$\boxed{6.6}$

① $\forall n \in \mathbb{Z}$, $3 \mid (n^3 - n)$, By division algorithm we have,

Case 1:

n is a multiple of 3 i.e $\forall m \in \mathbb{Z}$, $n = 3m$
So, for the equation $n^3 - n$,

$$(3m)^3 - 3m \Rightarrow 27m^3 - 3m$$
$$\Rightarrow 3(9m^3 - m) \Rightarrow 3k, \exists k \in \mathbb{Z} \text{ s.t } k = 9m^3 - m$$

So, $3 \mid n^3 - n$

Case 2:

$\forall m \in \mathbb{Z}$, $n = 3m + 1$
So, for $n^3 - n$,

$$(3m+1)^3 - (3m+1) \Rightarrow 27m^3 + 1 + 27m^2 + 9m - 3m - 1$$
$$\Rightarrow 3(9m^3 + 9m^2 + 2m)$$

So, $3 \mid n^3 - n$ $\Rightarrow 3k, \exists k \in \mathbb{Z}$ s.t $k = 9m^3 + 9m^2 + 2m$

Case 3:

$\forall m \in \mathbb{Z}$, $n = 3m + 2$
So, for $n^3 - n$,

$$(3m+2)^3 - (3m+2) \Rightarrow 27m^3 + 8 + 54m^2 + 36m - 3m - 2$$
$$\Rightarrow 3(9m^3 + 18m^2 + 11m + 2)$$

So, $3 \mid n^3 - n$
$\Rightarrow 3k, \exists k \in \mathbb{Z}$ s.t $k = 9m^3 + 18m^2 + 11m + 2$

Hence $3 \mid n^3 - n$ by proof by cases $\forall n \in \mathbb{Z}$ ∎

② $\forall n, k \in \mathbb{Z}, (k \mid (2n+1) \wedge k \mid 4n^2+1) \Rightarrow k \in \{-1, 1\}$

Now, $\forall a, b \in \mathbb{Z}$, $2n+1 = a \cdot k$

$4n^2+1 = b \cdot k$

$(k+1)(k-1)$

$k^2-1$

Now, $(a \cdot k)^2 = (2n+1)^2$

$$a^2 \cdot k^2 - b \cdot k = 4n$$

$$a^2 \cdot k^2 - b \cdot k = 2 \cdot ak - 2$$

$$a^2 \cdot k^2 - (2a+b) k + 2 = 0$$

$$2 = (2a+b)k - a^2 \cdot k^2$$

$$2 = k(2a+b-a^2k)$$

So, $2 \mid k$

However $k \mid 2n+1$ so we know that it's odd $k$ is an odd divisor of 2

i.e $\pm 1$

(3) $\forall n \in \mathbb{Z}$, $\exists a,b \in \mathbb{Z}$ st $a^2+b^2 = n \Rightarrow n \not\equiv 3 \pmod 4$

Contoapositive

if $n \equiv 3 \pmod 4 \Rightarrow a^2+b^2 \neq n$

Case 1:

$a$ & $b$ both even,

$$a^2+b^2 = 4(k^2+l^2)$$

$$a^2+b^2 \equiv 0 \pmod 4$$

Case 2:

$a$ & $b$ are odd

$$a^2+b^2 = 4(k^2+l^2+k+l)+2$$

$$a^2+b^2 \equiv 2 \pmod 4$$

Case 3:

one of $a$ & $b$ is even

$$a^2+b^2 = 4(k^2+l^2+l)+1$$

$$a^2+b^2 \equiv 1 \pmod 4$$

Hence $a^2+b^2 \neq n$ //

## Lecture 1:

$\{n^2 \mid n \in \mathbb{Z}\} = \{0, 1, 4, 16 \dots\}$

$\{x \in \mathbb{Z} \mid x^2 - 2 = 0\} = \{\ \}$

$\{x \in \mathbb{R} \mid x^2 - 2 = 0\} = \{-\sqrt{2}, \sqrt{2}\}$

---

## Lecture 2: ☆

- If $x \in \mathbb{R}$ and the sequence $(x_n)$ converges to $x$, then $(1/x_n)$ converges to $1/x$.

  $P(x)$: $x \in \mathbb{R}$

  $Q(y, z)$: $y$ converges to $z$

  $$P(x) \wedge Q(x_n, x) \Rightarrow Q(1/x_n, 1/x)$$

- $\forall y$ multiple of 2 & 3, $y$ is also a multiple of 6.

