① given curve $x^2y+xy^2=2e^{x-y}$

Eqn of tangent line at $(1,1)$

differentiate the curve on both sides.

$$\frac{d}{dx}(x^2y)+\frac{d}{dx}(xy^2)=\frac{d}{dx}(2e^{x-y})$$

$$x^2\frac{dy}{dx}+y(2x)+x2y\frac{dy}{dx}+y^2=2\left[e^{x-y}\left(1-\frac{dy}{dx}\right)\right]$$

$$x^2\frac{dy}{dx}+2xy+2xy\frac{dy}{dx}+y^2=2e^{x-y}-2e^{x-y}\frac{dy}{dx}$$

$$x^2\frac{dy}{dx}+2xy\frac{dy}{dx}+2e^{x-y}\frac{dy}{dx}=2e^{x-y}-2xy-y^2$$

$$\frac{dy}{dx}\left[x^2+2xy+2e^{x-y}\right]=2e^{x-y}-2xy-y^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{2e^{x-y}-2xy-y^2}{x^2+2xy+2e^{x-y}}$$

∴ the slope at $(1,1)$ is

$$m=\frac{dy}{dx}=\frac{2e^{1-1}-2(1)(1)-1^2}{1^2+2(1)(1)+2e^{1-1}}$$

$$m=\frac{2-2-1}{1+2+2}$$

$$\boxed{m=-\frac{1}{5}}$$

using slope form.

$$y-y_1=m(x-x_1)$$

$$y-1=\frac{-1}{5}(x-1)$$

$$\boxed{y=\frac{-1}{5}(x-1)+1}$$ is the eqn of the tangent line.

(2) Let length of the rectangular prism be 'L'
width of the rectangular prism be 'W'
Depth of the rectangular prism be 'D'

i. given that depth is geometric mean of its length and width

so, $D = \sqrt{L \times W} \rightarrow (1)$

ii. the ~~depth~~ volume of the prism is equal to its surface area.

$$LWD = 2(LW + WD + DL) \rightarrow (2)$$

Substitute (1) in (2).

$$LW\sqrt{LW} = 2[LW + L\sqrt{LW} + W\sqrt{LW}]$$

$$LW\sqrt{\cancel{LW}} = 2\sqrt{\cancel{LW}}[\sqrt{LW} + L + W]$$

$$LW = 2[\sqrt{LW} + L + W]$$

differentiate with respect to 'w' on both sides. and apply chain rule.

$$(uv)' = uv' + vu'$$

$$L'W + L(1) = 2\left[\frac{1}{2\sqrt{LW}}[L'W + L] + L' + 1\right]$$

$$L'W + L = \frac{1}{\sqrt{LW}}[L'W] + \frac{L}{\sqrt{LW}} + 2L' + 2$$

$$L'\left[W - \frac{W}{\sqrt{LW}} - 2\right] = \frac{L}{\sqrt{LW}} + 2 - L$$

$$L'\left[\frac{W\sqrt{LW} - W - 2\sqrt{LW}}{\sqrt{LW}}\right] = \frac{L - L\sqrt{LW} + 2\sqrt{LW}}{\sqrt{LW}}$$

$$\therefore \frac{dL}{dW} = \frac{L - L\sqrt{LW} + 2\sqrt{LW}}{W\sqrt{LW} + W - 2\sqrt{LW}}$$

$\therefore \frac{dL}{dW}$ is the rate of change of length of prism with respect to width.

therefore, $$\frac{dL}{dW} = \frac{L - LD + 2D}{WD - W - 2D}$$

③ given $H : [0, 24] \to \mathbb{R}$

$t \in [0, 24]$ measured in hours.

$t = 0$ corresponds to midnight

$H(t)$ measures temperatures in degrees Fahrenheit at time t.

A) given data set is $D = \{(t_0, H_0), \dots, (t_N, H_N)\}$

to estimate the average temperature given in the data set D.

To get the average of left & right Riemann sum, we use trapezoidal rule for numerical approximation.

$$\int_a^b f(x)\,dx \simeq \sum_{i=0}^{N-1} \frac{f(x_i) + f(x_{i+1})}{2} \Delta x$$

$$\simeq \frac{\Delta x}{2}\left(f(x_0) + 2\left(\sum_{i=1}^{N-1} f(x_i)\right) + f(x_n)\right)$$

So, from the given data $\Delta t = 15 \text{ min.} = \frac{1}{4}$ hours

$a = 0$ & $b = 24$

$$\int_a^b f(x)\,dx = \frac{1}{24} \sum_{i=0}^{N-1} \left(\frac{H_i + H_{i+1}}{2}\right)\left(\frac{1}{4}\right)$$

here, $N \to$ no. of measurements

$H_i, H_{i+1} \to$ Consecutive temperatures.

So, $\Delta t = \frac{24}{N} \Rightarrow N = \frac{24}{\Delta t}$

$N = \frac{24}{\frac{1}{4}} \Rightarrow \boxed{N = 96}$

Now, average temperature $= \bar{H}$

$$\bar{H} = \frac{1}{24} \sum_{i=0}^{95} \frac{H_i + H_{i+1}}{2} \left(\frac{1}{4}\right)$$

$$\therefore \bar{H} = \frac{1}{96} \sum_{i=0}^{95} \frac{H_i + H_{i+1}}{2}$$

B) $f: R \to R$ on an interval $[a, b] = \frac{1}{b-a} \int_a^b f(x)\,dx$

This formula should be used for a large no. of sample values of f because of following reasons:

* The avg value integration gives an underestimate of the area. ~~If~~ the lesser the no. of division, the more area under curve is left out.
* ~~if the~~ the more no. of division, the lesser area under the curve is leftout, which is more accurat.

