Math 220
Section 108
Lecture 14

25th October 2022

Power Sets

Definition

Let S be any set. Then the **power set** $\mathcal{P}(S)$ of S is the set of all subsets of S.

Note: this includes S itself as well as the empty set!

Example

Let $S = \{a, b, c\}$. Then

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

$S \cap P(S) = \phi$

$a \in S$

$\{a\} \in P(S)$

1. Let $A = \{1, 2\}$. Find $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A) - \{\emptyset\})$.
2. Show that if $p, q \in \mathbb{N}$, then $\{pn \mid n \in \mathbb{N}\} \cap \{qn \mid n \in \mathbb{N}\} \neq \emptyset$.

Hint: Think about a couple of concrete examples first.

① $P(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

$P(P(A) - \{\phi\}) = \{\phi, \{1\}, \{2\}, \{\{1\}, \{2\}\}, \{1,2,3\}, \{\{1\}, \{1,2,3\}\}, \{\{2\}, \{1,2,3\}\}, \{\{1\}, \{2\}, \{1,2,3\}\}\}$

② if $p, q \in \mathbb{N} \Rightarrow \{pn \mid n \in \mathbb{N}\} \cap \{qn \mid n \in \mathbb{N}\} \neq \emptyset$

Let $p \in \mathbb{N}$ & $n \in \mathbb{N}$ So, $pn \in \mathbb{N}$ ie $pq \in \{pn \mid n \in \mathbb{N}\}$

Let $q \in \mathbb{N}$ & $n \in \mathbb{N}$ So, $qn \in \mathbb{N}$ i.e $qp \in \{qn \mid n \in \mathbb{N}\}$

So, $pn \cap qn \in \mathbb{N}$ $\Box$

(Continued)

(Continued) 1. Let $A = \{1, 2\}$. Find $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A) - \{\emptyset\})$.
2. Show that if $p, q \in \mathbb{N}$, then $\{pn \mid n \in \mathbb{N}\} \cap \{qn \mid n \in \mathbb{N}\} \neq \emptyset$.

Sets (old final question)

3. Let $p_1, p_2, p_3, \ldots, p_n, \ldots$ be the infinite sequence of prime numbers listed in increasing order (so that $p_1 = 2, p_2 = 3, p_3 = 5$, etc.). For $k \in \mathbb{N}$, let $A_k = \{a \in \mathbb{N} \mid a \geq 2 \text{ and } p_k \text{ does not divide a}\}$, and for $n \in \mathbb{N}$, define

$$B_n = \bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n.$$

(a) Find the smallest element of the set B_4.
(b) Consider whether for every $n \in \mathbb{N}$, the set B_n is infinite or not.
(c) Find the intersection of all A_k's.

(a) $B_n = \bigcap_{k=}^{4} = A_1 \cap A_2 \ldots \cap A_n$

$A_1 = \{3, 5, 7, 9, 11 \ldots\}$
$A_2 = \{2, 4, 5, 7, 8, 10, 11 \ldots\}$
$A_3 = \{2, 3, 4, 6, 7, 8, 9, 11 \ldots\}$
$A_4 = \{2, 3, 4, 5, 6, 8, 9, 10, 11 \ldots\}$
Smallest element $= 11$

(b) B_n is not $| \; p_1 \ldots p_n$ so, $p_{n+1}, p_{n+2} \ldots \in B_n$. So, B_n is ∞ $\forall n \in \mathbb{N}$

(c) $\bigcap_{k=1}^{\infty} A_k = \phi$

(Continued) 3. Let $p_1, p_2, p_3, \ldots, p_n, \ldots$ be the prime numbers listed in increasing order. For $k \in \mathbb{N}$, let $A_k = \{a \in \mathbb{N} \mid a \geq 2 \text{ and } p_k \nmid a\}$, and for $n \in \mathbb{N}$, define $B_n = \bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$.

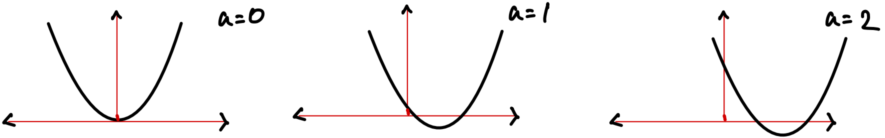
(a) Find the smallest element of the set B_4.

(b) Consider whether for every $n \in \mathbb{N}$, the set B_n is infinite or not.

(c) Find the intersection of all A_k's.

4. Let $a \in \mathbb{R}$.
(a) On the xy-plane, draw the set $S_a = \{(x, x^2 - ax),\ x \in \mathbb{R}\}$ when $a = 0$, $a = 1$ and $a = 2$.
(b) Show that $\bigcap_{a\in\mathbb{R}} S_a = \{(0,0)\}$.
Hint: What does it mean for a point to be in the intersection?

(a) $y = x^2 - ax$ [Parabola Eq]



(b) $\bigcap_{a\in\mathbb{R}} S_a = \{(0,0)\}$

Consider $S_0 \cap S_1$: An el (x,y) in this must satisfy $y = x^2$ & $y = x^2 - x$, so $x = 0$.
$\therefore y = 0$. So, $S_0 \cap S_1 = \{(0,0)\}$. So far $\{(0,0)\} = S_0 \cap S_1 \subseteq \bigcap_{a\in\mathbb{R}} S_a$
Now, $\forall S_a$, note that $y = x^2 - ax$ we have $(0,0)$ as a soln.
So, $(0,0) \in S_a$ $\forall a \in \mathbb{R}$ $\therefore \bigcap_{a\in\mathbb{R}} S_a \supseteq \{(0,0)\}$.
In conclu $\bigcap_{a\in\mathbb{R}} S_a = \{(0,0)\}$ $\square$

(Continued)

(Continued) 4. Let $a \in \mathbb{R}$.
(a) Draw $S_a = \{(x, x^2 - ax),\ x \in \mathbb{R}\}$ when $a = 0, 1, 2$.
(b) Show that $\bigcap_{a \in \mathbb{R}} S_a = \{(0, 0)\}$.

Sets

5. (a) Show that for every $k \in \mathbb{Z}$, $\exists x, y \in \mathbb{Z}$, such that $k = 4x + 5y$.
(b) What does it say about the set $A = \{4x + 5y \mid x, y \in \mathbb{Z}\}$: is it a subset of, superset of, or equal to $\mathbb{Z}$?

(Continued)

(Continued) 5. Show: $\forall k \in \mathbb{Z}$, $\exists x, y \in \mathbb{Z}$, such that $k = 4x + 5y$.
(b) Is $A = \{4x + 5y \mid x, y \in \mathbb{Z}\}$ a subset of, superset of, or equal to $\mathbb{Z}$?

6. Let A, B and C be sets. For each of the following statements, either prove it is true or give a counterexample.

(a) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$,

(b) $\mathcal{P}(A \cup B) \supseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Hint: First try this with small sets.

(Continued) 6. Prove or give a counterexample:
(a) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$,
(b) $\mathcal{P}(A \cup B) \supseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.