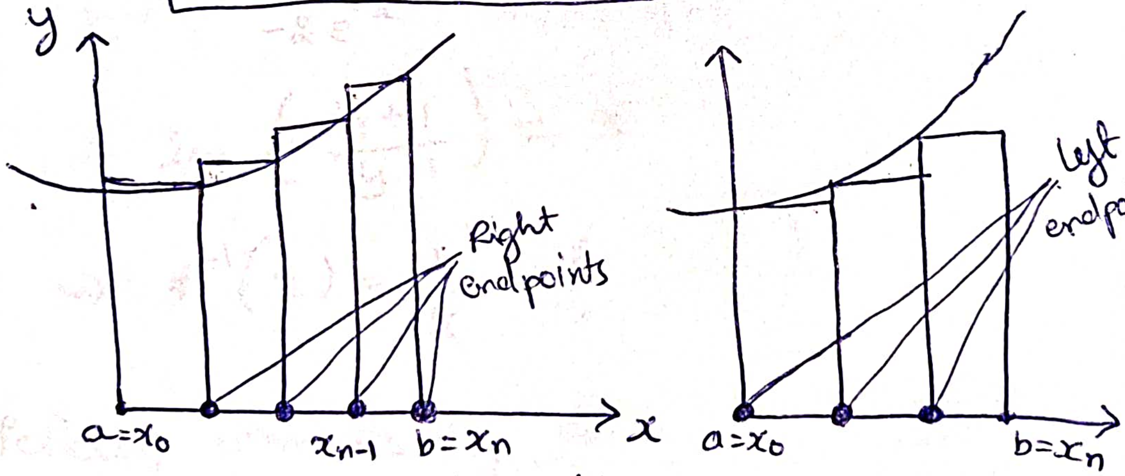
① Definite Integrals

To approximate the area under a curve we have been using Rectangles, The heights of these Rectangles have been determined by evaluating the functions at either the right or left endpoints of the sub-interval $[x_{i-1}, x_i]$. We could evaluate the function at any point (x_i^*) in the Sub interval $[x_{i-1}, x_i]$ & use $f(x_i^*)$ as the height of ~~e~~ the rectangle.

The estimate for the area of the form is given by:

$$\boxed{A \approx \sum_{i=1}^{n} f(x_i^*) \Delta x}$$



The above equation is called as Riemann Sum eqn.

It Can be defined as, Let $f(x)$ be defined on a closed interval $[a, b]$ & let 'P' be a regular partition of $[a, b]$. Let 'Δx' be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$.

Let $f(x)$ be a continuous, non-negative function on an interval $[a,b]$ and let $\sum_{i=1}^{n} f(x_i^*)\Delta x$ be a Riemann sum of $f(x)$.

Then area under the curve $y = f(x)$ on $[a,b]$ is:

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

To meet the restrictions that Riemann Sum eqn has which is $f(x)$. ~~should~~ should be a continous & non- negative ~~so~~ we use definite Integrals.

<u>Definite Integrals</u>.

If $f(x)$ is a function defined on an interval $[a,b]$, the definite integral of 'f' from a to b is given by

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.}$$

Provided that the limit exists.

If the limit exists, $f(x)$ is said to integratable on $[a,b]$ or is an Integrable function.

② Properties of Definite Integrals:

i) If the limits of integration are the same, the integral is just a line and contains no area.

i.e. $$\boxed{\int_a^a f(x)\,dx = 0}$$

Proof: from the definition of definite Integrals,

$$\int_a^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\left\{ \text{we know, } \Delta x = \frac{b-a}{n} \Rightarrow \Delta x = \frac{a-a}{n} \Rightarrow \Delta x = 0 \right\}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} f(x_i^*)\,(0)$$

$$= \lim_{n\to\infty} (0)$$

$$\therefore \boxed{\int_a^a f(x)\,dx = 0}$$

ii) If the limits are reversed, then place a '-ve' sign in front of the integral.

i.e $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

proof/Explaination:

from the definition of definite integral,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \;; \quad \Delta x = \frac{b-a}{n}$$

Also,

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \quad ; \quad \Delta x = \frac{a-b}{n}$$

