Pre-class Assignment - 21

Reading

① **Taylor Series**:- A taylor series is a representation of a fn. as an infinite sum of terms calculated from the function's derivatives at a single point. It can be used to approximate the function near that point.

the taylor series expansion of a fn. $f(x)$ around the point $x=a$ is:

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

where $f'(a)$ is the first derivative of 'f' evaluated at $x=a$,
$f''(a)$ is the 2^{nd} derivative evaluated at $x=a$,
& so on for the higher order derivatives.

Taylor Polynomial: It is a truncation of the taylor series to a finite number of terms. So it uses just the first few derivative terms to approximate the fn. near the chosen point.

② $$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

This inequality gives an upper bound on the error when approximating a function by its taylor polynomial.

$x \rightarrow$ upper bound, $c \rightarrow$ lower bound
$n \rightarrow$ order of the polynomial

$|R_n(x)|$ ⇒ This represents the remainder or error when approximating the function by its degree n Taylor polynomial centered at 'a'.

M ⇒ Max value of the function at a given interval.

$|x-a|$ ⇒ the distance from the centre point 'a' where the taylor polynomial approximation is centered.

This is useful for computer programming because it gives a way to bound the error when calculating a function like sine with a Taylor poly. approximation.

The Inequality describes exactly how good the approximation will be. By choosing 'n' appropriately, the programmer can make the error as small as needed.

Exercise

① given, $f(x) = \sin x$ centred at $x = \pi/2$

Taylor Polynomials are given by:

$$P_n(x) = f(a) + \frac{f'(a)(x-a)^1}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^n(a)(x-a)^n}{n!}$$

$$f(x) = \sin x \Rightarrow f(\pi/2) = 1$$

$$f'(x) = \cos x \Rightarrow f'(\pi/2) = 0$$

$$f''(x) = -\sin x \Rightarrow f''(\pi/2) = -1$$

$$f'''(x) = -\cos x \Rightarrow f'''(\pi/2) = 0$$

$$f^4(x) = \sin x \Rightarrow f^4(\pi/2) = 1$$

$$P_4(x) = 1 + \frac{0(x-\pi/2)}{1!} + \frac{(-1)(x-\pi/2)^2}{2!} + \frac{(0)(x-\pi/2)^3}{3!} + \frac{(1)(x-\pi/2)^4}{4!}$$

$$\Rightarrow P_0(x) = 1$$

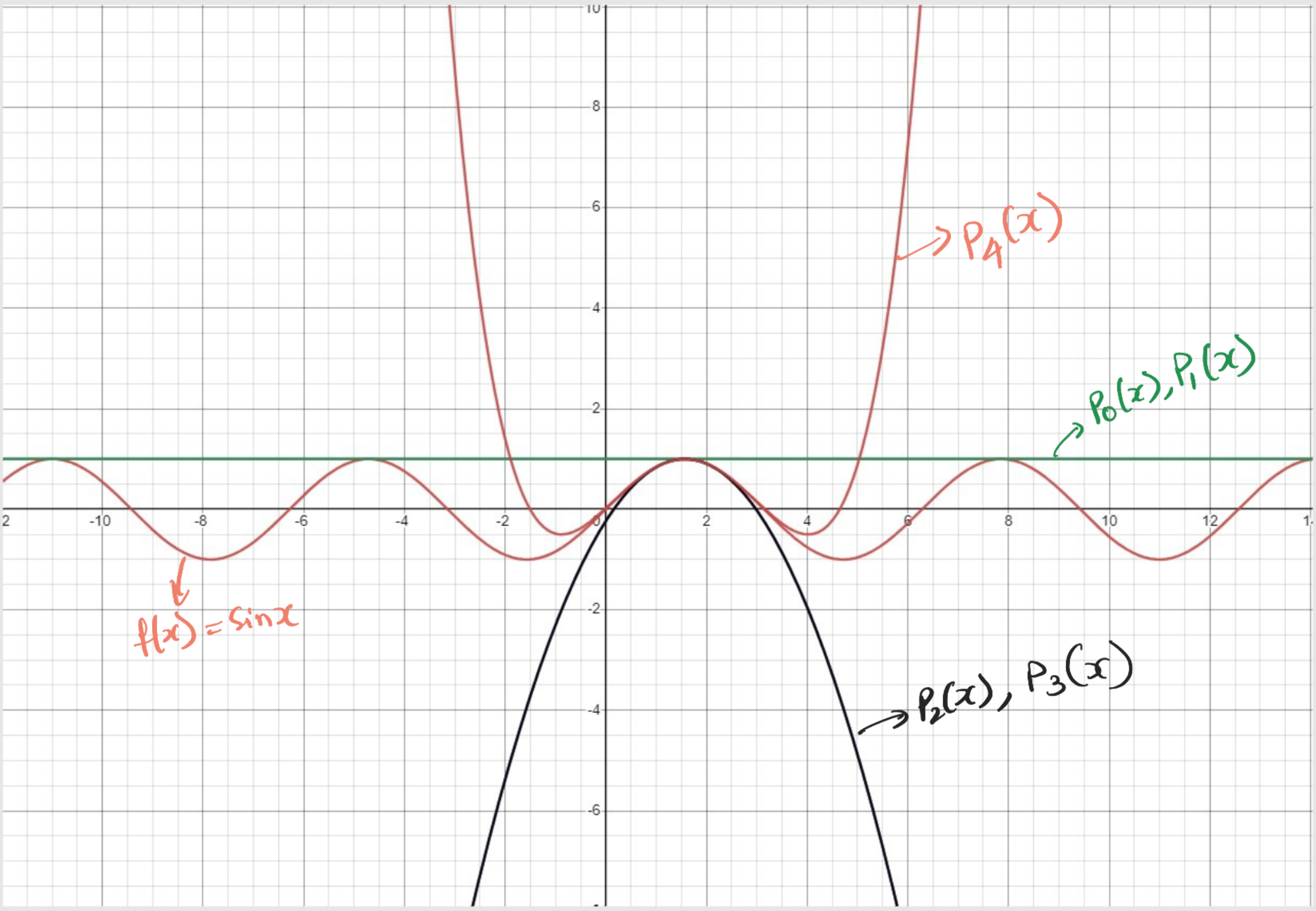
$$\Rightarrow P_1(x) = 1 + \frac{(0)(x-\pi/2)}{1} = 1$$

