SOMAIYA VIDYAVIHAR UNIVERSITY

<table>
<tr><td colspan="3">Maximum Marks: 100   Semester: October 2022 – January 2023
Examination: ESE Examination   Duration:3 Hrs.</td></tr>
<tr><td>Programme code: 06
Programme: B. Tech</td><td>Class: FY</td><td>Semester: I (SVU 2020)</td></tr>
<tr><td colspan="2">Name of the Constituent College:
K. J. Somaiya College of Engineering</td><td>Name of the department:
COMP/IT/EXCP/EXTC/MECH</td></tr>
<tr><td>Course Code: 116U06C101</td><td colspan="2">Name of the Course: Applied Mathematics-I</td></tr>
<tr><td colspan="3">Instructions: 1) Draw neat diagrams 2) All questions are compulsory
3) Assume suitable data wherever necessary</td></tr>
</table>

Que. No.	Question	Max. Marks
Q1	Solve any **Four** of the following	**20**
i)	If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2.2^{n/2} \cos n\,\pi/4$, Hence, deduce that $\alpha^8 + \beta^8 = 32$ **Solution:** The given equation is $x^2 - 2x + 2 = 0$ $\therefore x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$ $\therefore \alpha = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ $\beta = 1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$ $\therefore \alpha^n + \beta^n = \left[\sqrt{2}\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\left(cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^n$ $= 2^{n/2}\left(cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + 2^{n/2}\left(cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$ $= 2^{n/2}\left(cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$ $= (\sqrt{2})^n\left(2cos\frac{n\pi}{4}\right) \quad = 2 \cdot 2^{n/2} cos\frac{n\pi}{4}$ Putting $n = 8$ $\quad \alpha^8 + \beta^8 = 2 \cdot 2^4 \cos 2\pi = 2^5 = 32$	2 5
ii)	Is the matrix $A = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$ orthogonal? If not, can it be converted into orthogonal matrix? **Solution:** For orthogonality $AA' = I$ Now, $AA' = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Since $AA' \neq I$ but 6 I, A is not orthogonal But the matrix A can be converted into an orthogonal matrix. Since, $AA' = 6I$, $\frac{1}{6}AA' = I$ $\quad \therefore \frac{1}{\sqrt{6}}A.\frac{1}{\sqrt{6}}A' = I$	3

$$\therefore \frac{1}{\sqrt{6}}A = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$$ is the required orthogonal matrix.

5

iii) Check whether the vectors $X_1 = [1 \quad 1 \quad 1]$, $X_2 = [1 \quad 2 \quad 4]$, $X_3 = [-2 \quad 3 \quad 8]$ are linearly dependent or independent.

Solution: Consider $k_1X_1 + k_2X_2 + k_3X_3 = 0$

$\therefore k_1[1\ 1\ 1] + k_2[1\ 2\ 4] + k_3[-2\ 3\ 8] = 0$

$\therefore k_1 + k_2 - 2k_3 = 0$..................(i) $\quad k_1 + 2k_2 + 3k_3 = 0$ ....(ii)

$k_1 + 4k_2 + 8k_3 = 0$................(iii)

This is a homogeneous system of equations $\begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ 1 & 4 & 8 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2

By $R_2 - R_1,\ R_3 - R_1$, we get $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 5 \\ 0 & 3 & 10 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_3 - 3R_2$, we get $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here Rank of A (r) = 3 = n Hence System has unique solution.

Hence trivial solution By solving, $k_1 = k_2 = k_3 = 0$

Therefore the vectors X_1, X_2, X_3 are linearly independent.

5

iv) Find Eigen values of $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ and hence find Eigen values of A^{10}

Solution: By solving characteristic equation $|A - \lambda I| = 0$, i.e. $\lambda^3 - 6\lambda^2 = 0$

Eigenvalues will be 6, 0, 0

3

Hence By properties of eigenvalues, we get eigenvalues of A^{10} to be $6^{10}, 0, 0$

5

v) If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$.

