9.7

1. Let $A = \{1, 2, 3, 4, 5, 6\}$. Write out the relation R that expresses "$\nmid$" (does not divide) on A as a set of ordered pairs. That is, $(x, y) \in R$ if and only if $x \nmid y$. Is the relation reflexive? Symmetric? Transitive?

$A = \{1,2,3,4,5,6\}$

$R = \{(2,1), (2,3), (2,5), (3,1), (3,2), (3,4), (3,5), (4,1), (4,2),$
$(4,3), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,6), (6,1),$
$(6,2), (6,3), (6,4), (6,5)\}$

$R \Rightarrow$ not Reflexive, $(1,1) \notin R$
$\Rightarrow$ not Symmetric, $(2,1) \in R$ but $(1,2) \notin R$
$\Rightarrow$ not Transitive, $(2,3), (3,2) \in R$ but $(2,2) \notin R$

2. Define a relation on $\mathbb{R}$ as $x \mathcal{R} y$ if $|x - y| < 1$. Is R reflexive? Symmetric? Transitive?

xRy if $|x-y| < 1$

Reflexive

$|x-x| = 0$
So, xRx
Hence Reflexive

Symmetric

$|x-y| < 1$
So, $|y-x| < 1$
So, xRy & yRx
Hence Symmetric

Transitive

$|x-y| < 1$
$|y-z| < 1$
$|x-z| < 2$
So, xRy, yRz but $(x,z) \notin R$
Hence not transitive

3. For each of the following relations, determine whether or not they are reflexive, symmetric, and transitive.

(a) $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\} \subseteq \{1,2,3\}\times\{1,2,3\}$

(b) For $a, b \in \mathbb{N}$, $a\,\mathcal{R}\,b$ if and only if $a \mid b$.

(c) $R = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x - y \in \mathbb{Q}\}$

(d) For $A, B \subseteq \mathbb{R}$, $A\,\mathcal{R}\,B$ if and only if $A \cap B = \emptyset$.

(e) For functions $f, g : \mathbb{R} \to \mathbb{R}$, $f\,\mathcal{R}\,g$ if and only if $f - g$ is linear, that is, there are constants $m, b \in \mathbb{R}$ so that $f(x) - g(x) = mx + b$ for all $x \in \mathbb{R}$. The constants m, b depend on f and g, but not on x.

(a) $R=\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\} \subseteq \{1,2,3\}\times\{1,2,3\}$

It's reflexive
It's symmetric
It's not transitive since $(2,1),(1,3) \in R$ but $(2,3) \notin R$

(b) Reflexive since $a|a$
Not symmetric since $a|b$ but $b \nmid a$
Transitive

(c) Reflexive $x - x = 0$
Symmetric $x-y = -(y-x)$
Transitive $x-y$, $y-z$, $x-z$

(d) Not reflexive as $A \cap A = A$
Symmetric as $A \cap B = B \cap A$
Not transitive as if $A=\{1,2\}$ $B=\{3,4\}$
$C=\{2,5\}$
then $A \cap B = \phi$, $B \cap C = \phi$ but $A \cap C = \{2\}$ i.e $\neq \phi$

(e) Reflexive
Symmetric
Transitive since for (f,g) (g,h) we have for (f,h)
$f-h = (m_1+m_2)x + b_1 + b_2$

$x \in \mathbb{R}$. The constants m, b depend on f and g, but not on x.

4. Determine whether the following relations are reflexive, symmetric and transitive. Prove your answers.

(a) On the set X of all functions $\mathbb{R} \to \mathbb{R}$, we define the relation:

$$f \mathcal{R} g \text{ if there exists } x \in \mathbb{R} \text{ such that } f(x) = g(x).$$

(b) Let $\mathcal{S}$ be a relation on $\mathbb{Z}$ defined by:

$$x \mathcal{S} y \text{ if } xy \equiv 0 \pmod 4.$$

(a) Reflexive

Consider (f, f)

So, by definition $f(x) = f(x)$

Hence it's reflexive

Symmetric

Consider fRg

So, $f(x) = g(x)$

Now, by defination $g(x) = f(x)$

So, gRf

Hence it's symmetric

Transitive

Consider fRg & gRh

So, $f(x) = g(x)$, $g(x) = h(x)$

So, $f(x) = g(x) = h(x)$

So, fRh

Hence it's transitive

So R is an equivalence relation.

