MATH 220 Final Review

UBC Undergraduate Math Society

Procedure

- introduction
- set theory
- relations
- functions
- cardinalities
- the pigeonhole principle
- ~~pizzas~~

(1 hr, 10 min break, 1hr, 10 min break, 40min)

UBC Math Undergraduate Society

Location: MATH ANNEX 1119

What we do: board games (we have oh so many board games), putnam practice, math circle, exam packs, louging around, and sometimes, math.

Instagram: ums.ubc

email:ums.ubc@gmail.com

https://discord.gg/jhc6sAff

Your instructor
uki (she/they)

- 4th year biomedical engineering (bioinformatics), minor in honors mathematics

www.ukistatemachine.com

Ask questions

- Unmute and ask, or ask during break
- If you are uncomfortable speaking, send the message privately to Yuqi and she will read it for you

Raffle Prizes

five $20 ubereats giftcard

Attendance - 1 entry

Answer/Ask meaningful questions - max 3 entries

Feedback form - 1 entry

Set theory

$$(A = B) \equiv ((A \subseteq B) \wedge (B \subseteq A))$$

$$(A \subseteq B) \equiv (\forall a \in A, a \in B) \equiv (a \in A \implies a \in B).$$

Set operations

$$(x \in A \cup B) \iff (x \in A) \vee (x \in B)$$

$$(x \in A \cap B) \iff (x \in A) \wedge (x \in B)$$

$$A - B = \{x \in A \mid x \notin B\} \quad = \quad A \wedge \bar{B}$$

$$\bar{A} = \{x \in U | x \notin A\}$$

Set operations

Theorem 8.4.2. DeMorgan's laws. *Let A, B be sets contained in a universal set U. Then*

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$
$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

Theorem 8.4.3. Distributive laws. *Let A, B, C be sets then*

$$A \cup (B \cap C) = (A \cup C) \cap (A \cup B)$$
$$A \cap (B \cup C) = (A \cap C) \cup (A \cap B)$$

and

$$A \times (B \cap C) = (A \times C) \cap (A \times B)$$
$$A \times (B \cup C) = (A \times C) \cup (A \times B)$$

$a \times (b + c)$

$a \times b + a \times c$

*

Let A, B and C be nonempty sets.

(a) (10 marks) Prove: $A \times (B - C) = (A \times B) - (A \times C)$.

(b) (5 marks) Prove: $A = (A - B) \cup (A \cap B)$

(c) (10 marks) Prove: if $|A - B| = |B - A|$, then $|A| = |B|$.

a) $x \in A \times (B - C)$ $\quad x = (m, n)$

$(m, n) \in A \times (B - C) \Rightarrow m \in A, n \in (B - C)$

$\underbrace{n \in B, n \notin C}$

$x \in A \times B - \underbrace{A \times C}$

$x = (m, n)$

$\rightarrow \underline{m \in A}, n \in B$

$\rightarrow \neg(m \in A, n \in C)$

$\underbrace{m \in A, n \in B, n \notin C}$

b) $x \in A$,

Case 1: $x \in B \Rightarrow x \in A \cap B$

case 2: $x \notin B \Rightarrow x \in A - B$

**

c) bijection:

$|A-B| = |B-A| \Rightarrow \exists g : A-B \to B-A$

4. (a) Let A, B be sets. Prove that

$$\bigl(A - (A \cap B) = B - (A \cap B)\bigr) \implies A = B.$$

$id : A \cap B \to A \cap B$

(b) Let A, B be sets. Prove or disprove that

$$\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B).$$

$A = (A - B) \cup (A \cap B)$

(c) Let C, D be sets. Prove or disprove that

$$\mathcal{P}(C) - \mathcal{P}(D) \subseteq \mathcal{P}(C - D).$$

g

$B = (B - A) \cup (B \cap A)$

id

Power set

Definition 8.1.5. Let A be a set. The ***power set*** of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Note that the elements of $\mathcal{P}(A)$ are themselves sets, and

$$X \in \mathcal{P}(A) \iff X \subseteq A.$$

Relations

Definition 9.1.1. Let A be a set. Then a relation, R, on A is a subset $R \subseteq A \times A$. If the ordered pair $(x, y) \in R$, we denote this as $x \; R \; y$, while if $(x, y) \notin R$ we write $x \not\!R \; y$.