Solution: Let $l = x^2 - y^2, m = y^2 - z^2, n = z^2 - x^2$

$$\frac{\partial l}{\partial x} = 2x, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = -2x \qquad \frac{\partial l}{\partial y} = -2y, \quad \frac{\partial m}{\partial y} = 2y, \quad \frac{\partial n}{\partial y} = 0$$

$$\frac{\partial l}{\partial z} = 0, \quad \frac{\partial m}{\partial z} = -2z, \quad \frac{\partial n}{\partial z} = 2z$$

$$u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2) = f(l, m, n)$$

2

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x} = \frac{\partial u}{\partial l}\cdot 2x + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\cdot(-2x)$$

$$\frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\frac{\partial u}{\partial n} \qquad \text{................ (1)}$$

Also, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y} = \frac{\partial u}{\partial l}(-2y) + \frac{\partial u}{\partial m}(2y) + \frac{\partial u}{\partial n}(0)$

$$\frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial l} + 2\frac{\partial u}{\partial m} \qquad \text{................ (2)}$$

and $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z} = \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}(-2z) + \frac{\partial u}{\partial n}(2z)$

$$\frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial m} + 2\frac{\partial u}{\partial n} \qquad \text{................ (3)}$$

	Adding Eqs (1), (2) and (3), $\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{z}\frac{\partial u}{\partial z}=0$	5
vi)	If $u=x^2tan^{-1}\frac{y}{x}+y^2sin^{-1}\frac{x}{y}$, prove that $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}=2u$ **Solution:** Putting $X=xt, Y=yt$, we get, $f(X,Y)=x^2t^2tan^{-1}\left(\frac{yt}{xt}\right)+y^2t^2sin^{-1}\left(\frac{xt}{yt}\right)=t^2\left[x^2tan^{-1}\frac{y}{x}+y^2sin^{-1}\frac{x}{y}\right]$ $=t^2\cdot f(x,y)$ Thus, u is a homogeneous function of degree $n=2$ Hence, by the corollary-1, we get, $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}=n(n-1)u=2\cdot1\cdot u=2u$	3 5
Q2 A	Solve the following	**10**
i)	Solve the equation $7cosh\ x+8sinh\ x=1$ for real values of x **Solution:** $7cosh\ x+8sinh\ x=1$ Putting the values of $coshx$ and $sin\ hx$, we get $\therefore 7\left(\frac{e^x+e^{-x}}{2}\right)+8\left(\frac{e^x-e^{-x}}{2}\right)=1$ $\therefore 7e^x+7e^{-x}+8e^x-8e^{-x}=2$ $\therefore 15e^x-e^{-x}=2$ $\therefore 15e^{2x}-2e^x-1=0$ Solving it as a quadratic equation in e^x, $e^x=\frac{2\pm\sqrt{4-4(15)(-1)}}{2(15)}=\frac{2\pm8}{30}=\frac{1}{3}\ or -\frac{1}{5}$ $\therefore x=\log\left(\frac{1}{3}\right)\ or\ x=\log\left(-\frac{1}{5}\right)$ Since x is real, $x=log\left(\frac{1}{3}\right)=-\log 3$	3 5
ii)	If $x=e^u\tan v, y=e^u\sec v$, prove that $\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=0$ **Solution:** $x=e^u\tan v, y=e^u\sec v$, $y^2-x^2=e^{2u}(\sec^2v-\tan^2v)=e^{2u}$ and $\frac{x}{y}=\sin v$ $v=\sin^{-1}\left(\frac{x}{y}\right)$ v is a homogenous function of degree 0 By Euler's theorem, $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0$ Hence, $\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=0$	3 5
	OR	
Q2 A	Find the Eigenvalues and Eigenvectors of matrix $A=\begin{bmatrix}-9&4&4\\-8&3&4\\-16&8&7\end{bmatrix}$ **Solution:** The characteristic equation is $\begin{vmatrix}-9-\lambda&4&4\\-8&3-\lambda&4\\-16&8&7-\lambda\end{vmatrix}=0$	

