Math 220
Section 108
Lecture 11

13th October 2022

Midterm Info

- **Date:** Thursday 20th October (for Tu/Th Lectures).
- **Length:** 45 minutes.
- **Location:** The midterm will be held in our regular lecture room and at the usual class time.
- **Syllabus:** Everything up to and including induction.
- **Resources:** Good study resources include your homework, the lecture videos, your notes from class, class slides, the PLP textbook itself, and the 'auxiliary material' worksheets for PLP.
- **What's allowed:** This is a closed-book exam. No notes, formula sheets, calculators, or phones are permitted. You may not ask for or receive help from other people.
- **What to bring:** You need to bring things to write with (pencil and pen are both OK) and your ID. UBC ID is preferred, but any government issued photo-ID is acceptable.

2. The Fibonacci numbers are defined by the recurrence

$$F_1 = F_2 = 1 \quad \text{and } F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 2.$$

Show that for every $n \in \mathbb{N}$, F_{4n} is a multiple of 3.

Base Case

$n=1$

Now, $F_3 = F(2) + F(1)$
$= 2$

so, $F_{4\cdot 1} = F(3) + F(2)$
$= 2+1$
$= 3$ i.e $3\cdot(1)$ ✓

Assume

$F_{4\cdot K}$ is true & multiple of 3 i.e

$F_{4\cdot K} = F_{4K-1} + F_{4K-2} = 3\cdot n \quad \forall n \in \mathbb{Z}$

Prove K+1 → Induction step

$F_{4(K+1)} = F_{4K+4-1} + F_{4K+4-2} = F_{4K+3} + F_{4K+2}$

$F_{4K+2} + 2F_{4K+1} + \underline{F_{4K}}$

$= 3F_{4K+1} + 2F_{4\cdot K}$

(Continued)

$= 3F_{4k+1} + 2F_{4\cdot k}$

$= 3F_{4k+1} + 2\cdot(3n)$

$= 3(F_{4k+1} + 2n)$

$= 3m$ $\quad \exists m \in \mathbb{Z}$ s.t $m = F_{4k+1} + 2n$

Hence $F_{4(k+1)}$ is a multiple of 3 & hence it's true for $n = k+1$

So, F_{4n} is a multiple of 3 by induction $\forall n \in \mathbb{N}$

3. Prove that $7^n - 2^n$ is divisible by 5 for all $n \in \mathbb{N}$.

Base Case

$n=1$

$7^1 - 2^1 = 5$ is divisible by 5 hence true for $n=1$

Assume it's true for $n=k$

$7^k - 2^k = 5 \cdot m \quad \forall m \in \mathbb{Z}$

For $n=k+1$

$$
\begin{aligned}
7^{k+1} - 2^{k+1} &= 7^k \cdot 7 - 2^k \cdot 2 \\
&= 7(5m + 2^k) - 2^k \cdot 2 \\
&= 7 \cdot 5m + 2^k(7-2) \\
&= 7 \cdot 5m + 2^k \cdot 5 \\
&= 5(7m + 2^k)
\end{aligned}
$$

So, $\Rightarrow 5(7m+2^k)$

$\Rightarrow 5p$ $\quad \exists p \in \mathbb{Z}$, s.t $p = 7m+2^k$

So, $7^n - 2^n$ is divisible by 5 for $n = k+1$

So, $7^n - 2^n$ is divisible by 5 $\forall n \in \mathbb{N}$ by induction

4. Let $n \in \mathbb{N}$. Prove that $\forall n \geq 7, n! > 3^n$.

Base Case

$n=7$

~~that~~

$7! = 5040$

$3^7 = 2187$

Hence $7! > 3^7$ so true for $n=7$

Assume $n = K$ is true

$K! > 3^K$, W.K.t $K \geq 7$

so, $K+1 \geq 8$ i.e > 3 — ①

Induction step

$K! > 3^K$ — ②

So, ① × ② $= (K+1)K! > 3^K \cdot 3$

$= (K+1)! > 3^{K+1}$

Hence it's true for $n=k+1$

& $\forall n \geq 7$, $n! > 3^n$ $\forall n \in \mathbb{N}$ by induction

5. Let $f(x) = x \ln x, x > 0$ and $n \in \mathbb{N}$. Let $f^{(n)}(x)$ denote the nth derivative of $f(x)$. Use Calculus to prove that if $n \geq 3$, then

$$f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}.$$

Base Case

$n=3$ $f^3(x) = -1/x^2$

Now, $f^3(x) = (-1)^3 \cdot \frac{(3-2)!}{x^{3-1}}$

$= -\frac{1 \cdot 1}{x^2} = -1/x^2$

Hence true for $n=3$

Assume true for $n=K$

$f^K(x) = (-1)^K \frac{(K-2)!}{x^{K-1}}$

Induction step i.e $n=K+1$

Differentiate $F^K(x) \Rightarrow F^{K+1}(x) = \frac{d}{dx}(-1)^K \frac{(K-2)!}{x^{K-1}}$

$(-1)^K (K-2)! \cdot \frac{d}{dx}\left(x^{1-K}\right)$

$= (-1)^{K+1} \frac{(K-1)!}{x^K}$

$= f^{(K+1)}(x)$

Hence $f(x) = x\ln x$ holds ~~true~~ for $n = k+1$ as well.

So, for $f(x), x\ln x, x > 0$,

$$f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}$$

$\forall n \in \mathbb{N}$ by induction

Induction (strong induction)

Define a_n as follows, for all natural numbers n:

$$a_1 = 1, a_2 = 3, a_n = 2a_{n-1} - a_{n-2} \text{ for } n \geq 3.$$

Prove that $a_n = 2n - 1$ for all $n \in \mathbb{N}$.

(continued)