MATH 220 Midterm Review

UBC Undergraduate Math Society

Procedure

Introduction

Logic

Some number theory

Proof by induction

Limits

Negation

UBC Math Undergraduate Society

Location: MATH ANNEX 1119

What we do: board games (we have oh so many board games), putnam practice, math circle, exam packs, louging around, and sometimes, math.

Instagram: ums.ubc

email:ums.ubc@gmail.com

https://discord.gg/E6AYdZC3

Exam packs

$15 today

Your instructors
uki (she/they) + ryan (he/him)

- final year engineering and honors math minor students

Logic

$P \wedge Q$ → true iff $P, Q = T$

$P \vee Q$ → true if either $P, Q = T$

$P \Rightarrow Q$ → False if $P = T$, $Q = F$; True o/w

$\neg P$ → $T \to F$; $F \to T$

P	Q	R	$(P \wedge Q) \implies R$	$(P \implies R) \vee (Q \implies R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

A Logic Puzzle

Select the probability of randomly choosing the correct answer for this question:

~~a) 1/4~~
b) 1/2
c) 0
d) 1/4

No correct Answer

Number theory

$m = np + r \rightsquigarrow r \sim$ remainder

$m \equiv r \pmod{p}$

$\gcd(a,b) = 1 \iff \exists\, x, y \in \mathbb{Z}$ s.t. $ax + by = 1$

modular arithmetic

$a \in [b]_m$

$a \equiv b \pmod{m} \Rightarrow a = b + \alpha m$

$2 \cdot 4 \equiv 8 \equiv 0 \pmod{8}$

$2 \equiv 0 \pmod{8}$

*

Assume $a, b \in \mathbb{Z}$. Prove that if $\boxed{ax + by = 1}$ for some $x, y \in \mathbb{Z}$, then $\gcd(a, b) = 1$.

Pf. $m \in \mathbb{Z}$, $m \mid a$, $m \mid b$

$\hookrightarrow \boxed{m \mid by}$

$ax \equiv 1 - by \pmod{m}$

$ax \equiv 1 \pmod{m}$

$ax \equiv 0 \pmod{m}$

$\Rightarrow$ Same equivalence class implies $m = 1$

$\square$

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Assume $a, b \in \mathbb{Z}$ and p is prime. Using Bézout's identity from homework 1, prove that if $p \mid ab$, then $p \mid a$ or $p \mid b$.

Proof by induction

P is true for all n >= a

- Base case: P(a) is true
- Inductive step: If P(k) is true for k>=a, then P(k+1) is true

Caution: Take finite steps. For instance, you can't use proof by induction to show that the cardinality of integers is finite.

$P: |\mathbb{Z}| = \text{finite}$

base case: 1 is finite ✓

Inductive: If $k \in \mathbb{Z}$ is finite

$\Downarrow$

$k+1$ is finite ✓.

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$a = \sum_{k=1}^{n} k$$

$$b = n+1$$

Prove that, $\forall n \in \mathbb{N}$, $\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2$.

Base case: prove the statement is true for $n=1$

$$\sum_{k=1}^{1} k^3 = 1 = \left(\sum_{k=1}^{1} k\right)^2 \quad \checkmark$$

Inductive step: suppose the statement is true for an arbitrary natural number, k >= 1

Suppose $P(n)$ is true. for arbitrary $n \in \mathbb{N}$.

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3$$

$$\left(\sum_{k=1}^{n+1} k\right)^2 = \left(\sum_{k=1}^{n} k + n+1\right)^2 = \left(\sum_{k=1}^{n} k\right)^2 + (n+1)^2 + 2(n+1)\sum_{k=1}^{n} k$$

$$= \left(\sum_{k=1}^{n} k\right)^2 + (n+1)^2 + n(n+1)^2$$

$$= \sum_{k=1}^{n} k^3 + (n+1)^3$$

$\square$

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Prove, using induction, that $\forall n \in \mathbb{N}, 3 \mid (n^3 - n)$.

Base case: $n = 1$

$3 \mid 0$

$P(k+1)$

Inductive step:

Suppose $3 \mid k^3 - k \Rightarrow 3 \mid (k-1)k(k+1)$

Note that $(k+1)^3 - (k+1)$
$= k(k+1)(k+2)$

Case 1: $3 \mid k-1$

$\Downarrow$

$3 \mid k - 1 + 3 = k+2$

Case 2 & 3:

$3 \mid k$ or $3 \mid k+1$

$\square$

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Theorem: A statement of the form $\forall n \in \mathbb{N}; P(n)$" is true if

- The statement P(1) is true,
  and,
- given $k \geq 1$, $P(1) \wedge P(2) \wedge P(3) \wedge ... \wedge P(k) \implies P(k+1)$.

This procedure is called the *strong induction*.
Use strong induction to prove the following statement: Suppose you begin with a pile of n stones ($n \geq 2$) and split this pile into n separate piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have p and q stones in them, respectively, you compute pq. Show that no matter how you split the piles (eventually into n piles of one stone each), the sum of the products computed at each step equals $\frac{n(n-1)}{2}$.