$\therefore (1+\lambda)(1+\lambda)(3-\lambda)=0$

$\therefore \lambda = -1, -1, 3$ 4

(i) For $\lambda = -1$, $[A-\lambda_1 I]X = 0$ gives

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \\ -(1/4)R_1 \end{matrix}$ $\begin{bmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore 2x_1 - x_2 - x_3 = 0$

The rank of coefficient of matrix is 1. The number of unknowns is 3. Hence, there are $3-1=2$ linealry independent solutions

Putting $x_2 = 0$ and $x_1 = 1$, we get $x_3 = 2$

Putting $x_3 = 0$ and $x_1 = 1$, we get $x_2 = 2$

$\therefore$ Corresponding to the eigen values -1, we get the following two linearly independent eigen vectors $X_1 = [1, 0, 2]'$ and $X_2 = [1, 2, 0]'$

(ii) For $\lambda = 3$, $[A-\lambda_2 I]X = 0$ gives

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Crammer's rule on 1st and 2nd row we get the values as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\therefore$ Corresponding to Eigen value 3, we get the Eigen vector $X = [1, 1, 2]'$ 10

Q 2 B	Solve any **One** of the following	**10**

i) Find the value of k (unknown) such that following homogeneous system of equations will have non-trivial solutions and find solution for each such value of k.

$$3x + y - kz = 0, \quad 4x - 2y - 3z = 0, \quad 2kx + 4y + kz = 0$$

Solution: The system can be written as $\begin{bmatrix} 3 & 1 & -k \\ 4 & -2 & -3 \\ 2k & 4 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

System will possess non trivial solution if rank of coefficient matrix is less than number of variables i.e., $r < 3$ if $|A| = 0$

$$\therefore \begin{bmatrix} 3 & 1 & -k \\ 4 & -2 & -3 \\ 2k & 4 & k \end{bmatrix} = 0$$

$\therefore 3(-2k+12) - (4k+6k) - k(16+4k) = 0$

$\therefore k^2 + 8k - 9 = 0 \qquad \therefore (k+9)(k-1) = 0$

$\therefore k = -9 \; and \; k = 1$ for which the system possesses a non – trivial solution. 3

For $k = -9$ the system can be written as $\begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & -3 \\ -18 & 4 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 - \frac{4}{3}R_1, R_3 + 6R_1$, we have $\begin{bmatrix} 3 & 1 & 9 \\ 0 & -10/3 & -15 \\ 0 & 10 & 45 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

	Applying $R_3 + 3R_2$, we have $\begin{bmatrix} 3 & 1 & 9 \\ 0 & -10/3 & -15 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ∴ The reduced form of system of equations can be written as $3x + y + 9z = 0$ and $-\left(\frac{10}{3}\right)y - 15z = 0$ $\therefore y = -(9/2)z$ $\therefore 3x + y + 9z = 0 \Rightarrow 3x = (9/2)z - 9z = -(9/2)z$ $\therefore x = -(3/2)z$ Let $z = a$ (arbitrary) Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -(3/2)a \\ -(9/2)a \\ a \end{bmatrix}$ has infinite solutions as 'a' varies	7
	For $k = 1$ the system can be written as $\begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Applying $R_1 - R_2$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Applying $R_2 - 4R_1, R_3 - 2R_1$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 0 & 10 & 5 \\ 0 & 10 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Applying $R_3 - R_2$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 0 & 10 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ∴ The reduced form of system of equations can be written as $x - 3y - 2z = 0$, $10y + 5z = 0$, $\therefore y = -(1/2)z$ $\therefore x - 3y - 2z = 0 \Rightarrow x = -(3/2)z + 2z = (1/2)z$ $\therefore x = (1/2)z$ Let $z = b$ (arbitrary) Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (1/2)b \\ -(1/2)b \\ b \end{bmatrix}$ has infinite solutions as 'b' varies.	10
ii)	If $u = f(r), r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$. **Solution:** $u = f(r)$ Differentiating u partially w.r.t. x, $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r) = \frac{\partial}{\partial r} f(r) \cdot \frac{\partial r}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$ .... (1) But $r^2 = x^2 + y^2 + z^2$ Differentiating r^2 partially w.r.t. x, $2r \frac{\partial r}{\partial x} = 2x$ $\frac{\partial r}{\partial x} = \frac{x}{r}$ Substituting in Eq. (1), $\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}$ Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r) \cdot \frac{x}{r}\right]$ $= f''(r) \frac{\partial r}{\partial x} \cdot \frac{x}{r} + \frac{f'(r)}{r} + x f'(r)\left(-\frac{1}{r^2}\right) \cdot \frac{\partial r}{\partial x}$	3

