Mathematics 220 — Homework 7

- Contains 8 questions on 2 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

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1. We define a relation $\mathcal{R}$ on $\mathcal{P}(\{1,2\})$ (the power set of $\{1,2\}$) by

$$S\mathcal{R}T \iff S \cap T = \emptyset.$$

Write down the all the elements in $\mathcal{R}$.

2. Let R be a relation on a nonempty set A. Then $\overline{R} = (A \times A) - R$ is also a relation on A. Prove or disprove each of the following statements:

    1. If R is reflexive, then $\overline{R}$ is reflexive.
    2. If R is symmetric, then $\overline{R}$ is symmetric.
    3. If R is transitive, then $\overline{R}$ is transitive.

3. Let R be a relation on a set A. Suppose that R is reflexive and satisfy $(aRc \wedge bRc) \implies aRb$ for any $a, b, c \in A$. Prove that R is symmetric and transitive.

4. Let $\mathcal{R}$ be a relation on a set A and $f : A \to B$ a function. We define a relation $\mathcal{R}'$ on B as

$$\mathcal{R}' = \{(f(x), f(y)) : (x, y) \in \mathcal{R}\}.$$

Determine (with proof) whether the following are true:

   (a) If $\mathcal{R}$ is reflexive then $\mathcal{R}'$ is reflexive.
   (b) If $\mathcal{R}$ is symmetric then $\mathcal{R}'$ is symmetric.

5. We define a relation $\mathcal{R}$ on the real numbers as

$$\mathcal{R} = \{(x, x + n) : x \in \mathbb{R}, n \in \mathbb{N}\}.$$

Determine (with proof) whether the following holds:

   (a) If $x_1\mathcal{R}y_1$ and $x_2\mathcal{R}y_2$ then $(x_1 + x_2)\mathcal{R}(y_1 + y_2)$.
   (b) If $x_1\mathcal{R}y_1$ and $x_2\mathcal{R}y_2$ then $(x_1 \cdot y_1)\mathcal{R}(x_2 \cdot y_2)$.

6. We define a relation T on $\mathbb{R} - \{0\}$ by

$$aTb \iff \frac{a}{b} \in \mathbb{Q}.$$

Show that T is symmetric, reflexive, and transitive.

7. Let a relation $\mathcal{R}$ on $\{0, 1, 2, 3\}$ be such that $x\mathcal{R}y$ if $(x + y)$ is a multiple of 3.
   (a) Write out $\mathcal{R}$ as a set.
   (b) Is this relation reflexive?
   (c) Is it symmetric?
   (d) What is the fewest number of elements that you need to add to $\mathcal{R}$ so as to obtain a transitive relation?

8. A relation on a set A is called **circular** if for all $a, b, c \in A$, aRb and bRc imply cRa. Prove that a relation is an equivalence relation if and only if it is reflexive and circular.

HW. 7

① $P(\{1,2\}) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

$sRT \iff S \cap T = \phi$

$R = \{(\phi,\phi), (\phi,\{1\}), (\phi,\{2\}), (\{1\},\{2\}),$
$(\phi,\{1,2\}), (\{1\},\phi), (\{2\},\phi),$
$(\{2\},\{1\}), (\{1,2\},\phi)\}$

② If R is a relation on A then $\bar{R} = (A \times A) - R$ is also a relation on A.

① Let $A = \{1,2\}$

And, $R = \{(1,1), (2,2)\}$

So, $A \times A = \{(1,2), (1,1), (2,1), (2,2)\}$

So, $\bar{R} = (A \times A) - R = \{(1,2), (2,1)\}$

So, $\therefore$ not reflexive

② $\bar{R}$ is not symmetric then R is not symmetric

Let $(a,b) \in \bar{R}$ but $(b,a) \notin \bar{R}$

Now, $\bar{R} = (A \times A) - R$

So, if $x \in \bar{R}$, then $x \in (A \times A) \wedge x \notin R$

So, $(a,b) \notin R$

Furthermore, similarly, if $(b,a) \notin \bar{R}$ then $(b,a) \in R$

So, since $(b,a) \in R \wedge (a,b) \notin R$

R is not symmetric, $\forall a,b \in A$

③ $\bar{R}$ is not transitive then R is transitive,

$(a,b), (b,c) \in \bar{R} \wedge (a,c) \notin \bar{R}$

So, $(a,b), (b,c) \notin R \wedge (a,c) \in R$

$\forall a, b, c \in A$

R is transitive

(3) $(aRc \wedge bRc) \Rightarrow aRb$

- W.K.t R is reflexive
  so, $(a,a) \in R$

Now, let $bRa \in R$
so, $aRa \wedge bRa \Rightarrow aRb$

& hence R is symmetric

- W.k.t R is reflexive & symmetric

so, let aRc & cRb be true, hence

$aRc \wedge cRb \Rightarrow aRc$

Now, $cRb \Rightarrow bRc$

so, $aRc \wedge bRc \Rightarrow aRb$

so, since, aRc, cRb, $aRb \in R$, it's transitive

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(4) R on $f : A \to B$

R' on B as $R' = \{(f(x), f(y)) : (x,y) \in R\}$

$(x,y_1),(x,y_2) \neq f$

$(x_1,y),(x_2,y) = f$

(a) If R is reflexive then R' is reflexive

(a,b)
(c,b) ✓
(a,b)
(a,c) ✗

Let $(a,a) \in R$

Hence, $f(a) = a$

So, $f(a), f(a) \in R'$

Hence, it's reflexive

(b) If R is symmetric then R' is symmetric

$(a,b) \in R$

$(b,a) \in R$

So, $(f(a), f(b))$, $(f(b), f(a)) \in \bar{R}$

Now, $f(a) = b$

$f(b) = a$