Homework - 8

① given, $f(x) = 2x^3 - 3x^2 - 12x + 8$

A) Critical points of $f(x)$

$$f'(x) = 0$$

$$6x^2 - 6x - 12 = 0$$

$$6(x^2 - x - 2) = 0$$

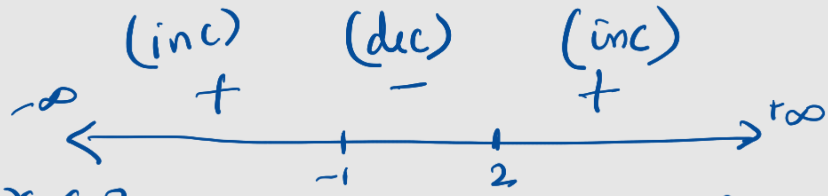
$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \ \& \ x = -1$$

Hence, the critical points are $\underline{x = 2}$ & $x = -1$.

B)



If $-1 \leq x \leq 2$,

$f'(1) = 6 - 6 - 12 = -12$

$f'(0) = 0 - 0 - 12 = -12$

$f'(-1) = 6 + 6 - 12 = 0$

$f'(2) = 24 - 12 - 12 = 0$

If $x < -1$,

$f'(-2) = 24 + 12 - 12 = 24$

$f'(-3) = 54 + 18 - 12 = 60$

If $x > 2$,

$f'(3) = 54 - 18 - 12 = 24$

$f'(4) = 96 - 24 - 12 = 60$

$\Rightarrow$ Here, $f(x)$ is increasing $\Rightarrow (-\infty, -1) \cup (2, \infty)$

$f(x)$ is decreasing $\Rightarrow (-1, 2)$

c) As, $f'(x)$ has sign change from +ve to −ve at $x=-1$,
$\Rightarrow$ $f(x)$ has local maxima at $x=-1$

$\Rightarrow$ $f'(x)$ has sign change from −ve to +ve at $x=2$,
$f(x)$ has local minima at $x=2$.

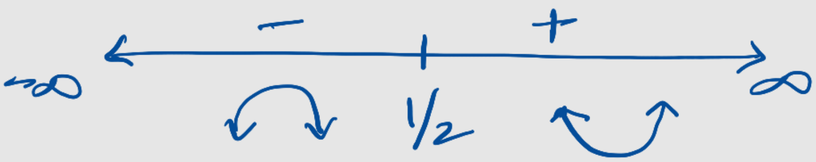
D) $f(x) = 2x^3 - 3x^2 - 12x + 8$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

when $f''(x) = 0$

$12x - 6 = 0 \Rightarrow x = \frac{1}{2}$



if $x=0$, $f''(0) = -6$

if $x=2$, $f''(2) = 24 - 6 = 18$

Hence, $f(x)$ is concave up on $\left(\frac{1}{2}, \infty\right)$

$f(x)$ is concave down on $\left(-\infty, \frac{1}{2}\right)$

$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{8}\right) - 3\left(\frac{1}{4}\right) - 12\left(\frac{1}{2}\right) + 8 = \frac{1}{4} - \frac{3}{4} + 2 = \frac{3}{2}$

Inflection point $= \left(\frac{1}{2}, \frac{3}{2}\right)$

E) $f(x) = 2x^3 - 3x^2 - 12x + 8$

given interval $[-2, 3]$

the critical points are $x = 2$ & $x = -1$

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 8 = 16 - 12 - 24 + 8 = -12$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 8 = -2 - 3 + 12 + 8 = 15$

$f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 8 = 54 - 27 - 36 + 8 = -1$

$f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 8 = -16 - 12 + 24 + 8 = 4$

Hence, the global maximum is 15 at $x = -1$

the global minimum is -12 at $x = 2$

(2) given, $f(x, y) = 2x^3 + 2y^3 - 3xy^2 - 12x - 20$

find partial derivatives,

$f_x = 6x^2 - 3y^2 - 12$ & $f_y = 6y^2 - 6xy$

when $f_x = 0$,

$6x^2 - 3y^2 - 12 = 0$

If $y = 0$,

$6x^2 - 0 - 12 = 0$

$6x^2 = 12$

$x^2 = 2$

$\boxed{x = \pm\sqrt{2}}$

If $y = x$,

$6x^2 - 3x^2 - 12 = 0$

$3x^2 = 12$

$\boxed{x = \pm 2}$

when $f_y = 0$

$6y^2 - 6xy = 0$

$6y(y - x) = 0$

$6y = 0$, $y - x = 0$

$\boxed{y = 0}$ & $\boxed{y = x}$

Hence, the critical points are:

$(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, $(2, 2)$ & $(-2, -2)$

Find 2^{nd} order partial derivative in order to classify:

