Q1)

A) $$\int_1^3 \int_0^1 \frac{2xy}{x^2+1}\, dx\, dy.$$

Integrating with respect to 'x' keeping 'y' constant.

$$= \int_1^3 \left[2y \int_0^1 \frac{x}{x^2+1}\, dx \right] dy.$$

Let $x^2+1 = u$

$du = 2x\,dx.$

$\frac{du}{2} = x\,dx.$

$$= \int_1^3 \left[2y \int_0^1 \frac{1}{u} \cdot \frac{du}{2} \right] dy$$

$$= \int_1^3 \left[\frac{2y}{2} \int_0^1 \frac{1}{u}\, du \right] dy$$

$$= \int_1^3 \left(y \left[\ln(u) \right]_0^1 \right) dy.$$

$$= \int_1^3 \left[y \left(\ln(x^2+1) \right)_0^1 \right] dy$$

$$= \int_1^3 \left(y \left[\ln(2) - \ln(1) \right] \right) dy$$

$$= \int_1^3 y \ln(2)\, dy.$$ {Integrate with respect to 'y'}

$$= \ln(2) \int_1^3 y\, dy.$$

$$= \ln(2)\left[\frac{y^2}{2}\right]_1^3$$

$$= \ln(2)\left(\frac{9}{2} - \frac{1}{2}\right)$$

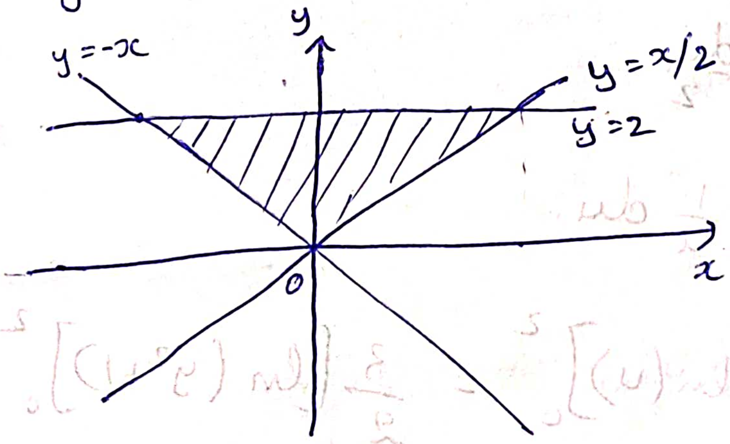
$$= \ln(2)\,\frac{8}{2}$$

$$= 4\ln(2)$$

Hence, $\displaystyle\int_1^3\int_0^1 \frac{2xy}{x^2+1}\,dx\,dy = 4\ln(2)$

B) $\displaystyle\iint_R \frac{1}{y^2+1}\,dA \qquad \left\{R: y=\frac{x}{2},\ y=-x,\ y=2\right\}$

Sketching the region:



limits of x: $-y \le x \le 2y$

limits of y: $0 \le y \le 2$

So the integral can be defined as:

$$\iint_R \frac{1}{y^2+1}\,dA = \int_0^2\int_{-y}^{2y} \frac{1}{y^2+1}\,dx\,dy.$$

$$\int_0^2 \int_{-y}^{2y} \frac{1}{y^2+1}\, dx\, dy.$$

Let $y^2+1 = u.$

$2y\,dy = du$

$y\,dy = \frac{du}{2}.$

Integrate with respect to 'x' keeping 'y' constant

$$\int_0^2 \left[\frac{1}{y^2+1} \int_{-y}^{2y} dx\right] dy. = \int_0^2 \left[\frac{1}{y^2+1} \left[x\right]_{-y}^{2y}\right] dy$$

$$= \int_0^2 \left[\frac{1}{y^2+1} (2y-(-y))\right] dy.$$

$$= \int_0^2 \left[\frac{1}{y^2+1} (3y)\right] dy.$$

$$= 3\int_0^2 \frac{1}{u} \frac{du}{2}$$

$$= \frac{3}{2} \int_0^2 \frac{1}{u}\, du.$$

$$= \frac{3}{2} \left[\ln(u)\right]_0^2 = \frac{3}{2} \left[\ln(y^2+1)\right]_0^2$$

