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Multi-Qubit Quantum Systems

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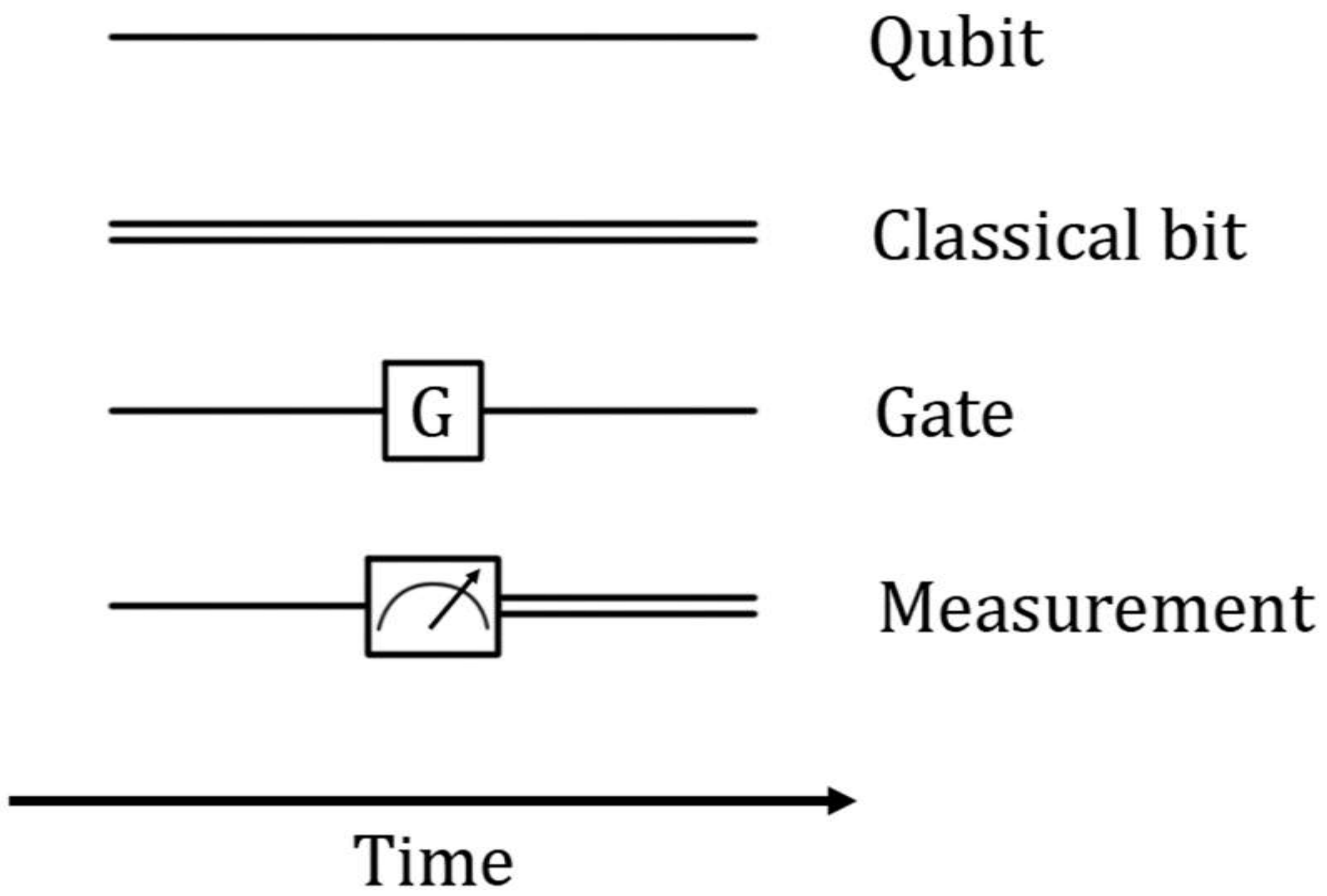

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Lecture outline

- Multi-qubit quantum states
- Multi-qubit quantum gates
- Notations for quantum algorithms
- Measuring multi-qubit systems