  $\rightarrow$ No, it's just a true statement

# Lecture 9

- $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, (n > N) \Rightarrow (|x_n - 2| < \varepsilon)$

S.T $\left(\frac{n}{n^2+1}\right)$ converges to $0$

**Scratchwork**

$$\left|\frac{n}{n^2+1} - 0\right| < \varepsilon \Rightarrow \left|\frac{n}{n^2+1}\right| < \varepsilon$$

$$\Rightarrow \frac{n}{n^2+1} < \frac{1}{n} < \varepsilon$$

---

(0) Prove that $\frac{n^3}{2n^3+5}$ converges to $\frac{1}{2}$ $(n \to \infty)$

Proof

Given $\varepsilon > 0$, let $N = \left\lceil \sqrt[3]{\frac{5}{4\varepsilon}} \right\rceil$

then for any $n > N \geq \sqrt[3]{\frac{5}{4\varepsilon}}$, we have,

$$n > \sqrt[3]{\frac{5}{4\varepsilon}}$$

$$n^3 > \frac{5}{4\varepsilon}$$

$$\varepsilon > \frac{5}{4n^3} > \frac{5}{4n^3+10}$$

$$\varepsilon > \left|-\frac{5}{4n^3+10}\right|$$

$$\varepsilon > \left|\frac{2n^3 - (2n^3+5)}{2(2n^3+5)}\right|$$

$$\varepsilon > \left|\frac{n^3}{2n^3+5} - \frac{1}{2}\right|$$

$\blacksquare$

⑥ $a_1 = 1$

$a_2 = 3$

$a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 3$

P.T $a_n = 2n - 1$ for all $n \in \mathbb{N}$

Scratch

Inductive step

$$a_{k+1} = 2a_k - a_{k-1}$$
$$= 2(2k-1) - a_{k-1}$$
$$= 2(2k-1) - (2k-3)$$
$$= 2k+1$$
$$= 2(k+1) - 1$$

We want to assume $a_k = 2k-1$ & $a_{k-1} = 2(k-1)-1$

Proof

Base case

$n=1$: $a_1 = 2(1) - 1$
$= 1$

$n=3$: $a_2 = 2(2) - 1$
$= 3$

$\therefore$ holds true for base cases

Induction Step

Assume $a_n = 2n-1$ for $n = k$ & $k-1$ for some $k \geq 2$

Then $a_{k+1} = 2a_k - a_{k-1}$

$= 2(2k-1) - (2(k-1)-1)$

$= 4k-2 - 2k+3$

$= 2k+1$

$= 2(k+1)-1$

$\Rightarrow a_{k+1}$

Therefore holds true for $a_{k+1}$ as well.

So, $a_n = 2n-1$ for all $n \in \mathbb{N}$ by strong induction.

---

Q) Find all real $x$ s.t $|x-3| \leq |2x+5|$

Case 1:

$x > 3$ So, $x-3 \leq |2x+5|$

Case a

$x > -\frac{5}{2}$

So, $x-3 \leq 2x+5$

$x \geq -8$

Case b

$x < -\frac{5}{2}$

so, $x-3 \leq -2x-5$

$3x \leq -2$

$x \leq -\frac{2}{3}$

Case 2:

$$x < 3 \Rightarrow -x+3 \leq |2x+5|$$

Case a:

$x > -\frac{5}{2}$

$$-x+3 \leq 2x+5$$

$$3x \geq -2$$

$$x \geq -\frac{2}{3}$$

Case b:

$x < -5/2$

$$-x+3 \leq -2x-5$$

$$x \leq -8$$

---

P.T if $n^2 \equiv 1 \text{ or } 5 \pmod 6 \Rightarrow 3 \nmid n$ & $2 \nmid n$

So, if $3 \mid n \vee 2 \mid n \Rightarrow n^2 \not\equiv 1 \text{ or } 5 \pmod 6$

So, for $p, q \in \mathbb{Z}$

$3 \cdot p = n$ or $2 \cdot q = n$

$$n^2 = 9p^2$$

- if $p^2$ is even then

$$n^2 = 9 \cdot (2m)^2$$

$$n^2 = 18m^2$$

$$n^2 = 6(3m^2)$$ So, $n^2 \equiv 0 \pmod 6$

· If $p^2$ is odd

$$n^2 = 9(2m+1)$$
$$n^2 = 18m+9$$
$$n^2 = 6(3m+1)+3$$
$$n^2 \equiv 3 \pmod 6$$

Case 2:

$$2 \cdot q = n$$
$$n^2 = 4q^2$$

· If $q^2$ a multiple of 3

$$n^2 = 4(3m)$$
$$n^2 = 12m$$
$$n^2 = 6(2m)$$
$$n^2 \equiv 0 \pmod 6$$

· If $q^2 = 3m+1$

$$n^2 = 4(3m+1)$$
$$n^2 = 12m+4$$
$$n^2 = 6(2m)+4 \quad n^2 \equiv 4 \pmod 6$$

· If $q^2 = 3m+2$

$$n^2 = 4(3m+2)$$
$$n^2 = 12m+8$$
$$n^2 = 6(2m+1)+2$$
$$n^2 \equiv 2 \pmod 6$$

(4) $\forall a, b \in \mathbb{Z}$, if $3 \mid (a^2+b^2)$, then $3 \mid a$ & $3 \mid b$

