Mathematics 220 — Homework 5

- Contains 8 questions on 1 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

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1. Prove that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n}(2k-1)\cdot 2^k = 6 + 2^n(4n-6).$$

2. Let $n \in \mathbb{N}$. Prove that if $a_{n+2} = 5a_{n+1} - 6a_n$ and $a_1 = 1, a_2 = 5$, then $a_n = 3^n - 2^n$ for all $n \geq 3$.

3. Let $n \in \mathbb{N}$ and suppose that $a_0 = 1$, $a_1 = 3$, $a_2 = 9$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$. Show that $a_n \leq 3^n$.

4. Prove that for all integers $n > 1$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{13}{24}.$$

5. Prove that $7^{4n+3} + 2$ is a multiple of 5 for all non-negative integers n.

6. We define a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_1 = 3$, and for every $n \geq 1$, $a_{n+1} = a_n^2 - a_n$. Show that (a_n) is increasing, which means that for all $n \in \mathbb{N}$, $a_n < a_{n+1}$.

   Hint: It is actually easier to prove that $a_{n+1} > a_n > 1$. Also to show $a_{n+1} > a_n$ it might be easier to show that $a_{n+1} - a_n > 0$.

7. Let $x \in \mathbb{R}$ with $x \neq 1$ and let $N \in \mathbb{N}$. Use mathmematical induction to show that

$$\sum_{k=1}^{N} k \cdot x^{k-1} = \frac{1-x^N}{(1-x)^2} - \frac{Nx^N}{1-x}$$

8. Find all positive integers n so that $n^3 > 2n^2 + n$. Prove your result using mathematical induction.

   Note: It is possible to prove this without induction, but the point of this question is to get you to practice using induction.

① $\sum_{K=1}^{n} (2K-1)\cdot 2^{K} = 6 + 2^{n}(4n-6)$

Base Case

$n=1$

So, $(2(1)-1)\cdot 2^{1}$ | $6+2^{1}(4-6)$

$6-4$

2 | 2

So, holds true

Inductive Step

Assume it holds true for $n=p$

So, $\sum_{K=1}^{p} (2p-1)\cdot 2^{p} = 6+2^{p}(4p-6)$

For $n=p+1$ we have,

$\sum_{K=1}^{p} (2p-1)\cdot 2^{p} + (2(p+1)-1)\cdot 2^{p+1}$

$\Rightarrow 6 + 2^{p}(4p-6) + (4p+2)\cdot 2^{p}$

$\Rightarrow 6+2^{p}(4p-6+4p+2)$

$\Rightarrow 6+2^{p}(8p-4)$

$\Rightarrow 6+2^{p+1}(4p-2)$

$\Rightarrow 6+2^{p+1}(4(p+1)-6)$

So, holds true for $n=p+1$

(2) if $a_{n+2} = 5a_{n+1} - 6a_n$ & $a_1 = 1$, $a_2 = 5$

then $a_n = 3^n - 2^n$ for all $n \geq 3$

Proof

Base Case

$a_1 = 1 = 3^1 - 2^1 = 1$

$a_2 = 5 = 3^2 - 2^2 = 9 - 4 = 5$

Hence verified for $n = 1 \& 2$

Induction Step

Consider $k \geq 2$ and assume that $a_i = 3^i - 2^i$ for all $0 \leq i \leq k$.

$$a_{(k-1)+2} = 5a_k - 6_{k-1}$$

$$a_{k+1} = 5 \cdot (3^k - 2^k) - 6 \cdot (3^{k-1} + 2^{k-1})$$

$$a_{k+1} = 5(3^k - 2^k) - 6\left(\frac{3^k}{3} - \frac{2^k}{2}\right)$$

$$a_{k+1} = 5(3^k - 2^k) - 6\left(\frac{2 \cdot 3^k - 3 \cdot 2^k}{6}\right)$$

$$a_{k+1} = 5(3^k - 2^k) - 6\left(\frac{3^k}{3} - \frac{2^k}{2}\right)$$

$$a_{k+1} = 5(3^k - 2^k) - 6\left(\frac{2\cdot 3^k - 3\cdot 2^k}{6}\right)$$

$$a_{k+1} = 5\cdot 3^k - 5\cdot 2^k - 2\cdot 3^k + 3\cdot 2^k$$

$$a_{k+1} = 3\cdot 3^k - 2\cdot 2^k$$

$$a_{k+1} = 3^{k+1} - 2^{k+1}$$

$\therefore$ Hence Proved

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(3) $a_0 = 1$, $a_1 = 3$, $a_2 = 9$ &

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$

S.T $a_n \leq 3^n$

Proof

Base Case

$a_0 = 1 = 3^0 = 1$ $\qquad a_2 = 9 = 3^2 = 9$

$a_1 = 3 = 3^1 = 3$

Hence true for base case

$n = 0, 1 \& 2$

Inductive step

For $k \geq 2$ we have $a_i \leq 3^i$ for $0 \leq i \leq k$.

