Math 220
Section 108
Lecture 25

6th December 2022

Sources: https://secure.math.ubc.ca/Ugrad/pastExams
https://www.people.vcu.edu/~rhammack/BookOfProof
https://personal.math.ubc.ca/~PLP/auxiliary.html

Finals Reminders!

- **Date:** Thursday 15th December at noon.
- **Length:** 2.5 hours.
- **Location:** OSB1 A
- **Syllabus:** Everything (including topics before the midterm).
- **Resources:** Good study resources include your homework, the lecture videos, your notes from class, class slides, the PLP textbook itself, and the 'auxiliary material' worksheets for PLP.
- **What's allowed:** This is a closed-book exam. No notes, formula sheets, calculators, or phones are permitted. You may not ask for or receive help from other people.
- **What to bring:** You need to bring things to write with (pencil and pen are both OK) and your UBC ID.

1. (a) Prove true or false: If R is an equivalence relation on some set A, then the equivalence classes of R are all the same size.
(b) Prove true or false: If R is an equivalence relation on an infinite set B, then R has infinitely many equivalence classes.
(c) Define a relation R on $\mathbb{Z}$ as xRy if and only if $4|(x+3y)$. Prove R is an equivalence relation and describe its equivalence classes.

(a) if R is an equivalence relation on $A \Rightarrow$ eq classes of R are all the same size

$\boxed{\text{FALSE}}$

let $A = \{1,2,3\}$ & let $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

$[1]$ i.e $aR1 = \{1, 2\}$
$[2]$ i.e $aR2 = \{2, 1\}$
$[3]$ i.e $aR3 = \{3\}$
$\forall a \in A$

(b) if R is eq on ∞ set B then R has inf eq classes.

$\boxed{\text{FALSE}}$ w.r.t congruence modulo 2 is an equivalence relation on $\mathbb{Z}$. We have exactly two equivalence classes: $[0] = \{2k \mid k \in \mathbb{Z}\}$, $[1] = \{2k+1 \mid k \in \mathbb{Z}\}$

(c) Define a relation R on $\mathbb{Z}$ as xRy if and only if $4|(x+3y)$. Prove R is an equivalence relation and describe its equivalence classes.

(c) $4 \mid x+3y$, so, $4n = x+3y$

Reflexive

Now, let's consider (a,a)

So, $a+3a = 4a \Rightarrow 4 \mid 4a$ Hence it's reflexive

Symmetric

Now, $(a,b) \in R$, so,

$4n = a+3b$

So, $a = 4n-3b$

So, Now, $3a+b = 12n-9b+b$

$= 12n-8b$

$= 4(n-2b)$

$\Rightarrow 4 \mid 3a+b$

So, $(b,a) \in R$

& R is symmetric

(c) Define a relation R on $\mathbb{Z}$ as xRy if and only if $4|(x+3y)$. Prove R is an equivalence relation and describe its equivalence classes.

Transitive

Let, $(a,b), (b,c) \in R$ so,

$4n = a+3b$

$4m = b+3c$

So, $4(n+m) = a+4b+3c$

$4(n+m-b) = a+3c$

$4 \mid a+3c$

Hence $(a,c) \in R$

So, R is transitive

And hence R is an equivalence relation.

2. Let $f : A \to B$ be a function and let $C \subseteq A$.
(a) Give an example for which $f^{-1}(f(C)) \not\subseteq C$.
(b) Now let $D \subseteq A$. Prove that if f is injective then $f(C \cap D) = f(C) \cap f(D)$.

