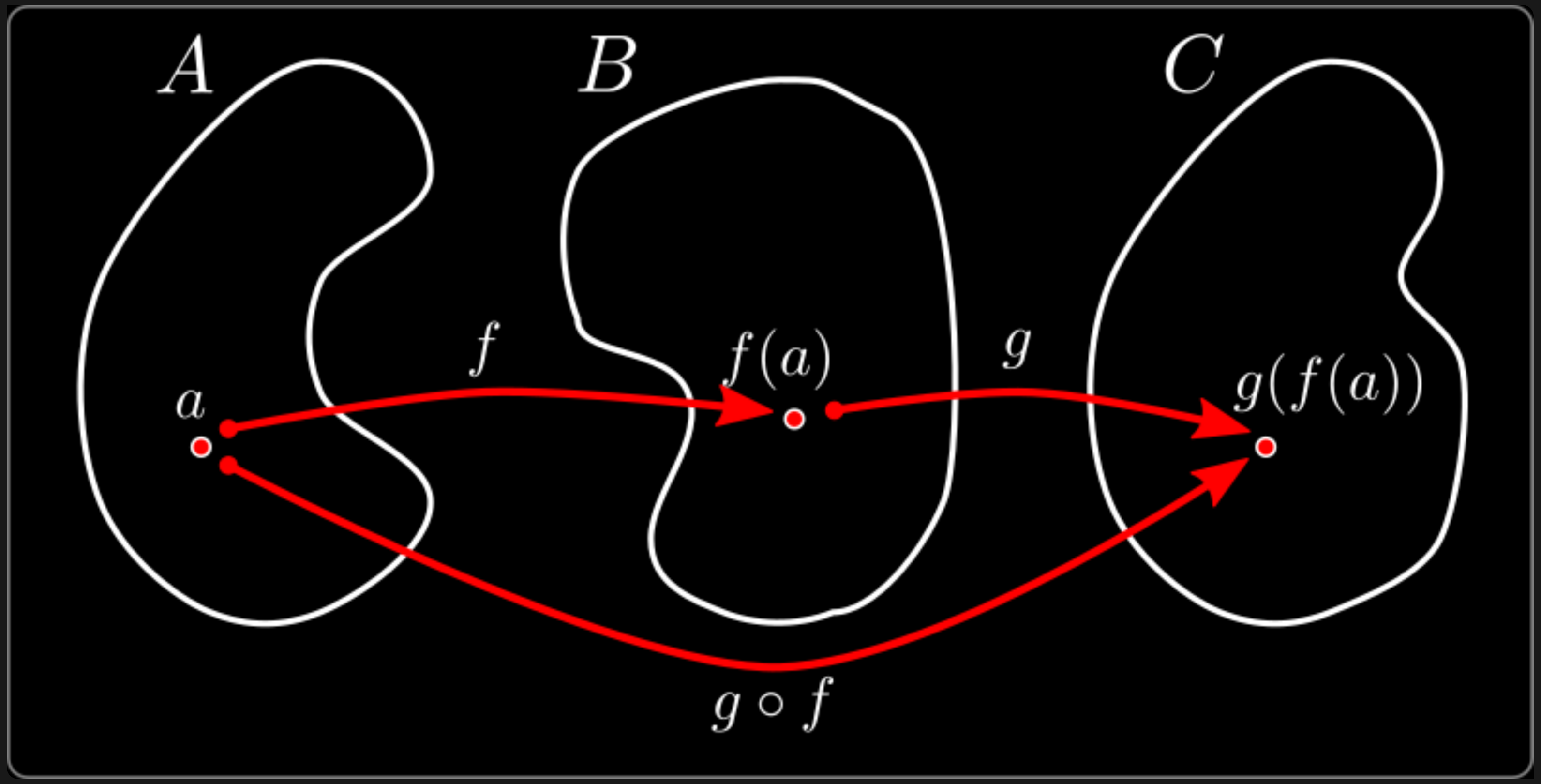
PLP - 34

TOPIC 34—COMPOSITIONS

Demirbaş & Rechnitzer

COMPOSITIONS

CHAINING FUNCTIONS TOGETHER



DEFINITION:

Let $f : A \to B$ and $g : B \to C$.

The **composition** of f and g, denoted $g \circ f$, defines a new function

$$g \circ f : A \to C \qquad (g \circ f)(a) = g(f(a)) \qquad \forall a \in A$$

Note composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$.

COMPOSITIONS, INJECTIONS AND SURJECTIONS

Compositions play nicely with injections and surjections.

