Solutions to Homework 7:

1. $P(\{1, 2\} = \{\{\phi\}, \{\ 1\}, \{\ 2\}, \{\ 1, 2\})$
   $R = \{(\phi, \phi), (\phi, \{1\}), (\phi, \{2\}), (\phi, \{1, 2\}), (\{1\}, \{1, 2\}), (\{2\}, \{1, 2\}), (\{1\}, \{2\}), (\{2\}, \{1\})\}$

2. (a) This statement is false
   Let $A = \{1, 2\}$ such that R $= \{(1, 1), (2, 2)\}$
   This implies R is reflexive
   Now $\bar{R} =$ (A x A) - $R = \bar{R} = \{(1, 2), (2, 1)\}$
   So $\bar{R}$ is not reflexive

   (b) This statement is true

   Proof. Let $a, b \in A$ such that relation R is symmetric and we know that $(a, b) \in \bar{R}$
   This implies, $(a, b) \notin R$
   Since R is symmetric, $(b, a) \notin R$
   Now, $\bar{R}$ is the complement of R which implies $(b, a) \in \bar{R}$
   Therefore, $\bar{R}$ is symmetric □

   (c) This statement is false
   Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$
   This implies, $\bar{R} = \{(2, 2), (3, 3)\}$
   So, $\bar{R}$ is not transitive

3. *Proof.* We know that R is reflexive and so ,
   $(a, a) \in R$
   Now let bRa $\in R$ and so (aRa $\wedge$ bRa) $\implies$ aRb
   Hence R is symmetric

   We now know that R is reflexive and symmetric so,
   let aRc and cRb be true
   Let cRb $\implies$ bRc since R is symmetric
   So, (aRc $\wedge$ cRb) $\implies$ aRb
   So now aRc, cRb and aRb $\in R$, R is transitive

   □

4. Let R be a relation on A and R' be a relation on B such that
   $R' = \{(f(x), f(y)) : (x, y) \in R\}$ where $f : A \to B$
   We can write this as
   $(x, y) \in R \implies (f(x), f(y)) \in R'$

   (a) This statement is true
   Let x be any element in A
   This implies, $f(x) \in B$
   Now, if R is reflexive, then $(x, x) \in R$

So we can say that $(f(x), f(x)) \in R'$
Since, $f(x) = f(x), R'$ is reflexive.

(b) This statement is true
Let x, y be any element in A such that R is symmetric
This implies, $xRy \wedge yRx$ is true
So ,$(f(x), f(y)) \in R'$ and $(f(y), f(x)) \in R'$
Therefore, R' is symmetric.

5. Let R be a relation on $\mathbb{R}$ such that
$R = \{(x, x+n) : x \in \mathbb{R}, n \in \mathbb{N}\}$
This can be written as
$(x, y) \in R \implies (y - x) \in \mathbb{N}$

(a) This statement is true
Proof. Let $x_1 R y_1$ and $x_2 R y_2$
Therefore, $y_1 - x_1 = n_1$ and $y_2 - x_2 = n_2$ where n_1 and $n_2 \in \mathbb{N}$
Adding the 2 equations,
$y_1 + y_2 - (x_1 + x_2) = n_1 + n_2 = n$ where $n = n_1 + n_2 \in \mathbb{N}$
Thus, $(x_1 + x_2)\, R\, (y_1 + y_2)$

(b) This statement is false
Let $x_1 = 1.6, y_1 = 2.6, x_2 = 3.4, y_2 = 4.4$
Now, $y_1 - x_1 = 1$ and $y_2 - x_2 = 1$
Thus, $x_1 R y_1$ and $x_2 R y_2$
But, $x_1 \cdot y_1 = 4.16$ and $x_2 \cdot y_2 = 14.96$
Now, $x_2 \cdot y_2 - x_1 \cdot y_1 = 10.8 \notin \mathbb{N}$
Thus, the statement $x_1 \cdot y_1 R x_2 \cdot y_2$ is false

6. *Proof.* Let T be a relation on $\mathbb{R} - \{0\}$ such that
$(a, b) \in T \implies \frac{a}{b} \in \mathbb{Q}$
Reflexive: Consider aTa where $a \in \mathbb{R} - \{0\}$

$$\frac{a}{a} = 1 \in \mathbb{Q} \tag{1}$$

Therefore, T is reflexive Symmetric: Let aTb where $a, b \in \mathbb{R} - \{0\}$

$$\begin{aligned} \text{By definition of rational numbers } \tfrac{a}{b} = \tfrac{p}{q} \in \mathbb{Q}, \forall p, q \in \mathbb{Z} \\ \text{Therefore, } \tfrac{b}{a} = \tfrac{q}{p} \in \mathbb{Q} \end{aligned} \tag{2}$$

Therefore, T is symmetric
Transitive: Let aTb and bTc where $a, b, c \in \mathbb{R} - \{0\}$

$$\begin{aligned} \frac{a}{b} \in \mathbb{Q} \wedge \frac{b}{c} \in \mathbb{Q} \\ \frac{a \cdot b}{b \cdot c} = \frac{a}{c} \text{ Multiplying the 2 equations together} \\ \frac{a}{c} \in \mathbb{Q} \end{aligned} \tag{3}$$

Therefore, T is transitive □

7. (a) $R = \{(0,0), (0,3), (1,2), (2,1), (3,0), (3,3)\}$
   (b) No, the relation is not reflexive since $(1,1), (2,2) \notin R$
   (c) Yes, the relation is symmetric
   (d) No elements need to be added. The relation is already transitive

8. *Proof.* To prove the statement that a relation is an equivalence relation if and only if the relation is reflexive and circular.

   We need to prove 2 statements.

   (a) (R is an equivalence relation) $\implies$ (R is reflexive and circular)
   (b) (R is reflexive and circular) $\implies$ (R is an equivalence relation)

   Case 1:
   Proof. Let R be an equivalence relation
   Therefore, R is reflexive, transitive and symmetric.
   If R is transitive, then
   $aRb \wedge bRc \implies aRc$
   Now, R is also symmetric
   Therefore, $aRc \implies cRa$
   Therefore, $aRb \wedge bRc \implies cRa$
   Thus, R is circular
   Hence, R is both reflexive and circular

   Case 2:
   Proof. Let R be a relation such that it is reflexive and circular
   To prove that it is an equivalence relation, we need to prove that it is symmetric and transitive(we already know that it is reflexive)
   Symmetric: Let aRb for some arbitrary a, b
   We also know that aRa since R is reflexive
   Therefore, $aRa \wedge aRb \implies bRa$ (From circular property)
   Since, aRa is always true
   $aRb \implies bRa$

Hence, R is symmetric
Transitive: Let aRb and bRc for some a, b, c
From circular property, $aRb \wedge bRc \implies cRa$
We already proved that R is symmetric,
Therefore, $cRa \implies aRc$
Thus, $aRb \wedge bRc \implies aRc$
Hence, R is transitive
Therefore, R is an equivalence relation
From the 2 cases the statement is proved.

□