Fuzzy Logic : Introduction

What is Fuzzy logic?

- Fuzzy logic is a mathematical language to express something.

  This means it has grammar, syntax, semantic like a language for communication.

- There are some other mathematical languages also known
  - **Relational algebra** (operations on sets)
  - **Boolean algebra** (operations on Boolean variables)
  - **Predicate logic** (operations on well formed formulae (wff), also called predicate propositions)

- **Fuzzy logic deals with Fuzzy set.**

First time introduced by Lotfi Abdelli Zadeh (1965), University of California, Berkley, USA (1965).

What is fuzzy?

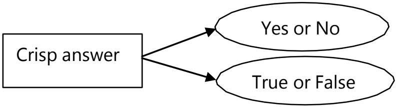
1. Dictionary meaning of **fuzzy** is not clear, noisy etc.

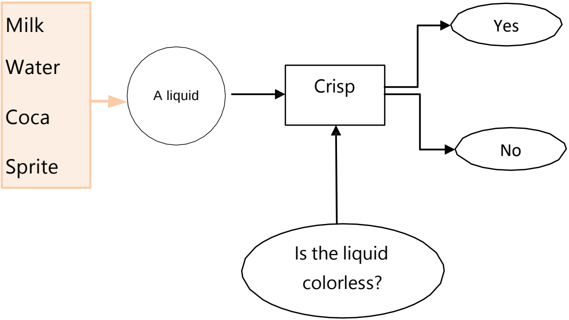
   Example: Is the picture on this slide is fuzzy?

2. Antonym of fuzzy is **crisp**

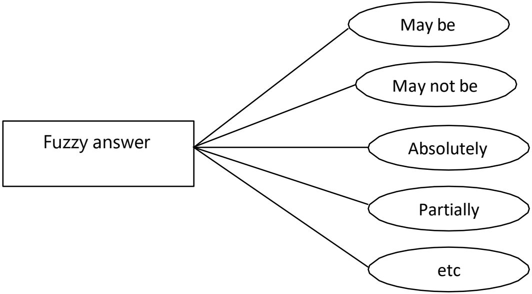
   Example: Are the chips crisp?

Example : Fuzzy logic vs. Crisp logic

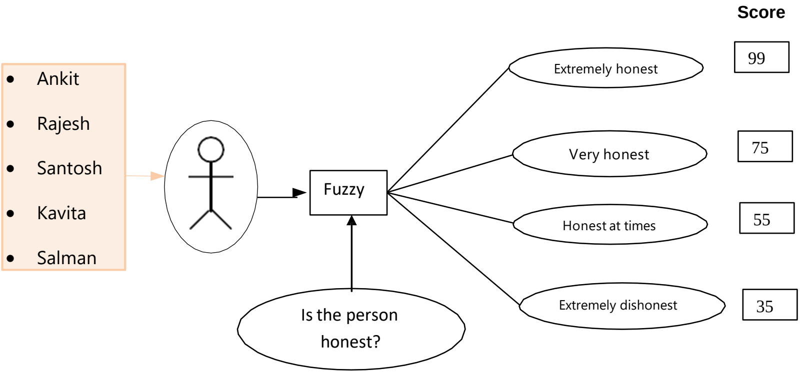




Example : Fuzzy logic vs. Crisp logic



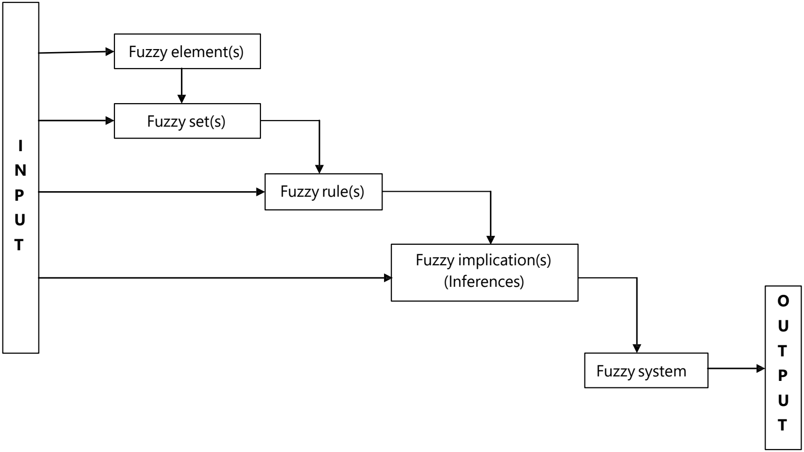
Example : Fuzzy logic vs. Crisp logic



World is fuzzy!

Our world is better described with fuzzily!

Concept of fuzzy system



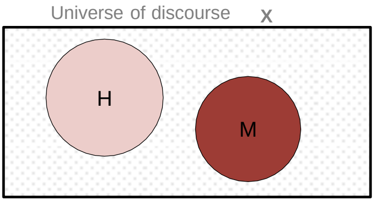
Concept of fuzzy set

To understand the concept of **fuzzy set** it is better, if we first clear our idea of **crisp set**.

X = The entire population of India.

H = All Hindu population = $\{ h_1, h_2, h_3, \ldots, h_L \}$

M = All Muslim population = $\{ m_1, m_2, m_3, \ldots, m_N \}$



Here, All are the sets of finite numbers of individuals.

Such a set is called crisp set.

Example of fuzzy set

Let us discuss about fuzzy set

X = All students in **116U01E514**

S = All **Good students**.
$S = \{ (s, g) \mid s \in X \}$ and $g(s)$ is a measurement of goodness of the student s.

Example:

$S = \{$ (Rajat, 0.8), (Kavita, 0.7), (Salman, 0.1), (Ankit, 0.9) $\}$ etc.

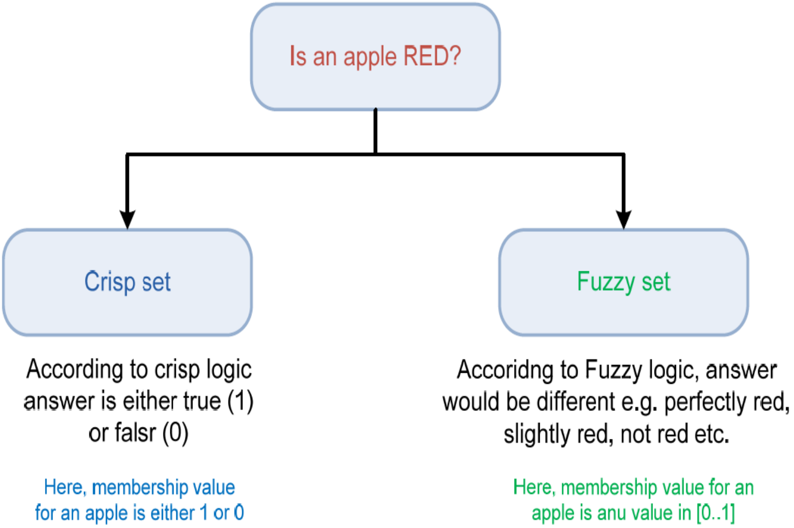


Figure 1: Crisp vs. Fuzzy sets

Fuzzy set vs. Crisp set

Crisp Set	Fuzzy Set
1. S = { s | s $\in$ X }	1. F = (s, μ) | s $\in$ X and μ(s) is the degree of s.
2. It is a collection of elements.	2. It is collection of ordered pairs.
3. Inclusion of an element s $\in$ X into S is crisp, that is, has strict boundary **yes** or **no**.	3. Inclusion of an element s $\in$ X into F is fuzzy, that is, if present, then with a degree of **membership**.

Fuzzy set vs. Crisp set

Note: A crisp set is a fuzzy set, but, a fuzzy set is not necessarily a crisp set.

Example:
H = $\{ (h_1, 1), (h_2, 1), \ldots, (h_L, 1) \}$

Person = $\{ (p_1, 1), (p_2, 0), \ldots, (p_N, 1) \}$

In case of a crisp set, the elements are with extreme values of degree of membership namely either 1 or 0.

How to decide the degree of memberships of elements in a fuzzy set?

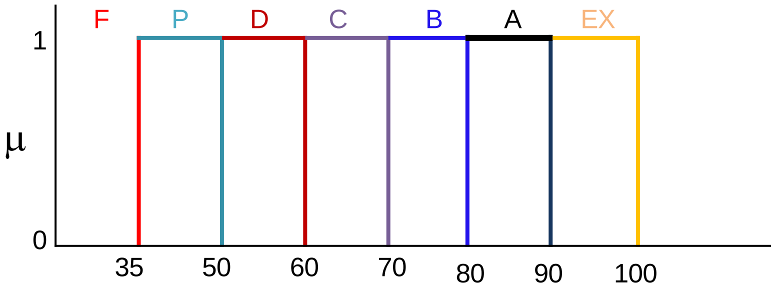
City	Bangaluru	Mumbai	Hyderabad	Kharagpur	Chennai	Delhi
DoM	0.95	0.90	0.80	0.01	0.65	0.75

How the cities of comfort can be judged?

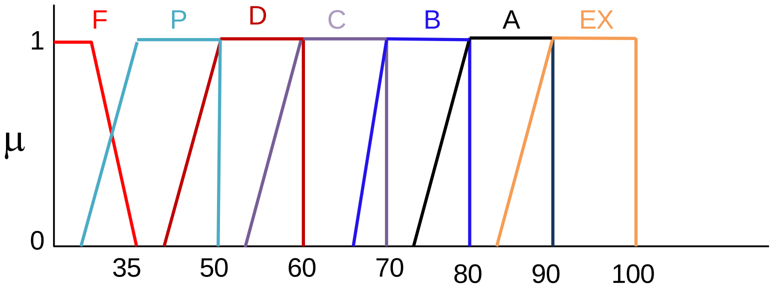
Example: Course evaluation in a crisp way

1. EX = Marks $\geq$ 90
2. A = 80 $\leq$ Marks $<$ 90
3. B = 70 $\leq$ Marks $<$ 80
4. C = 60 $\leq$ Marks $<$ 70
5. D = 50 $\leq$ Marks $<$ 60
6. P = 35 $\leq$ Marks $<$ 50
7. F = Marks $<$ 35

Example: Course evaluation in a crisp way



Example: Course evaluation in a fuzzy way



Few examples of fuzzy set

- High Temperature
- Low Pressure
- Color of Apple
- Sweetness of Orange
- Weight of Mango

Note: Degree of membership values lie in the range [0...1].

Some basic terminologies and notations

Definition 1: Membership function (and Fuzzy set)

If X is a universe of discourse and x $\in$ X, then a fuzzy set A in X is defined as a set of ordered pairs, that is

$A = \{(x, \mu_A(x)) | x \in X\}$ where μ_A(x) is called the membership function for the fuzzy set A.

Note:
μ_A(x) map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

Question:
How (and who) decides μ_A(x) for a Fuzzy set A in X?

Some basic terminologies and notations

Example:

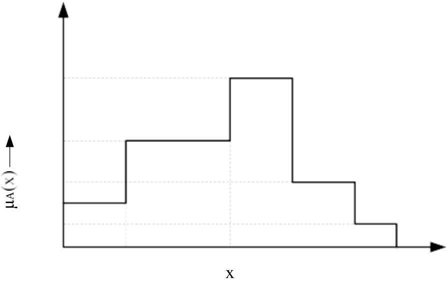
X = All cities in India

A = City of comfort

A={(New Delhi, 0.7), (Bangalore, 0.9), (Chennai, 0.8), (Hyderabad, 0.6), (Kolkata, 0.3), (Kharagpur, 0)}

Membership function with discrete membership values

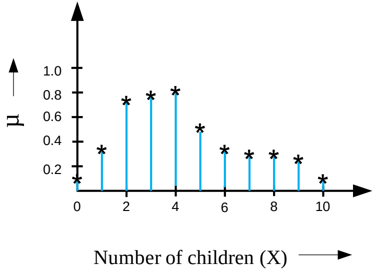
The membership values may be of discrete values.



A fuzzy set with discrete values of μ

Membership function with discrete membership values

Either elements or their membership values (or both) also may be of discrete values.



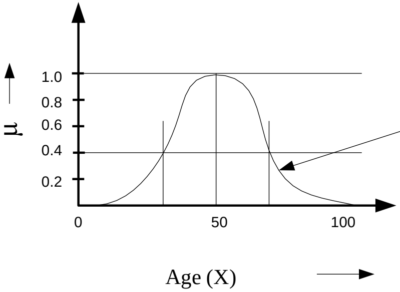
A = “Happy family”

$A = \{(0,0.1),(1,0.30),(2,0.78)\ldots\ldots(10,0.1)\}$

Note : X = discrete value

How you measure happiness ??

Membership function with continuous membership values



B = “Middle aged”

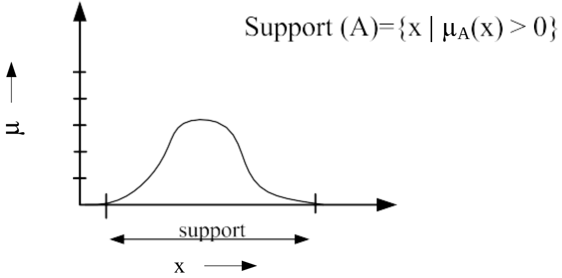
$$\mu_B(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^4}$$

$$B=\{(x,\mu_B(x))\}$$

Note : x = real value
= R^+

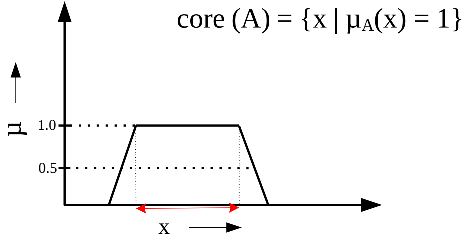
Fuzzy terminologies: Support

Support: The support of a fuzzy set A is the set of all points $x \in X$ such that $\mu_A(x) > 0$



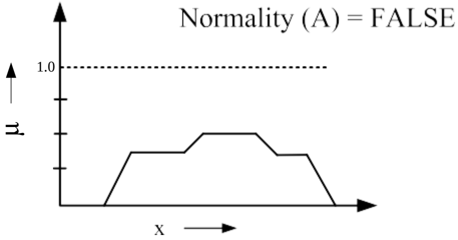
Fuzzy terminologies: Core

Core: The core of a fuzzy set A is the set of all points x in X such that $\mu_A(x) = 1$



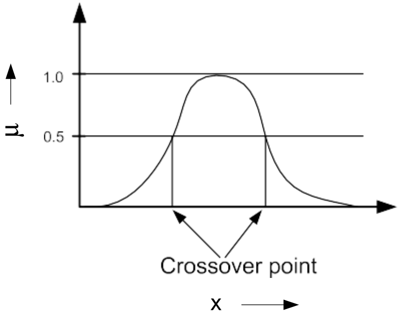
Fuzzy terminologies: Normality

Normality : A fuzzy set A is a normal if its core is non-empty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.