$f_{xx} = 12x \quad , f_{yy} = 12y - 6x \quad , f_{xy} = -6y = f_{yx}$

Hessian Matrix $H_f = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

$H_f = \begin{bmatrix} 12x & -6y \\ -6y & 12y-6x \end{bmatrix}$

$D = 12x(12y-6x) - 36y^2 \Rightarrow 144xy - 72x^2 - 36y^2$

$D(\sqrt{2},0) = 0 - 72(\sqrt{2})^2 - 36(0) = -144 < 0$

$D(-\sqrt{2},0) = 0 - 72(-\sqrt{2})^2 - 36(0) = -144 < 0$

$D(2,2) = 144(4) - 72(2)^2 - 36(2) = 144 > 0$

$f_{xx}(2,2) = 12(2) = 24 > 0$

$D(-2,-2) = 144(4) - 72(-2)^2 - 36(-2) = 144 > 0$

$f_{xx}(-2,-2) = 12(-2) = -24 < 0$

Hence, from 2^{nd} derivative test

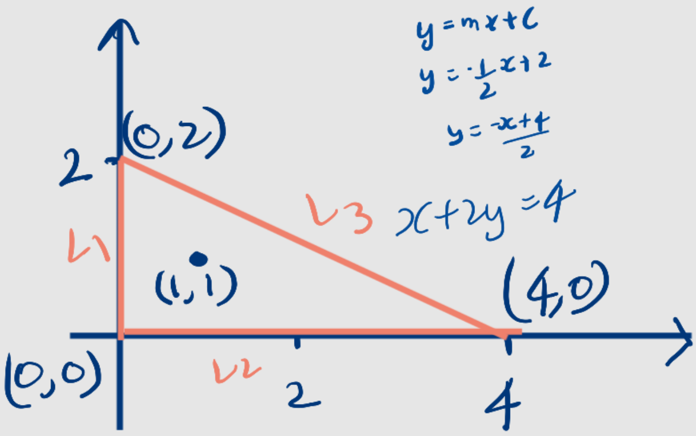
→ $(2,2)$ is the local minimum as $D(2,2) > 0$ & $f_{xx} > 0$

→ $(-2,-2)$ is the local maximum as $D(-2,-2) > 0$ & $f_{xx} < 0$

→ $(\sqrt{2},0)$ & $(-\sqrt{2},0)$ are the saddle points as $D(\sqrt{2},0) < 0$ & $D(-\sqrt{2},0) < 0$

③ given, $f(x) = x + y - xy$

vertices: $(0,0)$, $(0,2)$ and $(4,0)$



partial derivatives,

$f_x = 1 - y$

$f_y = 1 - x$

$\Rightarrow f_x = 0 \Rightarrow 1 - y = 0 \Rightarrow \boxed{y = 1}$

$\Rightarrow f_y = 0 \Rightarrow 1 - x = 0 \Rightarrow \boxed{x = 1}$

So, there is only one critical pt. inside the region $(1,1)$.

$f(1,1) = 1 + 1 - 1 = 1$

On L_1:- $f(x,y) = f(0,y) = y \; ; \; 0 \le y \le 2$

Max Value $= 2$ at $y = 2$

Min Value $= 0$ at $y = 0$

On L_2:- $f(x,y) = f(x,0) = x \; ; \; 0 \le x \le 4$

Max value $= 4$ at $x = 4$

Min value $= 0$ at $x = 0$

On L_3:-

$x + 2y = 4 \; ; \; 0 \le y \le 2$

$f(x,y) = (4 - 2y) + y - (4 - 2y)y = 4 - y - 4y + 2y^2$

$= 2y^2 - 5y + 4$

$f'(y) = 0 \Rightarrow 4y - 5 = 0 \Rightarrow y = \frac{5}{4}$

$f\left(\frac{5}{4}\right) = 2\left(\frac{5}{4}\right)^2 - 5\left(\frac{5}{4}\right) + 4 = \cancel{2}\left(\frac{25}{\cancel{16}_{8}}\right) - \frac{25}{4} + 4 = \frac{7}{8}$

$f(0) = 2(0) - 5(0) + 4 = 4 \rightarrow$ Max value

$f(2) = 2(2)^2 - 5(2) + 4 = 2 \rightarrow$ Min Value

Hence, the minimum value is 0

the maximum value is 4

④ For the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(x, y) \longmapsto (x^2 - xe^y, 1 + y\sin x)$$

the scaling factor also known as Jacobian determinant is given using the Jacobian Matrix:

$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x} = 2x - e^y \quad \frac{\partial f_1}{\partial y} = -xe^y$$

$$\frac{\partial f_2}{\partial x} = y\cos x \quad \frac{\partial f_2}{\partial y} = \sin x$$

$$\Rightarrow J_f = \begin{bmatrix} 2x - e^y & -xe^y \\ y\cos x & \sin x \end{bmatrix}$$

$$\text{Det}(J_f) = (2x - e^y)\sin x + xye^y\cos x$$

Lets choose 5 different points in $\mathbb{R}^2$

At $(0,0) \Rightarrow \text{Det}(J_f) = (2(0) - e^0)\sin 0 + 0 = 0$

At $(1,1) \Rightarrow \text{Det}(J_f) = (2(1) - e^1)\sin(1) + e^1\cos(1) = 2.7$

At $(0,\pi) \Rightarrow \text{Det}(J_f) = (2(0) - e^\pi)\sin(0) + 0 = 0$

At $(\pi,0) \Rightarrow \text{Det}(J_f) = (2\pi - e^0)\sin(\pi) + \pi e^0\cos(\pi)(0) = 0$

