PLP - 17

TOPIC 17—DISPROOFS

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DISPROOFS

DISPROVING A STATEMENT

To disprove the statement P, we prove that $(\sim P)$ is true.

universal quantifier

Since

$$\sim (\forall x, P(x)) \qquad \equiv \qquad \exists x \text{ s.t. } \sim P(x)$$

our disproof can be a **counter example**

existential quantifier

Since

$$\sim (\exists x \text{ s.t. } P(x)) \qquad \equiv \qquad \forall x, \sim P(x)$$

we have to work harder to show that $(\sim P(x))$ is true *for all* x.

Counter examples *do not* disprove existential quantifiers

DISPROVE A UNIVERSAL QUANTIFIER

For every $n \in \mathbb{N}$, $2^n - 1$ or $2^n + 1$ is prime.

scratch work — smells false

- $n = 1$ gives $1, 3$ and $n = 2$ gives $3, 5$
- $n = 3$ gives $7, 9$ and $n = 4$ gives $15, 17$
- $n = 5$ gives $31, 33$ and $n = 6$ gives $63, 65$

PROOF.

Pick $n = 6$. Since neither $2^n - 1 = 63$ or $2^n + 1 = 65$ are prime, the statement is false.

Our counter-example proves "$\exists n \in \mathbb{N}$ **s.t. neither** $2^n - 1, 2^n + 1$ **are prime.**"

ANOTHER EXAMPLE

For all $a, b, c \in \mathbb{N}$, if $(a \mid bc)$ then $(a \mid c)$ or $(a \mid b)$

scratch work — again smells false.

- Since is universal quantifier, a counter example is sufficient
- Negation is "$\exists a, b, c$ s.t. $(a \mid bc) \wedge (a \nmid b \wedge a \nmid c)$ "
- Something about prime-factors feels like the right thing here
- Pick $a = 4$ and $b = 2, c = 2$. Then $(4 \mid 2 \cdot 2)$ but $4 \nmid 2$.

PROOF.

The statement is false. Let $a = 4$ and $b = c = 2$. Then $a \mid bc$ but $a \nmid b$ and $a \nmid c$.

DISPROVING AN EXISTENTIAL QUANTIFIER

There exist prime numbers p, q so that $p - q = 999$

Typically this is much harder. Sometimes we can reduce to a finite number of cases.

scratch work

- Since odd-odd = even, we must have that $q = 2$
- Then since $1001 = 7 \times 11 \times 13,$ no such primes exist

PROOF.

This is false. Either q is even or odd.

- If q is even, then $q = 2$. Since $999 + 2 = 1001$ is divisible by 7 it is not prime.
- Now assume that q is odd. Then we must have that $q = 2k + 1$ for some $k \in \mathbb{Z}$. But then $p = 2k + 1000$ which is divisible by 2 and so not prime.

Hence no such primes exist.