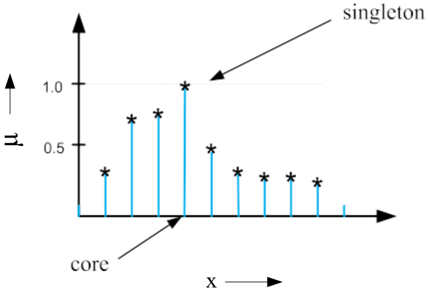
Fuzzy terminologies: Crossover points

Crossover point : A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$. That is
Crossover $(A) = \{x \,|\, \mu_A(x) = 0.5\}$.



Fuzzy terminologies: Fuzzy Singleton

Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = |\{ x \mid \mu_A(x) = 1\}| = 1$. Following fuzzy set is not a fuzzy singleton.



Fuzzy terminologies: α-cut and strong α-cut

α-cut and strong α-cut :

The α-cut of a fuzzy set A is a crisp set defined by

$$A_\alpha = \{ x \mid \mu_A(x) \geq \alpha \}$$

Strong α-cut is defined similarly :

$$A_\alpha' = \{ x \mid \mu_A(x) > \alpha \}$$

Note : Support(A) = A_0' and Core(A) = A_1.

Fuzzy terminologies: Bandwidth

Bandwidth :

For a normal and convex fuzzy set, the bandwidth (or width) is defined as the distance the two unique crossover points:

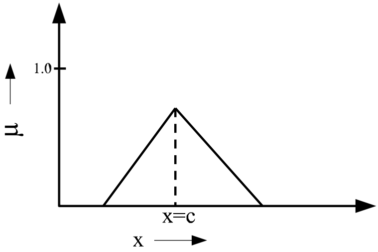
Bandwidth(A) = $|\, x_1 - x_2 \,|$

where $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Fuzzy terminologies: Symmetry

Symmetry :

A fuzzy set A is symmetric if its membership function around a certain point $x = c$, namely μ_A(x + c) = μ_A(x - c) for all x $\in$ X.



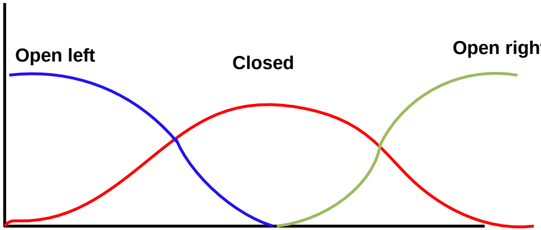
Fuzzy terminologies: Open and Closed

A fuzzy set A is

Open left
If $\lim_{x \to -\infty} \mu_A(x) = 1$ and $\lim_{x \to +\infty} \mu_A(x) = 0$

Open right:
If $\lim_{x \to -\infty} \mu_A(x) = 0$ and $\lim_{x \to +\infty} \mu_A(x) = 1$

Closed
If : $\lim_{x \to -\infty} \mu_A(x) = \lim_{x \to +\infty} \mu_A(x) = 0$



Fuzzy Membership Functions

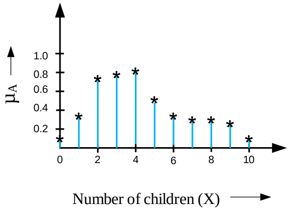
Fuzzy membership functions

A fuzzy set is completely characterized by its membership function (sometimes abbreviated as *MF* and denoted as μ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

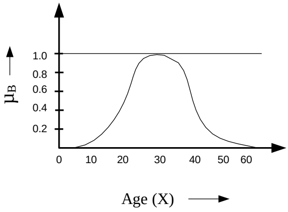
Note: A membership function can be on

(a) a discrete universe of discourse and
(b) a continuous universe of discourse.

Example:



A = Fuzzy set of "Happy family"

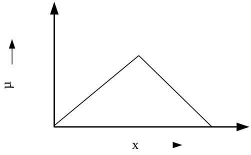


B = "Young age"

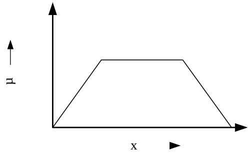
Fuzzy membership functions

So, membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

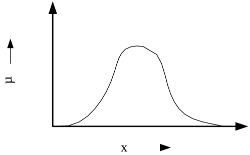
Following figures shows a typical examples of membership functions.



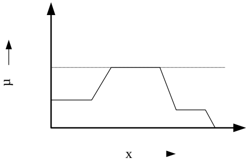
< triangular >



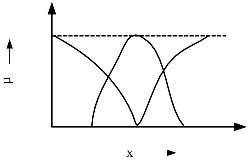
< trapezoidal >



< curve >



< non-uniform >



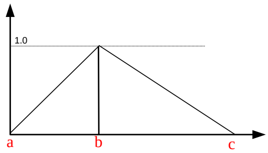
< non-uniform >

Fuzzy MFs : Formulation and parameterization

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

Triangular MFs : A triangular MF is specified by three parameters $\{a, b, c\}$ and can be formulated as follows.

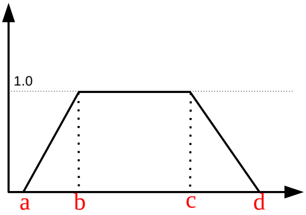
$$triangle(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } c \leq x \end{cases} \tag{1}$$



Fuzzy MFs: Trapezoidal

A trapezoidal MF is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

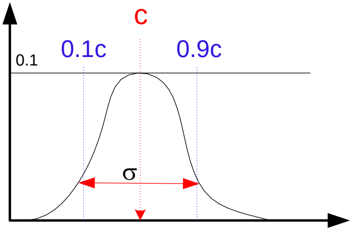
$$trapeziod(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases} \quad (2)$$



Fuzzy MFs: Gaussian

A Gaussian MF is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

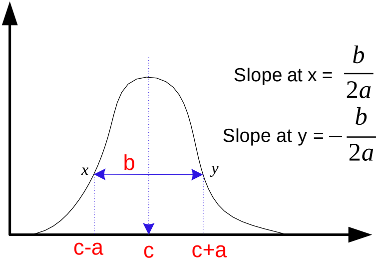
$$\text{gaussian}(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}.$$



Fuzzy MFs: Generalized bell

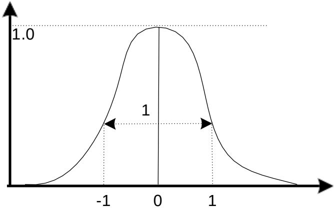
It is also called Cauchy MF. A generalized bell MF is specified by three parameters $\{a, b, c\}$ and is defined as:

$$\text{bell}(x;\, a,\, b,\, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$

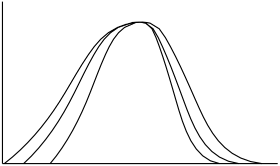


Example: Generalized bell MFs

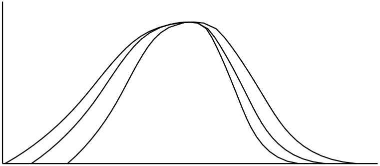
Example: μ(x)= $\frac{1}{1+x^2}$;
$a = b = 1$ and $c = 0$;



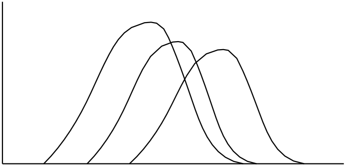
Generalized bell MFs: Different shapes



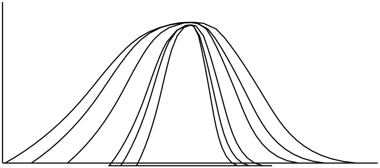
Changing a



Changing b



Changing a

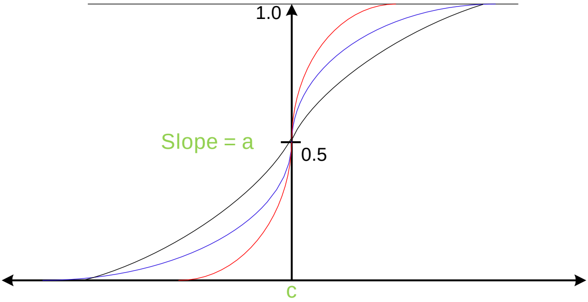


Changing a and b

Fuzzy MFs: Sigmoidal MFs

Parameters: $\{a, c\}$; where c = crossover point and a = slope at c;

$$\text{Sigmoid}(x;a,c) = \frac{1}{1+e^{-[\frac{a}{x-c}]}}$$



Fuzzy MFs : Example

Example : Consider the following grading system for a course.

Excellent = Marks $\leq$ 90

Very good = 75 $\leq$ Marks $\leq$ 90

Good = 60 $\leq$ Marks $\leq$ 75

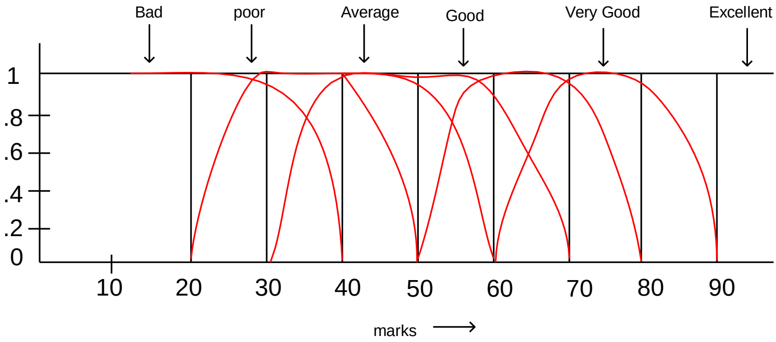
Average = 50 $\leq$ Marks $\leq$ 60

Poor = 35 $\leq$ Marks $\leq$ 50

Bad= Marks $\leq$ 35

Grading System

A fuzzy implementation will look like the following.



You can decide a standard fuzzy MF for each of the fuzzy garde.

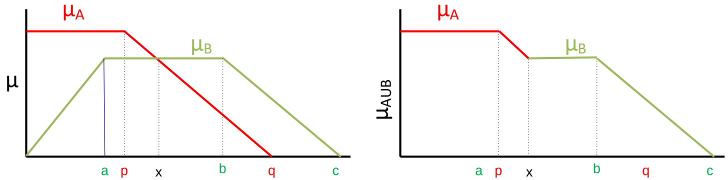
Operations on Fuzzy Sets

Basic fuzzy set operations: Union

Union ($A \cup B$):

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Example:
$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and
$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;
$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$

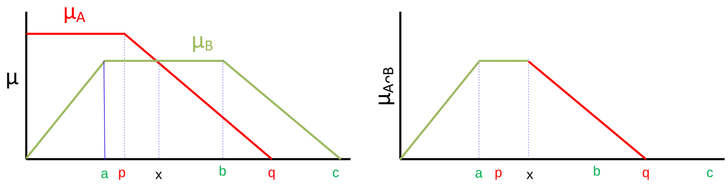


Basic fuzzy set operations: Intersection

Intersection ($A \cap B$):

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Example:
$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and
$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;
$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$



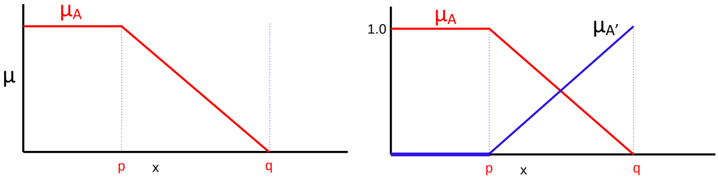
Basic fuzzy set operations: Complement

Complement (A^C):

$$\mu_{A_{A^C}}(x) = 1\text{-}\mu_A(x)$$

Example:
$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$C = A^C = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$



Basic fuzzy set operations: Products

Algebric product or Vector product (A·B):

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Scalar product $(a \times A)$:

$$\mu_{\alpha A}(x) = a \cdot \mu_A(x)$$

Basic fuzzy set operations: Sum and Difference

Sum ($A + B$):

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Difference ($A - B = A \cap B^C$):

$$\mu_{A-B}(x) = \mu_{A \cap B^C}(x)$$

Disjunctive sum: $A \oplus B = (A^C \cap B) \cup (A \cap B^C)$)

Bounded Sum: $| A(x) \oplus B(x) |$

$$\mu_{|A(x) \oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

Bounded Difference: $| A(x) \text{ g } B(x) |$

$$\mu_{|A(x) \text{g} B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Basic fuzzy set operations: Equality and Power

Equality ($A = B$):

$$\mu_A(x) = \mu_B(x)$$

Power of a fuzzy set A^α:

$$\mu_{A^\alpha}(x) = \{\mu_A(x)\}^\alpha$$

- If $a < 1$, then it is called *dilation*
- If $a > 1$, then it is called *concentration*

Basic fuzzy set operations: Cartesian product

Caretsian Product ($A \times B$):

$$\mu_{A \times B}(x, y) = min\{\mu_A(x), \mu_B(y)$$

Example 3:
A(x) = {(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)}
B(y) = {(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)}

$$A \times B = \min\{\mu_A(x), \mu_B(y)\} = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix} \end{array}$$

Properties of fuzzy sets

Commutativity :

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Properties of fuzzy sets

Idempotence :

$$A \cup A = A$$
$$A \cap A = \mathrm{A}$$
$$A \cup \varnothing = A$$
$$A \cap \varnothing = \mathrm{A}$$

Transitivity :

$$\text{If } A \subseteq B, B \subseteq C \text{ then } A \subseteq C$$

Involution :

$$(A^c)^c = A$$

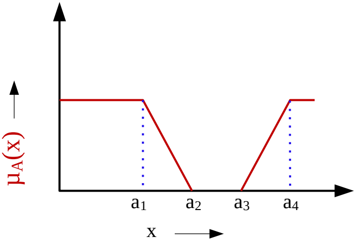
De Morgan's law :

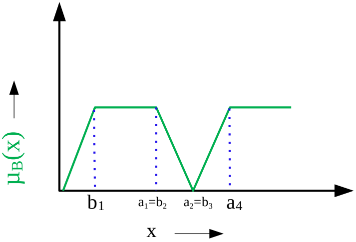
$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Few Illustrations on Fuzzy Sets

Example 1: Fuzzy Set Operations

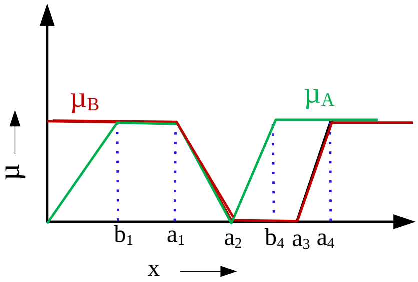
Let A and B are two fuzzy sets defined over a universe of discourse X with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Two MFs $\mu_A(x)$ and $\mu_B(x)$ are shown graphically.





