Mathematics 220 — Homework 9

- Contains 8 questions on 1 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

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1. Suppose $f : A \to A$ such that $f \circ f$ is bijective. Is f necessarily bijective?

2. Suppose that $f : A \to B$ is a surjection and $Y \subseteq B$. Show that
$$f(f^{-1}(Y)) = Y.$$

3. Let $f : E \to F$ be a function. We recall that for any $A \subseteq E$, the image $f(A)$ of A by f is defined as
$$f(A) = \{f(x) : x \in A\}.$$
Show that f is surjective if and only if
$$\forall A \subseteq E,\ F - f(A) \subseteq f(E - A).$$

4. (a) Prove that the function $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = x^2 - y^2$, is surjective.
   (b) Find $g^{-1}(\{0\})$.
   (c) Let $A := \{a \in \mathbb{R} : a \geq 0\}$ and consider the function $h : A \to A$, $h(x) = x^4 + 3$. Find $h^{-1}(\{c\})$ for each c in the codomain.

5. For a function $f : A \to B$ and subsets E and F of B, prove
$$f^{-1}(E - F) = f^{-1}(E) - f^{-1}(F).$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + ax + b$, where $a, b \in \mathbb{R}$. Determine whether f is injective and/or surjective.

7. For $n \in \mathbb{N}$, let $A = \{a_1, a_2, a_3, \cdots, a_n\}$ be a fixed set and let F be the set of all functions $f : A \to \{0, 1\}$.
   (a) What is $|F|$, the cardinality of F?
   (b) Now, for $\mathcal{P}(A)$, the power set of A, consider the function $g : F \to \mathcal{P}(A)$, defined as
   $$g(f) = \{a \in A : f(a) = 1\}.$$
   Is g injective? Is g surjective?

8. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ that are injective and such that for all $n \in \mathbb{N}$ we have $f(n) \leq n$.

① $f: A \to A$ $\quad$ $f \circ f$ is bijective. Is f necessarily bijective?

$$f \circ f : A \to A$$

- w.r.t $f \circ f$ is injective,

So, let $x_1, x_2 \in A$ such that
$f(x_1) = f(x_2)$
So, $f \circ f(x_1) = f \circ f(x_2)$
& since $f \circ f$ is injective we have $x_1 = x_2$
& f is injective.

- w.r.t $f \circ f$ is surjective,

Let $y_1 \in A$ it suffices to show that for $x_1 \in A$, $f(x_1) = y_1$
Since $f \circ f$ is surjective $a \in A$ so, $f(f(a)) = y_1$
Now, let $f(a) = x_1$ then $f(x_1) = f(f(a)) = y_1$

& f is surjective

Hence f is bijective

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② $f: A \to B$ is a surjection & $Y \subseteq B$ s.t
$$f(f^{-1}(Y)) = Y$$

So, for $y_1 \in Y$
we have $f^{-1}(y_1) = \{x_1 \in A \mid f(x_1) = y_1\}$

Now, $x_1 \in f^{-1}(y_1)$
so, $f(x_1) = y_1 \in Y$

So, $x_1 \in f^{-1}(Y)$

So, $f(x_1) \in f(f^{-1}(Y))$

Now, $f(x_1) = y_1$

So, $y_1 \in f(f^{-1}(Y))$

But $y_1 \in Y$

So, $Y \in f(f^{-1}(Y))$

or

$Y = f(f^{-1}(Y))$

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(3) f is surjective iff $\forall A \subseteq E, F - f(A) \subseteq f(E-A)$

$$\boxed{\text{Qs } 11}$$

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(4) (a) $g : \mathbb{R}^2 \to \mathbb{R}$

$g(x,y) = x^2 - y^2$ is surjective

Case 1

$x^2 > y^2$, so, $x^2 - y^2 > 0$

$z = x^2 - y^2$

$x = \sqrt{y^2 + z}$

$y = \sqrt{x^2 - z}$

Now, $z = x^2 - y^2$

so, $z > 0$

Case 2

$x^2 < y^2$ so, $x^2 - y^2 < 0$

⋮

$z < 0$

Case 3

$x^2 = y^2$

$$z = x^2 - y^2$$
$$z = 0$$

So, z is $\mathbb{R}$

So, since the range = codomain $g(x,y) = x^2 - y^2$ is surjective.

