MATH 220 Midterm Review

UBC Undergraduate Math Society

Procedure

Introduction

Logic

Some number theory

Proof by induction

Limits

Negation

What happens after MATH 220?

UBC Math Undergraduate Society

Location: MATH ANNEX 1119

What we do: board games (we have oh so many board games), putnam practice, math circle, exam packs, louging around, and sometimes, math.

Instagram: ums.ubc

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https://discord.gg/E6AYdZC3

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*EARLY BIRD DISCOUNT AND BUNDLES END NOV 26, 2021
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Your instructor
uki (she/they)

- 4th year biomedical engineering (bioinformatics), honors math minors
- cats

- why take MATH 220

Logic

P	Q	R	$(P \wedge Q) \implies R$	$(P \implies R) \vee (Q \implies R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

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A Logic Puzzle

Select the probability of randomly choosing the correct answer for this question:
a) 1/4
b) 1/2
c) 0
d) 1/4

Number theory

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Assume $a, b \in \mathbb{Z}$**. Prove that if** $ax + by = 1$ **for some** $x, y \in \mathbb{Z}$**, then gcd**$(a, b) = 1$**.**

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Assume $a, b \in \mathbb{Z}$ and p is prime. Using Bézout's identity from homework 1, prove that if $p \mid ab$, then $p \mid a$ or $p \mid b$.

Proof by induction

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Prove that, $\forall n \in \mathbb{N}$, $\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2$.

Base case

$n=1$

$1^3 = (1)^2$

Proved

$(1+2+3)^2 + 4^2$

36

Ind.

$n=K \Rightarrow$ true assume

$n=K+1$

$$\boxed{\sum_{k=1}^{n} k^3 + (K+1)^3 \qquad \left(\sum_{k=1}^{n} K\right)^2 + (K+1)^3}$$

Σn^2

"

$\frac{(n)(n+1)(2n+1)}{6}$

$$\sum k^3 = \left(\frac{(n)(n+1)}{2}\right)^2$$

**

Prove, using induction, that $\forall n \in \mathbb{N}, 3 \mid (n^3 - n)$.

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Theorem: A statement of the form $\forall n \in \mathbb{N}; P(n)$" is true if

- The statement P(1) is true,
  and,
- given $k \geq 1$, $P(1) \wedge P(2) \wedge P(3) \wedge ... \wedge P(k) \implies P(k+1)$.

This procedure is called the *strong induction*.
Use strong induction to prove the following statement: Suppose you begin with a pile of n stones ($n \geq 2$) and split this pile into n separate piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have p and q stones in them, respectively, you compute pq. Show that no matter how you split the piles (eventually into n piles of one stone each), the sum of the products computed at each step equals $\frac{n(n-1)}{2}$.

The Rainbow Lemma

For any positive integer x with exactly n divisors, where n is even, we write $1 = d_1 < d_2 < \cdots < d_{n-1} < d_n = x$. d_i are distinct divisors of x for distinct $1 \leq i \leq n$. Prove that

$$x = d_1 d_n = d_2 d_{n-1} = d_3 d_{n-2} \cdots = d_{\frac{n}{2}} d_{\frac{n}{2}+1}.$$

Limits of sequences

Definition 6.4.2. Let (x_n) be a sequence of real numbers. We say that (x_n) has a ***limit*** $L \in \mathbb{R}$ when

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n > N) \implies (|x_n - L| < \varepsilon).$$

In this case we say that the sequence ***converges*** to L and write

$$x_n \to L \qquad \text{or} \qquad \lim_{n\to\infty} x_n = L.$$

If the sequence doesn't converge to any number L, we say that the sequence ***diverges***.

**

We say that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to L if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \varepsilon.$$

Using the definition, prove that the sequence $(x_n)_{n\in\mathbb{N}} = ((-1)^n + \frac{1}{n})_{n\in\mathbb{N}}$ does not converge to 0.

**

Let $a_n = \dfrac{2n^2 + n + 14}{2n^2 + 11}$. Show, using the definition of convergence, that $a_n \to 1$ L

$$\left| \frac{2n^2 + n + 11 + 3}{2n^2 + 11} - 1 \right| \leq \frac{2n^2 + n + 14}{2n^2} - 1 \leq \varepsilon$$

$$1 + \frac{n+3}{2n^2+11} - 1 < \frac{n+3}{2n^2} < \frac{n+3}{n}$$

$$< 1 + \frac{3}{n}$$

Limits of functions

Definition 6.4.8. Let $a, L \in \mathbb{R}$ and let f be a real-valued function. We say that the ***limit*** of f as x approaches a is L when

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < |x-a| < \delta) \implies (|f(x) - L| < \varepsilon).$$

In this case we write

$$\lim_{x \to a} f(x) = L \qquad \text{or sometimes} \qquad f(x) \xrightarrow[x \to a]{} L$$

and say that f ***converges*** to L as x approaches a. We also sometimes say the limit of f as x goes to a is L, which we denote by

$$f(x) \to L \text{ as } x \to a.$$

If f does not converge to any finite limit L as x approaches a, then we say that f ***diverges*** as x approaches a.

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Limit Problem

Suppose a real-valued function f satisfies $f(x) = f(x+1)$ for all $x \in \mathbb{R}$. Prove that $f'(x) = f'(x+1)$ for all $x \in \mathbb{R}$ using $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

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10. **[8 pts]** Suppose that $f : \mathbb{N} \to \mathbb{R}$ is a bounded function and that $\{a_n\}$ is a sequence that converges to 0. Prove that $\lim_{n\to\infty} f(n)a_n = 0$.

Negation

- Writing sentences in symbolic logic notations

  "8 is even and 5 is prime"
  "If a function f is differentiable everywhere then whenever $x \in \mathbb{R}$ is a local maximum of f we have $f'(x) = 0$"

- Negating sentences

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Function types

In this question, we are going to call a function, $f : \mathbb{R} \to \mathbb{R}$, *type A*, if $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $y \geq x$ and $|f(y)| \geq 1$. We also say that a function, g, is *type B* if $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, if $y \geq x$, then $|g(y)| \geq 1$.

Prove or find a counterexample for the following statements.

a) If a function is type A, then it is type B.

b) If a function is type B, then it is type A.

Feedback form