Results

- Name = Joshipura, Kashish
- ID = 27745629
- Test number = 382

question	version	mark	out of
Q1	4	4	5
Q2	4	5	5
Q3	1	4	5
Q4	2	5	5
Q5	3	1	5
total	·	19	25

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Mathematics 220 — Midterm — 45 minutes

19th and 20th October 2022

- The test consists of 12 pages and 5 questions worth a total of 25 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number	2	7	7	4	5	6	2	9
Section	1	0	8					
Name	KASHISH JOSHIPURA							
Signature								

Q1: 4 out of 5

1. 5 marks Answer the following:

(a) Negate the statement

$$P \implies Q$$

$P \Rightarrow Q$

$\sim P \vee Q$ [Implication]

$\sim(\sim P \vee Q)$ [Negation]

$\sim(\sim P) \wedge \sim Q$ [De Morgan's Law]

$P \wedge \sim Q$ [D.Neg]

+1 (a) correct

$\boxed{P \wedge \sim Q} \equiv \sim(P \Rightarrow Q)$

(b) Negate the following statement

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } (x < y) \vee (\sin(x) = \cos(y))$$

Org: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $(x < y) \vee (\sin(x) = \cos(y))$

Negation: $\sim(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $(x < y) \vee (\sin(x) = \cos(y)))$

$\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, \sim((x<y) \vee (\sin(x) = \cos(y)))$

$\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$, $\sim(x<y) \wedge \sim(\sin(x) = \cos(y))$

$\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$, $x \geq y \wedge (\sin(x) \neq \cos(y))$

So, Negation is: $\exists x \in \mathbb{R}$ s.t $\forall y \in \mathbb{R}$, $x \geq y \wedge (\sin(x) \neq \cos(y))$

+1 (b) correct

(c) Determine whether the following is true or false. Prove your answer.

(c) +0

$$\forall a \in \mathbb{N}, \exists b \in \mathbb{N} \text{ s.t. } (a \leq b) \implies (ab = b + a)$$

The statement is true, because consider that for all natural number a we let $\boxed{b = a-1}$ where $b \in N$.

not in N if a=1

Since, $a > a-1$, we have $a > b$.

This proves the hypothesis False & since this is an implication it's always True when the hypothesis is false.

So, the statement is True

(d) Let P, Q, R be statements. Assume that

$$(P \iff Q) \implies R$$

+2 (d) correct

is true and Q is false. What are the possible truth values of P and R?

$(P \Leftrightarrow Q) \Rightarrow R$ is true

Q is false

Case 1:

IF Q is false, $P \Leftrightarrow Q$ is true when P is also False and hence $(P \Leftrightarrow Q) \Rightarrow R$ is true when R is true

So, P = False, R = True

Case 2:

If P is true then $(P \Leftrightarrow Q)$ is False, $(P \Leftrightarrow Q) \Rightarrow R$ is true for any truth value of R, so, P = True, R = True/False

Q2: 5 out of 5

2. $\boxed{\text{5 marks}}$ Let $n \in \mathbb{N}$. Prove or disprove the following:

(a) If $5 \mid (n+4)$ then $5 \mid (2n^2 - 3)$.

(b) If $n \equiv 1 \pmod 4$, then $n \equiv 1, 5,$ or $9 \pmod{12}$.

(a) if $5 \mid (n+4) \Rightarrow 5 \mid (2n^2-3)$

Let $5 \cdot K = n + 4$, $\forall k \in \mathbb{Z}$

So, $n = 5k - 4$

So, $n^2 = 25K^2 + 16 - 40K$

$2n^2 = 50K^2 + 32 - 80K$

$2n^2 - 3 = 50K^2 + 32 - 80K - 3$

$2n^2 - 3 = 50K^2 - 80K + 29$

$2n^2 - 3 = 5(10K^2 - 16K + 5) + 4$

So, $2n^2 - 3 \equiv 4 \pmod 5$, i.e $5 \nmid (2n^2-3)$

Hence this statement is False

+2 good

(b) if $n \equiv 1 \pmod 4 \Rightarrow n \equiv 1, 5,$ or $9 \pmod{12}$

Let $n = 4 \cdot K + 1$, $\forall k \in \mathbb{Z}$

Let $k = 3m + r$, $r \in \{0, 1, 2\}$

So, Case 1: $r = 0$

$K = 3m$

So, $n = 4(3m) + 1$

$n = 12m + 1$

$n \equiv 1 \pmod{12}$

Case 2: $r = 1$

$K = 3m + 1$

$n = 4(3m+1) + 1$

$n = 12m + 4 + 1 = 12m + 5$

$n \equiv 5 \pmod{12}$

Case 3: $r = 2$

$K = 3m + 2$

$n = 4(3m+2) + 1$

$n = 12m + 8 + 1$

$n = 12m + 9$

$n \equiv 9 \pmod{12}$

Hence, the statement is True

+3 good

*This blank page is for your solution to **Question 2** if you need more space.*

Q3: 4 out of 5

3. [5 marks] Find all $x \in \mathbb{R}$ such that $|x+7| + |3x+1| \leq 44$.

Please prove your answer.