	$= f''(r)\frac{x}{r}\frac{x}{r} + \frac{f'(r)}{r} - \frac{x}{r^2}f'(r)\cdot\frac{x}{r}$ $= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3}f'(r)$ .... (2) Similarly, $\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - \frac{y^2}{r^3}f'(r)$ .... (3) and $\frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - \frac{z^2}{r^3}f'(r)$ .... (4) Adding Eqs (2), (3) and (4), $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{r^2}(x^2+y^2+z^2) + \frac{3f'(r)}{r} - \frac{(x^2+y^2+z^2)}{r^3}f'(r)$ $= \frac{f''(r)}{r^2}\cdot r^2 + \frac{3f'(r)}{r} - \frac{r^2}{r^3}f'(r)$ $= f''(r) + \frac{2f'(r)}{r}$	7 10
Q3	Solve any **Two** of the following	**20**
i)	A) Find the values of p for which the following matrix A will have **(i)** rank 1, **(ii)** rank 2, **(iii)** rank 3, where $A = \begin{bmatrix} p & 2 & p \\ p & p & 2 \\ 2 & p & p \end{bmatrix}$ **Solution:** Let us first find the determinant of A. $|A| = \begin{vmatrix} p & 2 & p \\ p & p & 2 \\ 2 & p & p \end{vmatrix} = P(P^2 - 2P) - 2(P^2 - 4) - P(P^2 - 2P)$ $= 2P^3 - 6P^2 + 8$ $= 2(P+1)(P-2)^2$ If $|A| = 0$, i.e if $P = -1$ or 2, then the rank of A is either 1 or 2 Consider, if $P = 2$, then $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ all minors of order 2 are zero. Hence rank of A is 1, when $P = 2$, ...(i) If $P = -1$, then $A = \begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 2 & -1 & -1 \end{bmatrix}$ Consider the minor of order of 2, $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0$ Hence rank of A is 2, when $P = -1$ ....(ii) For rank 3, $|A|$ should not be equal to zero. Hence rank of A is 3, when P can take any value other than 3 or -1 ......(iii) Thus (i), (ii) and (iii) determine the required result.	3 5
	B) If $u = sinh^{-1}\left(\frac{x^3+y^3}{x^2+y^2}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\,tanh^3 u$ **Solution:** $u = \sinh^{-1}\left(\frac{x^3+y^3}{x^2+y^2}\right)$ Replacing x by xt and y by yt, $u = \sinh^{-1}\left[t\left(\frac{x^3+y^3}{x^2+y^2}\right)\right]$ u is a nonhomogenous function.	

