① $\|A\|_F \Rightarrow$ Square root of the sum of square root of the sum of squares of the entries.

A)

$$A = \begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 3 & 9 & 1 & -7 & 23 & 8 \\ 1 & 3 & 1 & -3 & 9 & 2 \\ 1 & 3 & -1 & -1 & 5 & 4 \end{bmatrix}$$

$$\|A\|_F = \sqrt{\sum |A_{ij}|^2}$$

$$= \sqrt{1^2+3^2+0^2+(-2)^2+7^2+3^2+3^2+9^2+1^2+(-7)^2+23^2+8^2+1^2+3^2+1^2+(-3)^2+9^2+2^2+1^2+3^2+(-1)^2+(-1)^2+5^2+4^2}$$

$$\Rightarrow \|A\|_F = \sqrt{963}$$

$$\|A\|_\infty = \max_{i=1,\dots n}\left(\sum_{j=1}^{n} |A_{ij}|\right)$$

$$= \max\big((1+3+2+7+3), (3+9+1+7+23+8), (1+3+1+3+9+2), (1+3+1+1+5+4)\big)$$

$$= \max(16, 51, 19, 15)$$

$$\Rightarrow \|A\|_\infty = 51$$

$$\|A\|_1 = \max_{j=1,2\dots n}\left(\sum_{j=1}^{n} |A_{ij}|\right)$$

$$= \max\big(1+3+1+1, 3+9+3+3, 0+1+1+1, 2+7+3+6, 7+23+9+5, 3+8+2+4\big)$$

$= Max(6, 18, 3, 14, 44, 17)$

$\Rightarrow \underline{\underline{||A||_1 = 44}}$

hence, $||A||_F = \sqrt{963}$, $||A||_\infty = 51$, $||A||_1 = 44$

B) $A^T = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 3 & 9 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ -2 & -7 & -3 & -1 \\ 7 & 23 & 9 & 5 \\ 3 & 8 & 2 & 4 \end{bmatrix}$

To Calculate the Eigen Value, we can use Python script.

we get $\lambda = \dfrac{963 + 3\sqrt{99021}}{2}$

$\lambda = 953.53$

Spectral norm. $||A||_2 = \sqrt{\lambda} = \sqrt{953.53} = \underline{\underline{30.879}}$

c) Finding the cosine of the angle between the first & the last row of vectors of A.

$V_1 = \langle 1, 3, 0, -2, 7, 3 \rangle$

$V_2 = \langle 1, 3, -1, -1, 5, 4 \rangle$

$\|V_1\| = \sqrt{1^2 + 3^2 + 0^2 + (-2)^2 + 7^2 + 3^2} = \sqrt{72}$

$\|V_2\| = \sqrt{1^2 + 3^2 + (-1)^2 + (-1)^2 + 5^2 + 4^2} = \sqrt{53}$

to calculate the angle,

we have, $V_1 \cdot V_2 = \|V_1\| \cdot \|V_2\| \cos\theta$.

$$\cos\theta = \frac{V_1 \cdot V_2}{\|V_1\| \cdot \|V_2\|}$$

$$\cos\theta = \frac{\langle 1, 3, 0, -2, 7, 3 \rangle \cdot \langle 1, 3, -1, -1, 5, 4 \rangle}{\sqrt{72} \cdot \sqrt{53}}$$

$$\cos\theta = \frac{1 + 9 + 2 + 35 + 12}{\sqrt{53}\ \sqrt{72}}$$

$$\cos\theta = \frac{59}{\sqrt{53 \times 72}}$$

$$\theta = \cos^{-1}\left(\frac{59}{\sqrt{3816}}\right)$$

$$\boxed{\theta \approx 25^\circ}$$

② given $x = (1,2,3,4)^T$, $U = span\{(1,-2,2,0)^T, (-1,1,1,-1)^T\}$

A) $Proj_x U = \frac{x \cdot \vec{u_1}}{\|u_1\|^2}\vec{u_1} + \frac{x \cdot \vec{u_2}}{\|u_2\|^2}\vec{u_2}$

$$\frac{x \cdot \vec{u_1}}{\|\vec{u_1}\|^2}\vec{u_1} = \frac{(1,2,3,4)(1,-2,2,0)}{\left(\sqrt{1^2+(-2)^2+(2)^2+0^2}\right)^2} \langle 1,-2,2,0 \rangle$$

$$= \frac{1-4+6}{9} \langle 1,-2,2,0 \rangle$$

$$= \frac{3}{9} (1,-2,2,0)$$

~~$Proj_x U$~~ =

$$\frac{x \cdot \vec{u_2}}{\|\vec{u_2}\|^2}\vec{u_2} = \frac{\langle 1,2,3,4 \rangle \langle -1,1,1,-1 \rangle}{\left(\sqrt{(-1)^2+(1)^2+(1)^2+(-1)^2}\right)^2} \langle -1,1,1,-1 \rangle$$

$$= \frac{-1+2+3-4}{4} \langle -1,1,1,-1 \rangle$$

$$= 0$$

$$Proj_x U = \frac{3}{9}(1,-2,2,0) + 0$$

$$= \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}, 0\right)$$

B) To find the projection matrix that performs above projection.

