Solutions to Homework 4:

1. *Proof.* $\forall n \geq 0 \in \mathbb{Z}, 9 \mid n^3 + (n+1)^3 + (n+2)^3$
   We know that if a number is divisible by 9 then it is also divisible by 3.
   So if a number is not divisible by 3 then it is not divisible by 9 as well by contraposition

   Case 1: Number is divisible by 3 i.e $n = 3 * k$ where $\exists k \in \mathbb{Z}$
   Substituting the value of n in the equation $n^3 + (n+1)^3 + (n+2)^3$ we get,

$$
\begin{aligned}
(3k)^3 + (3k+1)^3 + (3k+2)^3 \quad (1)\\
27k^3 + (27k^3 + 9k + 27k^2 + 1) + (27k^3 + 36k + 54k^2 + 8) \quad (2)\\
81k^3 + 81k^2 + 45k + 9 \quad (3)\\
9(9k^3 + 9k^2 + 5k + 1) \quad (4)\\
\quad (5)
\end{aligned}
$$

   So, $\exists m \in \mathbb{Z}$ where $m = 9k^3 + 9k^2 + 5k + 1$ such that

$$n^3 + (n+1)^3 + (n+2)^3 = 9m \quad (6)$$

   Case 2: Number is not divisible by 3 i.e $n = 3 * k + 1$ where $\exists k \in \mathbb{Z}$
   Substituting the value of n in the equation $n^3 + (n+1)^3 + (n+2)^3$ we get,

$$
\begin{aligned}
(3k+1)^3 + (3k+2)^3 + (3k+3)^3 \quad (7)\\
(27k^3 + 9k + 27k^2 + 1) + (27k^3 + 36k + 54k^2 + 8) + (27k^3 + 81k + 81k^2 + 27) \quad (8)\\
81k^3 + 162k^2 + 126k + 9 \quad (9)\\
9(9k^3 + 18k^2 + 14k + 1) \quad (10)\\
\quad (11)
\end{aligned}
$$

   So, $\exists m \in \mathbb{Z}$ where $m = 9k^3 + 18k^2 + 14k + 1$ such that

$$n^3 + (n+1)^3 + (n+2)^3 = 9m \quad (12)$$

   Case 3: Number is not divisible by 3 i.e $n = 3 * k + 2$ where $\exists k \in \mathbb{Z}$
   Substituting the value of n in the equation $n^3 + (n+1)^3 + (n+2)^3$ we get,

$$(3k+2)^3 + (3k+3)^3 + (3k+4)^3 \tag{13}$$

$$(27k^3 + 36k + 54k^2 + 8) + (27k^3 + 81k + 81k^2 + 27) + (27k^3 + 144k + 108k^2 + 64) \tag{14}$$

$$81k^3 + 243k^2 + 279k + 99 \tag{15}$$

$$9(9k^3 + 27k^2 + 31k + 11) \tag{16}$$

$$\tag{17}$$

So, $\exists m \in \mathbb{Z}$ where $m = 9k^3 + 27k^2 + 31k + 11$ such that

$$n^3 + (n+1)^3 + (n+2)^3 = 9m \tag{18}$$

Hence $9 \mid n^3 + (n+1)^3 + (n+2)^3$

$\square$

2. *Proof.* We need that $\exists a, b, c \in \mathbb{Z}$ such that gcd(a,b) = 1 $\implies ((a \mid bc) \implies (a \mid c))$
   Now, from Bezout's identity $\exists x, y \in \mathbb{Z}$ such that ax+by=gcd(a,b) $\exists a, b \in \mathbb{Z}$
   Since gcd(a,b) = 1 we have that $ax + by = 1$
   On multiplying the above equation with c we get,

$$c.ax + c.by = c \tag{19}$$

   Now, we also know that $a \mid bc$ so $\exists k \in \mathbb{Z}$ such that $ak = bc$
   Substituting the above equation in eq (19) we get,

$$c.ax + a.k.y = c \tag{20}$$

$$a(c.x + k.y) = c \tag{21}$$

   So $\exists m \in \mathbb{Z}$ such that $m = c.x + k.y$ and hence,

$$a.m = ca \mid c \tag{22}$$

$\square$

3. It is given that $P = \{2, 3, 5, 7, 11, ...\}$

(a) The statement $\forall x \in P, \forall y \in P, x + y \in P$ is false. We will prove this statement by taking a counter example.
Let x = 3 and y = 5, so here x+y = 8 which is not a prime number and hence this statement is false.