But $\sinh u = \frac{x^3+y^3}{x^2+y^2}$ is a homogeneous function of degree 1

Let $f(u) = \sinh u$

By Cor. 3, $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$ 3

where, $g(u) = n \frac{f(u)}{f'(u)} = 1 \cdot \frac{\sinh u}{\cosh u} = \tanh u$

$g'(u) = \operatorname{sech}^2 u$

Hence,

$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \tanh u\,(\operatorname{sech}^2 u - 1) = \tanh u\,(-\tanh^2 u) = -\tanh^3 u$ 5

ii) If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ then prove that $3 \tan A = A \tan 3$

Solution: The characteristic equation of A is $\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$\therefore -(1+\lambda)(1-\lambda) - 8 = 0$ $\quad \therefore \lambda^2 - 9 = 0$

$\therefore \lambda = 3, -3$

(i) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 + \frac{1}{2}R_1$ $\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore -4x_1 + 4x_2 = 0$ $\quad \therefore x_1 - x_2 = 0$

If $x_1 = 1$, we get $x_2 = 1$

$\therefore$ The eigen vector is $[1, \ 1]'$

(ii) For $\lambda = -3$, $[A - \lambda I]X = 0$ gives

$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 - R_1$ $\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore 2x_1 + 4x_2 = 0$ i.e. $x_1 + 2x_2 = 0$

Putting $x_2 = -1$, we get $x_1 = 2$

$\therefore$ The eigen vector is $[2, -1]'$

$\therefore M = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $|M| = -3$

$M^{-1} = \frac{\text{adj.} M}{|M|} = -\frac{1}{3} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$

Now $D = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ 7

If $f(A) = \tan A$, $f(D) = \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix} = \begin{bmatrix} \tan 3 & 0 \\ 0 & -\tan 3 \end{bmatrix}$

$\therefore \tan A = M f(D) M^{-1}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix} \left(-\frac{1}{3}\right) \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} \tan 3 & -2\tan 3 \\ -\tan 3 & -\tan 3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{-1}{3} \begin{bmatrix} \tan 3 & -4\tan 3 \\ -2\tan 3 & -\tan 3 \end{bmatrix}$$

	$\therefore 3\tan A = \begin{bmatrix} -\tan 3 & 4\tan 3 \\ 2\tan 3 & \tan 3 \end{bmatrix}$ $= \tan 3 \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} = \tan 3 \cdot A$ $= A\tan 3$	10
iii)	If $\tan z = \frac{i}{2}(1-i)$, prove that $z = \frac{1}{2}tan^{-1}2 + \frac{i}{4}log\left(\frac{1}{5}\right)$ **Solution:** $\tan z = \frac{i}{2}(1-i)$ $\tan z = \frac{1}{2}(i - i^2) = \frac{1}{2}i + \frac{1}{2}$ $Let\ z = x + iy$ $\therefore \tan(x+iy) = \frac{1}{2} + \frac{i}{2},$ $\tan(x-iy) = \frac{1}{2} - \frac{i}{2}$ $\therefore \tan(2x) = [(x+iy)+(x-iy)]$ $= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)} = \frac{\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]+\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]}{1-\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]} = \frac{1}{1-\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\right]} = \frac{1}{1/2} = 2$ $\therefore 2x = \tan^{-1}2$ $\therefore x = \frac{1}{2}\tan^{-1}2$ Now, $tan(2iy) = \tan[(x+iy)-(x-iy)]$ $= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)} = \frac{\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]-\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]}{1+\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]} = \frac{i}{1+\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\right]} = \frac{i}{1+(1/2)} = \frac{2}{3}i$ $\therefore i\tan h\ 2y = \frac{2}{3}i$ $\therefore \tan h\ 2y = \frac{2}{3}$ $\therefore 2y = tanh^{-1}\left(\frac{2}{3}\right) = \frac{1}{2}log\left[\frac{1+(2/3)}{1-(2/3)}\right] = \frac{1}{2}\log 5$ $\therefore y = \frac{1}{4}\log 5$ $\therefore z = x + iy = \frac{1}{2}tan^{-1}2 + \frac{i}{4}\log 5$	5 10
Q4	Solve any **Two** of the following	**20**
i)	Reduce the following matrix to the Normal form and find it's rank $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ **Solution:** By $R_2 \leftrightarrow R_1$ $\begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix},$ $\begin{matrix} R_2 - 2R_1 \\ R_4 - 4R_1 \end{matrix} \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix},$ $\begin{matrix} C_2 + 2C_1 \\ C_3 - C_1 \\ C_4 + 4C_1 \\ C_5 - 4C_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$ $\begin{matrix} R_4 - (R_2 + R_3) \\ R_2 \leftrightarrow R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$ $\begin{matrix} C_3 + C_2 \\ C_4 - 3C_2 \\ C_5 - C_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} C_4 - 9C_3 \\ C_5 + 4C_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ This is normal form Rank of A is 3	5 10

