Mathematics 220 — Homework 8

- Contains 7 questions on 2 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

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1. Prove or disprove. If a relation $\mathcal{R}$ on a set A is symmetric and transitive, then it is also reflexive.

2. Define a relation on $\mathbb{Z}$ as aRb if $3 \mid (5a - 8b)$. Is R an equivalence relation? Justify your answer.

3. Determine whether the following relations are reflexive, symmetric and transitive.

   (a) On the set X of all functions $\mathbb{R} \to \mathbb{R}$, we define the relation:
   $$f\mathcal{R}g \text{ if there exists } x \in \mathbb{R} \text{ such that } f(x) = g(x).$$

   (b) Let R be a relation on $\mathbb{Z}$ defined by:
   $$xRy \text{ if } xy \equiv 0 \pmod 4.$$

4. Let A be a non-empty set and $P \subseteq \mathcal{P}(A)$ and $Q \subseteq \mathcal{P}(A)$ partitions of A. Show that R defined as
   $$R = \{S \cap T : S \in P,\ T \in Q\} \setminus \{\emptyset\}$$
   is a partition of A.

5. Let E be a non-empty set and $x \in E$ be a fixed element of E. Consider the relation $\mathcal{R}$ on $\mathcal{P}(E)$ defined as
   $$A\mathcal{R}B \iff (x \in A \cap B) \vee (x \in \overline{A} \cap \overline{B}),$$
   where for any set $S \subseteq E$, we write $\overline{S} = E \setminus S$ for the complement of S in E. Prove or disprove that $\mathcal{R}$ an equivalence relation.

6. Let $n \geq 2$ and i be integers. For $0 \leq i \leq n-1$, define
   $$X_i = \{x \in \mathbb{Z} |\ x = nk + i, \text{ for some } k \in \mathbb{Z}\} \qquad \text{and}$$
   $$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} |\ a, b \in X_i \text{ for some } i\}.$$

   Show that

   (a) $S = \{X_0, \ldots, X_{n-1}\}$ forms a partition of $\mathbb{Z}$.

   (b) R is an equivalence relation on $\mathbb{Z}$.

(c) S equals the set of the equivalence classes of R.

7. Suppose that $n \in \mathbb{N}$ and $\mathbb{Z}_n$ is the set of equivalence class of congruence modulo n on $\mathbb{Z}$. In this question we will call an element $[u]_n$ invertible if there is another class $[v]_n$ so that

$$[u]_n \cdot [v_n] = [u \cdot v]_n = [1]_n.$$

That is, $[u_n]$ has a multiplicative inverse. For example, since $[3]_7 \cdot [5]_7 = [15]_7 = [1]_7$, we say that $[3]_7$ is invertible. However, $[2]_4$ is not invertible (you can check!).

Now, define a relation R on $\mathbb{Z}_n$ by xRy iff $xu = y$ for some invertible $[u]_n \in \mathbb{Z}_n$.

(a) Show that R is a equivalence relation.

(b) Compute the equivalence classes of this relation for $n = 6$.

Hint: Start by finding the invertible elements in $\mathbb{Z}_6$ — you can do this by making a table of $[u]_6 \cdot [v]_6$.

HW 8

① If relation R is symm & transi $\Rightarrow$ reflexive

Let aRb and bRa — eq 1 — as R is symmetric

So, we know that for a relation to be transitive xRy and $yRz \Rightarrow xRz$ and here R is transitive

Hence from eq 1, $(a,a) \in R$ & hence R is reflexive

② aRb if $3 \mid (5a-8b)$ is R an eqv relation

$R = \{(0,0), (1,1), (2,2), (3,6), (6,3), (6,9), (3,9) \ldots\ldots\}$

<u>Reflexive</u>

Let's take an ordered pair (a,a)

So, $5a-8b = 5a-8a = -3a$

So, $3 \mid 5a-8a$

Hence aRa so,

R is reflexive

<u>Symmetric</u>

Let $(a,b) \in R$

So, $3 \mid 5a-8b$

or $3n = 5a-8b$

So, $3n+3a+3b = 8a-5b$

$-3(n+a+b) = 5b-8a$

So, $3 \mid 5b-8a$

Hence, $(b,a) \in R$

R is symmetric

Transitive

Let aRb & bRc so,

$3n = 5a - 8b$ & $3m = 5b - 8c$

Adding them we get,

$$3(n+m) = 5a - 3b - 8c$$

$$3(n+m+b) = 5a - 8c$$

So, $3 \mid 5a - 8c$

Hence, $(a,c) \in R$ &

R is transitive

And since R is reflexive, symmetric & transitive it's an equivalent relation.

