Solutions to Homework 3:

1. Original Statement : $\forall \epsilon > 0, \exists M > 0$ such that, $|1 - \frac{x^2}{x^2+1}| < \epsilon$ for $x \geq M$

   Negated Statement : $\exists \epsilon > 0$ such that, $\forall M > 0, |1 - \frac{x^2}{x^2+1}| \geq \epsilon$ for $x \geq M$

   There exists $\epsilon > 0$ such that for any positive number M we have that $|1 - \frac{x^2}{x^2+1}| \geq \epsilon$

2. Original Statement : $\forall x \in \mathbb{Z}, \exists y \in \mathbb{R}$, $((x \geq y) \implies (\frac{x}{y} = 1)$

   Negated Statement : $\exists x \in \mathbb{Z}$ such that $\forall y \in \mathbb{R}$, $(x \geq y) \wedge (\frac{x}{y} \neq 1)$

   The Original Statement is **True**

3. $A = \{n \in \mathbb{N} : 3 \mid n \vee 4 \mid n\} \subset \mathbb{N}$
   So $\exists p, q \in \mathbb{Z}$ such that $n = 3 * p$ or $n = 4 * q$

   (a) $\exists x \in \mathbb{A}$ s.t., $\exists y \in \mathbb{A}$ s.t, $x + y \in \mathbb{A}$
   This statement is **True**
   Consider $x = 12$ and $y = 24$
   then in that case we have $x + y = 36$
   So, $x + y \in \mathbb{A}$

   (b) $\forall x \in \mathbb{A}, \forall y \in \mathbb{A}, x + y \in \mathbb{A}$
   This statement is **False**
   This is false because when x = 3 and y = 4 which belongs to $\mathbb{A}$, their sum x+y i.e $7 \notin \mathbb{A}$

   (c) $\exists x \in \mathbb{A}$ such that,$\forall y \in \mathbb{A}, x + y \in \mathbb{A}$
   This statement is **True**
   This statement is true because :

   Case 1 : y = 3p where $\exists p \in \mathbb{N}$
   Taking x = 3 we get,
   x + y = 3*p + 3 or simply x + y = 3(p + 1)
   which is a multiple of 3 so it's a part of $\mathbb{A}$

   Case 2 : y = 4q where $\exists q \in \mathbb{N}$
   Taking x = 4 we get,
   x + y = 4*q + 4 or simply x + y = 4(q + 1)
   which is a multiple of 4 so it's a part of $\mathbb{A}$

4. (a) Original Statement: $\forall n \in \mathbb{Z}, \exists y \in \mathbb{R} - \{0\}$ s.t., $y^n \leq y$
   Negation : $\exists n \in \mathbb{Z}$ s.t., $\forall y \in \mathbb{R} - \{0\}$, $y^n > y$

The original statement is **true** because let's take y=1,
$y^n = (1)^n$, and $1^n = 1, \forall n \in \mathbb{Z}$

(b) Original Statement: $\exists y \in \mathbb{R} - \{0\}$ s.t.,$\forall n \in \mathbb{Z}, y^n \leq y$
Negation : $\forall y \in \mathbb{R} - \{0\}$,$\exists n \in \mathbb{Z}$ s.t, $y^n \leq y$
This statement is **true** because let's take y = 1,
For any any arbitrary n , $y^n = (1)^n$, and $1^n = 1, \forall n \in \mathbb{Z}$

(c) Original Statement:$\forall x \in \mathbb{R}$, where $x \neq 0$, we have $x \leq 1$ or $\frac{1}{x} \leq 1$
Negation: $\exists x \in \mathbb{R}$ s.t., where $x \neq 0$, we have $x > 1$ and $\frac{1}{x} > 1$
Here it's given that $x \neq 0$ so,

Case 1: $x < 0$
This automatically means that $x \leq 1$

Case 1: $x > 0$
Now, we know that $x \leq 1$ so,
x is between (0,1],
$0 < x \leq 1$
Hence $\frac{1}{x} \leq 1$

5. (a)
- Reformulated Statement: $\exists x \in \mathbb{K}$ s.t., $\exists y \in \mathbb{L}$ s.t., x unlocks y
- Negation: $\forall x \in \mathbb{K}, \forall y \in \mathbb{L}$, x does not unlock y
- Reformulated negation: "No keys unlock all locks"