$$= \frac{3}{2} \left[\ln(5) - \ln(1)\right]$$

$$= \frac{3}{2} \ln(5)$$

hence, $\iint_R \frac{1}{y^2+1}\, dA = \frac{3}{2} \ln(5)$

Q2) $\iint_D xy\,dA$; D is the region enclosed by

parabola $y^2 = 2x+6$ & $\to$ (1)

line $y = x-1$ $\to$ (2)

substitute eqn (2) in (1)

$(x-1)^2 = 2x+6$

$x^2+1-2x = 2x+6.$

$x^2-4x-5=0$

$x^2-5x+x-5=0$

$x^2+x-5x-5=0$

$x(x+1)-5(x+1)=0$

$x-5=0$ and $x+1=0$

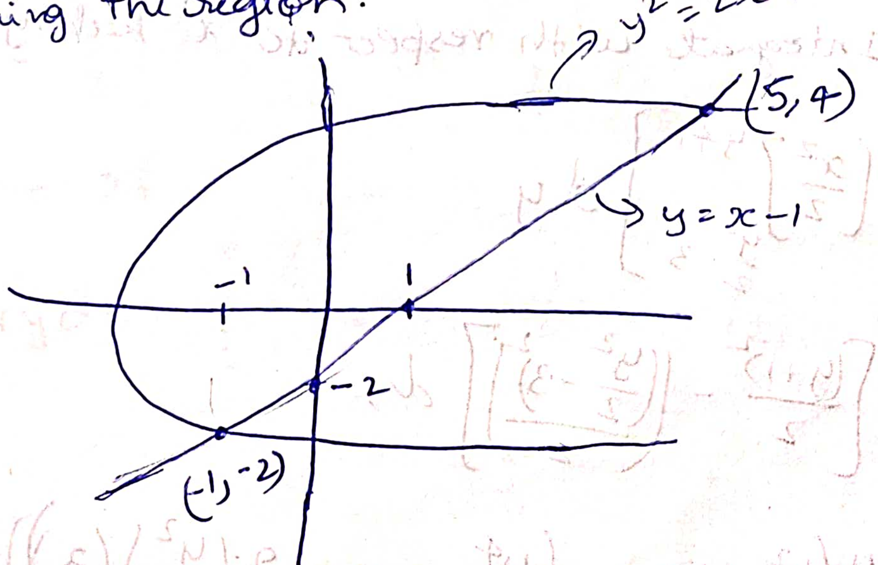
$x=5$ & $x=-1$

Substitute the values of x in (2)

if $x=-1 \Rightarrow y=-1-1 \Rightarrow \boxed{y=-2}$

if $x=5 \Rightarrow y=5-1 \Rightarrow \boxed{y=4}$

Sketching the region:



~~Writting ① & ②~~

eqn ① $\Rightarrow$ $y^2 = 2x+6$.

$$y^2 = 2(x+3)$$

$$\frac{y^2}{2} = x+3$$

$$\frac{y^2}{2} - 3 = x.$$

eq ② $\Rightarrow$ $y = x-1$

$$x = y+1$$

So the limits are.

$$\Rightarrow \quad \frac{y^2}{2} - 3 \leq x \leq y+1$$

$$\Rightarrow \quad -2 \leq y \leq 4.$$

$$\iint_D xy\, dA = \int_{-2}^{4} \int_{\frac{y^2}{2}-3}^{y+1} xy \, dx\, dy.$$

Integrate with respect to 'x' keeping 'y' const.

