{-4}. \longrightarrow (3)$$

Multiplying (3) into both sides of (2)

we get

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^{x}$$

$$\Rightarrow \frac{d}{dx}\left[x^{-4}y\right] = xe^{x}$$, integrating we have

$$x^{-4}y = \int xe^{x}\,dx$$

$$\Rightarrow x^{-4}y = xe^{x} + e^{x} + C$$

$$\Rightarrow y = x^{5}e^{x} + x^{4}e^{x} + Cx^{4}$$ Ans.

EXACT EQUATIONS

Definition:

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an exact differential in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an exact equation if the expression on the left-hand side is an exact differential.

Theorem

Criterion for an Exact differential.

Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient Condition that $M(x,y)\,dx + N(x,y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Method of solution ⇒

Given equation $M(x,y)dx + N(x,y)dy = 0$. → (1)

* Check the exactness via

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} . \longrightarrow (2)$$

Suppose it is exact & (2) is satisfied then there must exists a function $f(x,y)$ such that

$$df = M(x,y)dx + N(x,y)dy . \rightarrow (3)$$

Now $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \longrightarrow (4)$

Comparing 3 & 4 we have

$$\frac{\partial f}{\partial x} = M(x,y). \longrightarrow (5)$$

$$\frac{\partial f}{\partial y} = N(x,y). \longrightarrow (6)$$

Integrating 5 w.r.t x we get

$$f(x,y) = \int M(x,y)\,dx + g(y). \longrightarrow (7)$$

Now differentiating 7 w.r.t y we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y). \longrightarrow 8$$

Equating 6 & 8 will gives $g(y)$.

And so

$f(x,y) = c$ will be the solution of 1.

Example: Solve

$$2xy\,dx + (x^2-1)\,dy = 0. \longrightarrow 1$$

$M = 2xy$, $N = x^2-1$.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2-1) = 2x$$

So 1 is exact. Thus

$$\frac{\partial f}{\partial x} = M(x,y) \quad \& \quad \frac{\partial f}{\partial y} = N(x,y)$$

or $$\frac{\partial f}{\partial x} = 2xy \longrightarrow (2)$$

& $$\frac{\partial f}{\partial y} = x^2-1. \longrightarrow (3)$$

Consider equation (2) we have

$$\frac{\partial f}{\partial x} = 2xy \text{ , integrating we have}$$

$$f(x,y) = \int 2xy\,dx = \frac{2x^2y}{2} + g(y)$$

$$\Rightarrow f(x,y) = x^2y + g(y). \longrightarrow (4)$$

Differentiating (4) w.r.t y we have

$$\frac{\partial f}{\partial y} = x^2 + g'(y). \longrightarrow (5)$$

Comparing 5 & 3 we have

$$x^2-1 = x^2 + g'(y)$$

$$\Rightarrow \quad g'(y) = -1$$

Integrating we have

$$g(y) = -y + c_1$$

Thus $f(x,y) = x^2y - y + c_1$

So the solution is

$$f(x,y) = c$$

or $x^2y - y + c_1 = c$ or $\boxed{x^2y - y = c_2}$

Ans.

Solve

$$(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$$

Solution

The equation is exact because

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}.$$

Hence a function $f(x,y)$ exists for which

$$M(x,y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x,y) = \frac{\partial f}{\partial y}.$$

Now for variety we shall start with the assumption that $\partial f/\partial y = N(x,y)$; that is

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

$$f(x,y) = 2x\int e^{2y}\,dy - x\int \cos xy\,dy + 2\int y\,dy$$

Remember. The reason x can come out in front of the symbol $\int$ is that in the integration with respect to, y, x is treated as an ordinary constant. It follows that

$$f(x,y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = \underbrace{e^{2y} - y\cos xy}_{M(x,y)}.$$

and so $h'(x) = 0$ OR

$h(x) = C$

Hence a family of solutions is $\boxed{xe^{2y} - \sin xy + y^2 + c = 0}$

An initial value problem:

Solve:

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)},$$

$$y(0) = 2$$

Solution: By writing the equation in the form

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$$(\cos x \sin x - xy^2)\,dx + y(1-x^2)\,dy = 0$$

We recognize that the equation is exact b/c

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}.$$

Now

$$\frac{\partial f}{\partial y} = y(1-x^2)$$

$$f(x,y) = \frac{y^2}{2}(1-x^2) + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2$$

The last equation implies that $h'(x) = \cos x \sin x$.