Base: $P(2) = 1$

$$\frac{n(n-1)}{2} = \frac{2 \cdot 1}{2} = 1. \quad \checkmark$$

Inductive step: Suppose $P(2) \wedge P(3) \wedge \dots \wedge P(n-1)$ holds

Case 1: $(n) \to (1)(n-1)$

$$P(n) = 1 \times (n-1) + P(n-1)$$
$$= n-1 + \frac{(n-1)(n-2)}{2}$$
$$= \frac{n(n-1)}{2}$$

Case 2: $(n) \to (k)(l),\ k, l \geq 2$

$$P(n) = P(k) + P(l) + kl$$
$$= \frac{k(k-1)}{2} + \frac{l(l-1)}{2} + kl$$
$$= \frac{(k+l)^2 - (k+l)}{2} = \frac{n^2 - n}{2} = \frac{n(n-1)}{2}.$$

✓

$1 = d_1 \quad d_2 \quad d_3 \quad \cdots \quad d_{n-1} \quad d_n = x$

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The Rainbow Lemma (skip probably)

For any positive integer x with exactly n divisors, where n is even, we write $1 = d_1 < d_2 < \cdots < d_{n-1} < d_n = x$. d_i are distinct divisors of x for distinct $1 \leq i \leq n$. Prove that

$$x = d_1 d_n = d_2 d_{n-1} = d_3 d_{n-2} \cdots = d_{\frac{n}{2}} d_{\frac{n}{2}+1}.$$

Base: $d_1 d_n = 1 \cdot x = x$

Inductive: Suppose $d_2 d_{n-1} \neq x$ $\quad d_2 \cdot m = x \quad d_{n-1} \cdot l = x$

then case 1: $d_2 d_{n-1} > x$

$d_2 d_{n-1} > d_{n-1} l$

$d_2 > l$ ↯

case 2: $d_2 d_{n-1} < x$

$d_2 d_{n-1} < d_2 m$

$d_{n-1} < m$ ↯

$d_2 d_{n-1} = x$

Limits of sequences

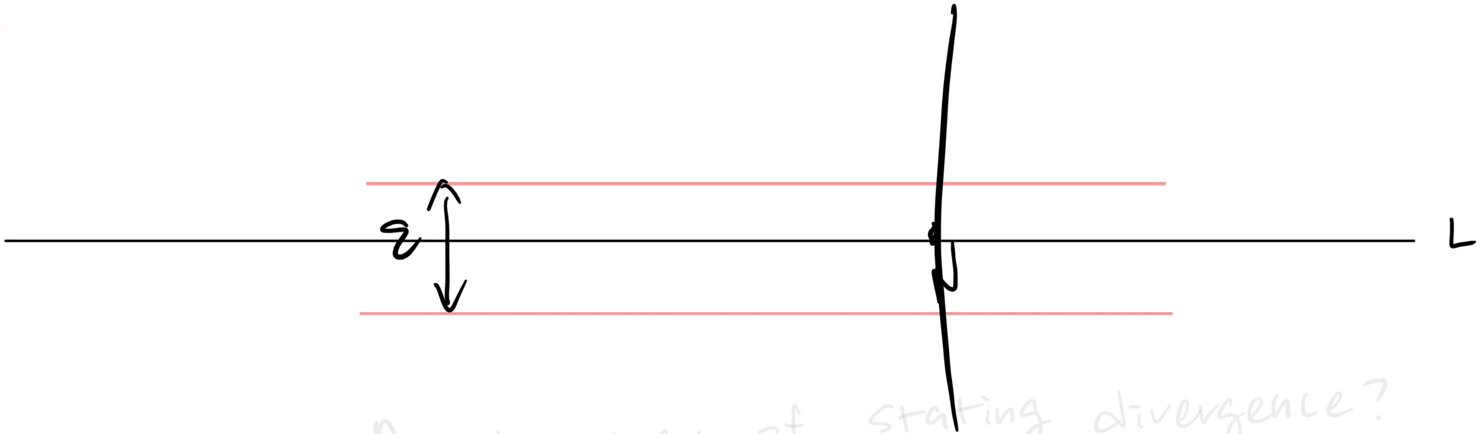
Definition 6.4.2. Let (x_n) be a sequence of real numbers. We say that (x_n) has a ***limit*** $L \in \mathbb{R}$ when

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n > N) \implies (|x_n - L| < \varepsilon).$$

In this case we say that the sequence ***converges*** to L and write

$$x_n \to L \qquad \text{or} \qquad \lim_{n\to\infty} x_n = L.$$

If the sequence doesn't converge to any number L, we say that the sequence ***diverges***.



Another way of stating divergence?

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We say that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to L if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \varepsilon.$$

Using the definition, prove that the sequence $(x_n)_{n\in\mathbb{N}} = ((-1)^n + \frac{1}{n})_{n\in\mathbb{N}}$ does not converge to 0.