Therefore,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\frac{-(a-b)}{n}$$

$$= \lim_{n\to\infty} \left(-\sum_{i=1}^{n} f(x_i^*)\left(\frac{a-b}{n}\right)\right)$$

$$= -\left(\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{a-b}{n}\right)\right)$$

$$\Rightarrow \boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

iii) The integral of a sum is the sum of the integrals.

i.e $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

from the definition of definite integrals,

$$\int_a^b [f(x)+g(x)]\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i^*)+g(x_i^*)\right)\Delta x$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\,\Delta x + \sum_{i=1}^{n} g(x_i^*)\,\Delta x\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*)\,\Delta x$$

$$\Rightarrow \boxed{\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx}$$

③ Fundamental theorem of Calculus (Part-1)

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

A) Role of x: 'x' is the upper limit of integration and it represents the variable with respect to which we are finding the accumulation or which means we are determining how the quantity described by the function f(t) accumulates as 't' ranges from lower bound 'a' to the upper bound 'x'.

B) Role of t: 't' is an integral variable or we can call it as a 'dummy' variable inside the integral. It serves as a placeholder. When we integrate the function with respect to 't', we are basically summing up the values of f(t) as 't' varies from 'a' to 'x'. After integrating, 't' disappears and we are left with a function of x, F(x).

where $F(x) = \int_a^x f(t)\,dt$

F(x) is continuous on the interval [a, x] & differentiable on the open interval (a, x).

Lower Bound of Integration 'a':

- The lower bound of integration is denoted as 'a', where 'a' is a constant. It represents the starting point of the integration.

- 'a' can be any real number within the domain of the function $f(x)$. Generally, it should be chosen such that interval $[a, x]$ lies within the domain of $f(x)$ so that the integral is well-defined.

Ex: Integrating a function $f(x)$ from $x = 1$ to $x = 5$, we can write it as:

$$\int_1^5 f(t)\,dt = F(5) - F(1)$$

Here, $a = 1$ represents the lower limit of the integration, and it's a specific number chosen to define the starting point of the integration.

④ The fundamental theorem of calculus (Part 2) is a crucial concept in calculus that unites the theories of differentiation and integration.

Let us consider a continuous function $f(x)$ defined on the closed interval $[a,b]$ and let $F(x)$ be an antiderivative of $f(x)$ on the same interval. Then the fundamental theorem of calculus (Part-2) ~~states~~ ~~that~~ is given by:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Let's prove the above:

Let $P = \{x_i\}$ where $i = 0, 1, \dots n$ be a regular partition of $[a,b]$

it can be written as,

$$F(x)\Big|_a^b = F(b) - F(a) = F(x_n) - F(x_0)$$

$$F(b) - F(a) = [F(x_n) - F(x_{n-1})] + F[(x_{n-1}) - F(x_{n-2})] + \dots + [F(x_1) - F(x_0)]$$

