PLP - 15

TOPIC 15—NESTED QUANTIFIERS

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Quantifiers do not commute

$$\forall x, \exists y \text{ s.t. } P(x,y) \quad \not\equiv \quad \exists y \text{ s.t. } \forall x, P(x,y)$$

Consider:

$$\forall z \in \mathbb{Z}, \exists w \in \mathbb{N} \text{ s.t. } z^2 < w$$

Must do quantifiers *in order* — like a 2 player game:

- Player 1: picks the value of z first
- Player 2: knows what Player 1 did, and chooses w

So

- Player 1 picks some integer z
- Player 2 needs w to be big enough so that $w > z^2$ – pick $w = z^2 + 1$

NESTED QUANTIFIERS

$\forall z \in \mathbb{Z}, \exists w \in \mathbb{N}$ **s.t.** $z^2 < w$

PROOF.

- Let z be any integer.
- Now choose $w = z^2 + 1$.
- We know that $w \in \mathbb{Z}$ and that $w \geq 1$, so $w \in \mathbb{N}$.
- Further we know that $w > z^2$ so the statement is true.

- Player 1 picks *any* $z \in \mathbb{Z}$ — universal quantifier
- Player 2 picks *a single* w based on that choice — existential quantifier
- We verify that $w \in \mathbb{N}$.
- We confirm that the inequality holds.

THE OTHER WAY AROUND

$$\exists w \in \mathbb{N} \text{ s.t. } \forall z \in \mathbb{Z}, z^2 < w$$

Must do quantifiers *in order* — like a 2 player game:

- Player 1: chooses *one* value of w first
- Player 2: knows what Player 1 did, but must check *all* z

Scratch work

- P1 picks $w = 1$, but then $z = 2$ is too big
- P1 picks $w = 2$, but then $z = 3$ is too big
- P1 picks $w = 3$, but then $z = 4$ is too big

Smells false, so check the negation.

LOOK AT NEGATION

$$\forall w \in \mathbb{N}, \exists z \in \mathbb{Z} \text{ s.t. } z^2 \geq w$$

- Player 1 picks *any* $w \in \mathbb{N}$
- Player 2 chooses *one* $z \in \mathbb{Z}$. What worked above?

PROOF.

We prove the statement is false by showing the negation is true.

- Let $w \in \mathbb{N}$.
- Now choose $z = w + 1 \in \mathbb{Z}$
- Then $z^2 = w^2 + 2w + 1 > w$ since $w^2 \geq 0$ and $w \geq 1$.

Since the negation is true, the original statement is false.

ANOTHER NESTED EXAMPLE

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } xy = x + y$$

Scratch work

- P1 picks any x they want.
- P2 needs to pick y so that $xy = x + y$
- We can solve that $xy - y = x$ so $y = \frac{x}{x-1}$

So is this true?

What happens when $x = 1$?

ANOTHER NESTED EXAMPLE — NEGATION

$$\exists x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{R}, xy \neq x + y$$

Scratch work. Failed last time when $x = 1$.

- P1 picks $x = 1$.
- Then no matter what $y \in \mathbb{R}$ we have $y \neq y + 1$.

PROOF.

The statement is false. Pick $x = 1$. Then no matter what $y \in \mathbb{R}$ we choose, we have $y \neq y + 1$ as required. Since the negation is true, the original statement is false.

ANOTHER ONE

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } (y \neq 0) \implies xy = 1$$

Scratch work

- P1 *first* picks one value of x
- P2 then picks y to make the implication true.
- If the hypothesis is false, implication is true. P2 just picks $y = 0$.

PROOF.

We prove the statement is true. Pick any $x \in \mathbb{R}$, and then set $y = 0$. Since the hypothesis of the implication is false, the implication is always true.

A SIMILAR ONE

$$\exists x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{R}, (y \neq 0) \implies xy = 1$$

Scratch work

- P1 *first* picks one value of x
- P2 then picks y to make the implication true.
- Implication is false when (H,C) = (T,F) — can that happen?
- Sure $x = 1$ then pick $y = 2$

Better look at the negation.

Recall: $\sim (P \implies Q) \equiv (P \wedge \sim (Q))$

A SIMILAR ONE — NEGATED

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } (y \neq 0) \wedge xy \neq 1$$

Scratch work

- P1 picks *any* x
- P2 knows x, so based on that picks $y \neq 0$ so that $xy \neq 1$.
- If P2 picks $y = 1$ that will work nicely unless $x = 1$
- If P1 has picked $x = 1$ then $P2$ can pick $x = 2$

PROOF.

We show the statement is false by proving the negation is true. Pick any $x \in \mathbb{R}$. Either $x = 1$ or $x \neq 1$

- If $x = 1$ then set $y = 2$.
- If $x \neq 1$ then set $y = 1$.

In both cases, $y \neq 0$ and $xy \neq 1$ as required.

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$, s.t. $y+5=x$ ✓

$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}$, s.t $y+5 \neq x$ ✗

(b) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$, s.t. $x^n > 0$ ✗

$\exists x \in \mathbb{R}, \forall n \in \mathbb{N}$, s.t $x^n \leq 0$ ✓

(c) $\forall n \in \mathbb{Z}$, (n^2 is even) $\Rightarrow$ (n is even) ✓

$\exists n \in \mathbb{Z}$, n^2 is even and n is odd

(d) $\exists x \in \mathbb{R}$, s.t. $\forall y \in \mathbb{R}, (y \leq x-1) \Rightarrow (y^2 - x^2 \geq 4)$ ✓

$\forall x \in \mathbb{R}$, s.t $\exists y \in \mathbb{R}$, $y \leq x-1$ and $y^2 - x^2 < 4$

$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (\forall z > 0, |x-y| < z) \Rightarrow (x=y)$

$(x \neq y) \Rightarrow (\exists z > 0, |x-y| \geq z)$

Proof

If $x \neq y$ then $x - y \neq 0$,

So, $|x-y| \neq 0$

Therefore, $|x-y| = c > 0$, for some $c \in \mathbb{R}$

Let, $z = c$, then $|x-y| \geq z$

∎

(8) Assume $\forall x \in I, (k \in \mathbb{Q} - \{0\} \Rightarrow \frac{x}{k} \in I)$.
Use this & fact that $\pi \in I$, to show,
$\forall n \in \mathbb{N}, \exists y \in I$ s.t. $1/n > y > 0$

Proof

$\forall n \in \mathbb{N}, \exists y \in I$ s.t. $1/n > y > 0$

Given, $n \in \mathbb{N}$, let $y = \frac{\pi}{4n}$

then, $0 < \frac{\pi}{4} < 1$ so, divide by n

$0 < \frac{\pi}{4n} < \frac{1}{n}$ & $4n \in \mathbb{Q} - \{0\}$

by the fact given that in q/s, $\frac{\pi}{4n} \in I$