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Multi-qubit quantum states

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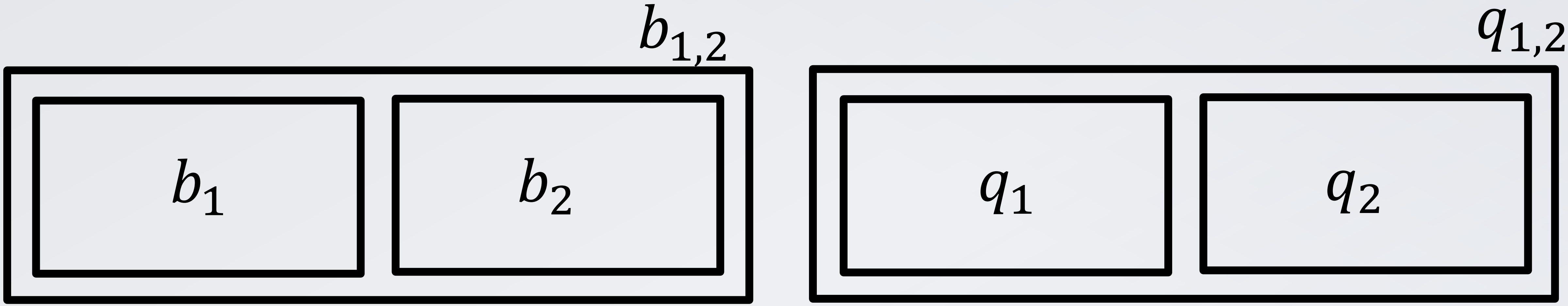
Scaling up: multiple qubits



$$b_{1,2} = 00 \text{ or } 01 \text{ or } 10 \text{ or } 11$$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \text{ or } \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{matrix}-\text{"00"}\\-\text{"01"}\\-\text{"10"}\\-\text{"11"}\end{matrix}$$

$$q_{1,2} = \begin{pmatrix}c_{00}\\c_{01}\\c_{10}\\c_{11}\end{pmatrix} \begin{matrix}-\text{"00"}\\-\text{"01"}\\-\text{"10"}\\-\text{"11"}\end{matrix}$$

$$|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1$$

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Dirac notation

"Ket" notation: $| \cdot \rangle$ denotes a column vector

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |c\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = c_0|0\rangle + c_1|1\rangle$$

More generally, let's consider an n-qubit quantum state:
$|\psi\rangle$ denotes a column vector of size 2^n, where $c_i \in \mathbb{C}$:

$$|\psi\rangle = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{2^n-2} \\ c_{2^n-1} \end{pmatrix} = \sum_{i=0}^{2^n-1} c_i|i\rangle$$

We will sometimes denote the i-th basis state in integer form for compactness, rather than in binary

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Dirac notation: examples

$$|\psi\rangle = \begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix} \begin{array}{l} -\text{"00"} \\ -\text{"01"} \\ -\text{"10"} \\ -\text{"11"} \end{array}$$

$$= c_{00}\,|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$
$$= c_0\,|0\rangle + c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$$

Ket notation:
Popular shorthand notation for sparse column vectors.

$$|\varphi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|110\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |6\rangle)$$

When working with integer basis states, double-check if the notation is big-endian or little-endian

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Tensor product of vectors

Denoted $|p\rangle \otimes |q\rangle$, $|p\rangle|q\rangle$, $|pq\rangle$, or $|p,q\rangle$

Input:

Vectors $|p\rangle, |q\rangle$ with dimensions m, n respectively: $|p\rangle = \begin{pmatrix} p_0 \\ \vdots \\ p_{m-1} \end{pmatrix}, |q\rangle = \begin{pmatrix} q_0 \\ \vdots \\ q_{n-1} \end{pmatrix}$

Output:

Vector $|p\rangle \otimes |q\rangle$ with dimension mn:

$$|p\rangle \otimes |q\rangle = \begin{pmatrix} p_0 q_0 \\ p_0 q_1 \\ \vdots \\ p_0 q_{n-1} \\ \vdots \\ p_{m-1} q_0 \\ \vdots \\ p_{m-1} q_{n-1} \end{pmatrix}$$

2 qubits cover a state space of dimension 4,
3 qubits - dimension 8,
n qubits - dimension 2^n

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Tensor product of vectors

Examples:

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle = |0\rangle$$

$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle = |2\rangle$$

big-endian

A (tensor) product state (a.k.a. "separable state"):

$$(a_0|0\rangle + a_1|1\rangle) \otimes (c_0|0\rangle + c_1|1\rangle) = a_0c_0|00\rangle + a_0c_1|01\rangle + a_1c_0|10\rangle + a_1c_1|11\rangle$$

In ket notation, just open the brackets to calculate the tensor product!