$$= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \rightarrow ①$$

we know, F is an antiderivative of f over $[a,b]$

From, Mean Value Theorem for $i = 0, 1, \dots, n$

we can find c_i in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$$

$$F(x_i) - F(x_{i-1}) = f(c_i)\Delta x \rightarrow ②$$

Substituting eq ② in eq ①

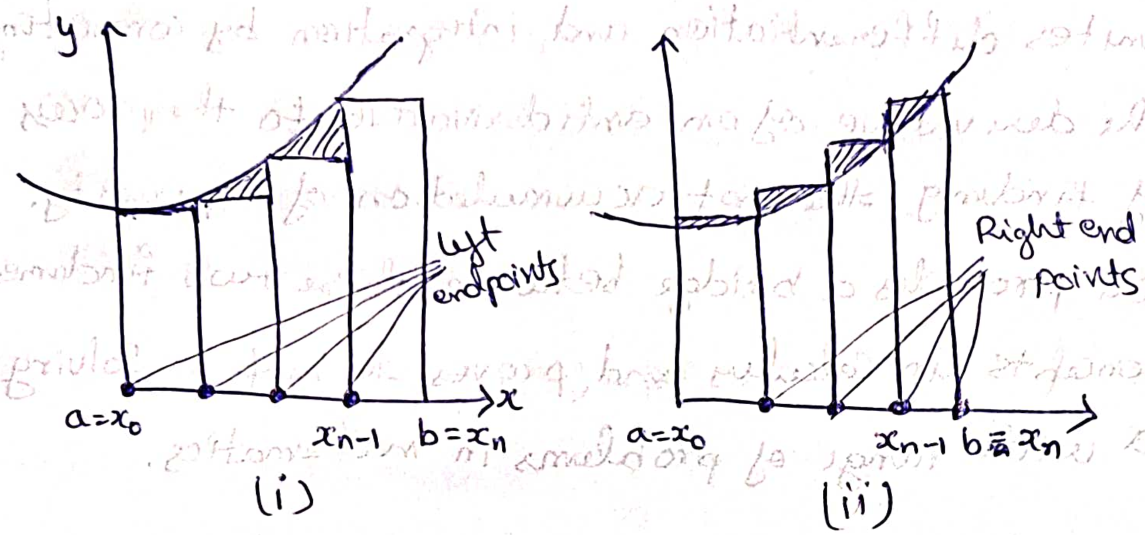
we get, $$\boxed{F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x}$$

Therefore, the fundamental theorem of Calculus (Part-2) unites differentiation and integration by connecting the derivative of an antiderivative to the process of finding the net accumulation of a quantity. It provides a bridge between these two fundamental concepts in calculus and proves to help in solving a wide range of problems in mathematics.

Exercises

① $f:[a,b] \to \mathbb{R}$ is increasing

$$A = \int_a^b f(x)\,dx$$



A) Since, for a left Riemann sum for $f(x)$, from the curve (i) we can see that some area between the rectangle and the curve are not being considered. So, the left Riemann sum for $f(x)$ gives the under estimate of A.

B) from the curve (ii) we can see that some extra area above the curve are also taken into consideration. so, the right Riemann sum for $f(x)$ gives over estimate of A.

c) Let $f(x)$ be a continuous function over $[a,b]$ & $\Delta x = \dfrac{b-a}{n}$. Let x_i^* be any point & calculate $f(x_i^*)$ for $i=1,2,\ldots,n$.

the average value of function may be approximated as;

$$\frac{f(x_1^*)+f(x_2^*)+\dots+f(x_n^*)}{n}.$$

we know $\Delta x = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{\Delta x}.$

so,

$$\frac{f(x_1^*)+f(x_2^*)+\dots+f(x_n^*)}{n} = \frac{f(x_1^*)+f(x_2^*)+\dots+f(x_n^*)}{\frac{(b-a)}{\Delta x}}$$

we can write the Numerator as:

$$f(x_1^*)+f(x_2^*)+\dots+f(x_n^*) = \sum_{i=1}^{n} f(x_i^*)$$

$$\Rightarrow \frac{\sum_{i=1}^{n} f(x_i^*)}{\frac{(b-a)}{\Delta x}} = \left(\frac{\Delta x}{b-a}\right)\sum_{i=1}^{n} f(x_i^*)$$

$$= \left(\frac{1}{b-a}\right)\sum_{i=1}^{n} f(x_i^*)\Delta x$$

This is a Riemann Sum. Then, to get the average value, take the limit as n goes to infinity.

Thus Average Value of function is given by:

$$= \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

2

a) $\int_3^{10} 2dx. = 2(x)_3^{10}$

$= 2(10-3)$

$= 2(7)$

$= \underline{14}$

b) $\int_0^5 (1+3x)dx = \int_0^5 1.dx + \int_0^5 3x\,dx$

$= (5-0) + \cancel{3(5-0)} \int_0^5 \frac{3x^2}{2} dx$

$= 5 + \cancel{5}\ \frac{3}{2}(25)$

$= 5 + \frac{75}{2} \Rightarrow \frac{85}{2}$

c) $\int_0^3 \sqrt{9-x^2}\,dx = y \cancel{\int_0^3 (9-(x^2))^{1/2} dx}$

Squaring both sides $\left(\int_0^3 \sqrt{9-x^2}dx\right)^2 = (y)^2$

Consider a ~~circ~~ Semi circle with radius 3 Centered at the origin.

eqn of semi circle is $x^2+y^2=9$

we want to find the area under this curve from $x=0$ to $x=3$, which is the same as finding the area of the semi circle in this interval.