So, $3 \nmid a \vee 3 \nmid b \Rightarrow 3 \nmid (a^2+b^2)$

Case 1:

$a = 3k+1$
$a^2 = 9k^2 + 1 + 6k$
$= 3m+1$

$a = 3k+2$
$a^2 = 9k^2 + 4 + 12k$
$= 3m+1$

So, $a^2 = 3m+1$

Now either $3 \mid b$ or $3 \nmid b$

⋮

16) S.T $\lim_{x \to 4} (-3x+5) = -7$

$0 < |x-a| < \delta$

$|f(x) - L| < \varepsilon$

$0 < |x-4| < \delta$

$|-3x+5+7| < \varepsilon$

$|-3x+12| < \varepsilon$

$|-3(x-4)| < \varepsilon$

$3|x-4| < \varepsilon$

$|x-4| < \varepsilon/3$

$\delta = \varepsilon/3$

(17) $\lim_{x \to 1} x^2 = 1$

$$0 < |x - a| < \delta$$

$$|f(x) - L| < \varepsilon$$

$$0 < |x - 1| < \delta$$

$$|x^2 - 1| < \varepsilon$$

$$|(x+1)(x-1)| < \varepsilon$$

$$|x - 1| < \varepsilon$$

so.

$$x - 1 > 0$$

$$x + 1 > 2$$

$$|(x+1)(x-1)| < \varepsilon/2$$

$$|x - 1| < 2 \cdot \varepsilon/2$$

$$|x - 1| < \varepsilon$$

18 $|x-0| < \delta$

$$\left|\frac{1}{n^2}\right| < \varepsilon$$

$$\left|\frac{1}{n^2}\right|$$

$$N = \lceil 1/\sqrt{\varepsilon} \rceil$$

$n > N$ ?

19 $0 < |x-0| < \delta$

$$\left|6x\sin\left(\frac{1}{x}\right)\right| < \varepsilon$$

$$|6x| < \varepsilon$$

$$|x| < \varepsilon/6$$

$\boxed{H.W\,1}$

① if $3 \mid n+1$ then $3 \nmid n^2+5n+5$

scratch

$n+1 = 3k$

$(n+1)^2 = 9k^2$

$n^2+1+2n = 9k^2$

Proof

Let $n+1 = 3k$, $\forall k \in \mathbb{Z}$

so, $(n+1)^2 = (3k)^2$

$n^2+2n+1 = 9k^2$

On adding $3(n+1)+1$ on both sides we get,

$n^2+5n+5 = 9k^2+3k+1$

$n^2+5n+5 = 3(k^2+k)+1$

$n^2+5n+5 = 3m+1$, $\exists m \in \mathbb{Z}$ s.t $m = k^2+k$

So, $3 \nmid n^2+5n+5$ //

② P.t if $5a+11$ is odd then $9a+3$ is odd.

Proof

$5a+11 = 2K+1, \ \forall K \in \mathbb{Z}$

Now, $5a+11+4a-8$

$= 2K+1+4a-8$

$9a+3 = 2K+4a-7$

$9a+3 = 2(K+2a-4)+1$

$9a+3 = 2m+1, \ \exists m \in \mathbb{Z}$ s.t $m = K+2a-4$

So, $9a+3$ is odd. ☑

---

③ $-1 < x < 2$ then $x^2 - x - 2 < 0$

Proof

$0 \leq x^2 < 4$

$1 \leq x^2 - x < 4-2$

$-1 \leq x^2 - x - 2 < 0$

So, $x^2 - x - 2 < 0$ ☑

(4) $a, c, b+d$ are odd
P.T $ab+cd$ is odd

Proof

$\forall p, q, r \in \mathbb{Z}$, $a = 2p+1$
$c = 2q+1$
$b+d = 2r+1$

$\Rightarrow b = 2r+1-d$

So, $(2p+1)(2r+1-d)$
$+ (2q+1)(d) = a\cdot b + c\cdot d$

$4pr + 2p - 2p\cdot d + 2r + 1 - d$
$+ 2q\cdot d + d = a\cdot b + c\cdot d$

$a\cdot b + c\cdot d = 2(2pr + p - p\cdot d + q\cdot d$
$+ r) + 1$

$a\cdot b + c\cdot d = 2m+1$, $\exists m \in \mathbb{Z}$ s.t $m = 2pr + p - pd + qd + r$