projection matrix $P = A \cdot (A^T \cdot A) \cdot A^T$

$$\left.\begin{array}{l} u_1 = \langle 1, -2, 2, 0 \rangle^T \\ u_2 = \langle -1, 1, 1, -1 \rangle^T \end{array}\right\} \text{Span of } A.$$

$$A = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 & 0 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 4/35 & 1/35 \\ 1/35 & 9/35 \end{bmatrix}$$

$$A(A^T A^{-1}) = \begin{bmatrix} 3/35 & -8/35 \\ -1/5 & 1/5 \\ 9/35 & 11/35 \\ -1/35 & -9/35 \end{bmatrix}$$

$$P = \begin{bmatrix} 11/35 & -2/5 & -2/35 & 8/35 \\ -2/5 & 3/5 & -1/5 & -1/5 \\ -2/35 & -1/5 & 29/35 & -11/35 \\ 8/35 & -1/5 & -11/35 & 9/35 \end{bmatrix}$$

c) Finding distance of vector x from subspace V,

distance $= ||x - \text{projection}(x, U)||$

$= || (-1, 1, 1, -1) - [(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}, 0)] ||.$

$\Rightarrow$ distance $= [-4/3, -1/3, 1/3, -1]$

Homework - 6.

(3) given, $n = \langle 1, 2, 3 \rangle$ & $b = \langle 5, 7, 4 \rangle$ in $\mathbb{R}^3$

A) The projection matrix 'P' onto a line 'L' spanned by the vector 'n' is given by $P = \dfrac{\vec{n} \cdot \vec{n}^T}{\vec{n}^T \cdot n}$

$$P = \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

$$P = \frac{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}}{14} \Rightarrow P = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

B) Applying RREF

$$P = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

$$P = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, Rank of $P = 1$

c) Projection of b onto line 'L' is given by

$$\text{Proj}_L \vec{b} = P \cdot b$$

$$= \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 5+14+12 \\ 10+28+24 \\ 15+42+36 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 31 \\ 62 \\ 93 \end{bmatrix}$$

D) Since V is orthogonal to n, any vector (x, y, z) that lies in 'V' must satisfy:

$n \cdot (x, y, z) = 0$.

$\Rightarrow \langle 1, 2, 3 \rangle \cdot \langle x\ y\ z \rangle = 0$

$1x + 2y + 3z = 0$

Hence, the equation of plane V at (x, y, z) is $x + 2y + 3z = 0$

E) To find the projection matrix Q that projects elements $\mathbb{R}^3$ onto V. is given by

$$Q = I - P.$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1/14 & 0-2/14 & 0-3/14 \\ 0-2/14 & 1-4/14 & 0-6/14 \\ 0-3/14 & 0-6/14 & 1-9/14 \end{bmatrix}$$

$$= \begin{bmatrix} 13/14 & -2/14 & -3/14 \\ -2/14 & 10/14 & -6/14 \\ -3/14 & -6/14 & 5/14 \end{bmatrix}$$

$$\Rightarrow Q = \frac{1}{14}\begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}$$

f) To find rank of Q, Apply RREF to Q

$$\begin{bmatrix} 13/14 & -2/14 & -3/14 \\ -2/14 & 10/14 & -6/14 \\ -3/14 & -6/14 & 5/14 \end{bmatrix} \xrightarrow[R_3 \to R_3 + \frac{3R_1}{13}]{R_2 \to R_2 + \frac{2R_1}{13}} \begin{bmatrix} 13/14 & -2/14 & -3/14 \\ 0 & 9/13 & -6/13 \\ 0 & -6/13 & 4/13 \end{bmatrix}$$

$$R_3 \to R_3 + \frac{2R_2}{13} \Rightarrow \begin{bmatrix} 13/14 & -2/14 & -3/14 \\ 0 & 9/13 & -6/13 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the rank = 2.

As there are 2 non-zero rows in the Reduced matrix, we can conclude that the rank of matrix is 2.

(i) Projection of $\vec{b}$ onto V

$$Q = \frac{1}{14}\begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix} \cancel{\begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}} \qquad b = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$$

$$\text{Proj}_V \vec{b} = Q \cdot b$$

$$= \frac{1}{14}\begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 13(5) - 2(7) - 3(4) \\ -2(5) + 10(7) - 6(4) \\ -3(5) - 6(7) + 5(4) \end{bmatrix}$$

$$\Rightarrow \text{Proj}_V \vec{b} = \frac{1}{14}\begin{bmatrix} 39 \\ 36 \\ -37 \end{bmatrix}$$

H) to chck if $b = \text{Proj}_L \vec{b} + \text{Proj}_V \vec{b}$

$$= \left[\frac{31}{14}, \frac{62}{14}, \frac{93}{14}\right] + \left[\frac{39}{14}, \frac{36}{14}, \frac{-37}{14}\right]$$

$$= \left[\frac{31}{14} + \frac{39}{14}, \frac{62}{14} + \frac{36}{14}, \frac{93}{14} - \frac{37}{14}\right]$$

$$= \left[\frac{70}{14}, \frac{98}{14}, \frac{56}{14}\right] = (5, 7, 4) = b$$

$\Rightarrow$ Hence, $b = \text{Proj}_L \vec{b} + \text{Proj}_V \vec{b}$