Homework-9

① given, $f(t) = t\sin(t^3)$, at $t=0$

A) let $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^4(x) = \sin x$

Taylor series for $f(x)$ about point 'a' is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

evaluating the derivatives $x=0$, since we are finding series around $a=0$

$f(0) = \sin(0) = 0$

$f'(0) = \cos(0) = 1$

$f''(0) = -\sin(0) = 0$

$f'''(0) = -\cos(0) = -1$

$f^4(0) = \sin(0) = 0$

Hence,

$$\sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Now, for $f(t) = \sin(t^3)$

$$\sin t^3 = t^3 - \frac{(t^3)^3}{3!} + \frac{(t^3)^5}{5!} - \frac{(t^3)^7}{7!} + \dots$$

$$\sin t^3 = t^3 - \frac{t^9}{3!} + \frac{t^{15}}{5!} - \frac{t^{21}}{7!} + \dots$$

$$\boxed{t \sin t^3 = t^4 - \frac{t^{10}}{3!} + \frac{t^{16}}{5!} - \frac{t^{22}}{7!} + \dots}$$

B) $$\int_0^1 f(t)\,dt = \int_0^1 \left[t^4 - \frac{t^{10}}{3!} + \frac{t^{16}}{5!} - \dots\right] dt$$

for degree-16 Taylor polynomial,

$$\int_0^1 f(t)\,dt = \int_0^1 \left[t^4 - \frac{t^{10}}{3!} + \frac{t^{16}}{5!}\right] dt$$

$$= \left[\frac{t^5}{5} - \frac{1}{3!}\left(\frac{t^{11}}{11}\right) + \frac{1}{5!}\left(\frac{t^{17}}{17}\right)\right]_0^1$$

$$= \frac{1}{5}(1-0) - \frac{1}{66}(1-0) + \frac{1}{2040}(1-0)$$

$$= \frac{1}{5} - \frac{1}{66} + \frac{1}{2040}$$

Hence, $\int_0^1 f(t)\,dt = 0.1853$

c) To prove: $\lim_{t \to 0} \frac{f(t)}{t^2} = 0$

$$\lim_{t \to 0} \frac{f(t)}{t^2} = \lim_{t \to 0} \frac{1}{t^2}\left[t^4 - \frac{t^{10}}{3!} + \frac{t^{16}}{5!} - \cdots\right]$$

$$= \lim_{t \to 0}\left[t^2 - \frac{t^8}{3!} + \frac{t^{14}}{5!} - \cdots\right]$$

$$= 0 - \frac{0}{3!} + \frac{0}{5!} - \cdots$$

$$= 0 - 0 + 0 - \cdots$$

Therefore, $\lim_{t \to 0} \frac{f(t)}{t^2} = 0$

(2) Given, the calculator will show only 6 digits after the decimal point, we should have an accuracy of 6 digits.

The taylor polynomial for e^x is given by

$$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \cdots$$

$$P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320}$$

$$+ \frac{x^9}{362880} + \frac{x^{10}}{3628800}$$

A) Given, $-1 \leq x \leq 1$

→ when $x = -1$, $e^x = e^{-1} = 0.367879$

$P(-1) = 0.367897$ when we consider the sum until the 9th polynomial. ie,

$$P(-1) = 1 + (-1) + \frac{(-1)^2}{2} + \frac{(-1)^3}{6} + \frac{(-1)^4}{24} + \frac{(-1)^5}{120} + \frac{(-1)^6}{720} + \frac{(-1)^7}{5040}$$

$$+ \frac{(-1)^8}{40320} + \frac{(-1)^9}{362880}$$

$$P(-1) = 0.367897$$

→ when $x = 1$, $e^1 = 2.718281$

$P(1) = 2.718281$ when we consider the sum until the 9th polynomial i.e,

$$P(1) = 1 + 1 + \frac{1^2}{2} + \frac{1^3}{6} + \frac{1^4}{24} + \frac{1^5}{120} + \frac{1^6}{720} + \frac{1^7}{5040} + \frac{1^8}{40320} + \frac{1^9}{362880}$$

$$P(1) = 2.718281$$

Hence, 9th degree is lowest-degree polynomial that we could use.

B) for, $-2 \leq x \leq 2$

→ when $x = -2$, $e^x = e^{-2} = 0.135335$

$P(-2) = 0.135335$ when we consider the sum until the 14^{th} polynomial. i.e,

$$P(-2) = 1 + (-2) + \frac{(-2)^2}{2} + \frac{(-2)^3}{6} + \frac{(-2)^4}{24} + \frac{(-2)^5}{120} + \frac{(-2)^6}{720} + \frac{(-2)^7}{5040}$$

$$+ \frac{(-2)^8}{40320} + \frac{(-2)^9}{362880} + \frac{(-2)^{10}}{3628800} + \frac{(-2)^{11}}{39916800} + \frac{(-2)^{12}}{479001600} + \frac{(-2)^{13}}{13!} + \frac{(-2)^{14}}{14!}$$