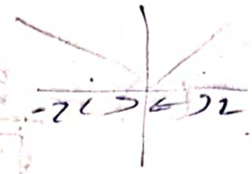
Area of circle = Area of Semi circle x 2.

Area of semicircle $= \frac{\pi r^2}{2}$

$= \frac{\pi(3^2)}{2} = \frac{9\pi}{2}$

$\int_0^3 \sqrt{9-x^2}dx = \frac{9\pi}{2}$

d) $\int_{-2}^{2} (1-|x|)\,dx$ $\quad |x| = \begin{cases} +x, \\ -x \end{cases}$



Consider the interval $(-2,0)$, $|x|$ is $-x$, as x is $-ve$

$$\int_{-2}^{0} (1-(-x))\,dx = \int_{-2}^{0} (1+x)\,dx$$

$$= [x]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{-2}^{0}$$

$$= [0-(-2)] + \left[0 - \frac{(-2)^2}{2}\right]$$

$$= 2 + (-2)$$

$$= \underline{\underline{0}}$$

Consider the interval $(0,2)$, $|x|$ is $+x$, as x is $+ve$.

$$\int_{0}^{2} (1-x)\,dx = \int_{0}^{2} (1-x)\,dx$$

$$= [x]_{0}^{2} - \left[\frac{x^2}{2}\right]_{0}^{2}$$

$$= 2 - \left(\frac{2^2}{2}\right)$$

$$= 2-2$$

$$= \underline{\underline{0}}$$

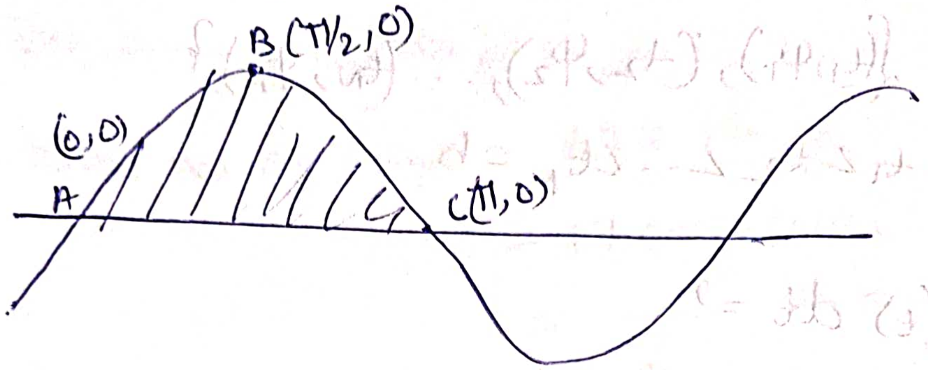
So, in the interval $(-2,2)$,

$$\int_{-2}^{2} (1-|x|)\,dx = \int_{-2}^{0} (1+x)\,dx + \int_{0}^{2} (1-x)\,dx$$

$$= 0+0$$

$$= \underline{\underline{0}}$$

③ $f(t) = \sin t$



as $\sin(0) = 0$, let point be $A = (0,0)$

as $\sin(\pi/2)$ is max value $= 1 \Rightarrow B = (\pi/2, 1)$

as $\sin(\pi) = 0 \Rightarrow C = (\pi, 0)$

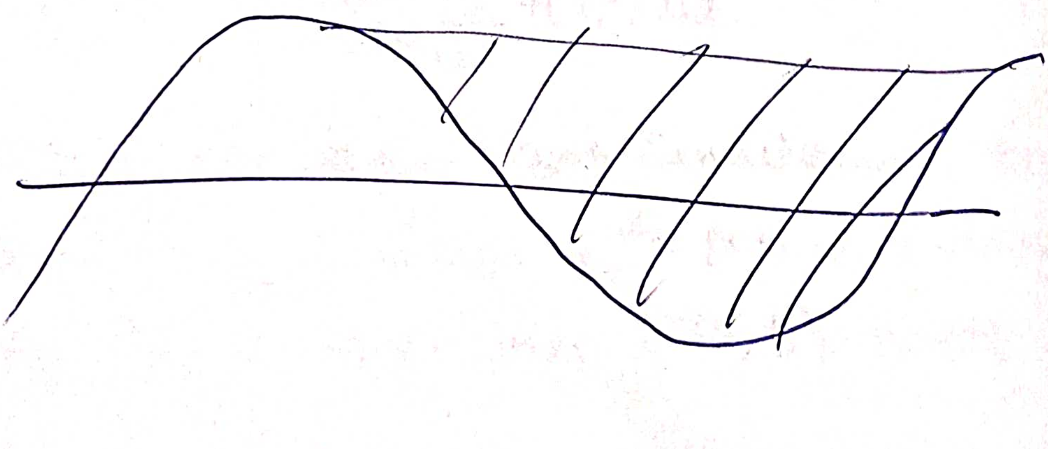
area under curve ABC $= \int_0^{\pi} \sin t \, dt = [-\cos t]_0^{\pi}$

$= -\cos \pi - (-\cos 0)$

$= -(-1) - (-1)$

$= \underline{2}$

Area under the curve ABC $= \underline{2}$



④

a) $F(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

