PLP - 24

TOPIC 24—SET PROOFS

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PROVING THINGS

WRITING SET OPERATIONS AS STATEMENTS

Let A, B be sets.

Subset and equality

- $(A \subseteq B) \quad \equiv \quad (\forall x \in A, x \in B) \quad \equiv \quad (x \in A \implies x \in B).$
- $(A = B) \quad \equiv \quad ((A \subseteq B) \wedge (B \subseteq A)) \quad \equiv \quad ((x \in A) \iff (x \in B))$

Intersection and union

- $(x \in A \cap B) \quad \equiv \quad (x \in A \wedge x \in B)$
- $(x \in A \cup B) \quad \equiv \quad (x \in A \vee x \in B)$

Complement and difference

- $(x \in \bar{A}) \quad \equiv \quad (x \notin A) \quad \equiv \quad \sim (x \in A)$
- $(x \in A - B) \quad \equiv \quad ((x \in A) \wedge (x \notin B)) \quad \equiv \quad ((x \in A) \wedge \sim (x \in B))$

A SUBSET EXAMPLE

PROPOSITION:

Let $A = \{n \in \mathbb{Z} \ : \ 6 \mid n\}$ and $B = \{n \in \mathbb{Z} \ : \ 2 \mid n\}$, then $A \subseteq B$

Scratch work

- We need to prove $a \in A \implies a \in B$
- So assume that $a \in A$. Hence a is an integer divisible by 6
- This means $a = 6k$ for some $k \in \mathbb{Z}$.
- We need to show that $a \in B$ which means we need to show that $2 \mid a$
- But since $a = 6k,$ we know $a = 2 \cdot 3k$ so, $2 \mid a$ as required.

WRITE IT UP NICELY

$$A = \{n \in \mathbb{Z} \; : \; 6 \mid n\} \subseteq \{n \in \mathbb{Z} \; : \; 2 \mid n\} = B$$

PROOF.

- Let the sets A, B be as stated and assume that $a \in A$.
- Hence we know that $6 \mid a$ and so $a = 6k$
- This implies that $a = 2(3k)$ and so $2 \mid a$
- By the definition of the set B, $a \in B$
- So $A \subseteq B$ as required

ANOTHER EXAMPLE

PROPOSITION:

Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Scratch work

- What do we assume? That $A \subseteq B$ and $B \subseteq C$.
- What do we need to prove? $A \subseteq C$, that is $$(x \in A) \implies (x \in C)$$
- A problem — what do we assume? Either $x \in A$ or $x \notin A$.
  - If $x \in A$ then since $A \subseteq B$, $x \in B$.

    Then since $x \in B$ and $B \subseteq C$, $x \in C$
  - If $x \notin A$ then the implication "$(x \in A) \implies (x \in C)$" is true.

WRITE IT UP

$$(A \subseteq B) \wedge (B \subseteq C) \implies (A \subseteq C)$$

PROOF.

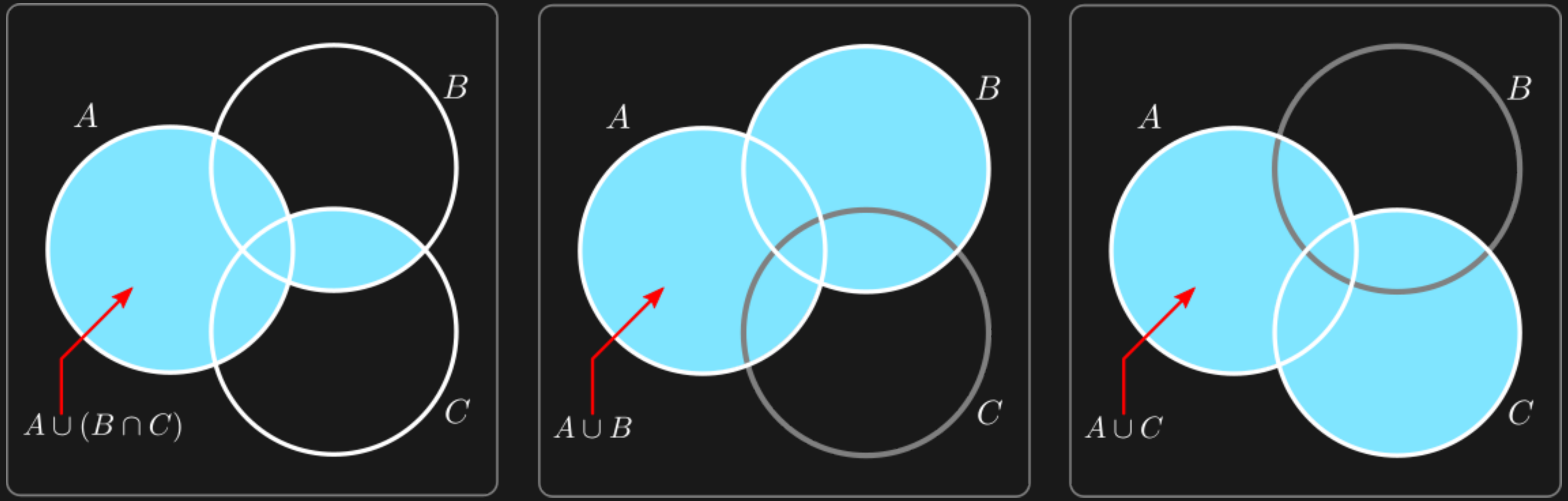
- Assume that $A \subseteq B$ and $B \subseteq C$.
- Further let $x \in A$
- Since $A \subseteq B$, we know that $x \in B$
- Then similarly, since $B \subseteq C$, we know that $x \in C$
- Hence $A \subseteq C$ as required

A DISTRIBUTIVE RESULT

PROPOSITION:

Let A, B, C be sets, then $A \cup (B \cap C) = (A \cup C) \cap (A \cup B)$.

Scratch work



JUST DO ONE INCLUSION

$$A \cup (B \cap C) \subseteq (A \cup C) \cap (A \cup B)$$

- We have to prove **LHS** is a subset of **RHS**
- Let $x \in$ **LHS**. Hence $x \in A$ or $x \in B \cap C$.
- So we have 2 cases to consider
  - Assume $x \in A$. Then $x \in A \cup C$ and $x \in A \cup B$
  - Now assume $x \in B \cap C$, then $x \in B$ and $x \in C$

    Since $x \in B$, we know $x \in B \cup A$.

    Similarly $x \in C$, so $x \in C \cup A$.
- In both cases, $x \in A \cup B$ and $x \in A \cup C$, so $x \in$ **RHS** as required

WRITE IT UP

$$A \cup (B \cap C) \subseteq (A \cup C) \cap (A \cup B)$$

PROOF.

Let $x \in A \cup (B \cap C)$, so that $x \in A$ or $x \in B \cap C$. We consider each case separately.

- Assume that $x \in A$, then we know that $x \in A \cup B$. Simiarly, we have $x \in A \cup C$.
- Now assume that $x \in B \cap C$, so that $x \in B$ and $x \in C$.

  Since $x \in B$ it follows that $x \in B \cup A$. Similarly, because $x \in C$, $x \in C \cup A$.

In both cases, $x \in (A \cup B)$ and $x \in (A \cup C)$. Hence $x \in (A \cup C) \cap (A \cup B)$ as required.