$$= \int_{-2}^{4} \left[y \left[\frac{x^2}{2} \right]_{\frac{y^2}{2}-3}^{y+1} \right] dy.$$

$$= \int_{-2}^{4} y \left[\frac{(y+1)^2}{2} - \left(\frac{\left(\frac{y^2}{2}-3\right)^2}{2} \right) \right] dy.$$

$$= \int_{-2}^{4} \frac{y}{2} \left(y^2 + 1 + 2y - \left(\frac{y^4}{4} + 9 - 2\cdot\left(\frac{y^2}{2}\right)(3) \right) \right) dy.$$

$$=\frac{1}{2}\int_{-2}^{4} y\left(y^2+1+2y-\frac{y^4}{4}+9+3y^2\right)dy$$

$$=\frac{1}{2}\int_{-2}^{4}\left(y^3+y+2y^2-\frac{y^5}{4}-9y+3y^3\right)dy$$

$$=\frac{1}{2}\int_{-2}^{4}\left(4y^3-8y+2y^2-\frac{y^5}{4}\right)dy.$$

$$=\frac{1}{2}\left[\frac{4y^4}{4}-\frac{8y^2}{2}+\frac{2y^3}{3}-\frac{y^6}{24}\right]_{-2}^{4}$$

$$=\frac{1}{2}\left[\frac{4(4)^4}{4}-\frac{8(4)^2}{2}+\frac{2(4)^3}{3}-\frac{4^6}{24}-\left(\frac{4(-2)^4}{4}-\frac{8(-2)^2}{2}+\frac{2(-2)^3}{3}-\frac{(-2)^6}{24}\right)\right]$$

$$=\frac{1}{2}\left[\frac{6144-1536+1024-4096}{24}-\left(\frac{16-64}{24}\right)\right]$$

$$=\frac{1}{2}\left[\frac{1536+192}{24}\right]$$

$$=\frac{1}{2}[72]$$

$$=36$$

$$\Rightarrow \iint_D xy\,dA=\int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xy\,dx\,dy=\underline{\underline{36}}$$

Q3) $\displaystyle\int_0^1 \int_{2\sqrt{y}}^{2} 12\sqrt{x^3+1}\, dx\, dy.$

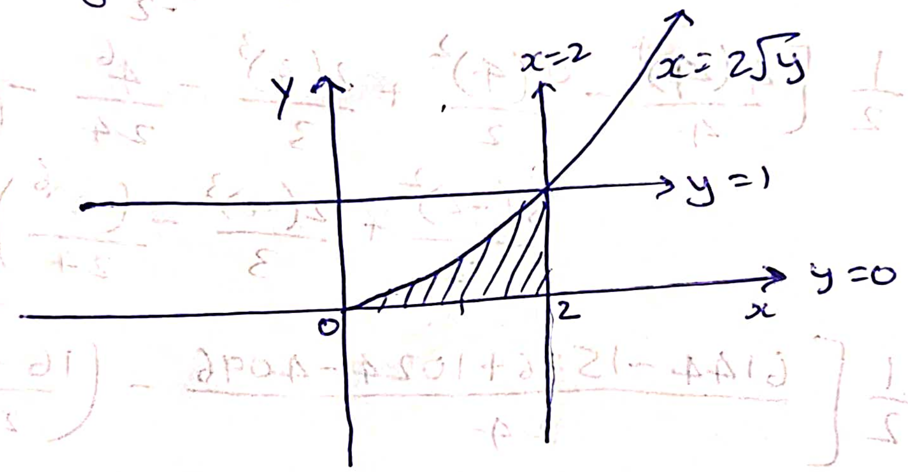
here, $2\sqrt{y} \le x \le 2$ and $0 \le y \le 1$

$x = 2\sqrt{y}$, $x = 2$ $\qquad y = 0$ & $y = 1$

$x^2 = 4y$

$y = \dfrac{x^2}{4}$

Sketching the region,



the new limits are:

$0 \le x \le 2$ & $0 \le y \le x^2/4$

So,

$$\int_0^2 \int_0^{x^2/4} 12\sqrt{x^3+1}\, dy\, dx$$

Integrate with respect to 'y' & keeping 'x' const.

$$= 12 \int_0^2 \left(\sqrt{x^3+1}\, (y) \Big|_0^{x^2/4} \right) dx = 12 \int_0^2 \sqrt{x^3+1} \left(\frac{x^2}{4}\right) dx.$$

Integrating with respect to 'x'.

