Math 220
Section 108
Lecture 22

24th November 2022

Sources: https://personal.math.ubc.ca/~PLP/auxiliary.html
https://secure.math.ubc.ca/Ugrad/pastExams

Final Question 8, 2016 WT1

8. Let $f : A \to B$ be a function. Prove:
(b) If f is injective, then there is a function $g : B \to A$ such that $g \circ f(x) = x$, for all $x \in A$.

(Continued) 8. Let $f : A \to B$ be a function. Prove:
(b) If f is injective, then there is a function $g : B \to A$ such that $g \circ f(x) = x$, for all $x \in A$.

Proof by Contradiction

Or: “How to be so wrong you end up being right”

Proof by contradiction

1. (a) Prove that $\sqrt{7}$ is irrational.
(b) If you try to repeat the same proof to show that $\sqrt{49}$ is irrational, where does the proof fail?

(a) Let's assume $\sqrt{7} \in \mathbb{Q}$ by proof by contradiction.

So, $\sqrt{7} = \frac{a}{b}$, $\exists a, b \in \mathbb{Z}$, $b \neq 0$

where a & b have 1 as the greatest common factor.

Now, $7 = \frac{a^2}{b^2}$

So, $7b^2 = a^2$, $7 \mid a^2$ or $7 \mid a$ (Euclid's lemma)

So, $a = 7k$, $\forall k \in \mathbb{Z}$, So, $7b^2 = 49k^2$ & $b^2 = 7k^2$ & $7 \mid b^2$ or $7 \mid b$

Contradicting the fact that they don't have anything in common. [7 in common] Hence $\sqrt{7}$ is irrational by proof by contradiction.

(Continued) 1. (a) Prove that $\sqrt{7}$ is irrational.
(b) If you try to repeat the same proof to show that $\sqrt{49}$ is irrational, where does the proof fail?

(b) $\sqrt{49}$ can be written as 7 or −7 which is a rational number, which we can't prove to be irrational using proof by contradiction.

$49 = \frac{a^2}{b^2}$

$49b^2 = a^2$

$49 | a^2$ → But 49 is not prime so, it fails by Euclid's Lemma

(Continued 2/2) 1. (a) Prove that $\sqrt{7}$ is irrational.
(b) If you try to repeat the same proof to show that $\sqrt{49}$ is irrational, where does the proof fail?

$\sqrt{22} \Rightarrow$ irrational

$22b^2 = a^2 \quad 22b^2 = 4k^2$

Take $a = 2k$ i.e even

Proof by contradiction

Procedure

- Assume the negation of the statement you want to prove.
- Do some math ...
- Obtain a contradiction (e.g., "t must be even and odd").
- Conclude that the assumption is false.
  This implies that the statement you wanted to prove is true!

Proof by contradiction

2. Let $n \in \mathbb{N}$, $n \geq 2$, and $a, b, c \in \mathbb{Z}$. Prove that if $ab \equiv 1 \pmod{n}$, then for all $c \not\equiv 0 \pmod{n}$ we have $ac \not\equiv 0 \pmod{n}$.

$\forall n \in \mathbb{N}, n \geq 2$ and $a, b, c \in \mathbb{Z}$, $ab \equiv 1 \pmod{n} \Rightarrow$
$\forall c \not\equiv 0 \pmod{n}$ we have
$ac \not\equiv 0 \pmod{n}$

By proof by contradiction, negation is,

$ab \equiv 1 \pmod{n}$ and $c \equiv 0 \pmod{n}$ we have
$ac \equiv 0 \pmod{n}$

$ab = n \cdot k + 1$, some $k \in \mathbb{Z}$

$c = n \cdot j$, some $j \in \mathbb{Z}$

$abc = n^2 kj + nj$

$abc = nj[nk+1]$

So, $abc \mid nk+1$

So, $ac \mid nk+1$ & $ac \equiv 1 \pmod{n}$
which contradicts the assumption

hence original stmt is true.

(Continued)

(Continued) 2. Let $n \in \mathbb{N}$, $n \geq 2$, and $a, b, c \in \mathbb{Z}$. Prove that if $ab \equiv 1 \pmod{n}$, then for all $c \not\equiv 0 \pmod{n}$ we have $ac \not\equiv 0 \pmod{n}$.

By proof by contradiction we have, $ac \equiv 0 \pmod{n}$

Then $bac \equiv 0 \pmod{n}$

$\rightarrow 1 \cdot c \equiv 0 \pmod{n}$

contradiction since $c \not\equiv 0 \pmod{n}$, assumption is false & opp is true.

3. Let $x \in \mathbb{R}$ satisfy $x^7 + 5x^2 - 3 = 0$. Prove that x is irrational.

x is rational $= \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b) = 1$

$x^7 = \frac{a^7}{b^7}$

$5x^2 = \frac{5a^2}{b^2}$

$\frac{a^7}{b^7} + \frac{5a^2}{b^2} = 3$

$\frac{a^7 b^2 + 5a^2 b^7}{b^9} = 3$

$a^7 b^2 + 5a^2 b^7 = 3b^9$

$a^2 [a^5 + 5b^7] = 3b^7$

So, $3 \mid a^2 \Rightarrow 3 \mid a$

So, $3k = a$, some $k \in \mathbb{Z}$

So, now, $9k^2 [(3k)^5 + 5b^2] = 3b^2$

$3k^2 [3k^5 + 5b^2] = b^2$

So, $3 \mid b^2 \Rightarrow 3 \mid b \rightarrow$ This contradicts assumption that $\gcd(a,b) = 1$

Hence orig is true

(Continued)

(Continued) 3. Let $x \in \mathbb{R}$ satisfy $x^7 + 5x^2 - 3 = 0$. Prove that x is irrational.

Proof by contradiction (if time)

4. Let $(x_n)_{n\in\mathbb{N}}$ be a real sequence. Then, recall that we say (x_n) converges to L if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \epsilon.$$

Prove that if a sequence (y_n) converges, then the limit is unique.

(Continued)

(Continued) 4. Let $(x_n)_{n\in\mathbb{N}}$ be a real sequence. Then, recall that we say (x_n) converges to L if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \epsilon.$$

Prove that if a sequence (y_n) converges, then the limit is unique.