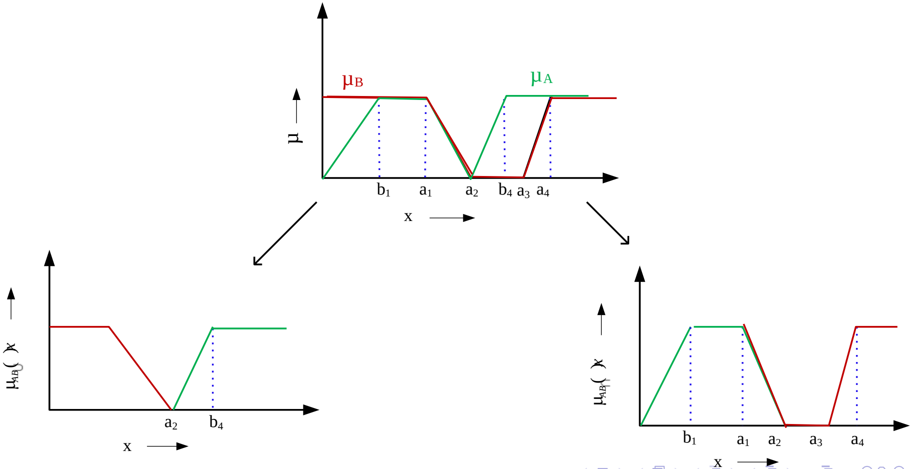
Example 1: Plotting two sets on the same graph

Let's plot the two membership functions on the same graph



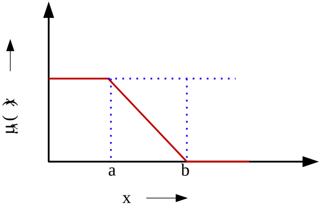
Example 1: Union and Intersection

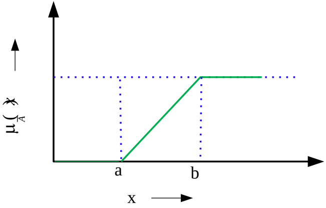
The plots of union $A \cup B$ and intersection $A \cap B$ are shown in the following.



Example 1: Intersection

The plots of union $\mu_{\bar{A}}(x)$ of the fuzzy set A is shown in the following.





Fuzzy set operations: Practice

Consider the following two fuzzy sets A and B defined over a universe of discourse [0,5] of real numbers with their membership functions

$$\mu_A(x) = \frac{x}{1+x} \text{ and } \mu_B(x) = 2^{-x}$$

Determine the membership functions of the following and draw them graphically.

i. $\overline{A}$, $\overline{B}$

ii. $A \cup B$

iii. $A \cap B$

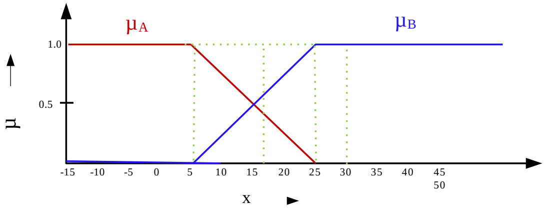
iv. $(A \cup B)^c$ [Hint: Use De' Morgan law]

Example 2: A real-life example

Two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively defined as below.

A = **Cold climate** with $\mu_A(x)$ as the MF.

B = **Hot climate** with $\mu_B(x)$ as the M.F.



Here, X being the universe of discourse representing entire range of temperatures.

Example 2: A real-life example

What are the fuzzy sets representing the following?

1. **Not cold climate**
2. **Not hold climate**
3. **Extreme climate**
4. **Pleasant climate**

Note: Note that ”Not cold climate” $/=$ ”Hot climate” and vice-versa.

Example 2 : A real-life example

Answer would be the following.

1. **Not cold climate**

   $\overline{A}$ with $1 - \mu_A(x)$ as the MF.

2. **Not hot climate**

   $\overline{B}$ with $1 - \mu_B(x)$ as the MF.

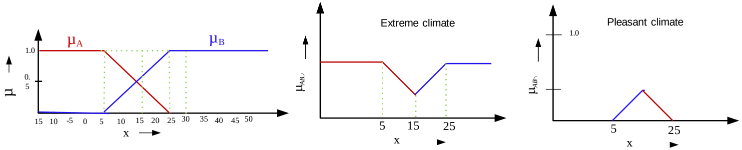
3. **Extreme climate**

   $A \cup B$ with $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ as the MF.

4. **Pleasant climate**

   $A \cap B$ with $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ as the MF.

The plot of the MFs of $A \cup B$ and $A \cap B$ are shown in the following.



Few More on Membership Functions

Generation of MFs

Given a membership function of a fuzzy set representing a linguistic hedge, we can derive many more MFs representing several other linguistic hedges using the concept of Concentration and Dilation.

- **Concentration:**

$$A^k = [\mu_A(x)]^k \; ; k > 1$$

- **Dilation:**

$$A^k = [\mu_A(x)]^k \; ; k < 1$$

Example : Age = { Young, Middle-aged, Old }

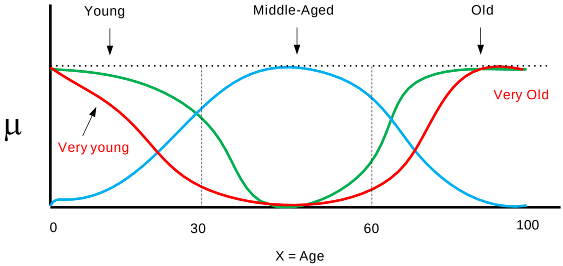
Thus, corresponding to Young, we have : Not young, Very young, Not very young and so on.

Similarly, with Old we can have : old, very old, very very old, extremely old etc.

Thus, Extremely old = $(((old)^2)^2)^2$ and so on

Or, More or less old = $A^{0.5} = (old)^{0.5}$

Linguistic variables and values



$$\mu_{young}(x) = bell(x, 20, 2, 0) = \frac{1}{1+(\frac{x}{20})^4}$$

$$\mu_{old}(x) = bell(x, 30, 3, 100) = \frac{1}{1+(\frac{x-100}{30})^6}$$

$$\mu_{middle-aged} = bell(x, 30, 60, 50)$$

Not young = $\overline{\mu_{young}(x)} = 1 - \mu_{young}(x)$

Young but not too young = $\mu_{young}(x) \cap \mu_{young}(x)$

Fuzzy Relations, Rules and Inferences

Fuzzy Relations

Crisp relations

To understand the fuzzy relations, it is better to discuss first crisp relation.

Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

Note :

(1) $A \times B \neq B \times A$

(2) $|A \times B| = |A| \times |B|$

(3)$A \times B$ provides a mapping from $a \in A$ to $b \in B$.

The mapping so mentioned is called a relation.

Crisp relations

Example 1:
Consider the two crisp sets A and B as given below. $A = \{1, 2, 3, 4\}$
$B = \{3, 5, 7\}$.

Then, $A \times B = \{(1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7), (3, 3), (3, 5), (3, 7), (4, 3), (4, 5), (4, 7)\}$

Let us define a relation R as $R = \{(a, b) | b = a + 1, (a, b) \in A \times B\}$

Then, $R = \{(2, 3), (4, 5)\}$ in this case.

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c c} & \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

Union:

$$R(x, y) \cup S(x, y) = max(R(x, y), S(x, y));$$

Intersection:

$$R(x, y) \cap S(x, y) = min(R(x, y), S(x, y));$$

Complement:

$$\overline{R(x, y)} = 1 - R(x, y)$$

Example: Operations on crisp relations

Example:
Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

1. $R \cup S$
2. $R \cap S$
3. $\overline{R}$

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y, Z. Then $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x, z) | (x, y) \in R \text{ and } (y, z) \in S \text{ and } \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S, the max-min composition is defined as $T = R \circ S$;

$$T(x, z) = max\{min\{R(x, y), S(y, z) \text{ and } \forall y \in Y\}\}$$

Example:

Given

$X = \{1,3,5\}; Y = \{1,3,5\}; R = \{(x,y)|y = x+2\}; S = \{(x,y)|x < y\}$

Here, R and S is on $X \times Y$.

Thus, we have

$R = \{(1,3),(3,5)\}$

$S = \{(1,3),(1,5),(3,5)\}$

R= $\begin{array}{c|c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ and S= $\begin{array}{c|c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

Using max-min composition $R \circ S$= $\begin{array}{c|c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set $X_1, X_2, ..., X_n$
- Here, n-tuples $(x_1, x_2, ..., x_n)$ may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the tuples.

Example:

$X = \{$ typhoid, viral, cold $\}$ and $Y = \{$ running nose, high temp, shivering $\}$

The fuzzy relation R is defined as

$$\begin{array}{cc} & \begin{array}{ccc} \textit{runningnose} & \textit{hightemperature} & \textit{shivering} \end{array} \\ \begin{array}{c} \textit{typhoid} \\ \textit{viral} \\ \textit{cold} \end{array} & \begin{bmatrix} 0.1 & 0.9 & 0.8 \\ 0.2 & 0.9 & 0.7 \\ 0.9 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

Fuzzy Cartesian product

Suppose
A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$

B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given by $\mu_R(x,y) = \mu_{A\times B}(x,y) = min\{\mu_A(x), \mu_B(y)\}$

Example :

$A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$and $B = \{(b_1, 0.5), (b_2, 0.6)\}$

$$R = A \times B = \begin{array}{c c} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \left[\begin{array}{cc} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{array}\right] \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

Union:

$$\mu_{R \cup S}(a, b) = max\{\mu_R(a, b), \mu_S(a, b)\}$$

Intersection:

$$\mu_{R \cap S}(a, b) = min\{\mu_R(a, b), \mu_S(a, b)\}$$

Complement:

$$\mu_{\overline{R}}(a, b) = 1 - \mu_R(a, b)$$

Composition

$$T = R \circ S$$
$$\mu_{R \circ S} = max_{y \in Y}\{min(\mu_R(x, y), \mu_S(y, z))\}$$

Operations on Fuzzy relations: Examples

Example:

$X = (x_1, x_2, x_3)$; $Y = (y_1, y_2)$; $Z = (z_1, z_2, z_3)$;

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$\mu_{R \circ S}(x_1, y_1) = max\{min(x_1, y_1), min(y_1, z_1), min(x_1, y_2), min(y_2, z_1)\}$
$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = 0.5$ and so on.

Fuzzy relation : An example

Consider the following two sets P and D, which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants
$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, then R can be stated as

$$R = \begin{array}{c c} & \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} & \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : An example

Also, consider T be the another relation on $D \times S$, which is given by

$$S = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of plants with the different symptoms of the disease using **max-min composition**.

$$R \circ S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy Propositions

Two-valued logic vs. Multi-valued logic

- The basic assumption upon which crisp logic is based - that every proposition is either TRUE or FALSE.
- The classical two-valued logic can be extended to multi-valued logic.
- As an example, three valued logic to denote true(1), false(0) and indeterminacy ($\frac{1}{2}$).

Two-valued logic vs. Multi-valued logic

Different operations with three-valued logic can be extended as shown in the following truth table:

a	b	$\wedge$	$\vee$	$\neg$a	$\Longrightarrow$	$=$
0	0	0	0	1	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	0	1	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	0	1	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1

Fuzzy connectives used in the above table are:

AND ($\wedge$), OR ($\vee$), NOT ($\neg$), IMPLICATION ($\Longrightarrow$) and EQUAL ($=$).

Three-valued logic

Fuzzy connectives defined for such a three-valued logic better can be stated as follows:

Symbol	Connective	Usage	Definition
$\neg$	NOT	$\neg$P	$1 - T(P)$
$\vee$	OR	$P \vee Q$	max{T(P), T(Q) }
$\wedge$	AND	$P \wedge Q$	min{ T(P),T(Q) }
$\Longrightarrow$	IMPLICATION	$(P \Longrightarrow Q)$ or $(\neg P \vee Q)$	max{(1 - T(P)), T(Q) }
$=$	EQUALITY	$(P = Q)$ or $[(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)]$	$1 - \lvert T(P) - T(Q) \rvert$

Fuzzy proposition

Example 1:

P : Ram is honest

1. T(P) = 0.0 : Absolutely false
2. T(P) = 0.2 : Partially false
3. T(P) = 0.4 : May be false or not false
4. T(P) = 0.6 : May be true or not true
5. T(P) = 0.8 : Partially true
6. T(P) = 1.0 : Absolutely true.

Example 2 :Fuzzy proposition

P : Mary is efficient ; T(P) = 0.8;

Q : Ram is efficient ; T(Q) = 0.6

1. **Mary is not efficient.**
   $T(\neg P) = 1 - T(P) = 0.2$
2. **Mary is efficient and so is Ram.**
   $T(P \wedge Q) = min\{T(P), T(Q)\} = 0.6$
3. **Either Mary or Ram is efficient**
   $T(P \vee Q) = max T(P), T(Q) = 0.8$
4. **If Mary is efficient then so is Ram**
   $T(P \Rightarrow Q) = max\{1 - T(P), T(Q)\} = 0.6$

Fuzzy proposition vs. Crisp proposition

- The fundamental difference between crisp (classical) proposition and fuzzy propositions is in the range of their truth values.
- While each classical proposition is required to be either true or false, the truth or falsity of fuzzy proposition is a matter of degree.
- The degree of truth of each fuzzy proposition is expressed by a value in the interval [0,1] both inclusive.

Canonical representation of Fuzzy proposition

- Suppose, X is a universe of discourse of five persons. Intelligent of $x \in X$ is a fuzzy set as defined below.

  Intelligent: $\{(x_1, 0.3), (x_2, 0.4), (x_3, 0.1), (x_4, 0.6), (x_5, 0.9)\}$

- We define a fuzzy proposition as follows:

  P : x is intelligent

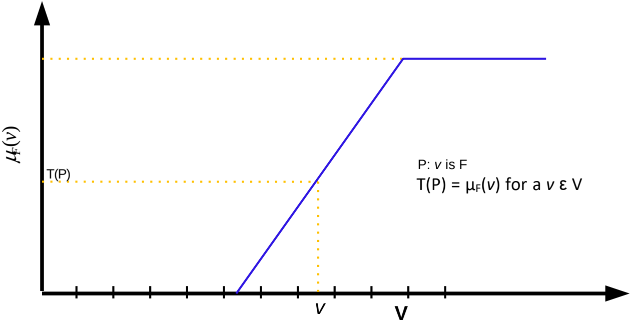
- The canonical form of fuzzy proposition of this type, P is expressed by the sentence P : v is F .

