① To find derivative of the inverse of a linear function $f(x) = ax+b$

$y = f(x) \Rightarrow x = f^{-1}(y)$

$y = ax+b.$

$ax = y-b$

$x = \frac{y-b}{a}.$

$f^{-1}(x) = \frac{x-b}{a}.$

Let $f^{-1}(x) = g(x)$

$\Rightarrow g(x) = \frac{x-b}{a}$

$\frac{d}{dx}[g(x)] = \frac{d}{dx}\left[\frac{x-b}{a}\right]$

$g'(x) = \frac{1}{a}\frac{d}{dx}\left(\frac{x-b}{}\right)$

$g'(x) = \frac{1}{a}(1-0)$

$= \frac{1}{a}$

$\therefore g'(x) = \frac{1}{a}$

The derivative of the inverse of the function $f(x) = ax+b$ is $\frac{1}{a}$

(2) To find the derivative of $\ln(x)$, $\ln(2x)$, $\ln(3x)$

we know, that limit definition is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

let $y = f(x) = \ln(x)$

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h}.$$

Since, $\left\{ \ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right) \right\}$

$$f'(x) = \lim_{h\to 0} \left( \frac{\ln\left(\frac{x+h}{x}\right)}{h} \right) = \lim_{h\to 0} \left( \frac{\ln\left(1+h/x\right)}{h} \right)$$

let, $\frac{h}{x} = z \Rightarrow h = zx$

$$f'(x) = \lim_{h\to 0} \frac{\ln(1+z)}{zx}$$

Now, Applying L'hopital's rule, {diff. w.r.t $z$}

$$f'(x) = \lim_{z\to 0} \frac{\frac{d}{dz}\left[\ln(1+z)\right]}{\frac{d}{dz}\left[(zx)\right]}$$

$$= \lim_{z\to 0} \frac{\frac{1}{1+z}}{x} = \lim_{z\to 0} \frac{1}{1+z}\left(\frac{1}{x}\right)$$

when $z$ approaches '0'

$$f'(x) = \frac{1}{1} \cdot \frac{1}{x}$$

$$f'(x) = \frac{1}{x} \rightarrow (1)$$

$\therefore$ The Derivative of $\ln(x) = \frac{1}{x}$

For, $\ln(2x)$ can be written as $\ln(2) + \ln(x)$

$$\ln(2x) = \ln(2) + \ln(x)$$

Differentiate w.r.t 'x' on both side.

$$\frac{d}{dx}(\ln(2x)) = \frac{d}{dx}(\ln(2)) + \frac{d}{dx}(\ln(x))$$

$$= 0 + \frac{1}{x} \quad \{\text{from eq } 1\}$$

$$\Rightarrow \frac{d}{dx}(\ln(2x)) = \frac{1}{x}$$

$\ln(3x)$ can be written as, $\ln(3x) = \ln(3) + \ln(x)$

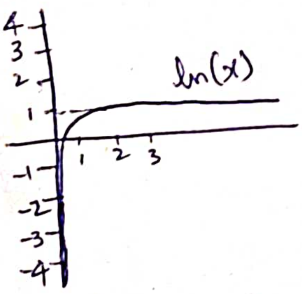
Differentiate w.r.t 'x' on both sides.

$$\frac{d}{dx}(\ln(3x)) = \frac{d}{dx}(\ln(3)) + \frac{d}{dx}(\ln(x))$$

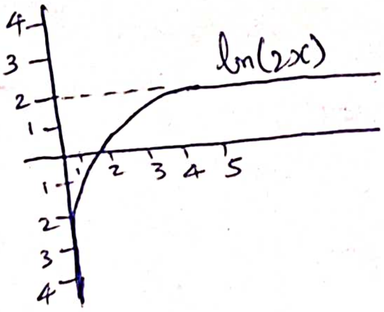
$$= 0 + \frac{1}{x} \quad \{\text{from eq } 1\}$$

