# Math 220
# Section 108
# Lecture 9

6th October 2022

# Worksheet 5 – Limits

## Definition (Definition 6.4.2 of PLP)

Let $(x_n)$ be a sequence of real numbers. We say that $(x_n)$ has a **limit** $L \in \mathbb{R}$ when

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n > N) \implies (|x_n - L| < \epsilon).$$

In this case we say that the sequence **converges** to $L$ and write

$$x_n \to L \quad \text{as } n \to \infty \qquad \text{or} \qquad \lim_{n \to \infty} x_n = L.$$

If the sequence doesn't converge to any number $L$, we say that the sequence **diverges**.

1. Show that $(x_n) = \left(\frac{n}{n^2+1}\right)$ converges to 0.

*Hint: For $n \in \mathbb{N}$, we have $\frac{n}{n^2+1} < \frac{1}{n}$.*

Want: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|x_n - L| < \varepsilon$.

$\left|\frac{n}{n^2+1} - 0\right| < \varepsilon$

$\frac{n}{n^2+1} < \varepsilon$.

Scratch:

Since $\frac{n}{n^2+1} < \frac{1}{n}$, if $\frac{1}{n} < \varepsilon$

then $\frac{n}{n^2+1} < \frac{1}{n} < \varepsilon$

If $\frac{1}{n} < \varepsilon$, then $\frac{1}{\varepsilon} < n$.

"ceiling function" (just round up)

Proof: Given some $\varepsilon > 0$, let $N = \left\lceil \frac{1}{\varepsilon} \right\rceil \in \mathbb{N}$.

For any $n > N = \left\lceil \frac{1}{\varepsilon} \right\rceil \geq \frac{1}{\varepsilon}$, we have $n > \frac{1}{\varepsilon}$, so $\varepsilon > \frac{1}{n}$.

Now $\varepsilon > \frac{1}{n} > \frac{n}{n^2+1} = \left|\frac{n}{n^2+1} - 0\right|$ $\forall n > N$. $\square$

# No no there's no limit

2. Show that $(x_n) = \left(\frac{n}{\sqrt{n^2+1}}\right)$ doesn't converge to 0.

*Hint: What does it mean for a sequence to NOT converge to a number?*

Negate: $\exists\, \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists\, n > N$ s.t. $|x_n - L| \geq \varepsilon$.

$$\left|\frac{n}{\sqrt{n^2+1}} - 0\right| \geq \varepsilon$$

$$\frac{n}{\sqrt{n^2+1}} \geq \varepsilon$$

Scratch: $n=1$: $\frac{1}{\sqrt{2}}$, $n=2$: $\frac{2}{\sqrt{5}}$, $n=3$: $\frac{3}{\sqrt{10}}$

Or note: $\frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}}$. An increasing function!

So it seems that $\frac{n}{\sqrt{n^2+1}} > \frac{1}{2}$ $\forall\, n \in \mathbb{N}$.

Proof: Let $\varepsilon = \frac{1}{2}$. Given any $N \in \mathbb{N}$, let $n = 2N$.

Then $|x_n - 0| = \left|\frac{n}{\sqrt{n^2+1}} - 0\right| = \frac{1}{\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{1+\frac{1}{(2N)^2}}} > \frac{1}{\sqrt{1+\frac{1}{1}}} = \frac{1}{\sqrt{2}} \geq \varepsilon$. $\square$

3. Let $(x_n), (b_n)$ be sequences. Prove that if $0 < x_n < b_n$ $\forall n \in \mathbb{N}$ and $b_n \to 0$, then $x_n \to 0$.

*Hint: If* $0 < x_n < b_n$*, then* $|x_n| < |b_n|$*. Thus, if we can make* $|b_n| < \epsilon$*, then we will have* $|x_n| < \epsilon$*.*

Want: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|x_n - 0| < \varepsilon$.

Scratch: $|x_n - 0| = |x_n| = x_n < \varepsilon$.

We know: $\forall \varepsilon$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|b_n - 0| < \varepsilon$

(i.e. $b_n < \varepsilon$).

Proof: Given $\varepsilon > 0$, we know that $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|b_n - 0| < \varepsilon$. So $b_n < \varepsilon$, $\forall n > N$.

Since $x_n < b_n$ $\forall n \in \mathbb{N}$, we have $x_n < \varepsilon$ $\forall n > N$.

Since $x_n > 0$, we have $|x_n - 0| < \varepsilon$ $\forall n > N$. $\square$