- Predicate in terms of fuzzy set.

  P : ***v* is F** ; where v is an element that takes values v from some universal set V and F is a fuzzy set on V that represents a fuzzy predicate.

- In other words, given, a particular element v , this element belongs to F with membership grade $\mu_F(v)$.

Graphical interpretation of fuzzy proposition



- For a given value v of variable V in proposition P, T(P) denotes the degree of truth of proposition P.

Fuzzy Implications

Fuzzy rule

- A fuzzy implication (also known as fuzzy If-Then rule, fuzzy rule, or fuzzy conditional statement) assumes the form :

  If x **is** A **then** y **is** B

  where, A and B are two linguistic variables defined by fuzzy sets A and B on the universe of discourses X and Y, respectively.

- Often, x **is** A is called the antecedent or premise, while y **is** B is called the consequence or conclusion.

Fuzzy implication : Example 1

- If pressure is High then temperature is Low
- If mango is Yellow then mango is Sweet else mango is Sour
- If road is Good then driving is Smooth else traffic is High
- The fuzzy implication is denoted as $R : A \rightarrow B$
- In essence, it represents a binary fuzzy relation R on the (Cartesian) product of $A \times B$

Fuzzy implication : Example 2

- Suppose, P and T are two universes of discourses representing pressure and temperature, respectively as follows.
- $P = \{1,2,3,4\}$ and $T = \{10, 15, 20, 25, 30, 35, 40, 45, 50\}$
- Let the linguistic variable High temperature and Low pressure are given as
- $T_{HIGH} = \{(20, 0.2), (25, 0.4), (30, 0.6), (35, 0.6), (40, 0.7), (45, 0.8), (50, 0.8)\}$
- $P_{LOW} = (1, 0.8), (2, 0.8), (3, 0.6), (4, 0.4)$

Fuzzy implications : Example 2

- Then the fuzzy implication **If temperature is High then pressure is Low** can be defined as

  $R : T_{HIGH} \rightarrow P_{LOW}$

where, R =

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 20 \\ 25 \\ 30 \\ 35 \\ 40 \\ 45 \\ 50 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.7 & 0.7 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

Note : If temperature is 40 then what about low pressure?

Zadeh max-min Rule to compute R

With the above mentioned implications, there are a number of fuzzy implication functions that are popularly followed in fuzzy rule-based system.

Zadeh's max-min rule :

$$R_{mm} = \bar{A} \cup (A \cap B) = \int_{X \times Y} (1 - \mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y)) |_{(x,y)}$$

or

$$f_{mm}(a, b) = (1 - a) \vee (a \wedge b)$$

Example 3: Zadeh's Max-Min rule

If x is A then y is B with the implication of Zadeh's max-min rule can be written equivalently as :

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y)$$

Here, Y is the universe of discourse with membership values for all $y \in Y$ is 1, that is , $\mu_Y(y) = 1 \forall y \in Y$.

Suppose $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$

and $A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$ are two fuzzy sets.

We are to determine $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

Example 3: Zadeh's min-max rule:

The computation of $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$ is as follows:

$$A \times B = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array} \quad \text{and}$$

$$\bar{A} \times Y = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 3: Zadeh's min-max rule:

Therefore,

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y) = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 3 :

$X = \{a, b, c, d\}$

$Y = \{1, 2, 3, 4\}$

Let, $A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$

Determine the implication relation :

If x is A then y is B

Here, $A \times B =$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 3 :

$$\text{and } \bar{A} \times Y = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y) = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

This R represents **If** x **is A then** y **is B**

Example 3 :

IF x is A THEN y is B ELSE y is C.

The relation R is equivalent to

$$R = (A \times B) \cup (\bar{A} \times C)$$

The membership function of R is given by

$\mu_R(x, y) = max[min\{\mu_A(x), \mu_B(y)\}, min\{\mu_{\bar{A}}(x), \mu_C(y)\}]$

Example 4:

$X = \{a, b, c, d\}$

$Y = \{1, 2, 3, 4\}$

$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$

$C = \{(1, 0), (2, 0.4), (3, 1.0), (4, 0.8)\}$

Determine the implication relation :

If x is A then y is B else y is C

$$\text{Here, } A \times B = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 4:

and $\bar{A} \times C =$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}$$

$R =$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix}
\end{array}$$

Fuzzy Inferences

Fuzzy inferences

Let's start with propositional logic. We know the following in propositional logic.

1. Modus Ponens : $P, P \Longrightarrow Q, \qquad \Leftrightarrow Q$
2. Modus Tollens : $P \Longrightarrow Q, \neg Q \qquad \Leftrightarrow, \neg P$
3. Chain rule : $P \Longrightarrow Q, Q \Longrightarrow R \qquad \Leftrightarrow, P \Longrightarrow R$

Inferring procedures in Fuzzy logic

Two important inferring procedures are used in fuzzy systems :

- **Generalized Modus Ponens (GMP)**

  If x is A Then y is B

  x is A'

  ---

  y is B'

- **Generalized Modus Tollens (GMT)**

  If x is A Then y is B

  y is B'

  ---

  x is A'

x is A'

Fuzzy inferring procedures

- Here, A, B, A' and B' are fuzzy sets.
- To compute the membership function A' and B' the max-min composition of fuzzy sets B' and A' ,respectively with $R(x, y)$ (which is the known implication relation) is to be used.
- Thus,

$$B' = A' \circ R(x, y) \qquad \mu_B(y) = max[min(\mu_{A'}(x), \mu_R(x, y))]$$

$$A' = B' \circ R(x, y) \qquad \mu_A(x) = max[min(\mu_{B'}(y), \mu_R(x, y))]$$

Generalized Modus Ponens (GMP)

$$P : \textbf{If } x \textbf{ is } A \textbf{ then } y \textbf{ is } B$$

Let us consider two sets of variables x and y be

$X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, respectively.

Also, let us consider the following.

$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$

$B = \{(y_1, 1), (y_2, 0.4)\}$

Then, given a fact expressed by the proposition x is A',
where $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$
derive a conclusion in the form y **is** B' (using generalized modus ponens (GMP)).

Example: Generalized Modus Ponens

If x is A Then y is B

x is A'

---

y is B'

We are to find $B' = A' \circ R(x,y)$ where $R(x,y) = max\{A \times B, \overline{A} \times Y\}$

$$A \times B = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.4 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} \text{ and } \overline{A} \times Y = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Note: For $A \times B$, $\mu_{A \times B}(x,y) = min(\mu_A x, \mu_B(y))$

Example: Generalized Modus Ponens

$$R(x,y) = (A \times B) \cup (\bar{A} \times y) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

Now, $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

Therefore, $B' = A' \circ R(x,y) =$

$$\begin{bmatrix} 0.6 & 0.9 & 0.7 \end{bmatrix} \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \end{bmatrix}$$

Thus we derive that y is B' where $B' = \{(y_1, 0.9), (y_2, 0.5)\}$

Example: Generalized Modus Tollens

Generalized Modus Tollens (GMT)

P: If x is A Then y is B

Q: y is B'

---

x is A'

Example: Generalized Modus Tollens

- Let sets of variables x and y be $X = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2\}$, respectively.
- Assume that a proposition **If** x **is** A **Then** y **is** B given where $A = \{(x_1, 0.5), (x_2, 1.0), (x_3, 0.6)\}$ and $B = \{(y_1, 0.6), (y_2, 0.4)\}$
- Assume now that a fact expressed by a proposition y **is** B is given where $B' = \{(y_1, 0.9), (y_2, 0.7)\}$.
- From the above, we are to conclude that x **is** A'. That is, we are to determine A'

Example: Generalized Modus Tollens

- We first calculate $R(x,y) = (A \times B) \cup (\overline{A} \times y)$

$$R(x,y) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

- Next, we calculate $A' = B' \circ R(x,y)$

$$A' = [0.9 \quad 0.7] \circ \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} = [0.5 \quad 0.9 \quad 0.6]$$

- Hence, we calculate that x is A' where $A' = [(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)]$

Practice

Apply the fuzzy GMP rule to deduce **Rotation is quite slow**

Given that :

- If temperature is High then rotation is Slow.
- temperature is Very High

Let,

$X = \{30, 40, 50, 60, 70, 80, 90, 100\}$ be the set of temperatures.

$Y = \{10, 20, 30, 40, 50, 60\}$ be the set of rotations per minute.

Practice

The fuzzy set High(H), Very High (VH), Slow(S) and Quite Slow (QS) are given below.

$H = \{(70, 1), (80, 1), (90, 0.3)\}$

$VH = \{(90, 0.9), (100, 1)\}$

$S = \{(30, 0.8), (40, 1.0), (50, 0.6)\}$

$QS = \{(10, 1), (20, 0.8)\}$

1. If temperature is High then the rotation is Slow.

$$R = (H \times S) \cup (\overline{H} \times Y)$$

2. temperature is Very High

Thus, to deduce "rotation is Quite Slow", we make use the composition rule $QS = VH \circ R(x, y)$

Defuzzification Techniques

What is defuzzification?

- Defuzzification means the fuzzy to crisp conversion.

- **Example 1:**

  Suppose, T_{HIGH} denotes a fuzzy set representing **temperature is High.**

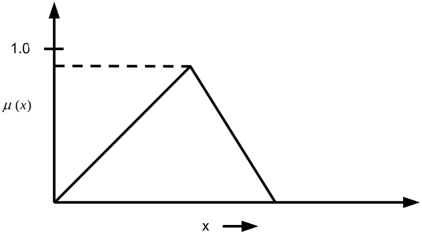
  T_{HIGH} is given as follows.

  T_{HIGH} = (15,0.1), (20, 0.4), (25,0.45), (30,0.55), (35,0.65), (40,0.7), (45,0.85),(50,0.9)

- What is the crisp value that implies for the high temperature?

Example 2: Fuzzy to crisp

As an another example, let us consider a fuzzy set whose membership finction is shown in the following figure.



What is the crisp value of the fuzzy set in this case?

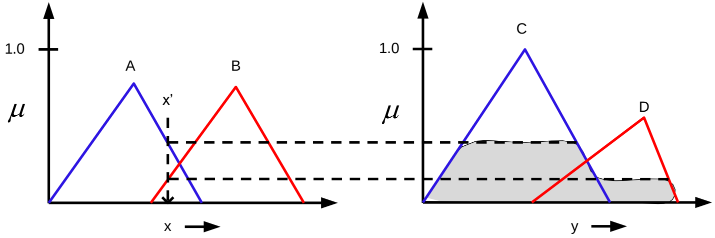
Example 3: Fuzzy to crisp

Now, consider the following two rules in the fuzzy rule base.

$$R1: \text{If } x \textbf{ is } A \text{ then } y \textbf{ is } C$$

$$R2: \text{If } x \textbf{ is } B \text{ then } y \textbf{ is } D$$

A pictorial representation of the above rule base is shown in the following figures.



What is the crisp value that can be inferred from the above rules given an input say x'?

Why defuzzification?

The fuzzy results generated can not be used in an application, where decision has to be taken only on crisp values.

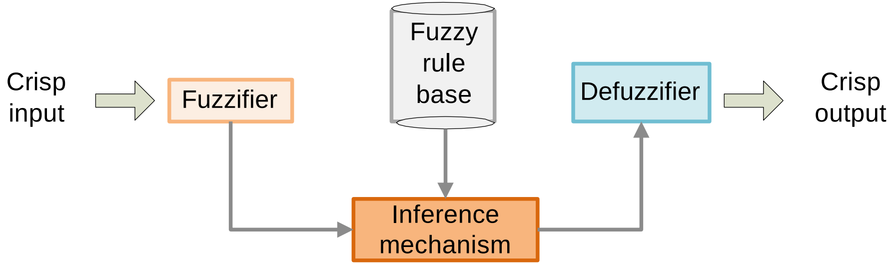
Example:

$$\text{If } T_{HIGH} \text{ then rotate } R_{FAST}.$$

Here, may be input T_{HIGH} is fuzzy, but action **rotate** should be based on the crisp value of R_{FAST}.

Generic structure of a Fuzzy system

Following figures shows a general fraework of a fuzzy system.



Defuzzification Techniques

Defuzzification methods

A number of defuzzification methods are known. Such as

1. **Lambda-cut method**
2. **Weighted average method**
3. **Maxima methods**
4. **Centroid methods**

Lambda-cut method

Lambda-cut method

Lambda-cut method is applicable to derive crisp value of a fuzzy set or relation. Thus

- Lambda-cut method for fuzzy set
- Lambda-cut method for fuzzy relation

In many literature, Lambda-cut method is also alternatively termed as *Alpha-cut method*.

Lamda-cut method for fuzzy set

1. In this method a fuzzy set A is transformed into a crisp set A_λ for a given value of λ $(0 \leq \lambda \leq 1)$
2. In other-words, $A_\lambda = \{x | \mu_A(x) \geq \lambda\}$
3. That is, the value of Lambda-cut set A_λ is x, when the membership value corresponding to x is greater than or equal to the specified λ.
4. This Lambda-cut set A_λ is also called alpha-cut set.

Lambda-cut for a fuzzy set : Example

$A_1 = \{(x_1, 0.9), (x_2, 0.5), (x_3, 0.2), (x_4, 0.3)\}$

Then $A_{0.6} = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 0)\} = \{x_1\}$

and

$A_2 = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.8), (x_4, 0.7)\}$

$A_{0.2} = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 1)\} = \{x_2, x_3, x_4\}$

Lambda-cut sets : Example

Two fuzzy sets P and Q are defined on x as follows.

$\mu(x)$	x_1	x_2	x_3	x_4	x_5
P	0.1	0.2	0.7	0.5	0.4
Q	0.9	0.6	0.3	0.2	0.8

Find the following :

(a) $P_{0.2}$, $Q_{0.3}$

(b) $(P \cup Q)_{0.6}$

(c) $(P \cup \overline{P})_{0.8}$

(d) $(P \cap Q)_{0.4}$

Lambda-cut for a fuzzy relation

The Lambda-cut method for a fuzzy set can also be extended to fuzzy relation also.

Example: For a fuzzy relation R

$$R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.5 & 0.9 & 0.6 \\ 0.4 & 0.8 & 0.7 \end{bmatrix}$$

We are to find λ-cut relations for the following values of $\lambda = 0, 0.2, 0.9, 0.5$

$R_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $R_{0.2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and

$R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $R_{0.5} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Summary: Lambda-cut methods

Lambda-cut method converts a fuzzy set (or a fuzzy relation) into crisp set (or relation).

