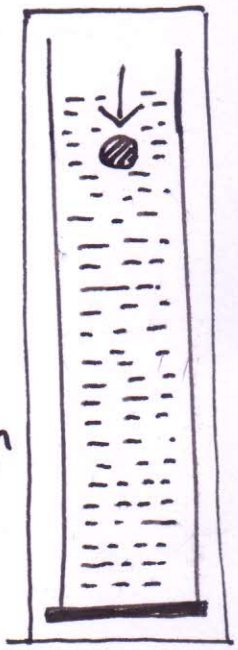
Examples and Applications



Stokes' Law of Terminal Velocity

For a heavy sphere (or any other shape) falling through a long column of viscous liquid, there are three forces acting on it, namely,

i) gravity, $\boxed{mg}$, ii) buoyancy, $\boxed{\rho_l V g}$, where ρ_l is the liquid density and V is the volume of the sphere, and iii) viscous drag, $\boxed{kv}$, where $\boxed{k = 6\pi\eta r}$, r being the radius of the sphere, η the viscosity and v the velocity.

Hence, $\boxed{m\frac{dv}{dt} = \overset{\downarrow}{mg} - \rho_l \overset{\uparrow}{V} g - k\overset{\uparrow}{v}}$.

Writing $\boxed{m = \rho V}$, where ρ is the density of the sphere, and dividing throughout by m we get, $\boxed{\frac{dv}{dt} = \bar{g} - \frac{k}{m}v}$, in which $\boxed{\bar{g} = g\left(1 - \frac{\rho_l}{\rho}\right)}$. The above equation is in the form $\boxed{\frac{dx}{dt} = a - bx}$, with the equivalence $\boxed{a \longrightarrow \bar{g}}$ and $\boxed{b \longrightarrow k/m}$.

Atomic Waste Disposal

$z \to$ Depth of the sea

Following the principle of the problem of Stokes' law of terminal velocity, we write $\boxed{m\frac{d^2z}{dt^2} = F = \underset{\downarrow}{W} - \underset{\uparrow}{B} - \underset{\uparrow}{D}}$, where $\boxed{W = mg}$ is the weight, $\boxed{B = (\rho_w V g)}$ is the buoyancy and $\boxed{D = kv}$ is the drag. Here ρ_w is the density of water, and k is the drag coefficient. The drag is proportional to the velocity, $\boxed{D \propto v}$. Noting $\boxed{\frac{dz}{dt} = v}$, we get $\boxed{\frac{dv}{dt} = g\left(1 - \frac{\rho_w V}{m}\right) - \frac{k}{m}v}$, in which we further write, $\boxed{m = \bar{\rho} V}$, where $\bar{\rho}$ is the average density of the fuel drum and V is its volume. Hence, $\boxed{\bar{g} = g\left(1 - \frac{\rho_w}{\bar{\rho}}\right)}$ using which we get $\boxed{\frac{dv}{dt} = \bar{g} - \frac{k}{m}v}$. The solution of this equation is $\boxed{v = v_T\left(1 - e^{-t/t_0}\right)}$ where $\boxed{v_T = \frac{m\bar{g}}{k}}$ and $\boxed{t_0 = \frac{m}{k}}$ under the initial condition at $\boxed{t=0,\ v=0}$.

Clearly $\boxed{v_T = \bar{g}\, t_0}$ which is the terminal velocity obtained when $t \to \infty$, in $\boxed{v = v_T\left(1 - e^{-t/t_0}\right)}$. Experimentally $\boxed{k = 0{\cdot}08 \text{ (in fps units)}}$, which gives the value of $\boxed{v_T \simeq 714 \text{ ft s}^{-1}}$. This is far greater than the tolerance velocity $\boxed{v_{tol} \simeq 40 \text{ ft s}^{-1}}$ at which the drums would break upon impact with the sea floor.

