① a) $S = \{(\lambda, \lambda+\mu^3, \lambda-\mu^3) \in R^3 \mid \lambda, \mu \in R\}$.

it is Subspace of R^3

b) $T = \{(\lambda^2, -\lambda^2, 0) \in R^3 \mid \lambda \in R\}$

It is subspace of R^3

c) $U = \{(x, y, z) \in R^3 \mid y \in Z\}$

It is not subspace of R3

D) $V = \{(x, y, z) \in R^3 \mid y \in Z\}$

It is subspace of R^3

② $S = \{ \langle 2,-1,3 \rangle, \langle 1,1,-2 \rangle, \langle 3,-3,8 \rangle \}$

$T = \{ \langle 1,2,1,0,0 \rangle, \langle 1,1,0,1,1 \rangle, \langle 1,0,0,1,1 \rangle \}$

A) $S_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad S_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad S_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$

Check $a_1 S_1 + a_2 S_2 + a_3 S_3 = 0$

Let $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \Rightarrow R_3 + 2R_2$

$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix}$

$R_2 \Rightarrow R_2 - R_3$

$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$

$R_1 \Rightarrow R_1 - R_3$

$\begin{bmatrix} 2 & 0 & 4 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 \Rightarrow R_1 + 2R_2} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$

So, $-a_1 - 2a_3 = 0$

$a_2 - a_3 = 0$

We can see that a_1, a_2 & $a_3 \neq 0$

Hence, 'S' is not linearly independent.

B) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_3 - R_5]{R_4 \to R_4 - R_5} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Let $T_x = 0$

$a_1 = 0$

$2a_1 + a_2 = 0$ $\quad \therefore a_2 = 0$

$a_2 + a_3 = 0$

hence, $a_3 = 0$.

here, $a_1 T_1 + a_2 T_2 + a_3 T_3 = 0$

Hence, T is linearly independent.

(3) given, A is $m \times n$ matrix

$A: R^n \to R^m$

Consider Matrix 'A' as a linear transformation that takes an n-dimensional vector 'u' and maps it to an m-dimensional vector 'V'.

$$V = Au$$

to show 'V' is a vector space, It should follow the below conditions:

i) V is closed under vector addition.

i.e, If V_1 & V_2 are in 'V' then

$V_1 + V_2$ is also in 'V'.

Let $V_1 = Au_1$ and $V_2 = Au_2$ for some u_1, u_2 in R^n

then, $V_1 + V_2 = Au_1 + Au_2$

$= A(u_1 + u_2)$

Since, $u_1 + u_2$ is in R^n & $V_1 + V_2$ is in 'V'.

ii) 'V' is closed under scalar multiplication. That is if 'v' is in 'V' and 'c' is a scalar, then cv is also in 'V'.

Let $v = Au$ for some 'u' in R^n

then $cv = c(Au) = A(cu)$

Since, cu is in R^n, cv in V.

iii) V contains the zero vector.

Let $u = 0$, the zero vector in R^n

then $Au = 0$, the zero vector in R^m

So, 0 is in V.

Since, V satisfies all the vector space axioms,

V is a vector space.