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Entanglement

Unentangled

Alice

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

Bob

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Entangled

Alice **Bob**

cannot write separately
(as a product state)

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

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Superposition vs entanglement

Superposition is relative to the basis, entanglement is absolute

- A state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is in superposition with respect to the basis $\{|0\rangle, |1\rangle\}$, but is a basis state with respect to the basis $\{|+\rangle, |-\rangle\}$
- Any state that is in superposition with respect to some basis is a basis state in some other basis
- A state that is entangled (cannot be represented as a tensor product of two states) is entangled in all bases

Both superposition and entanglement are properties of quantum states, not specific states

There are lots of states in superposition and lots of entangled states

States can be separable even if they're not "neatly" separable

State $\frac{1}{\sqrt{2}}(|010\rangle + |111\rangle)$ is separable: qubit 2 in state $|1\rangle$ and qubits 1 and 3 in state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

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Multi-qubit quantum gates

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Review: matrices and vectors

$$n \times m \text{ matrix } A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0,m-1} \\ a_{10} & a_{11} & \cdots & a_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,m-1} \end{pmatrix} \quad \text{Vector } x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix}$$

Multiplying a vector by a matrix

$$\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0,m-1} \\ a_{10} & a_{11} & \cdots & a_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,m-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix} = \begin{pmatrix} a_{00}x_0 + a_{01}x_1 + \ \ldots + a_{0,m-1}x_{m-1} \\ a_{10}x_0 + a_{11}x_1 + \ \ldots + a_{1,m-1}x_{m-1} \\ \vdots \\ a_{n-1,0}x_0 + \cdots + a_{n-1,m-1}x_{m-1} \end{pmatrix}$$

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Multi-qubit quantum gates

A **quantum gate** that acts on n qubits is a $2^n \times 2^n$ unitary matrix U:

$$U = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0,2^n-1} \\ a_{10} & a_{11} & \cdots & a_{1,2^n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2^n-1,0} & a_{2^n-1,1} & \cdots & a_{2^n-1,2^n-1} \end{pmatrix}$$

Reminder: unitary matrix means that $U^\dagger U = UU^\dagger = I$

n-qubit state is a vector of size 2^n

To apply a gate to n qubits, multiply the state vector by the matrix

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Dirac notation: bra

"Bra" notation: $\langle \cdot |$ denotes a row vector (adjoint of corresponding ket vector $| \cdot \rangle$)

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} = |0\rangle^\dagger$$
$$\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix} = |1\rangle^\dagger$$
$$\langle c| = \begin{pmatrix} c_0 & c_1 \end{pmatrix} = c_0\langle 0| + c_1\langle 1| = |c\rangle^\dagger$$

Inner product of vectors $|\varphi\rangle$ and $|\psi\rangle$: **bra-ket**

$$\langle \varphi|\psi\rangle = \begin{pmatrix} \varphi_0^* & \dots & \varphi_{n-1}^* \end{pmatrix} \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{n-1} \end{pmatrix} = \varphi_0^*\psi_0 + \dots + \varphi_{n-1}^*\psi_{n-1}$$

Bra-ket notation is used to denote orthogonal vectors (their bra-ket will be zero) and to calculate the probability of measurement outcomes (the probability of getting 0 when measuring a state $|\mathrm{c}\rangle$ is $\langle 0|\mathrm{c}\rangle$).