let $G(x) = \int_0^x e^{-t^2} dt$

$\Rightarrow F(x) = \frac{2}{\sqrt{\pi}} G(x)$

Now, diff. F(x) with respect to 'x'

$$\frac{d}{dx}\left(\frac{2}{\sqrt{\pi}} G(x)\right) = \frac{2}{\sqrt{\pi}} \cdot \frac{d\,G(x)}{dx} + G(x) \frac{d}{dx}\left(\frac{2}{\sqrt{\pi}}\right)$$

$$\frac{d}{dx} G(x) = \frac{d}{dx}\left[\int_0^x e^{-t^2} dt\right] = e^{-x^2}$$

$$\frac{d}{dx}\left(\frac{2}{\sqrt{\pi}}\right) = 0$$

$$\frac{d}{dx}\left(\frac{2}{\sqrt{\pi}} G(x)\right) = \frac{2}{\sqrt{\pi}} e^{-x^2} + 0$$

$$\therefore \boxed{F'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}}$$

b) $G(x) = \frac{2}{\sqrt{\pi}} \int_{100}^x e^{-t^2} dt$

~~derivat~~ diff with respect to 'x'

$$G'(x) = \frac{2}{\sqrt{\pi}} \cdot \frac{d}{dx}\left(\int_{100}^x e^{-t^2} dt\right)$$

Apply chain rule,

$$G'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \cdot \frac{d}{dx}(x)^{1} - \frac{2}{\sqrt{\pi}} \cdot e^{-100^2} \cdot \frac{d}{dx}(100)^{0}$$

$$G'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} - 0$$

$$\therefore \boxed{G'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}}$$

c) $H(x) = \int_{x}^{\pi} \frac{1}{1+e^{-t}} dt$

we can use Leibniz integral rule,

$$\frac{\partial}{\partial z} \int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z} dx + f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z}$$

Let $f(x,t) = \frac{1}{1+e^{-t}}$ & $g(x) = \pi$.

from Leibniz rule:

$$\frac{d}{dx}\left(\int_{g(x)}^{f(x)} f(x,t)dt\right) = -\frac{d}{dx}\left(\int_{f(x)}^{g(x)} f(x,t)\, dt\right)$$

