Math 220
Section 108
Lecture 17

3rd November 2022

Source: `https://personal.math.ubc.ca/~PLP/auxiliary.html`

Introduction to Equivalence Classes

Definition (Definition 9.3.3 of PLP)

Given an equivalence relation $\boldsymbol{R}$ defined on a set A, we define the equivalence class of $x \in A$ (with respect to $\boldsymbol{R}$) to be the set of elements related to x:

$$[x] = \{y \in A \ : \ y\boldsymbol{R}x\}.$$

Example

Let $\boldsymbol{R}$ be the equivalence relation on $\mathbb{N}$ where $x\boldsymbol{R}y$ iff x and y have the same last digit (in base 10). Find $[0]$ and $[1]$.

Answer:

$$[0] = \{10, 20, 30, 40, \ldots\}, \text{ and}$$
$$[1] = \{1, 11, 21, 31, \ldots, 91, 101, \ldots\}.$$

2. A relation R on $\mathbb{Z}$ is defined by aRb if $7a^2 \equiv 2b^2 \pmod 5$.
(a) Prove that R is an equivalence relation.
(b) Determine the distinct equivalence classes [0] and [1], simplify your answer as much as possible.

$7a^2 = 5n + 2b^2$

$7a^2 - 2b^2 = 5n$ $\forall n \in \mathbb{Z}$

(a) Reflexive

Consider $a = b$

So, $7a^2 - 2a^2 = 5a^2$

$5 \mid 5a^2$, $\because a^2 \in \mathbb{Z}$

So, R is reflexive

Symmetric

Consider aRb

So, $7a^2 - 2b^2 = 5n$

$2b^2 - 7a^2 = -5n$

$7b^2 - 7a^2 = 5(-n + 5b^2)$

$7b^2 - 2a^2 = 5(-n + 5b^2 + 5a^2)$

So, $5 \mid 7b^2 - 2a^2$, $\because 7b^2 - 2a^2 \in \mathbb{Z}$

& hence bRa. So, R is symmetric

Transitive

Let aRb & bRc

So, $7a^2 - 2b^2 = 5n$ — ①

$7b^2 - 2c^2 = 5m$, $\forall m \in \mathbb{Z}$ — ②

So, adding ① + ② $\Rightarrow$ $7a^2 - 2c^2 = 5(n + m + b^2)$

$5 \mid 7a^2 - 2c^2$, $\because 7a^2 - 2c^2 \in \mathbb{Z}$

So, aRc & hence R is transitive

So, it's equivalent relation

(Continued) 2. $\boldsymbol{R}$ on $\mathbb{Z}$ is defined by $a\boldsymbol{R}b$ if $7a^2 \equiv 2b^2 \pmod 5$.
(a) Prove that $\boldsymbol{R}$ is an equivalence relation.
(b) Determine the distinct equivalence classes [0] and [1].

(b) If $aR0$, then $7a^2 \equiv 0 \pmod 5$

So, $5 \mid 7a^2$

So, $5 \mid a^2 \Leftrightarrow 5 \mid a$ or $5 \mid a$

So, $5 \mid a$

$[0] = \{5k \mid k \in \mathbb{Z}\}$

If $aR1$, then $7a^2 \equiv 2 \pmod 5$

$\Updownarrow$

So, $a^2 \equiv 1 \pmod 5$

$5 \mid a^2 - 1 = (a-1)(a+1)$

So, $5 \mid a-1$ or $5 \mid a+1$

$[1] = \{5k-1 \mid k \in \mathbb{Z}\} \cup \{5j+1 \mid j \in \mathbb{Z}\}$

(Continued 2/2)

(Continued) 2. $\boldsymbol{R}$ on $\mathbb{Z}$ is defined by $a\boldsymbol{R}b$ if $7a^2 \equiv 2b^2 \pmod 5$.
(a) Prove that $\boldsymbol{R}$ is an equivalence relation.
(b) Determine the distinct equivalence classes [0] and [1].

Old final question

3. Let $\boldsymbol{R}$ be a relation on $\mathbb{R}$ defined as

$$\boldsymbol{R} = \{(a, b) : \cos^2(a) + \sin^2(b) = 1\}.$$

(a) Prove that $\boldsymbol{R}$ is an equivalence relation.
(b) For $\theta \in \mathbb{R}$, write the equivalence class $[\theta]$.
Hint: For part (b), you can try to visualize it on the unit circle.