ex: $\mathbb{Z}$: "$=$"

"$>$" "$<$"

"$\geq$"

$1 = 1$ ✓ $\Rightarrow (1,1) \in$ "$=$"

$1 = 2$ ✗ $(1,2) \notin$ "$=$"

Equivalence relations

Definition 9.2.1. Let R be a relation on a set A.

- We say that the relation R is ***reflexive*** when $a\ R\ a$ for every $a \in A$.
- The relation R is ***symmetric*** when for any $a, b \in A$, $a\ R\ b$ implies $b\ R\ a$.
- The relation R is ***transitive*** when for any $a, b, c \in A$, $a\ R\ b$ and $b\ R\ c$ implies $a\ R\ c$.

Notice that in the definition of transitive we do not require that a, b, c are different.

$\sim \quad a, b, c$

$\forall a \in A \Rightarrow (a,a) \in R \qquad$ Neg: $\exists a \in A, (a,a) \notin R$

$(a,b) \in R \Rightarrow (b,a) \in R \qquad$ Neg: $\exists (a,b) \in R . (b,a) \notin R$

$(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$; Neg $\exists (a,b), (b,c) \in R$

$1 \ (a,c) \notin R$

Equivalence class

Definition 9.3.3. Given an equivalence relation R defined on a set A, we define the equivalence class of $x \in A$ (with respect to R) to be the set of elements related to x:

$$[x] = \{y \in A \ : \ y \, R \, x\}$$

i.e. $(y,x) \in R \Rightarrow y \in [x]$

This is sometimes also written as "E_x".

Lemma 9.3.4. *Let R be an equivalence relation on a set A. Then for any $x \in A$, $x \in [x]$.*

$(x,x) \in R$

Theorem 9.3.5. *Let R be an equivalence relation on A and let $x, y \in A$. Then*

$$x \, R \, y \iff [x] = [y]$$

*

Define a relation on $\mathbb{R}$ as xRy if $|x-y|<1$. Is R reflexive? Symmetric? Transitive? Justify your answers.

Reflexive: $(x,x) \in R$? $|x-x| = 0 < 1$ ✓ yes

Symmetric: $(x,y) \in R \Rightarrow (y,x) \in R$

$\hookrightarrow \underbrace{|x-y|<1}_{} \Rightarrow |y-x| < 1$ ✓ yes

$|y-x| = |x-y| < 1$

$(0, \frac{1}{2}) \in R$

$(\frac{1}{2}, 1) \in R$

$(0, 1) \notin R$

Transitive? NO

**

Let R be a symmetric and transitive relation on a set A. (These assumptions apply to both parts (a) and (b) of this problem.)

(a) Show that R is not necessarily reflexive.

(b) Suppose that for every $a \in A$, there exists $b \in A$ such that aRb. Prove that R if reflexive.

a) $\{1,2,3\}$ $\quad R = \{(1,2),(1,1),(2,2),(2,1)\}$

b) $\forall a \in A \ \exists \ aRb \in R \Rightarrow bRa \in R$

symmetry

$aRb,\ bRa \Rightarrow aRa \in R$

$a,\ b,\ a$

**

Let R be a relation on a nonempty set A. Then $\overline{R} = (A \times A) - R$ is also a relation on A. Prove or disprove each of the following statements:

(a) If R is reflexive, then $\overline{R}$ is reflexive.

(b) If R is symmetric, then $\overline{R}$ is symmetric.

(c) If R is transitive, then $\overline{R}$ is transitive.

