Solutions to Homework 2:

1. *Proof.* if $a \in \mathbb{Z}$ then $4 \nmid (a^2+1)$

Case 1: a is odd

$$a = 2k+1 \qquad \text{where } \exists k \in \mathbb{Z} \tag{1}$$

So,

$$(a^2+1) = (2k+1)^2 + 1 \tag{2}$$
$$(a^2+1) = 4k^2 + 1 + 4k + 1 \tag{3}$$
$$(a^2+1) = 4(k^2+k) + 2 \tag{4}$$
$$\exists j \in \mathbb{Z} \text{ where } j = k^2+k \qquad (a^2+1) = 4j + 2 \tag{5}$$

So, $4 \nmid (a^2+1)$

Case 2: a is even

$$a = 2k \qquad \text{where } \exists k \in \mathbb{Z} \tag{6}$$

So,

$$(a^2+1) = (2k)^2 + 1 \tag{7}$$
$$(a^2+1) = 4k^2 + 1 \tag{8}$$
$$(a^2+1) = 4k^2 + 1 \tag{9}$$
$$\exists i \in \mathbb{Z} \text{ where } i = k^2 \qquad (a^2+1) = 4i + 1 \tag{10}$$

So, $4 \nmid (a^2+1)$

Hence, by Proof of Cases $4 \nmid (a^2+1)$ □

2. *Proof.* if $2x - \frac{1}{x} > 1$ then $x > 1$

$$2x - \frac{1}{x} > 1 \tag{11}$$
$$\frac{(2x^2-1)}{x} > 1 \tag{12}$$
$$(2x^2-1) > x \tag{13}$$
$$2x^2 - x - 1 > 0 \tag{14}$$

We then factorize the equation so obtained to get,

$$2x^2 - 2x + x - 1 > 0 \tag{15}$$
$$2x(x-1) + 1(x-1) > 0 \tag{16}$$
$$(2x-1)(x-1) > 0 \tag{17}$$
$$\tag{18}$$

We obtain $x - 1/2$ and $x - 1$ and 1 is greater,
So, $x - 1 > 0$ or $x > 1$

$\square$

3. *Proof.* if $k \in \mathbb{Z}$ then $3 \mid (k(2k+1)(4k+1))$

Case 1: k is divisible by 3

$$k = 3m \qquad \text{where } m \in \mathbb{Z} \tag{19}$$

For the equation $(k(2k+1)(4k+1))$,

$$((3m)(2(3m)+1)(4(3m)+1)) \tag{20}$$
$$(3m)(6m+1)(12m+1) \tag{21}$$
$$(18m^2 + 3m)(12m+1) \tag{22}$$
$$(216m^3 + 18m^2 + 36m^2 + 3m) \tag{23}$$
$$(216m^3 + 54m^2 + 3m) \tag{24}$$
$$3(72m^3 + 18m^2 + m) \tag{25}$$

So, $3 \mid (k(2k+1)(4k+1))$

Case 2: k is not divisible by 3

$$k = 3m + 1 \qquad \text{where } m \in \mathbb{Z} \tag{26}$$

For the equation $(k(2k+1)(4k+1))$,

$$((3m+1)(2(3m+1)+1)(4(3m+1)+1)) \tag{27}$$
$$(3m+1)(6m+3)(12m+5) \tag{28}$$
$$(18m^2 + 15m + 3)(12m+5) \tag{29}$$
$$(216m^3 + 90m^2 + 180m^2 + 90m + 36m + 15) \tag{30}$$
$$(216m^3 + 270m^2 + 126m + 15) \tag{31}$$
$$3(72m^3 + 90m^2 + 42m + 5) \tag{32}$$