Since $\boxed{v_T > v_{tol}}$, the v-t equation does not guarantee that v_{tol} may not be overcome. Hence, we need to look at the v-z equation, which can be obtained from $\boxed{\frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz}}$

$$\therefore \boxed{v\frac{dv}{dz} = \bar{g} - \frac{v}{t_0}} \quad \text{Since } \boxed{t_0 = m/k},$$

$$\Rightarrow \boxed{t_0\, v\frac{dv}{dz} = \bar{g}\, t_0 - v = v_T - v}.$$

$$\Rightarrow \boxed{-\frac{v\, dv}{v_T - v} = -\frac{dz}{t_0}} \quad \text{Separation of Variables.}$$

$$\Rightarrow \quad \frac{v_T - v - v_T}{v_T - v}\, dv = -\frac{dz}{t_0}$$

$$\Rightarrow \quad \boxed{\int dv + \int \frac{v_T\, d(-v)}{v_T - v} = -\int \frac{dz}{t_0}}$$

$$\Rightarrow \quad v + v_T \int \frac{d(-v/v_T)}{1 + (-v/v_T)} = -\frac{z}{t_0}$$

$$\Rightarrow \quad \boxed{v + v_T \ln\left(1 - \frac{v}{v_T}\right) = -\frac{z}{t_0} + c}\,.$$

When $\boxed{z = 0}$ (at the surface of the sea), $\boxed{v = 0}$. For this initial condition, $\boxed{c = 0}$.

$$\Rightarrow \quad \boxed{z = -t_0\left[v + v_T \ln\left(1 - \frac{v}{v_T}\right)\right]}\,.$$

This is a transcendental equation and a solution of $\boxed{v = v(z)}$ cannot be found in closed form. Therefore, we invert the problem. First we write $\boxed{v = v_{tol} = 40\ ft\,s^{-1}}$. The depth at which this velocity is to be reached is z_{tol}. The weight of a drum, $\boxed{W = 527.4\,lbs}$. Hence, $\boxed{m = \frac{W}{g} = \frac{527.4}{32.2} = 16.38\,slugs}$.

$\Rightarrow$ $\boxed{t_0 = \frac{m}{k} = \frac{16.38}{0.08}}$ in fps unit, $\boxed{V_T = 714\ ft\,s^{-1}}$.

Hence, $\boxed{z_{tol} = -\frac{16.38}{0.08}\left[40 + 714 \ln\left(1 - \frac{40}{714}\right)\right]}$ (ft)

$$\Rightarrow \boxed{Z_{tol} = \frac{-16.38}{0.08} \times -1.1644 = 238\,ft}.$$

Since the actual sea depth is 300 ft, at the point of impact, $V > V_{tol}$ $\Rightarrow$ $\boxed{\text{Drums will break}}$

To check if the depth, z, is a monotonic function of t, we consider $\boxed{V = V_T(1-e^{-t/t_0})}$, in which we write $\boxed{V = \frac{dz}{dt} = V_T(1-e^{-t/t_0})}$.

$$\Rightarrow Z = V_T t - \frac{V_T e^{-t/t_0}}{-1/t_0} + c = V_T t + V_T t_0 e^{-t/t_0} + c$$

when $\boxed{t=0,\ Z=0}$ $\Rightarrow$ $\boxed{C = -V_T t_0}$.

$$\Rightarrow \boxed{Z = V_T t + V_T t_0 (e^{-t/t_0} - 1)}$$

define $\boxed{\zeta = Z/V_T t_0}$ and $\boxed{\tau = t/t_0}$.

$\boxed{\frac{d^2\zeta}{d\tau^2} = e^{-\tau} = 1 > 0}$

Hence, we get $\boxed{\zeta = (\tau - 1) + e^{-\tau}}$. $\Rightarrow$ $\boxed{\frac{d\zeta}{d\tau} = 1 - e^{-\tau}}$

(minimum)

$\boxed{\frac{d\zeta}{d\tau} = 0}$ only when $\boxed{\tau = 0}$. Hence ζ (or z) increases monotonically for τ (or t) > 0.

