Math 220
Section 108
Lecture 24

1st December 2022

Sources: https://personal.math.ubc.ca/~PLP/auxiliary.html
https://secure.math.ubc.ca/Ugrad/pastExams

Final Q4 2009 WT1

1.(a) Prove that $f : [3, \infty) \to [5, \infty)$, defined by $f(x) = x^2 - 6x + 14$, is a bijection.

Assume that $f:[3,\infty) \to [5,\infty)$, $f(x) = x^2 - 6x + 14$ is not a bijection.

For $x_1, x_2 \in [3,\infty)$ we have

$$f(x_1) = x_1^2 - 6x_1 + 14 \ \&$$
$$f(x_2) = x_2^2 - 6x_2 + 14$$

$$f(x_1) = x_1^2 - 6x_1 + 9 + 5$$
$$= (x_1 - 3)^2 + 5$$
$$f(x_2) = (x_2 - 3)^2 + 5$$

Now, we know ~~that~~ $x_1, x_2 \in [3,\infty)$

So, $(x_1 - 3)^2 \geq 0$

Hence let's take $f(x_1) = f(x_2)$

we have, $(x_1-3)^2 = (x_2-3)^2$

OD $x_1 = x_2$, Hence it's injective

(Continued)

(Continued) 1.(a) Prove that $f : [3, \infty) \to [5, \infty)$, defined by $f(x) = x^2 - 6x + 14$, is a bijection.

Now, let $y = x^2 - 6x + 14$ for $y \in [5, \infty)$

Scratch $\rightarrow$ So, $y - 5 = (x-3)^2$

$\sqrt{y-5} + 3 = x$ $\rightarrow$ Here this exists when $y \geq 5$ as required

Given $y \in [5, \infty)$ let, $x = \sqrt{y-5} + 3$

For $x^2 - 6x + 14$ we have,

$f(x) = y - 5 + 9 - 6\sqrt{y-5} - 18 + 14 + 6\sqrt{y-5}$

$f(x) = y$

So, f is surjective

Hence f is bijective & our assumption is false.

Final Q4 2009 WT1

1.(b) Prove that if A is a denumerable subset of $\mathbb{Z}$ then there exists a set B such that B is a proper subset of A and $|A| = |B|$.

So, $\exists f : \mathbb{N} \to A$ is bijective

since $|A| = |\mathbb{N}|$

Let $B = A - \{f(1)\}$

Let, $g : \mathbb{N} \to B$ s.t. $g(n) = f(n+1)$

We claim that g is a bijection:

<u>Injective</u>: If $g(n_1) = g(n_2)$

$f(n_1+1) = f(n_2+1)$

Since f is bijective

we have $n_1 = n_2$

Hence injective

<u>Surjective</u>:

Given $b \in B$, note since f is surjective

$\exists n \in \mathbb{N}$ s.t. $f(n) = b$. Note that $n \neq$ since

$f(1) \notin B$

(Continued)

(Continued) 1.(b) Prove that if A is a denumerable subset of $\mathbb{Z}$ then there exists a set B such that B is a proper subset of A and $|A| = |B|$.

Pigeonhole Principle (PHP)

Theorem (Theorem 12.1.3 of PLP)

If n objects are placed in k boxes then:

- *If $n < k$ then at least one box has zero objects in it.*
- *If $n > k$ then at least one box has at least two objects in it.*

Variation of Final Q4 2016 WT1

2. Prove or disprove: For every $a \in \mathbb{N}$, there exist distinct $k, \ell \in \mathbb{N}$ such that 11 divides $a^k - a^\ell$.

(Continued)

(Continued) 2. Prove or disprove: For every $a \in \mathbb{N}$, there exist distinct $k, \ell \in \mathbb{N}$ such that 11 divides $a^k - a^\ell$.

More cardinality results

Theorem (Result 12.2.6 of PLP)

Let A and B be denumerable sets. Then $A \times B$ is denumerable.

Definition (Definition 12.4.1 of PLP)

Let A and B be sets.

- We write $|A| \leq |B|$ if there is an injection from A to B.
- We write $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$.

Theorem (Cantor-Schröder-Bernstein theorem – Theorem 12.5.1 in PLP)

Let A and B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

3. Let $\mathbb{Z}(\sqrt{2})$ be the set of numbers of the form $a + b\sqrt{2}$. where a and b are integers.
(a) Prove that $\mathbb{Z}(\sqrt{2}) \cap \mathbb{Q} = \mathbb{Z}$.
(b) Prove that if $x \in \mathbb{Z}(\sqrt{2})$, then for all natural numbers n, we have $x^n \in \mathbb{Z}(\sqrt{2})$.
(c) Prove that $\mathbb{Z}(\sqrt{2})$ is denumerable.
Hint: Knowing that $\sqrt{2}$ is irrational, can you relate $\mathbb{Z}(\sqrt{2})$ to $\mathbb{Z}^2$?

(Continued)

(Continued) 3. $\mathbb{Z}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$.
(a) Prove that $\mathbb{Z}(\sqrt{2}) \cap \mathbb{Q} = \mathbb{Z}$.
(b) Prove that if $x \in \mathbb{Z}(\sqrt{2})$, then for all $n \in \mathbb{N}$, we have $x^n \in \mathbb{Z}(\sqrt{2})$.
(c) Prove that $\mathbb{Z}(\sqrt{2})$ is denumerable.

(Continued 2)

(Continued)
(b) Prove that if $x \in \mathbb{Z}(\sqrt{2})$, then for all $n \in \mathbb{N}$, we have $x^n \in \mathbb{Z}(\sqrt{2})$.
(c) Prove that $\mathbb{Z}(\sqrt{2})$ is denumerable.

(Continued 3)

(Continued)
(b) Prove that if $x \in \mathbb{Z}(\sqrt{2})$, then for all $n \in \mathbb{N}$, we have $x^n \in \mathbb{Z}(\sqrt{2})$.
(c) Prove that $\mathbb{Z}(\sqrt{2})$ is denumerable.

(Continued 4)

(Continued)
(c) Prove that $\mathbb{Z}(\sqrt{2})$ is denumerable.

(Continued 5)

(Continued)
(c) Prove that $\mathbb{Z}(\sqrt{2})$ is denumerable.

Cardinality (if time)

4. Prove that $|(0,1]| = |(0,1)|$ using Cantor-Schröder-Bernstein theorem.

(Continued)

(Continued) 4. Prove that $|(0,1]| = |(0,1)|$ using Cantor-Schröder-Bernstein theorem.