Now, $a_{k+1} = a_k + a_{k-1} + a_{k-2}$

So $a_{k+1} = 3^k + 3^{k-1} + 3^{k-2}$

$$a_{k+1} = 3^k\left(1 + \frac{1}{3} + \frac{1}{9}\right)$$

$$a_{k+1} = 3^k\left(\frac{9+3+1}{9}\right)$$

$$a_{k+1} = 3^{k-2}(13)$$

Now, $3^{k-2}(13) < (27)\,3^{k-2}$

& $a_{k+1} < 27 \cdot 3^{k-2}$

$$a_{k+1} < 3^{k+1}$$

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4

$$5^{2K} + 3K = 9m + 1$$

$K \geq 1$

<u>Base Case</u>

$K = 1$

$$5^2 + 3 = 28$$
$$= 9(3) + 1$$

<u>Induction</u>

Assume that,

$K = n$

$$5^{2n} + 3n = 9m + 1$$

$$5^{2(n+1)} + 3(n+1)$$

$5^{2(n+1)} + 3(n+1)$

$5^{2n} 25 + 3n + 3$

$(9m+1-3n) \cdot 25 + 3n + 3$

$9m \cdot 25 + 25 - 25 \cdot 3n + 3n + 3$

$9m \cdot 25 + 28 - 72 \cdot n$

$9(25m + 3 - 8n) + 1$

since $5^{2(n+1)} + 3(n+1)$

$\equiv 1 \pmod 9$

it's proved //

(5) P.T $7^{4n+3}+2$ is a multiple of 5 $\forall n \in \mathbb{Z}, n>0$

Base Case

$n = 1$

$7^{4n+3}+2 \Rightarrow 7^{4+3}+2 \Rightarrow 7^7+2$

$\Rightarrow 823543+2$

$\Rightarrow 823545$

$\Rightarrow$ Hence holds true

Inductive Step

Consider true for $n = k$

So, $7^{4k+3}+2 = 5\cdot m$

For, $7^{4k+7}+2$

$7^4(5m-2)+2 \Rightarrow 5m\cdot 7^4 - 2\cdot 7^4 + 2$

$\Rightarrow 5m\cdot 7^4 - 4800$

$\therefore$ Hence true

(6) $(a_n)_{n\in\mathbb{N}}$; $a_1 = 3$ $\forall n \geq 1$
$a_{n+1} = a_n^2 - a_n$

S.T a_n is inc i.e $\forall n \in \mathbb{N}, a_n < a_{n+1}$

Base Case

For $n = 1$,

$$a_2 = a_1^2 - a_1$$
$$a_2 = 9 - 3$$
$$= 6$$

Now, $6 > 3 > 2$ So, a_i is increasing.

Inductive Case

For $K \geq 2$, we have $a_i < a_{i+1}$
for $0 \leq i \leq K$

So, $a_{K+1} = a_K^2 - a_K$ i.e $a_K < a_{K+1}$

Now, $a_K = a_{K-1}^2 - a_{K-1}$ i.e $a_{K-1} < a_K$

$$a_{K+2} = a_{K+1}^2 - a_{K+1}$$

Now, $a_K < a_{K+1}$
& so, $a_K^2 < a_{K+1}^2$

So, $a_K^2 - a_K < a_{K+1}^2 - a_{K+1}$
Since $a_{K+1+1} = a_{K+1}^2 - a_{K+1}$
We have, $a_{K+1} < a_{K+2}$

$\blacksquare$

7) Let, $x \in \mathbb{R}$ with $x \neq 1$ & $N \in \mathbb{N}$, s.t,

$$\sum_{k=1}^{N} k \cdot x^{k-1} = \frac{1-x^N}{(1-x)^2} - \frac{N x^N}{1-x}$$

Base Case

$N = 1$

So, $\sum_{k=1}^{1} 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$

& $\frac{1-x^1}{(1-x)^2} - \frac{1 \cdot x^1}{1-x} \Rightarrow \frac{1}{1-x} - \frac{x}{1-x}$

$\Rightarrow \frac{1-x}{1-x} \Rightarrow 1$

So, the base case holds true.

Inductive Step

Assume $N = i$ holds true

hence,
$$\sum_{k=1}^{i} k \cdot x^{k-1} = \frac{1-x^i}{(1-x)^2} - \frac{i \cdot x^i}{(1-x)}$$

$$(i+1) \cdot x^i + \frac{1-x^i}{(1-x)^2} - \frac{i \cdot x^i}{(1-x)}$$

$$(i+1)\cdot x^i + \frac{1-x^i}{(1-x)^2} - \frac{i\cdot x^i}{(1-x)}$$

$$(1-x)^2\cdot(i+1)\cdot x^i + (1-x^i) - (i\cdot x^i(1-x))$$

$$(i+1)x^i + (i+1)x^{i+2} - 2(i+1)x^{i+1} + 1 - x^i - ix^i + ix^{i+1}$$

$$\underline{(i+1)x^i} + (i+1)x^{i+2} - 2(i+1)x^{i+1} + 1 \underline{- x^i(1+i)} + ix^{i+1}$$

$$(i+1)x^{i+2} - x^{i+1}(2i+2-i) + 1$$

$$(i+1)x^{i+2} - x^{i+1}(i+2) + 1$$

$$\frac{(i+1)x^{i+2} - x^{i+1} - x^{i+1}(i+1) + 1}{(1-x)^2}$$

$$\frac{1 - x^{i+1} - (i+1)x^{(i+1)}(1-x)}{(1-x)^2}$$

$$\frac{1-x^{i+1}}{(1-x)^2} - \frac{(i+1)x^{i+1}}{(1-x)}$$

So true for $N=i+1$ as well

Hence proved by induction $\forall N \in \mathbb{N}$

$\blacksquare$

(8) $n^3 > 2n^2 + n$

Base Case

$n = 3$

$n^3 = 27$

$2n^2 + n = 21$

So, $27 > 21$ i.e $n^3 > 2n^2 + n$

Inductive Step

Consider this to be true for $n = k$

So, $k^3 > 2k^2 + k$