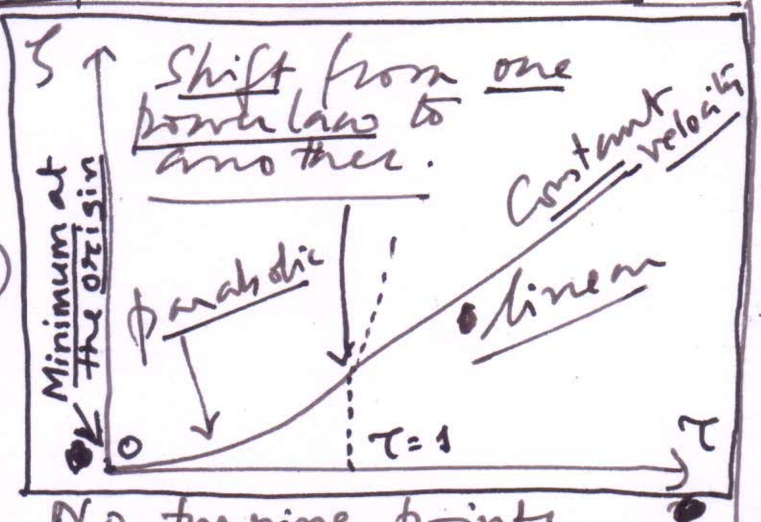
i.) When $\tau \to 0$,

$$\zeta = \cancel{\tau} \cancel{-1} + (\cancel{1} \cancel{-\tau} + \frac{\tau^2}{2} + \cdots)$$

$$\Rightarrow \boxed{\zeta \simeq \frac{\tau^2}{2}} \quad \text{(parabolic)}$$

ii.) When $\tau \to \infty$,

$$\boxed{\zeta \simeq \tau} \quad \text{(linear)}$$



No turning points.

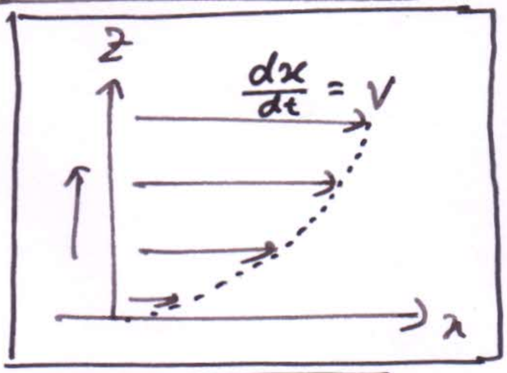
Monotonic → Also velocity increases with time.

Kelvin's Viscoelastic Deformation of Rocks

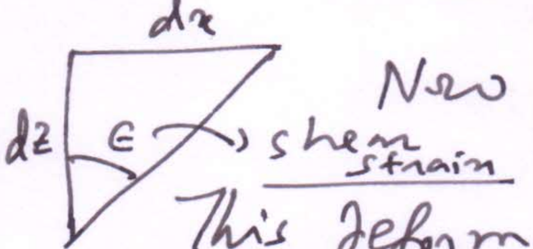
$\boxed{\sigma \to \text{Stress}}$, $\boxed{\epsilon \to \text{Strain}}$. For a Solid $\boxed{\sigma \propto \epsilon} \Rightarrow \boxed{\sigma = Y\epsilon}$ where Y is the Young's modulus (an elastic property)

For a liquid $\boxed{\sigma = \eta \frac{dv}{dz}}$

where $\eta \to$ Coefficient of viscosity.



Now $\boxed{\sigma = \eta \frac{d}{dz}\left(\frac{dx}{dt}\right) = \eta \frac{d}{dt}\left(\frac{dx}{dz}\right)}$



Now $\boxed{\frac{dx}{dz} = \tan\epsilon \approx \epsilon}$ for small deformation.

This deformation of a highly viscous liquid is named as FUGITIVE ELASTICITY by Maxwell. $\Rightarrow \boxed{\sigma = \eta \frac{d\epsilon}{dt}}$. Hence for a constant stress, σ, we can write

$\boxed{\sigma = Y\epsilon + \eta \frac{d\epsilon}{dt}}$ → Viscoelastic (Both viscosity and elasticity)

$\Rightarrow \boxed{\frac{d\epsilon}{dt} = \frac{\sigma}{\eta} - \frac{Y}{\eta}\epsilon}$ like $\boxed{\frac{dx}{dt} = a - bx}$

$\boxed{a \to \sigma/\eta, \; b \to Y/\eta}$.

Solid rocks FLOW OUT under the weight of the Earth matter above it.