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③ (a) fRg if there exists $x \in \mathbb{R}$ such that $f(x) = g(x)$

Reflexive

$\forall a \in \mathbb{R}$, $f(a) = f(a)$

Hence fRf

R is reflexive

Check TB Q4

Symmetric

Let fRg so, $f(a) = g(a)$, $\forall a \in \mathbb{R}$

Hence, $g(a) = f(a)$

So, gRf

R is symmetric

$f(x) = |x|$

$g(x) = |f(x)|$

$h(x) = x$

(b) xRy if $xy \equiv 0 \pmod 4$

Reflexive

Let, $x = 3$

So, $3 \cdot 3 = 9$

$9 \equiv 1 \pmod 4$

So, $(x, x) \notin R$

Hence R is not reflexive

Symmetric

Let xRy and $xy \equiv 0 \pmod 4$

So, $xy = 4n$

and $y = \frac{4n}{x}$

So, $y \cdot x = 4n$

So, $4 \mid y \cdot x$

So, $y \cdot x \equiv 0 \pmod 4$

So, yRx & R is symmetric

Transitive

let $3R4$ and $4R5$ exist

Since $12 \equiv 0 \pmod 4$ &

$20 \equiv 0 \pmod 4$

But $3 \times 5 = 15 \equiv 3 \pmod 4$

So, $(3,5) \notin R$

Hence, R is not transitive

(4)

$A = \{1, 2\}$

$P(A) = \{\Phi, \{1\}, \{2\}, \{1,2\}\}$

$P(A) = \{\{1\}, \{2\}\}$

Check H.W Qs 14

(5)

5. Let E be a non-empty set and $x \in E$ be a fixed element of E. Consider the relation $\mathcal{R}$ on $\mathcal{P}(E)$ defined as

$$A\mathcal{R}B \iff (x \in A \cap B) \vee (x \in \overline{A} \cap \overline{B}),$$

where for any set $S \subseteq E$, we write $\overline{S} = E \setminus S$ for the complement of S in E. Prove or disprove that $\mathcal{R}$ an equivalence relation.

$ARB \iff (x \in A \cap B) \vee (x \in \bar{A} \cap \bar{B})$

Check to qs 7 → similar

(6)

6. Let $n \geq 2$ and i be integers. For $0 \leq i \leq n-1$, define

$$X_i = \{x \in \mathbb{Z} | \; x = nk + i, \text{ for some } k \in \mathbb{Z}\} \qquad \text{and}$$
$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | \; a, b \in X_i \text{ for some } i\}.$$

Show that

(a) $S = \{X_0, \ldots, X_{n-1}\}$ forms a partition of $\mathbb{Z}$.

(b) R is an equivalence relation on $\mathbb{Z}$.

(a) $S = \{X_0, \ldots X_{n-1}\}$

$= \{nk, nk+1, nk+2 \ldots \ldots nk+(n-1)\}$

Now, we need to prove that

$\bigcup_{X\in S} X = \mathbb{Z}$ &&

$X, Y \in S$ then $\Rightarrow$ $X \cap Y = \phi$ or $X = Y$

- $\bigcup_{X\in S} X = \mathbb{Z} \iff \bigcup_{X\in S} X \subseteq \mathbb{Z} \cap \mathbb{Z} \subseteq \bigcup_{X\in S} X$

Now, $\bigcup_{X\in S} X = X_0 \cup X_1 \cup X_2 \cup \dots X_{n-1}$

Now, since $X_i = nk+i$

$x \in X_i$

$x \in \mathbb{Z}$

So, $X_i \subseteq \mathbb{Z}$

So, $\bigcup_{X\in S} X \subseteq \mathbb{Z}$

Now, $\forall\, x \in \mathbb{Z}$ in $\bigcup_{X\in S} X$ there exists X_i such that $x \in X_i$ b/c