So, $a\cdot b + c\cdot d$ is odd

(5) S.T $xy \leq \frac{1}{2}(x^2+y^2)$

Proof

Now, $x \in R$ & $y \in R$

So, $x-y \in R$

$(x-y)^2 \geq 0$

$x^2+y^2-2x\cdot y \geq 0$

$x^2+y^2 \geq 2x\cdot y$

$x\cdot y \leq \frac{(x^2+y^2)}{2}$

---

(6) $x<y$ & $y^2<x^2$

Proof S.T $x+y<0$

Now, $y^2<x^2$

So, $x^2-y^2>0$

$(x+y)(x-y)>0$

& $x<y$ so, $x-y<0$

So, $-(x-y)>0$

So, $-(x+y)\cdot 1>0$ so $x+y<0$

(7) if $5|(n+7) \Rightarrow 5|(n^2+1)$

Proof

$n+7 = 5 \cdot k \ , \forall k \in \mathbb{Z}$

So, $n = 5k - 7$

$n^2 = 25k^2 + 49 - 70k$

$n^2+1 = 25k^2 + 50 - 70k$

$n^2+1 = 5(5k^2 + 10 - 14k)$

$n^2+1 = 5m \ , \exists m \in \mathbb{Z}$ s.t $m = 5k^2+10-14k$

So, $5 | n^2+1$ $\blacksquare$

---

(8) $n, a, b, x, y \in \mathbb{Z}$ with $n > 0$

p.t $n|a$ & $n|b$ then

$n \mid (ax + by)$ $\quad ax+by = n \cdot m$

Proof $\quad \exists m \in \mathbb{Z}$ s.t, $m = xp + yq$

$\forall p, q \in \mathbb{Z}$, $n \cdot p = a$

$n \cdot q = b$

multiply both sides with $x$, $y$

$n \cdot x \cdot p = a \cdot x$

$n \cdot q \cdot y = by$

so $ax + by = n(x \cdot p + y \cdot q)$

$n | ax+by$ $\blacksquare$

# H·W 2

① If $a \in \mathbb{Z} \Rightarrow 4 \nmid a^2+1$

Proof

Case 1:

a is even

$a = 2K \quad \forall k \in \mathbb{Z}$

So, $a^2 = 4K^2$

$a^2+1 = 4K^2+1$

which is not divisible by 4 i.e

$4 \nmid a^2+1$

Case 2:

a is odd

$a = 2k+1 \quad \forall k \in \mathbb{Z}$

So, $a^2 = 4k^2+1+4K$

$a^2+1 = 4K^2+4K+2$

$a^2+1 = 4(k^2+k)+2$

$a^2+1 = 4m+2$, $\exists m \in \mathbb{Z}$ s.t $m = K^2+K$

So, not divisible by 4 i.e

$4 | a^2+1$

$\therefore$ Proved by proof by cases $\blacksquare$

② Proof $2x - \frac{1}{x} > 1 \Rightarrow x > 1$

where $x > 0$

Now, $2x - \frac{1}{x} > 1$

So, $2x^2 - 1 > x$

$2x^2 - 1 - x > 0$

---

③ if $K \in \mathbb{Z} \Rightarrow 3 \mid (K(2K+1)(4K+1))$

Proof

<u>Case 1:</u>

K is multiple of 3, $K = 3 \cdot n$, $\forall n \in \mathbb{Z}$

So,

$(K(2K+1)(4K+1))$

$\Rightarrow (2K^2 + K)(4K+1)$

$\Rightarrow 8K^3 + 2K^2 + 4K^2 + K$

$\Rightarrow 8 \cdot 27 \cdot n^3 + 2 \cdot 9 \cdot n^2 + 4 \cdot 9 \cdot n^2 + 3 \cdot n$

$\Rightarrow 3(8 \cdot 9 \cdot n^3 + 2 \cdot 3n^2 + 4 \cdot 3n^2 + n)$

$\Rightarrow 3 \mid (K(2K+1)(4K+1))$

Case 2:

$K = 3n+1$

So, for $8k^3 + 6k^2 + k$

$\Rightarrow 8(27n^3 + 1 + 9n + 27n^2) + 6(9n^2 + 1 + 6n) + (3n+1)$

$\Rightarrow 8 \cdot 27n^3 + 8 + 8 \cdot 9n + 8 \cdot 27n^2 + 6 \cdot 9n^2 + 6 + 36n + 3n + 1$

$\Rightarrow 3(8 \cdot 9n^3 + 5 + 38n + 9 \cdot 8n^2 + 2 \cdot 9n^2 + 12n + 1)$

$\Rightarrow 3 \mid (k(2k+1)(4k+1))$

Case 3:

$K = 3n+2$

So, for $8k^3 + 6k^2 + k$

$\Rightarrow 8(27n^3 + 8 + 36n + 54n^2) + 6(9n^2 + 4 + 12n) + 3n + 2$

$\Rightarrow 8 \cdot 27n^3 + 64 + 8 \cdot 36n + 8 \cdot 54n^2 + 6 \cdot 9n^2 + 24 + 6 \cdot 12n + 3n + 2$

$\Rightarrow 3(8 \cdot 9n^3 + 32 + 12 \cdot 8n + 18 \cdot 8n^2 + 2 \cdot 9n^2 + 2 \cdot 12n + n + 9)$