Duckworth - Lewis Method (in cricket)

$$\boxed{Z(u, w) = Z_0(w)\left[1 - e^{-b(w)u}\right]}$$

$w \to$ No. of wickets lost. $u \to$ No. of overs left.
$Z(u,w) \longrightarrow$ No. of runs obtainable.
(Compare with $\boxed{x = x_0\left(1 - e^{-t/\tau}\right)}$).
w is to be treated as a parameter.

Reduce the Duckworth - Lewis equation to an autonomous system. We write

$$\boxed{\frac{dz}{du} = -z_0 e^{-bu} \times -b = \left(z_0 e^{-bu}\right) b}.$$

But $\boxed{z_0 e^{-bu} = z_0 - z}$. Therefore,

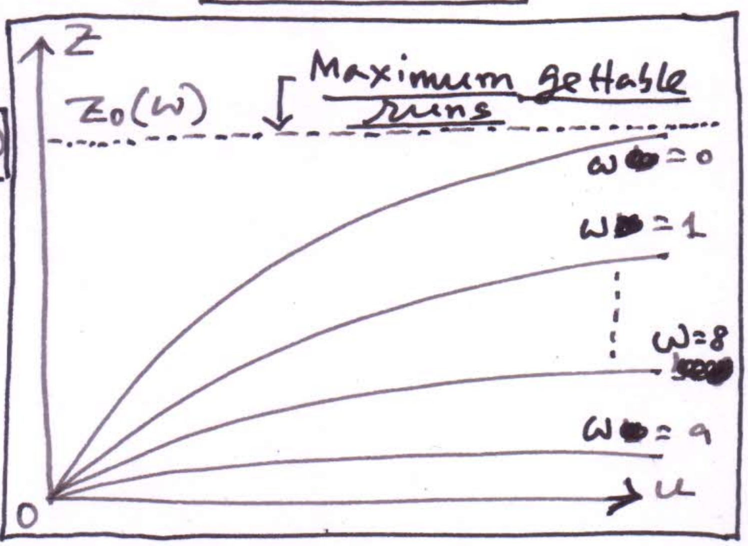
$$\boxed{\frac{dz}{du} = b(z_0 - z) = bz_0 - bz} \longrightarrow \boxed{\frac{dz}{du} = f(z) \quad \text{Autonomous}}$$

Now compare with $\boxed{\frac{dx}{dt} = a - bx}$.

We see $\boxed{a \to bz_0}$ and $\boxed{b \to b(w)}$.

The limiting value is $\boxed{a/b \to bz_0/b = z_0(w)}$

As b increases, (and also w) more wickets are lost. Hence less will be the gettable runs.



Van Meegeren Art Forgery Case

Radioactivity: $\boxed{\text{Rate} \propto \text{State}}$

$$\boxed{\frac{dN}{dt} = -\lambda N}$$

$\boxed{\lambda > 0}$, radioactive DECAY; $\lambda \rightarrow$ Decay constant

Integrate: $\Rightarrow$ $\boxed{\ln N = -\lambda t + c}$.

Initial condition is when $\boxed{t = t_0,\ N = N_0}$.

$\therefore$ $\boxed{c = \ln N_0 + \lambda t_0}$ $\Rightarrow$ $\boxed{\ln N - \ln N_0 = -\lambda(t - t_0)}$

$\Rightarrow$ $\boxed{N = N_0\, e^{-\lambda(t - t_0)}}$ $\rightarrow$ Exponential decay.

Half-life: $\Rightarrow$ $\boxed{N = N_0/2}$

Time taken to decay to half the initial amount.

$\Rightarrow$ $\frac{N}{N_0} = 2^{-1} = e^{-\lambda(t - t_0)}$

Write $t - t_0 = T_{1/2}$

$\Rightarrow$ $-\lambda(t - t_0) = -\ln 2$

$\Rightarrow$ $\boxed{t - t_0 = T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}}$

Eg.
$T_{1/2}\left(\begin{matrix}\text{Carbon}\\ C_{14}\end{matrix}\right) = 5568$ years, $T_{1/2}\left(\begin{matrix}\text{Uranium}\\ U_{238}\end{matrix}\right) = 4.5 \times 10^9$ years

Actual Age: $\boxed{t - t_0 = \frac{1}{\lambda}\ln\left(N_0/N\right)}$

OR $\boxed{t - t_0 = \frac{T_{1/2}}{\ln 2}\ln\left(N_0/N\right)}$

1/. N and λ can be measured.

