Solutions to Homework 9:

1. *Proof.* (a) Assume that $f \circ f$ is injective, and let $a1, a2 \in A$ so that f(a1) = f(a2). To show that f is injective it suffices to show that a1 = a2. Since f(a1) = f(a2), we know that g(f(a1)) = g(f(a2)), and since $g \circ f$ is injective we have that a1 = a2. So, f is injective

   (b) Now assume that $f \circ f$ is surjective and let $c \in A$. To prove that g is surjective it suffices to find $b \in A$ so that g(b) = c. Since $g \circ f$ is surjective, we know that there is $a \in A$ so that g(f(a)) = c. Now set b = f(a). Then g(b) = g(f(a)) = c as required.
   So, f is surjective

   □

2. *Proof.* $f : A \to B$ is a surjection and $Y \subseteq B$
   Let $y_1 \in Y$
   So we have $f^{-1}(y_1) = \{x_1 \in A | f(x_1) = y_1\}$
   Now, $x_1 \in f^{-1}(y_1)$
   So, $f(x_1) = y_1 \in Y$
   This means , $x_1 \in f^{-1}(Y)$ and so, $f(x_1) \in f(f^{-1}(Y))$
   Now $f(x_1) = y_1$ and so $y_1 \in f(f^{-1}(Y))$
   But we know that $y_1 \in Y$
   So, $Y = f(f^{-1}(Y))$

   □

3. *Proof.* (a) Now we let f be surjective. So, let $A \in E$, and let $y \in F - f(A)$, so $y \notin f(A)$. We need to prove that $y \in f(E - A)$.
   Since f is surjective y = f(x) for some $x \in F$. Now, by definition of the image, if $x \in A$ then $f(x) \in f(A)$. Since we know, by assumption that $f(x) = y \notin f(A)$, it means that $x \notin A$.
   Hence $x \in E - A$, so that $y \in f(E - A)$.
   Therefore, $F - f(A) \subseteq f(E - A)$.

   (b) Now, let $\forall A \in E, F - f(A) \subseteq f(E - A)$.
   We then take $A = \phi$ to get
   $F - f(\phi) \subseteq f(E - \phi)$ so $F \subseteq f(E)$.
   So now, let $y \in F$. We know that, we have $y \in f(E)$ and so y = f(x) for some $x \in E$.
   Hence f is surjective.

   □

4. (a) In order to prove this we take three cases into consideration

   Case 1: $z > 0$
   Let's take $x = z^2 + 1$ and $y = z^2$
   $z = x^2 - y^2$

$z^2 = z^2 + 1 - z^2$
$z^2 = 1$
So $z > 0$ i.e it is positive
Case 2: $z < 0$
Let's take $x = z^2$ and $y = z^2 + 1$
$z = x^2 - y^2$
$z^2 = z^2 - z^2 - 1$
$z^2 = -1$
So $z < 0$ i.e it is negative
Case 3: $z = 0$
Let's take $x = z^2$ and $y = z^2$
$z = x^2 - y^2$
$z^2 = z^2 - z^2$
$z^2 = 0$
So $z = 0$

(b) $g(\{0\}) = \{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\}$
$g(x, y) = z = 0$
$x^2 - y^2 = 0$
$x^2 = y^2$
So, $(\pm x, \pm y)$
$g^{-1}(\{0\}) = \{p \in \mathbb{R} : (\pm p, \pm p)\}$

(c) **Case 1:** $0 \leq c < 3$
$h^{-1}(\{c\}) = \phi$ as c is positive since $x^4 \geq 0$,
therefore c will always be greater than 3.
Case 2: $c \geq 3$
$h^{-1}(\{c\}) = c \in A : (c - 3)^{1/4}$

5. *Proof.* $f : A \to B$ and $E, F \subseteq B$
Let $x \in f^{-1}(E - F)$ where $x \in A$
So, $f(x) \in E - F$
We can also write this as $f(x) \in E$ and $f(x) \notin F$
So, $x \in f^{-1}(E)$ and $x \notin f^{-1}(F)$
Which means, $x \in f^{-1}(E) - f^{-1}(F)$
Hence we have that $f^{-1}(E - F) = f^{-1}(E) - f^{-1}(F)$ □

6. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + ax + b$ where $a, b \in \mathbb{R}$
Injective:
For $x_1, x_2 \in \mathbb{R}$ assume that $f(x_1) = f(x_2)$

$$x_1^2 + ax_1 + b = x_2^2 + ax_2 + b \tag{1}$$
$$x_1^2 + ax_1 = x_2^2 + ax_2 \tag{2}$$
$$x_1^2 + ax_1 + \frac{a^2}{4} - \frac{a^2}{4} = x_2^2 + ax_2 + \frac{a^2}{4} - \frac{a^2}{4} \tag{3}$$
$$(x_1 + \frac{a}{2})^2 - \frac{a^2}{4} = (x_2 + \frac{a}{2})^2 - \frac{a^2}{4} \tag{4}$$
$$(x_1 + \frac{a}{2})^2 = (x_2 + \frac{a}{2})^2 \tag{5}$$
$$\tag{6}$$

On simplifying this we get that x_1 is not always equal to x_2 i.e $x_1 \neq x_2$
Hence f is not injective.

Surjective:
Let $f(x) = y$ and $y \in \mathbb{R}$
So for f(x)=y we know that
$f(x) = (x + \frac{a}{2})^2 + (b - \frac{a^2}{4})$
Now we know that $(x + \frac{a}{2})^2 \geq 0$ as sqaures are always positive so the minimum value f(x) can have is $(b - \frac{a^2}{4})$
Now let's say the value of y is $b - a^2$, so in that case
$f(x) \geq b - a^2$
In this case $b - a^2 \neq f(x), \forall x \in \mathbb{R}$
Hence f is not surjective

7. (a) Let $h(n) = |F| = 2^n$
We will try to prove this using induction
Base Case: n = 1
$h(1) = 2^1 = 2$
When n = 1, $f(a_1)$ can be mapped to $\{0, 1\}$ i.e two values and hence the base case holds true

Inductive Step: Let $h(k) = 2^k$ be true
So, for $h(k+1)$ we have $A = \{a_1, a_2, ..., a_k, a_{k+1}\}$
We can write this as $A = \{a_1, a_2, ..., a_k\} \cup \{a_{k+1}\}$
Now we know that $f(a_{k+1})$ can be mapped to two values i.e $\{0, 1\}$, and $h(k) = 2^k$
So F will have twice the number of elments as $h(k)$ that is $h(k+1) = 2 \cdot 2^k = 2^(k+1)$
Hence proved by induction hypothesis that $|F| = 2^n$

(b) $g(f) = \{a \in A : f() = 1\}$

Proof. **Injective:**
We let $f_1, f_2 \in F$ and $g(f_1) = g(f_2)$

We can now divide it into cases since we map it to two values i.e 0 and 1 so,

i. **Case 1:** $f_1(x) = 0, \forall x \in A$
This means that $x \notin g(f_1)$ which means that $x \notin g(f_2)$ since they are equal
This further means that $f_2(x) = 0$
Hence $f_1 = f_2$

ii. **Case 2:** $f_1(x) = 1, \forall x \in A$
This means that $x \in g(f_1)$ which means that $x \in g(f_2)$ since they are equal
This further means that $f_2(x) = 1$
Hence $f_1 = f_2$

So, we can say that g is injective
Surjective:
Let X be any set from $\mathcal{P}(A)$.
We define a function $h : A \to \{0, 1\}$ as:
When $x \in A - X, h(a) = 0$
When $x \in X, h(a) = 1$
Now, consequently
$g(h) = \{a \in A : h(a) = 1\} = \{a \in A : a \in X\} = a \in X = X$.
So, we can say that g is surjective. □

8. *Proof.* We will prove this using strong induction that $f(n) = n$

(a) **Base case:** Let's take n = 1
Since $f(1) = 1$ which also means $f(1) \leq 1$ the base case holds true

(b) **Induction step:** Assume that f(k) = k for all $k \leq n$ holds true.
So, $f(n + 1) \in \{1, 2, 3, ..., n + 1\}$.
Since $f(k) = k$ and $k \neq n + 1$ we have that
$f(n + 1) = n + 1$

So, by strong induction we have that $f(n) = n, \forall n \in N$ □