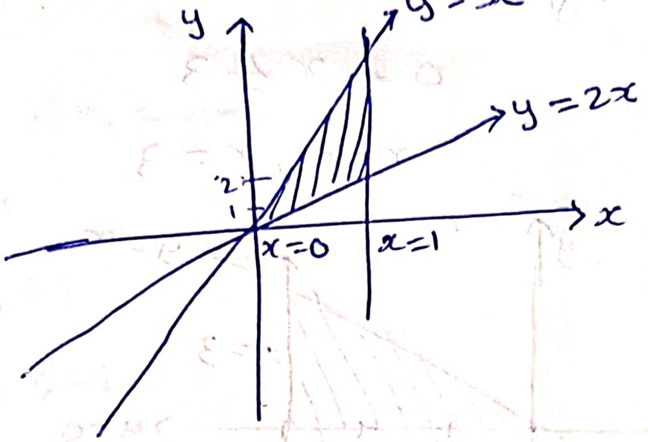
① A) $\int_0^1 \int_{2x}^{3x} (x+y^2)\, dy\, dx.$

Sketching the region:



$2x \leq y \leq 3x$ & $0 \leq x \leq 1$

$y = 2x$ & $y = 3x$ $x = 0$ & $x = 1$

evaluating the integral:

$$\int_0^1 \int_{2x}^{3x} (x+y^2)\, dy\, dx = \int_0^1 \left[xy + \frac{y^3}{3} \right]_{2x}^{3x} dx$$

$$= \int_0^1 \left(\left[x(3x) + \frac{(3x)^3}{3} \right] - \left[x(2x) + \frac{(2x)^3}{3} \right] \right) dx$$

$$= \int_0^1 \left(3x^2 + \frac{27x^3}{3} - 2x^2 - \frac{8x^3}{3} \right) dx$$

$$= \int_0^1 \left(x^2 + \frac{19x^3}{3} \right) dx$$

$$= \left[\frac{x^3}{3} + \frac{19x^4}{12} \right]_0^1 = \frac{1}{3} + \frac{19}{12} - 0$$

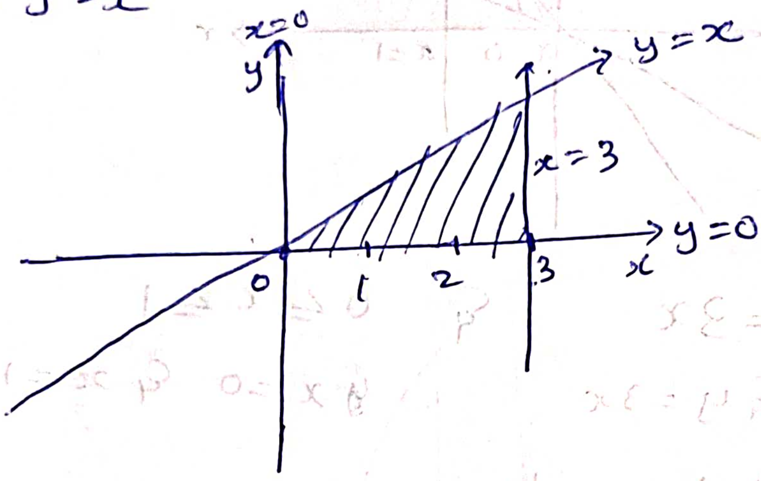
$$\Rightarrow \int_0^1 \int_{2x}^{3x} (x+y^2)\, dy\, dx = \frac{23}{12}$$

B) $\displaystyle\int_0^3 \int_0^x e^{x^2}\, dy\, dx$

Sketching the region:

$0 \leq y \leq x$ $\quad$ $0 \leq x \leq 3$

$y=0,\ y=x$ $\quad$ $x=0,\ x=3$



Evaluating the integral:

$$\int_0^3 \int_0^x e^{x^2}\, dy\, dx = \int_0^3 \left[y e^{x^2}\right]_0^x dx$$

$$= \int_0^3 (x e^{x^2} - 0)\, dx$$

let $x^2 = u$ & $du = 2x\, dx$

$x\, dx = \frac{du}{2}$

$$= \int_0^3 \frac{1}{2} e^u\, du = \frac{1}{2} \int_0^3 e^u\, du$$

$$= \frac{1}{2} \left[e^u\right]_0^3 = \frac{1}{2} \left[e^{x^2}\right]_0^3$$

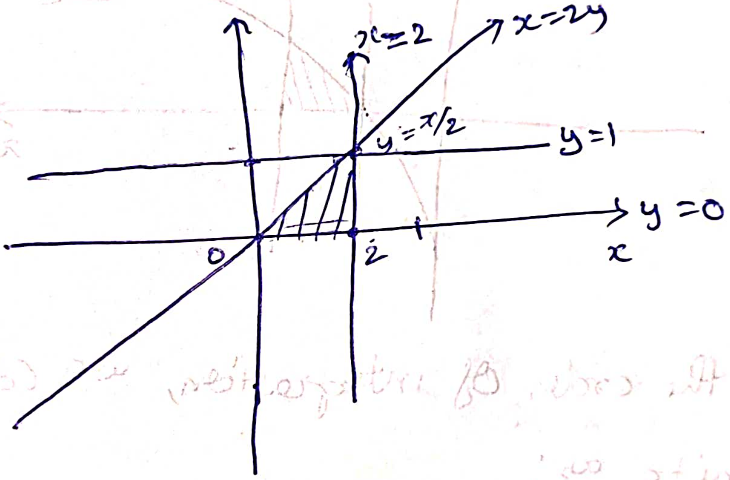
$$= \frac{1}{2}\left(e^{3^2} - e^0\right)$$

$$\therefore \int_0^3 \int_0^x e^{x^2}\,dy\,dx = \frac{1}{2}(e^9 - 1)$$

② A) $\int_0^1 \int_{2y}^2 f(x,y)\,dx\,dy$.

$0 \le y \le 1$ $\quad$ $2y \le x \le 2$

$y = 0,\ y = 1$ $\quad$ $x = 2y,\ x = 2$



Since, $x = 2y \Rightarrow y = \frac{x}{2}$

~~So~~, the reversing the order of integration is:

$0 \le x \le 2.$ & $0 \le y \le x/2$

Hence, the integral can be rewritten as

$$\int_0^2 \int_0^{x/2} f(x,y)\,dy\,dx$$

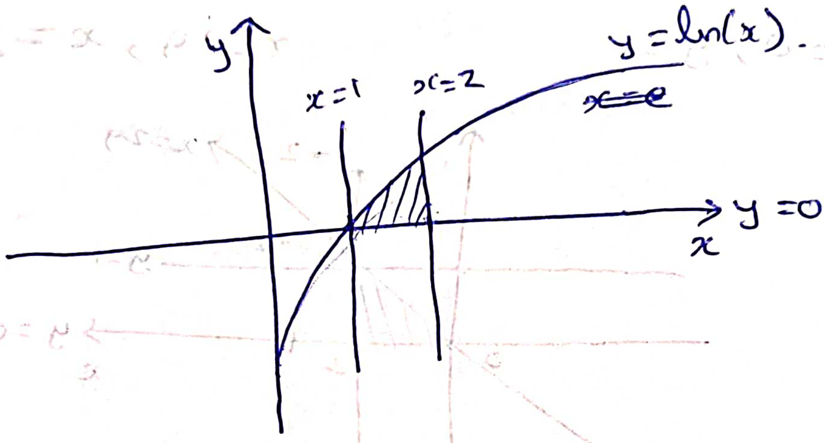
B) $\displaystyle\int_1^2 \int_0^{\ln(x)} g(x,y)\,dy\,dx$

$0 \le y \le \ln(x)$ $\qquad 1 \le x \le 2$

$y=0,\ y=\ln(x)$ $\qquad x=1,\ x=2$

$x=e^y$

Sketching the region:



Reversing the order of integration, we can consider the limits as:

$e^y \le x \le 2$ & $0 \le y \le \ln(2)$.