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Dirac notation: gates

Outer product of vectors $|\varphi\rangle$ and $|\psi\rangle$: **ket-bra**

$$|\psi\rangle\langle\varphi| = \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{n-1} \end{pmatrix} \begin{pmatrix} \varphi_0^* & \dots & \varphi_{n-1}^* \end{pmatrix} = \begin{pmatrix} \psi_0\varphi_0^* & \dots & \psi_0\varphi_{n-1}^* \\ \vdots & \ddots & \vdots \\ \psi_{n-1}\varphi_0^* & \dots & \psi_{n-1}\varphi_{n-1}^* \end{pmatrix}$$

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We can use ket-bra notation to write gates

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = a_{00}|0\rangle\langle 0| + a_{01}|0\rangle\langle 1| + a_{10}|1\rangle\langle 0| + a_{11}|1\rangle\langle 1|$$

Convenient for sparse gates (gates with a lot of zero elements in the matrix)

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1|$$

(read: $|0\rangle$ becomes $|1\rangle$, and $|1\rangle$ becomes $|0\rangle$)

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Controlled NOT gate

$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$CX|\psi\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_3 \\ c_2 \end{pmatrix}$$

In ket-bra notation:

$$CX = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$$

$$CX = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

"Controlled X" – the first qubit is "control", the second qubit is "target"

- Apply gate X on target qubit if control qubit is in $|1\rangle$ state
- Otherwise, do nothing

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Controlled NOT gate

Quantum equivalent of classical XOR gate

$$|a, b\rangle \rightarrow |a, a \oplus b\rangle$$

$$00 \rightarrow 00$$
$$01 \rightarrow 01$$
$$10 \rightarrow 11$$
$$11 \rightarrow 10$$

Can entangle qubits (does not always!)

$$CX\,|+, 0\rangle = CX\,\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$CX\,|1, +\rangle = CX\frac{1}{\sqrt{2}}(|10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|11\rangle + |10\rangle) = |1, +\rangle$$

"Phase kickback":

Apply the CX gate with control qubit in any superposition and target qubit in state $|-\rangle$.

Keeps qubits not entangled but propagates the -1 phase to control in state $|1\rangle$.

$$CX\,|+, -\rangle = CX\,\frac{1}{2}(|00\rangle + |10\rangle - |01\rangle - |11\rangle) = \frac{1}{2}(|00\rangle + |11\rangle - |01\rangle - |10\rangle) = |-, -\rangle$$

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Endianness in multi-qubit gates

$$\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1 \\ 11 & 0 & 0 & 1 & 0 \end{array}$$

First (most significant bit) as control

$$\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 1 & 0 \\ 11 & 0 & 1 & 0 & 0 \end{array}$$

Second (least significant bit) as control

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SWAP gate

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$SWAP|\psi\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_2 \\ c_1 \\ c_3 \end{pmatrix}$$

In ket-bra notation:

$$SWAP = |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|$$

Swaps the states of the first and the second qubits

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Toffoli gate (double-controlled NOT)

Toffoli gate (CCX) flips the state of the third qubit if and only if the first two qubits are both in the $|1\rangle$ state
(the quantum equivalent of AND gate)

$$|a, b, c\rangle \to |a, b, (a \wedge b) \oplus c\rangle$$

In ket-bra notation:

$$CCX = (I_2 - |11\rangle\langle 11|) \otimes I_1 + |11\rangle\langle 11| \otimes X$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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Controlled gates

If you have a quantum gate, you can always define its **controlled** variant using one or several qubits as controls

The gate is applied if all control qubits are in the $|1\rangle$ state, and nothing happens otherwise

$$C_n U_m = (I_n - |1 \dots 1\rangle\langle 1 \dots 1|) \otimes I_m + |1 \dots 1\rangle\langle 1 \dots 1| \otimes U_m$$

Example: controlled Z gate

$$CZ\,|+,+\rangle = CZ\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

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Gates, controlled on patterns

You can also define controlled variants of gates with patterns other than $|1 \dots 1\rangle$ as controls!
The gate is applied if the control qubits are in the given state $|ctrl\rangle$, and nothing happens otherwise

$$C_n U_m = (I_n - |ctrl\rangle\langle ctrl|) \otimes I_m + |ctrl\rangle\langle ctrl| \otimes U_m$$

Example: zero-controlled Z gate

$$C_0 Z\,|+,+\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$$

How to implement? (ControlledOnInt and ControlledOnBitstring functions in Q#)

