Pre-class Assignment - 22

Reading:

① The objective function mathematically represents the quantity to optimize, mapping variables to a real number. For ex: maximizing profit in an economic model.

The constraints define limitations or conditions the optimization must satisfy.

A constrained optimization aims to find the variable values that optimize the objective while satisfying constraints.

② By Lagrange Multipliers

$$\nabla f = \lambda \nabla g$$

$\lambda \rightarrow$ marginal utility of the constraint. Suppose the gpa is function of studying maths and economic $g(m,e)$ and the constraint is time (t) i.e $m+e=t$.

Now $L = g(m,e) + \lambda(t-m-e)$.

In this case, λ will represent the marginal utility of t, i.e additional points in gpa that could be gained by increasing t by an hour.

Exercises:

① Based on the image provided, the level curves of the Cobb-Douglas production function $f(x,y) = x^{2/3} y^{1/3}$ are convex to origin.

this indicates that $f(x,y)$ exhibits increasing returns to scale.

this relates to the method of Lagrange multipliers because we can set up the following Lagrangian function to find this maximum production level analytically:

$$L(x, y, \lambda) = f(x,y) + \lambda(3.78 - x - y)$$

from the method of Lagrange multipliers,

$$\nabla f = \lambda \nabla g$$

$$f_x = \lambda g_x = \frac{2}{3} x^{-1/3} y^{1/3} \cdot \lambda \qquad f_y = \lambda g_y = \frac{1}{3} x^{2/3} y^{-2/3} \cdot \lambda$$

$$\cancel{\lambda} \cdot \frac{2}{\cancel{3}} x^{-1/3} y^{1/3} = \frac{1}{\cancel{3}} x^{2/3} y^{-2/3} \cdot \cancel{\lambda}$$

On simplifying, $2Y = X$

from, $x + y = 3.78$

$$2y + y = 3.78 \Rightarrow 3y = 3.78 \Rightarrow y = 1.26$$

hence, $x = 2(1.26) \Rightarrow x = 2.52$

The budget constraint line $x+y=3.78$ intersects the highest achievable level curve at a point approximately given by $(x,y) = (2.52, 1.26)$.

This indicates that the maximum possible production level, given the constraint $x+y <= 3.78$ occurs when $x = 2.52$ & $y = 1.26$

(2) Max & min values of $f(x,y) = x+y$

Constraint function $g(x,y) = x^2+y^2$

For Lagrange multiplication take partial derivatives

$$f_x = 1 \qquad f_y = 1$$

$$g_x = 2x \qquad g_y = 2y$$

lagrange multiplier eqns are:

$f_x = \lambda g_x \Rightarrow 1 = \lambda 2x \rightarrow (1)$ & $f_y = \lambda g_y \Rightarrow 1 = \lambda 2y \rightarrow (2)$

& Constraint $\Rightarrow x^2+y^2 = 4 \rightarrow (3)$

Divide eq (1) & eq (2), we get

$$\frac{1}{1} = \frac{\lambda 2x}{\lambda 2y} \Rightarrow \frac{x}{y} = 1$$

$$\Rightarrow x = y \rightarrow (4)$$

from eq ③ & ④ $\Rightarrow x^2+x^2=4$

$$2x^2=4$$

$$x=\pm\sqrt{2}$$

Since $x=y$, the solns $(x,y)=(\sqrt{2},\sqrt{2})$

& $(x,y)=(-\sqrt{2},-\sqrt{2})$

Substituting $(x,y)=(\sqrt{2},\sqrt{2})$ in $f(x,y)$:

$$f(x,y)_{(\sqrt{2},\sqrt{2})}=\sqrt{2}+\sqrt{2}=2\sqrt{2}$$

& for point $(x,y)=(-\sqrt{2},-\sqrt{2})$

$$f(x,y)_{(-\sqrt{2},-\sqrt{2})}=-\sqrt{2}-\sqrt{2}=-2\sqrt{2}$$

Hence, Max value of f on $x^2+y^2=4$ is $2\sqrt{2}$ at $(\sqrt{2},\sqrt{2})$

& Min value of f on $x^2+y^2=4$ is $-2\sqrt{2}$ at $(-\sqrt{2},-\sqrt{2})$

③ given, $f(x,y) = x^2 + y^2$

Constraint $xy = 1$

Let $g(x,y) = xy$

The Lagrange equations are: $2x = \lambda y$

$2y = \lambda x$

$xy = 1$

Multiplying the corresponding sides of the 1st two eqn gives: $4xy = \lambda^2 xy$

Since, $xy = 1 \Rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$

If, $\lambda = -2 \Rightarrow x = -y$ which is impossible

Hence, $\lambda = 2 \Rightarrow x = y$.

