① Integration by Substitution.

A) $\int x^2 \sqrt{1+x^3}\, dx$

let $u = 1+x^3$

$$\frac{du}{dx} = \frac{d}{dx}(1) + \frac{d}{dx}(x^3)$$

$$\frac{du}{dx} = 0 + 3x^2$$

$$du = 3x^2\, dx. \Rightarrow dx = \frac{du}{3x^2}$$

$$\Rightarrow \int x^2 \sqrt{1+x^3}\, dx = \int \cancel{x^2} \sqrt{u}\, \frac{du}{3\cancel{x^2}}$$

$$= \frac{1}{3}\int \sqrt{u}\, du$$

$$= \frac{1}{3}\int (u)^{1/2}\, du$$

$$= \frac{1}{3}\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + C$$

$$= \frac{1}{3}\left(\frac{u^{3/2}}{(3/2)}\right) + C$$

$$\therefore \int x^2 \sqrt{1+x^3}\, dx = \frac{2}{9}(1+x^3)^{3/2} + C$$

B) $\int_0^{\pi} \sin\theta \cos\theta \, d\theta$.

Let $u = \sin\theta$.

$\frac{du}{d\theta} = \cos\theta. \Rightarrow du = \cos\theta \, d\theta.$

$d\theta = \frac{du}{\cos\theta}.$

$\int_0^{\pi} \sin\theta \cos\theta \, d\theta = \int_0^{\pi} u \, du.$

$\int u \, du = \frac{u^2}{2} + C.$

Converting back to original Variable 'θ'.

$\frac{u^2}{2} + C = \frac{\sin^2\theta}{2} + C.$

evaluating the expression from 0 to π,

$\left[\frac{\sin^2\theta}{2}\right]_0^{\pi} + C = \left(\frac{\sin^2\pi}{2} - \frac{\sin^2 0}{2}\right) + C$

$= (0 - 0) + C$

$= 0$

$\therefore \int_0^{\pi} \sin\theta \cos\theta \, d\theta = 0$

(2) Integration by parts

A) $\int \ln x \, dx$.

Integration by parts can be solved using the below expression

$$\int u \, dv = uv - \int v \, du.$$

Let $u = \ln(x) \Rightarrow du = \frac{1}{x} dx.$

$dx = dv \Rightarrow x = v.$

$$\int \ln x \, dx = \cancel{\int \ln x \, (x)}$$

$$= \ln x \, (x) - \int x \cdot \left(\frac{1}{x}\right) dx$$

$$\therefore \int \ln x \, dx = x \ln(x) - x + C$$

B) $\int_0^{2\pi} t \sin t \, dt$.

Let $u = t \Rightarrow du = dt$

$dv = \sin t \, dt \Rightarrow v = -\cos t$.

using Integration by parts, $\int u \, dv = uv - \int v \, du$

$$\int_0^{2\pi} t \sin t \, dt = \cancel{t \sin t \, dt - \int (-\cos t) \, dt}$$

$$\cancel{= t \sin t \, dt +}$$

$$= -t \cos t - \int (-\cos t) \, dt$$

$$= -t \cos t + \int \cos t \, dt$$

$$= \left[t \cos t + \sin t \right]_0^{2\pi} + C$$

$$\int_0^{2\pi} t \sin t \, dt = \left[-2\pi \cos(2\pi) + \sin(2\pi)\right] - \left[0 + \sin(0)\right]$$

$\cos 2\pi = 1$, $\sin(2\pi) = 0$.

$\cos 0 = 1$, $\sin(0) = 0$.

$$\int_0^{2\pi} t \sin t \, dt = \left[-2\pi(1) + 0\right] - \left[0 + 0\right]$$

$$\Rightarrow \int_0^{2\pi} t \sin t \, dt = -2\pi.$$

c) $\int e^{-t} \cos t \, dt$.

let $u = e^{-t} \Rightarrow du = -e^{-t} dt$.

