# Pre-class Assigment - 16.

① given, $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

to find eigen Values,

we know, $\det(A - \lambda I) = 0$

$$\det\left(\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - (-1)(-1) = 0$$

$$4 - 2\lambda - 2\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - \lambda - 3\lambda + 3 = 0$$

$$\lambda(\lambda-1) - 3(\lambda-1) = 0$$

$$(\lambda-1)(\lambda-3) = 0$$

hence, the eigen values are $\lambda_1 = 1$ & $\lambda_2 = 3$.

to find the eigen vectors,

$$\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $\lambda_1 = 1$

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x - y = 0$ & $-x + y = 0$.

$\Rightarrow x = y$.

$\Rightarrow x = y$ is the ~~eigen~~ Solution, so the eigen vector corresponding to $\lambda_1 = 1$ is any non-zero Scalar multiple of $[1, 1]$

for $\lambda_2 = 3$,

$$\begin{pmatrix} 2-3 & -1 \\ -1 & 2-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-x - y = 0$ & $-x - y = 0$

$\Rightarrow x = -y$

$\rightarrow$ hence, $x = -y$ is Solution, the eigen vector corresponding to $\lambda_2 = 3$ is any non zero. Scalar multiple of $[1, -1]$.

We can ~~\(\not\)~~ use. eigen values & eigen vectors of 'A' to find. eigenvalues & eigen vectors of $A^2$.

here, $A^2$ has Same eigen vectors as A, but the eigen values are Squared,

eigen values for $A^2 = (\lambda_1^2, \lambda_2^2) = (1^2, 3^2) = (1, 9)$

~~eigen vectors~~ .

trace of $A$ is $Tr(A) = 2+2 = 4$.
Sum of eigen values $= \lambda_1 + \lambda_2 = 1+3 = 4$
Hence, the trace is indeed the sum of eigen values of $A$.

determinant of $A$ $\det(A) = (2)(2) - (-1)(-1)$
$= 4 - 1$
$= 3$
product of eigen values $= \lambda_1 \cdot \lambda_2$
$= 1 \times 3$
$= 3$
Hence, the determinant is indeed the product of the eigen values ~~of~~ for $A$.

(2) Let '$\lambda$' be the eigen value of '$A$' ~~and~~ & $A\vec{x} = \lambda\vec{x}$ where $\vec{x} \neq 0$. $\rightarrow$ (1)

Multiply '$A$' on both the sides of eqn (1)
$A \cdot (A\vec{x}) = A(\lambda\vec{x})$
$(A \cdot A)\vec{x} = \lambda(A \cdot \vec{x})$ (from associative property)
$A^2\vec{x} = \lambda(\lambda\vec{x})$ {from eqn (1)}
$A^2\vec{x} = \lambda^2\vec{x}$.
Hence, from the above eqn it is clear that $\lambda^2$ is eigen value of $A^2$ with same eigen vetor $\vec{x}$.

3) A) $\begin{pmatrix} 0 & 1 \\ -8 & 4 \end{pmatrix} = A$.

$|A - \lambda I| = 0$.

$\left| \begin{pmatrix} 0 & 1 \\ -8 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$

$\begin{vmatrix} -\lambda & 1 \\ -8 & 4-\lambda \end{vmatrix} = 0$

$-\lambda(4-\lambda) + 8 = 0$

$\lambda^2 - 4\lambda + 8 = 0$. ; $a = 1$, $b = -4$, $c = 8$.

$\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{4 \pm \sqrt{16-32}}{2} = \dfrac{4 \pm \sqrt{-16}}{2}$.

$\lambda_1 = 2+2i$ & $\lambda_2 = 2-2i$

for eigen vector; $[A - \lambda_1 I]\, V = 0$

$\begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(-2-2i) V_1 + V_2 = 0 \rightarrow (1)$

$-8 V_1 + (2-2i) V_2 = 0 \rightarrow (2)$.

from eqn (1) & (2), $-8V_1 + (2-2i)\cdot(2+2i)V_1 = 0$

$-8V_1 + (4 - (2i)^2)V_1 = 0$

$-8V_1 + 8V_1 = 0$

$V_1 = 1$

$\Rightarrow \underline{V_2 = 2+2i}$

for $\lambda_2$: $[A - \lambda_2 I][v_2] = 0$.

$$\begin{bmatrix} -2+2i & 1 \\ -8 & 2+2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore (-2+2i)v_1 + v_2 = 0 \quad \rightarrow ①$$

$$-8v_1 + (2+2i)v_2 = 0 \quad \rightarrow ②$$

from eq ① & ②,

$$-8v_1 + (2+2i)(2-2i)v_1 = 0.$$

$$-8v_1 + (4-(2i)^2)v_1 = 0$$

$$-8v_1 + 8v_1 = 0.$$

$$v_1 = 1$$

$$\Rightarrow v_2 = 2-2i.$$

$$\therefore X = [v_1 \ v_2] = \begin{bmatrix} 1 & 1 \\ 2+2i & 2-2i \end{bmatrix}$$

Now, diagnalization matrix $\Rightarrow X \Lambda X^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ 2+2i & 2-2i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} (1+i)/2 & -i/4 \\ (1-i)/2 & i/4 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 5 \\ -16+16i & 10-6i \end{bmatrix}\begin{bmatrix} (1+i)/2 & -i/4 \\ (1-i)/2 & i/4 \end{bmatrix}$$

$$= \begin{bmatrix} -1.5-6.5i & 3.25i \\ -14-8i & 5.5+6.5i \end{bmatrix}$$

Hence, the given matrix is diagonalizable with above diagonalization matrix.

