Solutions to Homework 8:

1. *Proof.* Relation R is symmetric and transitive
   So, let $a\mathcal{R}b$ and $b\mathcal{R}a$ since R is symmetric
   Now, for a relation to be transitive we know that if $x\mathcal{R}y$ and $y\mathcal{R}z \implies x\mathcal{R}z$
   So, now since R is transitive, we have $a\mathcal{R}b$ and $b\mathcal{R}a \implies a\mathcal{R}a$
   Hence R is reflexive □

2. R is an equivalence relation

   Proof. To prove that it is an equivalence relation we must show that R is reflexive, symmetric, and transitive.

   - Reflexive:For any $a \in \mathbb{Z}$, we have $(5a - 8a) = 3(-a)$, which implies $3 \mid (5a - 8a)$. Thus $a\mathcal{R}a$.
   - Symmetric: Let $a, b \in \mathbb{Z}$ and assume $a\mathcal{R}b$. Then we see $3 \mid (5a - 8b)$, and so $5a - 8b = 3k$ for some $k \in \mathbb{Z}$. Then

   $$\begin{aligned} 5a + 3a + 3b - 8b &= 3k + 3a + 3b && (1)\\ 8a - 5b &= 3(k + a + b) && (2)\\ 5b - 8a &= 3(-k - a - b) && (3)\\ 3 &\mod (5b - 8a) && (4) \end{aligned}$$

   Since $(-k - a - b) \in \mathbb{Z}$ we see that $3 \mid (5b - 8a)$. Therefore R is symmetric.
   - Transitive: Let $a, b, c \in \mathbb{Z}$ and assume $a\mathcal{R}b$ and $b\mathcal{R}c$. Then we see $3 \mid (5a - 8b)$ and $3 \mid (5b - 8c)$
   So,

   $$\begin{aligned} 5a - 8b + 5b - 8c &= 3n + 3m, \qquad \forall n, m \in \mathbb{Z} && (5)\\ 5a - 8c &= 3(n + m + b) && (6)\\ 3 &\mod (5a - 8c) && (7) \end{aligned}$$

   Since $(n + m + b) \in \mathbb{Z}$ we see that $3 \mid (5a - 8c)$. Therefore R is transitive.

   Thus R is an equivalence relation.

   □

3. (a)

   - We see that by definition of the relation, it is reflexive since f(x) = f(x) so, $f\mathcal{R}f$
   - If $f\mathcal{R}g$, then we know that $\exists x \in \mathbb{R}$ such that $f(x) = g(x)$ and so this means that $g(x) = f(x)$. Hence the relation is also symmetric i.e if $f\mathcal{R}g \implies g\mathcal{R}f$

- The relation is not transitive.
  Let f, g and h such that $f(x) = 0, g(x) = x$ and $h(x) = 1$. We have $f\mathcal{R}g$ and $g\mathcal{R}h$ but it is not true that $f\mathcal{R}h$.

(b)
Let $\mathcal{R}$ be a relation on $\mathbb{Z}$ defined by:

$$x\mathcal{R}y \text{ if } xy \equiv 0 \pmod{4}. \tag{8}$$

- We see that $(3, 3) \notin \mathcal{R}$, since $3 \cdot 3 = 9 \not\equiv 0(\text{mod}4)$. Therefore, the relation is not reflexive.
- This relation is symmetric since if $xy \equiv 0(\text{mod}4)$, then $yx = xy \equiv 0 \pmod{4}$, that is, if $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$.
- This relation is not transitive. For a counterexample, we can take, $a = 3, b = 4$, and $c = 5$. Then, we see that $(3, 4), (4, 5) \in \mathcal{R}$, whereas, $(3, 5) \notin \mathcal{R}$.