THEOREM:

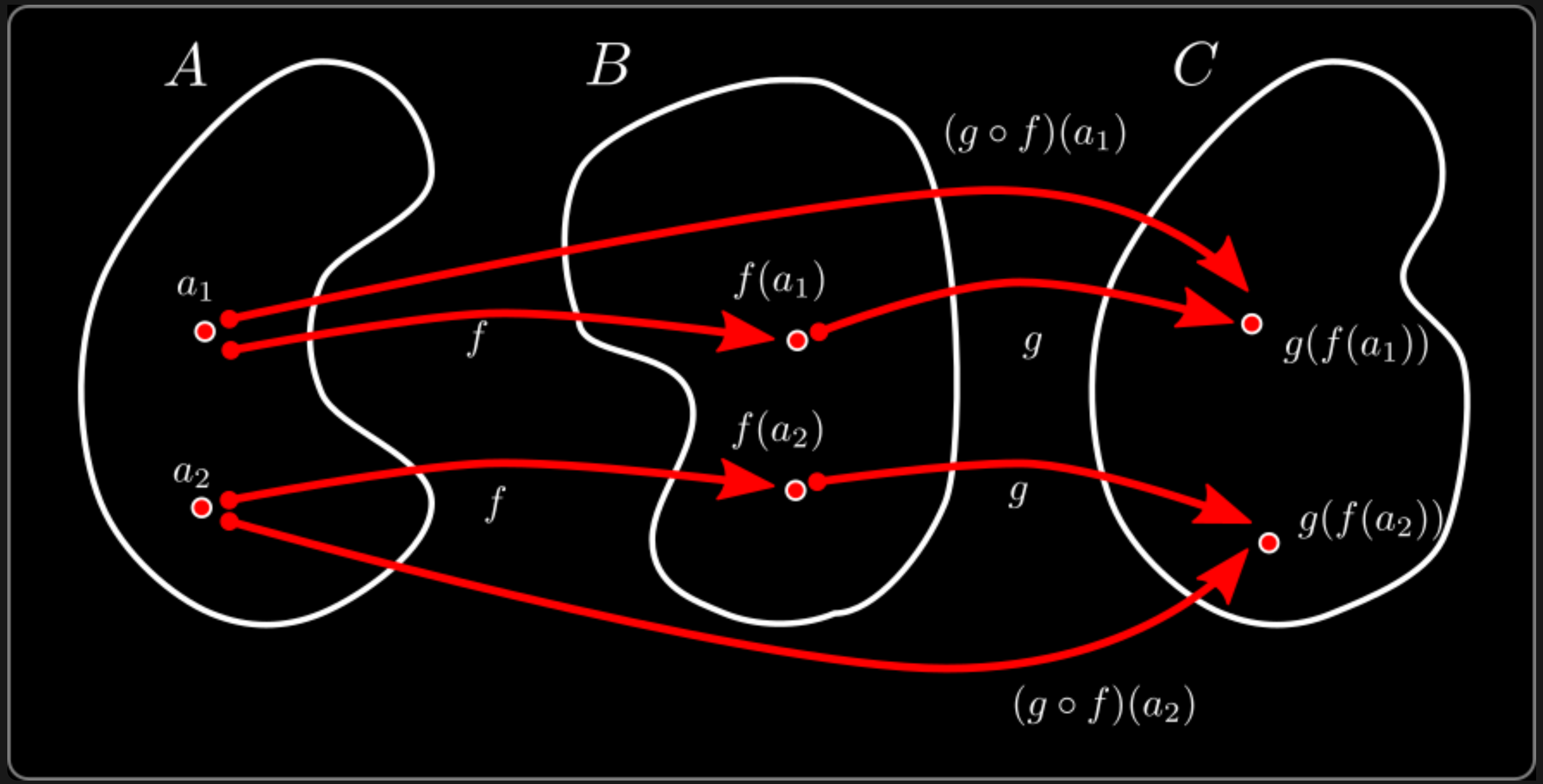
Let $f : A \to B$ and $g : B \to C$ be functions.

- If f and g are injective then so is $g \circ f$.
- If f and g are surjective then so is $g \circ f$.

Consequently if f, g are bijective then so is $g \circ f$.

COMPOSITION OF INJECTIONS

Use injection property — different map to different



PROOF.