So, $3 \mid (k(2k+1)(4k+1))$

Case 3: k is not divisible by 3

$$k = 3m + 2 \qquad \text{where } m \in \mathbb{Z} \tag{33}$$

For the equation $(k(2k+1)(4k+1))$,

$$((3m+2)(2(3m+2)+1)(4(3m+2)+1)) \tag{34}$$
$$(3m+2)(6m+5)(12m+9) \tag{35}$$
$$(18m^2+27m+10)(12m+9) \tag{36}$$
$$(216m^3+162m^2+324m^2+243m+120m+90) \tag{37}$$
$$(216m^3+486m^2+363m+90) \tag{38}$$
$$3(72m^3+162m^2+121m+30) \tag{39}$$

So, $3 \mid (k(2k+1)(4k+1))$

$\square$

Hence, by Proof by Cases $3 \mid (k(2k+1)(4k+1))$

4. *Proof.* For $n \in \mathbb{Z}$

(a) If $3 \mid n$ and $4 \mid n$, then $12 \mid n$,
Let's consider $\exists p, q \in \mathbb{Z}$ such that,

$$n = 3 * p \tag{40}$$
$$n = 4 * q \tag{41}$$

We multiply eq(40) with 4 and eq(41) with 3 in order to obtain

$$4 * n = 12 * p \tag{42}$$
$$3 * n = 12 * q \tag{43}$$

On subtraction eq(43) from eq(42)

$$n = 12(p - q) \tag{44}$$

We know that subtraction of two integers results in an integer so let $\exists r \in \mathbb{Z}$ such that $r = p - q$,

$$n = 12 * r \tag{45}$$

Hence, $12 \mid n$

(b) If $n > 3$ is a prime then $n^2 \equiv 1(\text{mod } 12)$
Now, we know that if $3 \mid n$ and $4 \mid n$, then $12 \mid n$

Case 1: When $n > 3$ all prime numbers are odd,
So, for $\exists m \in \mathbb{Z}$, $n = 2 * m + 1$ Now,

$$n^2 = (2m+1)^2 \tag{46}$$
$$n^2 = 4m^2 + 4m + 1 \tag{47}$$
$$n^2 - 1 = 4m^2 + 4m + 1 - 1 \tag{48}$$
$$n^2 - 1 = 4m^2 + 4m \tag{49}$$
$$n^2 - 1 = 4(m^2 + m) \tag{50}$$

So $\exists p \in \mathbb{Z}$ such that $p = m^2 + m$,

$$n^2 - 1 = 4 * p \tag{51}$$

Hence, $4 \mid n^2 - 1$ or $n^2 \equiv 1 (\text{mod } 4)$

Case 2: n is not divisible by 3

i. **Case 2i:**

$$n = 3m + 1 \qquad \text{where } m \in \mathbb{Z} \tag{52}$$

Now,

$$n^2 = (3m+1)^2 \tag{53}$$
$$n^2 = 9m^2 + 6m + 1 \tag{54}$$
$$n^2 - 1 = 9m^2 + 6m + 1 - 1 \tag{55}$$
$$n^2 - 1 = 9m^2 + 6m \tag{56}$$
$$n^2 - 1 = 3(3m^2 + 2m) \tag{57}$$

For $\exists j \in \mathbb{Z}$ such that $j = 3m^2 + 2m$,

$$n^2 - 1 = 3 * j \tag{58}$$

So, $n^2 - 1$ is divisible by 3

ii. **Case 2ii:**

$$n = 3m + 2 \qquad \text{where } m \in \mathbb{Z} \tag{59}$$

Now,

$$n^2 = (3m+2)^2 \tag{60}$$
$$n^2 = 9m^2 + 12m + 4 \tag{61}$$
$$n^2 - 1 = 9m^2 + 12m + 4 - 1 \tag{62}$$
$$n^2 - 1 = 9m^2 + 12m + 3 \tag{63}$$
$$n^2 - 1 = 3(3m^2 + 4m + 1) \tag{64}$$

For $\exists i \in \mathbb{Z}$ such that $j = 3m^2 + 4m + 1$,

$$n^2 - 1 = 3 * i \tag{65}$$

So, $n^2 - 1$ is divisible by 3

Hence, $3 \mid n^2 - 1$ or $n^2 \equiv 1 (\text{mod } 3)$

So, since if $3 \mid n^2 - 1$ and $4 \mid n^2 - 1$ then $12 \mid n^2 - 1$
Which is the same as $n^2 \equiv 1 (\text{mod } 12)$ □