Now, $H(x) = -\int_{f(x)}^{g(x)} \frac{\partial}{\partial x} f(x,t)\, dt$

first, lets find $\frac{\partial}{\partial x} f(x,t)$

$$\frac{\partial}{\partial x} f(x,t) = \frac{\partial}{\partial x}\left(\frac{1}{1+e^{-t}}\right) = 0 - \frac{e^{-t}}{(1+e^{-t})^2} (-1)\cdot e^{-x}$$

$$= \frac{e^{-x}}{(1+e^{-t})^2}. \Rightarrow H(x) = -\int_{f(x)}^{g(x)} \frac{e^{-x}}{(1+e^{-t})^2} dt$$

$$\therefore \boxed{\frac{d}{dx}(H(x)) = -\frac{d}{dx}\int_{f(x)}^{g(x)} \frac{e^{-x}}{(1+e^{-t})^2} dt}$$

d) $I(x) = \int_{0.1}^{x^2} \ln\left(\frac{t}{1-t}\right) dt$

from leibniz rule,

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt + f(x, b(x))\frac{db}{dx}$$

$$- f(x, a(x))\frac{da}{dx}$$

here, $f(x,t) = \ln\left(\frac{t}{1-t}\right)$; $a = 0.1$ & $b(x) = x^2$

* $\frac{\partial}{\partial x} f(x,t) \Rightarrow$.

$$\frac{\partial}{\partial x} \ln\left(\frac{t}{1-t}\right) = \frac{1}{\frac{t}{1-t}} \cdot \frac{d}{dx}\left(\frac{t}{1-t}\right)$$

to find $\frac{d}{dx}\left(\frac{t}{1-t}\right)$ we use qotient rule,

$$\frac{d}{dx}\left(\frac{t}{1-t}\right) = \frac{(1-t)\frac{dt}{dx} - t\cdot\frac{d(1-t)}{dx}}{(1-t)^2}$$

$$= \frac{(1-t)\frac{dt}{dx} + t\cdot\frac{dt}{dx}}{(1-t)^2}$$

$$= \frac{\frac{dt}{dx}}{(1-t)^2}$$

$$\frac{d}{dx}\ln\left(\frac{t}{1-t}\right) = \frac{1}{\frac{t}{1-t}}\left(\frac{\frac{dt}{dx}}{(1-t)^2}\right) = \frac{\frac{dt}{dx}}{t(1-t)^2}$$

Substitute all in Leibniz Rule,

$$\frac{dI}{dx} = \int_{0.1}^{x^2} \frac{\frac{dt}{dx}}{t(1-t)^2} dt + \ln\left(\frac{x^2}{1-x^2}\right) \cdot \frac{d}{dx}(x^2) - \ln\left(\frac{0.1}{1-0.1}\right) \frac{d}{dx}(0.1)$$

Since, $\frac{d}{dx}(x^2) = 2x$ & $\frac{d}{dx}(0.1) = 0$.

$$\frac{dI}{dx} = \int_{0.1}^{x^2} \frac{\frac{dt}{dx}}{t(1-t)^2} dt + 2x \ln\left(\frac{x^2}{1-x^2}\right)$$

In the function $G(x) = \frac{2}{\sqrt{\pi}} \int_{100}^{x} e^{-t^2} dt$,

e^{-t^2} is the probability density function (PDF) of standard normal distribution in statistics

The integral of this PDF over an interval gives the cummulative dist. fn (CDF).

(5) The given date range is from January 23, 2020 to January 19, 2022.

total no. of days = 727 days.

= 17448 hours

= . . . . . min

The more we split it, the more accurate result we get.

let $n = 727$.

Let $f(x)$ be a function which denote the no. of cases on a day 'x'.

~~Hence~~, we can use definite integral to represent this,

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

here, $a = 0$ $b = 727$

$$\int_0^{727} f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

~~f(x)~~ the above eqn can determine the total no. of cases during the period of time.

As, $f(x)$ is continuous & (a, b) exists.

⑥ $\psi : [a, b] \to R$.

$S = \{(t_1, \psi_1), (t_2, \psi_2), \ldots (t_N, \psi_N)\}$

$a = t_1 < t_2 < \ldots < t_N = b$.

$$\int_a^b \psi(t)\, dt = ?$$

We can use definite integrals to approximate the above equation. and it can be written as

$$\int_a^b \psi(t) dt = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

where, $f(x_i^*)$ can take values $t_1, t_2, \ldots t_n$ for different values of N.

$\Delta x = t_2 - t_1 = t_3 - t_2 = t_N - t_{N-1} \ldots$