(b) $xy \equiv 0 \pmod 4$

Reflexive

Consider $(1,1)$ So, $1 \cdot 1 \not\equiv 0 \pmod 4$

So, not reflexive

Symmetric

$xy \equiv 0 \pmod 4$

$xy = 4n \quad , \forall n \in \mathbb{Z}$

By definition $yx = 4n$

So, yRx

Hence it's symmetric

Transitive

Consider $(1,0)$ & $(0,3) \in R$

but $(1,3) \notin R$

So, it's not transitive

$x \; S \; y$ if $xy \equiv 0 \pmod 4$.

5. For each of the following relations, show that they are equivalence relations, and determine their equivalence classes.

(a) $R = \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_1^2 + y_1^2 = x_2^2 + y_2^2\}$

(b) Let L be the set of all lines on the Euclidean plane, $\mathbb{R}^2$. For $\ell_1, \ell_2 \in L$, $\ell_1 \mathcal{R} \ell_2$ if and only if ℓ_1 and ℓ_2 have the same slope, or they are both vertical lines.

(c) Let R be a relation on $\mathbb{Z}^2$ defined by $x \mathcal{R} y$ if and only if $3 \mid x^2 - y^2$.

6. Define a relation on $\mathbb{Z}$ as

(a) $R = \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_1^2 + y_1^2 = x_2^2 + y_2^2\}$

Reflexive

Let's take $((x_1, y_1), (x_1, y_1))$ and so,

$$x_1^2 + y_1^2 = x_1^2 + y_1^2$$

By definition it is reflexive

Symmetric

let $(x_1, y_1) R (x_2, y_2)$

so, $x_1^2 + y_1^2 = x_2^2 + y_2^2$

By definition this is symmetric

Transitive

Let $(x_1, y_1) R (x_2, y_2)$ & $(x_2, y_2) R (x_3, y_3)$

So, $x_1^2 + y_1^2 = x_2^2 + y_2^2$ & $x_2^2 + y^2 = x_3^2 + y_3^2$

So, $x_1^2 + y_1^2 = x_3^2 + y_3^2$

So, it's transitive

Eq Class

$[(a,b)] = \quad x_1^2 + y_1^2 = a^2 + b^2$

$\quad\quad x_1^2 + y_1^2 = \sqrt{a^2+b^2}$

So, $[(a,b)] = \{(0, \sqrt{a^2+b^2}) \circ \delta (\sqrt{a^2+b^2}, 0)\}$

(b) <u>Reflexive</u>

$l_1 R l_1$ by definition

It's reflexive

<u>Symmetric</u>

$l_1 R l_2$ then $l_2 R l_1$ again by definition

It's symmetric

<u>Transitive</u>

$l_1 R l_2$ & $l_2 R l_3$

So, $m_1 = m_2$ & $m_2 = m_3$

So, $m_1 = m_3$

So, $l_1 R l_3$

It's transitive

Let $\{[l_m] : m \in R, m \geq 0\}$

(c) $3 \mid x^2 - y^2$

<u>Reflexive</u>

$x^2 - x^2 = 0$ & $3 \mid 0$

So, it's reflexive

Symmetric

Let, xRy So, $3 \mid x^2 - y^2$ or $3 \mid -(y^2 - x^2)$

or $3 \mid y^2 - x^2$

so, yRx &

it's symmetric

Transitive

Let xRy & yRz, so, $3n = x^2 - y^2$ &

$3m = y^2 - z^2$

So, $3(n+m) = x^2 - z^2$

$3 \mid x^2 - z^2$

So, xRz

& it's transitive

$x^2 \equiv y^2 \pmod 3$

$[0] = \{x \mid x = 3k\}$

$[1] = \{x \mid x = 3k+1 \text{ or } 3k+2\}$

---

6. Define a relation on $\mathbb{Z}$ as

$$a \mathcal{R} b \iff 3 \mid (2a - 5b).$$

Is R an equivalence relation? Prove your answer.

Reflexive

$2a - 5a = -3a \Rightarrow 3 \mid -3a$

So, aRa

It's reflexive

Symmetric

Let, aRb

So, $3 \mid 2a - 5b$

$3n = 2a - 5b$

So, $3n + 3b + 3a = 5a - 2b$

$3(-n - b - a) = 2b - 5a$

So, $3 \mid 2b - 5a$

So, it's symmetric

Transitive

Let, aRb, bRc

So, $3n = 2a - 5b$

$3m = 2b - 5c$

So, $3(n + m + b) = 2a - 5c$

$3 \mid 2a - 5c$

So, aRc

So, it's transitive

Hence it's an equivalence relation

7. Let E be a non-empty set and $q \in E$ be a fixed element of E. Consider the relation $\mathcal{R}$ on $\mathcal{P}(E)$ (power set of E) defined as

$$A \,\mathcal{R}\, B \iff (q \in A \cap B) \vee (q \in \overline{A} \cap \overline{B}),$$

where for any set $S \subseteq E$, we write $\overline{S} = E - S$ for the complement of S in E. Prove or disprove that $\mathcal{R}$ an equivalence relation.

