Mathematics 220 — Homework 3

- Contains 8 questions on 2 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

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1. Negate the following statement: For every positive number ϵ there is a positive number M for which
$$\left|1 - \frac{x^2}{x^2+1}\right| < \epsilon,$$
whenever $x \geq M$.

2. Write down the negation of the statement
$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{R}, \left((x \geq y) \implies \left(\frac{x}{y} = 1\right)\right)$$
and determine whether the *original* statement is true or false.

3. Let $A = \{n \in \mathbb{N} : 3 \mid n \text{ or } 4 \mid n\} \subset \mathbb{N}$. Note that all numbers in A are positive.
Determine whether the following four statements are true or false — explain your answers (“true” or “false” is not sufficient).

   (a) $\exists x \in A$ s.t. $\exists y \in A$ s.t. $x + y \in A$.

   (b) $\forall x \in A, \forall y \in A, x + y \in A$.

   (c) $\exists x \in A$ s.t. $\forall y \in A, x + y \in A$.

4. Negate the following statements and determine whether the original statements are true or false. Justify your answer.

   (a) $\forall n \in \mathbb{Z},\ \exists y \in \mathbb{R} - \{0\}$ such that $y^n \leq y$.

   (b) $\exists y \in \mathbb{R} - \{0\}$ such that $\forall n \in \mathbb{Z},\ y^n \leq y$.

   (c) $\forall x \in \mathbb{R}$ where $x \neq 0$, we have $x \leq 1$ or $\dfrac{1}{x} \leq 1$.

5. After cleaning your basement, you find a set of keys K and a set of locks L. For every one of the following statements (a), (b) and (c),

   1. re-express the statement in a mathematical form using quantifiers $\forall$ and/or $\exists$,
   2. negate this mathematical statement,
   3. re-express the negation in standard english.

E.g.: "All keys unlock all locks" gives us:

- Reformulated statement: $\forall k \in K, \forall l \in L, k$ unlocks l.
- Negation: $\exists k \in K, \exists l \in L, k$ does not unlock l.
- Reformulated negation: "Some key does not unlock some lock."

(a) "At least one of the keys unlocks one of the locks."

(b) "Some key unlocks all the locks."

(c) "Some lock is not unlocked by any key."

6. Prove that $\forall a \in \mathbb{Z},\ \exists b \in \mathbb{Z},\ a^2 + b^2 \equiv 1 \bmod 3$.

7. Prove or disprove:

$$\forall x, y, z \in \{3, 6\}, \left(x = y = z \text{ or } \frac{x+y+z}{3} > \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right).$$

Hint: think carefully about reducing the number of cases.

8. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is increasing if

$$\forall a, b \in \mathbb{R},\ (a < b \Rightarrow f(a) < f(b))$$

Show that

(a) $f(x) = x^3 + 3x + 4$ is increasing.
Hint: Completing the square may help you show something is positive.
Warning: Do not use calculus to answer this problem.

(b) $g(x) = \sin x$ is not increasing.

① For every +ve no. ε there is a +ve number M for which $\left|1-\frac{n^2}{n^2+1}\right| < \varepsilon$ whenever $n \geq M$

Org -> $\forall \varepsilon > 0, \exists M > 0$ s.t $\left|1-\frac{n^2}{n^2+1}\right| < \varepsilon$ for $\forall n \geq M$

Neg -> $\exists \varepsilon > 0$ s.t, $\forall M > 0$ $\left|1-\frac{n^2}{n^2+1}\right| \geq \varepsilon$ for $n \geq M$

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② Org -> $\forall n \in \mathbb{Z}, \exists y \in \mathbb{R}, \left((n \geq y) \Rightarrow \left(\frac{n}{y} = 1\right)\right)$

neg -> $\exists n \in \mathbb{Z}, \forall y \in \mathbb{R}, \left(\frac{n}{y} = 1\right) \wedge (n < y)$

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③ $A = \{n \in \mathbb{N} : 3 | n \text{ or } 4 | n\} \subset \mathbb{N}$ A ⇒ +ve

(a) $\exists n \in A$ s.t. $\exists y \in A$ s.t. $n + y \in A$ -> true

(b) $\forall n \in A, \forall y \in A, n + y \in A$ -> false

(c) $\exists n \in A$ st. $\forall y \in A, n + y \in A$ => false

4 (a) org $\Rightarrow \forall n \in \mathbb{Z}, \exists y \in \mathbb{R}-\{0\}$ s.t $y^n \leq y \Rightarrow$ true

neg $\rightarrow \exists n \in \mathbb{Z}$ s.t. $\forall y \in \mathbb{R}-\{0\}$ $y^n > y$