(b) The statement $\forall x \in P, \exists y \in P$ such that, $x + y \in P$ is false. This is because when you take a prime number such as 7 you find that there exists no prime number such that their sum is a prime number as well. This is due to the fact that every prime number greater than 2 is odd and sum of odd prime number computes to even numbers which are not primes.
So if $x = 7$ and $y = 2$ then $x + y = 9$ which is not a prime number.

(c) The statement $\exists x \in P$ such that, $\forall x \in P, x + y \in P$ is false. This is because similar to the previous case there exists no prime that when you add it to another prime number say 7 to get a prime number as well. This is also due to the fact that every prime number greater than 2 is odd and sum of odd number is an even number which are not primes.
So if $x = 5$ and $y = 7$ then $x + y = 12$ which is not a prime number.

(d) This statement is true. Let's consider two prime number x and y such that,
$x = 5$ and $y = 2$, so $x + y = 7$ which is a prime number as well.

4. *Proof.* We need to prove that $\forall \epsilon > 0, \exists M > 0$ such that, $|\frac{2x^2}{x^2+1} - 2| < \epsilon$ whenever $x \geq M$
So, let's consider M = $\sqrt{(\frac{2}{\epsilon})}$, so $x \geq \sqrt{(\frac{2}{\epsilon})}$,
Consider $|\frac{2x^2}{x^2+1} - 2|$, here we will substitute M for x because of the inequality after simplifying the equation. So,

$$|\frac{2x^2}{x^2+1} - 2| \tag{23}$$

$$|\frac{2x^2 - (2x^2+2)}{x^2+1}| \tag{24}$$

$$|\frac{-2}{x^2+1}| \tag{25}$$

$$|\frac{2}{x^2+1}| \tag{26}$$

$$|\frac{2}{(\sqrt{(\frac{2}{\epsilon})})^2+1}| \tag{27}$$

$$|\frac{2}{\frac{2}{\epsilon}+1}| \tag{28}$$

$$|\frac{2\epsilon}{2+\epsilon}| \tag{29}$$

Now, we know that $2 + \epsilon$ is greater than 2 so $\frac{1}{2+\epsilon} < \frac{1}{2}$

$$\frac{2\epsilon}{2+\epsilon} < \frac{2\epsilon}{2} \tag{30}$$
$$\frac{2\epsilon}{2} = \epsilon \tag{31}$$

So,

$$|\frac{2x^2}{x^2+1} - 2| < \epsilon \tag{32}$$

□

5. *Proof.* We know that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$
We know by first principles of limits that ,

$$\forall \epsilon > 0, \exists \delta > 0, (0 < |x - a| < \delta) \implies (|f(x) - L| < \epsilon) \tag{33}$$

Let's consider $\delta = \sqrt{\epsilon}$,
We also have that $0 < |x - 0| < \delta$ and so,
$x < \delta$ or $x^2 < \delta^2$
Consider the function f(x) = $x^2 sin(\frac{1}{x})$ when $x \neq 0$

$$|f(x) - 0| = |x^2 sin(\frac{1}{x}) - 0| \tag{34}$$
$$|x^2 sin(\frac{1}{x})| \tag{35}$$

Now since the value of Sin lies between -1 and 1 we have that,

$$0 < |x^2 sin(\frac{1}{x})| < x^2 \tag{36}$$
$$0 < |x^2 sin(\frac{1}{x})| < \delta^2 \tag{37}$$

But we have $\delta = \sqrt{\epsilon}$ so,

$$0 < |x^2 sin(\frac{1}{x})| < (\sqrt{\epsilon})^2 \tag{38}$$
$$0 < |x^2 sin(\frac{1}{x})| < \epsilon \tag{39}$$
$$0 < |x^2 sin(\frac{1}{x}) - 0| < \epsilon \tag{40}$$
$$0 < |f(x) - 0| < \epsilon \tag{41}$$

Hence function f is continuous at x=0 □

6. **Case 1:** n ¿ N
   Now we know that x_n converges to 0 so $|x_n - 0| < \epsilon$ and hence we can take the value of $M = \epsilon$
   This you result in $|x_n - 0| \leq M$ and therefore this case holds true.