ii)	Check whether the matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ is diagonalisable or not. If Yes then find the transforming matrix M and the diagonal matrix D	
	Solution: The characteristic equation of A is $\begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$ $\therefore (1-\lambda)[(4-\lambda)(-3-\lambda)+12] = 0$ $\therefore (1-\lambda)(\lambda^2-\lambda) = 0$ $\therefore (\lambda-1)^2(\lambda) = 0$ $\therefore \lambda = 0, 1, 1$	3
	(i) For $\lambda = 0$, $[A-\lambda I]X = 0$ gives $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ By $\begin{matrix} R_2/2 \\ R_3/(-3) \end{matrix}$ $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ By $R_3 - R_2$ $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\therefore x_1 - 6x_2 - 4x_3 = 0 \qquad 2x_2 + x_3 = 0$ Putting $x_2 = -1$, we get $x_3 = 2$ and then we get $x_1 = 2$ Hence, corresponding $\lambda = 0$, we get the Eigen vector $X = [2, -1, 2]'$ **(ii)** For $\lambda = 1$, $[A-\lambda I]X = 0$ gives $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ By $\begin{matrix} R_1/(-2) \\ R_3/(-2) \end{matrix}$ $\begin{bmatrix} 0 & 3 & 2 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ By $\begin{matrix} R_2 - R_1 \\ R_3 - R_2 \end{matrix}$ $\begin{bmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Since, the rank of the coefficient matrix is 1 and the number of variables is 3, there are $3 - 1 = 2$ independent solutions $0x_1 + 3x_2 + 2x_3 = 0$ Putting $x_2 = -2, x_3 = 3$ and $x_1 = 1$, (Say) we get one solution Putting $x_2 = -2, x_3 = 3$ and $x_1 = 2$, (Say) we get another solution Hence, corresponding to $\lambda = 1$, we get the following two linearly independent solutions $X_1 = [1, -2, 3]'$ and $X_2 = [2, -2, 3]'$ Although the Eigen values of A are not distinct, the geometric multiplicity of each eigen value is equal to its algebraic multiplicity, A is diagonalisable	8
	Since $M^{-1}AM = D$, the given matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ will be diagonalised to	

