Math 220
Section 108
Lecture 10

11th October 2022

Limit definition

Definition (Definition 6.4.8 of PLP)

Let $a, L \in \mathbb{R}$ and let f be a real-valued function. We say that the **limit** of f as x approaches a is L when

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } (0 < |x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon).$$

In this case we write

$$\lim_{x \to a} f(x) = L \qquad \text{or} \qquad f(x) \to L \text{ as } x \to a$$

and say that f **converges** to L as x approaches a. We also sometimes say that the limit of f as x goes to a is L.
If f does not converge to any finite limit L as x approaches a , then we say that f **diverges** as x approaches a.

4. Show that $\lim_{x\to 1}(5x+3) = 8$.

Hint: Be careful that δ may depend on ϵ.

$\forall x \in \mathbb{R}$,

$\forall \varepsilon > 0, \exists \delta > 0$, s.t $(0 < |x-a| < \delta) \Rightarrow (|f(x) - L| < \varepsilon)$

Scratch

we want: $|5x+3-8| < \varepsilon$, if $0 < |x-1| < \delta$

s, $|5x-5| < \varepsilon$ so, $5(x-1) < \varepsilon$

Proof

Let's consider $5\delta = \varepsilon$, $\delta > |x-a|$

we have $0 < |x-a| < \delta$

So, $|x-1| < \delta$

so, $5|x-1| < 5\delta$

So, $\varepsilon > 5|(x-1)|$ [since $5\delta = \varepsilon$]

$\varepsilon > |5x-5|$

$\varepsilon > |5x+3-8|$

So, $\varepsilon > |5x+3-8|$ $\blacksquare$

(Continued)

5. Prove that $\lim_{x \to 2} \left(\frac{1}{x}\right) = \frac{1}{2}$.

Hint: You may need to have more than one condition on δ.

scratch we need $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$, if $0 < |x-2| < \delta$

$$\left|\frac{2-x}{2x}\right| < \varepsilon$$

$2\delta = \varepsilon$ $\qquad \left|\frac{x-2}{2x}\right| \leq |x-2| < \varepsilon$

Proof

Given, $\varepsilon > 0$, we have $\delta \leq 2\varepsilon$ & $\delta \leq 1$ then,

So, $\varepsilon > |x-2|$

$\varepsilon > \left|\frac{x-2}{2x}\right|$ [since $|x-2| \geq \left|\frac{x-2}{2x}\right|$, for $x > 1$ bcus $\delta \leq 1$]

$\blacksquare$

(Continued)

Induction

1. Consider the sequence defined by

$$u_1 = \frac{1}{2}, \quad u_{n+1} = \frac{1+u_n}{2+u_n} \text{ for } n \in \mathbb{N}.$$

Prove that $0 < u_n < 1$ for every $n \in \mathbb{N}$.

Base $\to$ $u_1 = \frac{1}{2}$ $\Rightarrow$ $0 < u_1 < 1$

Assume $\to$ $0 < u_K < 1$

W.T.S. $\to$ $0 < u_{K+1} < 1$.

Now, $0 < u_K < 1$ $\Rightarrow$ $1 < u_K + 1 < 2$

& $2 < u_K + 2 < 3$

So, $\frac{1}{2} > \frac{1}{u_K + 2} > \frac{1}{3}$

So, $1 > \frac{u_K + 1}{u_K + 2} > \frac{1}{3}$ So $1 > u_{K+1} > \frac{1}{3}$ $\Rightarrow$ $1 > u_{K+1} > 0$ $\boxed{}$

(Continued)

2. The Fibonacci numbers are defined by the recurrence

$$F_1 = F_2 = 1 \quad \text{and } F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 2.$$

Show that for every $n \in \mathbb{N}$, F_{4n} is a multiple of 3.

(Continued)

3. Prove that $7^n - 2^n$ is divisible by 5 for all $n \in \mathbb{N}$.

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