Output of a Fuzzy System

Output of a fuzzy System

The output of a fuzzy system can be a single fuzzy set or union of two or more fuzzy sets.

To understand the second concept, let us consider a fuzzy system with n-rules.

R_1: If x is A_1 then y is B_1
R_2: If x is A_2 then y is B_2
....
....
R_n: If x is A_n then y is B_n

In this case, the output y for a given input $x = x_1$ is possibly $B = B_1 \cup B_2 \cup .....B_n$

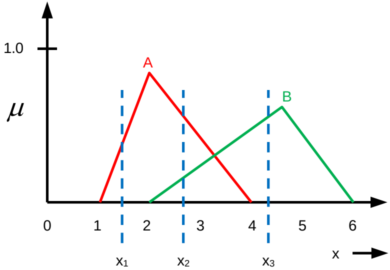
Output fuzzy set : Illustration

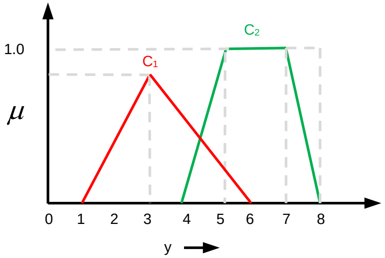
Suppose, two rules R_1 and R_2 are given as follows:

1. R_1: If x is A_1 then y is C_1
2. R_2: If x is A_2 then y is C_2

Here, the output fuzzy set $C = C_1 \cup C_2$.

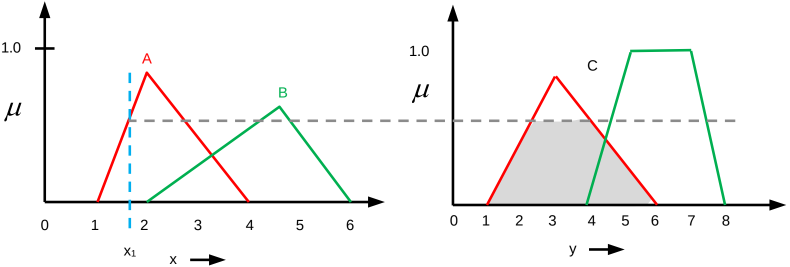
For instance, let us consider the following:





Output fuzzy set : Illustration

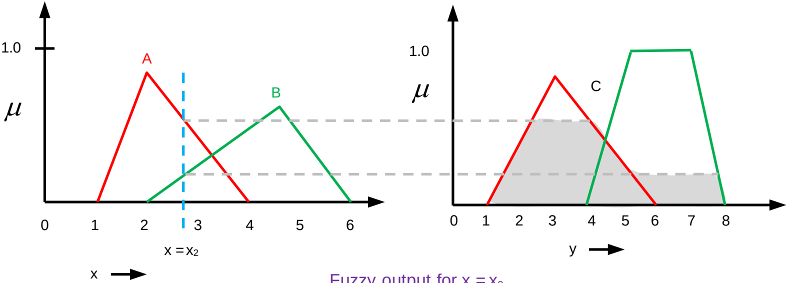
The fuzzy output for $x = x_1$ is shown below.



Fuzzy output for $x = x_1$

Output fuzzy set : Illustration

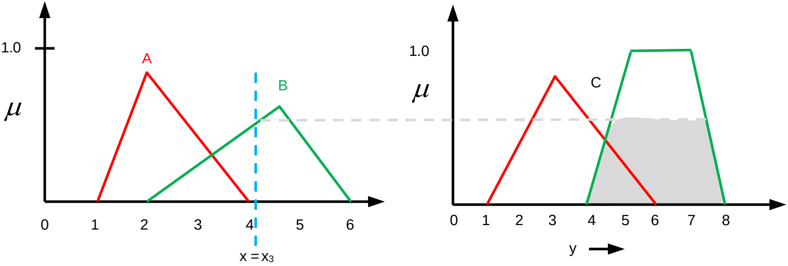
The fuzzy output for $x = x_2$ is shown below.



Fuzzy output for $x = x_2$

Output fuzzy set : Illustration

The fuzzy output for $x = x_3$ is shown below.



Fuzzy output for $x = x_3$

Defuzzification Methods

Following defuzzification methods are known to calculate crisp output in the situations as discussed in the last few slides

- **Maxima Methods**
  1. Height method
  2. First of maxima (FoM)
  3. Last of maxima (LoM)
  4. Mean of maxima(MoM)
- **Centroid methods**
  1. Center of gravity method (CoG)
  2. Center of sum method (CoS)
  3. Center of area method (CoA)
- **Weighted average method**

Defuzzification Technique
Maxima Methods

Maxima methods

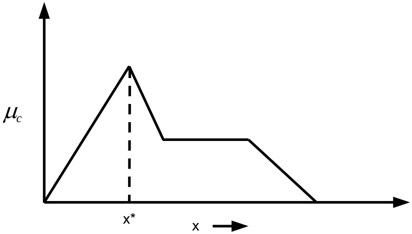
Following defuzzification methods are known to calculate crisp output.

- **Maxima Methods**
  1. Height method
  2. First of maxima (FoM)
  3. Last of maxima (LoM)
  4. Mean of maxima(MoM)
- **Centroid methods**
  1. Center of gravity method (CoG)
  2. Center of sum method (CoS)
  3. Center of area method (CoA)
- **Weighted average method**

Maxima method : Height method

This method is based on Max-membership principle, and defined as follows.

$$\mu_C(x^*) \geq \mu_C(x) \text{ for all } x \in X$$

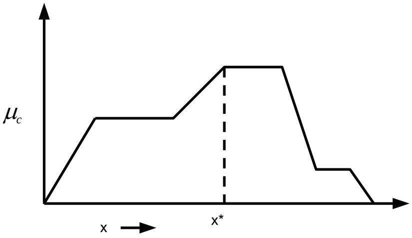


Note:

1. Here, x^* is the height of the output fuzzy set C.
2. This method is applicable when height is unique.

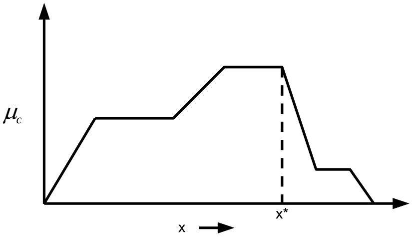
Maxima method : FoM

FoM: First of Maxima : $x^* = min\{x|C(x) = max_W C\{w\}\}$



Maxima method : LoM

LoM : Last of Maxima : $x^* = max\{x|C(x) = max_w C\{w\}\}$



Maxima method : MoM

$$X^* = \frac{\sum_{x_i \in M}(x_i)}{|M|}$$

where, $M = \{x_i | \mu(x_i) = h(C)\}$ where $h(C)$ is the height of the fuzzy set C

MoM : Example 1

Suppose, a fuzzy set **Young** is defined as follows:

Young = $\{(15,0.5), (20,0.8), (25,0.8), (30,0.5), (35,0.3)\}$

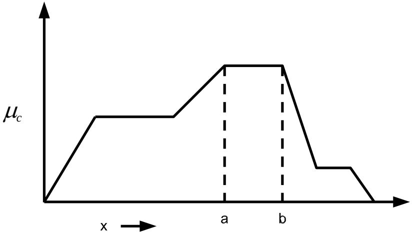
Then the crisp value of **Young** using MoM method is

$x^* = \frac{20+25}{2} = 22.5$

Thus, a person of 22.5 years old is treated as young!

MoM : Example 2

What is the crisp value of the fuzzy set using MoM in the following case?



$$x^* = \frac{a+b}{2}$$

Note:

- Thus, MoM is also synonymous to middle of maxima.
- MoM is also general method of Height.

Defuzzification Technique
Centroid Methods

Cenroid methods

Following defuzzification methods are known to calculate crisp output.

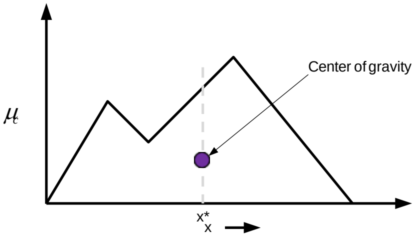
- **Maxima Methods**
  1. Height method
  2. First of maxima (FoM)
  3. Last of maxima (LoM)
  4. Mean of maxima(MoM)
- **Centroid methods**
  1. Center of gravity method (CoG)
  2. Center of sum method (CoS)
  3. Center of area method (CoA)
- **Weighted average method**

Centroid method : CoG

1. The basic principle in CoG method is to find the point x^* where a vertical line would slice the aggregate into two equal masses.
2. Mathematically, the CoG can be expressed as follows :

$$x^* = \frac{\int x.\mu_C(x)dx}{\int \mu_C(x)dx}$$

3. Graphically,



Centroid method : CoG

Note:

1. x^* is the x-coordinate of center of gravity.
2. $\int \mu_C(x)dx$ denotes the area of the region bounded by the curve μ_C.
3. If μ_C is defined with a discrete membership function, then CoG can be stated as :

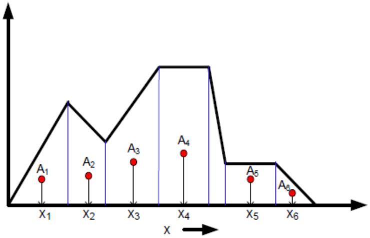
   $$x^* = \frac{\sum_{i=1}^{n} x_i.\mu_C(x_i)}{\sum_{i=1}^{n} \mu_C(x_i)} \; ;$$

4. Here, x_i is a sample element and n represents the number of samples in fuzzy set C.

CoG : A geometrical method of calculation

Steps:

1. Divide the entire region into a number of small regular regions (e.g. triangles, trapizoid etc.)

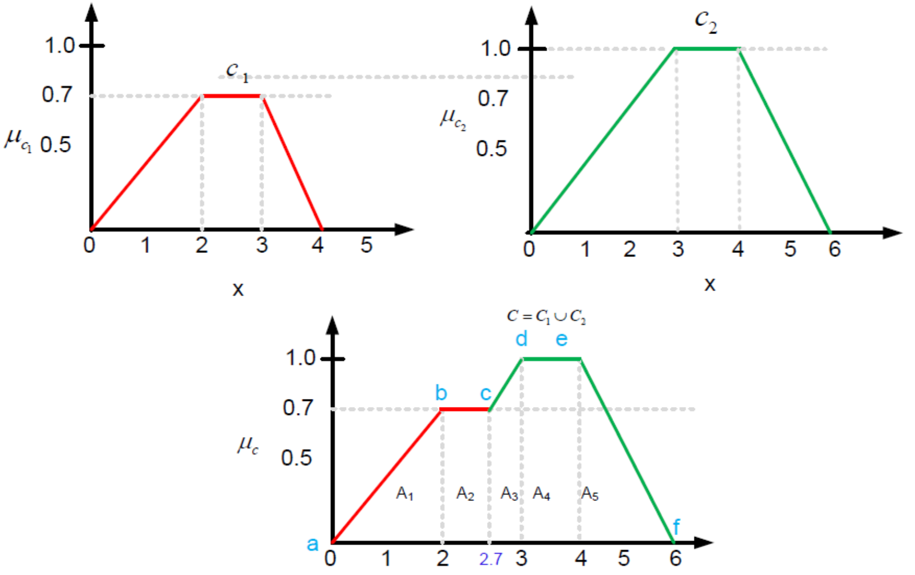


2. Let A_i and x_i denotes the area and c.g. of the i-th portion.
3. Then x^* according to CoG is

$$x^* = \frac{\sum_{i=1}^{n} x_i.(A_i)}{\sum_{i=1}^{n} A_i}$$

where n is the number of smaller geometrical components.

CoG: An example of integral method of calculation



CoG: An example of integral method of calculation

$$\mu_c(x) = \begin{cases} 0.35x & 0 \leq x < 2 \\ 0.7 & 2 \leq x < 2.7 \\ x - 2 & 2.7 \leq x < 3 \\ 1 & 3 \leq x < 4 \\ (-0.5x + 3) & 4 \leq x \leq 6 \end{cases}$$

For $A_1 : y - 0 = \frac{0.7}{2}(x - 0)$, or $y = 0.35x$

For $A_2 : y = 0.7$

For $A_3 : y - 0 = \frac{1-0}{3-2}(x - 2)$, or $y = x - 2$

For, $A_4 : y = 1$

For, $A_5 : y - 1 = \frac{0-1}{6-4}(x - 4)$, or $y = -0.5x + 3$

CoG: An example of integral method of calculation

Thus, $x^* = \frac{\int x.\mu_c(x)dx}{\int \mu_c(x)dx} = \frac{N}{D}$

$N = \int_0^2 0.35x^2dx + \int_2^{2.7} 0.7x^2dx + \int_{2.7}^3 (x^2 - 2x)dx + \int_3^4 xdx + \int_4^6 (-0.5x^2 + 3x)dx$

= 10.98

$D = \int_0^2 0.35xdx + \int_2^{2.7} 0.7xdx + \int_{2.7}^3 (x-2)dx + \int_3^4 dx + \int_4^6 (-0.5x+3)dx$

= 3.445

Thus, $x^* = \frac{10.98}{3.445} = 3.187$

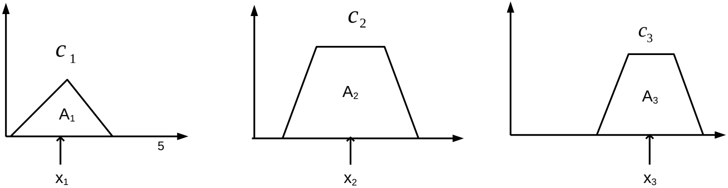
Centroid method : CoS

If the output fuzzy set $C = C_1 \cup C_2 \cup \ldots . C_n$, then the crisp value according to CoS is defined as

$$x^* = \frac{\sum_{i=1}^{n} x_i . A_{c_i}}{\sum_{i=1}^{n} A_{c_i}}$$

Here, A_{c_i} denotes the area of the region bounded by the fuzzy set C_i and x_i is the geometric center of the area A_{c_i}.

Graphically,



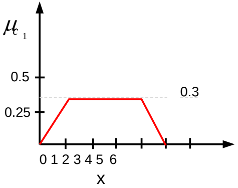
Centroid method : CoS

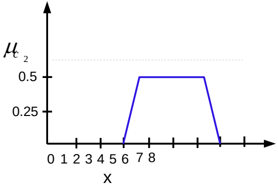
Note:

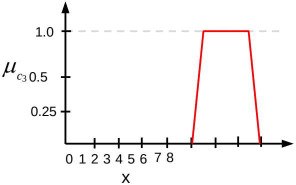
1. In CoG method, the overlapping area is counted once, whereas, in CoS , the overlapping is counted twice or so.
2. In CoS, we use the center of area and hence, its name instead of center of gravity as in CoG.