$\Rightarrow$ Let $x^3+1 = u$

$3x^2\, dx = du$

$x^2\, dx = \dfrac{du}{3}$

$$= \frac{12}{4} \int_0^2 \frac{1}{\sqrt{u}} \cdot \frac{du}{3}$$

$$= \frac{12}{12} \int_0^2 u^{1/2} \, du$$

$$= \cancel{8} \left[\frac{u^{3/2}}{3/2} \right]_0^2$$

$$= \frac{2}{3} \left[u^{3/2} \right]_0^2$$

$$= \frac{2}{3} \left[(x^3+1)^{3/2} \right]_0^2$$

$$= \frac{2}{3} \left[(8+1) - (0+1) \right]^{3/2}$$

$$= \frac{2}{3} \left(9^{3/2} - 1 \right)$$

$$= \frac{2}{3} \left(3^3 - 1 \right)$$

$$= \frac{2}{3} \times 26$$

$$= \frac{52}{3}$$

Hence,

$$\int_0^1 \int_{2\sqrt{y}}^2 12\sqrt{x^3+1} \, dx \, dy = \frac{52}{3}$$

Q4)

A) $\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2} \, dy \, dx$.

here,

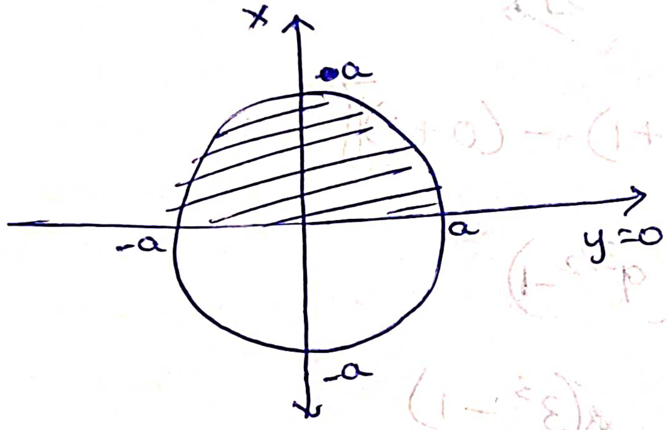
$-a \le x \le a$ and $0 \le y \le \sqrt{a^2-x^2}$

$x=-a$ & $x=a$ $\qquad$ $y=0$ & $y=\sqrt{a^2-x^2}$

$y^2 = a^2 - x^2$

$x^2+y^2=a^2$

Sketching the region:



for polar co-ordinates,

$x = r\cos\theta$ & $y = r\sin\theta$.

therefore, $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$.

$= r^2(\cos^2\theta + \sin^2\theta)$

$x^2+y^2 = r^2 \rightarrow (1)$ $\qquad$ $\{\because \cos^2\theta + \sin^2\theta = 1\}$

for polar limits

$0 \le r \le a$ & $0 \le \theta \le \pi$.

$$\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2} \, dy\,dx = \int_{0}^{\pi} \int_{0}^{a} (r^2)^{3/2} \, r \, dr \, d\theta.$$

{from (1)}

$$= \int_{0}^{\pi} \int_{0}^{a} r^4 \, dr \, d\theta.$$

Integrate with respect to 'r'

$$= \int_0^{\pi} \left[\frac{r^5}{5}\right]_0^a d\theta.$$

$$= \frac{1}{5} \int_0^{\pi} \left[r^5\right]_0^a$$

$$= \frac{1}{5} \int_0^{\pi} \left[a^5 - a^0\right] d\theta.$$

Integrate with respect to θ.

$$= \cancel{\tfrac{1}{5}}$$

$$= \frac{1}{5} \left[a^5 . \theta\right]_0^{\pi}.$$

$$= \frac{1}{5} \left[a^5 (\pi) - \frac{a^5}{5} 0\right]$$

$$= \frac{a^5 \pi}{5}$$

Hence, $\displaystyle\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2} \, dy \, dx = \frac{\pi a^5}{5}$

B) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

x limits : $0 \leq x \leq 2$

$x = 0 , \ x = 2$

y limits : $0 \leq y \leq \sqrt{2x-x^2}$

$y = 0$ & $y^2 = 2x - x^2$

$x^2 - 2x + y^2 = 0$

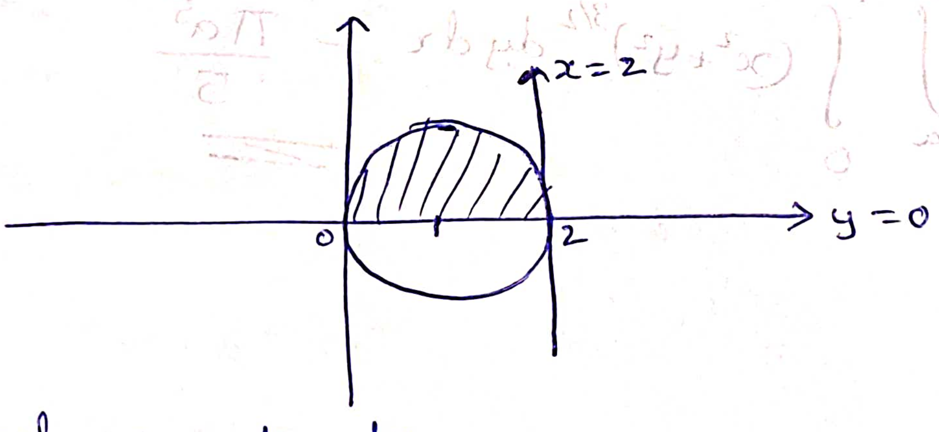
$x^2 - 2x + y^2 + 1 = 1$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

the above eqn gives the equation of circle with origin at (0, 1) and with radius = 1

Sketching the region:



for polar coordinates,

$x = r\cos\theta \qquad y = r\sin\theta$

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$x^2 + y^2 = r^2$

The polar limits are:

$0 \leq r \leq 2\cos\theta$ & $0 \leq \theta \leq \pi/2$.

{from the figure}

the integral can be written as :

$$\cancel{\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta \cdot (x^2+y^2)^{1/2}}.$$

$$\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta. = \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2\cos\theta}.$$

$$= \int_0^{\pi/2} \frac{8\cos^3\theta}{3}$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta$$

$$\left\{\begin{aligned} \cos^3\theta &= \cos^2\theta \cdot \cos\theta. \\ \cos^3\theta &= (1-\sin^2\theta)\cdot\cos\theta \end{aligned}\right\}$$

Let $\sin\theta = u$

$du = \cos\theta\, d\theta.$

So, new limits ~~are~~ for 'u' are:

when $\theta = 0 \Rightarrow \sin(0) = 0$

$\theta = \pi/2 \Rightarrow \sin(\pi/2) = 1$

Substitute the values

$$\frac{8}{3}\int_0^1 (1-u^2)\,du = \frac{8}{3}\left[u - \frac{u^3}{3}\right]_0^1$$

$$= \frac{8}{3}\left[\left(1-\frac{1}{3}\right) - 0\right]$$

$$= \frac{8}{3} \times \frac{2}{3} = \frac{16}{9}.$$

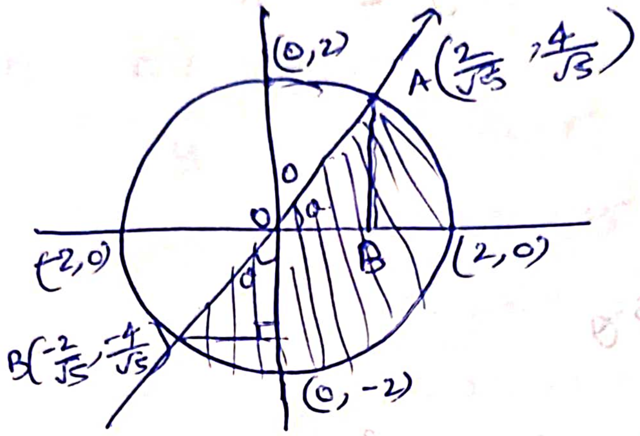
hence, $$\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \frac{16}{9}$$

