Math 220
Section 108
Lecture 16

1st November 2022

Source: https://personal.math.ubc.ca/~PLP/auxiliary.html

$R = \{(1,2)\}$ is transitive !!

bcs there is no

(b,c) & (a,c)

to prove it false

Relations

Introduction to Relations

Definition (Definition 9.1.1 of PLP)

Let A be a set. Then a relation $\mathbf{R}$ on A is a subset $R \subset A \times A$. If the ordered pair $(x, y) \in \mathbf{R}$, we denote this as $x\mathbf{R}y$, while if $(x, y) \notin \mathbf{R}$, we denote this as $x\not{\mathbf{R}}y$

Example (1)

The symbol $>$ can be thought of as a relation $\mathbf{R}$ on $\mathbb{Z}$.
For example, since we know that $5 > 4$, this means that we can write $5\mathbf{R}4$ and also $(5, 4) \in \mathbf{R} \subset \mathbb{Z} \times \mathbb{Z}$.
In contrast, since $4 \not> 4$, we have $4\not{\mathbf{R}}4$ and $(4, 4) \notin \mathbf{R}$.

Example (2)

Another example of a relation on $\mathbb{Z}$ is simply

$$\mathbf{R} = \{(1, 2), (18, 37), (1, -80)\}.$$

1. Let $A = \{1, 2, 3, 6\}$. Write out the relation **R** that expresses "$\nmid$" (does not divide) on A as a set of ordered pairs.
Note: We need to make sure that we are not leaving any one of the ordered pairs out. If we do, it is not the same relation anymore.

$A = \{1, 2, 3, 6\}$

$A = \{(2,1), (3,1), (6,1), (3,2), (6,2), (6,3), (2,3)\}$

(Continued)

(Continued) 1. Let $A = \{1, 2, 3, 6\}$. Write out the relation **R** that expresses "$\nmid$" (does not divide) on A as a set of ordered pairs.

Old final question

2. Determine which of the following relations, **R**, are reflexive, symmetric and transitive on the given set A. (We call a relation that satisfies all 3 properties, an **equivalence relation**.) Prove your answers.

- (a) $\mathbf{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \colon y = x^2\}$ on $A = \mathbb{R}$.
- R → (b) $\mathbf{R} = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)\}$ on $A = \{1, 2, 3\}$.
- (c) $\mathbf{R} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \colon 3 \text{ divides } a - b\}$ on $A = \mathbb{Z}$.

(c) $R = \{(1,1), (2,2), \ldots\ldots (6,3), (3,6), (3,9), (9,6) \ldots\ldots\}$

All of the elements must satisfy in the given relation

- Reflexive, Transitive, Symmetric !!!

(Continued)

(Continued) 2. Determine which **R** are equivalence relations on A.

a. $\mathbf{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \colon y = x^2\}$ on $A = \mathbb{R}$.
b. $\mathbf{R} = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)\}$ on $A = \{1, 2, 3\}$.
c. $\mathbf{R} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \colon 3 \text{ divides } a - b\}$ on $A = \mathbb{Z}$.

3. Define a relation on $\mathbb{Z}$ as $a\mathbf{R}b$ if $3 \mid (2a - 5b)$. Is $\mathbf{R}$ reflexive, symmetric, transitive? Justify your answer.

$R = \{(1,1), (2,2) \ldots$

Reflexive

$3 \mid (2a - 5b)$

let $b = a$

so, $3 \mid 2a - 5a$

$3 \mid -3(a)$

Symmetric

$3m = 2a - 5b$

so, $3m + 5b = 2a$

& $2a - 3m = 5b$

so, $5b - 2a = -3(m)$

$2b - 2a = -3(m+b)$

$2b - 5a = -3(m+b+a)$

So, $3 \mid 2b - 5a$

(Continued)

(Continued) 3. Define a relation on $\mathbb{Z}$ as $a\mathbf{R}b$ if $3 \mid (2a - 5b)$. Is $\mathbf{R}$ reflexive, symmetric, transitive? Justify your answer.

Transitive

$3 \mid 2a - 5b$

$3 \mid 2b - 5c$

So, $3 \mid 2a - 3b - 5c$

$3 \mid 2a - 5c$

So, it's an equivalent relation

4. Let $\mathcal{R}$ and $\mathcal{R}'$ be two relations on the same set A. Prove or disprove the following.

- (a) If $\mathcal{R}$ and $\mathcal{R}'$ are transitive, then the relation $\widehat{\mathcal{R}}$ defined as $\widehat{\mathcal{R}} = \mathcal{R} \cup \mathcal{R}'$ is transitive.
- (b) If $\mathcal{R}$ and $\mathcal{R}'$ are transitive, then the relation $\widehat{\mathcal{R}}$ defined as $\widehat{\mathcal{R}} = \mathcal{R} \cap \mathcal{R}'$ is transitive.

Yes, transitive.
Proof on next page.

(a) Let $A = \{1,2,3\}$
$R = \{(1,2)\}$
$R' = \{(2,3)\}$
Then $R \cup R'$
$= \{(1,2),(2,3)\}$ but $(1,3) \notin R \cup R'$ so it's not transitive

(Continued) 4. Let $\mathcal{R}$ and $\mathcal{R}'$ be two relations on A. Prove or disprove:

- a. If $\mathcal{R}$ and $\mathcal{R}'$ are transitive, then $\widehat{\mathcal{R}} = \mathcal{R} \cup \mathcal{R}'$ is transitive.
- b. If $\mathcal{R}$ and $\mathcal{R}'$ are transitive, then $\widehat{\mathcal{R}} = \mathcal{R} \cap \mathcal{R}'$ is transitive.

If $(a,b), (b,c) \in R \cap R'$, then $(a,b), (b,c) \in R$

so, $(a,c) \in R$ similarly for R'

& hence $(a,c) \in R \cap R'$ & it's transitive

5. Let p be a prime number and let a, b be integers. Then prove that $p|ab \Rightarrow p|a$ or $p|b$.

Hint: Bézout's lemma may be useful:

Lemma (Lemma 9.5.7 of PLP)

Given $a, b \in \mathbb{Z}$, not both zero, there exist $x, y \in \mathbb{Z}$ such that

$$ax + by = \gcd(a, b).$$

Assume $p|ab$

- If $p|a$, we are done
- If $p \nmid a$, $\gcd(p,a)=1$. Bezout's lemma implies $\exists x,y,z \in \mathbb{Z}$ s.t. $ax+py=1$. So, $abx+pby=b$.

Since $p|ab$, we have $ab=pk$ for some $k \in \mathbb{Z}$. So, $pkx+pby=b$ and so $p(kx+by)=b$. So, $p|b$ since $kx+by \in \mathbb{Z}$

(Continued)

(Continued) 5. Let p be a prime number and let a, b be integers. Then prove that $p|ab \Rightarrow (p|a)$ or $(p|b)$.

Relations - old exam question

6. A relation $\boldsymbol{R}$ on $\mathbb{Z}$ is defined by $a\boldsymbol{R}b$ if $7a^2 \equiv 2b^2 \pmod 5$.

(a) Prove that $\boldsymbol{R}$ is an equivalence relation.

(b) Determine the distinct equivalence classes [0] and [1], simplify your answer as much as possible.

Hint: The previous example will be useful here.

(Continued)

(Continued) 6. A relation $\boldsymbol{R}$ on $\mathbb{Z}$ is defined by $a\boldsymbol{R}b$ if $7a^2 \equiv 2b^2 \pmod{5}$.
(a) Prove that $\boldsymbol{R}$ is an equivalence relation.
(b) Determine the distinct equivalence classes [0] and [1].