CoS: Example

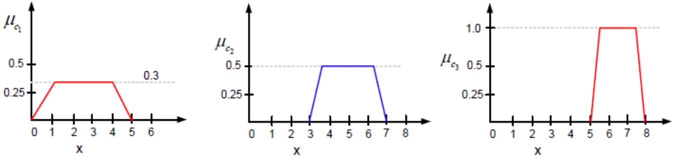
Consider the three output fuzzy sets as shown in the following plots:







CoS: Example



In this case, we have

$A_{C_1} = \frac{1}{2} \times 0.3 \times (3+5), x_1 = 2.5$

$A_{C_2} = \frac{1}{2} \times 0.5 \times (4+2), x_2 = 5$

$A_{C_3} = \frac{1}{2} \times 1 \times (3+1), x_3 = 6.5$

Thus, $x^* = \frac{\frac{1}{2}\times 0.3\times(3+5)\times 2.5+\frac{1}{2}\times 0.5\times(4+2)\times 5+\frac{1}{2}\times 1\times(3+1)\times 6.5}{\frac{1}{2}\times 0.3\times(3+5+\frac{1}{2}\times 0.5\times(4+2)+\frac{1}{2}\times 1\times(3+1)} = 5.00$

Note:

The crisp value of $C = C_1 \cup C_2 \cup C_3$ using CoG method can be found to be calculated as $x^* = 4.9$

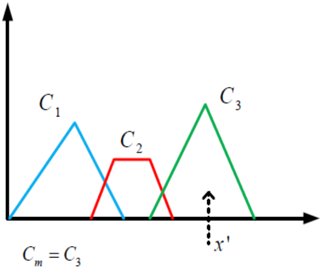
Centroid method: Certer of largest area

If the fuzzy set has two subregions, then the center of gravity of the subregion with the largest area can be used to calculate the defuzzified value.

Mathematically, $x^* = \frac{\int \mu_{c_m}(x).x' dx}{\int \mu_{c_m}(x) dx}$;

Here, C_m is the region with largest area, x' is the center of gravity of C_m.

Graphically,



Weighted Average Method

Cenroid methods

Following defuzzification methods are known to calculate crisp output.

- **Maxima Methods**
  1. Height method
  2. First of maxima (FoM)
  3. Last of maxima (LoM)
  4. Mean of maxima(MoM)
- **Centroid methods**
  1. Center of gravity method (CoG)
  2. Center of sum method (CoS)
  3. Center of area method (CoA)
- **Weighted average method**

Weighted average method

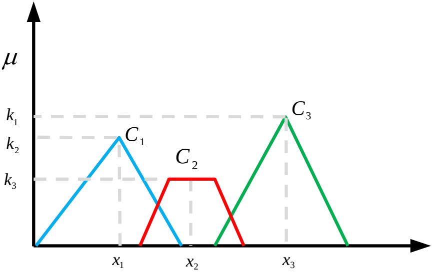
1. This method is also alternatively called "Sugeno defuzzification" method.
2. The method can be used only for symmetrical output membership functions.
3. The crisp value accroding to this method is

$$x^* = \frac{\sum_{i=1}^{n} \mu_{C_i}(x_i).(x_i)}{\sum_{i=1}^{n} \mu_{C_i}(x_i)}$$

where, $C_1, C_2, ...C_n$ are the output fuzzy sets and (x_i) is the value where middle of the fuzzy set C_i is observed.

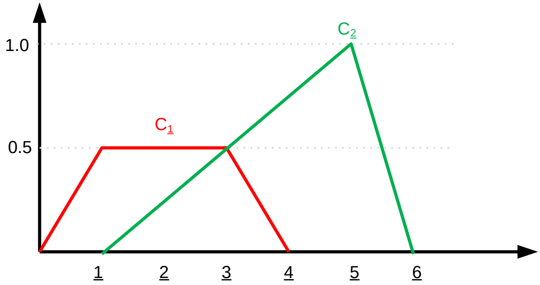
Weighted average method

Graphically,



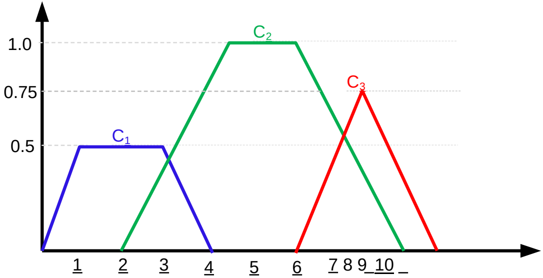
Exercise 1

Find the crisp value of the following using all defuzzified methods.



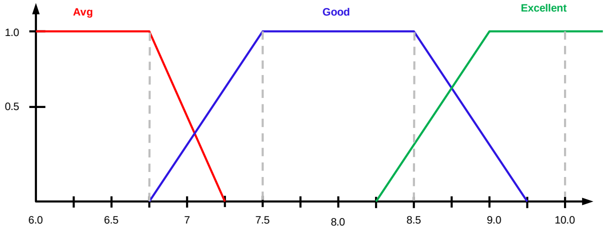
Exercise 1

Find the crisp value of the following using all defuzzified methods.



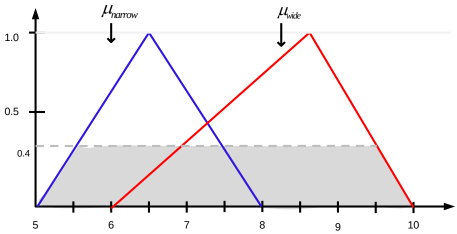
Exercise 3

- The membership function defining a student as Average, Good, and Excellent denoted by respective membership functions are as shown below.



- Find the crisp value of ”Good Student”
- Hint: Use CoG method to the portion ”Good” to calculate it.

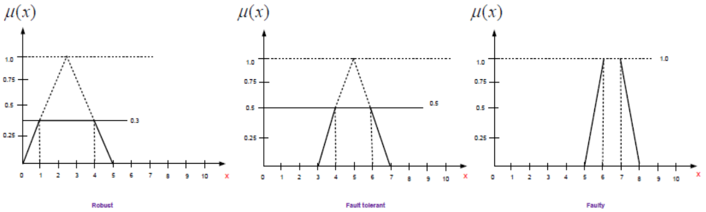
Exercise 4



- The width of a road as narrow and wide is defined by two fuzzy sets, whose membership functions are plotted as shown above.
- If a road with its degree of membership value is 0.4 then what will be its width (in crisp) measure.
- Hint: Use CoG method for the shadded region.

Exercise 5

- The faulty measure of a circuit is defined fuzzily by three fuzzy sets namely Faulty(F), Fault tolerant (FT) and Robust(R) defined by three membership functions with number of faults occur as universe of discourses and is shown below.



- Reliability is measured as $R^* = F \cup FT \cup R$. With a certain observation in testing $(x, 0.3) \in R, (x, 0.5) \in FT, (x, 0.8) \in F$.
- Calculate the reliability measure in crisp value.
- Calculate with 1) CoS 2) CoG .

Fuzzy Logic Controller

Applications of Fuzzy Logic

Fuzzy Systems : Fuzzy Logic Controller

- Concept of fuzzy theory can be applied in many applications, such as fuzzy reasoning, fuzzy clustering, fuzzy programming etc.
- Out of all these applications, fuzzy reasoning, also called ”fuzzy logic controller (FLC)” is an important application.
- Fuzzy logic controllers are special expert systems. In general, a FLC employs a knowledge base expressed in terms of a fuzzy inference rules and a fuzzy inference engine to solve a problem.
- We use FLC where an exact mathematical formulation of the problem is not possible or very difficult.
- These difficulties are due to non-linearities, time-varying nature of the process, large unpredictable environment disturbances etc.

Fuzzy Systems : Fuzzy Logic Controller

A general scheme of a fuzzy controller is shown in the following figure.

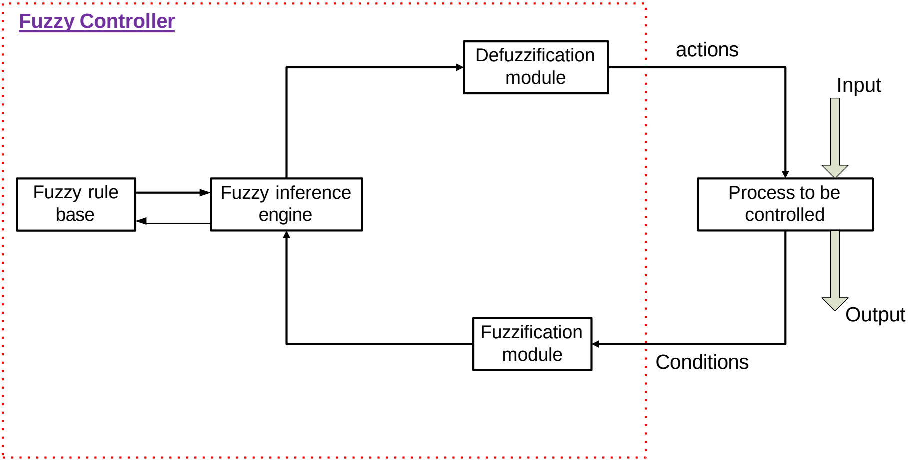


Figure 1

Fuzzy Systems : Fuzzy Logic Controller

A general fuzzy controller consists of four modules:

1. a fuzzy rule base,
2. a fuzzy inference engine,
3. a fuzzification module, and
4. a defuzzification module.

Fuzzy Systems : Fuzzy Logic Controller

As shown in Figure 1, a fuzzy controller operates by repeating a cycle of the following four steps :

1. Measurements (inputs) are taken of all variables that represent relevant condition of controller process.
2. These measurements are converted into appropriate fuzzy sets to express measurements uncertainties. This step is called fuzzification.
3. The fuzzified measurements are then used by the inference engine to evaluate the control rules stroed in the fuzzy rule base. The result of this evaluation is a fuzzy set (or several fuzzy sets) defined on the universe of possible actions.
4. This output fuzzy set is then converted into a single (crisp) value (or a vector of values). This is the final step called defuzzification. The defuzzified values represent actions to be taken by the fuzzy contoller.

Fuzzy Systems : Fuzzy Logic Controller

There are two approaches of FLC known.

1. Mamdani approach

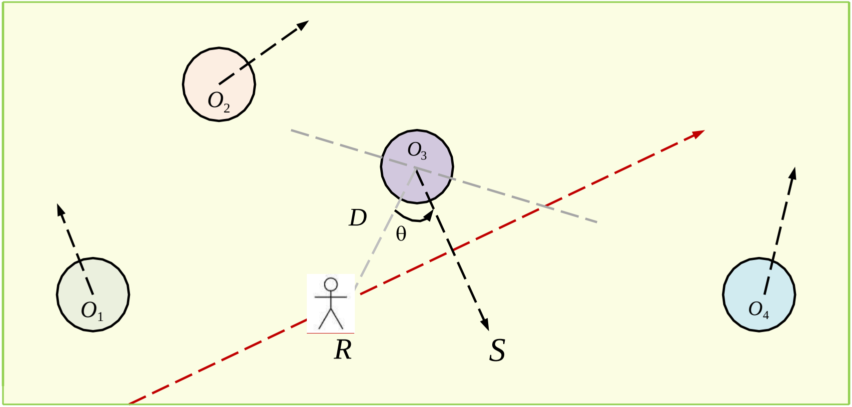
2. Takagi and sugeno's approach

- Mamdani approach follows linguistic fuzzy modeling and chacterized by its high interpretability and low accuracy.
- On the other hand, Takagi and Sugeno's approach follows precise fuzzy modeling and obtains high accuracy but at the cost of low interpretabily.
- We illustrate the above two approaches with two examples.

Mamdani approach : Mobile Robot

- Consider the control of navigation of a mobile robot in the pressure of a number of moving objects.
- To make the problem simple, consider only four moving objects, each of equal size and moving with the same speed.
- A typical scenario is shown in Figure 2.

Mamdani approach : Mobile Robot



Mamdani approach : Mobile Robot

- We consider two parameters : D, the distance from the robot to an object and θ the angle of motion of an object with respect to the robot.
- The value of these parameters with respect to the most critical object will decide an output called deviation (δ).
- We assume the range of values of D is [0.1, ... 2.2] in meter and θ is [-90, ... , 0, ... 90] in degree.
- After identifying the relevant input and output variables of the controller and their range of values, the Mamdani approach is to select some meaningful states called "linguistic states" for each variable and express them by appropriate fuzzy sets.

Linguistic States

For the current example, we consider the following linguistic states for the three parameters.

Distance is represented using four linguistic states:

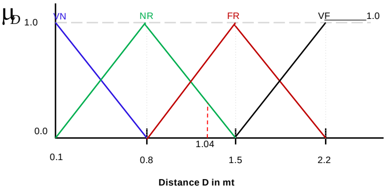
1. VN : Very Near
2. NR : Near
3. VF : Very Far
4. FR : Far

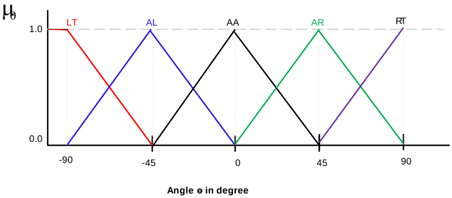
Angle (for both angular direction (θ) and deviation (δ)) are represented using five linguistic states:

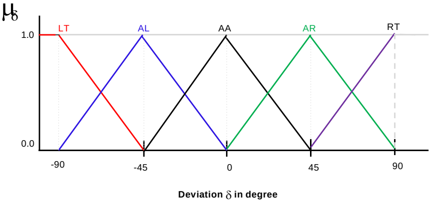
1. LT : Left
2. AL : Ahead Left
3. AA: Ahead
4. AR : Ahead Right
5. RT : Right

Linguistic States

Three different fuzzy sets for the three different parameters are given below (Figure 3).







Fuzzy rule base

Once the fuzzy sets of all parameters are worked out, our next step in FLC design is to decide fuzzy rule base of the FLC.

The rule base for the FLC of mobile robot is shown in the form of a table below.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} LT & AL & AA & AR & RT \end{array} \\
\begin{array}{c} VN \\ NR \\ FR \\ VF \end{array} & \begin{bmatrix} AA & AR & AL & AL & AA \\ AA & AA & RT & AA & AA \\ AA & AA & AR & AA & AA \\ AA & AA & AA & AA & AA \end{bmatrix}
\end{array}
$$

Fuzzy rule base for the mobile robot

Note that this rule base defines 20 rules for all possible instances. These rules are simple rules and take in the following forms.

