Pre-class Assignment - 17

① Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. ~~Determine~~

$$A - \lambda I = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (2-\lambda)^2 - (1) = 0$$

$$4 + \lambda^2 - 4\lambda - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda^2 - 3\lambda - \lambda + 3 = 0$$

$$\cancel{\lambda^2 - 3\lambda + \lambda + 3 = 0} \qquad \lambda(\lambda - 3) - 1(\lambda - 3) = 0$$

$$\cancel{\lambda(\lambda - 3) - 1(\lambda - 3)} \qquad \lambda = 3,\ \lambda = 1$$

Since, the Eigen values are positive, the given Matrix A is positive semi-definite.

② Cholesky decomposition of $A = \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix}$

$$A = L \cdot L^T$$

$$L \cdot L^T = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ ab & b^2 + d^2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ ab & b^2 + d^2 \end{bmatrix}$$

$$a = 3,\ b = 2,\ d = 1$$

$$\Rightarrow L = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \ \& \ L^T = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

(3) A) $A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$

Eigen values of A is:

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -\lambda & 4 \\ 0 & -\lambda \end{bmatrix} \right| = 0$$

$$\lambda^2 = 0$$

$$\boxed{\lambda = 0}$$

Finding Singular values from $A^T \cdot A$.

$A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$ & $A^T = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$

$\cancel{As}$ $A^T \cdot A = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix}$

~~we kno~~ Now, $|A^T \cdot A - \lambda I| = 0$.

$$\left| \begin{pmatrix} 0 & 0 \\ 0 & 16 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 0 \\ 0 & 16-\lambda \end{vmatrix} = 0$$

$$-\lambda(16-\lambda) = 0 \Rightarrow \cancel{-16\lambda + \lambda^2 = 0}$$

$$\cancel{\lambda^2 = 16\lambda} \quad \lambda(16-\lambda) = 0$$

$$\lambda = 0 \text{ \& } \lambda^2 = 16 \Rightarrow \lambda = \pm 4$$

The Singular values ~~are~~ is $\sigma_1 = 4$

'V' matrix from eigenvector of $A^T A$.

$\lambda = 4$, $\begin{bmatrix} -\lambda & 0 \\ 0 & 16-\lambda \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 12 \end{bmatrix}$.

$A\vec{x} = \vec{0}$.

$\begin{bmatrix} -4 & 0 \\ 0 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} -4 & 0 & 0 \\ 0 & 12 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cc|c} -4 & 12 & 0 \\ 0 & 12 & 0 \end{array}\right]$

$-4x_1 + 12x_2 = 0$.

$12x_2 = 0 \Rightarrow \boxed{x_2 = 0.}$

$\boxed{x_1 = 0}$

$V = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $\lambda = 4$.

$V = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $\lambda = -4$.

$A \cdot A^T = \begin{bmatrix} 16 & 0 \\ 0 & 0 \end{bmatrix}$

$|A \cdot A^T - \lambda I| = 0$.

$\left| \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$

$\begin{vmatrix} 16-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$

$\lambda = \pm 4$.

for $\lambda = 4$, $\begin{bmatrix} 16-4 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & -4 \end{bmatrix}$

$\begin{bmatrix} 12 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$12x_1 = 0 \Rightarrow x_1 = 0.$

$-4x_2 = 0 \Rightarrow x_2 = 0$

$U = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $\lambda = 4$

$U = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ for $\lambda = -4$

checking $A = U \Sigma V^T$

$A = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$

$A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

hence, $\underline{\underline{A \neq U \Sigma V^T}}$

B) $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \left| \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$

$\begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4 = 0.$

$\lambda^2 = 4 \Rightarrow \lambda = \pm 2$

~~singular value is 2~~

Singular value from $A^T.A$

$$\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0 \\ 0 & 16+0 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$$

$$|A^TA - \lambda I| = 0.$$

$$\left| \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 16-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(16-\lambda) = 0$$

$$\lambda = 1, \lambda = 16.$$

Singular value of $A^T.A$ is $\sigma_1 = \sqrt{1} = 1$

$\sigma_2 = \sqrt{16} = 4$

'V' matrix is given by

$\lambda = 1$, $\begin{bmatrix} 1-\lambda & 0 \\ 0 & 16-\lambda \end{bmatrix}$ = 0

$$\begin{bmatrix} 0 & 0 \\ 0 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = V.$$

$$x_1 = 0, \quad x_2 = 0$$

for $\lambda = 4$, $x_1 = 0$, $x_2 = 0$.

$$V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

~~U~~

$$A \cdot A^T = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|A \cdot A^T - \lambda I| = 0$$

$$\left| \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 16-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0.$$

$$\lambda = 16, \quad \lambda = 1.$$

for $\lambda = 16$, $\begin{bmatrix} 16-16 & 0 \\ 0 & 1-16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 \\ 0 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0, \quad x_2 = 0.$$

$$U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

similarly, for $\lambda = 1$, $U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Checking $A = U \Sigma V^T$

$$A = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ hence, $\underline{A \neq U \Sigma V^T}$

4) SVD of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

$A \cdot A^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Eigen Values of $A \cdot A^T$

$|A \cdot A^T - \lambda I| = 0$.

$\left| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 1 = 0$

$2^2 + \lambda^2 - 4\lambda - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 3, \lambda = 1$

When $\lambda_1 = 1 \Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 3 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Singular values,

$$\sigma_1 = \sqrt{1} = 1 \quad \& \quad \sigma_2 = \sqrt{3}$$

for $\sigma_1 = 1$,

$$U_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

for $\sigma_2 = \sqrt{3}$,

$$U_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cancel{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}} \quad V_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

SVD $\Rightarrow$ $A = U \Sigma V^T$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix},$$

$$V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$