$dv = \cos t \, dt \Rightarrow v = \sin t$

we know $\int u\,dv = uv - \int v\,du$

$$\int e^{-t} \cos t \, dt = e^{-t} \sin t - \int \sin t(-e^{-t}) \, dt.$$

$$= e^{-t} \sin t + \underbrace{\int e^{-t} \sin t \, dt}_{\text{Again, Apply Integration by parts}}$$

here $u = e^{-t} \Rightarrow du = -e^{-t} dt$

$dV = \sin t \, dt \Rightarrow V = -\cos t$

$$= e^{-t} \sin t + \left[e^{-t}(-\cos t) - \int (-\cos t)(-e^{-t} dt)\right]$$

$$= e^{-t} \sin t + \left[-e^{-t} \cos t - \int e^{-t} \cos t \, dt\right]$$

$$= e^{-t} \sin t - e^{-t} \cos t - \int e^{-t} \cos t \, dt$$

$$\int e^{-t} \cos t \, dt = e^{-t} \sin t - e^{-t} \cos t - \int e^{-t} \cos t \, dt$$

$$2 \int e^{-t} \cos t \, dt = e^{-t} \sin t - e^{-t} \cos t + C$$

$$\therefore \int e^{-t} \cos t \, dt = \frac{e^{-t} \sin t - e^{-t} \cos t}{2} + C$$

③ Relationship between PDF & CDF

Probability density function (PDF):

It is denoted as $f(x)$, where x is a continuous random variable. A PDF must be a non-negative i.e., $f(x) \geq 0$, for all $x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Cummulative density function (CDF):

It is denoted as $F(x)$. CDF applies to both discrete and continuous random variables.
It can be mathematically written as.

$$F(x) = \int_{a}^{x} f(t)\,dt \quad \text{~~where~~, for } x \in \mathbb{R}$$

where, a is lower bound (often $-ve\ \infty$)
x is upper bound and
$f(t)$ is PDF.

We can use Fundamental theorem of Calculus to relate PDF & CDF. It states that, If $f(x)$ is a continuous function & its anti-derivative is $F(x)$ then $\int_{a}^{b} f(x)\,dx = F(b) - F(a)$

* It implies that the PDF of a continuous variable can be found by differentiating CDF i.e.,

$$f(x) = \frac{d}{dx}\left[F(x)\right]$$

* Similarly, CDF can be found by integrating PDF, i.e,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

(4) the expectation of a random Variable 'x' for a continuous Random variable is given by:

$$E[g(x)] = \int_x g(x)\, p(x)\, dx .$$

where $g: \mathbb{R} \rightarrow \mathbb{R}$.

for a discrete Random variable, the expectation of a random variable is given by:

$$E[g(x)] = \sum_{x \in X} x \cdot P(X=x)$$

where, $E[g(x)]$ or $E(X)$ is the expectation.

Eg. Expectation is a weighted average. It gives more weight to the outcomes with higher probability. Each value 'x' is multiplied by its probability $P(X=x)$ in the discrete case & 'x' is multiplied by its density function $f(x)$ in the continuous case. This reflects the fact that outcomes with greater probability contribute more to the overall average.

(5) $P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\mu = 0$ & $\sigma = 1$

Substitute in $P(x)$

$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-0}{1}\right)^2}$

$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

A) $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

square on both sides

$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2$

Converting it to polar co-ordinates.

$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy.$

Let $x = r\cos\theta$ & $y = r\sin\theta$.

Changing the limits to 0 to 2π & 0 to ∞, As it covers all values of $-\infty$ to ∞

$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-(r^2\sin^2\theta + r^2\cos^2\theta)} . r\, dr\, d\theta.$

$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} . r\, dr\, d\theta$ $\{\sin^2\theta + \cos^2\theta = 1\}$ $\rightarrow$ (1)

Integrate with ~~ref~~ respect to ~~r~~ θ.

$= \int_0^{2\pi} \cancel{\theta}\, r\, d\theta = [2\pi - 0]$

$= 2\pi \rightarrow$ eq (a)

integrate with respect to 'r'

$$\int_0^\infty e^{-r^2} \cdot r\,dr$$

$u = r^2 \Rightarrow du = 2r\,dr$

$$= \frac{1}{2}\int_0^\infty e^{-u} \cdot du.$$

$$= \frac{1}{2}\left[-e^{-u}\right]_0^\infty$$

$$= \frac{1}{2}\left[-e^{-\infty} - (-e^{0})\right]$$

$$= \frac{1}{2} \Rightarrow eq\ (3)$$

sub eq (2) & (3) in (1)

$$I^2 = (2\pi)\,\frac{1}{2} = \pi.$$

$$I = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \underline{\sqrt{\pi}}$$

B) $\int_{-a}^{a} x\,P(x)\,dx \ ; a > 0$

$\mu = 0$ & $\sigma = 1$

$p(x)$ is even function $\Rightarrow P(x) = P(-x)$.