$x = nk+i$

So, $\mathbb{Z} \subseteq \bigcup_{X\in S} X$

Now let $X, Y \in S$

$x \in X$, $y \in Y$

So, $x = nk+i$

$y = nk+j$

Let $X = X_i$ & $Y = X_j$

<u>Case 1:</u> and $i > j$

So, let's consider $x \in X_i$ and $x \notin X_j$

So, $x \in X_i - X_j$

So, $X_i - X_j \subseteq X_i$

Now, let $x \in X_i - X_j$

So, $x \in X_i \cap x \notin X_j$

So, $X_i \subseteq X_i - X_j$

So, $X_i - X_j = X_i$

or

$X_i \cap X_j = \phi$

So, $X \cap Y = \phi$

<u>Case 2:</u> $i = j$

So, since $x \in X_i$ where $x = nk + i$

$0 \le i \le n-1$

$y \in X_j \Rightarrow 0 \le j \le n-1$

Th 9.3.13

So, they'll have same elements

i.e $X_i = X_j$ or $X = Y$

(b) R is an equivalence relation on $\mathbb{Z}$

(c) S equals set of the equivalence class of R

$x \in X_i$

$[i]_n = [x]_n$

$x = nk$

$[0]$ yRO

$y, 0 \in X_0$

xRO $x \in X_0$

$[0]$ $[x]_n = [0]_n$

$[0] = X_0$

$y \in X_r$ $\qquad\qquad\qquad [j] = X_j$

$$[y]_n = \underline{[i]_n}$$

$$a = x$$

$$[x]_n, [y]_n$$

$$([0]_n, [i]_n)$$

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Now, define a relation R on $\mathbb{Z}_n$ by xRy iff $xu = y$ for some invertible $[u]_n \in \mathbb{Z}_n$.

(a) Show that R is a equivalence relation.

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Hint: Start by finding the invertible elements in $\mathbb{Z}_6$ — you can do this by making a table of $[u]_6 \cdot [v]_6$.

(a) Reflexive: $\forall [a]_n \in \mathbb{Z}_n$, we know that

$$[a]_n = [a \cdot 1]_n$$
$$= [a]_n \cdot [1]_n \quad \left([1]_n \text{ is invertible as } [1]_n \cdot [1]_n = [1]_n\right)$$

$\therefore$, R is reflexive

(b) Symmetric: let $[a]_n$ & $[b]_n \in \mathbb{Z}_n$ and aRb be true.
So, for some invertible class $[w]_n \in \mathbb{Z}_n$,

$$[a]_n \cdot [w]_n = [b]_n$$

let $[v]_n \in \mathbb{Z}_n$ be multiplicative inverse of $[w]_n$

$$\therefore [a]_n \cdot [w]_n \cdot [v]_n = [b]_n \cdot [v]_n$$
$$[a]_n \cdot [1]_n = [b]_n \cdot [v]_n$$
$$\therefore [b]_n \cdot [v]_n = [a]_n$$
$\therefore$ bRa is true

$\therefore$ R is symmetric

(C) Transitive : let $[a]_n, [b]_n$ & $[c]_n \in \mathbb{Z}_n$ and aRb & bRc be true. So, for some invertible $[u]_n$ & $[p]_n \in \mathbb{Z}_n$,

$$[a]_n \cdot [u]_n = [b]_n \qquad [b]_n \cdot [p]_n = [c]_n$$

let $[v]_n$ & $[q]_n \in \mathbb{Z}_n$ be multiplicative inverse of $[u]_n$ & $[p]_n$ resp

$$[c]_n = [a]_n \cdot [u]_n \cdot [p]_n$$
$$= [a]_n \cdot [u \cdot p]_n$$

Now,

$$[u \cdot p]_n \cdot [v \cdot q]_n = [u]_n \cdot [p]_n \cdot [v]_n \cdot [q]_n$$
$$= [u]_n \cdot [v]_n \cdot [p]_n \cdot [q]_n$$
$$= [1]_n \cdot [1]_n$$
$$= [1]_n$$

$\therefore$ $[u \cdot p]_n$ is invertible.

$\therefore$ $[c]_n = [a]_n \cdot [u \cdot p]_n$

$\therefore$ if aRb & bRc are true, then aRc is true

$\therefore$ R is transitive.

(b)

	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$
$[0]_6$	0	0	0	0	0	0
$[1]_6$	0	1	2	3	4	5
$[2]_6$	0	2	4	0	2	4
$[3]_6$	0	3	0	3	0	3
$[4]_6$	0	4	2	0	4	2
$[5]_6$	0	5	4	3	2	1

Invertible classes: $[1]_6$, $[5]_6$

$[[1]_6] = \{[1]_6, [5]_6\}$ $\quad [[0]_6] = \phi$

$[[2]_6] = \{[2]_6, [10]_6\}$

$[[3]_6] = \{[3]_6, [15]_6\}$

$[[4]_6] = \{[4]_6, [20]_6\}$

$[[5]_6] = \{[5]_6, [25]_6\}$