At $(1,0) \Rightarrow \text{Det}(J_f) = (2(1) - e^0)\sin(1) + e^0\cos(1)(0) = 0.017$

Hence, the answers are not same. The reason is that the function is not linear, so it does not scale a small patch of the area near a point in a constant manner.

⑤

A) $f(x,y,z) = (xyz, x^2y^2+z^2, x^2+y^2z^2)$

The Jacobian Matrix $J_f(x)$ is given by

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}$$

$\frac{\partial f_1}{\partial x} = yz \qquad \frac{\partial f_1}{\partial y} = xz \qquad \frac{\partial f_1}{\partial z} = xy$

$\frac{\partial f_2}{\partial x} = 2xy^2 \qquad \frac{\partial f_2}{\partial y} = 2x^2y \qquad \frac{\partial f_2}{\partial z} = 2z$

$\frac{\partial f_3}{\partial x} = 2x \qquad \frac{\partial f_3}{\partial y} = 2yz^2 \qquad \frac{\partial f_3}{\partial z} = 2y^2z$

$$\Rightarrow J_f(x,y,z) = \begin{bmatrix} yz & xz & xy \\ 2xy^2 & 2x^2y & 2z \\ 2x & 2yz^2 & 2y^2z \end{bmatrix}$$

B) given, $f = \tanh(z) \in \mathbb{R}^M$

where, $z = Ax + b$ $\quad x \in \mathbb{R}^N$, $A \in \mathbb{R}^{M \times N}$, $b \in \mathbb{R}^M$

$$[\tanh(z)]_i = \tanh(z_i)$$

The Jacobian Matrix $J_f(x)$ is given by

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \cdots & \frac{\partial f_3}{\partial x_N} \end{bmatrix}$$

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j}(\tanh(z_i)) = \operatorname{sech}^2(z_i)\frac{\partial}{\partial x_j}(z_i)$$

$$= \operatorname{Sech}^2(z_i)\frac{d}{dx_j}(Ax+b)$$

$$= \operatorname{Sech}^2(z_i) A_{ij}$$

Hence, the Jacobian Matrix is

$$J_f(x) = \begin{bmatrix} \operatorname{sech}^2(z_1)A_{11} & \operatorname{sech}^2(z_1)A_{12} & \cdots & \operatorname{sech}^2(z_1)A_{1N} \\ \operatorname{sech}^2(z_2)A_{21} & \operatorname{sech}^2(z_2)A_{22} & \cdots & \operatorname{sech}^2(z_2)A_{2N} \\ \vdots & \vdots & & \vdots \\ \operatorname{sech}^2(z_M)A_{M1} & \operatorname{sech}^2(z_3)A_{M2} & \cdots & \operatorname{sech}^2(z_M)A_{MN} \end{bmatrix}$$

⑥ Given, $f(x,y) = x^3 - 6xy^2 + y^3$ about the point $(1,1)$

The 2^{nd} order Taylor polynomial for a function $f(x,y)$ about the point (a,b) is given by:

$$P_2(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)(x-a)^2}{2} + \frac{f_{yy}(a,b)(y-b)^2}{2} + f_{xy}(a,b)(x-a)(y-b)$$

Calculate the partial derivatives,

$f_x = 3x^2 - 6y^2 \Rightarrow f_x(1,1) = 3-6 = -3$

$f_y = -12xy + 3y^2 \Rightarrow f_y(1,1) = -12+3 = -9$

$f_{xx} = 6x \Rightarrow f_{xx}(1,1) = 6$

$f_{yy} = -12x + 6y \Rightarrow f_{yy}(1,1) = -6$

$f_{xy} = -12y \Rightarrow f_{xy}(1,1) = -12$

$f(1,1) = 1-6+1 = -4$

Substituting the above values in 2^{nd} order Taylor poly.

$$P_2(x,y) = f(1,1) - 3(x-1) - 9(y-1) + \frac{6(x-1)^2}{2} - \frac{6(y-1)^2}{2} - 12(x-1)(y-1)$$

$$= -4 - 3x + 3 - 9y + 9 + 3(x^2+1-2x) - 3(y^2+1-2y) - 12[xy - x - y + 1]$$

$$= -3x - 9y + 8 + 3x^2 + 3 - 6x - 3y^2 - 3 + 6y - 12xy + 12x + 12y - 12$$

$$\Rightarrow P_2(x,y) = 3x^2 - 3y^2 + 3x + 9y - 12xy - 4$$