Reflexive

Let Now, $q \in A$ or $q \in \bar{A}$

i.e $q \in A \cap A \vee q \in \bar{A} \cap \bar{A}$

So, ARA

It's reflexive

Symmetric

Let ARB

So, $q \in (A \cap B) \vee q \in (\bar{A} \cap \bar{B})$

This is same as

$q \in (B \cap A) \vee q \in (\bar{B} \cap \bar{A})$

So, BRA

It's symmetric

Transitive

Let ARB & BRC

So, $q \in A \cap B \vee q \in \bar{A} \cap \bar{B}$

$\wedge$

$q \in B \cap C \vee q \in \bar{B} \cap \bar{C}$

So, $(q \in A \cap B \wedge q \in B \cap C) \vee$
$(q \notin A \cap B \wedge q \in \bar{B} \cap \bar{C}) \vee$
$(q \notin \bar{A} \cap \bar{B} \wedge q \in B \cap C) \vee$
$(q \notin \bar{A} \cap \bar{B} \wedge q \in \bar{B} \cap \bar{C})$

So, $q \in A \cap B \cap C \vee q \in \bar{A} \cap \bar{B} \cap \bar{C}$

$q \in A \cap C \vee q \in \bar{A} \cap \bar{C}$

So, ARC

& it's transitive,

So, it's an equivalence relation

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8. Let R be a relation on $\mathbb{Z}$ defined by $a \mathcal{R} b$ if $7a^2 \equiv 2b^2 \pmod 5$. Prove that R is an equivalence relation. Determine its equivalence classes.

It is an equivalence relation

$2a^2 = 2b^2 \pmod 5$

$[0] = \{a \mid a = 5k\}$
$[1] = \{a \mid a = 5k+1 \text{ or } 5k+4\}$
$[2] = \{a \mid a = 5k \pm 2\}$

---

9. Let $n \in \mathbb{N}$ with $n > 1$ and let P be the set of polynomials with coefficients in $\mathbb{R}$. We define a relation, T, on P as follows:

Let $f, g \in P$. Then we say fTg if $f - g = c$ for some $c \in \mathbb{R}$. That is, if there exists a $c \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f(x) - g(x) = c$.

Show that T is an equivalence relation on P.

It's equivalent bcz,

R $f - f = c = 0$

S $f - g = c$, $g - f = -c = d$ (let's say)

T $f - h = c + d$

---

10. Prove or disprove the following statements:

(a) If R and S are two equivalence relations on a set A, then $R \cup S$ is also an equivalence relation on A.

(b) If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

(a) $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

$S = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$

$R \cup S = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)\}$

Not a Relation as not transitive bcz $(2,1), (1,3) \in R \cup S$ but $(2,3) \notin R \cup S$

(b) Reflexive

$(a,a) \in R$ & S

So reflexive

Symmetric

Let $(a,b) \in R \cap S$

so, $(a,b) \in R$ and $(a,b) \in S$

So, $(b,a) \in R$ & $(b,a) \in S$

So, $(b,a) \in R \cap S$

So, symmetric,

<u>Transitive</u>

similar to above two

So, it's an equivalence relation

---

an equivalence relation on A.

11. Let R be a symmetric and transitive relation on a set A.

(a) Show that R is not necessarily reflexive.

(b) Suppose that for every $a \in A$, there exists $b \in A$ such that $a \mathcal{R} b$. Prove that R is reflexive.

(a) R is symmetric & transitive

$A = \{1, 2, 3, 4\}$

$R = \{(1,2), (2,1), (1,1), (3,1), (1,3), (2,3), (3,2)\}$

Here R is symmetric & transitive but not reflexive

(b) R is symmetric & transitive

So, let aRb & bRa

So, aRa by transitivity

& this holds true for every a

i.e $\{aRa \mid \forall a \in A\} \subseteq R$

So, it's reflexive

---

12. Let R be a relation on a nonempty set A. Then $\overline{R} = (A \times A) - R$ is also a relation on A. Prove or disprove each of the following statements:

(a) If R is reflexive, then $\overline{R}$ is reflexive.

