PLP - 35

TOPIC 35—INVERSE FUNCTIONS

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INVERSE FUNCTIONS

INVERSES AND ONE-SIDED INVERSES

DEFINITION:

Let $f : A \to B$ and $g : B \to A$ be functions.

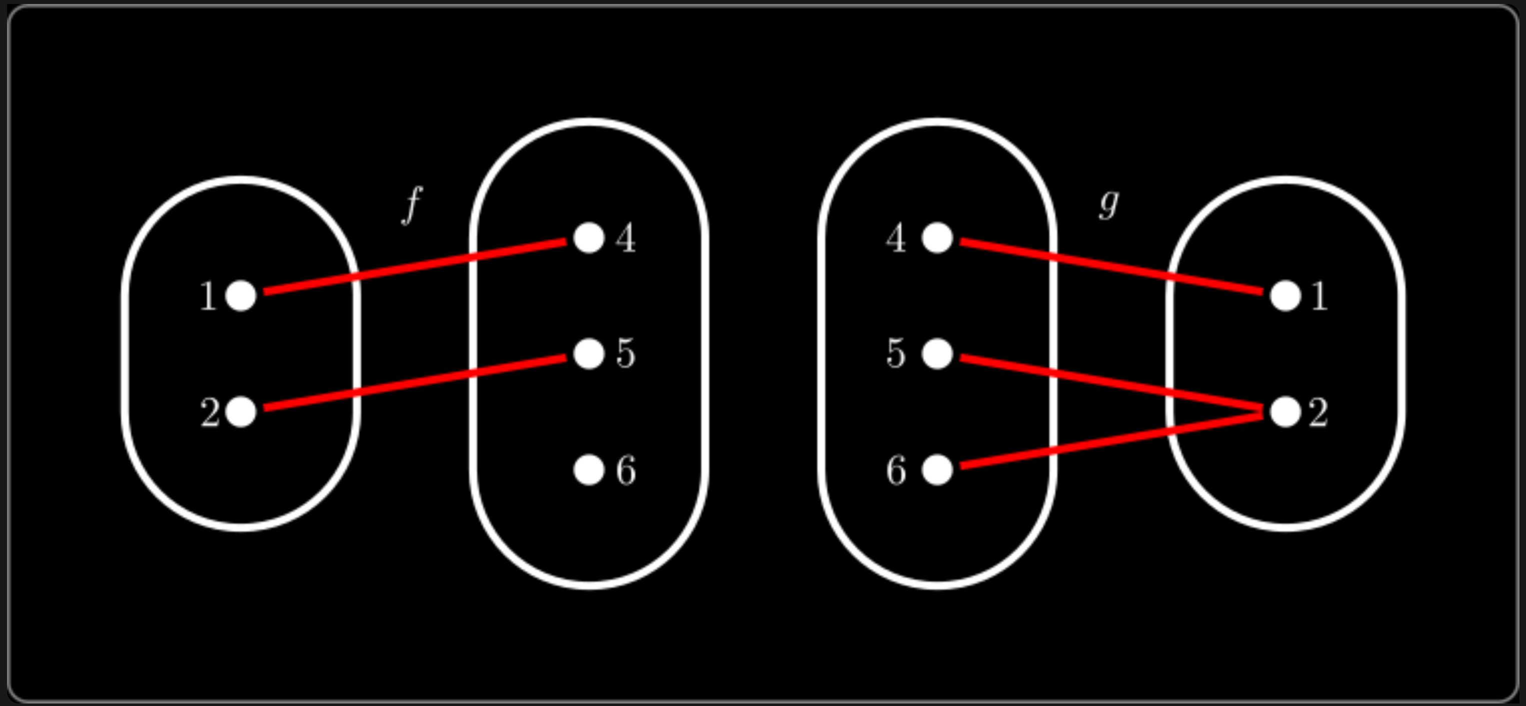
- If $g \circ f = i_A$ then we say that g is a **left-inverse** of f.
- Similarly, if $f \circ g = i_B$ then we say that g is a **right-inverse** of f.
- If g is both a **left-inverse** and **right-inverse**, then we call it an **inverse** of f.

Note that one can prove that if *an* inverse exists, then it is *unqiue*.

So we can say *the* inverse and denote it f^{-1}.

LEFT- BUT NOT RIGHT-INVERSE

Consider the functions f, g defined below



Notice that $g(f(1)) = 1$ and $g(f(2)) = 2$ so g is a left-inverse of f.

Then $f(g(4)) = 4$, $f(g(5)) = 5$ but $f(g(6)) = 5 \neq 6$ so g is not a right-inverse of f.

The non-injectiveness of g is to blame.

A similar example gives a right-inverse that is not a left-inverse (non-surjectiveness is to blame)

EXISTENCE OF ONE-SIDED INVERSES

LEMMA:

Let $f : A \to B$ be a function. Then

- f has a left-inverse iff f is injective.
- f has a right-inverse iff f is surjective.

The proofs of these statements make very good exercises. We'll do the forward implications.

ONE SIDED INVERSE

If f has a left-inverse then it is injective

PROOF.

Assume that f has a left-inverse g, so that $g(f(x)) = x$.

Now let $a_1, a_2 \in A$ so that $f(a_1) = f(a_2)$. Then we know that $g(f(a_1)) = g(f(a_2))$. But since g is a left-inverse, $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. Thus f is injective.

If f has a right-inverse then it is surjective

PROOF.

Assume that f has a right-inverse g, so that $f(g(y)) = y$.

Let $b \in B$ and set $a = g(b)$. Then $f(a) = f(g(b)) = b$, since g is a right-inverse. Thus f is surjective.

JOINING INVERSES

LEMMA:

Let $f : A \to B$ have a left-inverse g and a right-inverse h. Then $g = h$.

PROOF.

Let f, g and h be as stated. Thus $g \circ f = i_A$ and $f \circ h = i_B$. Then

$$\begin{aligned} g &= g \circ i_B = g \circ (f \circ h) \\ &= (g \circ f) \circ h && \text{assoc of compositions} \\ &= i_A \circ h = h \end{aligned}$$

as required.

EXISTENCE OF INVERSE

THEOREM:

Let $f : A \to B$. Then f has an inverse iff f is bijective. Further, that inverse, if it exists, is unique.

PROOF.

- Assume that f has an inverse g. Then g is both a left-inverse and a right-inverse. Lemma: since f has a left-inverse, f is injective, and then since f has a right-inverse, f is surjective. Hence f is bijective.
- Now assume that f is bijective. Lemma: since f is injective, it has a left inverse, and since f is surjective, it has a right inverse. Lemma: those one-sided inverses are the same function, g. Hence g is an inverse of f.
- Finally, assume that g, h are inverses of f, then $g = g \circ (f \circ h) = (g \circ f) \circ h = h$. Thus the inverse function is unique.

EXAMPLE

PROPOSITION:

The function $f : \mathbb{R} \to \mathbb{R}, f(x) = 7x - 3$ is bijective and so has an inverse.

PROOF.

Previously we showed that f is injective and surjective, and so is bijective. Hence its inverse exists.

In this case we can find the inverse explicitly: $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(y) = \frac{y+3}{7}$

Since the function is bijective, enough to prove this is a left-inverse

$$(f^{-1} \circ f)(x) = f^{-1}(7x - 3) = \frac{(7x - 3) + 3}{7} = x$$

as required.