Mathematics 220 — Homework 2

- Contains 8 questions on 1 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

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1. Prove that if $a \in \mathbb{Z}$, then $4 \nmid (a^2+1)$.
2. Let x be a positive real number. Prove that if $2x - \dfrac{1}{x} > 1$, then $x > 1$.
3. Prove that if $k \in \mathbb{Z}$ then $3 \big| \big(k(2k+1)(4k+1)\big)$.
4. Let $n \in \mathbb{Z}$.
   (a) Show that if $3 \mid n$ and $4 \mid n$, then $12 \mid n$.
   (b) Use the previous part to show that if $n > 3$ is a prime, then $n^2 \equiv 1 \pmod{12}$.
5. Let $n \in \mathbb{Z}$. Prove that if $n^3 + n^2 - n + 3$ is a multiple of three, then n is a multiple of three.
6. Let $x \in \mathbb{R}$. Then, prove that $x^2 + |x-6| > 5$.
7. Let $x, y \in \mathbb{Z}$. Prove that

   $$3 \nmid (x^3+y^3) \text{ if and only if } 3 \nmid (x+y).$$

8. **Bézout's identity**: Let $a, b \in \mathbb{Z}$ such that a and b are not both zero. Then there exists $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a,b)$.

   For example, for $a = 5$ and $b = 7$, we see $\gcd(a,b) = 1$ and we can take $x = 10$ and $y = -7$.

   Now, let $a, b, k \in \mathbb{Z}$ and assume that a, b are not both zero. Then, using Bézout's identity, show that if $k \nmid \gcd(a,b)$, then $k \nmid a$ or $k \nmid b$.

(1) if $a \in \mathbb{Z}$, then $4 \nmid (a^2+1)$

Case 1: a is odd

$a = 2k+1$

And $a^2+1 = 4j$

Now, $(2k+1)^2+1$

$= 4k^2 + 4k + 1 + 1$

$= 4(k^2+k) + 2$

$\therefore$ NO

Case 2: a is even

$a = 2i$

Now, $(2i)^2+1$

$= 4i^2+1$

$= 4(i^2)+1$

$\therefore$ NO

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(2) $2n - \frac{1}{n} > 1$ then $n > 1$

$2n^2 - 1 > n$

$2n^2 - n - 1 > 0$

$2n^2 - 2n + n - 1 > 0$

$2n(n-1) + 1(n-1) > 0$

$(2n+1)(n-1) > 0$

So, either $n > -\frac{1}{2}$ or $n > 1$

so, $n > 1$

③ $k \in \mathbb{Z} \rightarrow 3 \mid (k(2k+1)(4k+1))$

$3m = k(2k+1)(4k+1)$

Case 1: $k = \text{odd} = 2p+1$

$(2p+1)(4p+3)(8p+5)$

$(8p^2 + 10p + 3)(8p+5)$

$(64p^3 + 80p^2 + 24p + 40p^2 + 50p + 15)$

$(64p^3 + 120p^2 + 74p + 15)$

$\therefore$ No,

Case 2: $k = \text{even} = 2q$

$2q(4q+1)(8q+1)$

$(8q^2 + 2q)(8q+1)$

$(64q^3 + 16q^2 + 8q^2 + 2q)$

$(64q^3 + 24q^2 + 2q)$

$\therefore$ No,

$m^3 + 2m$

$\left(\frac{k-1}{2}\right)^3 + k - 1$

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④ $n \in \mathbb{Z}$

(a) if $3 \mid n$ & $4 \mid n \Rightarrow 12 \mid n$

$n = 3p$ & $n = 4q \Rightarrow n = 12r$

Let, $3n = 12q$ - ①

Let, $4n = 12p$ - ②

② - ① $\rightarrow n = 12(p-q)$

$n = 12r$

(b) if $n > 3$ = prime $\Rightarrow n^2 \equiv 1 \pmod{12}$

when $n > 3$ all prime numbers are odd

So, $n = 2m+1$

Now, $12 \mid n^2 - 1$ (cas $n^2 \equiv 1 \pmod{12}$)

$n^2 - 1 = 12 \cdot k$

So, $(2m+1)^2 - 1$

$4m^2 + 4m + 1 - 1$

$4m^2 + 4m$

$4(m^2 + m)$

So, $n^2 - 1$ is divisible by 4

Now, no prime > 3 is divisible by 3,

So, $n = 3m+1$ & $n = 3m+2$

$(3m+1)^2 - 1 = 9m^2 + 1 + 6m - 1$
$= 3(3m^2 + 2m)$

$(3m+2)^2 - 1 = 9m^2 + 12m + 4 - 1$
$= 3(3m^2 + 4m + 1)$

So, $n^2 - 1$ divisible by 3

So, $n^2 - 1 = 4p \Rightarrow 3(n^2-1) = 12p$
$n^2 - 1 = 3q \Rightarrow 4(n^2-1) = 12q$

$n^2 - 1 = 12i$

$\blacksquare$

(5) n^3+n^2-n+3 is a multiple of 3, then n is a multiple of 3.