   Case 2: $n \leq N$
   Now we know that x_n converges to 0 so $|x_n - 0| < \epsilon$ and hence we can take the value of $M = max\{x_1, x_2.....x_N\}$
   This you result in $|x_n - 0| \leq M$ and therefore this case holds true.

7. *Proof.* **Case 1:** $M \geq 0$
   Let's consider $t = e^{-2M}$ where M is any arbitrary real number,
   So, consider the equation,

$$f(t) = log(t) \tag{42}$$
$$f(t) = log(e^{-2M}) \tag{43}$$
$$f(t) = -2M \tag{44}$$

   Now, the absolute value of f(t) is,

$$|f(t)| = |-2M| \tag{45}$$
$$|f(t)| = 2M \tag{46}$$
$$\tag{47}$$

   Case 2: $M < 0$
   Let's consider $t = e^{2M}$ where M is any arbitrary real number,
   So, consider the equation,

$$f(t) = log(t) \tag{48}$$
$$f(t) = log(e^{2M}) \tag{49}$$
$$f(t) = 2M \tag{50}$$

   Now, the absolute value of f(t) is,

$$|f(t)| = |2M| \tag{51}$$
$$|f(t)| = 2M \tag{52}$$
$$\tag{53}$$

   Now since, $2M > M$
   $|f(t)| > M$
   Hence, log(x) is unbounded on (1,0) □

8. *Proof.* Since we want to prove that x_n does not converge we essentially prove that,

$$\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n > N, |x_n - L| \geq \epsilon \tag{54}$$

Case 1: $L \geq 0$
Now, let us consider $\epsilon = 1 - L$ and $n = 2N$
So, for the equation $(-1)^n + \frac{1}{n}$ we have that,

$$(-1)^{2N} + \frac{1}{2N} \tag{55}$$

$$1 + \frac{1}{2N} \tag{56}$$

We know that as N gets bigger the value of $\frac{1}{2N}$ will approach 0 and hence

$$1 + \frac{1}{2N} \leq 1 \tag{57}$$

$$1 + \frac{1}{2N} - L \leq 1 - L \tag{58}$$

$$1 + \frac{1}{2N} - L \leq \epsilon \tag{59}$$

$$|1 + \frac{1}{2N} - L| \leq \epsilon \tag{60}$$

$$|(-1)^n + \frac{1}{n} - L| \leq \epsilon \tag{61}$$

Case 2: $L < 0$
Now, let us consider $\epsilon = 1 - L$ and $n = 2N + 1$
So, for the equation $(-1)^n + \frac{1}{n}$ we have that,

$$(-1)^{2N+1} + \frac{1}{2N+1} \tag{62}$$

$$-1 + \frac{1}{2N+1} \tag{63}$$

We know that as N gets bigger the value of $\frac{1}{2N+1}$ will approach 0 and hence
Now for $|x_n - L|$

$$|x_n| - |L| \tag{64}$$

$$|-1 + \frac{1}{2N+1}| - L \tag{65}$$

$$|-1 + \frac{1}{2N+1}| - L \leq 1 - L \tag{66}$$

$$|-1 + \frac{1}{2N+1}| - L \leq \epsilon \tag{67}$$

$$|(-1)^n + \frac{1}{n} - L| \leq \epsilon \tag{68}$$

Hence $|(-1)^n + \frac{1}{n} - L| \leq \epsilon$ or $|x_n - L| \leq \epsilon$ and so $x_n = (-1)^n + \frac{1}{n}$ does not converge to any $L \in \mathbb{R}$ using Proof by Cases $\square$