PLP - 39

TOPIC 39 — CARDINALITY OF FINITE SETS

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FINITE SETS AND PIGEONS

SIZE OF A FINITE SET

We defined $|A|$ to be the number of elements in A

When we count the elements in A we count off “one, two, three, … six”

We build a function $f : A \to \{1, 2, \ldots, 6\}$ so that

- Injective — *different* objects counted by *different* numbers
- Surjective — each number is used to count an object

So that function is a *bijection*.

EQUAL CARDINALITIES AND BIJECTIONS

DEFINITION:

Let A, B be sets. They have the same **cardinality** if $A = B = \varnothing$ or if there is a bijection from A to B.

In this case we write $|A| = |B|$ and say the sets are **equinumerous**.

If A and B are not equinumerous, so no bijection between them, then we write $|A| \neq |B|$

- Special case: $A = \varnothing$ we write $|A| = 0$. No bijection between empty sets.
- Finite: $|A| = n \in \mathbb{N}$ then have a bijection $f : A \to \{1, 2, \ldots, n\}$.
- If $|A| = n = |B|$ then we have

$$f : A \to \{1, \ldots, n\} \qquad g : B \to \{1, \ldots, n\} \qquad \text{so} \qquad (g^{-1} \circ f) : A \to B$$

- Definition in terms of bijection allows us to handle infinite sets.

FINITE NON-EQUINUMEROUS SETS

Consider $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$

- Since $5 = |A| \neq |B| = 3$, so no bijection between them
- Easy to build a surjection $f : A \to B$
- But cannot build an injection $g : A \to B$:

$$g(a_1) = b_1 \qquad g(a_2) = b_2 \qquad g(a_3) = b_3 \qquad g(a_4) = ? \qquad g(a_5) = ?$$

- Use the pigeonhole principle to formalise this.

PIGEONHOLE PRINCIPLE

THEOREM: (DIRICHLET).

If n objects are placed in k boxes then

- If $n < k$ then at least one box has zero objects in it
- If $n > k$ then at least one box has at least two objects in it

Can refine $n > k$ case: at least one box has at least $\lceil n/k \rceil$ objects in it.

PROOF.

Prove contrapositive of each:

- Assume each box contains at least one object, then total number of objects $n \geq k$.
- Assume each box contains at most one object, then the total number of objects $n \leq k$.

FINITE NON-EQUINUMEROUS SETS — CONTINUED

COROLLARY:

Let A, B be *finite* sets and let $f : A \to B$. Then

- If $|A| > |B|$ then f is not an injection
- If $|A| < |B|$ then f is not a surjection

If f is an injection then $|A| \leq |B|$

If f is a surjection then $|A| \geq |B|$

PROOF.

We prove each point in turn.

- Assume that $|A| > |B|$. Then, by PHP, when the images of elements of A are placed into B by the function, at least one element of B is the image of two elements of A. Hence there are $a_1, a_2 \in A$ so that $f(a_1) = f(a_2)$, and so f is not injective.
- Now assume that $|A| < |B|$. Then, by PHP, when the images of elements of A are placed into B by the function, at least one element of B is not the image of any element of A. That is, there is $b \in B$ so that for every $a \in A$, $f(a) \neq b$, and so f is not surjective.

PIGEON FLAVOURED EXAMPLE

PROPOSITION:

There exist two powers of 3 whose difference is divisible by 220.

PROOF.

- Consider the sequence of 221 numbers

$$3^0, 3^1, 3^2, 3^3, \ldots, 3^{219}, 3^{220}$$

  and compute their remainders when divided by 220.

- There are at most 220 possible remainders, but 221 numbers in the sequence
- Hence two numbers have the same remainder: $3^i = 220k + r, 3^j = 220\ell + r$
- So their difference is a multiple of 220 as required

Checking remainders modulo 220 you can quickly find that $220 \mid (3^{20} - 3^0)$

PIGEON FLAVOURED EXAMPLE #2

PROPOSITION:

Place 5 points in an equilateral triangle of side-length 1. There is a pair at distance no greater than 0.5

PROOF.

- Split the triangle into 4 sub-triangles as shown
- The subtriangle side-length is $\frac{1}{2}$
- One sub-triangle must contain 2 points
- So those points are at distance $\leq \frac{1}{2}$

BIG DATA $\Longrightarrow$ SPURIOUS CORRELATIONS

Consider

- How many function “shapes” you can draw
- How many time-series data sets exist
- By PHP, each “shape” will fit many data-sets
- Those data-sets appear correlated

Search-engine your way to “Spurious Correlations” by Tyler Vigen