(b) If R is symmetric, then $\overline{R}$ is symmetric.

(c) If R is transitive, then $\overline{R}$ is transitive.

(a) Let $A = \{1, 2, 3\}$

$A \times A \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$R = \{(1,1), (2,2), (3,3)\}$

So, $\overline{R} = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$

Hence $\overline{R}$ is not reflexive.

(b) Let aRb, so,

Assume $(a,b) \in \overline{R}$. Now this means $(a,b) \notin R$ & hence $(b,a) \notin R$ since it's symmetric & hence $(b,a) \in \overline{R}$.

So, it's symmetric.

(c) $A = \{1, 2\}$

$R = \{(1,1), (2,2)\}$

$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

$\overline{R} = \{(1,2), (2,1)\}$

but $(1,1) \notin \overline{R}$

So, it's not transitive.

(c) If R is transitive, then R is transitive.

13. In this question we will call a relation $R \subset \mathbb{Z} \times \mathbb{Z}$ *sparse* if $(a, b) \in R$ implies that $(a, b+1)$ and $(a+1, b)$ are NOT elements of R.

(a) Prove that for all $n \in \mathbb{N}$ the equivalence relation $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : n \mid (a - b)\}$ is sparse if and only if $n \neq 1$.

(b) Prove or disprove that every equivalence relation R on $\mathbb{Z}$ is sparse.

(a) $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : n \mid (a-b)\}$

Let's assume it is sparse at $n = 1$,

but for $(a, b+1)$ we have

$n \mid a-b-1$

So, $1 \mid a-(b+1)$

Similarly $1 \mid a+1-b$

But w.r.t its not possible as it is sparse

(b) Let $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\}$

is equivalent

but not sparse

14. Let A be a non-empty set and $P \subseteq \mathcal{P}(A)$ and $Q \subseteq \mathcal{P}(A)$ partitions of A. Prove that the set R defined as

$$R = \{S \cap T : S \in P,\ T \in Q\} - \{\emptyset\}$$

is also a partition of A.

$P \subseteq P(A)$

$Q \subseteq P(A)$

$R = \{ S \cap T : S \in P, T \in Q \} - \{ \phi \}$

Now, $S \in P \subseteq P(A)$ & $T \in Q \subseteq P(A)$

Case 1: $S \cap T = \phi$

Let $P(A) = \{ [x] \mid x \in A \}$

Let $x \in A$ & w.k.t $x \in [x]$

Also $[x] \in P(A)$

So, for $x \in A$, $x \in [x]$ for $[x] \in P(A)$

Let, $[x], [y] \in P(A)$ then w.k.t either

$[x] = [y]$ or $[x] \cap [y] = \phi$

Thus satisfying the two conditions for partitions i.e

$$\bigcup_{[x] \in R} [x] = P(A)$$

& $[x] \cap [y] = \phi$ or $[x] = [y]$

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15. Suppose that $n \in \mathbb{N}$ and $\mathbb{Z}_n$ is the set of equivalence class of congruent modulo n on $\mathbb{Z}$. In this question we will call an element $[u]_n$ *invertible* if it has a multiplicative inverse. That is,

$$[u]_n \text{ is invertible} \iff \text{there exists } [v]_n \in \mathbb{Z}_n \text{ so that } [u]_n[v]_n = [1]_n.$$

Now, define a relation R on $\mathbb{Z}_n$ by

$$[x]_n \dagger \ _n \iff [x]_n[u]_n = [y]_n \text{ for some invertible } [u]_n \in \mathbb{Z}_n.$$

(a) Show that R is a equivalence relation.

(b) Compute the equivalence classes of this relation for $n = 6$.

16. Let $n \in \mathbb{Z}$ and $p \geq 5$ be prime.

(a) Prove, using Bézout's identity that if $3 \mid n$ and $8 \mid n$, then $24 \mid n$.

(b) Use the result in part (a) to show that $p^2 \equiv 1 \pmod{24}$.

