PLP - 40

TOPIC 40 — TOWARDS INFINITE SETS

Demirbaş & Rechnitzer

TOWARDS INFINITY

CARDINALITY COMPARISONS BY FUNCTION

In previous lecture we saw for *finite* sets A, B

- If $f : A \to B$ is an injection then $|A| \leq |B|$
- If $g : A \to B$ is an surjection then $|A| \geq |B|$
- If $h : A \to B$ is an bijection then $|A| = |B|$

and we use the last one to *define* equinumerous for *any* sets

Need to prove that this is well defined

THEOREM:

Let A, B, C be sets, then “being equinumerous” is an equivalence relation

- reflexive: $|A| = |A|$
- symmetric: $|A| = |B| \implies |B| = |A|$
- transitive: $|A| = |B|$ and $|B| = |C| \implies |A| = |C|$

EQUINUMEROUS IS AN EQUIVALENCE RELATION

“Being equinumerous” is an equivalence relation

PROOF.

We have to prove that it is reflexive, symmetric and transitive. It suffices to construct appropriate bijections.

- The identity $i_A : A \to A$ is a bijection, so $|A| = |A|$
- If $|A| = |B|$ then there is a bijection $f : A \to B$. Theorem: since f is a bijection, its inverse $f^{-1} : B \to A$ exists and is also a bijection. Hence $|B| = |A|$.
- If $|A| = |B|$ and $|B| = |C|$ then there exist bijections $f : A \to B$ and $g : B \to C$. Theorem: the composition $(g \circ f) : A \to C$ is a bijection. Hence $|A| = |C|$.

Let's put this bijection definition to work on *infinite* sets

TWO EXAMPLES

PROPOSITION:

Let $\mathcal{E} = \{n \in \mathbb{N}$ s.t. n is even$\}$ and $\mathcal{O} = \{n \in \mathbb{N}$ s.t. n is odd$\}$, then $|\mathcal{O}| = |\mathcal{E}|$.

Further, $|\mathbb{N}| = |\mathcal{E}|$.

PROOF.

The function $f : \mathcal{O} \to \mathcal{E}$ defined by $f(n) = n + 1$ is a bijection.

The function $g : \mathbb{N} \to \mathcal{E}$ defined by $g(n) = 2n$ is a bijection.

$\mathcal{E} \subset \mathbb{N}$ but
$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 6 & 8 & \dots \end{array}$$

FINITE AND INFINITE BEHAVE VERY DIFFERENTLY

Consider sets A, B with $A \subset B$

- If A, B are *finite* then PHP tells us $|A| \neq |B|$ — no bijection possible
- If A, B are *infinite* then a bijection may be possible

$$
\begin{aligned}
|\mathcal{E}| &= |\mathbb{N}| \qquad & f &: \mathbb{N} \to \mathcal{E} \qquad & f(n) &= 2n \\
|\{1, 4, 9, 16, \ldots\}| &= |\mathbb{N}| \qquad & g &: \mathbb{N} \to \{1, 4, 9, 16, \ldots\} \qquad & g(n) &= n^2
\end{aligned}
$$

DEFINITION: INFINITE SET.

Informal: an infinite set keeps on going: $1, 2, 3, 4, \ldots$

Formal: a set A is infinite if there is a bijection from A to a proper subset of A

This definition is due to Dedekin, but we will use more precise ones.

A FIRST INFINITY

First infinite set we meet is the natural numbers.

DEFINITION:

- A set A is called **denumerable** if there is a bijection $f : \mathbb{N} \to A$
- We denote the cardinality of any denumerable set by $\aleph_0$ — "aleph-null"
- When a set A is finite or denumerable we say that it is **countable**
- When a set is not countable it is **uncountable**

- Since bijection has bijective inverse $f : \mathbb{N} \to A \iff f^{-1} : A \to \mathbb{N}$
- We now know that $\mathcal{E}, \mathcal{O}, \{1, 4, 9, 16, \dots\}$ are all denumerable, and so countable
- Are all subsets of $\mathbb{N}$ countable?
- Are supersets of $\mathbb{N}$ such as $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denumerable?
- Do uncountable sets exist?