Mathematics 220 — Homework 11

- Contains 8 questions on 1 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

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1. Determine if the following sets are countable, and prove your answers.
   (a) The set of all functions $f : \{0, 1\} \to \mathbf{N}$.
   (b) The set of all functions $f : \mathbf{N} \to \{0, 1\}$.

2. Prove the following statements
   (a) If A is countable but B is uncountable, then $B - A$ is uncountable.
   (b) Between any real numbers a, b such that $a < b$ there are uncountably many irrationals.

3. Prove that $\mathbb{R}$ and $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ are equinumerous.

4. Let S, T be sets. Prove the following
   (a) If $|S| \leq |T|$ then $|\mathcal{P}(S)| \leq |\mathcal{P}(T)|$.
   (b) If $|S| = |T|$ then $|\mathcal{P}(S)| = |\mathcal{P}(T)|$.

5. Show that there exist infinitely many pairs of distinct natural numbers a, b such that $17^a - 17^b$ is divisible by 2022.

   Hint: Pigeons can help.

6. Prove that $(-\infty, -\sqrt{29})$ and $\mathbb{R}$ are equinumerous by constructing an explicit bijection.

7. Prove or disprove: for any non-empty sets A, B, C, if $|A \times B| = |A \times C|$ then $|B| = |C|$.

8. Let A be a finite set and $f : \mathbb{R} \to A$. Show that there exists some $a \in \mathbb{A}$ such that $f^{-1}(\{a\})$ is uncountable.

(1) Q9

(6) Q8

(5) Similar to lectures

(3)

Proof. We show that both sets are equinumerous to the interval $(1, \infty)$.

- First we find a bijection $f : \mathbb{R} \to (1, \infty)$. A function that almost works is the exponential function, 2^x; the issue is its range is $(0, \infty)$ rather than $(1, \infty)$. We can resolve this by taking $f(x) = 2^x + 1$. Moreover, we know that f has an inverse, $f^{-1} : (1, \infty) \to \mathbb{R}$, given by $f^{-1}(x) = \log_2(x - 1)$. Then by Theorem 10.6.8, f is bijective. Thus $\mathbb{R}$ and $(1, \infty)$ are equinumerous.

- Next, we want to find a bijection $g : (1, \infty) \to (0, 1)$. Take $g(x) = 1/x$. This function is well-defined, as for $x > 1$, we have $0 < 1/x < \infty$. Moreover, its inverse is given by $g^{-1} : (0, 1) \to (1, \infty)$, with $g^{-1}(x) = 1/x$. Again by Theorem 10.6.8, g is bijective. Thus $(1, \infty)$ and $(0, 1)$ are equinumerous.

Combining the two statements we proved, and appealing to Theorem 12.1.8, we can conclude that $\mathbb{R}$ and $(0, 1)$ are equinumerous, as desired. Indeed, the function $g(f(x)) = 1/(2^x + 1)$ is a bijection from $\mathbb{R}$ to $(0, 1)$. ■

(2) (a) A is countable $\wedge$
B is uncountable $\wedge$
B-A is countable

P.T.⇒ B-A is uncountable

So, Now, B-A is countable
Now so, $B \subseteq B-A$

But w.r.t B is uncountable & B-A is countable so, $B \subseteq B-A$ is a contradiction
Hence B-A is uncountable

(b) We assume that nos. in (a,b) are countable & there are uncountable ∞ I b/w them.

So, $c = a - b$ = countable

however this is a contradiction & hence (a,b) is uncountable.

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(4) (a) if $|S| \le |T|$ then $|P(S)| \le |P(T)|$

There exists an injection $f: S \Rightarrow T$ since $|S| \le |T|$. So,

Let's take $g: P(S) \to P(T)$ & p.t it's injective.

Now, $g(A) = f(A) = \{f(a) \mid a \in A\}$

Assume $g(A) = g(B)$ since $g(A) = f(A)$ & $f(A)$ is injective

~~So~~ now, let $x \in A$ such that $f(x) = y \in f(A) = g(A)$

Similarly $p \in B$ $f(p) = q \in f(B) = g(B)$

But it's injective

so, $f(x) = f(p)$

so, $x = p$

so, $A = B$

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(5) $17^a - 17^b$ is divisible by 2022

Consider the set $\{17^1, 17^2, 17^3 \ldots\ldots 17^{2022}\}$

by Pigeon hole principle w.k.t

$$17^a \equiv 17^b \pmod{2022}$$

where $\exists\ a \geq 1,\ b \leq 2022$

So, $17^a - 17^b \equiv 0 \pmod{2022}$

OR

$$17^a - 17^b = 2022n$$

So, $2022 \mid 17^a - 17^b$

We can do this infinitely using pigeon hole principle

$\therefore$ Hence Proved