Solutions to Homework 11:

1. *Proof.* (a) Given that $f : \{0,1\} \to \mathbb{N}$ we know that S would be a set of functions which would equal the cartesian product of $\mathbb{N}$ since it is mapped from $\{0,1\}$.
   We know that the cartesian product of two countable set is countable hence $f : \{0,1\} \to \mathbb{N}$ is countable.

   (b) Let's consider a function $G : S \to \mathcal{P}(\mathbb{N})$ such that $G(n) = \{n | f(n) = 0\}$
   So, we try to prove that this is bijective
   Injective: Consider $a, b \in S$ so that $G(a) = G(b)$
   This is because the function produces 0 at every point.
   Hence $a = b$ and so this is injective.
   We can prove this for a function producing 1 similarly.

   Surjective: Consider a function
   $$f(n) = \begin{cases} 1 & n \in X \\ 0 & n \notin X \end{cases} \quad \text{Where } X \subseteq \mathbb{N}$$
   We know that $f : \mathbb{N} \to \{0,1\}$
   So by definition we have that it is surjective

   Now G is bijective so we can say that $|S| = |\mathcal{P}(\mathbb{N})|$
   However, we know that $\mathcal{P}(\mathbb{N})$ is uncountable and so $|S|$ is also uncountable

   □

2. *Proof.* (a) A is countable but B is uncountable then B-A is uncountable.
   By proof by contradiction it's negation would be
   A is countable but B is uncountable and B-A is countable.
   Now we know that $B \subseteq B - A$ by definition.
   But, B-A is countable and B is uncountable, so this is not possible and hence our assumption is false
   So by proof by contradiction we have that if A is countable but B is uncountable then B-A is uncountable.

   (b) We assume that all the numbers in the range (a,b) are uncountable but there are countable irrational numbers between them.
   However we know that if a and b are uncountable then a-b i.e the numbers between them is also uncountable. However here we have assumed that there are countable irrational numbers between them.
   Hence our assumption is false and there are uncountable irrational numbers between them.

   □

3. *Proof.* Here we need to prove that $\mathbb{R}$ and $\mathbb{R}^+$ are equinumerous
   Let's consider a function $f : \mathbb{R} \to \mathbb{R}^+$ such that $f(x) = e^x$
   Injective: Let's consider $x, y \in \mathbb{R}$

So, $f(x) = f(y)$
$e^x = e^y$
$x = y$
Hence f is injective

Surjective: Let's consider x = ln(y)
Hence $f(x) = e^x = e^{ln(y)}$
So, $f(x) = y$
Hence f is surjective

Now $f : \mathbb{R} \to \mathbb{R}^+$ is bijective and so $\mathbb{R}$ and $\mathbb{R}^+$ are equinumerous □

4. (a) There exists an injection $f : S \to T$ since $|S| \leq |T|$
So, let us take another function $G : \mathcal{P}(S) \to \mathcal{P}(T)$ and try to prove that it is injective
Now, let $G(A) = f(A) = \{f(a)|a \in A\}$
Now since $G(A) = f(A)$ and the fact that f is injective we can take $g(A) = g(B)$
Let $x \in A$ such that $f(x) = y \in f(A) = g(A)$
Similarly let $p \in B$ such that $f(p) = q \in f(B) = g(B)$
But we know that f is injective and so $f(x) = f(p)$
So, $x = p$ and consequently $A = B$. Therefore G is injective too and hence
If $|S| \leq |T|$ then $|\mathcal{P}(S)| \leq |\mathcal{P}(T)|$

(b) We just proved that $|S| \leq |T|$ then $|\mathcal{P}(S)| \leq |\mathcal{P}(T)|$ and so we just need to prove that G is surjective
So, now we also know that the set T is nothing but $\{f^{-1}(b)|b \in B\}$
Now let $x \in G(A)$ and so there exists $a \in A$ such that $f^{-1}(x) = a$
Now we know that $a = f^{-1}(b)$
So, $f^{-1}(x) = f^{-1}(b)$ or $x = b$,
now since $b \in B$ we can say that $x \in B$
So, $G(A) \subseteq B$

Now let's assume that $x \in B$ and so $f^{-1}(x) = a \in A$
So, $x = f(a) \in f(A)$
Now we know that $f(A) = G(A)$ and so $x \in G(A)$
Hence $B \subseteq G(A)$

Now since $G(A) \subseteq B$ and $B \subseteq G(A)$ we have that $G(A) = B$
Hence the function G is surjective and so we can say that
If $|S| = |T|$ then $|\mathcal{P}(S)| = |\mathcal{P}(T)|$

5. *Proof.* We need to show that there exists infinitely many pairs of a,b such that $17^a - 17^b$ is divisible by 2022.
Consider the set $\{17^1, 17^2, 17^3, ..., 17^{2022}, 17^{2023}\}$

Here there are 2022 equivalence classes of mod 2022.
By pigeon hole principle we know that
$17^a \equiv 17^b \pmod{2022}$ where $\exists a, b \in \mathbb{N}, a \geq 1, b \leq 2023, a \neq b$
we can also write this as $17^a - 17^b \equiv 0 \pmod{2022}$
and hence $2022|17^a - 17^b$
We can repeat this infinitely many times similarly using the Pigeon Hole Principle and hence there exists infinitely many pairs of a,b such that $17^a - 17^b$ is divisible by 2022.

$\square$

6. *Proof.* Let's consider the function $f(x) = log(-x - \sqrt{29})$ where $f : (-\infty, -\sqrt{29}) \to \mathbb{R}$ and the function $g(y) = -e^y - \sqrt{29}$ where $g : \mathbb{R} \to (-\infty, -\sqrt{29})$
So,

$$f \circ g = log(-(-e^y - \sqrt{29}) - \sqrt{29}) \tag{1}$$
$$f \circ g = log(e^y) \tag{2}$$
$$f \circ g = y \tag{3}$$

And let's consider $g \circ f$ so,

$$g \circ f = -e^{(log(-x-\sqrt{29}))} - \sqrt{29} \tag{4}$$
$$g \circ f = -(x - \sqrt{29}) - \sqrt{29} \tag{5}$$
$$g \circ f = x \tag{6}$$

Hence f and g are two sided inverses of each other and so they are bijective
This also means that $(-\infty, -\sqrt{29})$ and $\mathbb{R}$ are equinumerous

$\square$

7. This is false
Consider $A = \mathbb{N}, B = \{0, 1\}$, and $C = \{1, 2, 3\}$
Now $|A \times B| = |A \times C|$ because $|\mathbb{N} \cup \{0, 1\}|$ and $|\mathbb{N} \cup \{1, 2, 3\}|$ both are denumerable i.e their cardinality is equal to $|\mathbb{N}|$
However we can observe that $|B| = 2$ and $|C| = 3$ and hence they are not equal i.e $|B| \neq |C|$

8. Let's assume that $f^{-1}(\{a\})$ is countable
We also know that $f : \mathbb{R} \to A$ and $a \in A$ and so
we can write the real number set as

$$\bigcup_{a \in A} f^{-1}(\{a\}) = \mathbb{R} \tag{7}$$

However $\mathbb{R}$ is an uncountable set and hence even $f^{-1}(\{a\})$ is uncountable. So our assumption is false and by proof by contradiction we have that
$f^{-1}(\{a\})$ is uncountable