Mathematics 220 — Homework 1

- Contains 8 questions on 1 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

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1. Let $n \in \mathbb{Z}$. Prove that if $3 \mid n+1$ then $3 \nmid n^2+5n+5$.
2. Let $a \in \mathbb{Z}$. Prove that if $5a+11$ is odd then $9a+3$ is odd.
3. If $-1 < x < 2$, then $x^2 - x - 2 < 0$.
4. Let a, b, c, d be integers. Suppose that $a, c, b+d$ are all odd numbers. Prove $ab+cd$ is odd.
5. Let x and y be real numbers. Show that

$$xy \leq \frac{1}{2}(x^2 + y^2)$$

6. Let x and y be real numbers. Suppose that $x < y$ and $y^2 < x^2$. Show that $x + y < 0$.
7. For an integer n, prove that if $5|(n+7)$, then $5|(n^2+1)$.
8. Let $n, a, b, x, y \in \mathbb{Z}$ with $n > 0$. Prove that if $n|a$ and $n|b$ then $n|(ax+by)$.

1) let $n \in \mathbb{Z}$. Prove that if $3 \mid n+1$ then $3 \nmid n^2+5n+5$

Proof: Let there exist $k \in \mathbb{Z}$ such that

$$n+1 = 3k$$

On squaring both sides,

$$n^2+1+2n = 9k^2$$

Now, since $n+1$ is divisible by 3

$(n+1)^2$ is also divisible by 3

$$n^2+1+2n+3(n+1) = 9k^2+9k$$

$$n^2+4+5n = 9k^2+9k$$

$$n^2+5n+4 = 3(3k^2+3k)$$

Adding 1 on both sides,

$$n^2+5n+5 = 3(3k^2+3k)+1$$

For any $j \in \mathbb{Z}$, $n^2+5n+5 = 3j+1$

where $j = 3k^2+3k$

Hence, $3 \nmid n^2+5n+5$

$\blacksquare$

2) Let $a \in \mathbb{Z}$. Prove that if $5a+11$ is odd then $9a+3$ is odd.

Proof:

Since $5a+11$ is odd, Let $\exists k \in \mathbb{Z}$

$$5a+11 = 2k+1$$

$$5a = 2k+1-11 = 2k-10$$

$$5a = 2(k-5)$$

We add $4a+3$ on both

$$9a+3 = 2(k-5)+4a+3$$

$$9a+3 = 2(k-5+2a+2)+1$$

So, let there exist $j \in \mathbb{Z}$, where $j = k-5+2a+2$

So, $9a+3 = 2j+1$

Hence $9a+3$ is odd

$\blacksquare$

3) If $-1 < x < 2$, then $x^2-x-2<0$

Proof: $0 < x^2 < 4$

$1 < x^2 - x < 2$

$-1 < x^2-x-2 < 0$

$\underbrace{x^2-x-2<0}$ $\blacksquare$

4) Let a, b, c, d be integers. Suppose that $a, c, b+d$ are all odd. Prove $ab+cd$ is odd.

Proof: Let $\exists\ i, j, k \in \mathbb{Z}$ such that,

$a = 2i+1$

$c = 2j+1$

$b+d = 2k+1$

$b = 2k+1-d$

So, $(2i+1)(2k+1-d) + (2j+1)d$

$4ik + 2i - 2i \cdot d + 2k+1$

$+ 2jd$

$2(2ik + i - id + k + jd)+1 = ab+cd$

$\exists\ p \in \mathbb{Z}$ $p = \ldots$ $ab+cd = 2p+1$ $\blacksquare$

5) let x & y be real nos then show that

Proof :

$$xy \leq \frac{1}{2}(x^2 + y^2)$$

x & y are real nos

So, w.r.t

$$(x-y)^2 \geq 0$$

$$x^2 + y^2 - 2xy \geq 0$$

$$x^2 + y^2 \geq 2xy$$

$$\frac{x^2+y^2}{2} \geq xy$$

6) x & $y \in R$, $x < y$ & $y^2 < x^2$

S.T $x + y < 0$

W.r.t $y^2 < x^2$ & $x < y$

So, $y^2 - x^2 < 0$ &

$y - x > 0$

Now, $(y-x)(y+x) < 0$

Now, since $y - x > 0$

$y + x < 0$ will be less than 0

7) For an int n, prove that if $5 \mid (n+7)$ then $5 \mid (n^2+1)$

So, $\exists\, k \in \mathbb{Z}$, $\quad n+7 = 5k$

On squaring, $\quad n^2 + 49 + 14n = 25k^2$

$$n+2 = 5k-5$$

$$n^2 + 4n + 4 = 25k^2 + 25 - 50k$$

$$n^2+1 = 25(k^2+1-2k) - 4n - 3$$

$$n^2+1 = 25(k^2+1-2k)$$

$$-(4n+3)$$

$$-(4(5k-7)+3)$$

$$-(4(5k) - 28 + 3)$$

$$-(4(5k) - 25)$$

$$n^2+1 = 5\left(5(k^2+1-2k) - 4k + 5\right)$$

So, $5 \mid n^2+1$ //

$\blacksquare$

8) Let $n, a, b, x, y \in \mathbb{Z}$ with $n > 0$. Prove if $n | a$ & $n | b$ then $n | (ax + by)$

For $j, k \in \mathbb{Z}$

$$a = jn$$
$$b = kn$$

$$\boxed{np = ax + by}$$

Now $jxn + nky$

$n(jx + ky)$

np where $p = j\cancel{x} + ky$

$\blacksquare$