Let 'x' is an odd function.

$\Rightarrow f(x) = -f(-x)$

$f(-x) = -f(x)$

$$\int_{-a}^{a} x\,p(x)\,dx = \int_{-a}^{0} x\,p(x)\,dx + \int_{0}^{a} x\,p(x)\,dx$$

$$= -\int_{0}^{-a} x\,p(x)\,dx + \int_{0}^{a} x\,p(x)\,dx$$

Consider $g(x) = x\,p(x)$

Since, $x\,p(x)$ is -ve odd function.

$$g(-x) = -g(x)$$

$$g(x) = -g(-x)$$

$$\int_{-a}^{a} g(x)\,dx = \int_{-0}^{0} g(x)\,dx + \int_{0}^{a} g(x)\,dx$$

$$= 0.$$

therefore, the area of curve from '0' to 'a' towards right & left are always equal. for a normal distribution curve with $\mu = 0$ & $\sigma = 1$.

(6) $p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

here, λ is the average rate at which users log into the website.

we know,

~~to find~~ $E(x) = \int x\, p(x)\,dx$ over its entire range.

Since the PDF is only defined for $x > 0$, we take the lower bound & upper bound as 0 to ∞.

$$E(x) = \int_0^\infty x . \lambda e^{-\lambda x} dx.$$

$$= \lambda \int_0^\infty x . e^{-\lambda x} dx.$$

from integration by parts we have,

$$\int u dv = uv - \int v du.$$

Let $u = x \Rightarrow du = dx$

$dv = e^{-\lambda x} dx \Rightarrow v = -e^{-\lambda x}$

$$E(x) = \lambda \left[x(-e^{-\lambda x}) \right]_0^\infty - \lambda \int_0^\infty (-e^{-\lambda x})\, dx$$

$$= \lambda \left[-x e^{-\lambda x} \right]_0^\infty + \lambda \int_0^\infty e^{-\lambda x} dx$$

$$= \lambda \left[(-\infty) e^{-\infty(\lambda)} - (0) e^0 \right] + \lambda \int_0^\infty e^{-\lambda x} dx$$

$$= \lambda [0 - 0] + \frac{-1}{\lambda} \left[e^{-\lambda x} \right]_0^\infty$$

$$= 0 - \frac{1}{\lambda} \left[e^{-\infty(\lambda)} - e^0 \right]$$

$$= -\frac{1}{\lambda}(-1)$$

$\int e^{-\lambda x} dx$

$u = -\lambda x \Rightarrow \frac{du}{dx} = -\lambda$

$du = -\lambda dx$

$dx = -\frac{1}{\lambda} du$

$\int e^u (-\frac{1}{\lambda}) du$

$= -\frac{1}{\lambda} e^{-\lambda x}.$

$$E(x) = \frac{1}{\lambda}$$

therefore, the average rate at which users log into the website, it means the expected time between successive login is $\frac{1}{\lambda}$.

⑦ $\sigma(t) = \dfrac{1}{1+e^{-t}}$

let $u = -t \Rightarrow du = -dt$

A) $\int_0^1 \sigma(t)\,dt$

If $t = 0,\ u = 0$

$t = 1,\ u = -1$

the . change the limits of integration accordingly

$$\int_0^1 \sigma(t)\,dt = \int_0^{-1} \sigma(t)\,dt$$

$$= \int_0^{-1} \sigma(-u)\,du$$

$$= -\int_0^{-1} \frac{1}{1+e^{u}}\,du.$$

$$= -\int_0\left[-\ln[1+e^{u}]_0^{-1}\right]$$

$$= -\left[-\ln[1+e^{-1}] - [-\ln[1+e^{0}]]\right].$$

$$= \ln[1+e^{-1}] - \ln[1+1]$$

$$\therefore \int_0^1 \sigma(t)\,dt = \ln[1+e^{-1}] - \ln(2)$$

B) The integral in part A), $\int_0^1 \sigma(t)\,dt$ represents the accumulated area under the sigmoid function $\sigma(t)$ from $t=0$ to $t=1$