Pre-class Assignment-20

① given, $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$(x,y) \to (f_1(x,y), f_2(x,y))$ where $f_1(x,y) = x^2y$ &

$f_2(x,y) = 5x + \sin y$

to find Jacobian J_f and $\det(J_f)$

The jacobian matrix J_f of a vector-valued function $f: \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$f_1(x,y) = x^2y \qquad f_2(x,y) = 5x + \sin y$

$$\frac{\partial f_1}{\partial x} = \frac{\partial(x^2y)}{\partial x} = 2xy \qquad \frac{\partial f_2}{\partial x} = \frac{\partial(5x+\sin y)}{\partial x} = 5$$

$$\frac{\partial f_1}{\partial y} = \frac{\partial(x^2y)}{\partial y} = x^2 \qquad \frac{\partial f_2}{\partial y} = \frac{\partial(5x+\sin y)}{\partial y} = \cos y$$

$$\Rightarrow J_f = \begin{bmatrix} 2xy & x^2 \\ 5 & \cos y \end{bmatrix}$$

$$\Rightarrow \det(J_f) = 2xy(\cos y) - 5x^2$$

(2)

A) The given identities $x = r\cos\theta$ & $y = r\sin\theta$ describe a transformation from Cartesian Co-ordinates (x, y) to polar Coordinates (r, θ). we can define a mapping $f: R^2 \to R^2$ using these identities as:

$$f(r, \theta) = (r\cos\theta,\ r\sin\theta)$$

this mapping takes a point in polar co-ordinates (r, θ) and maps it to a point in cartesian co-ordinates (x, y)

B) The Jacobian matrix J_f of the mapping f is given by:

$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{bmatrix}$$

where, $f_1(r, \theta) = r\cos\theta$
$f_2(r, \theta) = r\sin\theta$

$$\frac{\partial f_1}{\partial r} = \frac{\partial(r\cos\theta)}{\partial r} = \cos\theta \qquad \frac{\partial f_2}{\partial r} = \frac{\partial(r\sin\theta)}{\partial \theta} = \sin\theta.$$

$$\frac{\partial f_1}{\partial \theta} = \frac{\partial(r\cos\theta)}{\partial \theta} = -r\sin\theta \qquad \frac{\partial f_2}{\partial \theta} = \frac{\partial(r\sin\theta)}{\partial \theta} = r\cos\theta.$$

the Jacobian matrix $J_f = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ //

c) the $\det(J_f) = \cos\theta(r\cos\theta) + r\sin(\sin\theta)$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(1)$$

$$\det(J_f) = r \ //$$

the determinant 'r' is significant because it represents the Scaling factor introduced by the change of co-ordinates from Cartesian to polar co-ordinates. When performing integration in polar coordinates, the $\det(J_f) = r$ is used to account for the Scaling effect, and it appears in the transformation of double integrals.

(3)

A) $f(x) = \sin(x_1)\cos(x_2)$

$$J_f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial(\sin(x_1)\cos(x_2))}{\partial x_1} \qquad \frac{\partial f}{\partial x_2} = \frac{\partial(\sin(x_1)\cos(x_2))}{\partial x_2}$$

$$\frac{\partial f}{\partial x_1} = \cos x_1 \cos x_2 \qquad \frac{\partial f}{\partial x_2} = -\sin x_1 \sin x_2.$$

$$\rightarrow J_f = \begin{bmatrix} \cos x_1 \cos x_2 & -\sin x_1 \sin x_2 \end{bmatrix}$$

B) $g(x, y) = x^T y$

$$J_g = \begin{bmatrix} \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial y_1}, \dots, \frac{\partial g}{\partial y_n} \end{bmatrix}$$

$$\frac{\partial g}{\partial x_i} = y_i \, , \frac{\partial g}{\partial y_i} = x_i \quad \text{where, } i = 1, 2, 3, \dots n$$

$$\Rightarrow J_g = \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ x_1 & x_2 & \dots & x_n \end{bmatrix}$$

c) $h(x) = xx^T$

$$J_h = \left[\frac{\partial h_{ij}}{\partial x_k}\right]$$

h_{ij} respresents $(i,j)^{th}$ entry of the matrix xx^T

$$\frac{\partial h_{ij}}{\partial x_k} = \frac{\partial (x_i x_j)}{\partial x_k} = \delta_{ik} x_j + \delta_{jk} x_i$$

here δ_{ik} is kronecker delta which is 1 when $i = k$ & 0 otherwise

$$J_h = \begin{bmatrix} x_1 & - & 0 \\ | & x_2 & | \\ 0 & - & \ddots \; x_n \end{bmatrix}$$

this is a diagonal matrix with the entries of x along the diagonal.

④ For a multivariable fn. $f: \mathbb{R}^n \to \mathbb{R}$ the 2nd order partial derivative can be organized into what is known as the Hessian matrix.

The Hessian Matrix, denoted by H is a square matrix where i,j^{th} entry is the 2nd mixed partial derivative of f with respect to x_i & x_j it is given by

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

2nd order partial derivative: the diagonal entries $\left(\frac{\partial^2 f}{\partial x_i^2}\right)$ indicates the concavity or convexity of the function along each axis.