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Define a relation on $\mathbb{Z}$ as aRb if $3 \mid (2a - 5b)$. Is R an equivalence relation? Justify your answer.

$3 \mid 2a - 5b \Rightarrow 2a \equiv 5b \pmod 3$

$2a \equiv 2b \pmod 3$

$2(a-b) \equiv 0 \pmod 3$

$a - b \equiv 0 \pmod 3$

$a \equiv b \pmod 3$

Reflex $a \equiv a \pmod 3$

Sym $\begin{cases} a \equiv b \pmod 3 \\ b \equiv a \pmod 3 \end{cases}$

Trans $\begin{cases} a \equiv b \equiv c \pmod 3 \\ a \equiv c \pmod 3 \end{cases}$

Partition

Definition 9.3.11. A partition of a set A is a collection $\mathcal{P}$ of non-empty subsets of A, so that

- if $x \in A$ then there exists $X \in \mathcal{P}$ so that $x \in X$, and
- if $X, Y \in \mathcal{P}$, then either $X \cap Y = \varnothing$ or $X = Y$

The elements of $\mathcal{P}$ are then called blocks, parts or pieces of the partition.

An equivalent definition is that a partition of a set A is a collection $\mathcal{P}$ of non-empty subsets of A, so that

- $\bigcup_{X \in \mathcal{P}} X = A$, and
- if $X, Y \in \mathcal{P}$, then either $X \cap Y = \varnothing$ or $X = Y$

Where the union $\bigcup_{X \in \mathcal{P}} X$ is the union of all the sets in the partition $\mathcal{P}$.

**

p_n = n^{th} prime number

Describe a partition of $\mathbb{N}$ that divides $\mathbb{N}$ into $\aleph_0$ countably infinite subsets.

$N_1 = \{x \in \mathbb{N} : p_1 \mid x\} \cup \{1\}$

$N_2 = \{x \in \mathbb{N} - N_1 : p_2 \mid x\}$

$\vdots$

$N_k = \{x \in \mathbb{N} - N_1 - N_2 - \cdots - N_{k-1} : p_k \mid x\}$

$N_1 = \{x \in \mathbb{N} : 2 \nmid x\}$

$N_2 = \{x \in \mathbb{N} - N_1 : 3 \nmid x\}$

$\vdots$

$N_3 = \{x \in \mathbb{N} - N_1 - N_2 : 4 \nmid x\}$

Functions

Definition 10.2.1. Let A, B be non-empty sets.

- A ***function*** from A to B, written $f : A \to B$ is a non-empty subset of $A \times B$ with two further properties
  - for every $a \in A$ there is some $b \in B$ so that $(a, b) \in f$.
  - if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.
- In this context we call the set A the ***domain*** of f, and the set B is called the ***co-domain***.
- If $(a, b) \in f$ then we write $f(a) = b$, and we call b the image of a. We also sometimes say that f maps a to b. With this notation the above two conditions are written as
  - for every $a \in A$ there is some $b \in B$ so that $f(a) = b$.
  - if $f(a) = b$ and $f(a) = c$ then $b = c$.
- We can further refine the co-domain to be exactly the set of elements of B that are mapped to by something in A; this set is called the ***range***:

$$\operatorname{rng} f = \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\}$$

codomain(abs) = $(-\infty, \infty)$

Range(abs) = $[0, \infty)$

Image and preimage

Definition 10.3.1. Image and preimage. Let $f : A \to B$ be a function, and let $C \subseteq A$ and let $D \subseteq B$.

- The set $f(C) = \{f(x) : x \in C\}$ is the ***image of C in B***.
- The set $f^{-1}(D) = \{x \in A : f(x) \in D\}$ is the ***preimage of D in A*** or ***f-inverse of D***.

