

Worksheet for Week 4

Note: Order of the quantifiers matters.

1. Consider the following statements and discuss which ones are true.

- $\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z} \text{ s.t. } m^2 < n.$
- $\exists m \in \mathbb{Z} \text{ s.t. } \forall n \in \mathbb{Z}, m < n.$
- $\exists m, n \in \mathbb{Z}, \text{ s.t. } m^2 < n.$
- $\forall m, n \in \mathbb{Z}, m^2 < n.$

2. Negate the statements below and determine whether the statements or their negations are true.

- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \text{ s.t. } y + 5 = x.$
- $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, \text{ s.t. } x^n > 0.$
- $\forall n \in \mathbb{R}, (n^2 \text{ is even}) \implies (n \text{ is even}).$
- $\exists y \in \mathbb{R}, \text{ s.t. } \forall x \in \mathbb{R}, x^2 \geq y.$
- $\exists x \in \mathbb{R}, \text{ s.t. } \forall y \in \mathbb{R}, (y \leq x - 1) \implies (y^2 - x^2 \geq 4).$

3. Prove the following statement:

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (\forall z > 0, |x - y| < z) \implies (x = y).$$

Hint: How can you rewrite the statement?

4. Let \mathbb{I} denote the set of irrationals. Prove that $\forall x \in \mathbb{I}, (k \in \mathbb{Q} \implies \frac{x}{k} \in \mathbb{I}).$

Now, use this result, and the fact that $\pi \in \mathbb{I}$ to show

$$\forall n \in \mathbb{N}, \exists y \in \mathbb{I} \text{ s.t. } x > y > 0.$$

5. Let A be a set. We say that a function $f : A \rightarrow \mathbb{R}$ is bounded if $\exists m \in \mathbb{R}, \text{ s.t. } \forall x \in A, |f(x)| \leq M.$

Use this definition to show that $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = 3x - 7$ is not bounded.

Hint: If the statement above is the definition of a bounded function, what should a function that is NOT bounded satisfy?

6. Fact: $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, \text{ s.t. } n \geq x.$

Use this fact to show

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \frac{1}{n} < \epsilon.$$

Hint: For $\epsilon, n > 0$, we have $(\frac{1}{n} < \epsilon) \iff (\frac{1}{\epsilon} < n).$

Before next week, watch videos 16 and 17 in <https://personal.math.ubc.ca/~PLP/auxiliary.html>.

