Solutions to Homework 5:

1. *Proof.*

$$\sum_{k=1}^{n}(2k-1)*2^k = 6+2^n*(4n-6) \tag{1}$$

Base Case:
Let's try to prove this for n = 1

$$\sum_{k=1}^{1}(2k-1)*2^k = 6+2^1*(4(1)-6) \tag{2}$$
$$(2(1)-1)*2^1 = 6+2*-2. \tag{3}$$
$$1*2 = 6-4 \tag{4}$$
$$2 = 2 \tag{5}$$
$$\tag{6}$$

So the base case holds true.

Inductive Step:
Let's assume that the equation holds true for n = i as well,

$$\sum_{k=1}^{i}(2k-1)*2^k = 6+2^i*(4i-6) \tag{7}$$

Now for n = i + 1 we have,

$$\sum_{k=1}^{i+1}(2k-1)*2^k = \sum_{k=1}^{i}(2k-1)*2^k + (2(i+1)-1)*2^{i+1} \tag{8}$$
$$6+2^i*(4i-6)+(2(i+1)-1)*2^{i+1} \tag{9}$$
$$6+2^i(4i-6+4i+2) \tag{10}$$
$$6+2^i(8i-4) \tag{11}$$
$$6+2^{i+1}(4i-2) \tag{12}$$
$$6+2^{i+1}(4(i+1)-6) \tag{13}$$

Hence the equation also holds true for n = i+1

So by induction $\sum_{k=1}^{n}(2k-1)*2^k = 6+2^n*(4n-6)$ holds true for all $n \in \mathbb{N}$. □

2. *Proof.* $a_{n+2} = 5a_{n+1} - 6a_n$ and $a_1 = 1, a_2 = 5$ then $a_n = 3^n - 2^n$ for all $n \geq 3$
Base Case:

For n = 1 and 2
$a_1 = 1$ and $3^1 - 2^1 = 1$
$a_2 = 5$ and $3^2 - 2^2 = 5$
So it holds true to the base case n = 1 and 2.

Inductive Step
Consider $k \geq 2$ and assume that $a_i = 3^i - 2^i$ for all $0 \leq i \leq k$
Now we that that it holds true for n = k-1 so,

$$a_{(k-1)+2} = 5a_{(k-1)+1} - 6_{k-1} \tag{14}$$
$$a_{k+1} = 5a_k - 6a_{k-1} \tag{15}$$
$$a_{k+1} = 5 * (3^k - 2^k) - 6 * (3^{k-1} + 2^{k-1}) \tag{16}$$
$$a_{k+1} = 5(3^k - 2^k) - 6(3^{k-1} + 2^{k-1}) \tag{17}$$
$$a_{k+1} = 5(3^k - 2^k) - 6(\frac{3^k}{3} + \frac{2^k}{2}) \tag{18}$$
$$a_{k+1} = 5(3^k - 2^k) - 6(\frac{2 * 3^k + 3 * 2^k}{6}) \tag{19}$$
$$a_{k+1} = 5 * 3^k - 5 * 2^k - 2 * 3^k + 3 * 2^k \tag{20}$$
$$a_{k+1} = 3 * 3^k - 2 * 2^k \tag{21}$$
$$a_{k+1} = 3^{k+1} - 2^{k+1} \tag{22}$$

Hence it holds true for n = k+1 as well

So, by using strong induction we proved $a_n = 3^n - 2^n$ for all $n \in \mathbb{N}$ □

3. *Proof.* $a_o = 1, a_1 = 3, a_2 = 9$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$ then show that $a_n \leq 3$

Base Case:
For n = 0, 1 and 2 $a_0 = 1$ and $3^0 = 1$ so $a_0 \leq 3^0$
$a_1 = 3$ and $3^1 = 3$ so $a_1 \leq 3^1$
$a_0 = 9$ and $3^2 = 9$ so $a_2 \leq 3^2$
Hence the base case holds true

Inductive Step:
For $k \geq 2$ we have $a_i \leq 3^i$ for all $0 \leq i \leq k$
Now, $a_{k+1} = a_k + a_{k-1} + a_{k-2}$
So,

$$a_{k+1} = 3^k + 3^{k-1} + 3^{k-1} \tag{23}$$
$$a_{k+1} = 3^k(1 + \frac{1}{3} + \frac{1}{9}) \tag{24}$$
$$a_{k+1} = 3^k(\frac{9+3+1}{9}) \tag{25}$$
$$a_{k+1} = 3^{k-2}(13) \tag{26}$$

Now, $3^{k-2}(13) < 3^{k-2}(27)$
So, $a_{k+1} < 27 * 3^{k-2}$
$a_{k+1} < 3^{k+1}$
Hence it holds true for n = k+1