So, n^3+n^2-n+3

$$3 \nmid n \Rightarrow 3 \nmid n^3+n^2-n+3$$

Case 1: $n = 3k+1$

$$(3k+1)^3 + (3k+1)^2 - 3k - 1 + 3$$

$$27k^3 + 1 + 9k + 27k^2 + 9k^2 + 1 + 6k - 3k + 2$$

$$27k^3 + 36k^2 + 12k + 3 + 1$$

$n^3+n^2-n+3 = 3m+1 \Rightarrow$ not divisible

Case 2: $n = 3k+2$

$$(3k+2)^3 + (3k+2)^2 - 3k - 2 + 3$$

$$27k^3 + 8 + 36k + 54k^2 + 9k^2 + 4 + 12k - 3k + 1$$

$$27k^3 + 63k^2 + 45k + 12 + 1$$

$n^3+n^2-n+3 = 3n+1 \Rightarrow$ not divisible

So, $3 \nmid n^3+n^2-n+3$

(6) $x \in \mathbb{R}$ then $x^2 + |x-6| > 5$

$x^2 + |x-6| \leq 5 \Rightarrow x \notin \mathbb{R}$

Case 1: $x^2 + x - 6 \leq 5$ $\Rightarrow$ next page

$x^2 + x - 11 \leq 0$

(7) $x, y \in \mathbb{Z}$

$3 \nmid x^3 + y^3 \Leftrightarrow 3 \nmid x + y$

If $3 \nmid x^3 + y^3 \Rightarrow 3 \nmid x + y$

If $3 \nmid x + y \quad \wedge \quad 3 \nmid x^3 + y^3$

Part 1: $3 \mid x+y \Rightarrow 3 \mid x^3 + y^3$

$x + y = 3k$

$(x+y)^3 = x^3 + y^3 + 3xy(x+y)$

$9k^3 - 3xy(x+y) = x^3 + y^3$

$x^3 + y^3 = 3m$ ✓

Part 2: $3 \mid x^3 + y^3 \Rightarrow 3 \mid x + y$

$3j = x^3 + y^3$

$(x+y)^3 = x^3 + y^3 + 3xy(x+y)$

$(x+y)^3 = 3n$

$x + y = 3i$ ✓

(6) $x \in \mathbb{R}$ then $x^2 + |x-6| > 5$

Case 1: $x > 6$

$x > 6$, so $x^2 > 36$

&, $x^2 + x - 6 > 36$

which is greater than 5

Case 2: $x \le 6$

$x^2 - x + 6$

$(x - 1/2) = \frac{23}{4} \approx 5.75$

which is > 5

$6_1 = 0.88 \times \frac{9}{12}$

(8) If $k \nmid \gcd(a,b) \Rightarrow k \nmid a$ or $k \nmid b$

$k \mid a$ & $k \mid b \Rightarrow k \mid \gcd(a,b)$

$a = pk$ $\quad$ $b = qk$

(3) $k \in \mathbb{Z} \Rightarrow 3 \mid (k(2k+1)(4k+1))$

Case 1: $k = 3m$

$$(3m)(6m+1)(12m+1)$$

$$(18m^2+3m)(12m+1)$$

$$18m^2 \cdot 12m + 18m^2 + 36m^2 + 3m$$

$\Rightarrow 3 \mid$ ✓

Case 2: $k = 3m+1$

$$(3m+1)(6m+3)(12m+5)$$

$$(18m^2+9m+6m+3)(12m+5)$$

$$(18 \cdot 12m^3 + 90m^2 + 108m^2 + 45m + 72m^2 + 30m + 36m + 15)$$

$\Rightarrow 3 \mid$ ✓

Case 3: $k = 3m+2$

$$(3m+2)(6m+5)(12m+9)$$

$$(18m^2+15m+12m+10)(12m+9)$$

$$(18 \cdot 12m^3 + 162m^2 + 180m^2 + 135m + 144m^2 + 108m + 120m + 90)$$

$\Rightarrow 3$ ✓

(8) $k \mid a$ & $k \mid b \Rightarrow k \mid \gcd(a,b)$

$a = x \cdot k$

$b = y \cdot k$

$ax = x^2 k$

$by = y^2 \cdot k$

$ax + by = \gcd(a,b) = x^2 y^2 k$

$ax + by = \gcd(a, b) = k \cdot m \; [\exists m \in \mathbb{Z}]$

So, $k \mid \gcd(a,b)$