- Rule 1: If (distance is VN) and (angle is LT) Then (deviation is AA)

  :

  :

- Rule 13: If (distance is FR) and (angle is AA) Then (deviation is AR)

  :

  :

- Rule 20: Rule 1: If (distance is VF) and (angle is RT) Then (deviation is AA)

Fuzzification of inputs

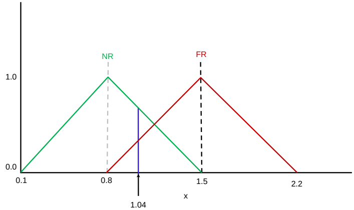
- The next step is the fuzzification of inputs. Let us consider, at any instantce, the object O_3 is critical to the Mobile Robot and distance D = 1.04 m and angle $\theta = 30^o$ (see Figure 2).
- For this input, we are to decide the deviation δ of the robot as output.

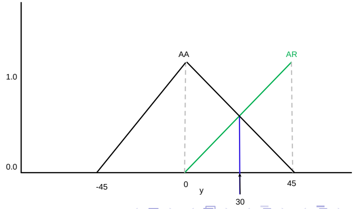
- From the given fuzzy sets and input parameters' values, we say that the distance D = 1.04m may be called as either NR (near) or FR (far).
- Similarly, the input angle $\theta = 30^o$ can be declared as either AA (ahead) or AR(ahead right).

Fuzzification of inputs

Hence, we are to determine the membership values corresponding to these values, which is as follows.

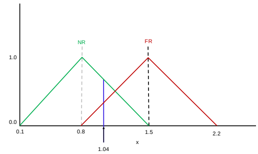
$$
\begin{aligned}
x &= 1.04m \\
\mu_{NR}(x) &= 0.6571 \\
\mu_{FR}(x) &= 0.3429 \\
y &= 30^{o} \\
\mu_{AA}(y) &= 0.3333 \\
\mu_{AR}(y) &= 0.6667
\end{aligned}
$$

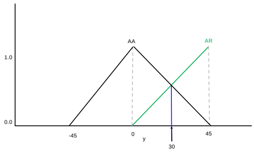




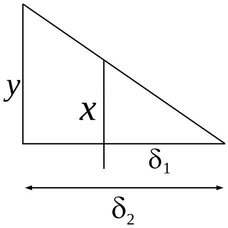
Fuzzification of inputs

Hint : Use the principle of similarity. $\frac{x}{y} = \frac{\delta_1}{\delta_2}$





Thus, $\frac{x}{1} = \frac{1.5 - 1.04}{1.5 - 0.8}$, that is, $x = 0.6571$



Rule strength computation

There are many rules in the rule base and all rules may not be applicable.

For the given $x = 1.04$ and $\theta = 30^o$, only following four rules out of 20 rules are firable.

- $R1$: If (distance is NR) and (angle is AA) Then (deviation is RT)
- $R2$: If (distance is NR) and (angle is AR) Then (deviation is AA)
- $R3$: If (distance is FR) and (angle is AA) Then (deviation is AR)
- $R4$: If (distance is FR) and (angle is AR) Then (deviation is AA)

Rule strength computation

The strength (also called α values) of the firable rules are calculated as follows.

- $\alpha(R1) = min(\mu_{NR}(x), \mu_{AA}(y)) = min(0.6571, 0.3333) = 0.3333$
- $\alpha(R2) = min(\mu_{NR}(x), \mu_{AR}(y)) = min(0.6571, 0.6667) = 0.6571$
- $\alpha(R3) = min(\mu_{FR}(x), \mu_{AA}(y)) = min(0.3429, 0.3333) = 0.3333$
- $\alpha(R4) = min(\mu_{FR}(x), \mu_{AR}(y)) = min(0.3429, 0.6667) = 0.3429$

In practice, all rules which are above certain threshold value of rule strngth are selected for the output computation.

Fuzzy output

The next step is to determine the fuzzified outputs corresponding to each fired rules.

The working principle of doing this is first discussed and then we illustrate with the running example.

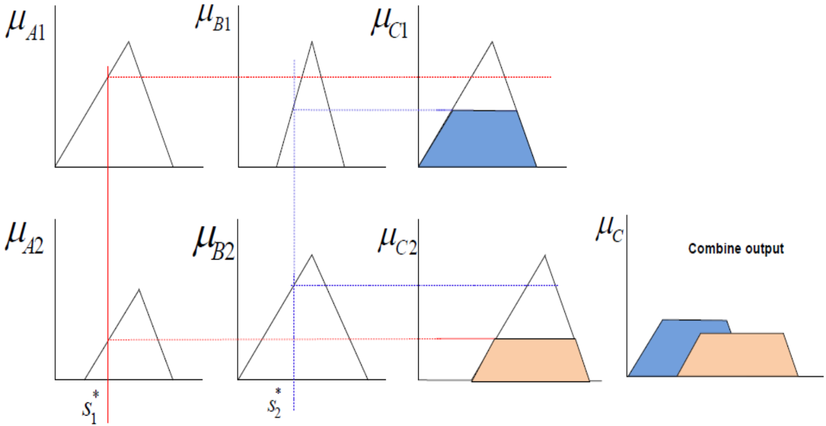
Suppose, only two fuzzy rules, $R1$ and $R2$, for which we are to decide fuzzy output.

- $R1$: IF (s_1 is A_1) AND (s_2 is B_1) THEN (f is C_1)
- $R2$: IF (s_1 is A_2) AND (s_2 is B_2) THEN (f is C_2)

Suppose, $s_1^*$and $s_2^*$are the inputs for fuzzy variables s_1 and s_2. μ_{A1}, μ_{A2}, μ_{B1}, μ_{B2}, μ_{C1} and μ_{C2} are the membership values for different fuzzy sets.

Fuzzy output

The fuzzy output computation is graphically shows in the following figure.



Fuzzy output

Note:

- We take min of membership function values for each rule.
- Output membership function is obtained by aggregating the membership function of result of each rule.
- Fuzzy output is nothing but fuzzy OR of all output of rules.

Illustration : Mobile Robot

For four rules, we find the following results.

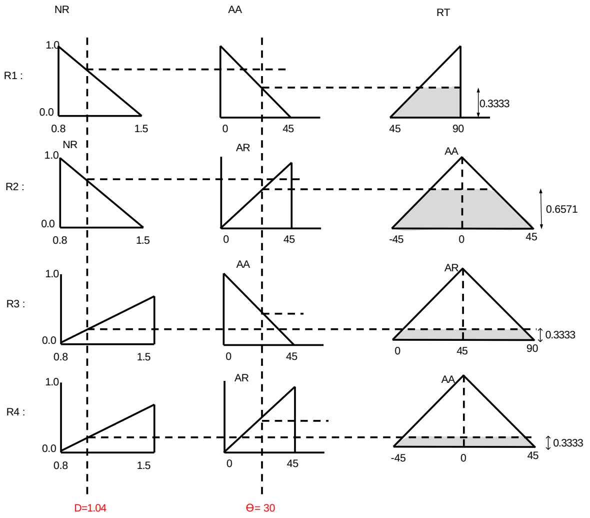
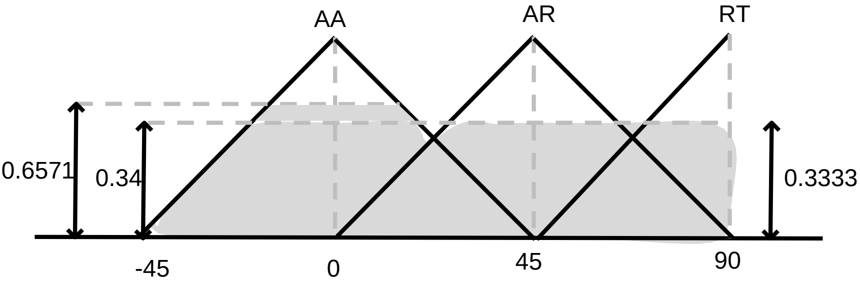


Illustration : Mobile Robot



Aggregation of all results

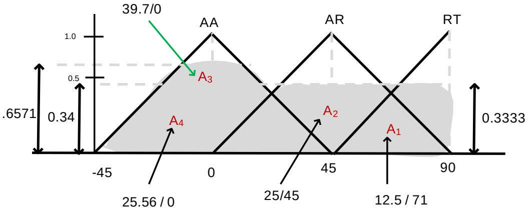
Defuzzification

The fuzzy output needs to be defuzzified and its crisp value has to be determined for the output to take decision.

Illustration : Mobile Robot

From the combined fuzzified output for all four fired rules, we get the crisp value using Center of Sum method as follows.

$$v = \frac{12.5 \times 71 + 25 \times 45 + 25.56 \times 0 + 25.56 \times 0}{12.5 + 39.79 + 25 + 25.56} = 19.59$$



Conclusion : Therefore, the robot should deviate by 19.58089 degree towards the right with respect to the line joining to the move of direction to avoid collision with the obstacle O_3.

Takagi and Sugeno's approach

- In this approach, a rule is composed of fuzzy antecedent and functional consequent parts.
- Thus, any i-th rule, in this approach is represented by
  If (x_1 is A_1^i) and (x_2 is A_2^i) ...... and (x_n is A_n^i)
- Then, $y^i = a_0^i + a_1^i x_1 + a_2^i x_2 + ... + a_n^i x_n$
  where, a_0, a_1, a_2, ... a_n are the co-efficients.
- The weight of i-th rule can be determined for a set of inputs x_1, x_2, ... x_n as follows.

  $$w^i = \mu^i_{A_1}(x_1) \times \mu^i_{A_2}(x_2) \times ..... \times \mu^i_{A_n}(x_n)$$

  where A_1, A_2, ... , A_n indicates membership function distributions of the linguistic hedges used to represent the input variables and μ denotes membership function value.
- The combined action then can be obtained as
  $y = \frac{\sum_i^k w^i y^i}{\sum_i^k w^i}$; where k denotes the total number of rules.

Illustration:

Consider two inputs I_1 and I_2. These two inputs have the following linguistic states :

I_1 : L(low), M(Medium), H(High)

I_2 : NR(Near), FR (Far), VF(Very Far)

Note:

The rule base of such as system is decided by a maximum of $3 \times 3 = 9$ feasible rules.

Illustration:

The output of any i-th rule can be expressed by the following.

$y^i = f(I_1, I_2) = a_j^i I_1 + b_k^i I_2$; where, j,k = 1,2,3.

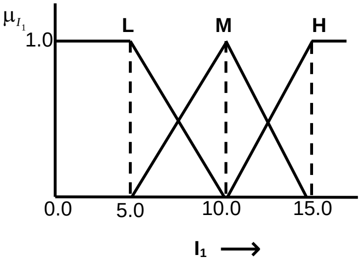
Suppose : $a_1^i = 1$, $a_2^i = 2$, $a_3^i = 3$ if $I_1 = L, M$ and H, respectively

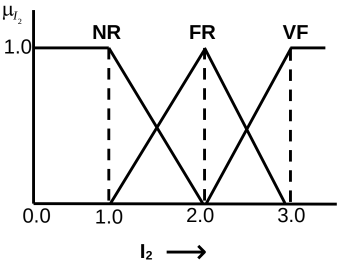
$b_1^i = 1$, $b_2^i = 2$, $b_3^i = 3$ if $I_2 = NR, FR$, and VF, respectively.

We have to calculate the output of FLC for $I_1 = 6.0$ and $I_2 = 2.2$

Illustration:

Given the distribution functions for I_1 and I_2 as below.





Solution

a) The input $I_1 = 6.0$ can be called either L or M. Similarly, the input $I_2 = 2.2$ can be declared either FR or VF.

b) Using the principal of similarity of triangle, we have the following.
$\mu_L(I_1) = 0.8$
$\mu_M(I_1) = 0.2$
$\mu_{FR}(I_2) = 0.8$
$\mu_{VF}(I_2) = 0.2$

c)For the input set, following four rules can be fired out of all 9 rules.
R1: I_1 is L and I_2 is FR
R2: I_1 is L and I_2 is VF
R3: I_1 is M and I_2 is FR
R4: I_1 is M and I_2 is VF

Solution

d)Now, the weights for each of the above rules can be determined as follows.

R1: $w^1 = \mu_L \times \mu_{FR} = 0.8 \times 0.8 = 0.6$
R2: $w^2 = \mu_L \times \mu_{VF} = 0.8 \times 0.2 = 0.16$
R3: $w^3 = \mu_M \times \mu_{FR} = 0.2 \times 0.8 = 0.16$
R4: $w^4 = \mu_M \times \mu_{VF} = 0.2 \times 0.2 = 0.6$

e)The functional consequent values for each rules can be calculated as below.

$y^1 = I_1 + 2I_2 = 6.0 + 2 \times 2.2 = 10.4$
$y^2 = I_1 + 3I_2 = 6.0 + 3 \times 2.2 = 12.6$
$y^3 = 2I_1 + 2I_2 = 2 \times 6.0 + 2 \times 2.2 = 16.4$
$y^4 = 2I_1 + 3I_2 = 2 \times 6.0 + 3 \times 2.2 = 18.6$

Solution

f) Therefore, the output y of the controller can be determined as follows.

$$y = \frac{w^1 y^1 + w^2 y^2 + w^3 y^3 + w^4 y^4}{w^1 + w^2 + w^3 + w^4} = 12.04$$

Chapter 5

Defuzzification Methods

Fuzzy rule based systems evaluate linguistic if-then rules using fuzzification, inference and composition procedures. They produce fuzzy results which usually have to be converted into crisp output. To transform the fuzzy results in to crisp, defuzzification is performed.

Defuzzification is the process of converting a fuzzified output into a single crisp value with respect to a fuzzy set. The defuzzified value in FLC (Fuzzy Logic Controller) represents the action to be taken in controlling the process.

Different Defuzzification Methods

The following are the known methods of defuzzification.

- Center of Sums Method (COS)
- Center of gravity (COG) / Centroid of Area (COA) Method
- Center of Area / Bisector of Area Method (BOA)
- Weighted Average Method
- Maxima Methods
  - First of Maxima Method (FOM)
  - Last of Maxima Method (LOM)
  - Mean of Maxima Method (MOM)

Center of Sums (COS) Method

This is the most commonly used defuzzification technique. In this method, the overlapping area is counted twice.

The defuzzified value x^* is defined as :

$$x^* = \frac{\sum_{i=1}^{N} x_i \cdot \sum_{k=1}^{n} \mu_{A_k}(x_i)}{\sum_{i=1}^{N} \quad \sum_{k=1}^{n} \mu_{A_k}(x_i)} \quad ,$$

Here, n is the number of fuzzy sets, N is the number of fuzzy variables, $\mu_{A_k}(x_i)$ is the membership function for the k-th fuzzy set.