(b) org $\rightarrow \exists y \in \mathbb{R}-\{0\}$ s.t. $\forall n \in \mathbb{Z}$, $y^n \leq y \Rightarrow$ true

neg $\rightarrow \forall y \in \mathbb{R}-\{0\}$ $\exists n \in \mathbb{Z}$ s.t., $y^n > y$

(c) org $\rightarrow \forall x \in \mathbb{R}$, where $x \neq 0$, we have $x \leq 1$ or $\frac{1}{x} \leq 1$ $\Rightarrow$ true

neg $\Rightarrow \exists y \in \mathbb{R}-\{0\}$ s.t, where $x \neq 0$, we have $x > 1$ and $\frac{1}{x} > 1$

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5 (a)
- $\exists x \in K, \exists y \in L$ s.t, x unlocks y
- $\forall x \in K, \forall y \in L$, x does not unlock y
- No keys unlock all locks

(b)
- $\exists x \in K$ s.t $\forall y \in L$, x unlocks y
- $\forall x \in K, \exists y \in L$ s.t, x does not unlock y
- All keys do not unlock some lock.

(c) · $\exists y \in L$ s.t, $\exists x \in K$ s.t, y is not unlocked by x

· $\forall y \in L$, $\forall x \in L$, y unlocked x

· All locks are unlocked by all keys.

(6) P.T $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}$ $a^2+b^2 \equiv 1 \mod 3$

Case 1: $a = 3k$

$(3k)^2 + (1)^2$

$9k^2+1$

$3(3k^2)+1 \Rightarrow 3m+1$ ✓

Case 2: $a = 3k+1$

$(3k+1)^2 + 3$

$3(3k^2+2k+1)+1 \Rightarrow 3m+1$ ✓

Case 3: $a = 3k+2$

$(3k+2)^2 + 3$

$3(3k^2+4k+2)+1 \Rightarrow 3m+1$ ✓

(7) $\forall x, y, z \in \{3, 6\}\left(x = y = z \text{ or } \frac{x+y+z}{3} > \frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)$

Since it's "$\forall$" we need to prove 1 is true

Case: 1

$x = y = z = 3/6$ then true

Case - 2

(a) if 2 of them are 3 & third is 6

(b) if 2 of them are 6 & thirst is 3

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(8) $f: R \Rightarrow R$ is inc if

$\forall a, b \in R, (a < b \Rightarrow f(a) < f(b))$

(a) S.T $f(x) = x^3 + 3x + 4$ is inc

$$f(x) = (x^3 + 3x + 1 + 3x^2) - 3x^2 + 3$$

$$= (x+1)^3 - 3(x^2+1)$$

$$f(a) = (a+1)^3 - 3(a^2+1)$$

$$f(b) = (b+1)^3 - 3(b^2+1)$$

Let $b > a$

$$\underbrace{(b+1) > (a+1)}$$

$$(b+1)^3 > (a+1)^3$$

Let $b > a$

$$\underbrace{(b+1) > (a+1)}$$

$$\underline{(b+1)^3 > (a+1)^3}$$

$$\text{---} \times \text{---}$$

$$b > a$$

$a, b > 0$	$a, b < 0$
$b^2 > a^2$	$b^2 < a^2$
$(b^2+1) > (a^2+1)$	$(b^2+1) < (a^2+1)$
$3(b^2+1) > 3(a^2+1)$	$3(b^2+1) < 3(a^2+1)$

$f(b) > f(a)$

$b^3 + 3b + 4 > a^3 + 3a + 4$

$(b+1)^3 - 3($

x^3

$- x -$

$f(x) = x^3 + x^2 - x^2$

$+ 3x + 4 + x - x$

$b^3 > a^3$

$3b > 3a$

$4 = 4$

$= (x+2)^2 + x^3 - x^2 - x$

$+ x(x^2 - x - 1)$

$x^3 + 3x + 4$

$x^3 + 3x + x^2 - x^2 + 4 + \frac{1}{36} - \frac{1}{36}$

$x^3 + \left(x + \frac{1}{6}\right)^2 - x^2 - \frac{1}{36} + 4$

$f(b) > f(a)$

$f(b) - f(a) > 0$

$b^3 + 3b + 4 - (a^3 + 3a + 4)$

$b^3 - a^3 \quad + 3(b-a)$

+ve

+ve

$b = -3$

$a = -4$

$(1)^2 - (2)^2$

$b^2 + 4 - 4b$

$\sin(b) - \sin(a)$

$-4+2$

$(-2)^2$

$-3+2$

$(-1)^2$

$1 - 4$

-3

$\sin(b) > \sin(a)$

$\sin(b) - \sin(a) > 0$

w.r.t $b > a$

$\sin(80)$

$\sin(40)$