Homework - 10

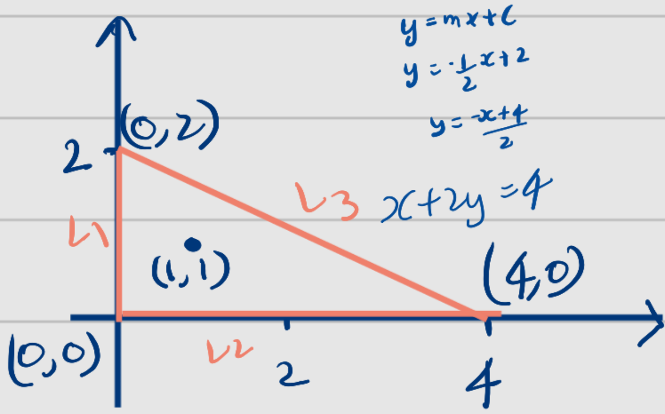
① given, $f(x,y) = x+y-xy$

with vertices $(0,0)$, $(0,2)$ & $(4,0)$

$f_x = 1-y$ & $f_y = 1-x$

$f_x = 0$ $\quad\quad$ $f_y = 0$

$\boxed{y=1}$ $\quad\quad$ $\boxed{x=1}$



Hence, the critical point is $(1,1)$

$f(1,1) = 1+1-1 = 1$

on L_1:- $f(x,y) = f(0,y) = y$; $0 \le y \le 2$

Max Value $= 2$ at $y=2$ $\quad\quad$ Min Value $= 0$ at $y=0$

on L_2:- $f(x,y) = f(x,0) = x$; $0 \le x \le 4$

Max value $= 4$ at $x=4$ $\quad\quad$ Min value $= 0$ at $x=0$

on L_3:- $x+2y=4$; $0 \le y \le 2$

$f(x,y) = (4-2y)+y-(4-2y)y = 4-y-4y+2y^2 = 2y^2-5y+4$

$f'(y) = 0 \Rightarrow 4y-5=0 \Rightarrow y = \frac{5}{4}$

$f\left(\frac{5}{4}\right) = 2\left(\frac{5}{4}\right)^2 - 5\left(\frac{5}{4}\right)+4 = 2\left(\frac{25}{16}\right) - \frac{25}{4} + 4 = \frac{7}{8}$

$f(0) = 2(0)-5(0)+4 = 4 \rightarrow$ Max value

$f(2) = 2(2)^2-5(2)+4 = 2 \rightarrow$ Min value

Hence, the minimum value is 0

the maximum value is 4

② Given, joint cost function $C(x, y) = 2x^2 + xy + y^2 + 500$

production constraint $g(x, y) \Rightarrow x + y = 200$

the lagrangian function is given by:

$$L(x, y, \lambda) = C(x, y) - \lambda(g(x, y) - 200)$$

$$L(x, y, \lambda) = 2x^2 + xy + y^2 + 500 - \lambda(x + y - 200)$$

partial derivatives of L: $\partial L/\partial x = 4x + y + \lambda \rightarrow ①$

$\partial L/\partial y = x + 2y + \lambda \rightarrow ②$

$\partial L/\partial \lambda = x + y - 200 \rightarrow ③$

from eq ① & ② $\Rightarrow$ $4x + y + \lambda = 0$

$$\frac{\begin{array}{r} 4x + y + \lambda = 0 \\ \underset{(-)}{x} \underset{(-)}{+2y} \underset{(-)}{+\lambda} = 0 \end{array}}{}$$

$$3x - y = 0$$

$$3x = y \Rightarrow x = y/3$$

Substitute $x = y/3$ in $g(x, y)$:

$$\frac{y}{3} + y = 200 \Rightarrow 4y = 600 \Rightarrow \boxed{y = 150}$$

$$\Rightarrow x = \frac{y}{3} \Rightarrow x = \frac{150}{3} \Rightarrow \boxed{x = 50}$$

$$C(x, y) = 2x^2 + xy + y^2 + 500$$

$$C(50, 150) = 2(50)^2 + 50(150) + (150)^2 + 500$$

$$= 35500$$

therefore, the minimum production cost is \$35,500 which occurs at $x = 50$ and $y = 150$.

3 Given Lagrangian function: $L(x,\lambda) = f(x) - \lambda(g(x))$

a) To show that (x_0, λ_0) is a critical point of L.