	$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by transforming matrix $M = \begin{bmatrix} 2 & 1 & 2 \\ -1 & -2 & -2 \\ 2 & 3 & 3 \end{bmatrix}$	10
iii)	A rectangular box open at the top is to have a volume of 108 cubic meters. Find the dimensions of the box if its total surface area is minimum. **Solution:** Let x, y and z be the dimensions of the box. Let V and S be its volume and surface area respectively. $108 = xyz$ $S = xy + 2xz + 2yz$ Substituting $z = \frac{108}{xy}$, $S = xy + 2x.\frac{108}{xy} + 2y.\frac{108}{xy} = xy + \frac{216}{y} + \frac{216}{x}$ **Step I:** For extreme values, $\frac{\partial S}{\partial x} = 0$, $y - \frac{216}{x^2} = 0$ .... (1) and $\frac{\partial S}{\partial y} = 0$, $x - \frac{216}{y^2} = 0$ .... (2) Substituting $y = \frac{216}{x^2}$ from Eq (1) in Eq (2) $x - 216\left(\frac{x^4}{216^2}\right) = 0 \quad x\left(1 - \frac{x^3}{216}\right) = 0$ $x = (216)^{\frac{1}{3}} = 6$ $\therefore y = \frac{216}{x^2} = \frac{216}{36} = 6$ [Since $x \neq 0$ being the side of the box] Stationary point is $[6, 6]$ **Step II:** $r = \frac{\partial^2 S}{\partial x^2} = \frac{432}{x^3}$, $s = \frac{\partial^2 S}{\partial x\, \partial y} = 1$, $t = \frac{\partial^2 S}{\partial y^2} = \frac{432}{y^3}$ **Step III:** At $[6, 6]$, $r = \frac{432}{216} = 2 > 0$, $s = 1, t = \frac{432}{216} = 2$ $rt - s^2 = (2)(2) - (1)^2 = 3 > 0$ and $r = 2 > 0$ Hence, S is minimum at $x = y = 6$ and $z = \frac{108}{xy} = \frac{108}{6 \times 6} = 3$ Hence, dimension of the box which make its total surface area S minimum are $x = 6, y = 6, z = 3$	3 6 10
Q5	Solve any **Four** of the following	**20**
i)	Solve $x^7 + x^4 + x^3 + 1 = 0$ **Solution:** $x^7 + x^4 + x^3 + 1 = 0 \quad \therefore x^4(x^3 + 1) + (x^3 + 1) = 0$ $\therefore (x^3 + 1)(x^4 + 1) = 0 \ \therefore x^3 = -1, x^4 = -1$ Consider $x^3 = -1$ $\therefore x = (-1 + i0)^{1/3} = (\cos\pi + i\sin\pi)^{1/3}$ $= [\cos(2k + 1)\pi - i\sin(2k + 1)\pi]^{1/3} \quad = \cos(2k + 1)\frac{\pi}{3} + i\sin(2k + 1)\frac{\pi}{3}$ Putting $k = 0, 1, 2$ we get the three roots	3

	Similarly from $x^4=-1$ we get the remaining four roots as $x=\cos(2k+1)\frac{\pi}{4}+i\sin(2k+1)\frac{\pi}{4}$ where $k=0,1,2,3$	5
ii)	Find the principal value of $(1+i)^{1-i}$ **Solution:** $z=(1+i)^{1-i}$ $\therefore \log z=(1-i)\log(1+i)$ $\therefore \log z=(1-i)[\log\sqrt{1+1}+i\tan^{-1}1]$ $=(1-i)\left[\frac{1}{2}\log 2+i.\frac{\pi}{4}\right]$ $=\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)+i\left(\frac{\pi}{4}-\frac{1}{2}log2\right)=x+iy\ say$ $\therefore z=e^{x+iy}=e^x.e^{iy}=e^x(\cos y+i\sin y)$ $=e^{\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)}\left[cos\left(\frac{\pi}{4}-\frac{1}{2}log2\right)+\ i\sin\left(\frac{\pi}{4}-\frac{1}{2}log2\right)\right]$ $=\sqrt{2}e^{\pi/4}\left[cos\left(\frac{\pi}{4}-\frac{1}{2}log2\right)+\ i\sin\left(\frac{\pi}{4}-\frac{1}{2}log2\right)\right]$ $\because e^{\frac{1}{2}\log 2}=e^{log\sqrt{2}}=\sqrt{2}$	3 5
iii)	Solve the following equations by Gauss – Seidel method (2 iterations) $20x+y-2z=17,\ \ 3x+20y-z=-18,\ \ 2x-3y+20z=25$ **Solution:** We first write the equations as $x=\frac{1}{20}[17-y+2z]$ .... (1) $y=\frac{1}{20}[-18-3x+z]$ .... (2) $z=\frac{1}{20}[25-2x+3y]$ .... (3) **(i)First Iteration:** We start with the approximation $y=0,z=0$ and then get from (1), $\therefore x_1=\frac{17}{20}=0.85$ We use this approximation to find y i.e., we put $x=0.85,z=0$ in (2) $\therefore y_1=\frac{1}{20}[-18-3(0.85)-0]=-1.0275$ We use these values of x_1 and y_1 to find z_1 i.e., we put $x=0.85,y=-1.0275$ in (3) $\therefore z_1=\frac{1}{20}[25-2(0.85)+3(-1.0275)]=1.0109$ **(ii)Second Iteration:** We use latest values of y and z to find x i.e., we put $y=-1.0275$, $z=1.0109$ in (1) $\therefore x_2=\frac{1}{20}[17-(-1.0275)+2(1.0109)]=1.0025$ We put $x=1.0025,z=1.0109$ in (2) $\therefore y_2=\frac{1}{20}[-18-3(1.0025)+1.0109]=-0.9998$ We put $x=1.0025,y=-0.9998$ in (3) $\therefore z_2=\frac{1}{20}[25-2(1.0025)+3(-0.9998)]=0.9998$	3 5