2/. The difficulty is in knowing N_0 (the initial amount).

All paints contain white lead (lead oxide).
White lead contains radioactive Pb-210, with a half life of approximately $\boxed{22 \text{ years}}$, in which time it decays to Pb-206 (non-radioactive).
Let $\boxed{x_0 = x(t_0)}$ be the amount of Pb-210 ~~per~~ contained per gram of white lead, at the time of manufacture of the pigment.
The decay rate of Pb-210 is given by

$$\boxed{\frac{dx}{dt} = -\lambda x + s(t)},$$

in which $s(t)$ is the rate at which Pb 210 is replenished due to the radioactive decay of Ra-226 per minute per gram of white lead. If R is the amount of radium (Ra-226) at time t, with a half life of $\boxed{T_{R1/2} = 1600 \text{ years}}$, we write the decay equation of Ra-226 as $\boxed{R = R_0\, e^{-\lambda_R (t - t_0)}}$.

We expand this as $\boxed{R = R_0 \left[1 - \lambda_R (t - t_0) + \cdots\right]}$.

Now, $\boxed{t - t_0 = 300 \text{ years}}$ at most, which is the age of the original painting. Further $\boxed{\lambda_R = \frac{\ln 2}{T_{R1/2}}}$

Hence, $\boxed{\lambda_R (t - t_0) = \frac{\ln 2}{T_{R1/2}} (t - t_0) \simeq 0.13 \ll 1}$.

Therefore, we neglect all the higher powers in the expansion and retain only,

$$\boxed{R \simeq R_0 \left[1 - \frac{\ln 2}{T_{R1/2}}(t - t_0)\right]}$$

. The decay rate of Ra-226 is

$$\boxed{\frac{dR}{dt} \simeq \frac{-R_0 \ln 2}{T_{R1/2}} = -r(t)}$$

, which is constant. Hence, the rate of replenishment of Pb 210, $r(t)$ is also constant.

$$\Rightarrow \boxed{r(t) = \frac{R_0 \ln 2}{T_{R1/2}}}$$

. The decay rate of Pb 210 is given now as

$$\boxed{\frac{dx}{dt} = r - \lambda x}$$

, which, with $\boxed{r, \lambda > 0}$, is now in the form $\boxed{\frac{dx}{dt} = a - bx}$.

Integration: $\boxed{\frac{dx}{r - \lambda x} = dt}$ Separation of variables.

$$\Rightarrow \boxed{\int \frac{d(-\lambda x)}{r - \lambda x} = -\lambda \int dt} \Rightarrow \boxed{\ln(r - \lambda x) = -\lambda t + C}$$

The initial condition is when $\boxed{t = t_0, \quad x = x_0}$.

$$\Rightarrow \boxed{C = \lambda t_0 + \ln(r - \lambda x_0)}$$

. Using this we get

$$\boxed{\ln\left(\frac{r - \lambda x}{r - \lambda x_0}\right) = -\lambda(t - t_0)}$$

$$\Rightarrow r - \lambda x = (r - \lambda x_0)\, e^{-\lambda(t - t_0)}$$

$$\Rightarrow r - \lambda x_0 = (r - \lambda x)\, e^{+\lambda(t - t_0)}$$

$$\Rightarrow \boxed{x_0 = \frac{r}{\lambda} - \left(\frac{r}{\lambda} - x\right) e^{\lambda(t - t_0)}}$$

Only x and t are variables.

$$\boxed{x_0 = \frac{1}{\lambda} + \left(x - \frac{1}{\lambda}\right) e^{\lambda(t-t_0)}}$$

. In this equation, both λ and 1 are fixed known quantities. x can be measured. For a new painting x is large and $t - t_0$ is small, and for an old painting, x is small and $t - t_0$ is large. x_0 is ALWAYS fixed.

i/. When $t - t_0 = 300$ years, $\lambda(t-t_0) = 9.45$

ii/. When $t - t_0 = 20$ years, $\lambda(t - t_0) = 0.63$.