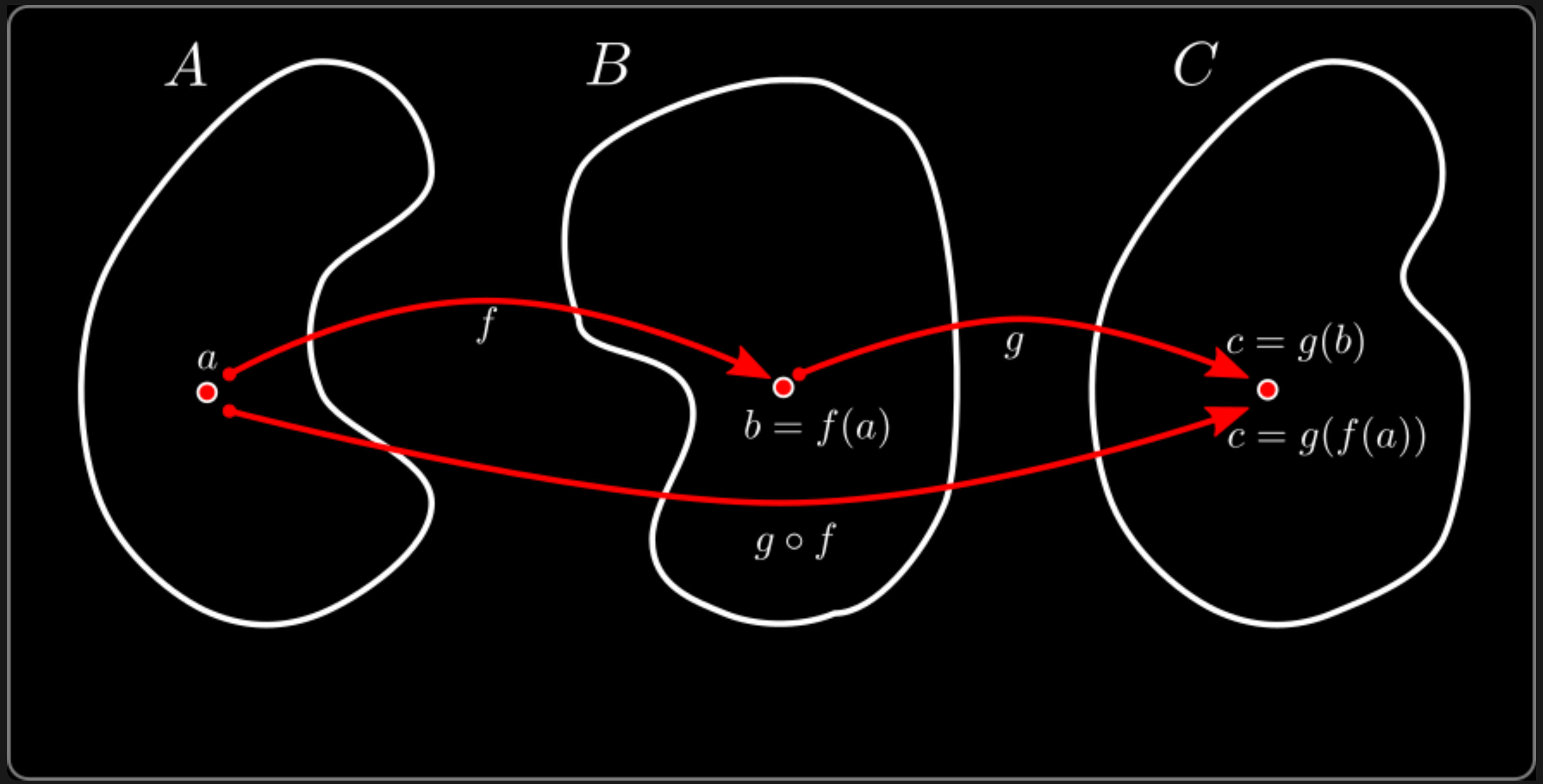
Let $a_1, a_2 \in A$ so that $a_1 \neq a_2$.

Since f is injective, we know that $f(a_1) \neq f(a_2)$. And thus since g is injective, we know that $g(f(a_1)) \neq g(f(a_2))$.

Thus $(g \circ f)(a_1) \neq (g \circ f)(a_2)$ as required.

COMPOSITION OF SURJECTIONS

Use surjection property — everything is mapped to by something



PROOF.

Let $c \in C$.

Since g is surjective, we know that there is $b \in B$ so that $g(b) = c$. Then since f is surjective, we have some $a \in A$ so that $f(a) = b$.

Thus $g(f(a)) = g(b) = c$, and so for any $c \in C$ we can find $a \in A$ so that $(g \circ f)(a) = c$ as required.