Integrating gives

$$h(x) = -\int ((\cos x)(-\sin x\,dx)) = -\frac{1}{2}\cos^2 x$$

thus

$$\frac{y^2}{2}(1-x^2) - \frac{1}{2}\cos^2 x = C_1$$

OR

$$y^2(1-x^2) - \cos^2 x = C,$$

Where $2C_1$ has been replaced by C.

The initial condition
$y = 2$ when $x = 0$
demands that $4(1) - \cos^2(0) = C$

and so $C = 3$
An implicit solution
of the problem is then

$y^2(1-x^2) - \cos^2 x = 3.$

Non exact Differential Equation ⇒

$M(x,y)dx + N(x,y)dy = 0$, is nonexact → (1)

when $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

* With the use of integrating factor, we can make eq (1) exact.

INTEGRATING FACTOR ⇒

(A) If $\frac{(M_y - N_x)}{N}$ is a function of x alone then integrating factor is

$$\mu(x) = \exp\left(\int \frac{(M_y - N_x)}{N} dx\right).$$

Note that, $M_y = \frac{\partial M}{\partial y}$, $N_x = \frac{\partial N}{\partial x}$.

(B) If $\frac{(N_x - M_y)}{M}$ is a function of y alone then integrating factor is

$$\mu(y) = \exp\left(\int \frac{(N_x - M_y)}{M}\, dy\right).$$

Examples.

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0. \rightarrow (1)$$

$$\frac{\partial M}{\partial y} = x, \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x^2 + 3y^2 - 20) = 4x$$

so $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, thus (1) is non-exact.

To find the integrating factor consider

$$\frac{M_y - N_x}{N} = \frac{\left(\partial M/\partial y - \frac{\partial N}{\partial x}\right)}{N}$$

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2+3y^2-20}$$

which is not function of x alone so $\mu(x) = \exp\left(\int \frac{(M_y - N_x)}{N} dx\right)$ is not integrating factor.

Now consider

$$\frac{N_x - M_y}{M} = \frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = \frac{(4x - x)}{xy}$$

$$= \frac{3x}{xy} = \frac{3}{y},$$ which is function of y-alone. Thus, integrating factor is

$$\mu(y) = \exp\left(\int \frac{(N_x - M_y)}{M} dy\right)$$

or $\mu(y) = \exp\left(\int \frac{3}{y} dy\right) = e^{3\ln y} = e^{\ln y^3} = y^3 \longrightarrow (2)$

Xing y^3 both sides of (1) we get

$\cancel{xy^4 dx + (2x^2+3y^2-20)y^3 dy}$

$$\underbrace{xy^4}_{M} dx + \underbrace{y^3(2x^2+3y^2-20)}_{N} dy = 0. \longrightarrow (3)$$

$$\frac{\partial M}{\partial y} = 4xy^3, \quad \frac{\partial N}{\partial x} = 4xy^3. \quad (4)$$

Thus, (3) is now exact & can be solved with the help of usual method of solving exact equation.

Solutions by Substitution

Homogenous function ⇒

If a function f possesses the property

$f(tx, ty) = t^{\alpha} f(x, y)$ for some real number α

then f is said to be a homogenous function of degree α.

For example, $f(x, y) = x^3 + y^3$ is homogenous function of degree 3 because

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3x^3 + t^3y^3$$

$$= t^3(x^3 + y^3) = t^3 f(x, y).$$

$f(x,y) = x^3+y^3+1$, is not homogenous, since

$f(tx,ty) = t^3x^3+t^3y^3+1 \neq t^3 f(x,y)$.

* A function is homogenous if the degree of each term is the same.

** We can also write if f is homogenous function then

$f(x,y) = x^\alpha f(1,u)$, $u = y/x$,

or $f(x,y) = y^\alpha f(v,1)$, $v = x/y$.

*** A differential Equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is homogenous if both M & N are homogenous of the same degree.

Method of Solution ⇒

Given that equation $M(x,y)dx + N(x,y)dy = 0$, → (1)
is homogenous. Then $M(x,y)$ & $N(x,y)$
are homogenous functions of the same degree.