So using Strong Induction $a_n \leq 3^n$ holds true for all $n \in \mathbb{N}$ □

4. *Proof.* $\sum_{n=1}^{k} \frac{1}{k+n} > \frac{13}{24}$
Base Case:
Let's take n = 2
$\sum_{n=1}^{2} \frac{1}{2+n} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{13}{24}$
Hence the statement holds true for the base case n = 2

Inductive Step:
Let's assume i holds true for n = k
So,

$$\sum_{n=1}^{k} \frac{1}{k+n} > \frac{13}{24} \tag{27}$$

For n = k + 1 we have

$$\sum_{n=1}^{k+1} \frac{1}{k+1+n} = \sum_{n=1}^{k} \frac{1}{k+n} + \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1} \tag{28}$$
$$\sum_{n=1}^{k+1} \frac{1}{k+1+n} = \sum_{n=1}^{k} \frac{1}{k+n} + \frac{2k+2+2k+1-2k-2}{(2k+1)(2k+2)} \tag{29}$$
$$\sum_{n=1}^{k+1} \frac{1}{k+1+n} = \sum_{n=1}^{k} \frac{1}{k+n} + \frac{1}{(2k+1)(2k+2)} \tag{30}$$
$$\sum_{n=1}^{k+1} \frac{1}{k+1+n} > \frac{13}{24} + \frac{1}{(2k+1)(2k+2)} \tag{31}$$
$$\sum_{n=1}^{k+1} \frac{1}{k+1+n} > \frac{13}{24} \tag{32}$$

Hence the statement holds true for n = k+1
So, by induction $\sum_{n=1}^{k} \frac{1}{k+n} > \frac{13}{24}$ holds true for all $n \in \mathbb{N}$ □

5. *Proof.* Show that $7^{4n+3} + 2$ is a multiple of 5 $\forall n \in \mathbb{Z}$ where n is positive.

Base Case:
For n = 1,
$7^{4*1+3} + 2 = 7^7 + 2 = 823543 + 2 = 823545$
823545 is a multiple of 5 and hence the base case holds true.

Induction Step:
Consider n=k to be true,
So, $7^{4k+3} + 2 = 5m, \qquad \forall m \in \mathbb{Z}$
Now, for n = k+1 we have,

$$7^{4(k+1)+3} + 2 \tag{33}$$
$$7^{4(k+1)+3} + 2 = 7^{4k+7} + 2 \tag{34}$$
$$7^{4(k+1)+3} + 2 = 7^4 * 7^{4k+3} + 2 \tag{35}$$
$$7^{4(k+1)+3} + 2 = 7^4(5m - 2) + 2 \tag{36}$$
$$7^{4(k+1)+3} + 2 = 7^4 * 5m - 2 * 7^4 + 2 \tag{37}$$
$$7^{4(k+1)+3} + 2 = 7^4 * 5m - 4802 + 2 \tag{38}$$
$$7^{4(k+1)+3} + 2 = 7^4 * 5m - 4800 \tag{39}$$
$$7^{4(k+1)+3} + 2 = 5(7^4 * m - 960) \tag{40}$$
$$7^{4(k+1)+3} + 2 = 5 * p \tag{41}$$

So it holds true for n = k+1

Hence using induction $7^{4n+3} + 2$ is a multiple of 5 $\forall n \in \mathbb{Z}$ where n is positive.

□

6. **Base Case:**
Let's take n = 1
$a_2 = a_1^2 - a_1$ $a_2 = 9 - 3 = 6$
Now $6 > 3 > 2$, so, a is an increasing

Induction Step:
For $k \geq 2$ we have $a_i < a_{i+1}$ for $0 \leq i \leq k$
So $a_{k+1} = a_k^2 - a_k$ that is $a_k < a_{k+1}$
Now for n = k + 1 we have
$a_{k+2} - a_{k+1}$

$a_{k+1}^2 - a_{k+1} - a_{k+1} = a_{k+1}^2 - 2a_{k+1}$
$(a_k^2 - 2a_k)^2 - 2(a_k^2 - 2a_k)$
$(a_k^2 - 2a_k)(a_k^2 - 2a_k - 2)$
Now we know that $(a_k^2 - 2a_k)$ is positive from the assumption and $(a_k^2 - 2a_k - 2)$ is also positive,
So, $a_{k+2} - a_{k+1} > 0$
Hence this holds true for n = k + 1
So by induction $a_{n+1} - a_n > 0$ holds true for all $n \in \mathbb{N}$

