① To prove that the derivative of a linear function $f: R \to R$ is equal to its slope.

The limit definition of a derivative is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \rightarrow ① \text{ eqn}$$

Consider a linear function $f: R \to R$ given by $f(x) = mx + c$. $\rightarrow$ ② eqn

where, m is the slope of the line

Substitute eqn ② in eqn ①

$$f'(x) = \lim_{h \to 0} \frac{(m(x+h)+c) - (mx+c)}{h}.$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{mx} + mh + \cancel{c} - \cancel{mx} - \cancel{c}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{m\cancel{h}}{\cancel{h}}$$

$$f'(x) = \lim_{h \to 0} m$$

$$f'(x) = m$$

Since, m is a constant, it does not depend on h.

Therefore, the derivative of a linear function $f(x) = mx + c$ is equal to its slope 'm'.

X

(2) To find the tangent line to the curve $y = \frac{1}{x}$ at point $a = 1$ using limit definition of the derivative.

we know that the limit definition of the derivative is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substitute $f(x)$ with $y = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$\cancel{= \lim_{h \to 0} x - x - h}$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x}\left(\frac{1}{1+h} - \frac{1}{1}\right)}{h}$$

The derivative at $a = 1$,

$$f'(1) = \lim_{h \to 0} \frac{\frac{1}{1+h} - \frac{1}{1}}{h}$$

$$f'(1) = \lim_{h \to 0} \left(\frac{1}{1+h} - 1\right) \frac{1}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{1 - 1 - h}{h(1+h)}$$

$$f'(1) = \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(1+h)}$$

$$f'(1) = \lim_{h \to 0} \frac{-1}{1+h}$$

As 'h' tends to 0 $\Rightarrow f'(1) = \frac{-1}{1+0}$

$$\boxed{f'(1) = -1}$$

So, the slope of the tangent line to the curve $y = \frac{1}{x}$ at the point $a=1$ is -1.

To find the equation of the tangent line, we can use the point slope form : $y - y_1 = m(x - x_1)$

where, m is the slope

(x_1, y_1) is the point of tangency

$y = \frac{1}{x}$; at $x=1 \Rightarrow y = \frac{1}{1} \Rightarrow y = 1$

So, $(x_1, y_1) = (1, 1)$

Substituting $m, (x_1, y_1)$ in point slope form

we get, $y - 1 = -1(x - 1)$

$y - 1 = -x + \cancel{2}\,1$

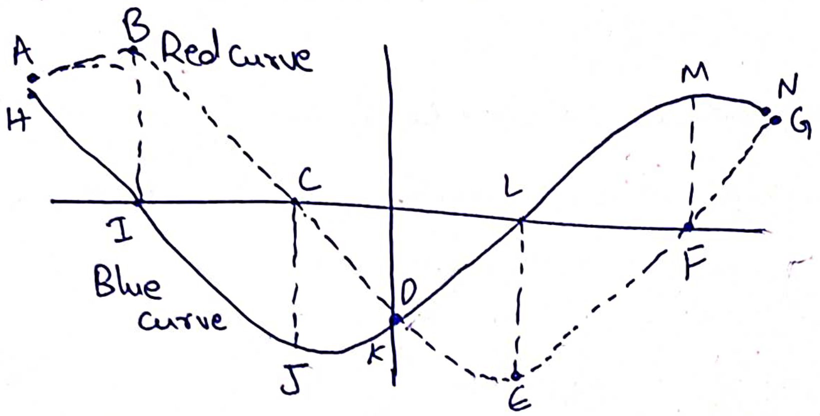
$x + y = 2$

$\boxed{y = -x + 2}$

therefore, the equation of the tangent to the curve $y = \frac{1}{x}$ at the point $(1,1)$ is given by $y = -x + 2$.

X

(3)



To ~~pr~~ find if Blue curve is a derivative of the red curve. & Is red curve derivative of Blue curve.

* Now, consider the Red curve, It is evident that slope is decreasing from point 'C' to 'D'. Now, The Blue curve to be its derivative, its y-coordinates must lie on the -ve y-axis, which is True for the Blue curve.

* In the Red curve, we can see that the slope increases from E to G. in the given graph. In the Blue curve we find the y-coordinates to lie on the positive side of the y-axis.

~~Hen~~ ⇒ From the Above analyzation it is clear that Blue curve is the derivative of Red curve.

* It is clear that when slope of Blue curve is decreasing from point 'I' to 'K', whereas in the Red curve the y-coordinates lies in the positive y-axis. when the slope of Blue curve increases from point '~~K~~' to 'M', ~~most~~ the coordinates of Red curve lies on the -ve y-axis

Hence, we can say that Blue curve is the derivative of Red curve But Red curve is not a derivative of Blue curve.

— X —

④ Given, Surge function $f(t) = \alpha t e^{-t}$ where $t \geq 0$ & $\alpha > 0$ is constant

we know that, the local maxima & minima occurs at $y' = 0$.

So, we have to differentiate the given Surge function $f(t) = \alpha t e^{-t}$

$$f'(t) = \frac{d}{dt}(\alpha t e^{-t})$$

Since, α is constant

$$f'(t) = \alpha \frac{d}{dt}(te^{-t})$$

Applying Chain rule, we get

$$f'(t) = \alpha\left(e^{-t}\frac{d}{dt}(t) + t\frac{d}{dt}(e^{-t})\right)$$

$$= \alpha\left(e^{-t} + t(-e^{-t})\right)$$

$$f'(t) = \alpha(e^{-t} - te^{-t})$$

Now, $\alpha e^{-t}(1-t) = 0$

$\alpha e^{-t} = 0$ And $1 - t = 0$

$e^{-t} = 0$ $\boxed{t = 1}$

$\boxed{t = \infty}$

$\Rightarrow$ At $t = \infty \Rightarrow \alpha t e^{-t}$

$= \alpha(\infty)(e^{-\infty}) = \underline{\underline{0}}$

So, $t = \infty$ is the point of Minima & $f(t) = 0$

$\Rightarrow$ At $t = 1 \Rightarrow f(t) = \alpha(1)e^{-1}$

$= \alpha \times 0.367$

So, $t = 1$ is the point of Maxima & $f(t) = \alpha(0.367)$

A) * As $t \geq 0$ & $t = 1$, t lies in ~~btw~~ between ~~to~~ 0 and 1

$0 \leq t \leq 1$ the function $f(t) = \alpha t e^{-t}$ surges.

