Math 220
Section 108
Lecture 23

29th November 2022

Sources: https://personal.math.ubc.ca/~PLP/auxiliary.html
https://secure.math.ubc.ca/Ugrad/pastExams

Proofs

1. Let $a, b, c \in \mathbb{Z}$. Show that if $a^2 + b^2 = c^2$, then a or b is even.

Proof by contradiction.

Assume a & b are both odd.

Then $a = 2k+1$ & $b = 2m+1$, for some $k, m \in \mathbb{Z}$.

So if $a^2+b^2=c^2$, we can write

$$(2k+1)^2 + (2m+1)^2 = c^2$$

$$4k^2+4k+1 + 4m^2+4m+1 = c^2$$

$$4k^2+4k+4m^2+4m+2 = c^2 \qquad (*)$$

Direction A: Take equation mod 4: $0+0+0+0+2 \equiv c^2 \pmod 4$

Since c must be even, $c \equiv 0$ or $2 \pmod 4$.

Then $c^2 \equiv 0 \pmod 4$. Contradiction. $\square$

(Continued) 1. Let $a, b, c \in \mathbb{Z}$. Show that if $a^2 + b^2 = c^2$, then a or b is even.

Direction B: Since the left-hand side of (*) is even, we have $2 \mid c^2$, so $2 \mid c$ (by Euclid's lemma). Write $c = 2\ell$, for some $\ell \in \mathbb{Z}$.

Then (*) becomes $4k^2 + 4k + 4m^2 + 4m + 2 = 4\ell^2$ and we see that the right is divisible by 4, but the left-hand side is not. Contradiction. $\square$

Cardinality

Definition

Two sets have the same **cardinality** if there exists a bijection between them. For example, given sets A, B, if we can find a function $f : A \to B$ that is a bijection, then $|A| = |B|$.

Definition

If a set S has the same cardinality as $\mathbb{N}$, we say that it is **denumerable**.
If a set T is either finite or denumerable, we say that it is **countable**.
If a set is not countable, we simply say that it is **uncountable**.

Cardinality

2. Show that $|(0,1)| = |(0,\infty)|$.

$f: (0,1) \to (1,\infty)$ $\qquad$ $g: (1,\infty) \to (0,\infty)$

$f(x) = \frac{1}{x}$ $\qquad$ $g(y) = y-1$

Then $h = g \circ f : (0,1) \longrightarrow (0,\infty)$

$$h(x) = g \circ f(x) = g\left(\frac{1}{x}\right) = \frac{1}{x} - 1.$$

We prove that h is bijective:

Injective: Say that $h(x_1) = h(x_2)$

$$\Rightarrow \frac{1}{x_1} - 1 = \frac{1}{x_2} - 1$$

$$\Rightarrow \frac{1}{x_1} = \frac{1}{x_2}$$

$$\Rightarrow x_2 = x_1.$$ So h is injective.

(Continued) 2. Show that $|(0,1)| = |(0,\infty)|$.

Surjective: Given $y \in (0,\infty)$, we want to find x s.t. $h(x) = y$.

[Scratch: $y = \frac{1}{x} - 1 \Rightarrow y+1 = \frac{1}{x} \Rightarrow x = \frac{1}{y+1}$.]

Proof: Given y, let $x = \frac{1}{y+1}$.

Then $y+1 = \frac{1}{x}$ (note that $x \neq 0$)

$\Rightarrow y = \frac{1}{x} - 1$, so $y = h(x)$.

So h is surjective. So h is bijective, therefore $|(0,1)| = |(0,\infty)|$.

Final Q8 2016 WT2

3. Let S and T be two arbitrary sets. Prove that if the sets $S - T$ and $T - S$ have the same cardinality, then the sets S and T have the same cardinality.

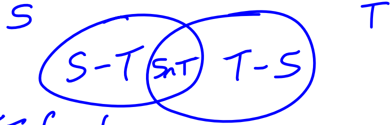
Since $|S-T| = |T-S|$,

$\exists$ bij. $f : S-T \to T-S$.

Define $g : S \to T$ s.t.

for $s \in S-T$, $g(s) = f(s)$

& for $s \in S \cap T$, $g(s) = s$.



Informal Note: $(S-T) \cup (S \cap T) = S$

& $(S-T) \cap (S \cap T) = \emptyset$,

and likewise for T.

We will show that g is bijective.

Injective: Assume $s_1 \neq s_2$, for some $s_1, s_2 \in S$.

(Continued)

(Continued) 3. Let S and T be two arbitrary sets. Prove that if the sets $S - T$ and $T - S$ have the same cardinality, then the sets S and T have the same cardinality.

Case 1: If $s_1, s_2 \in S \cap T$, then $g(s_1) = s_1$ & $g(s_2) = s_2$, so $g(s_1) \neq g(s_2)$.

Case 2: If $s_1, s_2 \in S - T$, then $g(s_1) = f(s_1)$ & $g(s_2) = f(s_2)$, and since f is injective, we have $f(s_1) \neq f(s_2)$, so $g(s_1) \neq g(s_2)$.

Case 3: If exactly one of s_1, s_2 is in $S \cap T$ & the other is not, WLOG ("without loss of generality") let $s_1 \in S \cap T$ & $s_2 \in S - T$. Then $g(s_1) \neq g(s_2)$ since $g(s_1) \in S \cap T$ & $g(s_2) \in T - S$.

Surjective: Given $t \in T$, want to find $s \in S$ s.t. $g(s) = t$.

Case 1: If $t \in S \cap T$, let $s = t$. Then $g(s) = t$. ✓

Case 2: If $t \in T - S$, then since f is surjective, $\exists\, s \in S - T$ s.t. $f(s) = t$ ✓ In summary, g is a bijection, so $|S| = |T|$. □