Solutions

- Name = Joshipura, Kashish
- ID = 27745629
- Test number = 382

question	version	mark out of
Q1	4	5
Q2	4	5
Q3	1	5
Q4	2	5
Q5	3	5
total	·	25

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1. $\boxed{\text{5 marks}}$ Answer the following:

(a) Negate the statement

$$P \implies Q$$

Solution: The negation is $P \wedge (\sim Q)$.

(b) Negate the following statement

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } (x < y) \vee (\sin(x) = \cos(y))$$

Solution: The negation is

$$\exists x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{R}, (x \geq y) \wedge (\sin(x) \neq \cos(y))$$

(c) Determine whether the following is true or false. Prove your answer.

$$\forall a \in \mathbb{N}, \exists b \in \mathbb{N} \text{ s.t. } (a \leq b) \implies (ab = b + a)$$

Solution: This is false. The negation is

$$\exists a \in \mathbb{N} \text{ s.t. } \forall b \in \mathbb{N}, (a \leq b) \wedge (ab \neq b + a)$$

Set $a = 1$ then for any $b \in \mathbb{N}$ we have $a \leq b$ and at the same time $ab = b \neq b + 1 = a + b$.

(d) Let P, Q, R be statements. Assume that

$$(P \iff Q) \implies R$$

is true and Q is false. What are the possible truth values of P and R?

Solution:

- If P is false, then $P \iff Q$ is true, and so R must be true.
- If P is true, then $P \iff Q$ is false, and so R can be true or false.

So $(P, R) = (F, T), (T, F), (T, T)$.
Alternatively

- If R is true, then $P \iff Q$ can be true or false, so P can be true or false.
- If R is false, then $P \iff Q$ must be false, and so P must be true.

Alternatively

$$\begin{aligned}(P \iff Q) \implies R &\equiv (P \iff F) \implies R \\ &\equiv (\sim P) \implies R \\ &\equiv P \vee R\end{aligned}$$

2. $\boxed{\text{5 marks}}$ Let $n \in \mathbb{N}$. Prove or disprove the following:

(a) If $5 \mid (n+4)$ then $5 \mid (2n^2 - 3)$.

Solution: This is false — we give a counter-example. Let $n = 1$. Then $n + 4 = 5$ but $2n^2 - 3 = 2 - 3 = -1$.

(b) If $n \equiv 1 \pmod 4$, then $n \equiv 1, 5,$ or $9 \pmod{12}$.

Solution: Assume $n \equiv 1 \pmod 4$. Then $n = 4k + 1$, for some integer k. We also have that $k = 3q + r$, where $r = 0, 1,$ or 2, so

$$n = 4k + 1 = 4(3q + r) + 1 = 12q + (4r + 1).$$

Case I: Assume $r = 0$. Then $n = 12q + 1$, so $n \equiv 1 \pmod{12}$.
Case II: Assume $r = 1$. Then $n = 12q + 5$, so $n \equiv 5 \pmod{12}$.
Case III: Assume $r = 2$. Then $n = 12q + 9$, so $n \equiv 9 \pmod{12}$.
We conclude that $n \equiv 1, 5,$ or $9 \pmod{12}$.

3. $\boxed{\text{5 marks}}$ Find all $x \in \mathbb{R}$ such that $|x+7| + |3x+1| \leq 44$.

Please prove your answer.

Solution: We consider three cases:

Case: $x \leq -7$. In this case $x + 7 < 0$ and $3x + 1 < 0$, so the inequality becomse

$$-(x+7) - (3x+1) \leq 44,$$

which is $-4x - 8 \leq 44$, or $x \geq -13$. Therefore all $x \in [-13, -7]$ satisfy the requirement.

Case: $-7 < x < -1/3$. In this case $x + 7 > 0$ and $3x + 1 < 0$, so the inequality becomse

$$(x+7) - (3x+1) \leq 44,$$

which is $-2x + 6 \leq 44$, or $x \geq -19$. Therefore all $x \in (-7, 1/3)$ satisfy the requirement.

Case: $x \geq -1/3$. In this case $x + 7 \geq 0$ and $3x + 1 \geq 0$, so the inequality becomse

$$(x+7) + (3x+1) \leq 44,$$

which is $4x + 8 \leq 44$, or $x \leq 9$. Therefore all $x \in [-1/3, 9]$ satisfy the requirement.

Combining the three cases we find that the inequality holds for $x \in [-13, 9]$.

4. $\boxed{\text{5 marks}}$ Let $n \in \mathbb{N}$. Use mathematical induction to prove that

$$2^{n+1}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq 2^{n-1}+1$$

Solution: We prove the result using induction.

- Base case: When $n = 1$ we have

  $$4\left(1-\frac{1}{2}\right) = 2 \geq 2 = 2^0 + 1$$

  and so the base case holds.

- Inductive step: Assume the result holds for $n = 1, \ldots, k$. Then

  $$\begin{aligned} 2^{n+2}\left(1-\frac{1}{2}\right)\cdots\left(1-\frac{1}{2^{n+1}}\right) &= 2^{n+1}\left(1-\frac{1}{2}\right)\cdots\left(1-\frac{1}{2^n}\right)\cdot 2\cdot\left(1-\frac{1}{2^{n+1}}\right) \\ &\geq (2^{n-1}+1)\cdot\left(2-\frac{1}{2^n}\right) \\ &= 2^n + 2 - 2^{-1} - 2^{-n} \\ &\geq 2^n + (2 - 1/2 - 1/2) \\ &= 2^n + 1 \end{aligned}$$

  So the inductive step holds.

By mathematical induction the result holds for all $n \in \mathbb{N}$.

5. $\boxed{\text{5 marks}}$ Recall that we say that a sequence (x_n) converges to L when

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n > N) \implies (|x_n - L| < \varepsilon).$$

Prove or disprove: The sequence (x_n), defined by $x_n = \dfrac{\sqrt{n^3+1}}{n}$, converges to 0.

Solution: The statement is false.

Proof. Set $\varepsilon = \frac{1}{2}$. For any $N \in \mathbb{N}$, set $n = N$. Then $n \geq N$ and:

$$|x_n - 0| = \frac{\sqrt{n^3+1}}{n} > \frac{\sqrt{n^3}}{n} = \sqrt{n} \geq 1 > \varepsilon$$

$\square$