# PLP - 16

## TOPIC 16—EXISTENCE PROOFS

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# EXISTENCE PROOFS

# AN EXAMPLE

*There exist integers $x, y$ so that $x^3 - y^2 = 13$.*

**PROOF.**

Consider $x = 17$ and $y = 70$. Since $17^3 - 70^2 = 4913 - 4900 = 13$ we are done.

Why is this sufficient?

- The statement is "$\exists x, y \in \mathbb{N}$ **s.t.** $x^3 - y^2 = 13$"
- So to prove it true we only need to give at least one instance that makes it true
- We *do not* have to explain how we found that example

This is a **constructive** proof.

# ANOTHER EXAMPLE

*There exists* $x \in \left[0, \frac{\pi}{2}\right]$ *so that* $\cos(x) = x$

**PROOF.**

Let $f(x) = \cos(x) - x$. Note that $f(0) = 1 > 0$ and $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2} < 0$ and that $f(x)$ is a continuous function. Then by the *Intermediate Value Theorem*, we know that there exists a point $c \in \left(0, \frac{\pi}{2}\right)$ so that $f(c) = 0$. From this we know that $\cos(c) = c$ as required.

Why is this sufficient?

- To prove this we only need to *infer* that an example exists
- We *do not* have to give the example explicitly

This is a **non-constructive** proof.

# EXISTENCE PROOFS

Proofs of existence results fall into 2 broad categories

**constructive proofs**

in which a specific example is given explicitly and verified

an explanation of *how* the example was found is *not required*

**non-constructive proofs**

in which the existence is *inferred* but an example is not explicitly stated

# UNIQUENESS PROOF

After demonstrating that a required object exists, one often also wants uniqueness

> *There exists a unique $x$ so that $P(x)$*

A simple way to approach such proofs is

- Let $x, y$ be objects so that $P(x)$ and $P(y)$ are true
- Do *stuff* to show that $x = y$

The *fun* is in working out what *stuff* is.

# AN EXAMPLE

*The equation $ax = b$ with $a, b \in \mathbb{R}$ and $a \neq 0$ has a unique real solution.*

**PROOF.**

First note that since $a \neq 0$, we can solve the equation by choosing $x = \frac{b}{a} \in \mathbb{R}$. Thus a solution exists.

Now assume that numbers $r, s$ both satisfy the equation. Hence

$$\begin{aligned} ar &= b \qquad as = b && \text{and so} \\ ar &= as && \text{and since } a \neq 0 \\ r &= s \end{aligned}$$

So both solutions are in fact equal and the solution must be unique.

Let $A$ be a set, we say a fnct $f : A \to \mathbb{R}$, is bounded if

$$\exists M \in R, \text{ s.t } \forall x \in A, |f(x)| \leq M$$

Q) Show that if $f : R \to R$, $f(x) = 3x - 7$ is not bounded

$\Rightarrow \forall M \in R, \exists x \in A \text{ s.t. } |f(x)| > M$

Let $M$ be any arbitary real number

Now, $f(x) = 3x - 7$

So, $|3x - 7|$

Let $x = \dfrac{8 + |M|}{3}$

$$3x - 7 > M$$
$$3x > 7 + M$$
$$x > \frac{7+M}{3}$$

So, $\left|3\left(\dfrac{8+|M|}{3}\right) - 7\right| = |8 + |M| - 7|$

$= ||M| + 1|$

$= M + 1$

which is $> M$

Hence $|f(x)| > M$

& so, $f(x)$ is unbounded

(6) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $n \geq x$ $\to$ Fact

Show that $\underline{\forall \varepsilon > 0}, \exists N \in \mathbb{N}$ s.t, $\forall n \in \mathbb{N}$ with $n > N$, we have $\frac{1}{n} < \varepsilon$

Hint : $\frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$

Consider a natural number $n$,

We have an arbitary real no. $\varepsilon$ which is +ve

<u>Scratch</u>

Want $\frac{1}{n} < \varepsilon$ , So want $\varepsilon < \frac{1}{n}$

if $N > \frac{1}{\varepsilon}$, then $n > N$, we get $n > N > \frac{1}{\varepsilon}$

<u>Proof</u>

Given some $\varepsilon > 0$ ( $\varepsilon \in \mathbb{R}$), let $N \in \mathbb{N}$ be such that $N \geq \frac{2}{\varepsilon}$

so, $N > \frac{1}{\varepsilon}$ [since $\frac{2}{\varepsilon} > \frac{1}{\varepsilon}$]

N exists due to "Fact".

So, $N \geq \frac{2}{\varepsilon} > \frac{1}{\varepsilon}$ , so for all $n > N$ we have $n > N > \frac{1}{\varepsilon}$ . since

$$\left(\frac{1}{n} < \varepsilon\right) \iff \left(\frac{1}{\varepsilon} < n\right)$$

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