The lagrangian function $C(x, y)$ is given by:

$$C(x,\lambda) = f(x) - \lambda g(x)$$

Taking the partial derivatives of C with respect to x & λ and setting them equal to zero.

$$\frac{\partial C}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}}$$

The gradient of f at the optimal point is parallel to the gradient of 'g' at that point.

$$\frac{\partial C}{\partial \lambda} = -g(x) = 0 \Rightarrow \boxed{-g(x) = 0}$$

This implies that the constraint $g(x)$ is satisfied at the optimal point.

for point (x_0, λ_0),

$$\frac{\partial f(x_0)}{\partial x} = \lambda \frac{\partial g}{\partial x_0} \quad \& \quad g(x_0) = 0$$

Hence, (x_0, λ_0) is a critical point of L.

b)

* the Lagrangian method is specifically designed to handle optimization problems with equality constraints like $g(x) = 0$.
* By introducting lagrange multiplier, the lagrangian method essentially converts the constrained optimization problem into an unconstrained problem with one or more variable (λ)
* the lagrangian multiplier λ_0 at the optimal solution provides valuable information about the sensitivity of the objective function to changes in the constraints.
* when dealing with complex problems with multiple constraints, the lagrangian method can be more efficient because it avoids the need to explicitly solve the constraint equations, which can be computationally expensive.

④ given $f(x,y) = e^{x^2+x+y^2}$

i) Gradient of f at origin:

gradient is given by: $\nabla f = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix}$

$\frac{\partial f}{\partial x} = (2x+1).e^{x^2+x+y^2}$ $\qquad$ $\frac{\partial f}{\partial y} = 2y.e^{x^2+x+y^2}$

At (0,0): $\nabla f(0,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Hessian of f at origin: $H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$f_{xx} = (2x+1)^2 e^{x^2+x+y^2} + 2.e^{x^2+x+y^2}$

$= (4x^2+1+4x+2)\, e^{x^2+x+y^2}$

$= (4x^2+4x+3)\, e^{x^2+x+y^2}$

$f_{yy} = (2+4y^2)\, e^{x^2+x+y^2}$

$f_{xy} = f_{yx} = (2x+1).(2y).e^{x^2+x+y^2}$

At (0,0) $\Rightarrow$ $f_{xx} = 1+2 = 3$

$f_{yy} = 2$

$f_{xy} = f_{yx} = 0$

Hence,

$\Rightarrow \quad H_f = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

ii) Quadratic Approximation using Taylor Polynomials:

$$f(x) \cong f(0) + \nabla f(0)^T \cdot x + \frac{1}{2} x^T H_f(0) \cdot x$$

$$f(0,0) = e^0 = 1$$

$$f(x,y) = 1 + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 1 + x + \frac{1}{2} \begin{bmatrix} 3x & 2y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 1 + x + \frac{1}{2} (3x^2 + 2y^2)$$

$$f(x,y) = \frac{3}{2} x^2 + y^2 + x + 1$$

iii) Min Value of 'f' near origin:

$f_x = 3x + 1$ & $f_y = 2y$

$f_x = 0$ $f_y = 0$

$\boxed{x = -1/3}$ $\boxed{y = 0}$

Hence, the minimum value of f is $\left(\frac{-1}{3}, 0\right)$.

iv) Since the eigen values are 3 & 2 which are positive, it means that the Hessian matrix of the given function is positive definite. This, in turn, indicates that the function is convex in the vicinity at the origin.

Lagrange Multipliers

In [1]:
```
import matplotlib.pyplot as plt
import numpy as np
```

The case study:

- Suppose we have a fixed budget for speding on marketing of 2500 dollar.
- Suppose we can choose to invest in two types of campaigns: Social Media and TV, where you have to decide.
- To simplify, let's say that one campaign on social media costs 25 dollar and one campaign on TV costs 250 dollar.
- Suppose we have experimented a lot in the past and that we have been able to define the Revenues as a function of the two types of media investments.

In this notebook, we will find the maximum revenue and the number of different types of campaigns one should buy using Lagrange Multiplier.

In [2]:
```
cost_social = 25
cost_tv = 250
budget = 2500
```

The equation for costs: 25 dollars times the number of social campains + 250 times the number of tv campaigns Since we want to spent exactly the budget we know that this is equal to 2500 dollar.

Otherwise, the constraint for our revenue optimization is: ***25* social + 250 *tv* = *2500*** -> Equation 1

In [3]:
```
#lets get the minimum and maximum number of campaigns:
social_min = 0
social_max = budget / cost_social

tv_min = 0
tv_max = budget / cost_tv
```

Task 1 - Solve for the variable 'social' in Equation 1 and return the value in the function.

In [4]:
```
# if we fix the number of tv campaings, we know the number of social campaigns left to buy by inver
def n_social(n_tv, budget):
    return (budget - cost_tv * n_tv) / cost_social
```

Task 2 - Solve for the variable 'tv' in Equation 1 and return the value in the function.