(b) $g^{-1}(\{0\})$

Now, $z = \sqrt{x^2 - y^2}$

$0 = \sqrt{x^2 - y^2}$

$x^2 = y^2$

$x = \pm y$

$y = \pm x$

So, $(\pm x, \pm y)$

Prove it properly

(c) $A = \{a \in \mathbb{R} : a \geq 0\}$

$h: A \to A$, $h(x) = x^4 + 3$

Find $h^{-1}(\{c\})$

$$c = x^4 + 3$$
$$c - 3 = x^4$$
$$x = (c-3)^{1/4} \text{ or } (3-c)^{1/4}$$

(5) $f: A \to B$ & subsets E & F of B, P.t

$$f^{-1}(E-F) = f^{-1}(E) - f^{-1}(F)$$

$E, F \subseteq B$

Let $x \in f^{-1}(E-F)$ where $x \in A$

So, $f(x) \in E-F$

or

$f(x) \in E$ and $f(x) \notin F$

So, $x \in f^{-1}(E)$ and $x \notin f^{-1}(F)$

So, $x \in f^{-1}(E) - f^{-1}(F)$

Hence $f^{-1}(E-F) = f^{-1}(E) - f^{-1}(F)$

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(6) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2 + ax + b$ where $a, b \in \mathbb{R}$

<u>Injective</u>

Let $x_1, x_2 \in \mathbb{R}$

So,

$$x_1^2 + ax_1 + b = x_2^2 + ax_2 + b$$

$$x_1^2 + ax_1 = x_2^2 + ax_2$$

$$\left(x_1 + \frac{a}{2}\right)^2 - \frac{a^2}{4} = \left(x_2 + \frac{a}{2}\right)^2 - \frac{a^2}{4}$$

$$\left(x_1 + \frac{a}{2}\right)^2 = \left(x_2 + \frac{a}{2}\right)^2$$

So, $x_1 \neq x_2$

Hence it's not injective

Surjective

$$f(x) = x^2 + ax + b$$

So, for $y \in \mathbb{R}$ such that $f(x) = y$
we have

$$y = \left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4}$$

$$y = \left(x + \frac{a}{2}\right)^2 + \frac{4b - a^2}{4}$$

$$y - \left(\frac{4b - a^2}{4}\right) = \left(x + \frac{a}{2}\right)^2$$

$$\sqrt{\frac{4y - 4b + a^2}{4}} - \frac{a}{2} = x$$

Put x in $f(x)$ & get y

$$\frac{4y - 4b + a^2}{4} + \frac{a^2}{4} - \frac{a}{2}\left(\sqrt{4y - 4b + a^2}\right)$$

$$+ \frac{a}{2}\sqrt{4y - 4b + a^2} - \frac{a^2}{2} + b$$

$$y - b + \frac{a^2}{2} - \frac{a^2}{2} + b$$

$$y = f(x)$$ Hence f is surjective

7 for $n \in \mathbb{N}$

$A = \{a_1, a_2, a_3 \dots a_n\}$

F is set of all fnct $f : A \to \{0, 1\}$

(a) $|F| = ?$

$|F| = 2^n$ ✓

a_1 has 2 choices
a_2 has 4 } proof by induction

⋮

a_n

Base

$n = 1$

a_1 has 2 choices

now $2^1 = 2$ ✓

Ind

Assume 2^k is true for $a_1, a_2 \dots a_k$

Now, for $a_1, a_2 \dots a_{k+1}$

we have

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(b) $P(A)$

$g : F \to P(A)$

$g(f) = \{a \in A : f(a) = 1\}$

is g injective and/or surjective?