7. *Proof.* $\sum_{k=1}^{N} k * x^{k-1} = \frac{1-x^N}{(1-x)^2} - \frac{Nx^N}{1-x}$
Base Case:
Let us consider N=1,
So,

$$\sum_{k=1}^{1} k * x^{k-1} = \frac{1-x^1}{(1-x)^2} - \frac{1x^1}{1-x} \quad (42)$$

$$1 * x^{1-1} = \frac{1-x}{(1-x)^2} - \frac{x}{1-x} = \frac{1}{1-x} - \frac{x}{1-x} \quad (43)$$

$$1 = 1 \quad (44)$$

So the base case holds true

Inductive Step:
Assume N=i holds true hence,
$\sum_{k=1}^{i} k * x^{k-1} = \frac{1-x^i}{(1-x)^2} - \frac{ix^i}{1-x}$
Now on adding $(i+1)x^i$ on both sides we get,

$$\sum_{k=1}^{i} k * x^{k-1} + (i+1)x^i = \frac{1-x^i}{(1-x)^2} - \frac{ix^i}{1-x} + (i+1)x^i \tag{45}$$

$$\sum_{k=1}^{i+1} k * x^{k-1} = \frac{1-x^i}{(1-x)^2} - \frac{(ix^i)(1-x)}{(1-x)^2} + \frac{(i+1)x^i(1-x)^2}{(1-x)^2} \tag{46}$$

$$\sum_{k=1}^{i+1} k * x^{k-1} = \frac{(i+1)x^i + (i+1)x^{i+2} - 2(i+1)x^{i+1} + 1 - x^i - ix^i + ix^{i+1}}{(1-x)^2} \tag{47}$$

$$\sum_{k=1}^{i+1} k * x^{k-1} = \frac{(i+1)x^i + (i+1)x^{i+2} - 2(i+1)x^{i+1} + 1 - x^i(1+i) + ix^{i+1}}{(1-x)^2} \tag{48}$$

$$\sum_{k=1}^{i+1} k * x^{k-1} = \frac{(i+1)x^{i+2} - x^{i+1}(2i+2-i) + 1}{(1-x)^2} \tag{49}$$

$$\sum_{k=1}^{i+1} k * x^{k-1} = \frac{(i+1)x^{i+2} - x^{i+1} - x^{i+1}(i+1) + 1}{(1-x)^2} \tag{50}$$

$$\sum_{k=1}^{i+1} k * x^{k-1} = \frac{1 - x^{i+1} - x^{i+1}(i+1)(1-x)}{(1-x)^2} \tag{51}$$

$$\sum_{k=1}^{i+1} k * x^{k-1} = \frac{1-x^{i+1}}{(1-x)^2} - \frac{(i+1)x^{i+1}}{1-x} \tag{52}$$

So it holds true for N = i+1 as well.
Hence by induction $\sum_{k=1}^{N} k * x^{k-1} = \frac{1-x^N}{(1-x)^2} - \frac{Nx^N}{1-x}$ holds true for all $N \in \mathbb{N}$ □

8. *Proof.* $n^3 > 2n^2 + n$
Base Case:
If we pluck the value of n=1,2 you'll notice that the equation $n^3 > 2n^2 + n$
$n = 1$, we have $1 \geq 3$ which is false
$n = 2$, we have $8 \geq 10$ which is false
If we take the value of n = 3, we get $27 \geq 21$ which is true
For n = 4, we have $64 \geq 36$ which is true as well
Hence it holds true for $n \geq 3$

Inductive step:
Let's assume that the statement is true for $n = k$. Hence

$$k^3 \geq 2k^2 + k \tag{53}$$

Now for n = k + 1 we have,

$$(k+1)^3 - 2(k+1)^2 - k - 1 = k^3 + 3k^2 + 1 + 3k - 4k - 2k^2 - k - 2 - 1 \tag{54}$$
$$(k+1)^3 - 2(k+1)^2 - k - 1 = k^3 + k^2 - 2k - 2 \tag{55}$$
$$(k+1)^3 - 2(k+1)^2 - k - 1 = k^3 - 2k^2 - k + 3k^2 - k - 2 \tag{56}$$
$$(k+1)^3 - 2(k+1)^2 - k - 1 > 3k^2 - k - 2 = (3k+2)(k-1) \tag{57}$$
$$k \geq 2(3k+2)(k-1) > 0 \tag{58}$$

So, $(k+1)^3 - 2(k+1)^2 - (k+1) > 0$
Which means $(k+1)^3 > 2(k+1)^2 + (k+1)$
So the statement holds true for n = k + 1
Hence by induction $n^3 > 2n^2 + n$ holds true for all $n \in \mathbb{N}$

$\square$