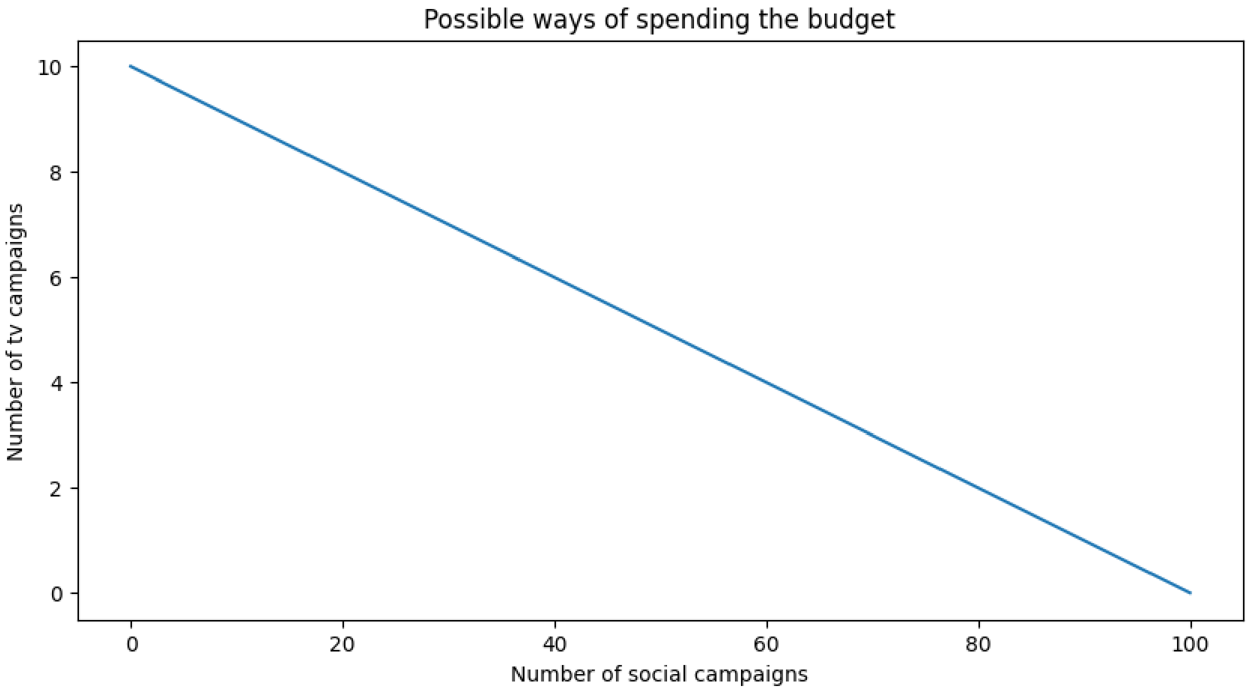
In [5]:
```
# if we fix the number of social campaings, we know the number of tv campaigns left to buy by inver
def n_tv(n_social, budget):
    return (budget - cost_social * n_social) / cost_tv
```

Task 3 a) - Set the social_x array using the linspace function. Ranging from 'social_min' to 'social_max' with 100 points in between.

Task 3 b) - Set the value 'tv_y' - number of TV campaigns required, by calling the 'n_tv' function and passing the parameters 'social_x' and 'budget' to it.

In [8]:
```
num_points=100
social_x = np.linspace(social_min, social_max, num_points)
tv_y = n_tv(social_x, budget)
```

```python
plt.figure(figsize=(10,5))
plt.plot(social_x, tv_y)
plt.xlabel('Number of social campaigns')
plt.ylabel('Number of tv campaigns')
plt.title('Possible ways of spending the budget')
plt.show()
```



Suppose that through eperimentation and analysis, somebody has been able to identify the revenue curve for your business and that it is defined as:

Revenue = 7 times the number of social campaigns to the power 3/4 times the number of tv campaigns to the power 1/4.

Task 4 - The function should return the value of the revenue curve.

In [30]:
```python
def revenues(social, tv):
    return 7 * (social ** (3/4)) * (tv ** (1/4))
```