B) $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$|B - \lambda I| = 0.$

$(1-\lambda)\left((1-\lambda)^2 - 1\right) = 0$

$(1-\lambda)(1-\lambda-1)(1-\lambda+1) = 0$

$(1-\lambda)(-\lambda)(2-\lambda) = 0$

$\lambda(\lambda^2 - 3\lambda + 2) = 0$

$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

for $\lambda_1 = 0$, $[B - \lambda_1 I]V = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y + z = 0.$

let $x = 1 \Rightarrow y = -1$ & $z = 0$

for $\lambda_2 = 1$, $[B - \lambda_2 I]V = 0$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y + z = 0, \quad x + z = 0, \quad x + y = 0.$

let $x = 0, \; y = 0, \; z = 0$

for $\lambda_3 = 2$; $[B - \lambda_3 I] V = 0$.

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x + y + z = 0$ & $x - y + z = 0$ & $x + y - z = 0$

eigenvectors $\gamma = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Since $\det(x)$ is zero, the matrix B is non-diagonalizable.

4) A) If all the eigen values $\lambda_i$ of A satisfy $|\lambda_i| < 1$ for $1 \leq i \leq n$ then as $k \to \infty$, ~~$A^k \to 0$~~. $A^k \to 0$.
This is because the eigen values of ~~$A^k$~~ ~~$A_k$~~ $A^k$ are $\lambda_i^k$. Since $|\lambda_i| < 1$ for all $i$, $\lambda_i^k \to 0$ as $k \to \infty$ for all eigen values. Therefore, all the eigenvalues of $A^k \to 0$ as $k \to \infty$.

B) If $\lambda_1 > 1$, then as $k \to \infty$, $A^k$ does not converge to '0'. This is because the largest eigenvalue $\lambda_1$ satisfies $\lambda_1^k \to \infty$ as $k \to \infty$ since $\lambda_1 > 1$. So the matrix $A^k$ is dominated by the eigen value $\lambda_1$ for large $k$. Since, $\lambda_1^k$ diverges, $A^k$ also diverges as $k \to \infty$.

(5) Given sequence definition: $F_1 = 0, F_2 = 1,$ & $F_{k+2} = F_{k+1} + F_k$ for $k \geq 1$.

A) So the fibonacci sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, ...

from the given sequence defn,

$$F_{k+1} = F_k + F_{k-1} \rightarrow (1)$$

$$F_k = F_k \rightarrow (2)$$

writting the above system of eqn in the form of matrix

$$\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = \begin{pmatrix} F_k + F_{k-1} \\ F_k + 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix}.$$

$$\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = A \cdot \begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix}.$$

hence, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

B) To find eigen values & eigen vectors of A.

we know $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)(-\lambda) - 1 = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0$$

to find the soln, we can use $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

here, $a = 1, b = -1, c = -1$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$\lambda_1 = \frac{1+\sqrt{5}}{2}$ & $\lambda_2 = \frac{1-\sqrt{5}}{2}$

let $\lambda_1 = K_1$ & $\lambda_2 = K_2$.

Hence, $\lambda_1$ & $\lambda_2$ are the eigen values.

to find the eigen vectors:

<u>for $\lambda_1$:</u> $(A-\lambda_1)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1-K_1 & 1 \\ 1 & -K_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x - xK_1 + y \\ x - K_1 y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x - xK_1 + y = 0 \rightarrow (3)$$

$$x - K_1 y = 0 \Rightarrow x = yK_1$$

Sub $x = yK_1$ in (3).

$$yK_1 - (yK_1)K_1 + y = 0$$

$$yK_1 - yK_1^2 + y = 0$$

$$y(K_1^2 - K_1 - 1) = 0$$

$$y = 1,\ x = 1(K_1) = K_1$$

$$\Rightarrow x = K_1,\ y = 1$$

eigen vectors for $\lambda_1 \Rightarrow \langle K_1, 1 \rangle$.

for $\lambda_2 \Rightarrow$ $[A - \lambda_2]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 1-K_2 & 1 \\ 1 & -K_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x - xk_2 + y = 0.$

$x - yK_2 = 0$

Solving the above 2 eqns we get.

$y = 1$ & $x = k_2$.

hence, eigen vectors for $\lambda_2 \Rightarrow \langle k_2, 1 \rangle$

finally, the eigen vectors are.

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$$

c) Let $A = PDP^{-1}$ ; $D \Rightarrow$ diagonal matrix

$P \Rightarrow$ eigen vector matrix.

to compute : $A^n = PD^nP^{-1}$

$$\text{Fibonacci}(n) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The above expression will give the $n^{th}$ fibonacci number without performing iterative addition explicitly.

This can also be computed as:

$$\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix} = A \begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix}.$$

$$= A\left( A . \begin{pmatrix} F_{k-1} \\ F_{k-2} \end{pmatrix} \right) = A^2 \begin{pmatrix} F_{k-1} \\ F_{k-2} \end{pmatrix} \cdots$$

$$A^n = \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$