4. To prove that R is a partition of A, we will prove that all the sets in R are not intersecting

   Proof. The sets in R are not intersecting which means

   $$(U_1, U_2 \in R, U_1 \neq U_2) \implies (U_1 \cap U_2 = \phi) \tag{9}$$

   Let U_1 and U_2 be sets in R
   We will use contrapositive here
   So, let $U_1 \cap U_2 \neq \phi$ Let $x \in U_1 \cap U_2$
   By definition,
   $U_1 = S_1 \cap T_1$ and $U_2 = S_2 \cap T_2$ where $S_1, S_2 \in P, T_1$ and $T_2 \in Q$
   Now, $x \in U_1$ and $x \in U_2$
   Therefore, $x \in S_1, T_1$ and $x \in S_2, T_2$
   Now, S_1, S_2 and T_1, T_2 are disjoint sets
   Thus, for the x to be in all the sets, $S_1 = S_2$ and $T_1 = T_2$
   Therefore, $U_1 = U_2$ □

5. *Proof.* We want to prove that the relation R is an equivalent realation

   - **Reflexive:** Let $A \in \mathcal{P}(E)$. Then $(x \in A) \vee (x \in \bar{A})$
     And so , $(x \in A \cap A) \vee (x \in \bar{A} \cap \bar{A})$
     Hence, $A\mathcal{R}A$.

- **Symmetry:** Let $A\mathcal{R}B$ This implies $(x \in A \cap B) \vee (x \in \overline{A} \cap \overline{B})$
  And so , $(x \in B \cap A) \vee (x \in \overline{B} \cap \overline{A})$
  Therefore, $A\mathcal{R}B \implies B\mathcal{R}A$ So, R is symmetric
- **Transitive:** Let $A, B, C \in \mathcal{P}(E)$ and assume that $A\mathcal{R}B$ and $B\mathcal{R}C$

$$((x \in A \cap B) \vee (x \in \bar{A} \cap \bar{B})) \wedge ((x \in B \cap C) \vee (x \in \bar{B} \cap \bar{C})). \tag{10}$$

  So,
  - **Case 1:** $(x \in A \cap B) \wedge (x \in B \cap C)$. Then $x \in A$ and $x \in C$ or
    $x \in A \cap C$
    which implies $A\mathcal{R}C$ and R is transitive
  - **Case 2:** $(x \in A \cap B) \wedge (x \in \bar{B} \cap \bar{C})$
    that is $x \in B \cap \bar{B}$
    But this case never happens
  - **Case 3:** $(x \in \bar{A} \cap \bar{B}) \wedge (x \in B \cap C)$
    that is $x \in A \cap \bar{A}$
    But this case never happens
  - **Case 4:** $(x \in \bar{A} \cap \bar{B})) \wedge (x \in \bar{B} \cap \bar{C})$. Which means
    $x \in \bar{A} \cap \bar{C}$
    So, $A\mathcal{R}C$ and R is transitive

Therefore it is an equivalence relation since R is reflexive, symmetric and transitive. □

6. (a) To prove that $\mathcal{S} = \{X_0, ...X_{n-1}$ forms a partition of $\mathbb{Z}$ we need to prove that :

   i. $\bigcup_{X \in \mathcal{S}} X = \mathbb{Z}$
   Any element $x \in X_i$ can be written as $x = nk + i$
   Also, by division algorithm, any integer x can be written as $nk + i$ where $0 \leq i < n - 1$
   Therefore, $x \in Z$ which implies
   $\bigcup_{X \in \mathcal{S}} X = \mathbb{Z}$

   ii. $X \cap Y = \phi$ or $X = Y$
   Let $X, Y \in \mathcal{S}$ and so,
   $x \in X, y \in Y$
   Here let $X = X_i, Y = X_j$ and so $x = nk + i$ and $y = nk + j$

   So when $i > j$ we let $x \in X_i$ and $x \notin X_j$
   So, $x \in X_i - X_j$
   So, $X_i - X_j \subseteq X_i$

   Now let, $x \in X_i - X_j$ and so $x \in X_i \wedge x \notin X_j$
   So, $X_i \subseteq X_i - X_j$

Therefore $Xi - X_j = X_i$ or $X_i \cap X_j = \phi$ or $X \cap Y = \phi$
Consider i = j so in this case we will have
$X_i = X_j$ or $X = Y$

So , $X \cap Y = \phi$ or $X = Y$

So, $\mathcal{S}$ is a partition of $\mathbb{Z}$

(b) *Proof.* $R = \{(a, b)\mathbb{Z}X\mathbb{Z}|a, b \in X_i\}$

Reflexive: Let, $a \in X_i$
Hence, $(a, a) \in \mathbb{Z}X\mathbb{Z}$ since $a \in \mathbb{Z}$
And so $\mathcal{R}$ is reflexive