* The value of 't' increases from 0 to 0.367α

* ~~As t=1 &~~ As 't' Ranges from '1' and '∞', the function

$f(t) = \alpha t e^{-t}$ decays. $(1 \leq t \leq \infty)$

The value of 't' decreases from $\alpha(0.367)$ to 0.

B) Since, the Maximum value occurs at $t = 1$, the function

$f(t) = \alpha t e^{-t}$ does not surge or decay.

The Max. value of Surge function is

$$f(t) = \alpha t e^{-t}$$

$$f(1) = \alpha (1)(e^{-t})$$

$$f(1) = \alpha e^{-1}$$

$$f(1) = \alpha(0.367)$$

(5) Given, exponential function $f(x) = e^x$

$f(x) = f'(x)$ & $f(x) = 0$ is a constant function

$g(x) = g'(x)$

the derivative of quotient $\frac{f(x)}{g(x)}$ is given by

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \quad \{\text{from chain rule}\}$$

$$= \frac{g(x)\,f(x) - f(x)\,g(x)}{(g(x))^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = 0$$

So, the derivative of quotient $\frac{f(x)}{g(x)}$ is always zero, which is a constant. if $g(x) \neq 0$.

So, for the function to be a derivative of itself has to be a constant multiple of '0' or 'e^x'.

$\Rightarrow f(x) = e^x$ (or) $f(x) = 0$ (or) $f(x) = Ce^x$

therefore, '0', 'e^x' & Ce^x are the only functions that can be a derivative of itself.

———— X ————

(6) given, Data Set $S = \{(x_1, y_1), (x_2, y_2), \dots, (x_N, y_N)\}$

where $0 < x_{n+1} - x_n < \Delta x$ for small increment $\Delta x > 0$

when $1 \leq n < N$

A) Finding the first derivative,

$$f'(x) = \frac{y_2 - y_1}{x_2 - x_1}$$ {As derivative of a function is equal to the slope}

~~here, $x_2 - x_1 = \Delta x$~~

let, y_n be a point & y_{n+1} is the point next to y_n

x_n be a point & x_{n+1} is the point next to x_n

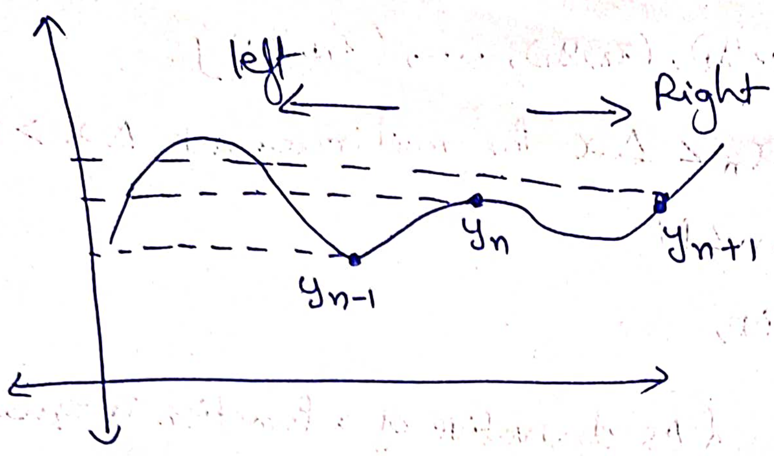
So, $x_2 - x_1 = x_{n+1} - x_n = \Delta x$

$$\therefore f'(x) = \frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \frac{y_{n+1} - y_n}{\Delta x}$$

$$f'(x) = \frac{y_{n+1} - y_n}{\Delta x}$$

hence, $f'(x) = \dfrac{\Delta y}{\Delta x}$ is the First Derivative

B) Lets consider point y_n to be the current point and Assume the point is discontinuous on the graph. The ~~left~~ point towards left of y_n is y_{n-1} & the point towards right side of y_n is y_{n+1}.



The Above method uses graph to determine the derivative at x_n. the Data points in 'S' are plotted on the graph. then the slope of the line joining the points (x_{n+1}, y_{n+1}) & (x_n, y_n) gives the value of the derivative at x_n.

the left derivative is given by $\Rightarrow f'(x) = \dfrac{y_n - y_{n-1}}{\Delta x}$

~~For left~~
the right derivative is given by $\Rightarrow f'(x) = \dfrac{y_{n+1} - y_n}{\Delta x}$.

c) To find the second derivative, we will be using the first derivative point.

$$f'(x) = \frac{y_{n+1} - y_n}{\Delta x}$$

$$\text{So, } f''(x) = \frac{f(x_{n+1}) - f(x_n)}{\Delta x} = \frac{\dfrac{y_{n+1+1} - y_{n+1}}{\Delta x} - \dfrac{(y_{n+1} - y_n)}{\Delta x}}{\Delta x}$$

$$\text{So, } f''(x) = \frac{y_{n+2} - 2y_{n+1} + y_n}{\Delta x^2}$$

$\Rightarrow$ The Above equation is used to estimate the value of Second derivative at x_n by using the data set 'S'.

—— X ——