$\Rightarrow 3 \mid (k(2k+1)(4k+1))$

So, proved by proof by cases

$\boxtimes$

④ Proof
(a) if $3|n$ & $4|n \Rightarrow 12|n$

$n = 3p$   $\forall p, q \in \mathbb{Z}$
$n = 4 \cdot q$

So, $4n = 12p$
$3n = 12q$

$n = 12(p \cdot q)$
$n = 12m$, $\exists m \in \mathbb{Z}$ s.t $m = p \cdot q$

$\boxed{\cancel{\mathbb{Z}}}$

(b) if $n > 3$ is prime then $n^2 \equiv 1 \pmod{12}$

Case 1:

prime nos $> 3$ are odd
So, $n = 2k+1$, $\forall k \in \mathbb{Z}$
So, $n^2 = 4k^2 + 1 + 4k$
$n^2 = 4(k^2 + k) + 1$
$n^2 = 4m + 1$, $\exists m \in \mathbb{Z}$ s.t $m = k^2 + k$
So, $n^2 \equiv 1 \pmod 4$

Case 2:

prime nos $> 3 \nmid 3$
So, case a
$n = 3k+1$, $\forall k \in \mathbb{Z}$
$n^2 = 9k^2 + 6k + 1$
$= 3(3k^2 + 2k) + 1$ $\Rightarrow$ $n^2 \equiv 1 \pmod 3$

Case 3:

$n = 3k+2$

$n^2 = 9k^2 + 4 + 12k$

$n^2 = 3(3k^2 + 1 + 4k) + 1$

$n^2 \equiv 1 \pmod 3$

So, $n^2 \equiv 1 \pmod 4$

$n^2 \equiv 1 \pmod 3$

So, $n^2 \equiv 1 \pmod{12}$ // $\boxed{\phantom{x}}$

---

⑤ $n \in \mathbb{Z}$, $3 \mid$ if $n^3 + n^2 - n + 3 \Rightarrow n \mid 3$

$n \nmid 3 \Rightarrow 3 \nmid n^3 + n^2 - n$

Proof

Case 1:

$n = 3k+1 \quad \forall k \in \mathbb{Z}$

$n^3 + n^2 - n + 3 = 27k^3 + 1 + 27k^2 + 9k + 9k^2 + 1 + 6k + 3 - 3k - 1$

$\Rightarrow 3(9k^3 + 9k^2 + 3k + 3k^2 + 2k + 1 - k) + 1$

So, $3 \nmid n^3 + n^2 - n + 3$

Case 2:

$n = 3k+2 \quad \forall k \in \mathbb{Z}$

$n^3+n^2-n+3 = 27k^3+8+36k+54k^2$
$\qquad + 9k^2+4+12k+3-3k-2$

$= 3(9k^3+2+12k+18k^2+3k^2$
$\qquad + \cancel{2} + 4k+1-3k-\cancel{1})$
$\qquad + 1$

So, $3 \nmid n^3+n^2-n+3$

Hence ~~by~~ proof by cases $3 \nmid n^3+n^2-n+3$ $\boxtimes$

---

(6) $x \in \mathbb{R}$ then P.T $x^2+|x-6| > 5$

Proof

Case 1:

$x > 6$

$x^2 + x-6 > 5$

Now, $x > 6$

So, $x^2 > 36$

So, $x^2+x-6 > 36+6-6$

$x^2+x-6 > 36$

So, $x^2+x-6 > 5$

Case 2

$x \leq 6$

$x^2 - x + 6 < 5$

$\left(x - \frac{1}{2}\right)^2 + \frac{23}{9} < 5$ //

---

(7) $x, y \in \mathbb{Z}$ $3 \nmid (x^3 + y^3) \Leftrightarrow 3 \nmid (x+y)$

Case 1:

Contrapositive: $3 \mid (x+y) \rightarrow 3 \mid (x^3+y^3)$

$3 \cdot K = x + y$

$(x+y)^3 = 9K^3$

$x^3 + y^3 + 3xy(x+y) = 9K^3$

$x^3 + y^3 = 3(3K^3 - xy(x+y))$

$3 \mid x^3 + y^3$ //

Case 2:

Contrapositive: $3 \mid (x^3+y^3) \rightarrow 3 \mid x+y$

$3K = x^3 + y^3 \rightarrow (x+y)^3 - 3xy(x+y) = 3K$

$\rightarrow (x+y) = 3(K + 3xy(x+y))^{1/3}$

So, $3 \mid x+y$ //

$\boxed{3}$

(8) : if $K \nmid \gcd(a,b)$ $\exists$ $K \nmid a$ or $K \nmid b$

$K \mid a$ & $K \mid a \Rightarrow K \mid \gcd(a,b)$

$\exists$ $K \cdot p = a$ $\forall p, q \in \mathbb{Z}$

$K \cdot q = b$

so, $\times x$

$\times y$

$K \cdot x \cdot p = a \cdot x$

$K \cdot y \cdot q = b \cdot y$ +

$\exists$ $ax + by = \gcd(a,b)$

$= k(xp + yq)$

Hence $K \mid \gcd(a,b)$

∴ proved

$K \mid \gcd(a,b)$ by using Bezout's Identity

$\boxed{\text{H.W 3}}$

① $\forall \varepsilon > 0, \exists M > 0$ s.t $\left| 1 - \frac{n^2}{n^2+1} \right| < \varepsilon$

whenever $n \geq M$

Scratch

$$\frac{n^2+1-n^2}{n^2+1} \to \left| \frac{1}{n^2+1} \right| < \varepsilon$$

$$\frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{M^2} \leq \frac{1}{\varepsilon}$$

