Math 220
Section 108
Lecture 20

17th November 2022

Source: `https://personal.math.ubc.ca/~PLP/auxiliary.html`

Old Final question

3. Suppose that $f : A \to B$ is a function and let C be a subset of A.

   (a) Prove that $f(A) - f(C) \subseteq f(A - C)$.

   (b) Find a counterexample for $f(A - C) \subseteq f(A) - f(C)$.

Hint: Think about for which type of functions part (b) fails.

(a) Let $y \in f(A) - f(C)$. So, $y \in f(A)$ and $y \notin f(C)$

$\exists x \in A$ s.t $f(x) = y$ & there is not $z \in C$ s.t $f(z) = y$. So, we must have $x \in A - C$

So, $f(x) \in f(A - C)$

$y \in f(A - C)$

(Continued)

(Continued) 3. Suppose that $f : A \to B$ is a function and let C be a subset of A.

1. Prove that $f(A) - f(C) \subseteq f(A - C)$.
2. Find a counterexample for $f(A - C) \subseteq f(A) - f(C)$.

(b) $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$

$C = \{3\}$ $\qquad f(A) = \{6, 7\}$

$A - C = \{1, 2\}$ $\qquad f(C) = \{7\}$

$f(A - C) = \{6, 7\}$

$f(A) - f(C) = \{6\}$

so, $f(A - C) \not\subseteq f(A) - f(C)$

Image and preimage

If $f(x)=x^2$ $a \mapsto f(a)$, $3 \mapsto 9$

Here $f(a)$ & 9 are the image of a & 3 respectively

So, $\{4,5\} \to \{16,25\}$
$\{-3,3\} \to \{9\}$

Definition (Definition 10.3.1 of PLP)

Image and preimage. Let $f : A \to B$ be a function, and let $C \subseteq A$ and let $D \subseteq B$.

- The set $f(C) = \{f(x) \mid x \in C\}$ is the **image** of C in B.
- The set $f^{-1}(D) = \{x \in A \mid f(x) \in D\}$ is the **preimage** of D in A or f-inverse of D.

Warning: f^{-1} does not necesarily represent the inverse function!

Example

For the function f represented by $y = x^2$, what is the preimage of $f^{-1}(\{1, 2, 3\})$? $\{-\sqrt{3}, -\sqrt{2}, -1, 1, \sqrt{2}, \sqrt{3}\}$

The preimage of $\{9\}$ is -3 & 3 i.e $\{-3,3\}$

So, $f^{-1}(\{9\}) = \{-3,3\}$, $f^{-1}(\{-1\}) = \phi$

Injective and Surjective

Definition (Definition 10.4.1 of PLP)

Let $a_1, a_2 \in A$ and let $f : A \to B$ be a function. We say that f is **injective** (or one-to-one) when

$$a_1 \neq a_2 \quad \Rightarrow \quad f(a_1) \neq f(a_2).$$

It is helpful to also write the contrapositive of this condition. We say that is **injective** (or one-to-one) when

$$f(a_1) = f(a_2) \quad \Rightarrow \quad a_1 = a_2.$$

Definition (Definition 10.4.5 of PLP)

Let $f : A \to B$ be a function. We say that f is **surjective** (or onto), when for every $b \in B$ there is some $a \in A$ such that $f(a) = b$.

This simply means that every element in B is mapped to by some element of A.

Functions

4.(a) Find a function $f : \mathbb{Z} \to \mathbb{Z}$ which is injective but not surjective.
(b) Find a function $g : \mathbb{Z} \to \mathbb{Z}$ which is surjective but not injective.
(c) Discuss what would happen if we replaced $\mathbb{Z}$ with a finite set.

(a) $f(x) = x^3$ or $\lfloor \ln(x) \rfloor$ or $2x$

(Continued)

4.(a) Find a function $f : \mathbb{Z} \to \mathbb{Z}$ which is injective but not surjective.
(b) Find a function $g : \mathbb{Z} \to \mathbb{Z}$ which is surjective but not injective.
(c) Discuss what would happen if we replaced $\mathbb{Z}$ with a finite set.

Injective

5. Suppose that $f : A \to B$ and C_1, C_2 are subsets of A. Show that if f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.

(Continued)

(Continued) 5. Suppose that $f : A \to B$ and C_1, C_2 are subsets of A. Show that if f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.

Bijections

Definition (Definition 10.4.10 of PLP)

Let $f : A \to B$ be a function. If f is both injective and surjective then we say that f is **bijective**, or a one-to-one correspondence.

Example

The function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ is bijective.
The function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^4$ is <u>not</u> bijective, since it is not injective (nor surjective).

6. Prove that the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$ given by $f(x) = \frac{2x}{x-1}$ is bijective.

(Continued)

(Continued) 6. Prove that $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$, $f(x) = \frac{2x}{x-1}$ is bijective.

Bijections (If time)

7. Let A, B be nonempty sets. Prove that if there is a bijection $f : A \to B$, then there is a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.
Hint: How can you send a subset of A to a subset of B, given that you know how to send each element of A to an element of B?

(Continued)

(Continued) 7. Let A, B be nonempty sets. Prove that if there is a bijection $f : A \to B$, then there is a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.