$\exists\, \varepsilon > 0$, s.t. $\forall N \in \mathbb{N}$, $\exists\, n \geq N$ s.t. $|x_n| \geq \varepsilon$.

if $2 \mid n$

$(-1)^n + \frac{1}{n} = 1 + \frac{1}{n} > 1$

Choose $\varepsilon = 1$

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Let $a_n = \dfrac{2n^2+n+14}{2n^2+11}$. Show, using the definition of convergence, that $a_n \to 1$

$$a_n = 1 + \frac{n+3}{2n^2+11}$$

Statement: $\forall \varepsilon > 0$,

$$|a_n - 1| = \frac{n+3}{2n^2+11} < 1 \quad \forall n \in \mathbb{N} \text{ takes cares of all } \varepsilon \geq 1$$

$$\frac{n+3}{2n^2+11} < \frac{n+3}{2n^2} < \frac{n+3}{n^2} = \frac{1}{n} + \frac{3}{n^2}$$

Take $N_1 = \frac{2}{\underline{\varepsilon}}$, $N_2 = \frac{\sqrt{6}}{\sqrt{\varepsilon}}$, $\underline{N} = \max\{N_1, N_2\}$.

$\frac{1}{n} < \frac{\varepsilon}{2}$ $\quad \frac{3}{n^2} < 3 \cdot \left(\frac{\sqrt{\varepsilon}}{\sqrt{6}}\right)^2 = \frac{\varepsilon}{2}$, Then $|a_n - 1| < \varepsilon$. $\square$

Limits of functions

Definition 6.4.8. Let $a, L \in \mathbb{R}$ and let f be a real-valued function. We say that the ***limit*** of f as x approaches a is L when

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < |x-a| < \delta) \implies (|f(x) - L| < \varepsilon).$$

In this case we write

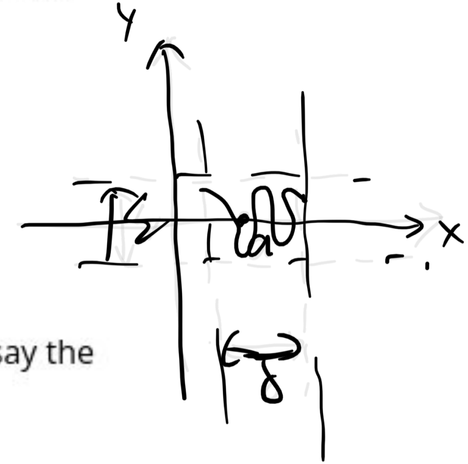
limit at a may exist
even if f is not defined at a.

$$\lim_{x \to a} f(x) = L \qquad \text{or sometimes} \qquad f(x) \xrightarrow[x \to a]{} L$$

and say that f ***converges*** to L as x approaches a. We also sometimes say the limit of f as x goes to a is L, which we denote by

$$f(x) \to L \text{ as } x \to a.$$

If f does not converge to any finite limit L as x approaches a, then we say that f ***diverges*** as x approaches a.



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Limit Problem

this may not be constant

Suppose a real-valued function f satisfies $f(x) = f(x+1)$ for all $x \in \mathbb{R}$. Prove that $f'(x) = f'(x+1)$ for all $x \in \mathbb{R}$ using $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad f'(x+1) = \lim_{h\to 0} \frac{f(x+1+h)-f(x+1)}{h}$$

$$f'(x+1) - f'(x) = \lim_{h\to 0} \frac{\cancel{f(x+h+1)} - \cancel{f(x+h)} + \cancel{f(x)} - \cancel{f(x+1)}}{h} = 0$$

$$f(x+h+1) = f(x+h) \qquad f(x) = f(x+1)$$

$$\therefore f'(x) = f'(x+1) \qquad \square$$

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10. **[8 pts]** Suppose that $f : \mathbb{N} \to \mathbb{R}$ is a bounded function and that $\{a_n\}$ is a sequence that converges to 0. Prove that $\lim_{n\to\infty} f(n)a_n = 0$.

$|f(n)| \le M \quad \forall n \in \mathbb{N}$

Restate definition: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a_n| < \varepsilon$.

arbitrary $\varepsilon > 0$, Take $N = \frac{\varepsilon}{M}$.

then $|f(n)a_n| < |M| \left|\frac{\varepsilon}{M}\right| = \varepsilon$. $\square$

Negation

- Writing sentences in symbolic logic notations

"8 is even and 5 is prime"
"If a function f is differentiable everywhere then whenever $x \in \mathbb{R}$ is a local maximum of f we have $f'(x) = 0$"

- Negating sentences

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Function types

In this question, we are going to call a function, $f : \mathbb{R} \to \mathbb{R}$, *type A*, if $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $y \geq x$ and $|f(y)| \geq 1$. We also say that a function, g, is *type B* if $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, if $y \geq x$, then $|g(y)| \geq 1$.
Prove or find a counterexample for the following statements.
a) If a function is type A, then it is type B.
b) If a function is type B, then it is type A.

Feedback form (1 min)