Math 220
Section 108
Lecture 21

22nd November 2022

Sources: https://personal.math.ubc.ca/~PLP/auxiliary.html
https://secure.math.ubc.ca/Ugrad/pastExams

Recall – Bijections

Definition (Definition 10.4.10 of PLP)

Let $f : A \to B$ be a function. If f is both injective and surjective then we say that f is **bijective**, or a one-to-one correspondence.

Example

The function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ is bijective.
The function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^4$ is <u>not</u> bijective, since it is not injective (nor surjective).

Bijections - Old Final Question

6. Prove that the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$ given by $f(x) = \frac{2x}{x-1}$ is bijective.

Let, $\frac{2x_1}{x_1-1} = \frac{2x_2}{x_2-1}$

$$x_1(x_2-1) = x_2(x_1-1)$$

$$x_1x_2 - x_1 = x_2x_1 - x_2$$

So, $-x_1 = -x_2$

or

$$x_1 = x_2$$

So, F is injective

(Continued)

(Continued) 6. Prove that $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$, $f(x) = \frac{2x}{x-1}$ is bijective.

Let $x = \frac{2}{y-2} + 1$

Then $\frac{x-1}{2} = \frac{1}{y-2}$

$\Rightarrow \frac{y-2}{2} = \frac{1}{x-1}$ (Note $x \neq 1$)

$\Rightarrow \frac{y}{2} - 1 = \frac{1}{x-1}$

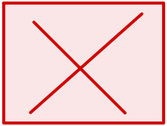
$\Rightarrow \frac{y}{2} = \frac{1}{x-1} + 1 \Rightarrow \frac{y}{2} = \frac{x}{x-1}$

$y = \frac{2x}{x-1} = f(x)$

So f is surjective

$\therefore f$ is bijective. $\blacksquare$

7. (a) Let $f : A \to B$ be a surjection and let $D_1, D_2 \subseteq B$. Show that if $f^{-1}(D_1) \subseteq f^{-1}(D_2)$, then $D_1 \subseteq D_2$.
(b) Construct an example that shows the above is not true when f is not a surjection.



(a) $D_1, D_2 \subseteq B$

$f^{-1}(D_1) \subseteq f^{-1}(D_2)$ then $D_1 \subset D_2$

For some $X_1, X_2 \subseteq A$ we have,

$f(X_1) = D_1$ & $f(X_2) = D_2$

Now, since f is surjective we have,

$X_1 = f^{-1}(D_1)$ & $X_2 = f^{-1}(D_2)$

Now, $f^{-1}(D_1) \subseteq f^{-1}(D_2)$ so, $X_1 \subseteq X_2$

So, $f(X_1) \subseteq f(X_2)$

Hence $D_1 \subseteq D_2$

(Continued)

(Continued) 7. (a) Let $f : A \to B$ be a surjection and let $D_1, D_2 \subseteq B$. Show that if $f^{-1}(D_1) \subseteq f^{-1}(D_2)$, then $D_1 \subseteq D_2$.
(b) Construct an example that shows the above is not true when f is not a surjection.

$$f^{-1}(D_1) \subseteq f^{-1}(D_2) \text{ but } D_1 \not\subseteq D_2$$

Compositions

Definition (Definition 10.5.1 in PLP)

Let $f : A \to B$ and $g : B \to C$ be functions. The **composition** of f and g is

$$g \circ f : A \to C$$
$$\text{where} \qquad (g \circ f)(a) = g(f(a)) \quad \forall a \in A$$

Example

Let $f(x) = x^3$ and $g(x) = 2x$ both be functions on $\mathbb{R}$. Then

$$(g \circ f)(x) = 2x^3.$$

In contrast, we have

$$(f \circ g)(x) = 8x^3.$$

Usually,

$$g \circ f \neq f \circ g.$$

Left, Right, and Two-sided Inverses

Definition (Definition 10.6.1 of PLP)

Let $f : A \to B$ and $g : B \to C$ be functions.

- If $g \circ f = i_A$ then we say that g is a **left-inverse** of f.
- If $f \circ g = i_B$ then we say that g is a **right-inverse** of f.

Definition (Definition 10.6.6 of PLP)

If g is both a left-inverse and a right-inverse of f, then we say it is **the inverse** of f.

Note that the inverse, if it exists, is unique.

Theorem (From Theorem 10.6.8 of PLP)

A function has an inverse if and only if it is bijective.

Final Question 8, 2016 WT1

8. Let $f : A \to B$ be a function. Prove:
(a) If there is a function $g : B \to A$ such that $g \circ f(x) = x$, for all $x \in A$, then f is injective.
(b) If f is injective, then there is a function $g : B \to A$ such that $g \circ f(x) = x$, for all $x \in A$.

Final Question 8, 2016 WT1 (Continued)

(Continued) 8. Let $f : A \to B$ be a function. Prove:
(a) If there is a function $g : B \to A$ such that $g \circ f(x) = x$, for all $x \in A$, then f is injective.
(b) If f is injective, then there is a function $g : B \to A$ such that $g \circ f(x) = x$, for all $x \in A$.

Injective (if time)

9. Suppose that $f : A \to B$ and C_1, C_2 are subsets of A. Show that if f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.

(Continued)

(Continued) 9. Suppose that $f : A \to B$ and C_1, C_2 are subsets of A. Show that if f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.