$$\Rightarrow \frac{d}{dx}(\ln(3x)) = \frac{1}{x}$$

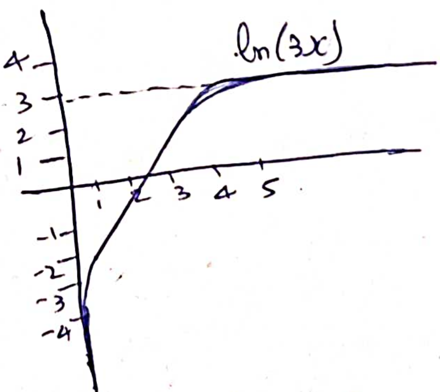
graph of $\ln(x)$




graph of $\ln(2x)$



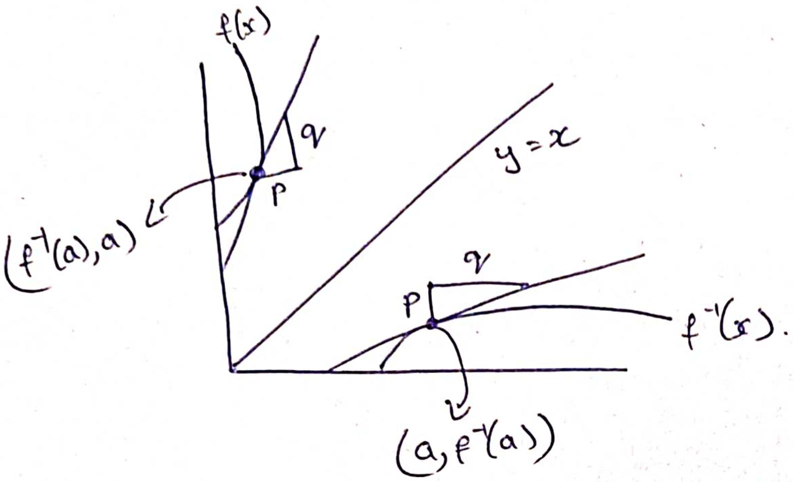

graph of $\ln(3x)$




from the above graphs its clear that All the curves derivative is equal to $\frac{1}{x}$. there are changes in the y-co-ordinates for each curve.

③ given, $f(x)$ is invertible and differentiable $\forall x$

$f^{-1}(x)$ is the inverse of $f(x)$ & is differentiable $\forall x$.




From the above figure, The relationship between a function $f(x)$ and its inverse $f^{-1}(x)$ is shown clearly.

The point $(a, f^{-1}(a))$ on the graph of $f^{-1}(x)$ having a tangent line with a slope of $(f^{-1})'a = \frac{p}{q}$.

Similarly this point corresponds to a point $(f^{-1}(a), a)$ on the $f(x)$ graph having a tangent line with a slope of $f'(f^{-1}(a)) = \frac{q}{p}$.

therefore, if $f^{-1}(x)$ is diffentiable at 'a' then

$$(f^{-1})'a = \frac{1}{f'(f^{-1}(a))}$$

Now, let $y = f^{-1}(x)$ be the inverse of $f(x)$

$\forall x$ satisfying $f'(f^{-1}(x)) \neq 0$

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x))$$

$$\frac{dy}{dx} = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}}$$

$\therefore$ Yes, it follows ~~the~~ from the inverse theorem

④ Use Implicit differentiation to find $\frac{dy}{dx}$

A) $x^2 - y^2 = 4$

Differentiate on both sides w.r.t 'x'

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = 0.$$

$$2x - 2y\frac{dy}{dx} = 0.$$

$$\cancel{-}2y\frac{dy}{dx} = \cancel{-}2x$$

$$\frac{dy}{dx} = \frac{2x}{2y}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{x}{y}}$$

B) $x^2y = y - 7$

Diff. on both sides w.r.t 'x'

$$\frac{d}{dx}(x^2y) = \frac{d}{dx}(y) - \frac{d}{dx}(7)$$

from chain rule,

$$2xy + x^2\frac{dy}{dx} = \frac{dy}{dx} - 0.$$

$$x^2\frac{dy}{dx} - \frac{dy}{dx} = -2xy$$

$$\frac{dy}{dx}(x^2 - 1) = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 - 1}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2xy}{1 - x^2}}$$

c) $e^{x/y} = x$

diff. w.r.t 'x'

$$\frac{d}{dx}(e^{x/y}) = \frac{d}{dx}(x)$$

$$e^{x/y} \cdot \frac{d}{dx}\left(\frac{x}{y}\right) = 1$$

from quotient rule,

$$e^{x/y} \cdot \left(\frac{y(1) - x \cdot \frac{dy}{dx}}{y^2}\right) = 1$$

$$e^{x/y} \cdot \left(y - x \cdot \frac{dy}{dx}\right) = y^2$$

$$\cancel{y - x\frac{dy}{dx} = \frac{y^2}{e^{x/y}}}$$

$$e^{x/y} \cdot y - e^{x/y} \cdot x \frac{dy}{dx} = y^2$$

$$-e^{x/y} \cdot x \frac{dy}{dx} = y^2 - e^{x/y} \cdot y$$

$$\frac{dy}{dx} = \frac{-(y^2 - e^{x/y}(y))}{x \cdot e^{x/y}}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{ye^{x/y} - y^2}{x \cdot e^{x/y}}}$$

