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Sabrinah Yonell C. Yap BSCS MAT 3101 Th 9:00-10:30 Limit of a Sequence

1. Theorem 1: If $\lim a_n = a$ and $\lim b_n = b$ then

(a) $\lim (a_n + b_n) = a + b$

Proof: Given $\epsilon > 0$, we may assume that there are numbers N' and N'' such that $|a_n - a| < \epsilon$ whenever $n > N'$ and $|b_n - b| < \epsilon$ whenever $n > N''$. Take N to be any number larger than both N' and N''. If we have $n > N$, then both the above inequalities hold, so that (by using Triangle inequality)

$$\begin{aligned} |a_n + b_n - (a+b)| &= |a_n - a + b_n - b| \\ &\le |a_n - a| + |b_n - b| \\ &< \epsilon + \epsilon = 2\epsilon \end{aligned}$$

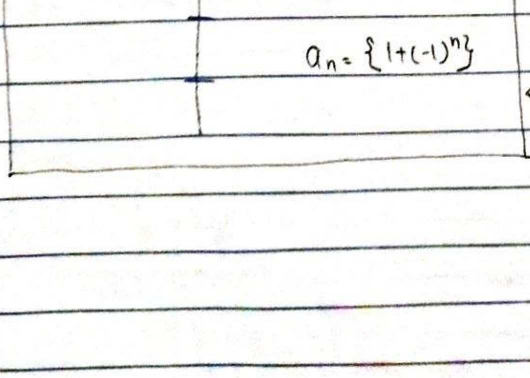
This says that, except for a finite number of terms at the start, all terms of the sequence $\{a_n + b_n\}$ lie in a neighborhood of radius 2ϵ, centered at the point $a+b$. Since ϵ can be any positive number, we have shown that this is true of every neighborhood of $a+b$, and thus $\lim (a_n + b_n) = a + b$. ■

2. Exercise 2: Show that the real sequence

$$a_n = \{1 + (-1)^n\}$$

has two limit points but no limit

$a_n = \{1 + (-1)^n\}$ has two limit points which are $[0, 2]$. To plot it



It is seen in the plane given by $a_n = \{1+(-1)^n\}$ has for its trace the infinite set.

The difference between "limit of a sequence" and "limit point of a sequence" is subtle; For limit point of a sequence to hold, a_n must lie in a neighborhood N only for some infinite sequence of values of n, which may in fact be widely separated with none consecutive. As a result, a sequence can be divergent and still have limit points.

② Exercise 4 : Give an example of a sequence with one limit but which diverges.

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd} \end{cases}$$

>> The sequence diverges if $\lim_{n\to\infty} a_n = \pm\infty$. Then such a sequence can't have another limit point that is not ∞.

>> The sequence is divergent and has 0 as a limit point.

③ Exercise 13 : Prove in general that

(a) $\lim_{n\to\infty} \sup (a_n + b_n) \leq \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n$

Proof: For all natural numbers k: $A_k = \sup\{a_n : n \geq k\}$, $B_k = \sup\{b_n : n \geq k\}$ where $A_k, B_k \in \mathbb{R} \cup \{+\infty\}$. By definition $\lim_{n\to\infty} \sup a_n = \lim_{k\to\infty} A_k$ and $\lim_{n\to\infty} \sup b_n = \lim_{k\to\infty} B_k$. Also, we consider $C_k = \sup\{(a_n + b_n) : n \geq k\}$, so,

$\lim_{k\to\infty} C_k = \lim_{n\to\infty} (a_n + b_n)$

For all $n \geq k$, we have $a_n + b_n \leq A_k + B_k$, because a_n is estimated by the supremum of all terms of (a_n) with $n \geq k$ and also for b_n. So,

$$C_k = \sup\{(a_n + b_n) : n \geq k\} \leq A_k + B_k$$

For all k, so we take the inf or lim on both sides as k tends to ∞, which preserves the inequality.

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