Hence, It is always a better practise to take sample points approaching infinity as the space between them nearly reaches zero & the function has a continuous set of samples.

c) $H(t) = 32 + 18\sin\left(\frac{\pi}{24}t\right)$

~~Integrate on both sides~~

$$\int_a^b H(t)\,dt = \frac{1}{b-a}\int_a^b H(t)\,dt \quad \left\{\text{Riemann average equation}\right\}$$

$$= \frac{1}{24-0}\int_0^{24} 32 + 18\sin\left(\frac{\pi}{24}t\right)dt$$

$$= \frac{1}{24}\left[32t - 18\cos\left(\frac{\pi}{24}t\right)\times\frac{24}{\pi}.\right]_0^{24}$$

$$= \left[\frac{32}{24}t - \frac{18}{24}\times\frac{24}{\pi.}\times\cos\left(\frac{\pi}{24}t\right)\right]_0^{24}$$

$$= \left[\frac{32\times\cancel{24}}{\cancel{24}} - \frac{18}{\cancel{24}}\times\frac{\cancel{24}}{\pi}\times\cos(\pi)\right] - \left[\left(\frac{32}{24}\times 0\right) - \frac{18}{\cancel{24}}\times\frac{\cancel{24}}{\pi}\times\cos 0\right]$$

$$= 32 - \frac{18}{\pi}\times(-1) - \left(\frac{-18}{\pi}(1)\right)$$

$$= 32 + \frac{18}{\pi} + \frac{18}{\pi.}$$

$$= 43.46\ ^\circ F$$

$\therefore$ The average temperature is $43.46\ ^\circ F$

(4) given pdf $p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

to Calculate variance $Var(x) = \int_{-\infty}^{\infty} (x-\lambda^{-1})^2 p(x) dx$

Splitting the limits from $-\infty$ to ∞ to $-\infty \to 0$ & $0 \to \infty$.

$$Var(x) = \cancel{\int_{-\infty}^{0} (x-\lambda^{-1})^2 p(x) dx}^{\,0} + \int_{0}^{\infty} (x-\lambda^{-1})^2 p(x) dx$$

$$Var(x) = \int_0^{\infty} \left(x - \frac{1}{\lambda}\right)^2 p(x) dx.$$

Subsitute ~~value~~ $p(x) = \lambda e^{-\lambda x}$

$$Var(x) = \int_0^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \left(\lambda e^{-\lambda x}\right) dx$$

$$= \int_0^{\infty} \left(x^2 + \frac{1}{\lambda^2} - 2x\frac{1}{\lambda}\right) \lambda e^{-\lambda x} dx.$$

$$= \int_0^{\infty} \left(x^2 \lambda e^{-\lambda x} + \frac{\lambda e^{-\lambda x}}{\lambda^2} - \frac{2x\lambda e^{-\lambda x}}{\lambda}\right) dx$$

$$= \lambda \int_0^{\infty} x^2 \cdot e^{-\lambda x} dx + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx - 2\int_0^{\infty} x \cdot e^{-\lambda x} dx$$

let us take $u = \lambda x \Rightarrow du = \lambda dx$.

$$\Rightarrow x = \frac{u}{\lambda}$$

$$Var(x) = \lambda \int_0^{\infty} \frac{u^2}{\lambda^2} e^{-u} \frac{du}{\lambda} + \frac{1}{\lambda} \int_0^{\infty} e^{-u} \frac{du}{\lambda} - 2\int_0^{\infty} \frac{u}{\lambda} e^{-u} \frac{du}{\lambda}$$

$$= \frac{\lambda}{\lambda^3} \int_0^\infty u^2 e^{-u}\, du + \frac{1}{\lambda^2} \int_0^\infty e^{-u}\, du - \frac{2}{\lambda^2} \int_0^\infty u e^{-u}\, du$$

use integration by parts.

$$\int f(x) g'(x)\, dx = f(x) g(x) - \int f'(x) g(x)\, dx$$

$$= \frac{1}{\lambda^2} \left[u^2 (-e^{-u}) - \int (2u)(-e^{-u})\, du \right] + \frac{1}{\lambda^2} (-e^{-u})$$

$$- \frac{2}{\lambda^2} \left(\int u e^{-u}\, du \right)$$

$$= -\frac{u^2}{\lambda^2} e^{-u} + \cancel{\frac{2}{\lambda^2} \int u e^{-u}\, du} - \frac{e^{-u}}{\lambda^2} - \cancel{\frac{2}{\lambda^2} \int u e^{-u}\, du}$$

$$= \left[\frac{-u^2}{\lambda^2} e^{-u} - \frac{e^{-u}}{\lambda^2} \right]_0^\infty$$

$$= \left[\frac{(-\lambda x)^2}{\lambda^2} e^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty$$

$$= \left[\frac{\cancel{\lambda^2} x^2}{\cancel{\lambda^2}} \cdot e^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty$$

$$= \left[x^2 e^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty$$

$$= (0 - 0) - \left(0 - \frac{e^0}{\lambda^2} \right)$$

$\{$ as $e^{-\infty} = 0 \}$.

$$= -\left(-\frac{1}{\lambda^2} \right)$$

$$\boxed{Var(X) = \frac{1}{\lambda^2}}$$