D) $y^3 - \ln(x^2 y) = 1$

diff w.r.t $x$

$\cancel{3y^2}\ \frac{d}{dx}(y^3) - \frac{d}{dx}(\ln(x^2 y)) = \frac{d}{dx}(1)$

$3y^2 \frac{dy}{dx} - \frac{1}{x^2 y}\left(2xy + x^2 \frac{dy}{dx}\right) = 0$

$3y^2 y' - \frac{2\cancel{x}\cancel{y}}{x^{\cancel{2}}\cancel{y}} - \frac{\cancel{x^2}}{\cancel{x^2}y} \cdot \cancel{\frac{dy}{dx}}\,y' = 0$

$3y^2 y' - \frac{2}{x} - \frac{1}{y} y' = 0$

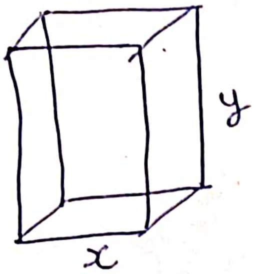
$3y^2 y' - \frac{1}{y} y' = \frac{2}{x}$

$y'\left(3y^2 - \frac{1}{y}\right) = \frac{2}{x}$

$y'\left(\frac{3y^3 - 1}{y}\right) = \frac{2}{x}$

$\Rightarrow \boxed{y' = \frac{2y}{3xy^3 - x}}$

(5) given a closed rectangular box with side length $= x$ cm. & height $= y$ cm. & Surface area $= 1000\ cm^2$




A) Surface area of Rectangular box is given by $= 2(lb + hl + hb)$

$l \rightarrow$ length of the side. $\quad l = x = b$

$b \rightarrow$ breadth or width $\quad h = y$

$h \rightarrow$ height

$$A = 2(lb + bh + hl)$$

$$\overset{500}{\cancel{1000}} = \cancel{2}(x^2 + xy + xy)$$

$$500 = x^2 + 2xy.$$

$$\boxed{x^2 + 2xy = 500} \rightarrow eq\ (1)$$

B) Rate of change of height of box when $h = 20$ cm. $= y$

Diff. eq (1) w.r.t $x$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) = \frac{d}{dx}(500)$$

$$2x + 2(xy' + y) = 0.$$

$$2x + 2xy' + 2y = 0$$

$$2(x + xy' + y) = 0$$

$$x + xy' + y = 0$$

$$xy' = -x - y$$

$$\boxed{y' = \frac{-(x+y)}{x}} \rightarrow (2)$$

Substitute $y = 20$ in eq ①

$$x^2 + 2xy - 500 = 0$$

$$x^2 + 2x(20) - 500 = 0$$

$$x^2 + 40x - 500 = 0$$

Solving for 'x'

$$x^2 + 50x - 10x - 500 = 0$$

$$x(x+50) - 10(x+50) = 0$$

$x + 50 = 0$ (or) $x - 10 = 0$ Can be the solution

$x = -50$ & $x = 10$

↓

Since 'x' has -ve value, it cannot be considered

$$\boxed{\therefore x = 10}$$

Now, Sub $x = 10$ & $y = 20$ in eq ②

$$y' = \frac{-(x+y)}{x} = \frac{-(10+20)}{10} = -3$$

$$\therefore \boxed{y' = -3}$$

The rate of change of height of the box w.r.t length of its base when height is 20 cm is '-3'.

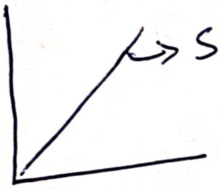
c) It makes sense to talk about the rate of change of height of the box w.r.t the length of its base because we know, the surface area can be same even with different combination of dimensions (length, height width).

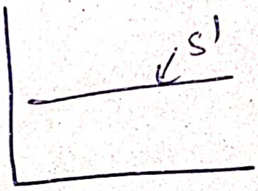
⑥ given data $S = \{x_1, x_2, \dots x_n\}$

$S' = \{y_1, y_2, \dots y_n\}$ ; $y_n = x_{n+1} - x_n$ for $1 \le n < N$

$\max\limits_{1 \le i,j < N} |y_i - y_j| < \epsilon$ for very small $\epsilon > 0$

A) From the above data, we can understand that the slope is not changing constantly b/w the points. So, It is linear below are the graphs for S & S'




~~slope~~ $y = mx + c$ ~~slope~~ $y = c$ are eqn of lines.

The curves S' is a linear function.

B) $S'' = \{z_1, z_2, \dots z_{N-2}\}$

where $z_n = y_{n+1} - y_n$ for $1 \le n < N-1$

$\max\limits_{1 \le i,j < N} |z_i - z_j| < \epsilon$ & $\epsilon > 0$

The rate of change of slope of S' is 0. this implies

S'' is a horizontal line parallel to x-axis.

the curve canbe in form $y = ax^2 + bx + c$