Pre-class Assignment - 19

① given, $f(x) = ax^2 + bx + c$, where $a > 0$

let's assume, $f(x)$ attains global minimum at 'x_0'.

then it satisfies $f'(x_0) = 0$ & $f''(x_0) > 0$

$f'(x) = 2ax + b$

if $f'(x_0) = 0 \Rightarrow 2ax_0 + b = 0 \Rightarrow \boxed{x_0 = \frac{-b}{2a}}$

$f''(x) = 2a$

if $f''(x_0) > 0 \Rightarrow 2a > 0 \Rightarrow \boxed{a > 0}$ {given}

Hence, at $x = x_0$, the given function satisfies both conditions of attaining minimum value at x_0.

②

a) $f(x) = x^2 + 3$ on $[-1, 4]$

$f'(x) = 0$

$f'(x) = 2x \Rightarrow 2x = 0 \Rightarrow \boxed{x = 0}$

Hence, the points are $x = 0, -1, 4$

$f(0) = 0^2 + 3 = 3$ $\qquad$ $f(4) = 4^2 + 3 = 19$

$f(-1) = (-1)^2 + 3 = 4$

Hence, the global maximum is 19 at $x = 4$

the global minimum is 3 at $x = 0$

b) $g(x) = (x - x^2)^2$ on $[-1, 1]$

$g(x) = x^2 + (x^2)^2 - 2x(x^2) \Rightarrow g(x) = x^4 - 2x^3 + x^2$

$g'(x) = 4x^3 - 6x^2 + 2x$

when $g'(x) = 0 \Rightarrow 4x^3 - 6x^2 + 2x = 0$

$2x(2x^2 - 3x + 1) = 0$

$2x(x-1)(x - 1/2) = 0$

So, $x = 0$, $x = 1$ & $x = 1/2$

at $x = -1$, $g(-1) = ((-1) - (-1)^2)^2 = (-1-1)^2 = 4$

at $x = 0$, $g(0) = 0$

at $x = 1$, $g(1) = (1 - 1^2)^2 = 0$

at $x = 1/2$, $g(1/2) = \left(\frac{1}{2} - \frac{1^2}{2}\right)^2 = \left(\frac{1}{2} - \frac{1}{4}\right)^2 = \frac{1}{16}$

Hence, the global maximum is 4 at $x = -1$

the global minimum is 0 at $x = 0, x = 1$

c) $h(x) = 4\sin(x) - 3\cos(x)$ on $[0, 2\pi]$

$h'(x) = 4\cos x + 3\sin x$

when $h'(x) = 0 \Rightarrow 4\cos x + 3\sin x = 0$

$4\cos x = -3\sin x$

$\frac{\sin x}{\cos x} = \frac{-4}{3} \Rightarrow \tan x = \frac{-4}{3}$

$\Rightarrow x = \tan^{-1}\left(\frac{-4}{3}\right) \Rightarrow \boxed{x = -53.13^\circ}$

at $x = 0 \Rightarrow 4\sin(0) - 3\cos(0) = 4(0) - 3(1) = -3$

at $x = 2\pi \Rightarrow 4\sin(2\pi) - 3\cos(2\pi) = 4(0) - 3(1) = -3$

at $x = -53.13 \Rightarrow 4\sin(-53.13) - 3\cos(-53.13) = -5$

Hence, the global maximum is -3 at $x = 0, x = 2\pi$

the global minimum is -5 at $x = -53.13$

③ for local Maxima of $h'(t)$ would occur when the rate of change of depth of the beer is greatest in positive direction momentarily (fastest rate of pouring beer in the glass).

for local minima of $h'(t)$ would occur when the rate of change of depth of the beer is smallest in positive direction momentarily (moment when the rate of pouring the beer is constant).

This can also happen at time $t=1$ when the beer is stopped being poured momentarily.

④ given, $f(x,y,z) = x^2 z + y^3 z^2 - xyz$

A) rate of change of f at $(1,1,1)$

gradient of $f \Rightarrow \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\nabla f = (2xz - yz, 3y^2 z^2 - xz, x^2 + 2y^3 z - xy)$

gradient at $(1,1,1)$

$\nabla f = \langle 2-1, 3-1, 1+2-1 \rangle$

$\nabla f = \langle 1, 2, 2 \rangle$

The maximum rate of change occurs along the gradient

Hence, Max change at $(1,1,1)$ is

$$|\nabla f_{(1,1,1)}| = \sqrt{1^2+2^2+2^2} = \underline{\underline{3}}$$

B) The rate of change is minimal along -ve gradient vector.