iv) Find the minimal polynomial of the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Comment whether A is derogatory or not?

Solution: The characteristic equation of A is $\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$

$\therefore (5-\lambda)[-(16-\lambda^2)+12] + 6[4+\lambda-6] - 6[6-3(4-\lambda)] = 0$

$\therefore (5-\lambda)[-4+\lambda^2] + 6[-2+\lambda] - 6[-6+3\lambda] = 0$

$\therefore \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0;\qquad \lambda^3 - 2\lambda^2 - 3\lambda^2 + 6\lambda + 2\lambda - 4 = 0$

$\therefore (\lambda-2)(\lambda^2 - 3\lambda + 2) = 0;\qquad \therefore (\lambda-2)(\lambda-2)(\lambda-1) = 0$

Hence, the roots of $|A - \lambda I| = 0$ are $2, 2, 1$ 2

Let us now find the minimal polynomial of A. We know that each characteristic root of A is also a root of the minimal polynomial of A. So if $f(x)$ is the minimal polynomial of A then $x-1$ and $x-2$ are the factors of $f(x)$.

Let us see whether the polynomial $(x-2)(x-1) = x^2 - 3x + 2$ annihilates A.

Now, $A^2 = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}^2 = \begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix}$

$\therefore A^2 - 3A + 2I = \begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix} - 3\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore f(x) = x^2 - 3x + 2$ annihilates A

Thus, $f(x)$ is the monic polynomial of lowest degree that annihilates A. Hence, $f(x)$ is the minimal polynomial of A. Since its degree is less than the order of A, A is derogatory 5

v) If $u = xyz,\ v = x^2 + y^2 + z^2, w = x + y + z$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ and hence using property find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Solution: If $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ and $J' = \frac{\partial(x,y,z)}{\partial(u,v,w)}$ then $JJ' = 1$

Now,

$$J = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$ 2

$= 2[yz(y-z) - xz(x-z) + xy(x-y)]$

$= 2[y^2z - yz^2 - zx^2 + z^2x + xy(x-y)]$

$= 2[z^2(x-y) - z(x^2-y^2) + xy(x-y)]$

$= 2(x-y)[z^2 - z(x+y) + xy]$

$= 2(x-y)[z^2 - zx - zy + xy]$

$= 2(x-y)[z(z-x) - y(z-x)]$

$= 2(x-y)(z-y)(z-x) = -2(x-y)(y-z)(z-x)$

$\therefore J' = \frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{1}{2(x-y)(y-z)(z-x)}$ 5

vi)	Verify Euler's Theorem for $u = ax^2 + 2hxy + by^2$ **Solution:** $u = ax^2 + 2hxy + by^2$ Replacing x by xt and y by $yt, u = t^2(ax^2 + 2hxy + by^2)$ Hence, u is homogeneous function of degree 2 By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ .... (1) Differentiating u partially w.r.t. x and y, $\frac{\partial u}{\partial x} = 2ax + 2hy \quad \frac{\partial u}{\partial y} = 2hx + 2by$ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2ax^2 + 2hxy + 2hxy + 2by^2 = 2(ax^2 + by^2 + 2hxy) = 2u$ .... (2) Hence, from Eqs. (1) and (2), theorem is verified	2 5