For measured values of x, using $\boxed{t - t_0 = 300 \text{ yrs}}$ makes the value of x_0 absurdly high. x_0 is acceptably small when $\boxed{t - t_0 = 20 \text{ years}}$.

Hence, the painting is a forgery.

Radio-Carbon Dating: Age of Ancient Cultures.

$$\boxed{{}^1n + {}^{14}N \longrightarrow {}^{14}C + {}^1p} \rightarrow \text{Willard Libby}$$

$$\boxed{N = N_0\, e^{-\lambda(t-t_0)}} \Rightarrow \boxed{\frac{N_0}{N} = e^{\lambda(t-t_0)}}.$$

$$\frac{dN}{dt} = \dot{N} = N_0\, e^{-\lambda(t-t_0)} \times -\lambda = -\lambda N. \left(\begin{array}{c}\text{late}\\ \text{a state}\end{array}\right)$$

At $\boxed{t = t_0}$, $\boxed{\frac{dN}{dt} = \dot{N}(t_0) = -\lambda N_0}$, $(N_0 = N(t_0))$

$$\Rightarrow t - t_0 = \frac{1}{\lambda} \ln\left(\frac{N_0}{N}\right) = \frac{1}{\lambda} \ln\left[\frac{\dot{N}(t_0)}{\dot{N}(t)}\right]$$

$$\Rightarrow \boxed{t - t_0 = \frac{T_{1/2}}{\ln 2} \ln\left[\frac{\dot{N}(t_0)}{\dot{N}(t)}\right]} \quad \boxed{T_{1/2} = 5568 \text{ years}}$$

Exercise 1: For living wood $\dot{N}(t_0) = 6.68$ unit

For a charcoal sample $\dot{N}(t) = 4.09$ unit

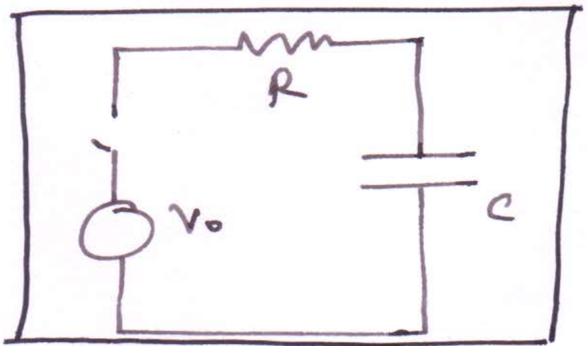
$$\Rightarrow \boxed{t - t_0 = \frac{5568}{\ln 2} \ln\left(\frac{6.68}{4.09}\right)} \quad t = 1950 \text{ A.D.}$$

$$\Rightarrow t_0 = (1950) - 3940 \simeq 2000 \text{ B.C.}$$

Exercise 2: $\dot{N}(t_0) = 6.68$ unit, $\dot{N}(t) = 0.47$ unit
$t = 1950$ A.D.

$$\Rightarrow \boxed{t_0 = 1950 - \frac{5568}{\ln 2} \ln\left(\frac{6.68}{0.97}\right) \simeq 13{,}500 \text{ B.C.}}$$

Q-R-C Circuit



$\boxed{Q = VC}$ $\Rightarrow$ $\boxed{V = Q/C}$

and $\boxed{V = IR}$

For the full circuit $\boxed{V_0 = IR + Q/C}$.

Further $\boxed{I = \frac{dQ}{dt}}$ $\Rightarrow$ $\boxed{R\frac{dQ}{dt} = V_0 - \frac{Q}{C}}$

$\Rightarrow \boxed{\frac{dQ}{dt} = \frac{V_0}{R} - \frac{Q}{RC}}$ in the form $\boxed{\frac{dx}{dt} = a - bx}$

$\boxed{a \to V_0/R}$, $\boxed{b \to \frac{1}{RC}}$

$\boxed{x = (a/b)\left[1 - e^{-bt}\right]}$

Solution is $Q = \frac{V_0}{\cancel{R}} \cdot \cancel{R}C\left(1 - e^{-t/RC}\right)$

$\Rightarrow \boxed{Q = Q_0\left(1 - e^{-t/RC}\right)}$

where $\boxed{Q_0 = CV_0}$ limiting value