(a) if $3|n$ & $8|n$ then $24|n$

$$ax+by=1$$

So, $ax = 1 - by$

$3p = n$ or
$8q = n$

So, $8n = 24p$
$3n = 24q$

$$8x+3y=1$$

$$8xn + 3xn = 24x(p+q)$$
$$n = 24x(p+q)$$
$$24|n$$

(b) Prove $3|p^2-1$ & $8|p^2-1$

mod 3 cases

odd & even cases

10.8

1. Is the set

$$\theta = \{((x, y), (5y, 4x, x + y)) : x, y \in \mathbb{R}\}$$

a function? If so, what is its domain and its range?

$$\theta = \{((x,y),(5y,4x,x+y)) : x,y \in \mathbb{R}\}$$

$$\theta : \mathbb{R}^2 \to \mathbb{R}^3$$

Domain is $\mathbb{R}^2$

Range $\Rightarrow$

$$5y + 4x + x+y$$

p, q, r

$p = 5y$

$q = 4x$

$r = \frac{q}{4} + \frac{p}{5}$

a function? If so, what is its domain and its range?

2. For which values of $a, b \in \mathbb{N}$ does the set $\phi = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : ax + by = 6\}$ define a function?

$$\phi = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : ax+by=6\}$$

$\phi : \mathbb{Z} \to \mathbb{Z}$ for a&b have gcd $= 6$

$\hookrightarrow$ to be an int

So, $b | 6$

$b | a$

$$ax + by = 6$$

$$y = \frac{6-ax}{b}$$

So, $\dfrac{6-ax}{b}$ must be an integer.

$\dfrac{6}{b} - \dfrac{ax}{b}$ is an int so,

$b|6$ & $b|a$ $\therefore$ Hence proved$_{//}$

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3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is defined by $f(x) = \dfrac{2x}{1+x^2}$. Show that $f(\mathbb{R}) = [-1, 1]$.

$f : R \to R$

$f(x) = \dfrac{2x}{1+x^2}$, S.T $f(R) = [-1, 1]$

$$(x+1)^2 \geq 0$$
$$x^2+1+2x \geq 0$$
$$2x \geq -(1+x^2)$$
$$\frac{2x}{1+x^2} \geq -1$$

Similarly

$$(x-1)^2 \geq 0$$
$$x^2+1-2x \geq 0$$
$$2x \leq x^2+1$$
$$\frac{2x}{x^2+1} \leq 1$$

So, $f(x) = \dfrac{2x}{x^2+1} \in [-1, 1]$

---

4. Consider the following functions and their images and preimages.

(a) Consider the function $f : \{1,2,3,4,5,6,7\} \to \{0,1,2,3,4,5,6,7,8,9\}$ given as

$$f = \{(1,3),(2,8),(3,3),(4,1),(5,2),(6,4),(7,6)\}.$$

Find: $f\big(\{1,2,3\}\big)$, $f\big(\{4,5,6,7\}\big)$, $f(\emptyset)$, $f^{-1}\big(\{0,5,9\}\big)$ and $f^{-1}\big(\{0,3,5,9\}\big)$.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 4x^2 - x - 3$.

Find: $g\left(\{\frac{1}{8}\}\right)$, $g^{-1}(\{0\})$, $g((-1,0) \cup [3,4])$, and, $g^{-1}([-10,-5))$.

(c) Define $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = \sin(2\pi t)$.

Find: $h(\mathbb{Z})$, $h\left(\{\frac{1}{4}, \frac{7}{2}, \frac{19}{4}, 22\}\right)$, $h^{-1}(\{1\})$, and $h^{-1}([0,1))$.

(a) $f : \{1,2,3,4,5,6,7\} \to \{0,1,2,3,4,5,6,7,8,9\}$

$f = \{(1,3),(2,8),(3,3),(4,1),(5,2),(6,4),(7,6)\}$

$f(\{1,2,3\}) = \{3,8\}$

$f(\{4,5,6,7\}) = \{1,2,4,6\}$

$f(\phi) = \phi$

$f^{-1}(\{0,5,9\}) = \phi$

$f^{-1}(\{0,3,5,9\}) = \{1,3\}$

(b) $g(x) = 4x^2 - x - 3$

(c)

5. Let A, B be sets and $f : A \to B$ be a function from $A$to B. Then prove that if E and F are subsets of B, then

$$f^{-1}(E - F) = f^{-1}(E) - f^{-1}(F).$$

Remember that since we do not know whether or not f is a bijection, f^{-1} denotes the preimage of f not its inverse.

