Mathematics 220 — Homework 10

- Contains 8 questions on 1 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

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1. Prove that there is no integer a so that $a \equiv 2 \pmod 6$ and $a \equiv 7 \pmod 9$.
2. The equation $5y^2 - 4x^2 = 7$ has no integer solutions.
   Hint: Consider the equation modulo 4.
3. Let $f : X \to Y$ be a function. Suppose that f admits an inverse function.
   (a) Prove the inverse function is unique.
   (b) Let $g : Y \to Z$ be another function with an inverse. Show that the inverse function of $g \circ f$ is given by $f^{-1} \circ g^{-1}$
4. Prove that $\sqrt[3]{25}$ is irrational.
5. Let $n \in \mathbb{N}$. Suppose that n is a perfect square, that is $n = m^2$ for some $m \in \mathbb{Z}$. Show that $2n$ is not a perfect square. You may use the fact that $\sqrt{2}$ is irrational without proof.
6. Let $f : \mathbb{Z} \to \mathbb{Z}$ defined so that

$$f(n) = \begin{cases} 3 - n & n \text{ is even} \\ 7 + n & n \text{ is odd} \end{cases}$$

   Prove that f is bijective and give its compositional inverse f^{-1}.
7. Let $A = \mathbb{R} - \{0, 1\}$ and let $f : A \to A$ be defined by $f(x) = 1 - \frac{1}{x}$.
   (a) Show that $f \circ f \circ f = i_A$ where i_A is the identity function on the set A.
   (b) Prove that any function $g : A \to A$ satisfying $g \circ g \circ g = i_A$ is bijective.
   (c) Use part (b) to conclude that f is bijective and determine f^{-1}.
8. Prove that if k is a positive integer and $\sqrt{k}$ is not an integer, then $\sqrt{k}$ is irrational.
   Hint: Bézout's identity will help you.

① Q1

② Q5

③ $f: X \to Y$

$f^{-1}: Y \to X$

(a) inverse function is not unique

i.e $f^{-1}(y_1) \neq f^{-1}(y_2)$

Let, $x_1, x_2 \in X$ & $y_1, y_2 \in Y$
such that $f(x_1) = y_1$
& $f(x_2) = y_2$

Now, w.r.t for a function to be inverse it should be bijective i.e

$x_1 = x_2$

So, $f^{-1}(y_1) = f^{-1}(y_2)$

contradicting our assumption

(b) $g: Y \to Z$, s.t inverse of $g \circ f$ is given by $f^{-1} \circ g^{-1}$.

(7) similar to Chap 1 Qs 27

(8) Qs 9

(4) $\sqrt[3]{25}$ is irrational

Assume $\sqrt[3]{25}$ is rational

So, $\sqrt[3]{25} = \frac{a}{b}$, $\gcd(a,b) = 1$, $b \neq 0$, $a, b \in \mathbb{Z}$

$$25 = \frac{a^3}{b^3}$$

$$25b^3 = a^3$$

$$25 \mid a^3$$

$$25 \mid a$$

So, $a = 25k$

So, $25b^3 = (25)^3 \cdot k^3$

$$b^3 = 25^2 \cdot k^3$$

$25 \mid b^3$ or $25 \mid b$

But w.k.t $\gcd(a,b) = 1$ hence it's not true

& $25^{1/3}$ is irrational.

(5) $n \in \mathbb{N}$, n is a perfect square i.e $n = m^2$ for some $m \in \mathbb{Z}$. S.T $2n$ is not a perfect square.

Assume $n = m^2$ & $2n$ is a perfect square

$$n = m^2$$
So, $2n = 2m^2$

$2n = (\sqrt{2}\, m)^2$ since $2n$ is a perfect square

However $\sqrt{2}\, m \notin \mathbb{Z}$ since $\sqrt{2} \notin \mathbb{Q}$

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(6) $f : \mathbb{Z} \to \mathbb{Z}$

$$f(n) = \begin{cases} 3-n & n \text{ is even} \\ 7+n & n \text{ is odd} \end{cases}$$

P.T f is bijective

Let's take

<u>Even</u>

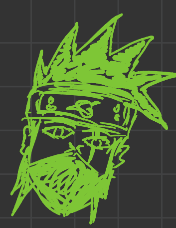
$n = 2k$

So, $f(2k) = 3 - 2k = \text{odd}$

<u>Odd</u>

$n = 2k+1$

So, $f(2k+1) = 7 + 2k+1 = \text{even}$



7 c

$$y = 1 - \frac{1}{u}$$

$$y_n = \quad n-1$$

$$\frac{1}{1-y}$$