*

Consider the function $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ given as

$$f = \{(1, 3), (2, 8), (3, 3), (4, 1), (5, 2), (6, 4), (7, 6)\}.$$

Find: $f(\{1, 2, 3\})$, $f(\{4, 5, 6, 7\})$, $f(\emptyset)$, $f^{-1}(\{0, 5, 9\})$ and $f^{-1}(\{0, 3, 5, 9\})$.

$\{3, 8\}$ $\quad$ $\{1, 2, 4, 6\}$ $\quad$ $\emptyset$ $\quad$ $\emptyset$ $\quad$ $\{1, 3\}$

Injection and surjection

Definition 10.4.1. Let $a_1, a_2 \in A$ and let $f : A \to B$ be a function. We say that f is injective or one-to-one when

$$\text{if } a_1 \neq a_2 \text{ then } f(a_1) \neq f(a_2).$$

It is helpful to also write the contrapositive of this condition. We say that f is injective or one-to-one when

$$\text{if } f(a_1) = f(a_2) \text{ then } a_1 = a_2.$$

Definition 10.4.4. Let $f : A \to B$ be a function. We say that f is surjective, or onto, when for every $b \in B$ there is some $a \in A$ such that $f(a) = b$. $\Leftrightarrow f(A) = B$

Definition 10.4.8. Let $f : A \to B$ be a function. If f is injective and surjective then we say that f is ***bijective***, or a ***one-to-one correspondence***.

**

Let A and B be nonempty sets. Prove that if f is an injection, then $f(A - B) = f(A) - f(B)$.

**

$\to y = x^2 + 2x = \underbrace{(x+1)^2}_{\geq 0} - 1 = y$

Consider the function $f : \mathbb{R} \to [-1, +\infty)$ defined by

$$f(x) = x^2 + 2x.$$

(a) (6 marks) Show $Range(f) = [-1, +\infty)$.

(b) (3 marks) What is $f^{-1}(\{0\})$? $f^{-1}(\{-4\})$? $f^{-1}(\{-1\})$?

$f^{-1}(0) = \{0, -2\}$, $f^{-1}(\{-4\}) = \emptyset$

$f^{-1}(-1) = \{-1\}$

(c) (2 marks) Is the function f injective, surjective, bijective?

(d) (4 marks) Now we consider the function $g : \mathbb{R} \to [-1, +\infty)$ defined by $g(x) = f(e^x)$. Is g surjective?

$g(x) = f(e^x) = \underbrace{e^{2x}}_{>0} + \underbrace{e^x \cdot 2}_{>0} > 0 \to g^{-1}(-1) = \emptyset \Rightarrow$ not surjective

Composition

Definition 10.5.1. Let $f : A \to B$ and $g : B \to C$ be two functions. The composition of f and g is denoted $g \circ f$ which we read "g of f". It defines a new function

$$g \circ f : A \to C \qquad (g \circ f)(a) = g(f(a)) \qquad \forall a \in A$$

Personal Note "$\circ$" = "after"

**

(a) Let A, B, C be non-empty sets so that $|A| \leq |B|$ and $B \subseteq C$. Prove that $|A| \leq |C|$.

(b) Let $g : A \to A$ and $h : A \to A$ so that $h \circ g$ is bijective.
Prove that g is injective and that h is surjective.

a) $\exists f : A \to B$ injective $\quad g : B \to C$

$g : b \mapsto b$

$h = g \circ f : A \to C$

$c_1, c_2 \in C$ s.t $h(x_1) = c_1$, $h(x_2) = c_2$

$\to g(f(x_1)) = g(f(x_2)) \to f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ ✓.

↳ $c_1 = c_2 \Rightarrow x_1 = x_2$ ~ injectivity

** b) h surjective so Range = Codomain possible

g injective to prevent h from mapping 2 points to a single output

Let A be a nonempty set and $f : A \to A$ be a function.

(a) Prove that f is bijective if and only if $f \circ f$ is bijective.
(Hint: You may need to show that if $f \circ g$ is injective then g is injective, and then show that if $f \circ g$ is surjective then f is surjective.)