$f : A \to B$

if $E \subseteq B$ & $F \subseteq B \Rightarrow f^{-1}(E-F) = f^{-1}(E) - f^{-1}(F)$

$\underline{f^{-1}(E-F) \subseteq f^{-1}(E) - f^{-1}(F)}$

let, $x \in f^{-1}(E-F)$

So, $f(x) \in E-F$

$f(x) \in E$ & $f(x) \notin F$

$x \in f^{-1}(E)$ & $x \notin f^{-1}(F)$

$x \in f^{-1}(E) - f^{-1}(F)$

So, $f^{-1}(E-F) \subseteq f^{-1}(E) - f^{-1}(F)$

$\underline{f^{-1}(E) - f^{-1}(F) \subseteq f^{-1}(E-F)}$

let, $x \in f^{-1}(E) - f^{-1}(F)$

So, $x \in f^{-1}(E)$ & $x \notin f^{-1}(F)$

So, $f(x) \in E$ & $f(x) \notin F$

So, $f(x) \in E - F$

$x \in f^{-1}(E-F)$

So, $f^{-1}(E) - f^{-1}(F) \subseteq f^{-1}(E-F)$

Hence $f^{-1}(E-F) = f^{-1}(E) - f^{-1}(F)$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + ax + b$, where $a, b \in \mathbb{R}$. Determine whether f is injective and/or surjective.

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2 + ax + b$ where $a, b \in \mathbb{R}$

Injective

Let $f(x_1) = f(x_2)$

So, $x_1^2 + ax_1 = x_2^2 + ax_2$

$(x_1 + a/2)^2 = (x_2 + a/2)^2$

So, $x_1 = \pm x_2$

& it's not injective.

Surjective

Now, $y = x^2 + ax + b$

$y - b + \frac{a^2}{4} = (x + a/2)^2$

$-\sqrt{y - b - \frac{a^2}{4}} - \frac{a}{2} = x$ or

$\sqrt{y - b - \frac{a^2}{4}} - \frac{a}{2} = x$

So, not surjective

7. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ that are injective and such that for all $n \in \mathbb{N}$ we have $f(n) \leq n$.

$f : \mathbb{N} \to \mathbb{N}$

$\forall\, n \in \mathbb{N}$ we have $f(n) \leq n$

$g(n) = n$

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8. Prove that $f : [3, \infty) \to [5, \infty)$, defined by $f(x) = x^2 - 6x + 14$ is a bijective function.

Proved in lecture

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9. For $n \in \mathbb{N}$, let $A = \{a_1, a_2, a_3, \cdots, a_n\}$ be a fixed set where $a_j \neq a_i$ for any $i \neq j$, and let F be the set of all functions $f : A \to \{0, 1\}$.

What is $|F|$, the cardinality of F?

Now, for $\mathcal{P}(A)$, the power set of A, consider the function $g : F \to \mathcal{P}(A)$, defined as

$$g(f) = \{a \in A : f(a) = 1\}.$$

Is g injective? Is g surjective?

$f : A \to \{0, 1\}$

$f = \{(a_1, 0), (a_1, 1), (a_2, 0), (a_2, 1) \ldots\ldots\}$

So, every a maps to two values

So, $2 + 2 + 2 \ldots\ldots n$

So, $|F| = 2^n$

$P(A)$, $g : F \to P(A)$

$g(F) = \{a \in A : f(a) = 1\}$

Injective

Let $g(F_1) = g(F_2)$

w.r.t f will always be 1

so, $f_1 = f_2 = 1$

so, it's injective

Surjective

....

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10. Let $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by $f(n) = (2n+1, n+2)$. Check whether this function is injective and whether it is surjective. Prove your answer.

Injective

$f(n) = (2n+1, n+2)$

11.3

4. Let $n \in \mathbb{N}$, $n \geq 2$, and $a, b \in \mathbb{Z}$. Prove that if $ab \equiv 1 \pmod{n}$, then $\forall c \in \mathbb{Z}, c \not\equiv 0 \pmod{n}$ we have $ac \not\equiv 0 \pmod{n}$.

$n \in \mathbb{N}$, $n \geq 2$, $a, b \in \mathbb{Z}$

P.T if $ab \equiv 1 \pmod{n} \Rightarrow c \not\equiv 0 \pmod{n}$ we have $ac \not\equiv 0 \pmod{n}$

$ab \equiv 1 \pmod{n}$ and $\forall c, \mathbb{Z}, c \not\equiv 0 \pmod{n}$ we have $ac \equiv 0 \pmod{n}$

$$ac \equiv 0 \pmod{n}$$
$$ab \equiv 1 \pmod{n}$$

$$ac = n \cdot p$$
$$ab = 1 + n \cdot q$$

$$a^2bc \equiv np(1+nq)$$

$$abc = c \pmod{n}$$

$$abc = c + nqc$$
$$npb = c + nqc$$
$$c = n(pb - qc)$$

so, $c \mid n$

so, $c \equiv 0 \pmod{n}$

which is a contradiction