- Apply X gates to each qubit in the control register that is in the $|0\rangle$ state in the $|ctrl\rangle$ state
- Apply the regular controlled gate
- Apply X gates to each qubit in the control register that is in the $|0\rangle$ state in the $|ctrl\rangle$ state again

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Tricks for applying gates to states in Dirac notation

Apply a gate to the relevant qubits of each basis state independently, then regroup terms if needed

Example: Hadamard gate acting on $\alpha|01\rangle + \beta|10\rangle$

$$H(\alpha|01\rangle + \beta|10\rangle) = \alpha(H|0\rangle)|1\rangle + \beta(H|1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(\alpha|01\rangle + \alpha|11\rangle + \beta|00\rangle - \beta|10\rangle)$$

If the gate acts on multiple qubits, don't spell out the math for it, think about effect

Example: CNOT gate with 3rd qubit as control and 1st qubit as target

$$CNOT_{3,1}\frac{1}{2}(|000\rangle + |001\rangle + |110\rangle + |101\rangle) =$$
$$= \frac{1}{2}(|000\rangle + |101\rangle + |110\rangle + |001\rangle)$$

Think of controlled gates as acting on basis states under classical conditions

Example: Controlled-on-zero Ry gate with 2nd qubit as control and 1st qubit as target

$$CRy_{2,1}\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + (Ry|1\rangle)|0\rangle)$$

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Universal sets of quantum gates

What set of gates allows to express an arbitrary gate?

- Any n-qubit gate can be represented using 2-qubit gates
- Any 2-qubit gate can be represented using CNOT and single-qubit gates (Krauss-Cirac decomposition)
- A limited set of single-qubit gates can **approximate** all single-qubit gates with arbitrary precision (unitary gate synthesis)

Example: {H, T, CNOT} is a universal set

Other sets exist; the choice of the best universal set depends on hardware architecture

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Notations for quantum algorithms

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Matrix notation

Quantum state on n qubits: a vector of 2^n numbers

$$\begin{pmatrix} c_0 \\ \vdots \\ c_{2^n-1} \end{pmatrix}$$

Quantum gate acting on n qubits: a $2^n \times 2^n$ matrix

$$\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0,2^n-1} \\ a_{10} & a_{11} & \cdots & a_{1,2^n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2^n-1,0} & a_{2^n-1,1} & \cdots & a_{2^n-1,2^n-1} \end{pmatrix}$$

Grows very inconvenient very quickly!

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Dirac notation

Quantum state on n qubits: a sum of up to 2^n ket vectors:

$$|\psi\rangle = \sum_{i=0}^{2^n-1} c_i |i\rangle$$

Quantum gate acting on n qubits a sum of up to 4^n ket-bra terms:

$$A = \sum_{i,j=0}^{2^n-1} a_{ij} |i\rangle\langle j|$$

Very convenient for sparse states and matrices and for orderly states and matrices (i.e., ones that allow to compress the sum into a formula)

Not very convenient for large dense notations

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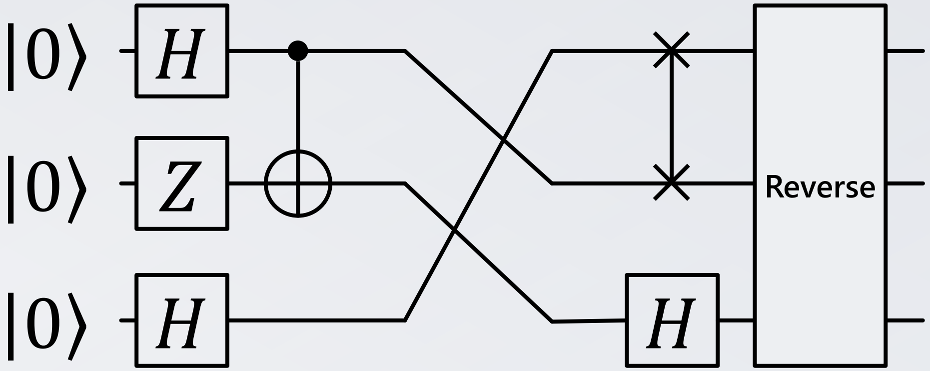
Circuit notation

Quantum state: a horizontal line (wire)

Quantum gate: an annotated box or symbol

Quantum measurement: an annotated box



Very popular in books and papers

Doesn’t represent quantum states (closer to real life)

Supports “procedures” but not loops or classical conditions

Grows quite unreadable very fast!