Find $x \in R$ s.t $|x+7| + |3x+1| \leq 44$

+1 Split into cases correctly

Case 1:
$x \in (-\infty, -7]$

So, Let, $x \geq -13$
So, $x+7 \geq -6$
And, $3x \geq -39$
$3x+1 \geq -38$

+1 case correct

Now, so, $-(x+7) \leq 6$ and $-(3x+1) \leq 38$

Adding the two eq we get,
$-(x+7) - (3x+1) \leq 44$
or $|x+7| + |3x+1| \leq 44$, for $x \in (-\infty, 7]$

Case 2:
$x \in (-7, -1/3)$

Now, since $x \in (-7, -1/3)$ we have,
For, $|x+7| + |(3x+1)| \leq 44$
$(x+7) - (3x+1) \leq 44$
$x+7 - 3x - 1 \leq 44$
$-2x + 6 \leq 44$
$-2x \leq 38$
$-x \leq 19$
$x \geq 19$

+1 case correct

Case 3:
$x \in [-1/3, \infty)$
consider $x \leq 9$
So, $x+7 \leq 16$
And $3x \leq 27$, $3x+1 \leq 28$
Adding these eq we get, $|x+7| + |3x+1| \leq 44$

Answer

+1 case correct

$x < -7$
$x > -7$
$x < -\frac{1}{3}$
$x > -\frac{1}{3}$
$(-\infty, -7], [-7, -\frac{1}{3}]$
$[-\frac{1}{3}, -\infty]$
$x+7 \quad -3x-1$
$-2x+6 \leq 44$
$-2x \leq 38$
$-x \leq 19$
$x \geq 19/2$

$x+7+3x+1$
$4x+8 \leq 44$
$4x \leq 36$
$x \leq 9$

So, $x \in [-13, 9] \cup [19, \infty)$

wrong final answer

This blank page is for your solution to ***Question 3*** *if you need more space.*

Be careful in your proof. You need to make sure that you are working out all possible x that make the inequality hold. In a few places the logic is a bit backward. See solutions.

Q4: 5 out of 5

4. $\boxed{\text{5 marks}}$ Let $n \in \mathbb{N}$. Use mathematical induction to prove that

$$2^{n+1}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq 2^{n-1}+1$$

$\boxed{5}$ full marks

Base Case :

Let $n=1$,

$$2^{1+1}\sum\left(1-\frac{1}{2^1}\right) \geq 2^2\cdot\left(1-\frac{1}{2}\right) \geq 4\cdot\frac{1}{2} \geq 2$$

$$\& \quad 2^{1-1}+1 \geq 1+2^0 \geq 1+1 \geq 2$$

Hence base case holds true,

Inductive step :

Assume this holds true for $n=k$,

$$\text{So, } 2^{k+1}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^k}\right) \geq 2^{k-1}+1$$

Now, consider $n=k+1$

$$\text{So, } 2^{k+2}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k+1}}\right)$$

$$\geq 2\cdot 2^{k+1}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k+1}}\right)$$

$$\text{Here } 2^{k+1}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^k}\right) = 2^{k-1}+1$$

$$\text{So, } 2\left(2^{k-1}+1\right)\cdot\left(1-\frac{1}{2^{k+1}}\right)$$

$$\left(2^k+2\right)\left(1-\frac{1}{2^{k+1}}\right) \Rightarrow 2^k+2-\frac{2^k}{2^{k+1}}-\frac{2}{2^{k+1}} \qquad 2^{(k+1)-1}+1$$

$$\geq 2^k+2-\frac{1}{2}-2^{-k}$$

$$\Rightarrow 2^k+\frac{3}{2}-\frac{1}{2^k} > 2^k+\frac{3}{2} > 2^k+1$$

Hence, using proof by induction,

$$2^{n+1}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq 2^{n-1}+1 \text{ for all } n \in \mathbb{N}.$$

*This blank page is for your solution to **Question 4** if you need more space.*

Q5: 1 out of 5

5. $\boxed{\text{5 marks}}$ Recall that we say that a sequence (x_n) converges to L when

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n > N) \implies (|x_n - L| < \varepsilon).$$

Prove or disprove: The sequence (x_n), defined by $x_n = \frac{\sqrt{n^3+1}}{n}$, converges to 0.

Scratch

$(n > N) \Rightarrow (x_n - 0) < \varepsilon$

$\Rightarrow \left(\sqrt{\frac{n^3+1}{n^2}}\right) < \varepsilon$

$\frac{n^3+1}{n^2} < \varepsilon^2$

$\varepsilon^2 > \frac{n^3+1}{n^2} > \frac{1}{n^2}$

$\varepsilon > \frac{1}{n}$ $\quad \varepsilon > \frac{1}{n^2} >$

$n > \frac{1}{\varepsilon} >$

$n^3 > n$

$n^3+1 > n$

$\frac{n^3+1}{n^2} > \frac{n^3+1}{1}$

Proof

Let's consider $N = \lceil 1/\varepsilon \rceil$, $\exists N \in \mathbb{N}$

Now, $n > N = \lceil 1/\varepsilon \rceil$

So, $n > N = 1/\varepsilon$

$n > \frac{1}{\varepsilon}$ or $\varepsilon > \frac{1}{n}$

So, $n > \frac{1}{\varepsilon}$, on squaring we get,

$n^3 + 1 > \frac{1}{\varepsilon}$

$\varepsilon > \frac{1}{n^3+1}$

+1 part marks for work with inequalities

*This blank page is for your solution to **Question 5** if you need more space.*

This page was flagged "Do No Mark" by the instructor. In most cases nothing here was marked.

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.
- Smoking is strictly prohibited during the test.