Mathematics 220 — Homework 4

- Contains 8 questions on 2 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.
- 1. Prove that for every integer $n \ge 0$, the sum $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.
- 2. Recall Bézout's identity: Let $a, b \in \mathbb{Z}$ such that a and b are not both zero. Then there exists $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$.

Use this result to prove the following result:

Let $a, b, c \in \mathbb{Z}$ such that gcd(a, b) = 1. Then

$$(a \mid bc) \implies (a \mid c).$$

- 3. Let $P \subset \mathbb{N}$ be the set of prime numbers $P = \{2, 3, 5, 7, 11, \ldots\}$. Determine whether the following statements are true or false. Prove your answers ("true" or "false" is not sufficient).
 - (a) $\forall x \in P, \forall y \in P, x + y \in P$.
 - (b) $\forall x \in P, \exists y \in P \text{ such that } x + y \in P.$
 - (c) $\exists x \in P$ such that $\forall y \in P$, $x+y \in P$. Again, we leverage the fact that every number in P is odd except 2.
 - (d) $\exists x \in P$ such that $\exists y \in P, x + y \in P$.
- 4. Prove the following statement: For every positive number ϵ there is a positive number M such that

$$\left| \frac{2x^2}{x^2 + 1} - 2 \right| < \epsilon$$

whenever $x \geq M$.

5. We say that a function $f: \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if $\lim_{x \to a} f(x) = f(a)$. Let

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & \text{if } x \neq 0\\ 0, & \text{if } x = 0. \end{cases}$$

Is f continuous at x = 0?

Warning: You must prove this from first-principles (ie with $\epsilon - \delta$).

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6. We say that a sequence (x_n) is bounded if

$$\exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M.$$

Prove that if a sequence (x_n) converges to 0, then (x_n) is bounded.

Hint: It helps to break the problem into two cases, small n and big n.

7. A function is said to be unbounded on the interval (a, b) if

$$\forall M \in \mathbb{R}, \exists t \in (a, b) \text{ s.t. } |f(t)| > M.$$

Prove that $\log x$ is unbounded on (0,1).

You may assume basic properties of logarithms and exponentials in your solution.

8. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to L if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \epsilon.$$

Using the definition, prove that the sequence (x_n) with $x_n = (-1)^n + \frac{1}{n}$ does not converge to any $L \in \mathbb{R}$.

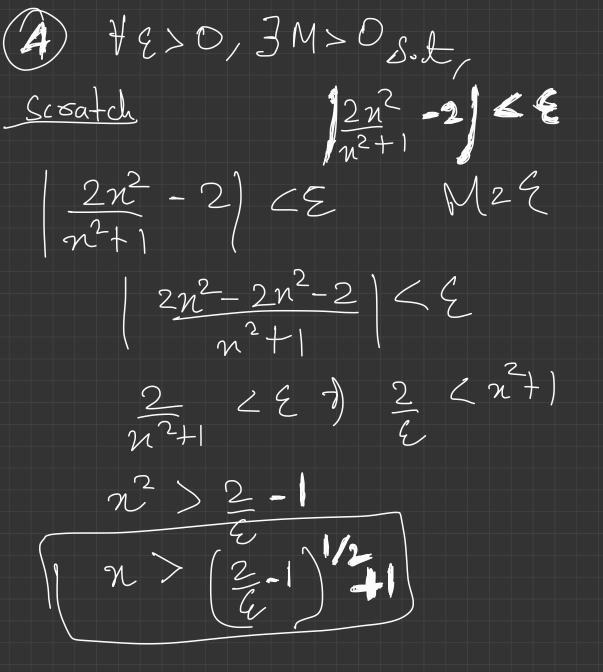
Hint: It is perhaps easier to split into cases depending on the sign of L.

9/nf(n+1)}+(n+2)3 (1) ANES USO Typed!. nz 3 (ase 1. ~=3K+1 (age 2: n=3K+2 Cash 3: b.(=a.} (2)]a,b,c&Z st,gcd(a,b)=1) antby=1 $a c \cdot n + b \cdot c \cdot y = k$ $a \cdot c \cdot n + a \cdot k \cdot y = k$ $a \cdot c \cdot n + a \cdot k \cdot y = k$ $a \cdot c \cdot n + a \cdot k \cdot y = k$ $a \cdot c \cdot n + a \cdot k \cdot y = k$ $a \cdot c \cdot n + a \cdot k \cdot y = k$ $a \cdot c \cdot n + a \cdot k \cdot y = k$ (a) HNEP, TYEP, NTYEP Take counter mample (b) In EP, 3y EP moh nat nty EP &(c) P>2 = old & old told: even & old + 2 7 P For all (d) 3 EP, s. L 3y EP, nty EP 2+3 = S

(8)
$$\forall E > 0, \exists N \in \mathbb{N}, \forall n \ge \mathbb{N}, |n_n - 1| \le n = (1)^n + 1$$
 $\exists E > 0, \forall N \in \mathbb{N}, \exists n \ge \mathbb{N}, |n_n - 1| \ge E$

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(1=0) $|n_n| \ge E \Rightarrow (-1)^n + 1 \ge E$
 $|n_n| \ge 1$
 $|n_n| \ge 1$