(b)
- Reformulated Statement: $\exists x \in \mathbb{K}$ s.t., $\forall y \in \mathbb{L}$, x unlocks y
- Negation: $\forall x \in \mathbb{K}, \exists y \in \mathbb{L}$ s.t, x does not unlock y
- Reformulated negation: "No key unlocks some lock"

(c)
- Reformulated Statement: $\exists x \in \mathbb{L}$ s.t., $\forall y \in \mathbb{K}$, x is not unlocked by y
- Negation: $\forall x \in \mathbb{L}, \exists y \in \mathbb{K}$ s.t., x unlocked y
- Reformulated negation: "All locks are unlocked by some keys"

6. *Proof.* **Case 1:** a is a multiple of 3 i.e $a = 3k$ where $\exists k \in \mathbb{Z}$
So, $a^2 + b^2 = (3k)^2 + b^2$
Now let b = 1
Hence $a^2 + b^2 = 3(3k^2) + 1$
So, $a^2 + b^2 \equiv 1 \bmod 3$

Case 2: a leaves a remainder of 1 with 3 i.e $a = 3k + 1$ where $\exists k \in \mathbb{Z}$
So, $a^2 + b^2 = (3k + 1)^2 + b^2$
Now let b = 3
Hence $a^2 + b^2 = (9k^2 + 6k + 1) + 9 = 3(3k^2 + 3 + 2k) + 1$
So, $a^2 + b^2 \equiv 1 \bmod 3$

Case 3: a leaves a remainder of 2 with 3 i.e $a = 3k + 2$ where $\exists k \in \mathbb{Z}$
So, $a^2 + b^2 = (3k + 2)^2 + b^2$

Now let b = 3
Hence $a^2 + b^2 = (9k^2 + 12k + 4) + 9 = 3(3k^2 + 4 + 4k) + 1$
So, $a^2 + b^2 \equiv 1 \bmod 3$ $\square$

7. *Proof.* Since there is an "or" in the question we can prove that the statement is true if we are able to prove just one part true. So,

   Case 1: If x = y = z = 3 or 6 then the statement is true

   Case 2:

   (a) If any two of them are 3 and one of them is 6 :
   So, $\frac{x+y+z}{3} = \frac{12}{3} = 4$ and,
   $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = \frac{3}{3} + \frac{3}{6} + \frac{6}{3} = 3.5$
   So, $\frac{x+y+z}{3} > \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

   (b) If any two of them are 6 and one of them is 3 :
   So, $\frac{x+y+z}{3} = \frac{15}{3} = 5$ and,
   $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = \frac{6}{6} + \frac{3}{6} + \frac{6}{3} = 3.5$
   So, $\frac{x+y+z}{3} > \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

   Hence if any of the case is true then the statement is proved true.

   $\square$

8. Fact : $F : \mathbb{R} \to \mathbb{R}$ is increasing if $\forall a, b \in \mathbb{R}, (a < b) \implies f(a) < f(b)$

   (a) *Proof.* For $f(x) = x^3 + 3x + 4$
   Let a and b be two arbitrary real numbers such that $b > a$.

   $f(b) - f(a) = (b^3 + 3b + 4) - (a^3 + 3a + 4)$
   $= (b^3 - a^3) + 3(b - a)$
   $= (b - a)(a^2 + ab + b^2 + 3) > 0,$

   Now we know that $b - a > 0$
   Similarly, $a^2 + ab + b^2 + 3 = (a + \frac{b}{4})^2 + \frac{3}{4}b^2 - 3 > 0$ because squares are always positive.
   So, $f(b) - f(a) > 0$ as a result $f(b) > f(a)$ and therefore from the fact we have that $f(x) = x^3 + 3x + 4$ is increasing

   $\square$

   (b) *Proof.* For $f(x) = sin(x)$
   Let a and b be two arbitrary real numbers such that $b > a$.

Case 1: a and b have the same sign
$f(b) - f(a) = sin(b) - sin(a)$
However, when b and a are between $pi/2 < a, b < pi$ we have that $sin(b) < sin(a)$
Which means $f(b) - f(a) < 0$ or simply $f(b) < f(a)$ and thus $f(x) = sin(x)$ is not increasing.

Case 2: a and b have different sign
$f(b) - f(a) = sin(b) - sin(-a) = sin(b) + sin(a)$
However, when b and a are between $pi/2 < a, b < pi$ we have that $sin(b) < sin(a)$
Which means $f(b) - f(a) < 0$ or simply $f(b) < f(a)$ and thus $f(x) = sin(x)$ is not increasing.

$\square$