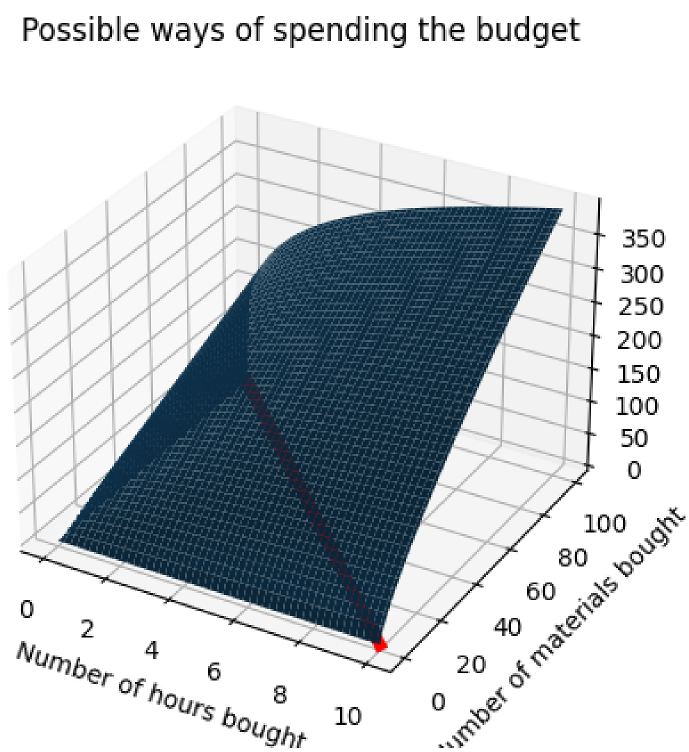
In [31]:
```python
from mpl_toolkits.mplot3d import Axes3D
social_axis = np.linspace(social_min, social_max, 100)
tv_axis = np.linspace(tv_min, tv_max, 100)
social_grid, tv_grid = np.meshgrid(social_axis, tv_axis)

fig = plt.figure()
ax = fig.add_subplot(projection = '3d')

ax.plot_surface(tv_grid, social_grid, revenues(social_grid, tv_grid))

ax.plot(tv_y, social_x, linewidth = 5, color = 'r')

ax.set_xlabel('Number of hours bought')
ax.set_ylabel('Number of materials bought')
ax.set_title('Possible ways of spending the budget')
plt.show()
```



The goal is to idenify the highest point on the 3D curve that is exactly on the constraint line.

We need to find the point where the revenue contour is tangent to the constraint line. The method we use for this is Lagrange Multiplier.

We can find the maximum at the point where the gradient of the Revenue contour is proportional to the gradient of the constraint line.

“gradient of revenues” = lambda times “gradient of constraint”

Since there are two variables in each, we need two partial derivatives to get a vector of those two derivatives.

Revenue Function

$$\mathrm{R(s,t)} = s^{3/4} * t^{\frac{1}{4}} * 7$$

Partial Derivative wrt s

$$\frac{\partial R}{\partial s} = \frac{3}{4} s^{-1/4} * t^{\frac{1}{4}} * 7 = \frac{21}{4} * \frac{t^{\frac{1}{4}}}{s^{1/4}}$$

Partial Derivative wrt t

$$\frac{\partial R}{\partial t} = s^{3/4} * \frac{1}{4} t^{-3/4} * 7 = \frac{7}{4} \frac{s^{3/4}}{t^{-3/4}}$$

System to pass to solver:

First Equation for Solver

$$\frac{21}{4} * \frac{t^{\frac{1}{4}}}{s^{1/4}} = 25 * lambda$$

Second Equation for Solver

$$\frac{7}{4} \frac{s^{3/4}}{t^{-3/4}} = 250 * lambda$$

Third Equation for Solver

$$25 * \mathrm{s} + 250 * \mathrm{t} = 2500$$

Constraint

$$\text{Constraint } \mathrm{C(s,t)}: 25 * \mathrm{s} + 250 * \mathrm{t} = 2500$$

Partial Derivative wrt s

$$\frac{\partial C}{\partial s} = 25$$

Partial Derivative wrt t

$$\frac{\partial C}{\partial t} = 250$$

We pass these equations in a Python solver.

```
In [32]: from sympy import *

         s, t, l = symbols('s t l')

         solution = solve([Eq((21/4)*((t**(1/4))/s**(1/4)) - 25*l, 0),
            Eq((7/4)*(s**(3/4)/t**(3/4)) - 250*l, 0),
            Eq(25*s+250*t - 2500, 0)], [s,t,l], simplify=False)
```

TASK 5 - Take the first and the second value of the array returned above and pass them as parameters in the function mentioned below.

```
In [36]: s_value = solution[0][0]
         t_value = solution[0][1]
         lagrange_multiplier = solution[0][2]

         optimal_revenue = revenues(s_value, t_value) #TD
         print(f"Optimal number of social campaigns: {round(s_value.evalf(),2)}")
         print(f"Optimal number of TV campaigns: {round(t_value.evalf(),2)}")
         print(f"Revenue at the optimal point: {round(optimal_revenue.evalf(),2)}")
```

```
Optimal number of social campaigns: 75.0000000000000
Optimal number of TV campaigns: 2.50
Revenue at the optimal point: 224.33
```