PLP - 12

TOPIC 12 —PROOF BY CASES

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PROOF BY CASES

ANOTHER EQUIVALENCE

PROPOSITION:

$$(P \vee Q) \implies R \quad \equiv \quad (P \implies R) \wedge (Q \implies R)$$

You can prove this with a truth-table (tedious) or via equivalences (good exercise).

Useful because we can split the **hypothesis** into cases.

$$\left(n \in \mathbb{N}\right) \implies \left(n^2 + 5n - 7 \text{ is odd}\right)$$
$$\left(n \text{ is even}\right) \vee \left(n \text{ is odd}\right) \implies \left(n^2 + 5n - 7 \text{ is odd}\right)$$
$$\underbrace{\left(n \text{ is even}\right) \implies \left(n^2 + 5n - 7 \text{ is odd}\right)} \text{ and } \underbrace{\left(n \text{ is odd}\right) \implies \left(n^2 + 5n - 7 \text{ is odd}\right)}$$

We can prove each **case** in turn — **proof by cases**

PROOF BY CASES

PROPOSITION:

Let $n \in \mathbb{Z}$ then $n^2 + 5n - 7$ is odd.

PROOF.

Assume the hypothesis is true, so that $n \in \mathbb{Z}$. Hence n is even or odd.

- *Case 1:* Assume that n is even, so that $n = 2k$ for some $k \in \mathbb{Z}$. Hence $n^2 + 5n - 7 = 4k^2 + 10k - 7 = 2(2k^2 + 5k - 4) + 1$. Thus $n^2 + 5n - 7$ is odd.
- *Case 2:* Assume that n is odd, so that $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$. Hence $n^2 + 5n - 7 = 4\ell^2 + 4\ell + 1 + 10\ell + 5 - 7 = 2(2\ell^2 + 7\ell - 1) + 1$. Thus $n^2 + 5n - 7$ is odd.

Since $n^2 + 5n - 7$ is odd in *both cases*, the result holds.

WHAT CAN GO WRONG

Proof by cases can be tricky

- tell the **reader** that you are doing **case analysis**
- make sure you get *all* the cases — a common mistake
- cases are often very similar — be very careful of skipping steps.

  without loss of generality or **WLOG** is a good source of errors

Dangerous phrases in mathematics

- Without loss of generality…
- Clearly…
- Obviously…
- A quick calculation shows that…
- It is easy to show that…

ANOTHER EXAMPLE

PROPOSITION:

Let $n \in \mathbb{Z}$. If $3 \mid n^2$ then $3 \mid n$.

Scratch work —This smells of the contrapositive: $(3 \nmid n) \implies (3 \nmid n^2)$.

- Recall Euclidean division — every integer n can be written uniquely as

$$n = 3a \qquad n = 3a + 1 \qquad n = 3a + 2 \qquad \text{for some } a \in \mathbb{Z}$$

- If $3 \nmid n$ we must have either $n = 3a + 1$ or $= 3a + 2$ — our cases.
- If $n = 3a + 1$ then $n^2 = 9a^2 + 6a + 1 = \ldots.$
- If $n = 3a + 2$ then $n^2 = 9a^2 + 12a + 4 = \ldots.$

Time to write up.

WRITE IT UP NICELY

Let $n \in \mathbb{Z}$. If $3 \mid n^2$ then $3 \mid n$.

PROOF.

We prove the contrapositive, so assume that $3 \nmid n$. By Euclidean division, we know that $n = 3a + 1$ or $n = 3a + 2$.

- *Case 1:* Let $n = 3a + 1$, then $n^2 = 9a^2 + 6a + 1 = 3(3a^2 + 2a) + 1$ and so is not divisible by 3.
- *Case 2:* Let $n = 3a + 2$, then $n^2 = 9a^2 + 12a + 4 = 3(3a^2 + 4a + 1) + 1$ and so is not divisible by 3.

Since $3 \nmid n^2$ in both cases, the result holds.

if $a \neq b$ then $\frac{a+b}{2} > a$ or $\frac{a+b}{2} > b$

$$a \neq b \implies \frac{a+b}{2} > a \lor \frac{a+b}{2} > b$$

$i \in \mathbb{Z}^+$ $\quad a > b \quad\quad\quad\quad b > a$

$a = b + i \quad\quad\quad\quad b = a + j \quad j \in \mathbb{Z}^+$

$\frac{\cancel{b}a + i + b}{2} \quad\quad\quad\quad a + \frac{\cancel{j}+a}{2}$

$\Rightarrow \frac{2b\ i}{2} \quad\quad\quad\quad \Rightarrow \frac{2a + j}{2}$

$\Rightarrow b + i/2 \quad\quad\quad\quad \Rightarrow a + j/2$

$> b \quad\quad\quad\quad > a$

$x \in R$, if x is irrational, then $x^{1/3}$ is irrational

$x^{1/3}$ is rational $\implies x$ is rational

$x^{1/3} = \frac{p}{q}$ $\quad\quad\quad\quad p, p^3, a \in \mathbb{Z}$

$x = \frac{p^3}{q^3}$ $\quad\quad\quad\quad q, q^3, b \in N$

for $\quad a = p^3 \quad\quad a, b \in R$
$\quad\quad\; b, = q^3$

so, $x = \frac{a}{b}$ $\quad \blacksquare$

$x > 0 \quad , \quad x - \frac{3}{x} > 2 \Rightarrow x > 3$

$x \in \mathbb{R}$

$$x^2 - 3 > 2x$$

$$x^2 - 3 - 2x > 0$$

$$x^2 - 2x - 3 > 0$$

$$x^2 - 3x + x - 3 > 0$$

$$(x+1)(x-3) > 0$$

So, $\quad x + 1 > 0 \text{ or } x - 3 \geq 0$

$$x > -1 \text{ or } x > 3$$

∎

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if $n \in \mathbb{Z} \Rightarrow n^2 + 3n + 8$ is even

Case 1: $n = 2k$

$$n^2 + 3n + 8 = 4k^2 + 6k + 8 = 2(2k^2 + 3k + 4)$$

$= \text{even}$

Case 2: $n = 2\cancel{k}j + 1$

$$n^2 + 3n + 8 = 4j^2 + 1 + 4j + 6j + 3 + 8$$

$$= 2(2j^2 + 5j + 6)$$

$= \text{even}$ ∎

$K, j \in \mathbb{Z}$

$n \in \mathbb{Z}$ $\rightarrow$ $2n^2+n+1$ is not divisible by 3

Case 1: $n = 3k$

$\Rightarrow 2(3k)^2 + 3k + 1$

$\Rightarrow 3(6k^2 + k) + 1 \Rightarrow 3y + 1$

not divisible

Case 2: $n = 3k+1$

$\Rightarrow 2(3k+1)^2 + 3k + 1 + 1$

$\Rightarrow 2(9k^2 + 6k + 1) + 3k + 2$

$\Rightarrow 3(6k^2 + 6k + 1 + k) + 1 \Rightarrow 3m + 1$

not divisible

Case 3: $n = 3k+2$

$\Rightarrow 2(3k+2)^2 + 3k + 2 + 1$

$\Rightarrow 2(9k^2 + 4 + 12k) + 3k + 3$

$\Rightarrow 3(6k^2 + 2 + 9k + 1) + 2 \Rightarrow 3i + 2$

not divisible

$\blacksquare$