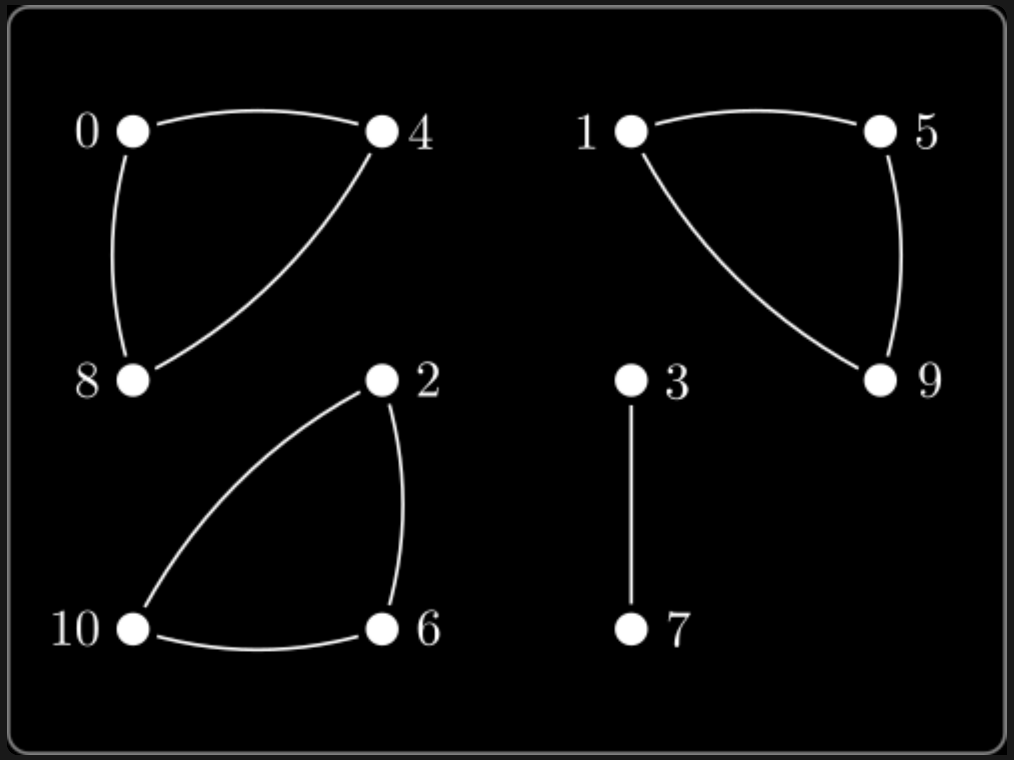
PLP - 29

TOPIC 29—SET PARTITIONS

Demirbaş & Rechnitzer

SET PARTITIONS

EQUIVALENCE CLASSES — EQUAL OR DISJOINT



COROLLARY:

Let R be an equivalence class on A and $a, b \in A$. Then

$$[a] = [b] \qquad \text{or} \qquad [a] \cap [b] = \varnothing$$

EQUAL OR DISJOINT

$[a] = [b]$ *or* $[a] \cap [b] = \varnothing$

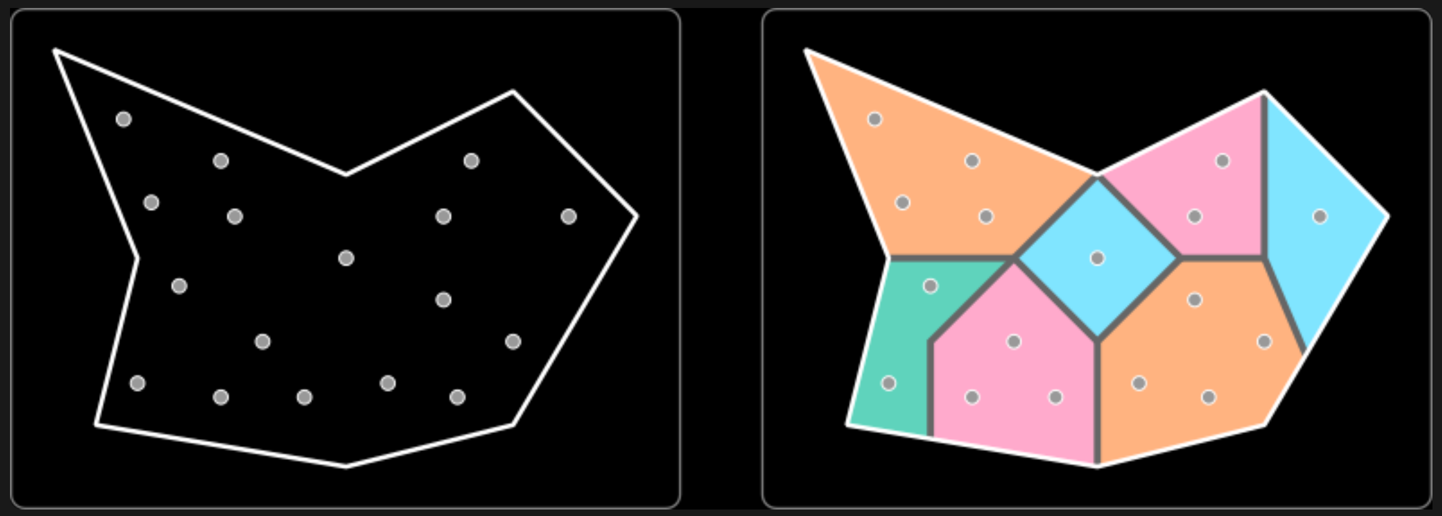
PROOF.

Let $a, b \in A$ and consider the intersection $C = [a] \cap [b]$. Either $C = \varnothing$ or $C \neq \varnothing$.

- If $C = \varnothing$ we are done.
- So now assume that $C \neq \varnothing$, which means there is some $c \in C$. Hence $c \in [a]$ and $c \in [b]$. Thus $c \; R \; a$ and $c \; R \; b$.

  By symmetry we know $a \; R \; c$, and then transitivity gives $a \; R \; b$. The previous theorem then implies $[a] = [b]$ as needed.

CUTTING UP A SET



DEFINITION:

A **partition** of the set A is a set, $\mathcal{P}$, of non-empty subsets of A so that

- if $x \in A$ then there is $X \in \mathcal{P}$ with $x \in X$
- if $X, Y \in \mathcal{P}$ then either $X \cap Y = \varnothing$ or $X = Y$

Elements of $\mathcal{P}$ are **parts** or **pieces** of the partition.

EQUIVALENCE CLASSES ARE A PARTITION

THEOREM:

Let R be an equivalence relation on A.

The set of equivalence classes of R forms a set partition.

Scratch work

- Let $\mathcal{P} = \{[x] \;:\; x \in A\}$
- Need to show that every $x \in A$ belongs to some $X \in \mathcal{P}$

  We already proved that each x belongs to $[x]$.
- Need to show that for each $X, Y \in \mathcal{P}$, either $X \cap Y = \varnothing$ or $X = Y$

  We just proved this!

PROOF

Equivalence classes form a set partition

PROOF.

Let $\mathcal{P} = \{[x] \ : \ x \in A\}$.

- Let $x \in A$ then we proved previously that $x \in [x]$. Since $[x] \in \mathcal{P}$, we know that x is in some piece of the partition.
- Let $X, Y \in \mathcal{P}$. By the previous corollary we know that either $X = Y$ or $X \cap Y = \varnothing$.

Thus $\mathcal{P}$ forms a set partition.

We can go further — a set partition can define an equivalence relation.

A SET PARTITION GIVES AN EQUIVALENCE RELATION

THEOREM:

Let $\mathcal{P}$ be a set partition of A. Now define a relation by

$$x \; R \; y \qquad \iff \qquad \exists X \in \mathcal{P} \text{ s.t. } x, y \in X$$

then R is an equivalence relation.

Scratch work / proof sketch — a good exercise