5. *Proof.* For $n \in \mathbb{Z}$ if $n^3 + n^2 - n + 3$ a multiple of three then n is a multiple of three
So, taking the contra-positive: if n is not a multiple of three then $n^3 + n^2 - n + 3$ is not a multiple of three
This statement can also be written as,

$$3 \nmid n \implies 3 \nmid (n^3 + n^2 - n + 3) \tag{66}$$

Case 1: n = 3k + 1 $\exists k \in \mathbb{Z}$
So for the equation $n^3 + n^2 - n + 3$ we replace n with 3k + 1

$$(3k+1)^3 + (3k+1)^2 - (3k+1) + 3 \tag{67}$$
$$(9k^3 + 1 + 9k + 27k^2) + (9k^2 + 1 + 6k) - 3k - 1 + 3 \tag{68}$$
$$9k^3 + 36k^2 + 12k + 1 \tag{69}$$
$$3(3k^3 + 12k^2 + 4k) + 1 \tag{70}$$

For $\exists p \in \mathbb{Z}$ where $p = 3k^3 + 12k^2 + 4k$,

$$n^3 + n^2 - n + 3 = 3p + 1 \tag{71}$$

Case 2: n = 3k + 2 $\exists k \in \mathbb{Z}$
So for the equation $n^3 + n^2 - n + 3$ we replace n with 3k + 2

$$(3k+2)^3 + (3k+2)^2 - (3k+2) + 3 \tag{72}$$
$$(9k^3 + 8 + 36k + 54k^2) + (9k^2 + 4 + 12k) - 3k - 2 + 3 \tag{73}$$
$$9k^3 + 63k^2 + 45k + 12 + 1 \tag{74}$$
$$3(3k^3 + 21k^2 + 15k + 4) + 1 \tag{75}$$

For $\exists q \in \mathbb{Z}$ where $q = 3k^3 + 21k^2 + 15k + 4$,

$$n^3 + n^2 - n + 3 = 3q + 1 \tag{76}$$

So, we can say that $3 \nmid n^3 + n^2 - n + 3$ or that $n^3 + n^2 - n + 3$ is not a multiple of 3 □

6. *Proof.* **Case 1:** $x > 6$
Here, we have that $x > 6$ so we can also say that $x^2 > 36$
So,

$$
\begin{align}
x^2 &> 36 \tag{77}\\
x^2 + x &> 36 + 6 \tag{78}\\
x^2 + x &> 42 \tag{79}\\
x^2 + x - 6 &> 42 - 6 \tag{80}\\
x^2 + x - 6 &> 36 \tag{81}
\end{align}
$$

Here we get that $x^2 - x + 6$ is greater than 36 which also means that $x^2 + x - 6$ is greater than 5 Thus, $x^2 + x - 6 > 5$ Since we are assuming $x > 6$ the value of $|x - 6|$ will be positive so we can write x - 6 as $|x - 6|$,

$$x^2 + |x - 6| > 5 \tag{82}$$

Case 2: $x \leq 6$
So, let's consider the equation $x^2 + |x - 6|$
Here we can write $|x - 6|$ as simply -(x - 6) because $x \leq 6$ We will try to simplify this equation using completing the square method

$$
\begin{align}
x^2 - (-(x - 6)) \tag{83}\\
x^2 + (x - 6) \tag{84}\\
x^2 + 2 * x * \frac{1}{2} - 6 + \frac{1}{4} - \frac{1}{4} \tag{85}\\
(x + \frac{1}{2})^2 - 6 - \frac{1}{4} \tag{86}\\
(x + \frac{1}{2})^2 - \frac{25}{4} \tag{87}\\
(x + \frac{1}{2})^2 = \frac{25}{4} \tag{88}
\end{align}
$$