Proof

$$1/\sqrt{\varepsilon} = M$$

Let's consider $M = \sqrt{\varepsilon}$

where $n \geq M$,

So, $1 - \frac{n^2}{n^2+1} = \frac{n^2+1-n^2}{n^2+1} = \frac{1}{n^2+1}$

Now, $\frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{M^2} = \varepsilon$

So, $\left|1-\dfrac{n^2}{n^2+1}\right| < \varepsilon$ //

→ Negation: there exists a positive number $\varepsilon$ such that for any +ve numbers $M$, $\left|1-\dfrac{n^2}{n^2+1}\right| \geq \varepsilon$, $n \geq M$

---

(2) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{R}, \left((x \geq y) \Rightarrow \left(\frac{x}{y}=1\right)\right)$

Negation: $\exists x \in \mathbb{Z}$ s.t, $\forall y \in \mathbb{R}$, $(x \geq y) \wedge \left(\frac{x}{y} \neq 1\right)$

Original is true,
For int $x$, let $y = x+1$,
So, $x \geq y$ = False & the stmt is true. //

(3) $A = \{ n \in \mathbb{N} : 3 | n \text{ or } 4 | n \} \subset \mathbb{N}$

(a) $\exists x \in A$ s.t, $\exists y \in A$ s.t, $x+y \in A$

True, let, $x = 3$, $y = 3$

so, $x+y = 6 \in A$

(b) $\forall x \in A$, $\forall y \in A$, $x+y \in A$

False, we can prove this by proving its negation true.

$\exists x \in A$, $\exists y \in A$, $x+y \notin A$.

So, Let $x = 3$, $y = 4$ so, $x+y = 7 \notin A$.

Hence negation is False.

(c) $\exists x \in A$ s.t $\forall y \in A$, $x+y \in A$.

True, Let $x = 12$, so, for any $y = 3k$, $x+y = 12+3k \in A$,

$y = 4k$, $x+y = 12+4k \in A$ //

④ (a) $\forall n \in \mathbb{Z}, \exists y \in \mathbb{R} - \{0\}$ s.t $y^n \leq y$

True, since for any int $n$, pick $y = 1$, So, $y^n \leq y$ //

Negation → $\exists n \in \mathbb{Z}, \forall y \in \mathbb{R} - \{0\}$ s.t $y^n > y$

(b) $\exists y \in \mathbb{R} - \{0\}$ s.t $\forall n \in \mathbb{Z}, y^n \leq y$

True, since pick $y = 1$, then for any int $n$, $y^n \leq y$,

Negation → $\forall y \in \mathbb{R} - \{0\}$ s.t, $\exists n \in \mathbb{Z}, y^n > y$

(c) $\forall x \in \mathbb{R}$ where $x \neq 0$, we have $x \leq 1$ or $\frac{1}{x} \leq 1$

This is true since, if number is +ve then $\frac{1}{x} \leq 1$ else if -ve then, $x \leq 1$

Negation → $\exists x \in \mathbb{R}$ $x \neq 0$, we have $x > 1$ and $\frac{1}{x} > 1$ //

$\blacksquare$

(5) (a) • $\exists x \in k, \exists y \in l$, k unlocks l

• $\forall x \in k, \forall y \in l$, k doest unlock l

• Any keys does not unlock any locks

$\approx$ No keys unlock any locks

(b) • $\exists x \in K, \forall y \in l$, k unlocks y

• $\forall x \in K, \exists y \in l$, k doest unlock y

• No key unlocks any lock.

(c) • $\exists x \in l, \forall y \in K$, k doest unlock l

• $\forall x \in l, \exists y \in K$, k unlocks l

• Every lock is unlocked by some key

(6) $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}, a^2+b^2 \equiv 1 \mod 3$

Proof

Using Euclid's division algorithm

$a$ is of the form $3k+r$, $\forall k \in \mathbb{Z}$, $r \in \{0, 1, 2\}$.

