PLP - 10

TOPIC 10 —LOGICAL EQUIVALENCE

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TAUTOLOGIES AND LOGICAL EQUIVALENCE

TAUTOLOGIES AND CONTRADICTIONS

Statements that are always true turn out to be very useful.

DEFINITION: TAUTOLOGIES AND CONTRADICTIONS.

A **tautology** is a statement that is always true

A **contradiction** is a statement that is always false.

The following are examples of tautologies

$$P \vee (\sim P) \qquad\qquad \sim (P \vee Q) \iff ((\sim P) \wedge (\sim Q))$$

The following are examples of contradictions

$$P \wedge (\sim P) \qquad\qquad (P \wedge Q) \wedge ((\sim P) \vee (\sim Q))$$

A VERY USEFUL TAUTOLOGY

- The statements $P \vee Q$ and $Q \vee P$ have the same truth-tables.
- The are *not the same* but they are *equivalent*
- We can express this by saying "$(P \vee Q) \iff (Q \vee P)$ *is a tautology*"

DEFINITION:

Two statements R and S are **logically equivalent** when "$R \iff S$" is a tautology.

In this case we write $R \equiv S$.

Showing logical equivalence

- build the truth tables, or
- think about when each side is true and false

A USEFUL EQUIVALENCE

Consider $(P \implies Q) \equiv (\sim P) \vee Q$.

Why are these equivalent — when true, when false?

- *Know your truth-tables!*
- **LHS** is false only when (P, Q) = (T,F). Otherwise true.
- **RHS** is false when both $(\sim P), Q$ are false, that is (P, Q) = (T,F). Otherwise false.

True at same time, false at same time. So equivalent.

Can also build the truth-tables — tedious but works.

P	Q	$P \implies Q$	$(\sim P) \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

USEFUL LOGICAL EQUIVALENCES

THEOREM: LOGICAL EQUIVALENCES.

Let P and Q be statements. Then

Implication

$$(P \implies Q) \equiv ((\sim P) \vee Q))$$

Contrapositive

$$(P \implies Q) \equiv ((\sim Q) \implies (\sim P))$$

Biconditional

$$(P \iff Q) \equiv ((P \implies Q) \wedge (Q \implies P))$$

Double negation

$$\sim (\sim (P)) \equiv P$$

Commutative laws

$$P \vee Q \equiv Q \vee P \qquad \text{and} \qquad P \wedge Q \equiv Q \wedge P$$

USEFUL LOGICAL EQUIVALENCES 2

THEOREM: LOGICAL EQUIVALENCES.

Let P, Q and R be statements. Then

Associative laws

$$P \vee (Q \vee R) \equiv (P \vee Q) \vee R \qquad \text{and} \qquad P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$$

Distributive laws

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R) \qquad \text{and} \qquad P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

DeMorgan's laws

$$\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q) \qquad \text{and} \qquad \sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$$

BACK TO THE CONTRAPOSITIVE

Show that $(P \implies Q) \equiv (\sim Q \implies \sim P)$ using equivalences

$$\begin{aligned}
(P \implies Q) &\equiv (\sim P \vee Q) && \text{implication as or} \\
&\equiv (Q \vee \sim P) && \text{commutes} \\
&\equiv (\sim\sim Q \vee \sim P) && \text{double negative} \\
&\equiv (\sim Q \implies \sim P) && \text{or as implication}
\end{aligned}$$

Why is this useful a useful equivalence?

- Contrapositive is *equivalent* to the original implication
- Proving one is true is *equivalent* as proving the other is true
- Sometimes the contrapositive is *easier* to prove than the original

BACK TO THE CONTRAPOSITIVE

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$p \Rightarrow (q \vee r)$

$\sim p \vee q \vee r$

$p \Rightarrow T$ $\boxed{T}$

$q \Rightarrow R$

p

T

$\sim(q \vee$

$(\sim q \wedge \sim r) \Rightarrow \sim p$

p	q	r	(p∨q)	¬R	p⇒R	q⇒R
F	F	F	F	T	T	T
F	F	T	F	T	T	T
F	T	F	T	F	T	F
F	T	T	T	T	T	T
T	F	F	T	F	F	T
T	F	T	T	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T