Here we get that $x^2 + x - 6$ is equal to $25/4$ which also means that $x^2 + x - 6$ is greater than 5 Thus, $x^2 + x - 6 > 5$ or,

$$x^2 - |x - 6| > 5 \tag{89}$$

Hence $x^2 - |x - 6| > 5$ $\square$

7. *Proof.* We want to prove that for $x, y \in \mathbb{Z}$, $3 \nmid (x^3 + y^3) \iff 3 \nmid (x + y)$ or,
$(3 \nmid (x^3 + y^3) \implies 3 \nmid (x + y)) \wedge (3 \nmid (x + y) \implies 3 \nmid (x^3 + y^3))$
Case 1: $(3 \nmid (x^3 + y^3) \implies 3 \nmid (x + y))$
Contra-positive of that statement would be $(3 \mid (x + y) \implies 3 \mid (x^3 + y^3))$

So, for $m \in \mathbb{Z}(x + y) = 3 * m$
On cubing that equation so found we obtain,

$$(x + y)^3 = (3m)^3 \tag{90}$$
$$x^3 + y^3 + 3xy(x + y) = 27 * m^3 \tag{91}$$
$$x^3 + y^3 = 27 * m^3 - 3xy(x + y) \tag{92}$$
$$x^3 + y^3 = 3(9 * m^3 + xy(x + y)) \tag{93}$$

So $\exists i \in \mathbb{Z}$ such that $i = 9 * m^3 + xy(x + y)$,

$$x^3 + y^3 = 3i \tag{94}$$

We can also say that $3 \mid x^3 + y^3$

Case 2: $(3 \nmid (x + y) \implies 3 \nmid (x^3 + y^3))$
Contra-positive of that statement would be $(3 \mid (x^3 + y^3) \implies 3 \mid (x + y))$
So, for $m \in \mathbb{Z}(x^3 + y^3) = 3 * m$
Now, let's consider the equation $(x + y)^3$

$$(x + y)^3 \tag{95}$$
$$(x + y)^3 = x^3 + y^3 + 3xy(x + y) \tag{96}$$
$$(x + y)^3 = 3 * m + 3xy(x + y) \tag{97}$$
$$(x + y)^3 = 3(m + xy(x + y)) \tag{98}$$

Now, since a number is divisible by three so is its cube root So $\exists j \in \mathbb{Z}$ such that $j = m + xy(x + y)$,

$$x + y = 3j \tag{99}$$

We can also say that $3 \mid x + y$

Now, since we have proved both $(3 \nmid (x^3 + y^3) \implies 3 \nmid (x + y))$ and $(3 \nmid (x + y) \implies 3 \nmid (x^3 + y^3))$
We can say that $3 \nmid (x^3 + y^3) \iff 3 \nmid (x + y)$ □

8. *Proof.* We need to prove that $k \nmid gcd(a, b) \implies (k \nmid a \vee k \nmid b)$
The contra-positive of which is $(k \mid a \wedge k \mid b) \implies k \mid gcd(a, b)$
So, we get two equations for $\exists x, y \in \mathbb{Z}$

$$a = x * k \tag{100}$$
$$b = y * k \tag{101}$$

We then multiply eq(100) with x and eq(101) with y to obtain,

$$a * x = x^2 * k \tag{102}$$
$$b * y = b^2 * k \tag{103}$$

On adding the two equations

$$a * x + b * y = x^2 * k + y^2 * k \tag{104}$$
$$a * x + b * y = k(x^2 + y^2) \tag{105}$$

Now, from Bezout's identity we know that ax + by is equal to gcd(a,b) so,

$$a * x + b * y = k(x^2 + y^2) = gcd(a, b) \tag{106}$$
$$\exists m \in \mathbb{Z} \qquad a * x + b * y = k * m = gcd(a, b) \tag{107}$$

We obtain that $gcd(a, b) = k * m$,
Hence $k \mid gcd(a, b)$

$\square$