Symmetric: Let $(a, b) \in R$
This implies $a, b \in X_i$
Since, $a, b \in Z$
So, $(b, a) \in \mathbb{Z}X\mathbb{Z}$
Therefore, $(a, b) \in R \implies (b, a) \in R$
So, R is symmetric

Transitive: Let $(a, b) \in R$ and $(b, c) \in R$
This implies, $a, b \in X_i$ $b, c \in X_j$
Now, $b \in X_i$ and $b \in X_j$
Since X_i and X_j are disjoint sets we have i = j
Therefore, $a \in Xi$ and $c \in Xi$
Therefore, $(a, c) \in R$
Thus, $(a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$
Hence, R is transitive

And so R is an equivalence relation since it is reflexive, symmetric and transitive

□

(c) Let Q denote the set of equivalence classes of R.
For any $X_i \in \mathcal{S}, X_i = \{x \in \mathbb{Z}|x = nk + i\}$
Now we know that , $[i] = \{x \in \mathbb{Z}|x \in X_j\} = \{x \in \mathbb{Z}|x \in X_i\} = X_i$
Therefore $S \subseteq Q$.
Also, for any $Y \in Q, Y = [x], \forall x \in \mathbb{Z}$
So, by the Euclidean division algorithm we have, $x = nk + i, \quad \forall k \in \mathbb{Z}$ and $0 \leq i < n - 1$
So, $Y = [x] = X_i$
And hence, $Q \subseteq S$
So, since $S \subseteq Q$ and $Q \subseteq S$ we have $Q = S$

7. (a) **Reflexive:** $\forall [a]_n \in \mathbb{Z}_n$ we know that,
$[a]_n = [a \cdot 1]_n$

$[a]_n = [a]_n \cdot [1]_n$
Hence R is reflexive

Symmetric: Let $[a]_n, [b]_n \in \mathbb{Z}_n$ and consequently let aRb be true,
So, for some invertible class $[u]_n, [v]_n \mathbb{Z}_n$ where $[v]_n$ is the inverse of $[u]_n$ we have

$$[a]_n \cdot [u]_n = [b]_n \tag{11}$$
$$[a]_n \cdot [u]_n \cdot [v]_n = [b]_n \cdot [v]_n \tag{12}$$
$$[a]_n \cdot 1 = [b]_n \cdot [v]_n \tag{13}$$
$$\tag{14}$$

So, bRa is true
and hence R is symmetric

Transitive: Let $[a]_n, [b]_n, [c]_n \in \mathbb{Z}_n$ and let aRb and bRc be true.
So, for some invertible $[u]_n, [v]_n, [p]_n, [q]_n \mathbb{Z}_n$ where $[v]_n$ is the multiplicative inverse of $[u]_n$ and $[q]_n$ is the multiplicative inverse of $[p]_n$
So, $[c]_n = [a]_n \cdot [u]_n \cdot [p]_n$
$[c]_n = [a]_n \cdot [u \cdot p]_n$
Now, we know that $[u \cdot p]_n = [u]_n \cdot [p]_n$

$$[u \cdot p]_n \cdot [v \cdot q]_n = [u]_n \cdot [p]_n \cdot [v]_n \cdot [q]_n \tag{15}$$
$$[u \cdot p]_n \cdot [v \cdot q]_n = [1]_n \tag{16}$$

Hence $[u \cdot p]_n$ is invertible and so
$[c]_n = [a]_n \cdot [u \cdot p]_n$
And so aRc is true
Hence R is transitive

So R is an equivalence relation since it is reflexive, symmetric and transitive.

(b) For n = 6,
R = (1, 1),(2, 2),(3, 3),(4, 4),(5, 5),(6, 6),(1, 5),(2, 4),(4, 2),(5, 1)
The equivalence classes are:-
$[0] = \{\phi\}$
$[1] = \{1, 5\}$
$[2] = \{2, 4\}$
$[3] = \{3\}$
$[4] = \{2, 4\}$
$[5] = \{1, 5\}$
$[6] = \{6\}$