PLP - 11

TOPIC 11 —CONTRAPOSITIVE PROOF

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CONTRAPOSITIVE PROOF

TRY THE CONTRAPOSITIVE

PROPOSITION:

Let $n \in \mathbb{Z}$. If n^2 is even then n is even.

Scratch work

- *Assume hypothesis is true* so n^2 is even
- Hence $n^2 = 2k$ for some integer k
- So $n = \sqrt{2k}$ and so …

Not sure where to go? Try the **contrapositive**

TRY THE CONTRAPOSITIVE 2

Let $n \in \mathbb{Z}$. If n^2 is even then n is even.

Scratch work

- Form the contrapositive: If n is not even then n^2 is not even
- Since $n \in \mathbb{Z}$: If n is odd then n^2 is odd
- Now we know what to do — just tell the **reader** that you are proving the contrapositive.

PROOF.

We prove the contrapositive: if n is odd then n^2 is odd.

- Assume that n is odd.
- Hence $n = 2\ell + 1$ and so $n^2 = 4\ell^2 + 4\ell + 1 = 2(2\ell^2 + 2\ell) + 1$.
- Since $2\ell^2 + 2\ell \in \mathbb{Z}$, it follows that n^2 is odd.

Since the contrapositive it true, the original statement is true.

ANOTHER EXAMPLE

PROPOSITION:

Let $n \in \mathbb{Z}$. If $3n + 7$ is odd then n is even.

Scratch work

- Assume $3n + 7$ is odd, so $3n + 7 = 2\ell + 1$
- Then $3n = 2\ell - 6$ and $n = \frac{2\ell - 6}{3}$ which is… **stuck**
- Start again with *contrapositive*: If n is odd then $3n + 7$ is even.
- Then $n = 2k + 1$ so $3n + 7 = 6k + 3 + 7 = 2(3k + 5)$ which is even.

Write it up nicely.

WRITE UP THE PROOF

PROOF.

We prove the contrapositive. Assume that n is odd, so $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then $3n + 7 = 2(3k + 5)$ and since $3k + 5 \in \mathbb{Z}$ it follows that $3n + 7$ is even.

Since the contrapositive is true, the result holds.

Usually more than 1 way to prove things. A direct proof:

PROOF.

Let $3n + 7$ be odd, so $3n + 7 = 2k + 1$ for some $k \in \mathbb{Z}$. But then

$$3n = 2k - 6 \qquad \text{and so} \qquad n = 2k - 6 - 2n = 2(k - 3 - n)$$

Now since $k - 3 - n \in \mathbb{Z}$ it follows that n is even.

Q $\sim(P \Leftrightarrow Q) \equiv (P \text{ XOR } Q)$

$\sim[(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$ [Bic]

$\sim[(\sim P \vee Q) \wedge (\sim Q \vee P)]$ [Imp]

$(\sim(\sim P) \wedge \sim Q) \rightarrow \sim(\sim P \vee Q) \vee \sim(\sim Q \vee P)$ [Dmg]

$\vee$

$(\sim(\sim Q) \wedge \sim P)$ $[P \wedge \sim Q] \vee [Q \wedge \sim P]$ [Dneg]

[Dmg]

$P \text{ XOR } Q$ [Def of XOR]

QED

Q Is $((P \Rightarrow Q) \Rightarrow R) \equiv (P \Rightarrow (Q \Rightarrow R))$

$P = F : 2|3$

F $Q = T : 3|3$ T

$R = F : 3|5$

Q Prove, Let $n \in \mathbb{Z}$, then $n \equiv 3 \pmod 5$ iff $5 \mid (3n+1)$

Prove both sides since iff

Assume $5 \mid (3n+1)$ $n \% 5 = 3$

$3n+1 = 5k$

$(3n = 5k-1) \times 3$

$9n = 15k - 3$

$10n - n = 15k - 3$

$n = 10n - 15k + 3$

$n = 5(2n - 3k) + 3$

$n = 3 \pmod 5 \Leftarrow n = 5q + 3$ where $q \in \mathbb{Z} = 2n - 3k$

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Q

$3 \mid n^2 \Rightarrow 3 \mid n$

$3 \nmid n \Rightarrow 3 \nmid n^2$

Case 1: $n = 3m+1$

$$n^2 = (3m+1)^2 = 9m^2 + 6m + 1$$
$$= 3(3m^2+2m)+1$$
$$= 3j+1 \text{ for } \exists j \in \mathbb{Z}$$

Case 2: $n = 3m+2$

$$n^2 = (3m+2)^2 = 9m^2 + 12m + 4$$
$$= 3(3m^2+4m+1)+1$$
$$= 3i+1 \text{ for } \exists i \in \mathbb{Z}$$

Q

9. Prove that for all $x \in \mathbb{R}$, $\big||x+1| - |x-3|\big| \le 4$.
Hint: How can we get rid of the absolute values in this expression?

$$||x+1| - |x-3|| \le 4$$
$$||x+1| - |x-3|| - 4 \le 0$$

Case 1: $x \ge 3$

$$|x+1-x+3| - 4$$
$$|4| - 4 = 0$$

Case 2: $-1 < x < 3$

Consider $\big||x+1| - |x-3|\big|$

$= |(x+1) + (x-3)|$

$= |2x - 2|$

$= 2|x-1|$

Case 2a: $-1 < x < 1$

$= -2(x-1)$

$\Rightarrow 2(1-x) < 2(1+1) = 4$ //

$\Rightarrow 1 - x < 1+1$

$\Rightarrow -x < 1$

$\Rightarrow x > -1$

Case 2b: $1 \le x < 3$

$2(x-1) < 2(3-1) = 4$ //

$x - 1 < 3 - 1$

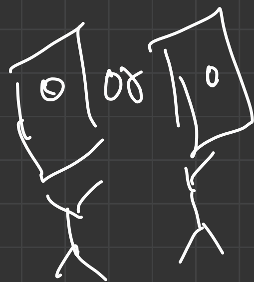
$x < 3$

Case 3: $x \le -1$ $\big||x+1| - |x-3|\big| \Rightarrow$

$|-(x+1) + (x-3)|$

$|-1-3| = |-4| = 4$ //

10. We have two doors and behind one of then there is a prize. There are two guardians guarding the doors. One of them always tells the truth and the other always tells a lie. You are allowed to ask one question to one of the guards to figure out where the prize is. What would that question be?



Truth $\Rightarrow$ T