(a) Let, $A = \{-1, -2, -3, 0, 1, 2, 3\}$

$B = \{0, 1, 4, 9\}$

$C = \{2, 3\}$

$f(x) = x^2$

$f : A \to B$

So, $f(C) = \{4, 9\}$

$f^{-1}(f(C)) = \{-2, -3, 2, 3\}$

So, $f^{-1}(f(C)) \not\subseteq C$

(b) Let $D \subseteq A$. f: injective P.T $f(C \cap D) = f(C) \cap f(D)$

$f(C \cap D) \subseteq f(C) \cap f(D)$

Let $x \in C \cap D$ & $y \in f(C \cap D)$ s.t $f(x) = y$

Since $x \in C$, $f(x) \in f(C)$ &

(Continued)

(Continued) Let $f : A \to B$ be a function and let $C \subseteq A$.
(a) Give an example for which $f^{-1}(f(C)) \not\subseteq C$.
(b) Now let $D \subseteq A$. Prove that if f is injective then $f(C \cap D) = f(C) \cap f(D)$.

Since $x \in D$, $f(x) \in f(D)$

So, $y = f(x) \in f(C) \cap f(D)$
So, $f(C \cap D) \subseteq f(C) \cap f(D)$

Now, $\underline{f(C) \cap f(D) \subseteq f(C \cap D)}$
Given $y \in f(C) \cap f(D)$, so we have
$y \in f(C)$ & $y \in f(D)$
So, $f^{-1}(y) \in C$ & $f^{-1}(y) \in D$ as they are injective
$f(x) = y$ $\quad$ $f(z) = y$
Since f is injective $f(x) = f(z)$
$\Rightarrow x = z$

So, $f^{-1}(y) \in C \cap D$
So, $y \in f(C \cap D)$ So, $f(C) \cap f(D) \subseteq f(C \cap D)$

Final Q10 2012 WT1

3. Prove that $|(0,1)| = |\mathbb{N} \cup (0,1)|$.

$|(0,1)| = |N \cup (0,1)|$

$|(0,1)| \to$ Consider fuction $f: N \cup (0,1) \to (0,1)$

$f: \frac{1}{x}$

Now, injective

$\frac{1}{x_1} = \frac{1}{x_2}$

$x_1 = x_2$ $\therefore$ injective

Surjective

Consider $x = \frac{1}{y}$

So, $f(x) = \frac{1}{1/y} = y$ $\therefore$ surjective

So, it's bijective

(Continued)

(Continued) Prove that $|(0,1)| = |\mathbb{N} \cup (0,1)|$.

Consider $g : (0,1) \to \mathbb{N} \cup (0,1)$

$f(x) = x$

So it's injective & surjective

Hence bijective

& so, $|(0,1)| = |\mathbb{N} \cup (0,1)|$

Limits - Q8 2008 WT2

Recall: A sequence (x_n) has a **limit** $L \in \mathbb{R}$ when

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n > N) \implies (|x_n - L| < \epsilon).$$

4. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Suppose $a_n \to a$ and $b_n \to b$.
(a) Prove that if $a \neq b$ then $\exists\ N \in \mathbb{N}$ so that if $k > N$ then $a_k \neq b_k$.
(b) Say $a = b$. Prove or disprove that there is some $k \in \mathbb{N}$ so that $a_k = b_k$.

(Continued)

(Continued) (a) Prove that if $a \neq b$ then $\exists\ N \in \mathbb{N}$ s.t. if $k > N$ then $a_k \neq b_k$.
(b) Say $a = b$. Prove or disprove that there is some $k \in \mathbb{N}$ so that $a_k = b_k$.

5. Consider a relation $\mathcal{R}$ defined on $\mathbb{R}$ by $x\mathcal{R}y$ if $|x-y|>1$ or $x=y$. Determine if it is reflexive, symmetric, or transitive.

(Continued)

(Continued) Consider a relation $\mathcal{R}$ defined on $\mathbb{R}$ by $x\mathcal{R}y$ if $|x-y|>1$ or $x=y$. Determine if it is reflexive, symmetric, or transitive.

Cardinality (if time)

6. Prove that $|(0,1]| = |(0,1)|$ using Cantor-Schröder-Bernstein theorem.

(Continued)

(Continued) Prove that $|(0,1]| = |(0,1)|$ using Cantor-Schröder-Bernstein theorem.