(Continued 1/2)

(Continued) 3. Define: $\boldsymbol{R} = \{(a, b) \in \mathbb{R} \times \mathbb{R} : \cos^2(a) + \sin^2(b) = 1\}$.
(a) Prove that $\boldsymbol{R}$ is an equivalence relation.
(b) For $\theta \in \mathbb{R}$, write the equivalence class $[\theta]$.

(Continued 2/2)

(Continued) 3. Define: $\boldsymbol{R} = \{(a, b) \in \mathbb{R} \times \mathbb{R} : \cos^2(a) + \sin^2(b) = 1\}$.
(a) Prove that $\boldsymbol{R}$ is an equivalence relation.
(b) For $\theta \in \mathbb{R}$, write the equivalence class $[\theta]$.

Partitions

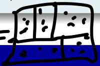
Definition (Definition 9.3.11 of PLP)

A **partition** of a set A is a collection $\mathcal{P}$ of non-empty subsets of A, so that
- if $x \in A$, then there exists $X \in \mathcal{P}$ so that $x \in X$, and
- if $X, Y \in \mathcal{P}$, then either $X \cap Y = \emptyset$ or $X = Y$.

$\mathcal{P}$ is different from $\mathcal{P}(A)$

No overlap of elements

Set A



Theorem (Theorem 9.3.12 of PLP)

Let $\boldsymbol{R}$ be an equivalence relation on A. The set of equivalence classes of $\boldsymbol{R}$ forms a partition of A. That is, $\mathcal{P} = \{[x] \mid x \in A\}$ is a partition of A.

Example

Define the equivalence relation $\boldsymbol{R}$ on $\mathbb{N}$ such that $a\boldsymbol{R}b$ if $a \equiv b \pmod 2$. Note that

$$[1] = \{1, 3, 5, 7, \dots\} \quad \text{and} \quad [2] = \{2, 4, 6, \dots\}.$$

The partition of $\mathbb{N}$ arising from $\boldsymbol{R}$ is

$$\mathcal{P} = \{[1], [2]\} = \Big\{\{1, 3, 5, 7, \dots\}, \{2, 4, 6, \dots\}\Big\}.$$

Partitions

4. Determine the partitions of $\mathbb{N}$ to which the following equivalence relations $\mathcal{R}$ correspond:

(a) $\mathcal{R} = \mathbb{N} \times \mathbb{N}$.

(b) $\mathcal{R} =$ The smallest possible equivalence relation on $\mathbb{N}$.

Continued

(Continued) 4. Determine the partitions of $\mathbb{N}$ to which the following equivalence relations $\mathcal{R}$ correspond:
(a) $\mathcal{R} = \mathbb{N} \times \mathbb{N}$.
(b) $\mathcal{R} =$ The smallest possible equivalence relation on $\mathbb{N}$.

Partitions

5. Suppose $\mathcal{P}$ is a partition of a set A. Define a relation $\boldsymbol{R}$ on A where $x\boldsymbol{R}y$ if $x, y \in X$ for some $S \in \mathcal{P}$. Prove $\boldsymbol{R}$ is an equivalence relation on A.

(Continued)

(Continued) 5. Suppose $\mathcal{P}$ is a partition of a set A. Define a relation $\boldsymbol{R}$ on A where $x\boldsymbol{R}y$ if $x, y \in X$ for some $S \in \mathcal{P}$. Prove $\boldsymbol{R}$ is an equivalence relation on A.

Modular arithmetic

6. Fix a natural number n. Define aRb if $n \mid (a - b)$. We write this as $a \equiv b \pmod{n}$; note that this is an equivalence relation.
(a) For $a, b \in \mathbb{N}$, show that if $x \in [a]$ and $y \in [b]$, then $x - y \in [a - b]$.
(b) Assume that $n = p$, a prime. Show that for $x \in [a]$, where $[a] \neq [0]$, there exists an integer d such that $xd \in [1]$.

Continued

(Continued) 6. Fix a natural number n. Define $a \equiv b \pmod{n}$ if $n \mid (a - b)$.
(a) For $a, b \in \mathbb{N}$, show that if $x \in [a]$ and $y \in [b]$, then $x - y \in [a - b]$.
(b) Assume that $n = p$, a prime. Show that for $x \in [a]$, where $[a] \neq [0]$, there exists an integer d such that $xd \in [1]$.