Direction along -ve gradient $\Rightarrow \langle -1, -2, -2 \rangle$

$\Rightarrow$ Unit vector $= \dfrac{\langle -1, -2, -2 \rangle}{\sqrt{1^2+2^2+2^2}} = \dfrac{\langle -1, -2, -2 \rangle}{\sqrt{9}}$

unit vector $= \underline{\underline{\left\langle \dfrac{-1}{3}, \dfrac{-2}{3}, \dfrac{-2}{3} \right\rangle}}$

c) Rate of change of f at $(1,1,1)$ in the direction towards the point $(1,2,3)$

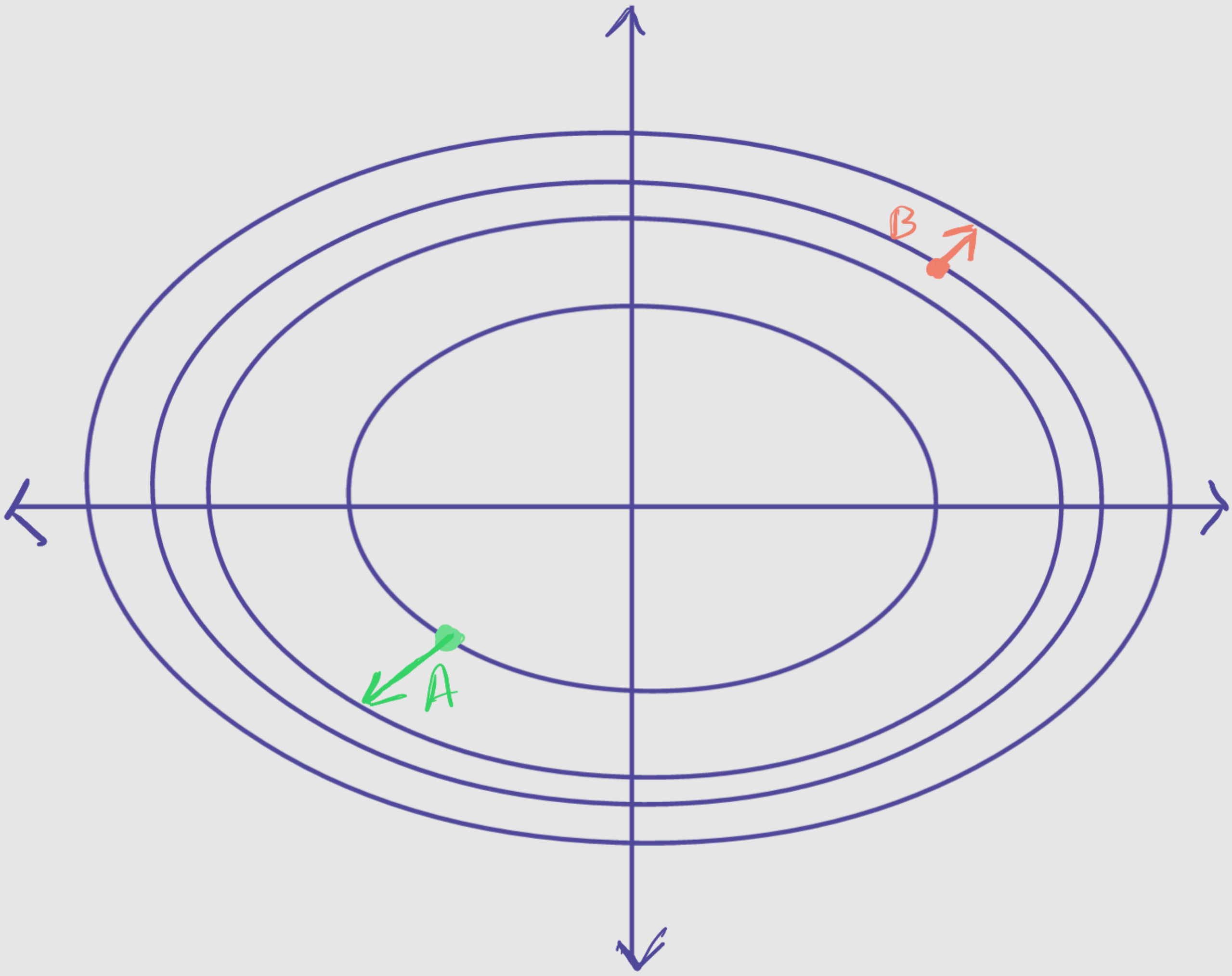
unit vector towards $(1,2,3) \Rightarrow \dfrac{(1,2,3)}{\sqrt{1^2+2^2+3^2}} = \left\langle \dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}} \right\rangle$

Rate of Change $= \nabla f_{(1,1,1)} \cdot$ unit vector towards $(1,2,3)$

$$= \langle 1, 2, 2 \rangle \cdot \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right\rangle = \frac{1}{\sqrt{14}} + \frac{4}{\sqrt{14}} + \frac{6}{\sqrt{14}}$$

$\Rightarrow$ rate of change $= \underline{\underline{\dfrac{11}{\sqrt{14}}}}$

(5)



Here is a sketch of a contour diagram, with 2 points A & B at which the gradient vectors point in opposite direction.

Also, the gradient vector at 'A' is longer than the gradient vector at B.

(6) to find critical point of $f(x,y) = x^3 - 3xy - y^3$

$\Rightarrow \frac{\partial f}{\partial x} = 3x^2 - 3y$

when $\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 = 3y \Rightarrow x = \pm\sqrt{y}$

$\Rightarrow \frac{\partial f}{\partial y} = -3x - 3y^2$

when $\frac{\partial f}{\partial y} = 0 \Rightarrow -3x = 3y^2 \Rightarrow x = -y^2$

for $x = +\sqrt{y}$,

$x = -y^2 \Rightarrow \sqrt{y} = -y^2$

only possible solution is $y = 0$.

for $x = -\sqrt{y}$,

$x = -y^2 \Rightarrow -\sqrt{y} = -y^2$

only possible solution is $y = 0$.

for $y = 0 \Rightarrow x = 0$

Hence, the critical points $(x,y) = (0,0)$.