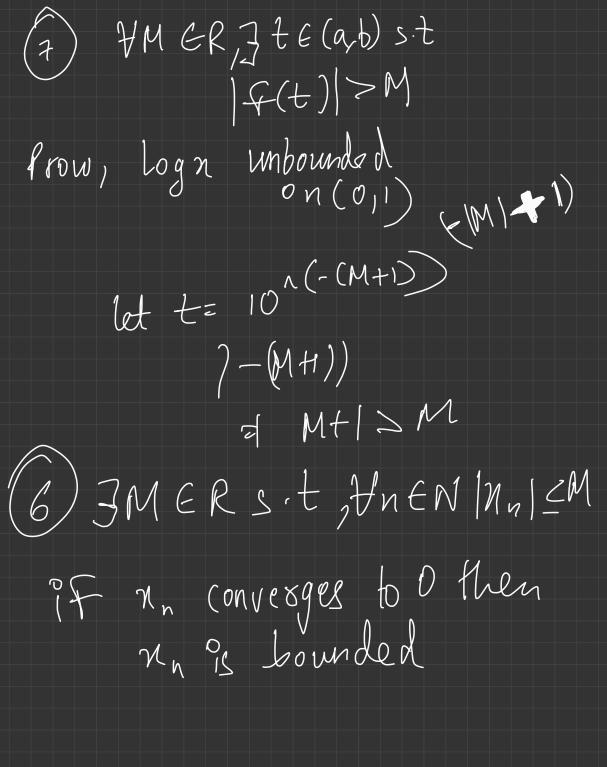
So, Consider
$$M = (\frac{7}{2} - 1)^{1/2} + 1$$

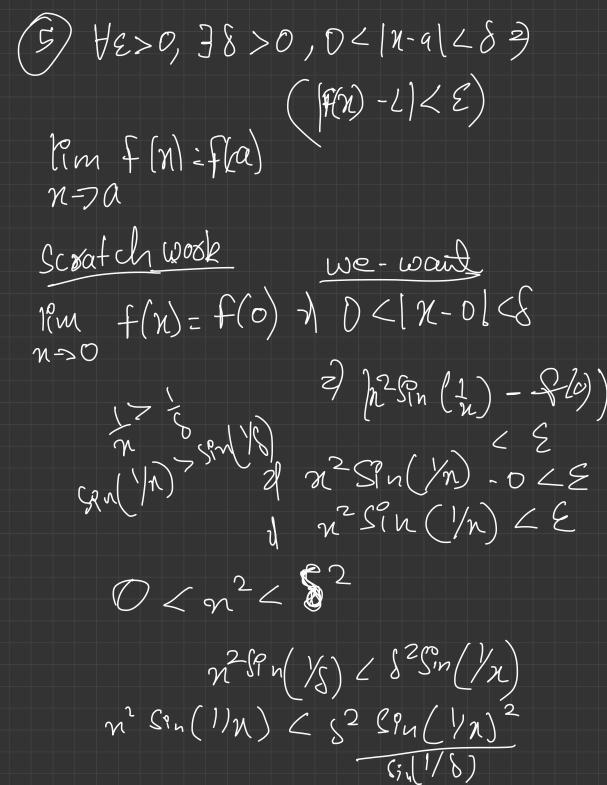
So, Consider $M = (\frac{7}{2} - 1)^{1/2} + 1$

Now, $n \ge M$

So, $\left| \frac{2n^2}{n^2 + 1} - 2 \right| = \left| \frac{2}{n^2 + 1} \right| = \frac{2}{n^2 + 1}$
 $= \left| \frac{2}{2} - 1\right|^{1/2} + 1 \right| = \frac{2}{2} + 2 = \frac{$

lim f(n)=f(a) n-, a ル ≠ 0 U=0 $f(n) = \begin{cases} n^2 Sin(1/n) \\ 0 \end{cases}$ Lonjinnond at n=0



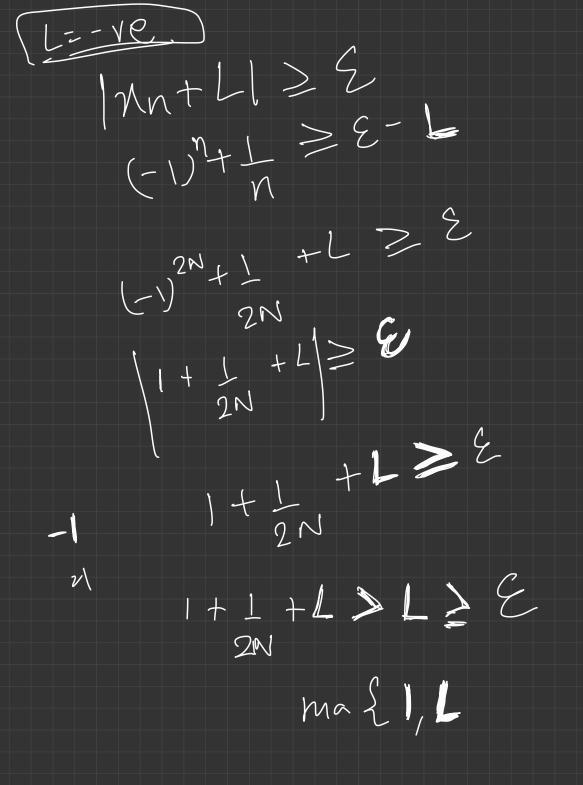


 $\left| \chi Siu \left(\frac{1}{\chi} \right) \right| < \xi$ 22/82 S = SE121 SPN/N 240-2 0 / 2 Sin (/n) / 2 But 2 (29/1/2) 282 8=79

(6) Bounded if BMER S. J. VnEN, InnIEM If nn converges to O i-e 7 6>0,3NEN, 7n>N, Nn-0)2E Then

JMER s.t. HneN, [nn] < M mn1 < E

(8) YESD, FNEN, YN >N, |nn-L/28 nn = (-1) n does not converge 6, 3 2>0, 4nEN, 3n >N, 1m-21>E Scartch L=0 we want $n \ge E$ (-1) $n + 1 \ge E$ $N=2\cdot N$ $(-1)^{2N}+1=1+1>1\geq E$ 2N



$$\frac{L=+ve}{|n_n-L|} \ge \frac{2}{|n_n-L|} \ge \frac{2}{|n_$$