Example

The defuzzified value x^* is defined as :

$$x^* = \frac{\sum_{i=1}^{k} A_i \times \overline{x_i}}{\sum_{i=1}^{k} A_i} \quad ,$$

Here, A_i represents the firing area of i^{th} rules and k is the total number of rules fired and $\overline{x_i}$ represents the center of area.

The aggregated fuzzy set of two fuzzy sets C_1 and C_2 is shown in Figure 1. Let the area of this two fuzzy sets are A_1 and A_2.

A_1 = ½ * [(8-1) + (7-3)] * 0.5 = ½ * 11 * 0.5 = 55/20=2.75

A_2 = ½ * [(9-3) + (8-4)] * 0.3 = ½ * 10 * 0.3 = 3/2 =1.5

Now the center of area of the fuzzy set C_1 is let say $\overline{x_1}$ = (7+3)/2= 5 and

the center of area of the fuzzy set C_2 is $\overline{x_2}$= (8+4)/2=6.

Now the defuzzified value $x^* = \frac{(A_1.\overline{x_1}+A_2.\overline{x_2})}{A_1+A_2} = \frac{(2.75 * 5 + 1.5 * 6)}{(2.75 + 1.5)}$ = 22.75/4.25 = 5.35

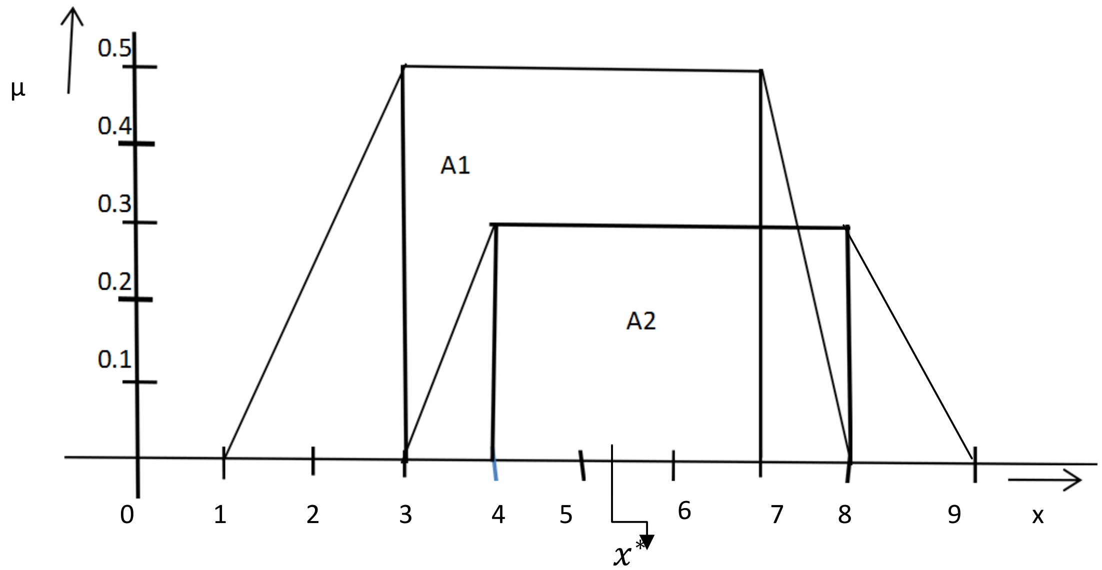


Figure 1 : Fuzzy sets $C_1$and C_2

Center of gravity (COG) / Centroid of Area (COA) Method

This method provides a crisp value based on the center of gravity of the fuzzy set. The total area of the membership function distribution used to represent the combined control action is divided into a number of sub-areas. The area and the center of gravity or centroid of each sub-area is calculated and then the summation of all these sub-areas is taken to find the defuzzified value for a discrete fuzzy set.

For discrete membership function, the defuzzified value denoted as x^* using COG is defined as:

$$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)}$$

, Here x_i indicates the sample element, $\mu(x_i)$ is the membership function, and n represents the number of elements in the sample.

For continuous membership function, x^* is defined as :

$$x^* = \frac{\int x\, \mu_A(\mathrm{x})\, \mathrm{dx}}{\int \mu_A(\mathrm{x})\, \mathrm{dx}}$$

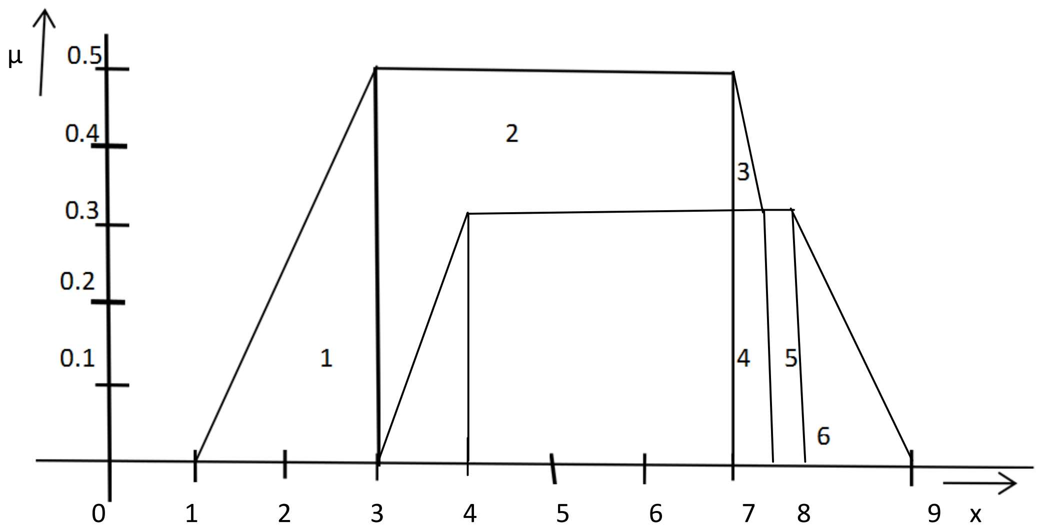


Figure 2 : Fuzzy sets C1 and C2

Example:

The defuzzified value x^* using COG is defined as:

$$x^* = \frac{\sum_{i=1}^{N} A_i \times \overline{x_i}}{\sum_{i=1}^{N} A_i}$$

, Here N indicates the number of sub-areas, A_i and $\overline{x_i}$ represents the area and centroid of area, respectively, of i^{th} sub-area.

In the aggregated fuzzy set as shown in figure 2. , the total area is divided into six sub-areas.
For COG method, we have to calculate the area and centroid of area of each sub-area.
These can be calculated as below.
The total area of the sub-area 1 is ½ * 2 * 0.5 = 0.5
The total area of the sub-area 2 is (7-3) * 0.5 = 4 * 0.5 = 2
The total area of the sub-area 3 is ½ * (7.5-7) * 0.2 = 0.5 * 0.5 *0.2 =.05
The total area of the sub-area 4 is 0.5* 0.3 = .15
The total area of the sub-area 5 is 0.5* 0.3 = .15
The total area of the sub-area 6 is ½ *1* 0.3 = .15
Now the centroid or center of gravity of these sub-areas can be calculated as

Centroid of sub-area1 will be (1+3+3)/3 = 7/3 =2.333
Centroid of sub-area2 will be (7+3)/2 = 10/2 = 5
Centroid of sub-area3 will be (7+7+7.5)/3 = 21.5/3 =7.166
Centroid of sub-area4 will be (7+7.5)/2 =14.5/2=7.25
Centroid of sub-are5 will be (7.5+8)/2 =15.5/2 = 7.75
Centroid of sub-area6 will be (8+8+9)/3 = 25/3 = 8.333
Now we can calculate $A_i \cdot \overline{x_i}$ and is shown in table 1.

Table 1

Sub-area number	Area(A_i)	Centroid of area($\overline{x_i}$)	$A_i \cdot \overline{x_i}$
1	0.5	2.333	1.1665
2	02	5	10
3	.05	7.166	0.3583
4	.15	7.25	1.0875
5	.15	7.75	1.1625
6	.15	8.333	1.2499

The defuzzified value x^* will be $\frac{\sum_{i=1}^{N} A_i \times \overline{x_i}}{\sum_{i=1}^{N} A_i}$

$$= \frac{(1.1665+10+0.3583+1.0875+1.1625+1.2499)}{(0.5+2+.05+.15+.15+.15)}$$

= (15.0247)/3 =5.008

$x^* = 5.008$

Center of Area / Bisector of Area Method (BOA)

This method calculates the position under the curve where the areas on both sides are equal. The BOA generates the action that partitions the area into two regions with the same area.

$\int_{\alpha}^{x^*} \mu_A(x)\ dx = \int_{x^*}^{\beta} \mu_A(x)dx$, where α = min {x| $x \in X$} and β = max {x| $x \in X$}

Weighted Average Method

This method is valid for fuzzy sets with symmetrical output membership functions and produces results very close to the COA method. This method is less computationally intensive. Each membership function is weighted by its maximum membership value. The defuzzified value is defined as :

$$x^* = \frac{\sum \mu(x).x}{\sum \mu(x)}$$

Here $\sum$ denotes the algebraic summation and x is the element with maximum membership function.

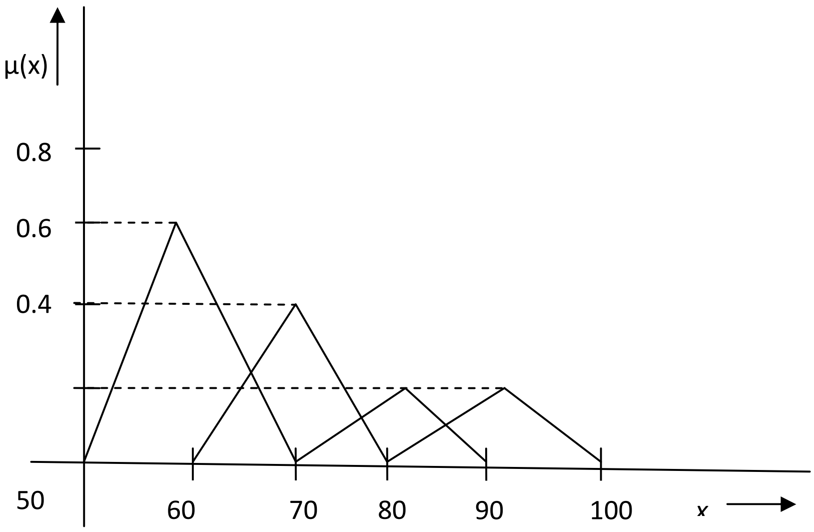


Figure 3: Fuzzy set A

Example:

Let A be a fuzzy set that tells about a student as shown in figure 3 and the elements with corresponding maximum membership values are also given.

A = {(P, 0.6), (F, 0.4),(G, 0.2),(VG, 0.2), (E, 0)}

Here, the linguistic variable P represents a Pass student, F stands for a Fair student, G represents a Good student, VG represents a Very Good student and E for an Excellent student.

Now the defuzzified value x^* for set A will be

$$x^* = \frac{(60*0.6+70*0.4+80*0.2+90*0.2+100*0)}{0.6+0.4+0.2+0.2+0}$$

= 98/1.4=70

The defuzzified value for the fuzzy set A with weighted average method represents a Fair student.

Maxima Methods

This method considers values with maximum membership. There are different maxima methods with different conflict resolution strategies for multiple maxima.

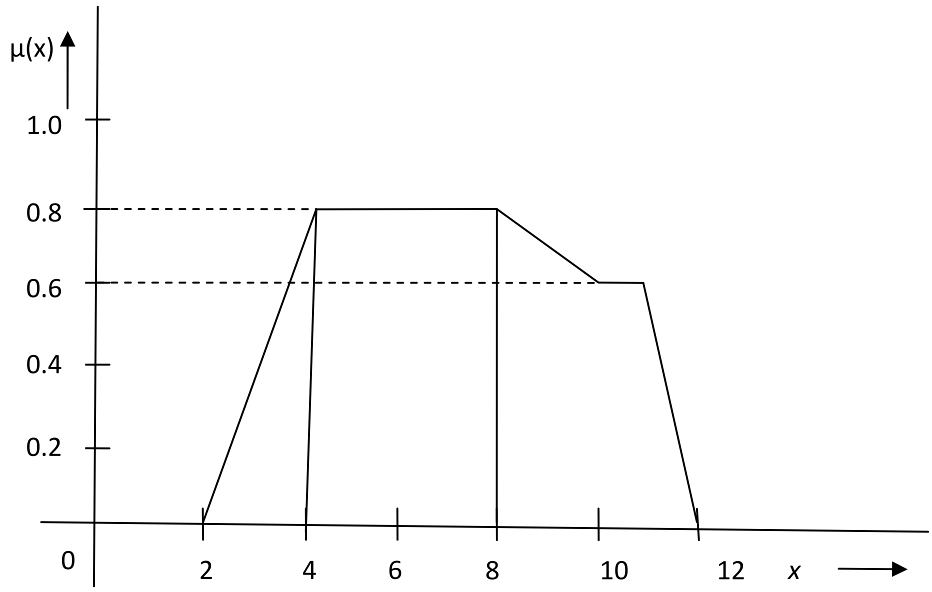
- o First of Maxima Method (FOM)
- o Last of Maxima Method (LOM)
- o Mean of Maxima Method (MOM)

First of Maxima Method (FOM)

This method determines the smallest value of the domain with maximum membership value.

Example:

The defuzzified value x^* of the given fuzzy set will be x^*=4.



Last of Maxima Method (LOM)

Determine the largest value of the domain with maximum membership value.

In the example given for FOM, the defuzzified value for LOM method will be x^*= 8

Mean of Maxima Method (MOM)

In this method, the defuzzified value is taken as the element with the highest membership values. When there are more than one element having maximum membership values, the mean value of the maxima is taken.

Let A be a fuzzy set with membership function $\mu_A(x)$ defined over $x \in X$, where X is a universe of discourse. The defuzzified value is let say x^* of a fuzzy set and is defined as,

$$x^* = \frac{\sum_{x_i \in M} x_i}{|M|},$$

Here, *M* = $\{x_i |$ $\mu_A(x_i)$ is equal to the height of the fuzzy set A} and |*M*| is the cardinality of the set *M.*

Example

In the example as shown in Fig. , x = 4, 6, 8 have maximum membership values and hence |M| = 3

According to MOM method, $x^* = \frac{\sum_{x_i \in M} x_i}{|M|}$

Now the defuzzified value x^* will be $x^* = \frac{4+6+8}{3} = \frac{18}{3} = 6.$

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