Case 1:

$r = 0$

$a = 3k$

So, $a^2 = 9k^2$

So, let $b = 1$ So $a^2+b^2 = 9k^2+1$

$= 3(3k^2)+1$

$a^2+b^2 \equiv 1 \mod 3$

Case 2:

$r = 1$

$a = 3k+1$

So, $a^2 = 9k^2+1+6k$

Let $b = 0$

So, $a^2+b^2 = 3(3k^2+2k)+1$

$a^2+b^2 \equiv 1 \mod 3$

Case 3:

$r = 2$

$a = 3k+2$

$a^2 = 9k^2+4+12k$

Let, $b=0$

so, $a^2+b^2 = 3(3k^2+1+4k)+1$

So, $a^2+b^2 \equiv 1 \mod 3$

So, hence by proof by cases

$a^2+b^2 \equiv 1 \mod 3$ $\square$

---

(7)

$$5^n + 3n \equiv 1 \mod 9$$

$n = 1$

$8 \equiv 1$ $\quad 8 = 9q + 1$

$8 \equiv 1 \mod 9$

$x_n = \dfrac{\sqrt{n^3+1}}{n}$ converges 0

scratch

$$\left| \sqrt{\frac{n^3+1}{n^2}} - 0 \right| < \varepsilon$$

$$\sqrt{n + \frac{1}{n^2}} < \varepsilon$$

$$n + \frac{1}{n^2} < \varepsilon^2$$

$$\frac{n^3+1}{n^2} < \varepsilon^2$$

$$\frac{1}{\varepsilon^2} < \frac{n^2}{n^3+1} < \frac{n^2}{n^3}$$

$$\frac{1}{\varepsilon^2} < \frac{1}{n} \qquad n > N$$

$$n < \varepsilon^2 \qquad n >$$

$\frac{n^3+1}{n^2} < \varepsilon^2$

$\frac{1}{\varepsilon^2} < \frac{n^2}{n^3+1} < \frac{n^2}{n^3}$

$\frac{1}{\varepsilon^2} < \frac{1}{n}$ $\quad n > N$

$n < \varepsilon^2$ $\quad n >$

$n > \frac{1}{\varepsilon^2}$

$\varepsilon^2 > \frac{\varepsilon^2}{\varepsilon^2+1} > \frac{1}{n}$

$n > \frac{\varepsilon^2+1}{\varepsilon^2}$

$\frac{\varepsilon^2 - \varepsilon^2 - 1}{\varepsilon^2+1} > n$

$\frac{-1}{\varepsilon^2+1}$ $\quad n > -(\varepsilon^2+1)$

So, $\frac{\varepsilon^2}{\varepsilon^2+1} > \frac{1}{n}$

Now, $\frac{\varepsilon^2}{1} > \frac{\varepsilon^2}{\varepsilon^2+1} > n$

$\varepsilon^2 > n$ $\quad n+1 \geq n + \frac{1}{n^2}$ $\quad n + \frac{1}{n^2}$

$\frac{\varepsilon^2}{\varepsilon^2+1} > n+1$

$n > 1 \quad n + \frac{1}{n} \geq n + \frac{1}{n^2}$

$n > \frac{1}{\varepsilon^2}$

$\varepsilon^2 > \frac{1}{n}$

Proof

Let $N = -(\varepsilon^2+1)$

So, $n > N$ & $n > -(\varepsilon^2+1)$

$$\frac{1}{n} < -\frac{1}{\varepsilon^2+1}$$

$$\frac{1}{n} <$$

---

$\boxed{\text{H.W. 4}}$

(7) $\forall M \in \mathbb{R}, \exists t \in (a,b)$ s.t. $|f(t)| > M$

$(0,1)$ $\log(n)$ is unbounded

scratch

$$|\log(\cancel{x})| > M$$

$$t = 10^{-(M+1)}$$

⑧ $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \varepsilon$

$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N, |x_n - L| \geq \varepsilon$

$\boxed{\text{P.E 1}}$

① (a) For every $n \in \mathbb{Z}$, $n \leq 1$ or $n^2 \geq 4$

$\exists n \in \mathbb{Z}$, $n > 1$ and $n^2 < 4$

(b) For every $x \in \mathbb{Q}$ there exists $r \in \mathbb{R}$ so that if $x \geq 0$ then $r^2 = x$

$\exists x$ $x \in \mathbb{Q}$ s.t. $\forall r \in \mathbb{R}$, $x \geq 0$ and $r^2 \neq x$

(c) If you do not speak Korean then understanding K-Pop is hard.

Converse ⇒ If understanding K-Pop is hard then you do not speak Korean

Contrapositive ⇒ In understanding K-PoP is not hard then you do speak Korean

(d) $a, b \in \mathbb{Z}$. What $a | b$ means

$\forall a, b \in \mathbb{Z}$, $\forall k \in \mathbb{Z}$, $b = a \cdot k$.

It means that b is a multiple of a or simply a is divisible by b.