⑤ $\iint_R (x+y)\,dA$ the upper region is bounded by

circle $\Rightarrow x^2+y^2=4 \Rightarrow ①$ &

line $\Rightarrow y=2x \Rightarrow ②$

The circle has a radius of 2.



to find the point where the circle & line intersects,

Substitute eq ② in eq ①

$$x^2+(2x)^2=4$$

$$x^2+4x^2=4$$

$$5x^2=4$$

$$x^2=\frac{4}{5} \Rightarrow x=\pm\frac{2}{\sqrt{5}}$$

Sub $x=\pm\frac{2}{\sqrt{5}}$ in ②

$$y=2\left(\pm\frac{2}{\sqrt{5}}\right) \Rightarrow y=\pm\frac{4}{\sqrt{5}}$$

A) Iterated integral by using cartesian co-ordinates

$$\int_{-2/\sqrt{5}}^{2/\sqrt{5}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x+y)\,dy\,dx$$

B) for polar co-ordinates :

$x = r\cos\theta \qquad y = r\sin\theta$

In ~~△ △~~ $\triangle OAB$,

$$\cos\theta = \frac{OB}{OA} = \frac{2/\sqrt{5}}{2} = \frac{1}{\sqrt{5}}$$

$$\cos\theta = \frac{1}{\sqrt{5}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$\theta = 63.43^{\circ}$$

$$= 63.43 \times \frac{\pi}{180}$$

$$\Rightarrow \theta_0 = 0.3523\,\pi.$$

Now, the integral is given by :

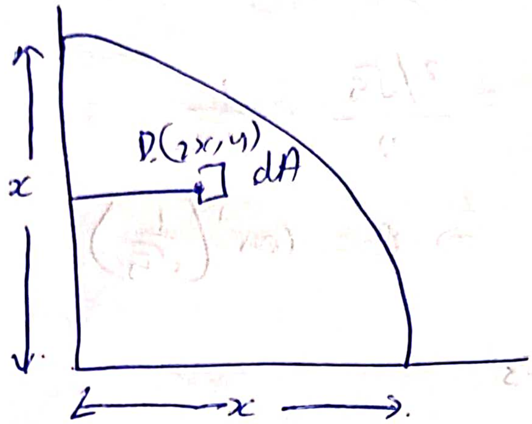
$$\int_{\cos^{-1}(1/\sqrt{5})}^{\cos^{-1}(-2/\sqrt{5})} \int_0^2 (r\cos\theta + r\sin\theta)\, r \, dr\, d\theta.$$

Hence, the iterated integral by using polar co-ordinates.

$$\int_{\cos^{-1}(1/\sqrt{5})}^{\cos^{-1}(-2/\sqrt{5})} \int_0^2 r^2(\cos\theta + \sin\theta)\, dr\, d\theta.$$

⑥ the radius of quater-circle $= 2m$.

the mass density $D = 2x$



Mass density of plate at given point

$D(x,y) = 2x \ kg/m^2$

Mass of plate $= \iint D(x,y)\,dA$

$= \iint 2x\,dA$

for polar co-ordinates.

$x = r\cos\theta$; $y = r\sin\theta$, $d\theta = r\,dr\,d\theta$.

Region (D) :

$0 \le r \le 2$

$0 \le \theta \le \pi/2$

$$M = \int_0^{\pi/2}\int_0^{2} 2(r\cos\theta)\, r\,dr\,d\theta.$$

$$= \int_0^{\pi/2} \left[2\cos\theta \left(\frac{r^3}{3}\right)_0^2\right] d\theta.$$

$$= \int_0^{\pi/2} 2\left(\frac{2^3}{3}\right) d\theta.$$

$$M = \frac{16}{3}(\sin\theta)_{0}^{\pi/2}$$

$$M = \frac{16}{3}$$

Hence, the Mass of the plate is $\frac{16}{3}$