Math 220
Section 108
Lecture 15

27th October 2022

Source: `https://personal.math.ubc.ca/~PLP/auxiliary.html`

5.(a) Show that for every $k \in \mathbb{Z}$, $\exists x, y \in \mathbb{Z}$, such that $k = 4x + 5y$.

(b) What does it say about the set $A = \{4x + 5y \mid x, y \in \mathbb{Z}\}$: is it a subset of, superset of, or equal to $\mathbb{Z}$?

(a) Given $k \in \mathbb{Z}$, let $x = -k$, $y = k$.
Then $4x + 5y = 4(-k) + 5(k) = k$,
as required.

(b) Since $x, y \in \mathbb{Z}$, we have $4x+5y \in \mathbb{Z}$, so
$A \subseteq \mathbb{Z}$.
By part (a), every integer $k \in \mathbb{Z}$ is in A,
so $A \supseteq \mathbb{Z}$.

So $A = \mathbb{Z}$.

6. Let A, B and C be sets. For each of the following statements, either prove it is true or give a counterexample.
(a) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$,
(b) $\mathcal{P}(A \cup B) \supseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.
Hint: First try this with small sets.

(a) $A = \{1\}$, $B = \{2\}$, $A \cup B = \{1,2\}$.
Note that $\{1,2\} \in \mathcal{P}(A \cup B)$.
But $\mathcal{P}(A) = \{\phi, \{1\}\}$, $\mathcal{P}(B) = \{\phi, \{2\}\}$,
so $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\phi, \{1\}, \{2\}\}$.
So $\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$, so (a) is false.

(b) Say $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$. So $S \in \mathcal{P}(A)$ or $S \in \mathcal{P}(B)$. "Without loss of generality", let $S \in \mathcal{P}(A)$.

(Continued)

(Continued) 6. Prove or give a counterexample:
(a) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$,
(b) $\mathcal{P}(A \cup B) \supseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

So $S \subseteq A$, so $S \subseteq A \cup B$.

Therefore $S \in \mathcal{P}(A \cup B)$.

Sets - An old final question

7. Let T be the set of all natural numbers that can be written as some nonnegative number of 3's plus some nonnegative number of 5's. For example, $9 = 3 + 3 + 3$ and $10 = 5 + 5$ and $17 = 3 + 3 + 3 + 3 + 5$ are all in T, but 4 is not. Find $\mathbb{N} - T$ (with justification).

Hint: Try to figure out which numbers are in the set, and then try to generalize your answer.

Claim: $\mathbb{N} - T = \{1, 2, 4, 7\}$.

We note that $3, 5 \in T$, $6 = (3+3) \in T$, $8 = (5+3) \in T$, $9 = (3+3+3) \in T$, $10 = (5+5) \in T$.

If $n \in T$, then $n+3 \in T$.

So since $8, 9, 10 \in T$, we have $11, 12, 13 \in T$.

So in general $m \in T$ $\forall m \in \mathbb{N}$ with $m \geq 8$.

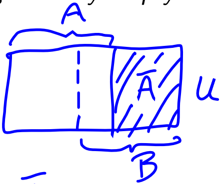
(Continued) 7. Let $T = \{n \in \mathbb{N} \mid n = 3a + 5b, a, b \in \mathbb{Z}, a, b \geq 0\}$. Find $\mathbb{N} - T$.

Lastly, we note that $1, 2 \notin T$, since they are too small. Also, $4, 7 \notin T$, since there are no small combinations of 3's & 5's to obtain these numbers.

So $\mathbb{N} - T = \{1, 2, 4, 7\}$.

9. We consider subsets A, B and C of a universal set U. Let $\bar{A}$ denote the complement of A.
(a) Prove that $\bar{A} \subseteq B$ if and only if $A \cup B = U$.
(b) Prove that $\bar{A} \subseteq B$ implies $(C \backslash B) \cup A = A$
Hint: A diagram may help you visualize.

(a)



Assume $\bar{A} \subseteq B$.
We know $A \cup B \subseteq U$. We want to show $A \cup B \supseteq U$.

Given $w \in U$: • If $w \in A$, then $w \in A \cup B$, so done.
• If $w \notin A$, then $w \in \bar{A} \subseteq B$, so $w \in A \cup B$. So $A \cup B = U$.

The complement of A is $\bar{A} = U - A$.

Notation: Can also write A^c for $\bar{A}$.