(2) (a) If $n \in \mathbb{Z}$ then $n^2+4n+8$ is even

False, because

Case 1: $n = 2k$

$4k^2 + 8k + 8$ = even

Case 2: $n = 2k+1$

$4k^2 + 1 + 4k + 8k + 4 + 8$ = odd

(b) If $n \in \mathbb{Z}$ then $n^2+3n+8$ is even

True, because

Case 1:

$4k^2 + 6k + 8$ = even

Case 2:

$4k^2 + 1 + 4k + 6k + 3 + 8$ = even

(3) Let $x \in \mathbb{R}$, if $x < 0$ then $\frac{x-2}{5} \leq \frac{2x+1}{2-x}$

scratch

$-\frac{(x-2)^2}{5} \leq 2x+1$

$\frac{(x-2)^2}{5} \geq -2x-1$

$(x-2)^2 \geq -10x-5$

$x^2+4-4x \geq -10x-5$

$x^2 \geq -6x-9$

$x^2+6x+9 \geq 0$

$(x+3)^2 \geq 0$

Proof

We know that $x < 0$,

Now, So, since $x < 0$,

$$x-3 < -3 < 0$$

So, $x-3 < 0$

On squaring $(x-3)^2 \geq 0$

Since squares are always +ve

So, $x^2+9-6x \geq 0$

$\vdots$

$$\frac{x-2}{5} \leq \frac{2x+1}{2-x}$$

(4) For any int $n \geq 1$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

Proof

Base Case

let's check for $n = 1$

$$1 \Rightarrow 2 - \frac{1}{1} \Rightarrow 2 - 1 \Rightarrow 1$$

Hence holds true

Inductive step

consider this holds true for $n=k$

So, $1+\frac{1}{4}+\frac{1}{9} \dots + \frac{1}{k^2} = 2-\frac{1}{k}$

Now, adding $\frac{1}{(k+1)^2}$ on both sides we get

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}\dots \frac{1}{k^2}+\frac{1}{(k+1)^2} \leq 2-\frac{1}{k}+\frac{1}{(k+1)^2}$$

So, $$2-\left[\frac{(k+1)^2+k}{k(k+1)^2}\right]$$

$$\leq 2-\frac{1}{(k+1)}$$

$\frac{(k+1)^2+k}{k(k+1)^2}$ so

$$> \frac{1}{k+1}$$

$$\frac{1}{(k+1)} < \frac{k^2+1+3k}{k(k+1)^2}$$

$$\frac{k^2+1+\cancel{2k}}{k(k+1)^2} > \frac{k^2+k}{k(k+1)^2}$$

$\boxed{\text{P.E 2}}$

① (a) $\forall x \in \{q \in \mathbb{Q} : q > 0\}$, $\frac{1}{x} \geq 1$ or $\ln(x) < 0$

$\exists x \in \{q \in \mathbb{Q} : q > 0\}$, $\frac{1}{x} < 1$ and $\ln(x) \geq 0$

③ $x \in \mathbb{R}$, if $0 < x < 4 \Rightarrow \frac{4}{x(4-x)} \geq 1$

$$4x - x^2 \leq 4$$

$$x^2 - 4x + 4 \geq 0$$

$$(x-2)^2 \geq 0$$

Now, $-2 < x - 2 < 2$

So $0 \leq (x-2)^2 < 4$

So, $(x-2)^2 \geq 0$

$\vdots$

④ P.T $4^{n-1} > n^2$ for $n \geq 3$

Base Case

$n = 3$

$4^{3-1} > 3^2$

$16 > 9$ ✓

Inductive

Let's true for $4^{k-1} > k^2$

Now, for $4^{(k+1)-1}$

we have

---

$$\sqrt{\frac{n^3+1}{n^2}}$$

$$\frac{n^3+1}{n^2} < \varepsilon^2$$

$$\frac{1}{n^2} < \frac{n^3+1}{n^2} < \varepsilon^2$$

So, $\frac{1}{n^2} < \varepsilon^2$

$\frac{1}{n} < \varepsilon$

$N = \left\lceil \frac{1}{\varepsilon} \right\rceil$

Proof

Let $N = \lceil 1/\varepsilon \rceil \leq n$,

So, $\frac{1}{\varepsilon} \leq n$

$\frac{1}{n} \leq \varepsilon$

$\frac{1}{n^2} \leq \varepsilon^2$

$\frac{1}{n^2} < \frac{n^3+1}{n^2}$

$\frac{n^3+1}{n^2} < \varepsilon^2$

$\sqrt{\frac{n^3+1}{n^2}} < \varepsilon$

WS-limits

(2) $\exists \varepsilon > 0, \forall N \in \mathbb{N}$ s.t $\exists n \geq N$,

$|x_n - L| > \varepsilon$

(4) $\lim_{x \to 1} (5x+3) = 8$

$0 < |x-1| < \delta$

$|5x+3-8| < \varepsilon$

$5|x-1| < \varepsilon$

$|x-1| < \varepsilon/5$

$\delta = \varepsilon/5$

(5)

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$$

$$0 < |x - 2| < \delta$$

$$\frac{|x-2|}{|2x|} < \varepsilon$$

$$|x-2| < \varepsilon \cdot |2x|$$