Mathematics 220 — Homework 4

- Contains 8 questions on 2 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

1. Prove that for every integer $n \geq 0$, the sum $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

2. Recall Bézout's identity: Let $a, b \in \mathbb{Z}$ such that a and b are not both zero. Then there exists $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$.

   Use this result to prove the following result:

   Let $a, b, c \in \mathbb{Z}$ such that $\gcd(a, b) = 1$. Then

   $$(a \mid bc) \implies (a \mid c).$$

3. Let $P \subset \mathbb{N}$ be the set of prime numbers $P = \{2, 3, 5, 7, 11, \ldots\}$. Determine whether the following statements are true or false. Prove your answers ("true" or "false" is not sufficient).

   (a) $\forall x \in P$, $\forall y \in P$, $x + y \in P$.

   (b) $\forall x \in P$, $\exists y \in P$ such that $x + y \in P$.

   (c) $\exists x \in P$ such that $\forall y \in P$, $x + y \in P$. Again, we leverage the fact that every number in P is odd *except* 2.

   (d) $\exists x \in P$ such that $\exists y \in P$, $x + y \in P$.

4. Prove the following statement: For every positive number ϵ there is a positive number M such that

   $$\left| \frac{2x^2}{x^2+1} - 2 \right| < \epsilon$$

   whenever $x \geq M$.

5. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if $\lim_{x \to a} f(x) = f(a)$. Let

   $$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

   Is f continuous at $x = 0$?

   Warning: You must prove this from first-principles (ie with $\epsilon - \delta$).

6. We say that a sequence (x_n) is bounded if
$$\exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M.$$
Prove that if a sequence (x_n) converges to 0, then (x_n) is bounded.

Hint: It helps to break the problem into two cases, small n and big n.

7. A function is said to be unbounded on the interval (a, b) if
$$\forall M \in \mathbb{R}, \exists t \in (a, b) \text{ s.t. } |f(t)| > M.$$
Prove that $\log x$ is unbounded on $(0, 1)$.

You may assume basic properties of logarithms and exponentials in your solution.

8. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to L if
$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \epsilon.$$
Using the definition, prove that the sequence (x_n) with $x_n = (-1)^n + \dfrac{1}{n}$ does not converge to any $L \in \mathbb{R}$.

Hint: It is perhaps easier to split into cases depending on the sign of L.

① $\forall n \in \mathbb{Z}\ n \geq 0,\ 9 \mid n^3 + (n+1)^3 + (n+2)^3$

Typed!!

Case 1: $n = 3k$

Case 2: $n = 3k+1$

Case 3: $n = 3k+2$

$b \cdot c = a \cdot k$

② $\exists a, b, c \in \mathbb{Z}$ st, $\gcd(a,b) = 1 \Rightarrow$

$$\left[(a \mid bc) \Rightarrow (a \mid c) \right]$$

Typed!!

$ax + by = 1$

$ac \cdot x + b \cdot c \cdot y = c$

$a \cdot c \cdot x + a \cdot k \cdot y = c$

$a \cdot (cx + ky) = c$

③ $P = \{2, 3, 5, 7, 11 \ldots\}$

Typed!!

(a) $\forall x \in P, \forall y \in P, x + y \in P$

Take counter example

(b) $\forall x \in P, \exists y \in P$ such that $x + y \in P$

& (c) $P > 2$ = odd & odd + odd = even

& odd + 2 $\neq P$ for all

(d) $\exists \in P$, s.t $\exists y \in P$, $x + y \in P$

$2 + 3 = 5$ ✓

(8) $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \varepsilon$ $\quad x_n = (-1)^n + \frac{1}{n}$

$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N, |x_n - L| \geq \varepsilon$

Scrathwork

$\boxed{L = 0}$, $\quad |x_n| \geq \varepsilon \rightarrow (-1)^n + \frac{1}{n} \geq \varepsilon$

$$n = 2N \quad \& \quad \varepsilon = 1$$

$1 + \frac{1}{2N} \geq 1$ $\quad$ so, $1 + \frac{1}{2N} \geq \varepsilon$

Hence $|x_n - L| \geq \varepsilon$

$\boxed{L = -1}$

$$(-1)^n + \frac{1}{n} + 1 \geq \varepsilon$$

$2 + \frac{1}{2N} \geq \varepsilon$ $\quad \varepsilon = 1 \;\&\; n = 2N$

so, $|x_n + 1| \geq \varepsilon$

$L = 16$

$(-1)^n + \frac{1}{n} - 16$ $\quad n = 2N$

$n = \max\{16, N\}$

$(-1)^{16} + \frac{1}{16} - 16 \geq -15 + \frac{1}{16}$

$15 - \frac{1}{16} \geq 14$

17

$(-1)^{17} + \frac{1}{17} - 16$

$-17 + \frac{1}{17}$

$17 - \frac{1}{17} \geq 15$

$n = 2\max\{L, N\}$

④ $\forall \varepsilon > 0, \exists M > 0$ s.t.,

<u>Scratch</u>

$$\left|\frac{2n^2}{n^2+1} - 2\right| < \varepsilon$$

$$\left|\frac{2n^2}{n^2+1} - 2\right| < \varepsilon \qquad M = \varepsilon$$

$$\left|\frac{2n^2 - 2n^2 - 2}{n^2+1}\right| < \varepsilon$$

$$\frac{2}{n^2+1} < \varepsilon \Rightarrow \frac{2}{\varepsilon} < n^2+1$$

$$n^2 > \frac{2}{\varepsilon} - 1$$

$$\boxed{n > \left(\frac{2}{\varepsilon} - 1\right)^{1/2} + 1}$$

$$n \geq \left(\frac{2}{\varepsilon}-1\right)^{1/2}+1 = M$$

Proof

So, Consider $M = \left(2/\varepsilon - 1\right)^{1/2} + 1$

Now, $n \geq M$

So, $$\left|\frac{2n^2}{n^2+1} - 2\right| = \left|\frac{2}{n^2+1}\right|$$

$$= \left|\frac{2}{\left(\left(2/\varepsilon - 1\right)^{1/2}+1\right)^2+1}\right|$$

