PLP - 21

TOPIC 21—GENERALISING INDUCTION (A BIT)

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TWO GENERALISATIONS OF INDUCTION

A GENERALISATION

THEOREM: MATHEMATICAL INDUCTION.

Let $\ell \in \mathbb{Z}$ and $S = \{n \in \mathbb{Z}$ s.t. $n \geq \ell\}$. Let $P(n)$ be a statement for all $n \in S$. Then if

- $P(\ell)$ is true, and
- $P(k) \implies P(k+1)$ is true for all integer $k \in S$

then $P(n)$ is true for all $n \in S$.

PROPOSITION:

For every integer $n \geq 5$, $2^n \geq n^2$

PROOF

PROOF.

We prove the result by induction. Since $2^5 = 32 > 25 = 5^2$, the result holds when $n = 5$. Now assume that $k \geq 5$ and that $2^k \geq k^2$. Then

$$\begin{aligned} 2^{k+1} &\geq 2k^2 = k^2 + k^2 \\ &\geq k^2 + 5k && \text{since } k \geq 5 \\ &= k^2 + 2k + 3k \\ &\geq k^2 + 2k + 1 && \text{since } k \geq 5 \end{aligned}$$

Thus the inductive step holds for $k \geq 5$.

The result follows for all integer $n \geq 5$ by induction.

ANOTHER GENERALISATION

THEOREM: STRONG MATHEMATICAL INDUCTION.

Let $\ell \in \mathbb{Z}$ and $S = \{n \in \mathbb{Z}$ s.t. $n \geq \ell\}$. Let $P(n)$ be a statement for all $n \in S$. Then if

- $P(\ell)$ is true, and
- $(P(\ell) \wedge P(\ell+1) \wedge P(\ell+2) \wedge \cdots \wedge P(k)) \implies P(k+1)$ is true for all integer $k \in S$

then $P(n)$ is true for all $n \in S$.

PROPOSITION:

Let $\theta \in \mathbb{R}$ be fixed.

Let $p_0 = 1, p_1 = \cos\theta$, and $p_n = 2p_1 p_{n-1} - p_{n-2}$. Then $p_n = \cos(n\theta)$ for all integer $n \geq 0$.

A LITTLE TRIGONOMETRIC REMINDER

Recall that

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \qquad \cos(a-b) = \cos a \cos b + \sin a \sin b$$

PROOF.

We prove the result by strong induction. When $n = 0$ we have $p_0 = \cos 0 = 1$ as required. Now assume that $p_j = \cos j\theta$ for $j = 0, 1, 2, \ldots k$. Now consider $p_{k+1} = 2p_1 p_k - p_{k-1}$

$$\begin{aligned} p_{k+1} &= 2\cos\theta\cos k\theta - \cos(k-1)\theta \\ &= 2\cos\theta\cos k\theta - (\cos k\theta\cos\theta + \sin\theta\sin k\theta) \\ &= \cos\theta\cos k\theta - \sin\theta\sin k\theta \\ &= \cos(k+1)\theta \end{aligned}$$

as required. So the result holds for all integer $n \geq 0$ by strong induction.