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Quantum code (Q#)

Quantum state: hidden state of Qubit objects

Quantum gate: an operation on Qubit arrays

```
operation WState_PowerOfTwo_Reference (qs : Qubit[]) : Unit is Adj {
    let N = Length(qs);
    if (N == 1) {
        // base of recursion: |1⟩
        X(qs[0]);
    } else {
        let K = N / 2;
        // create W state on the first K qubits
        WState_PowerOfTwo_Reference(qs[0 .. K - 1]);
        // the next K qubits are in |0...0⟩ state
        // allocate ancilla in |+⟩ state
        use anc = Qubit();
        H(anc);

        for i in 0 .. K - 1 {
            Controlled SWAP([anc], (qs[i], qs[i + K]));
        }
        for i in K .. N - 1 {
            CNOT(qs[i], anc);
        }
    }
}
```

Quantum programming languages don’t have native quantum state representation

But they can support loops, classical flow control, classical computations, and other capabilities

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Measuring multi-qubit systems

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Measuring multiple qubits

Consider a system of n qubits in state $|q\rangle$; the 2^n basis states of this system are $|b_0\rangle, \dots, |b_{2^n-1}\rangle$.

If we measure the system in the basis $\{|b_0\rangle, \dots, |b_{2^n-1}\rangle\}$,

- we'll get outcome b_i with probability $|\langle b_i|q\rangle|^2$
- and the system state will collapse to $|b_i\rangle$.

Example: measure $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$ in the computational basis.

- outcome 00 with probability $|\langle 00|q\rangle|^2 = \frac{1}{3}$
- outcome 01 with probability $|\langle 01|q\rangle|^2 = \frac{1}{3}$
- outcome 10 with probability $|\langle 10|q\rangle|^2 = \frac{1}{3}$

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Partial measurement

Consider a system of n qubits in the state $|q\rangle$;
measure the *first* qubit of this system in computational basis.

Same as measurements in Dirac notation: the basis of the measured part of the system is $\{|b_i\rangle\}$

- We'll get outcome b_i with probability $|\langle b_i|q\rangle|^2$,
  but this inner product will be a vector!
  Proper inner product is defined for vectors of the matching dimensions, so you need to split the basis vectors of $|q\rangle$ into tensor products of the first qubit and the rest of the system
  For example, $\langle 1|10\rangle = \langle 1|_1 |1\rangle_1 |0\rangle_2 = \langle 1|1\rangle_1 |0\rangle_2 = |0\rangle_2$
- The system state will collapse to $|b_i\rangle\langle b_i||q\rangle$, renormalized,
  but this expression will be a vector on n qubits with norm 1

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Example: partial measurement

Consider the state $|q\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$. Measure the first qubit of $|q\rangle$ in the $\{|0\rangle, |1\rangle\}$ basis.

$$\langle 0|q\rangle = \frac{1}{\sqrt{3}}(\langle 0|_1 |0\rangle_1 |0\rangle_2 + \langle 0|_1 |0\rangle_1 |1\rangle_2 + \langle 0|_1\ |1\rangle_1 |0\rangle_2) = \frac{1}{\sqrt{3}}(|0\rangle_2 + |1\rangle_2)$$

Not renormalized yet!

$$\langle 1|q\rangle = \frac{1}{\sqrt{3}}(\langle 1|00\rangle + \langle 1|01\rangle + \langle 1|10\rangle) = \frac{1}{\sqrt{3}}|0\rangle_2$$

Probability of measuring 0 is $|\langle 0|q\rangle|^2 = \frac{2}{3}$, and if we measure 0, the system collapses to

$$|0\rangle\langle 0|q\rangle = |0\rangle_1 \otimes \frac{1}{\sqrt{2}}(|0\rangle_2 + |1\rangle_2)$$

Probability of measuring 1 is $|\langle 1|q\rangle|^2 = \frac{1}{3}$, and if we measure 1, the system collapses to

$$|1\rangle\langle 1|q\rangle = |1\rangle_1 \otimes |0\rangle_2$$

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