$$< \frac{2}{2/\varepsilon + 2\left(2/\varepsilon - 1\right)^{1/2}+1}$$

$$= \frac{2\varepsilon}{2+2\varepsilon\left(\frac{2}{\varepsilon}-1\right)^{1/2}+\varepsilon} < \frac{2\varepsilon}{2}$$

so, $\frac{2\varepsilon}{2} < \varepsilon$

$$\left(2/\varepsilon\right)^{1/2} \qquad \frac{2^{n}}{2} < \varepsilon$$

$$\frac{2^{n}}{2+\varepsilon} <$$

5 $\lim_{x \to a} f(x) = f(a)$

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Is f continuous at $x = 0$

7 $\forall M \in \mathbb{R}, \exists t \in (a,b)$ s.t

$|f(t)| > M$

Prove, $\log x$ unbounded on $(0,1)$

let $t = 10^{(-(M+1))}$ $(-|M|+1)$

$? -(M+1))$

$\exists M+1 > M$

6 $\exists M \in \mathbb{R}$ s.t, $\forall n \in \mathbb{N}$ $|x_n| \leq M$

if x_n converges to 0 then x_n is bounded

(5) $\forall \varepsilon > 0, \exists \delta > 0, 0 < |x-a| < \delta \Rightarrow$

$(|f(x) - L| < \varepsilon)$

$\lim_{x \to a} f(x) = f(a)$

Scratch work

we - want

$\lim_{x \to 0} f(x) = f(0)$) $0 < |x - 0| < \delta$

$\Rightarrow |x^2 \sin(\frac{1}{x}) - f(0)|$

$\frac{1}{x} > \frac{1}{\delta}$

$< \varepsilon$

$\sin(1/x) > \sin(1/\delta)$

$\Rightarrow x^2 \sin(1/x) - 0 < \varepsilon$

$\Rightarrow x^2 \sin(1/x) < \varepsilon$

$0 < x^2 < \delta^2$

$x^2 \sin(1/\delta) < \delta^2 \sin(1/x)$

$x^2 \sin(1/x) < \frac{\delta^2 \sin(1/x)^2}{\sin(1/\delta)}$

$$\left| x^2 \sin\left(\frac{1}{x}\right) \right| < \varepsilon$$

$$x^2 < \delta^2$$

$$\delta = \sqrt{\varepsilon}$$

$|x^2| \; |\sin 1/x|$

$2 \times 0 - 2$

0.4

$$0 < \left| x^2 \sin(1/x) \right| < x^2$$

But $x^2 < \delta^2$

so, $|x^2 \sin(1/x)| < \delta^2$

$$\delta = \sqrt{\varepsilon}$$

⑥ Bounded if

$\exists M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $|x_n| \leq M$

If x_n converges to 0

i.e

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N,$

$|x_n - 0| < \varepsilon$

then

$\exists M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $|x_n| \leq M$

$|x_n| < \varepsilon$

(8) $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \varepsilon$

$x_n = (-1)^n + \frac{1}{n}$ does not converge

So, $\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N, |x_n - L| \geq \varepsilon$

<u>Scratch</u>

$L = 0$

we want $x_n \geq \varepsilon$

$$(-1)^n + \frac{1}{n} \geq \varepsilon$$

$\varepsilon = 1$

$n = 2 \cdot N$

$$(-1)^{2N} + \frac{1}{2N} \geq 1 + \frac{1}{2N} > 1 \geq \varepsilon$$

$\boxed{L = -ve}$

$$|x_n + L| \geq \varepsilon$$

$$(-1)^n + \frac{1}{n} \geq \varepsilon - L$$

$$(-1)^{2N} + \frac{1}{2N} + L \geq \varepsilon$$

$$\left| 1 + \frac{1}{2N} + L \right| \geq \varepsilon$$

$$1 + \frac{1}{2N} + L \geq \varepsilon$$

-1

≥ 1

$$1 + \frac{1}{2N} + L > L \geq \varepsilon$$

$ma\{1, L$

$\boxed{L = +ve}$

$$|x_n - L| \geq \varepsilon$$

$$\left|(-1)^n + \frac{1}{n} - L\right| \geq \varepsilon$$

$$1 + \frac{1}{2N} - L \geq \varepsilon$$

$$1 - L + \frac{1}{2N} \geq \varepsilon$$

⑥ $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N,$

$|x_n - 0| < \varepsilon$

then

$\exists M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, |x_n| \leq M$

<u>Case 1</u>

$n > N$

$x_n < \varepsilon$

<u>Case 2</u>

$n \leq N$

(8) $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N,$

$(-1)^n + \frac{1}{n}$ $\qquad |x_n - L| < \varepsilon$

$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N,$

$|x_n - L| \geq \varepsilon$

$\boxed{L \geq 0}$

$\varepsilon = 1 - L$

$n = 2N$

$\boxed{L < 0}$

$\varepsilon = 1 - L$

$n = 2N + 1$

$\boxed{L = +ve}$

$$\left| (-1)^n + \frac{1}{n} - L \right| \geq \varepsilon$$

$n = 2N$

$$\left| 1 + \frac{1}{2N} - L \right| \qquad \varepsilon = 1 - L$$

$$1 + \frac{1}{2N} - L \geq 1 - L$$

$\boxed{L = -ve}$

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