Math 220
Section 108
Lecture 18

8th November 2022

Source: `https://personal.math.ubc.ca/~PLP/auxiliary.html`

Recall - Equivalence Classes

Definition (Definition 9.3.3 of PLP)

Given an equivalence relation $\boldsymbol{R}$ defined on a set A, we define the **equivalence class** of $x \in A$ (with respect to $\boldsymbol{R}$) to be the set of elements related to x:

$$[x] = \{y \in A \ : \ y\boldsymbol{R}x\}.$$

Variation of an old final question

3. Let $\boldsymbol{R}$ be a relation on $\mathbb{R}$ defined as

$$\boldsymbol{R} = \{(a, b) : \cos^2(a) + \sin^2(b) = 1\}.$$

(a) Prove that $\boldsymbol{R}$ is an equivalence relation.
(b) For $\theta \in [0, \pi/2]$, find the equivalence class $[\theta]$.

$\cos^2 x + \sin^2 x = 1$

trig identity

(a) reflexive: $\forall a \in \mathbb{R}$, $\cos^2(a) + \sin^2(a) = 1$ since it is a trig identity. So aRa.

symmetric: Given aRb, we have

$$\cos^2(a) + \sin^2(b) = 1$$

$$\Rightarrow \left(1 - \sin^2(a)\right) + \left(1 - \cos^2(b)\right) = 1$$

$$\Rightarrow 2 = 1 + \sin^2 a + \cos^2 b$$

$$\Rightarrow 1 = \cos^2 b + \sin^2 a,$$

so bRa.

(Continued) 3. Define: $\boldsymbol{R} = \{(a,b) \in \mathbb{R} \times \mathbb{R} : \cos^2(a) + \sin^2(b) = 1\}$.
(a) Prove that $\boldsymbol{R}$ is an equivalence relation.
(b) For $\theta \in [0, \pi/2]$, find the equivalence class $[\theta]$.

(a) (ctd.) transitive: Given aRb & bRc, some $a,b,c \in \mathbb{R}$,
we have ① $\cos^2 a + \sin^2 b = 1$ & ② $\cos^2 b + \sin^2 c = 1$.

①+②: $\cos^2 a + \underbrace{\sin^2 b + \cos^2 b}_{1} + \sin^2 c = 2$

$\Rightarrow \cos^2 a + 1 + \sin^2 c = 2$

$\Rightarrow \cos^2 a + \sin^2 c = 1$,

$\Rightarrow aRc$.

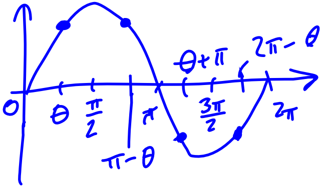
(b) Consider $[\theta] = \{t \in \mathbb{R} \mid tR\theta\}$.
$\Rightarrow \cos^2 t + \sin^2 \theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2 t = \sin^2 t$.

(Continued) 3. Define: $\boldsymbol{R} = \{(a,b) \in \mathbb{R} \times \mathbb{R} : \cos^2(a) + \sin^2(b) = 1\}$.

(a) Prove that $\boldsymbol{R}$ is an equivalence relation.

(b) For $\theta \in [0, \pi/2]$, find the equivalence class $[\theta]$.

(b) (Cont'd.) So $|\sin\theta| = |\sin t|$.

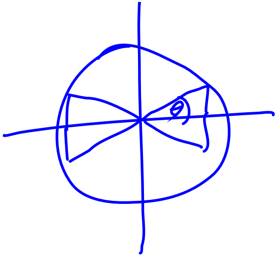


If $\theta \in (0, \frac{\pi}{2})$:

$$[\theta] = \{\theta + 2\pi k,\ \pi - \theta + 2\pi k,\ \theta + \pi + 2\pi k,\ 2\pi - \theta + 2\pi k \mid k \in \mathbb{Z}\}$$

If $\theta = 0$: $[\theta] = \{\pi k \mid k \in \mathbb{Z}\}$

If $\theta = \frac{\pi}{2}$: $[\theta] = \{\frac{\pi}{2} + \pi k \mid k \in \mathbb{Z}\}$.



Recall - Partitions

Definition (Definition 9.3.11 of PLP)

A **partition** of a set A is a collection $\mathcal{P}$ of non-empty subsets of A, so that

- if $x \in A$, then there exists $X \in \mathcal{P}$ so that $x \in X$, and
- if $X, Y \in \mathcal{P}$, then either $X \cap Y = \emptyset$ or $X = Y$.

$\{1,2,3\} = A$ $\qquad$ $1 = x$

Theorem (Theorem 9.3.12 of PLP)

Let $\boldsymbol{R}$ be an equivalence relation on A. The set of equivalence classes of $\boldsymbol{R}$ forms a partition of A. That is, $\mathcal{P} = \{[x] \mid x \in A\}$ is a partition of A.

$\{1,2\} = X$

$\{3\}$ $\qquad$ $X \cap Y = \emptyset$

4. (a) Give an example of a partition $\mathcal{P}$ of $A = \{1, 2, 3, \ldots, 9, 10\}$ that has exactly four elements.
(b) Is the power set $\mathcal{P}(A)$ a partition of A?
(c) Is a partition of A a subset of the power set $\mathcal{P}(A)$?

(a) $\mathcal{P} = \{\{1,10\}, \{2,3,4\}, \{8\}, \{5,6,7,9\}\}$.

(b) $\emptyset \in \mathcal{P}(A)$, so it can't be a partition.
$\mathcal{P}(A) - \emptyset$: This is still not a partition, since $\{1,2\}, \{2,3\} \in \mathcal{P}(A) - \emptyset$ and they overlap, i.e. $\{1,2\} \cap \{2,3\} \neq \emptyset$.

(c) Yes. Every element of $\mathcal{P}$ is a subset of A, so $\mathcal{P} \subseteq \mathcal{P}(A)$.

$|\mathcal{P}| = 4$

$\#\mathcal{P} = 4$

Partitions

5. Suppose $\mathcal{P}$ is a partition of a set A. Define a relation $\boldsymbol{R}$ on A where $x\boldsymbol{R}y$ if $x, y \in S$ for some $S \in \mathcal{P}$. Prove $\boldsymbol{R}$ is an equivalence relation on A.

reflexive: Given any $a \in A$, aRa since a is in the same element of $\mathcal{P}$ as itself.

symmetric: If aRb, then $a, b \in S$, for some $S \in \mathcal{P}$. So $b, a \in S$, some $S \in \mathcal{P}$, so bRa.

transitive: If aRb & bRc, $\exists\ S_1, S_2 \in \mathcal{P}$ s.t. $a, b \in S_1$ & $b, c \in S_2$. Since $b \in S_1 \cap S_2$, and so $S_1 \cap S_2$

(Continued)

(Continued) 5. Suppose $\mathcal{P}$ is a partition of a set A. Define a relation $\boldsymbol{R}$ on A where $x\boldsymbol{R}y$ if $x, y \in \cancel{X}$ for some $S \in \mathcal{P}$. Prove $\boldsymbol{R}$ is an equivalence relation on A.

is non-empty. Therefore, we must $S_1 = S_2$,

and so $a, b, c \in S_1$. So aRc.

So R is an equivalence relation.