$$\Rightarrow P_2(x) = 1 + 0 + \frac{(-1)(x-\pi/2)^2}{2} = 1 - \frac{(x-\pi/2)^2}{2}$$

$$\Rightarrow P_3(x) = 1 + 0 + \frac{(-1)(x-\pi/2)^2}{2} + 0 = 1 - \frac{(x-\pi/2)^2}{2}$$

$$P_4(x) = 1 + 0 + \frac{(-1)(x-\pi/2)^2}{2} + 0 + \frac{(1)(x-\pi/2)^4}{4!}$$

$$\Rightarrow P_4(x) = 1 - \frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{24}$$



We notice that, the Taylor polynomials have the same value for every two alternate polynomials.

(2) To find the taylor Series of $N(x) = e^{-x^2}$ centered at $x=0$ by manipulating the taylor series of $f(x) = e^x$ centred at $x=0$.

Taylor Series:

$$f(x) = f(a) + \frac{f'(a)(x-a)^1}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^n(a)(x-a)^n}{n!}$$

at $a=0$

$$f(x) = f(a) + \frac{f'(a)(x)^1}{1!} + \frac{f''(a)(x)^2}{2!} + \dots + \frac{f^n(a)(x)^n}{n!}$$

$f(x) = e^x \Rightarrow f'(0) = 1$

$f''(0) = 1$

$f'''(0) = 1$

$\vdots$

$f^n(0) = 1$

Now, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Substitute x by $-x^2$ which gives $N(x) = e^{-x^2}$

$$\Rightarrow \boxed{e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots}$$

(3) given, $f(x) = 2x^7 - 3x^4 + x^2 - 5x + 1$ Centred at $x = \sqrt{2}$

Taylor Series.

$$f(x) = f(\sqrt{2}) + f'(\sqrt{2})(x-\sqrt{2}) + \frac{f''(\sqrt{2})(x-\sqrt{2})^2}{2!} + \frac{f'''(\sqrt{2})(x-\sqrt{2})^3}{3!} + \cdots$$

$f(\sqrt{2}) = 2(\sqrt{2})^7 - 3(\sqrt{2})^7 + (\sqrt{2})^2 - 5(\sqrt{2}) + 1 = 11\sqrt{2} - 9$

$f'(x) = 14x^6 - 12x^3 + 2x - 5$

$f'(\sqrt{2}) = 14(\sqrt{2})^6 - 12(\sqrt{2})^3 + 2(\sqrt{2}) - 5 = 107 - 22\sqrt{2}$