Hence, the integral can be written as.

$$\int_0^{\ln(2)} \int_{e^y}^{2} g(x,y)\,dx\,dy.$$

(3) given, $y=0$ & $y=x-x^2$.

equate the 'y' region given.

$$0 = x - x^2$$

$$\cancel{x} \quad x^2 - x = 0$$

$$x(x-1) = 0$$

$$x=0 \quad \text{or} \quad x=1$$

The limit of Integration for 'x' over the region 'R' is $[0,1]$

to find average distance b/w a point in R & the x-axis is given by: $= \dfrac{\int_0^1 y\,dx}{A}$ $\left\{A = \int_0^1 dx\right\}$

here, $A \to$ area of region.

the distance from a point (x,y) to the x-axis is 'y'.

$$= \int_0^1 y\,dx. = \int_0^1 (x-x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} - 0.$$

$$= \frac{1}{6}$$

Hence the averag dist $= \dfrac{\frac{1}{6}}{\int_0^1 dx} = \dfrac{\frac{1}{6}}{[x]_0^1} = \dfrac{\frac{1}{6}}{1}$

$$= 1/6$$

④

A) $\int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x \, dy \, dx$

Let $y = \sqrt{1-x^2} \Rightarrow y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$

The above equation is an equation of circle with radius 1. So, range of x is

$-1 \leq x \leq 0.$

for polar co-ordinates,

let $x = r\cos\theta$ & $y = r\sin\theta$.

for the limits $-1 \leq x \leq 0$, the curve is a semi-circle with radius 1 centered at origin.

$\therefore$ the limits are, $0 \leq r \leq 1$ & $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\int_{-\pi/2}^{\pi/2} \int_{0}^{1} r\cos\theta \cdot r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_{0}^{1} r^2 \cos\theta \, dr \cdot d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3}\right]_0^1 \cos\theta \, d\theta$$

$$= \frac{1}{27} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{1}{27} \left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{27} [1 - (-1)]$$

$$= \frac{2}{27}$$

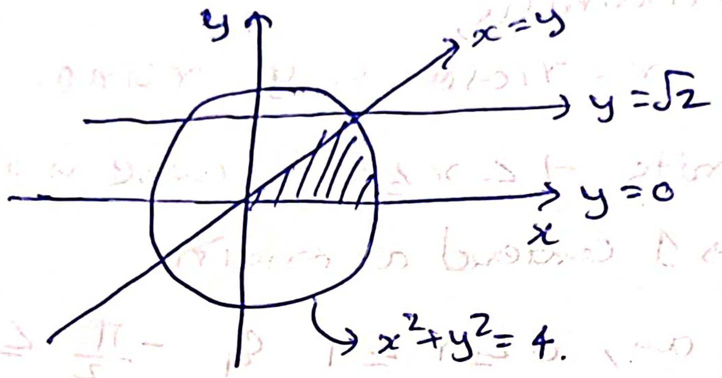
(B) $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} xy \, dx \, dy.$

Let $x = \sqrt{4-y^2}$

$x^2 = 4 - y^2 \Rightarrow x^2 + y^2 = 4$

It is eqn. of circle with radius 2. ~~at Centr~~

with line passing the origin $(x=y)$



Consider, $x = r\cos\theta$ & $y = r\sin\theta$

$xy = r^2(\cos\theta \cdot \sin\theta)$

Limits of $r \rightarrow 0 \leq r \leq 2$ & limit of θ is $0 \leq \theta \leq \pi/4$