Using $xy = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 = y$

observe that as $x \to \infty$ & $xy = 1$

$$f(x,y) = f\left(x, \frac{1}{x}\right) = x^2 + \frac{1}{x^2} \longrightarrow \infty$$

So, 'f' does not have a maximum under the constraint $xy = 1$. therefore, 'f' has a minimum at $(\pm 1, \pm 1)$

i.e, $f(\pm 1, \pm 1) = 2$

(4) given, $f(x,y,z) = xy^2z^3$ & $g(x,y,z) = x^2+y^2+z^2 \leq 1$

the gradient of $f(x,y,z)$ & $g(x,y,z)$ are:

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= y^2z^3\,\hat{i} + 2xyz^3\,\hat{j} + 3xy^2z^2\,\hat{k}$$

$$= \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$$

$$\nabla g(x,y,z) = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$$

$$= \langle 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \rangle$$

$$= \langle 2x, 2y, 2z \rangle$$

Consider the System,

$$\nabla f = \lambda \nabla g$$

$$y^2z^3 = 2x\lambda \rightarrow (1) \Rightarrow xy^2z^3 = 2x^2\lambda \rightarrow (5)$$

$$2xyz^3 = 2y\lambda \rightarrow (2) \Rightarrow 2xy^2z^3 = 2y^2\lambda \rightarrow (6)$$

$$3xy^2z^2 = 2z\lambda \rightarrow (3) \Rightarrow 3xy^2z^3 = 2z^2\lambda \rightarrow (7)$$

& $x^2+y^2+z^2 = 1 \rightarrow (4)$

Add eq (5), (6), (7) $\Rightarrow 6xy^2z^3 = 2\lambda(x^2+y^2+z^2)$

$$6xy^2z^3 = 2\lambda(1)$$

$$6x(2x\lambda) = 2\lambda \quad \{\text{from } (1)\}$$

$$12x^2 = 2 \Rightarrow x^2 = \frac{1}{6}$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{6}}$$

Similarly, $2\left(\frac{1}{\sqrt{6}}\right) y^2 z^3 = 2y^2$

$$2\left(\frac{1}{\sqrt{6}}\right) 2\left(\frac{1}{\sqrt{6}}\right)\lambda = 2y^2\lambda$$

$$\frac{2\cancel{4}}{\cancel{6}3} = 2y^2 \Rightarrow y^2 = \frac{1}{3}$$

$$y = \pm \frac{1}{\sqrt{3}}$$

Similarly, $3xy^2z^2 = 2z\lambda$

$$3\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{3}\right) z = 2$$

$$z = 2\sqrt{6}$$

Max value of $f(x,y,z) \Rightarrow x = \frac{1}{\sqrt{6}}, y = \frac{1}{\sqrt{3}}, z = 2\sqrt{6}$

$$\Rightarrow f(x,y,z) = xy^2z^3 = \frac{1}{\sqrt{6}}\left(\frac{1}{3}\right)\left(2\sqrt{6}\right)^3$$

Min value of $f(x,y,z) \Rightarrow x = -\frac{1}{\sqrt{6}}, y = -\frac{1}{\sqrt{3}}, z = -2\sqrt{6}$

$$\Rightarrow f(x,y,z) = xy^2z^3 = \left(\frac{-1}{\sqrt{6}}\right)\left(-\frac{1}{3}\right)\left(-2\sqrt{6}\right)^3$$

5 The lagrangian is given by:

$$L(x_1, x_2, \dots x_n) = x_1 + x_2 + \dots + x_n - \lambda(x_1^2 + x_2^2 + \dots + x_{n-1}^2)$$

taking partial derivatives

$$\frac{\partial L}{\partial x_1} = 1 - 2\lambda x_1 = 0 \rightarrow (1)$$

$$\frac{\partial L}{\partial x_2} = 1 - 2\lambda x_2 = 0 \rightarrow (2)$$

$$\vdots \qquad \vdots$$

$$\frac{\partial L}{\partial x_n} = 1 - 2\lambda x_n = 0 \rightarrow (n)$$

$$\frac{\partial L}{\partial \lambda} = -(x_1^2 + x_2^2 + \dots + x_{n-1}^2) = 0$$

Solving above eqn we get

$$2\lambda x_i = 1 \Rightarrow x_i = \frac{1}{2\lambda} \rightarrow (1) \ ; \quad i = 1, 2, 3, \dots, n$$

Substitute $x_i = \frac{1}{2}\lambda$ in constant eqn

$$(1/2\lambda)^2 + (1/2\lambda)^2 + \dots + (1/2\lambda)^2 = 1$$

$$n/4\lambda^2 = 1 \Rightarrow \lambda^2 = n/4 \Rightarrow \lambda = \sqrt{n}/2$$

Sub in (1) $\quad x_i = \frac{1}{2 \times \frac{\sqrt{n}}{2}} = \sqrt{1/n} = 1/\sqrt{n}$

Substitute in $f(x_1, x_2, \dots x_n)$

$$f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$$

$$= \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \dots + \frac{n}{\sqrt{n}} = \sqrt{n}$$

Hence, Max value of $f(x_1, x_2, \dots x_n) = \sqrt{n}$
when $x_i = 1/\sqrt{n}$ for $i = 1, 2, \dots, n$