$f''(x) = 84x^5 - 36x^2 + 2$

$f''(\sqrt{2}) = 84(\sqrt{2})^5 - 36(\sqrt{2})^2 + 2 = 336\sqrt{2} - 70$

$f'''(x) = 420x^4 - 72x$

$f'''(\sqrt{2}) = 420(\sqrt{2})^4 - 72(\sqrt{2}) = 1680 - 72\sqrt{2}$

$f^4(x) = 1680x^3 - 72$

$f^4(\sqrt{2}) = 1680(\sqrt{2})^3 - 72 = 6720\sqrt{2} - 72$

$$f^5(x) = 5040x^2$$

$$f^5(\sqrt{2}) = 5040(\sqrt{2})^2 = 10080$$

$$f^6(x) = 10080x$$

$$f^6(\sqrt{2}) = 10080\sqrt{2}$$

$$f^7(x) = 10080 = f^7(\sqrt{2})$$

$$P_7(x) = 11\sqrt{2} - 9 + \frac{107 - 22\sqrt{2}(x-\sqrt{2})}{1!} + \frac{336\sqrt{2} - 70(x-\sqrt{2})^2}{2!}$$

$$+ \frac{1680 - 72\sqrt{2}(x-\sqrt{2})^3}{3!} + \frac{6720\sqrt{2} - 72(x-\sqrt{2})^4}{4!}$$

$$+ \frac{10080(x-\sqrt{2})^5}{5!} + \frac{10080\sqrt{2}(x-\sqrt{2})^6}{6!} + \frac{10080(x-\sqrt{2})^7}{7!}$$

(4) given, $f(x) = e^x$ centered at $x = 0$

$$P_4(x) = f(a) + \frac{f'(a)(x-a)^1}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \frac{f^4(a)(x-a)^4}{4!}$$

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = 1$$

Similarly, $f''(0) = 1$, $f'''(0) = 1$, $f^4(0) = 1$

Now, $P_4(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

we know,

$$|R_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}$$ here, $M = \max|e^{x}|$
$M = e^{0.5}$

hence, maximum error is given by

$$|R_n(x)| \le \frac{f^{n+1}(K)}{(n+1)!} |x-a|^{n+1}$$

here, $x = 0.5$ & $a = -0.5$

$$|R_4(x)| = \frac{M}{5!} (0.5-(-0.5))^5$$

$$|R_4(x)| \le \left|\frac{e^{0.5}}{5!}(1)\right|$$

here $K=0.5$, as the max value of e^K is at $K=0.5$ in the given interval $[0.5,-0.5]$

$$|R_4(x)| = \left|\frac{1.648}{120}\right|$$

$$R_4(x) = 0.0137$$

therefore, the max possible error on interval $[0.5,-0.5]$ is $\underline{0.0137}$

5) given, $f(x,y) = \ln(1+x-2y)$ centered at $(0,0)$

the taylor expansion for a function of 2 variables upto 2nd order is given by.

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)(x-a)^2}{2!}$$
$$+ \frac{f_{yy}(a,b)(y-b)^2}{2!} + f_{xy}(a,b)(x-a)(y-b) + \cdots$$

here, $a=0$, $b=0$

Now, $f(0,0) = \ln(1+0-0) = \ln(1) = 0$

$$f_x = \frac{1}{1+x-2y} \Rightarrow f_x(0,0) = 1$$

$$f_y = \frac{1}{1+x-2y}(-2) \Rightarrow f_y(0,0) = -2$$

$$f_{xx} = \frac{-1}{(1+x-2y)^2} \Rightarrow f_{xx}(0,0) = -1$$

$$f_{yy} = \frac{(-2)(-2)}{(1+x-2y)^2} \Rightarrow f_{yy}(0,0) = 4$$

$$f_{xy} = f_{yx} = \frac{-(-2)}{(1+x-2y)^2} \Rightarrow f_{xy}(0,0) = 2$$

Hence,

$$f(x,y) = 0 + 1(x) + (-2)(y) + \frac{(-1)(x)}{2!} + \frac{4(y)}{2!} + 2(x)(y)$$

$$\Rightarrow f(x,y) = x - \cancel{2y} - \frac{x}{2} + \frac{\cancel{4y}}{\cancel{2}} + 2xy$$

$$\Rightarrow f(x,y) = \frac{x}{2} + 2xy$$