(b) Use part a) to show that $f(x) = \ln(\frac{e^x+1}{e^x-1})$ is a bijective function from $(0, \infty)$ to $(0, \infty)$.

Cardinalities

- If $f : A \to B$ is an injection then $|A| \leq |B|$,
- If $g : A \to B$ is a surjection then $|A| \geq |B|$, and
- If $h : A \to B$ is a bijection then $|A| = |B|$

$$(|A| \leq |B|) \wedge (|B| \leq |A|) \implies (|A| = |B|)$$

Countability

Definition 12.2.1. Let A be a set.

- The set A is called ***denumerable*** if there is a bijection $f : \mathbb{N} \to A$.
- The cardinal number of a denumerable set is denoted $\aleph_0$ (read "aleph naught" or "aleph null").
- If A is finite or denumerable it is called ***countable***.
- If A is not countable it is called ***uncountable***.

*

There is an injection from A to B if and only if there is a surjection from B to A.

**

q: Prove or disprove: If there is an injection $f : A \to B$, and a surjection $g : A \to B$, then there is a bijection between A and B.

$f \Rightarrow |A| \leq |B|$

$\Rightarrow |A| = |B| \Rightarrow \exists$ bijection

$g \Rightarrow |A| \geq |B|$

***

Given two sets A and B, such that $|A - B| = |B - A|$, prove that $|A| = |B|$.

**

Let A and B be sets. Let P be a partition of A, and let Q be a partition of B. Suppose that $h : P \to Q$ is a bijection. Suppose also that for each $X \in P$, the sets X and $h(X)$ have the same cardinality. Prove that A and B have the same cardinality.

The pigeonhole principle

If $|A| > |B|$, then f is not injective.
If $|A| < |B|$, then f is not surjective.

$f : A \to B$

***

Q8: Let n be an odd natural number and $a_1, a_2, \ldots, a_n \in \{1, 2, ..., n\}$ all distinct (that is $a_i \neq a_j$ for $i \neq j$). Prove, using the pigeonhole principle, that $x = (1 - a_1) \cdot (2 - a_2) \cdot \ldots (n - a_n)$ is even.

k and a_k have the same parity

n odd $\frac{n+1}{2}$ even $\frac{n-1}{2}$

$\{1, \ldots, n\}$ $\frac{n+1}{2}$

a_n $\{1, \ldots, n\}$. $\frac{n-1}{2}$

***

$\underline{1, \cdots, n-1}$

$\underline{s_1, s_2, \cdots, s_n}$

$\exists, s_\alpha, s_\beta$ s.t. $s_\alpha, s_\beta \equiv k \mod n$

$s_\beta - s_\alpha \equiv 0 \mod n$

$(a_{\alpha+1}, a_{\alpha+2}, \cdots a_\beta)$

$A = \{a_1, a_2, a_3, ..., a_n\}$ be a nonempty set of n distinct natural numbers. Prove that there exists a nonempty subset of A for which the sum of its elements is divisible by n.

Hint: Consider the sums $s_k = a_1 + a_2 + ... + a_k$.

case 1: if $S_h \equiv n \pmod n$ for any h, we're done

case 2: if $S_h \not\equiv n \pmod n$ $\forall h \in [1, 2, \dots, n]$

$S_j \equiv S_h \pmod n$ for some $j, h \in [1, 2, \dots, n]$

If $j > h$: $S_j - S_h \equiv 0 \pmod n \longrightarrow S_j - S_h = a_{k+1} + a_{k+2} + \cdots + a_j$

$h > j$: $S_h - S_j \equiv 0 \pmod n$

After MATH 220

If you like number theory ----> MATH 437 or MATH 312 (easy version of 437)

If you like logic ----> Read work of Bertrand Russell, Kurt Godel et al

If you want to do more rigorous math ----> MATH 320 + MATH 322

Fun and chanllenging problems ----> putnam practice

Feedback form (2 min)