PARTIAL CONVERSE

THEOREM:

Let $f : A \to B$ and $g : B \to C$ be functions, then

- if $g \circ f$ is an injection then f is an injection.
- if $g \circ f$ is a surjection then g is a surjection.

The proofs of these statements make excellent exercises.

Note that you *cannot* extend this to a full converse. There exist f, g so that

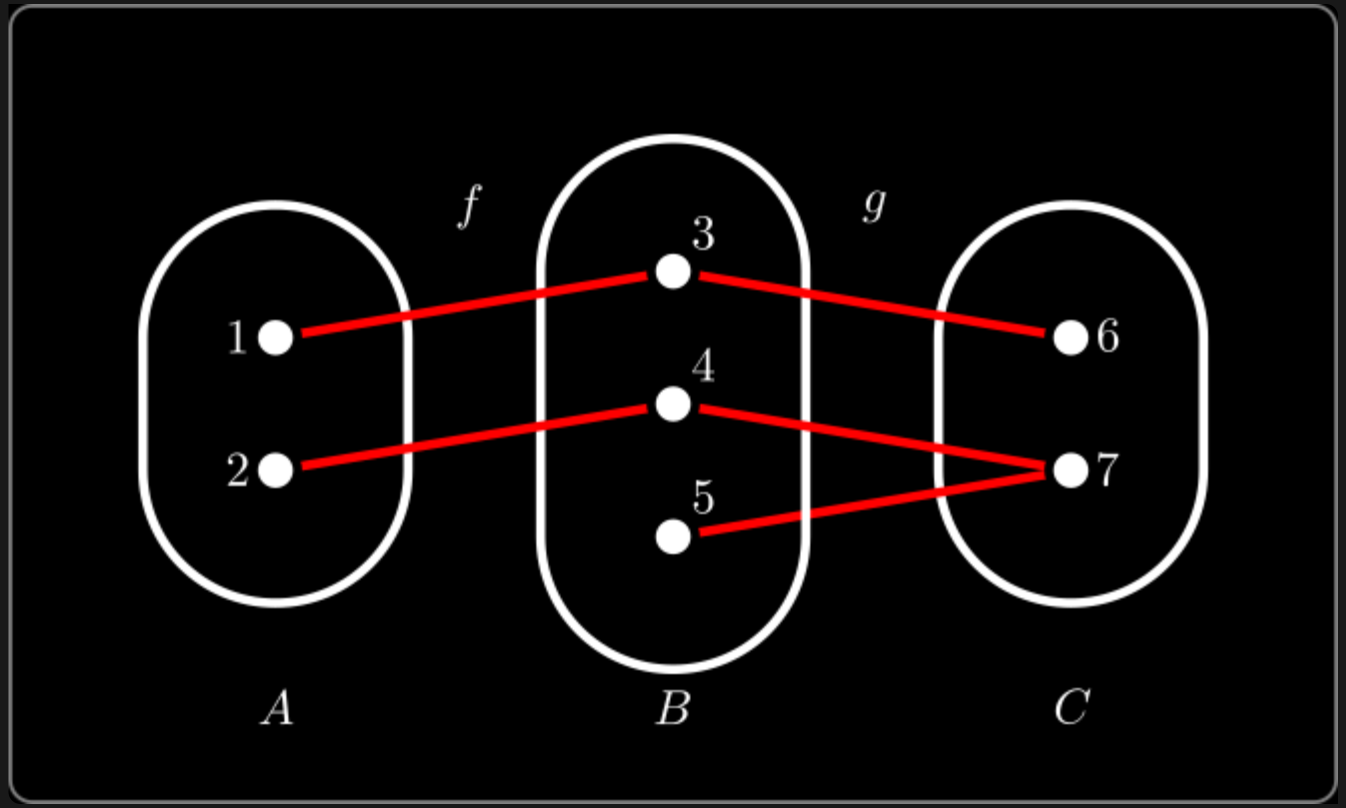
- $g \circ f$ is an injection, but g is *not* injective
- $g \circ f$ is a surjection, but f is *not* surjective

DISPROOF OF FULL CONVERSE

- $g \circ f$ is an injection, but g is *not* injective
- $g \circ f$ is a surjection, but f is *not* surjective

PROOF.

Consider functions f, g defined by the diagram below.



- Since $g(f(1)) \neq g(f(2))$, $g \circ f$ is injective. But $g(4) = g(5)$, so g not an injection.
- Since $6 = g(f(1)), 7 = g(f(2))$, $g \circ f$ is surjective. But, $f(1), f(2) \neq 5$ so f is not a surjection.