Solving the integral,

$$\int_0^{\pi/4} \int_0^2 r^2 \cos\theta \sin\theta \, dr \, d\theta = \int_0^{\pi/4} \left(\frac{r^3}{3}\right)_0^2 \cos\theta \sin\theta \, d\theta.$$

$$= \frac{8}{3} \int_0^{\pi/4} \cos\theta \sin\theta \, d\theta$$

{As, $\sin 2\theta = 2\sin\theta\cos\theta$ }

$$= \frac{8}{3} \times \frac{1}{2} \int_0^{\pi/4} \sin 2\theta \, d\theta.$$

$$= \frac{4}{3}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/4}$$

$$= \frac{2}{3}(1)$$

$$= \frac{2}{3}$$

hence, $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} xy\,dx\,dy = \frac{2}{3}$

(5) given, diameter of drill = 1 cm $\Rightarrow r = 0.5 cm$.

diameter of sphere = 2 cm. $\Rightarrow r = 1$ cm

Vol. of sphere is given $= \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi (1)^3$

$= \frac{4\pi}{3}$

Vol. of drilled area $= \pi r^2 h$ {as it is a cylinder}

$= \pi \cdot \left(\frac{1}{2}\right)^2 \times 2$

$= \pi \times \frac{1}{\cancel{4}_2} \times \cancel{2}$

$= \frac{\pi}{2}$

Volume of sphere that is left =

= Vol. of sphere − Vol. of drilled area

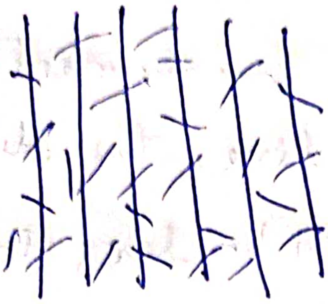
$= \frac{4}{3}\pi - \frac{\pi}{2}$

$= \frac{8\pi - 3\pi}{6}$

$= \frac{5\pi}{6}$

$= \underline{\underline{2.616 \text{ cm}^3}}$

⑤



given that, length of the needle is 1 inch.
the parallel lines are spaced 1 inch apart.

Let's consider ~~two~~ any two parallel lines where the needle is bounded with.

here, $\theta \rightarrow$ angle made by needle with respect to x-axis.
$l \rightarrow$ length of needle = 1 inch.
$x \rightarrow$ distance from midpoint of needle to the nearest parallel line.

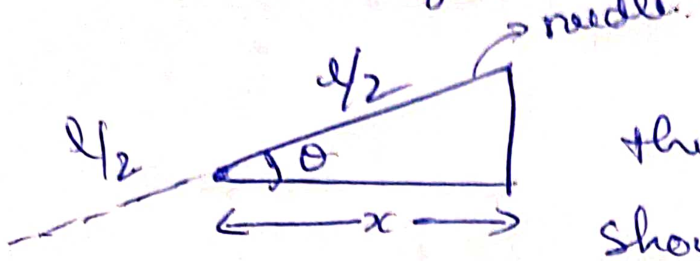
So, ~~Since x is the~~

range of x is $0 \leq x \leq 1/2$

range of θ is $0 \leq \theta \leq \pi/2$.

As per our consideration, the 'θ' is the I^{st} quadrant ~~so~~ $(0 \text{ to } \pi/2)$ is the angle 'θ' way more we can take it as multiples of 0 to $\pi/2$.

Consider the figure.



for the needle to go across the line, the distance 'x' should be less than length multiplied by cosine of the angle.

~~so, l cos θ~~.

Hence, probability of needle that it will cross one of the parallel lines after it lands is given by:

$$P(\text{needle crossing parallel lines}) = \frac{\int_0^{\pi/2} \cos\theta \, d\theta}{\pi/2}$$

here, $\cos\theta \, d\theta$ is where the needle crosses the line.

$$= \left(\frac{\sin\frac{\pi}{2}}{\pi/2} - \frac{\sin 0}{\pi/2}\right)$$

$$= \frac{1}{\pi/2}$$

$$= \frac{2}{\pi}$$