Thus
$$\boxed{\begin{aligned} M(x,y) &= x^{\alpha} M(1,u), \quad u = y/x \\ N(x,y) &= x^{\alpha} N(1,u). \end{aligned}}$$
&

Now $u = y/x$ implies $\boxed{y = ux}$ — (2)
so $\boxed{dy = u\,dx + x\,du}$

Using (2) in (1) we can write

$$x^{\alpha} M(1,u)\,dx + x^{\alpha} N(1,u)\,(u\,dx + x\,du) = 0$$

$$\Rightarrow M(1,u)\,dx + N(1,u)\,u\,dx + x N(1,u)\,du = 0$$

$$\Rightarrow [M(1,u) + u N(1,u)]\,dx = -x N(1,u)\,du$$

$$\Rightarrow \frac{dx}{x} = -\frac{N(1,u)}{(M(1,u)+uN(1,u))}\,du$$

Integrating on both sides the above equation we will get the solution in terms of x & u. Then substitute back $u = y/x$ and get the solution of (1).

Example: Solve

$$(x-y)\,dx + x\,dy = 0 \qquad \text{—— (1)}$$

clearly both $M = x-y$ & $N = x$ are homogenous function of degree 1. so (1) is homogenous equation. Thus, let $\boxed{y = ux}$

so $dy = u\,dx + x\,du$. Using this in (1) we have

$$(x - ux)\,dx + x(u\,dx + x\,du) = 0$$

$$x(1-u)\,dx + ux\,dx + x^2\,du = 0$$

$$\Rightarrow (1-u)\,dx + u\,dx + x\,du = 0$$

$$\Rightarrow (1-u+u)\,dx + x\,du = 0$$

$$\Rightarrow$$

$dx + x\,du = 0$ or $dx = -x\,du$

$\Rightarrow \frac{dx}{x} = -du$, integrating both sides

$$\int \frac{dx}{x} = -\int du$$

$\Rightarrow \ln x + \ln c = -u$ or $-u = \ln(cx)$

or $\boxed{e^{-u} = cx}$

putting back $u = y/x$ we have

$$\boxed{e^{-y/x} = cx}$$

Ans.

Example: Solve $(x^2+y^2)dx + (x^2-xy)dy = 0$

Solution ⇒

Inspection of $M(x,y) = x^2+y^2$ and $N(x,y) = x^2-xy$ shows that these coefficients are homogeneous functions of degree 2. If we let $y = ux$.

then $dy = u\,dx + x\,du$.

So after substituting, the given equation becomes

$$(x^2+u^2x^2)\,dx + (x^2 - ux^2)\,[u\,dx + x\,du] = 0$$

$$x^2(1+u)\,dx + x^3(1-u)\,du = 0$$

$$\frac{1-u}{1+u}\,du \div \frac{dx}{x} = 0$$

$$\left[-1 + \frac{2}{1+u}\right] du \div \frac{dx}{x} = 0$$

long division

After integration the last line gives

$$-u + 2\ln|1+u| + \ln|x| = \ln|c|$$

$$-\frac{y}{x} + 2\ln\left|1+\frac{y}{x}\right| + \ln|x| = \ln|c|$$

resubstituting $u = y/x$

Using the properties of logarithms, we can write the preceding solution as

$$\ln\left|\frac{(x+y)^2}{cx}\right| = \frac{y}{x}$$

OR $(x+y)^2 = cxe^{y/x}$.

Bernoulli's Equation:

The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Where n is any real number, is called Bernoulli's equation.

Note that for $n=0$ and $n=1$ equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is linear. For $n \neq 0$ and $n \neq 1$ the substitution $u = y^{1-n}$ reduces any equation of form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

to a linear equation

Bernoulli's DE

Solve $x\frac{dy}{dx}+y=x^2y^2$

<u>Solution</u>:
We first rewrite the equation as

$$\frac{dy}{dx}+\frac{1}{x}y=xy^2$$

by dividing by x. with $n=$
we have $u=y^{-1}$ OR
$y=u^{-1}$.

We then substitute

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx}$$

Chain Rule

into the given equation and simplify. The result is

$$\frac{du}{dx} - \frac{1}{x}u = -x$$

The integrating factor for this linear equation on, say $(0,\infty)$ is

$$e^{-\int dx/x} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

Integrating

$$\frac{d}{dx}\left[x^{-1}u\right] = -1$$

gives $x^{-1}u = -x + C$ OR $u = -x^2 + Cx$

Since $u = y^{-1}$, we have $y = \frac{1}{u}$ and so a solution of the given equation is

$$y = \frac{1}{(-x^2 + Cx)}$$