$P(-2) = 0.135335$

→ when $x = 2$, $e^2 = 7.389056$

$P(2) = 2.718281$ when we consider the sum until the 14^{th} polynomial i.e,

$$P(1) = 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \frac{2^4}{24} + \frac{2^5}{120} + \frac{2^6}{720} + \frac{2^7}{5040} + \frac{2^8}{40320} + \frac{2^9}{362880}$$

$$+ \frac{2^{10}}{10!} + \frac{2^{11}}{11!} + \frac{2^{12}}{12!} + \frac{2^{13}}{13!} + \frac{2^{14}}{14!}$$

$P(2) = 2.718281$

Hence, 14^{th} degree is lowest-degree polynomial that we could use.

c) If we program the Calculator to be centered at a number that is close to N, there will be a bias towards it. There will be a weighted shift towards the new Centre. Also, Calculating other Taylor approximations for different functions would be very hard & would require reprogramming of the Calculator. Hence, it is not a practical Solution.

3 Given, $f(x,y) = (x^2+y)e^{y/2}$

A) find the partial derivatives of f,

$\Rightarrow f_x = 2x e^{y/2}$

$f_x = 0 \Rightarrow 2x e^{y/2} = 0 \Rightarrow \boxed{x=0}$

$\Rightarrow f_y = e^{y/2} + (x^2+y)e^{y/2}\left(\frac{1}{2}\right)$

$f_y = 0 \Rightarrow e^{y/2}\left(1 + \frac{(x^2+y)}{2}\right) = 0$

$1 + \frac{x^2+y}{2} = 0 \quad \{as\ x=0\}$

$1 + \frac{y}{2} = 0 \Rightarrow \frac{y}{2} = -1 \Rightarrow \boxed{y=-2}$

$\Rightarrow$ Hence, the critical point is $\underline{(0,-2)}$

B) 2^{nd} degree Taylor poly. of 'f' centred at $x=c$

$$f(x,y) = f(c) + f_x(c)(x-c_1) + f_y(c)(y-c_2)$$
$$+ \frac{f_{xx}(c)}{2}(x-c_1)^2 + f_{xy}(c)(x-c_1)(y-c_2)$$
$$+ \frac{f_{yy}(c)}{2}(y-c_2)^2$$

$$f_{xx} = 2e^{y/2} \; ; \; f_{yy} = \frac{e^{y/2}}{2} + \frac{e^{y/2}}{2} + \frac{(x^2+y)}{4} e^{y/2}$$

$$f_{xy} = x e^{y/2}$$

Now, $f(0,-2) = (-2)e^{-1} = -2e^{-1}$

$$f_x(0,-2) = 0$$
$$f_y(0,-2) = 0$$
$$f_{xx}(0,-2) = 2e^{-1}$$
$$f_{xy}(0,-2) = 0$$
$$f_{yy}(0,-2) = \frac{e^{-1}}{2}$$

$$P_2(x,y) = -2e^{-1} + 0 + 0 + \frac{2e^{-1}}{2}(x)^2 + 0 + \frac{e^{-1}}{4}(y+2)^2$$

$$\Rightarrow \boxed{P_2(x,y) = -2e^{-1} + e^{-1}x^2 + \frac{e^{-1}}{4}(y+2)^2}$$

c) Hessian matrix is given by.

$$H_f = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2e^{y/2} & xe^{y/2} \\ xe^{y/2} & e^{y/2}\left(\frac{x^2+y}{4}+1\right) \end{bmatrix}$$

At $(0,-2)$: $H_f(0,-2) = \begin{bmatrix} 2e^{-1} & 0 \\ 0 & e^{-1}/2 \end{bmatrix}$

$$D(H_f) = 2e^{-1}\left(\frac{e^{-1}}{2}\right) = e^{-2}$$

As, $D(H_f) > 0$ & $f_{xx} > 0$

Hence, 'C' is a point of local Minimum

D) $P_2(x,y) = -2e^{-1} + e^{-1}x^2 + \frac{e^{-1}}{4}(y+2)^2$

taking partial derivatives,

$P_{2x} = 2xe^{-1}$. $\quad P_{2y} = \frac{e^{-1}}{2}(y+2)$.

when $P_{2x} = 0 \Rightarrow 2xe^{-1} = 0 \Rightarrow \boxed{x=0}$

when $P_{2y} = 0 \Rightarrow \frac{e^{-1}}{2}(y+2) = 0 \Rightarrow y+2=0 \Rightarrow \boxed{y=-2}$

$\Rightarrow$ Critical pts $(0,-2) = C$

$\Rightarrow$ Hence, Yes, C is a critical point for P_2 Also.
The critical pt. of $f(x,y)$ & $P_2(x,y)$ are same which means taylor polynomial is behaving like the function $f(x,y)$ around the critical point 'C' accurately upto 2^{nd} degree.