HW-5

3) A) given, the matrix A rotates the plane clockwise by 30°

Matrix 'A' is given by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

here $\theta = -30^\circ$ {as rotation is clockwise}

$$A = \begin{bmatrix} \cos(-30^\circ) & -\sin(-30^\circ) \\ \sin(-30^\circ) & \cos(-30^\circ) \end{bmatrix}$$

$$A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

here, $n=2$, $m=2$

Applying RREF : $R_2 \to R_2 + \frac{\sqrt{3}}{3} R_1$

$$\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 0 & 2\sqrt{3}/3. \end{bmatrix}$$

Hence, Rank $(r) = 2$

$\text{Dim}(C(A)) = r = 2$
$\text{Dim}(C(A^T)) = r = 2$
$\text{Dim}(N(A)) = n - r = 2 - 2 = 0$
$\text{Dim}(N(A^T)) = m - r = 2 - 2 = 0.$

B) given, the matrix B projects elements of the plane onto the line $x + y = 0$.

$$x = -y$$

$$\langle x \ y \rangle = \langle x \ -x \rangle = x \langle 1 \ -1 \rangle$$

span of vector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

hence, the normal vector $n = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The project matrix B is given by

$$B = \frac{n n^T}{n^T n}$$

$$B = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$B = \frac{\begin{bmatrix} 1 & -1 \\ -1 & +1 \end{bmatrix}}{(1+1)}$$

$$B = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$ here $n=2$, $m=2$

Performing RREF on B gives:

$R_1 \rightarrow R_2 + R_1$

$$\cancel{\begin{bmatrix} 1 & -1 \\ -1/2 & 1/2 \end{bmatrix}} \qquad \begin{bmatrix} 1/2 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$\cancel{R_2 \Rightarrow \frac{R_2 + R_1}{2}}$

$$\cancel{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}$$

hence, the rank of the matrix $= 1$

Columnspace dimension $= Dim(C(B)) = r = 1$

dimension of row space $= Dim(C(B^T)) = r = 1$

dimension of Null space $= Dim(N(B)) = n - r = 2-1 = 1$

dimension of left Null space $= Dim(N(B^T)) = m - r$

$= 2-1$

$= 1$

c) The matrix 'C' shears the plane horizontally

* when shearing is done, the x-axis column vector remains unaffected i.e $\begin{bmatrix}1\\0\end{bmatrix}$

* The Y-axis column vectors will ~~now~~ be changed where 'x' co-ordinated will be x + (shearing factor) and 'y' magnitude remains same ~~by given~~

$$C = \begin{bmatrix}1 & 0+0.5\\0 & 1\end{bmatrix}$$

the matrix $C = \begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}$ $n=2$, $m=2$

hence, this is already in the row echelon form,

the rank of the matrix $= 2$

$Dim(C(C)) = r = 2$ {Column Space}

$Dim(C(C^T)) = r = 2$ {Row Space}

~~Dim(N~~ $Dim(N(C)) = 2-2 = 0$ {Null Space}

$Dim(N(C^T)) = 2-2 = \underline{\underline{0}}$. {left Null space}

D) $A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$

$B = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

$ABC = (AB)C$

$$AB = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$AB = \begin{bmatrix} \frac{\sqrt{3}}{4} - \frac{1}{4} & -\frac{\sqrt{3}}{4} + \frac{1}{4} \\ \frac{-1}{4} - \frac{\sqrt{3}}{4} & \frac{1}{4} + \frac{\sqrt{3}}{4} \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} \frac{\sqrt{3}}{4} - \frac{1}{4} & -\frac{\sqrt{3}}{4} + \frac{1}{4} \\ \frac{-1}{4} - \frac{\sqrt{3}}{4} & \frac{1}{4} + \frac{\sqrt{3}}{4} \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}-1}{4} & \frac{\sqrt{3}-1}{8} - \frac{\sqrt{3}+1}{4} \\ -\frac{(\sqrt{3}+1)}{4} & -\frac{(\sqrt{3}+1)}{8} + \frac{\sqrt{3}+1}{4} \end{bmatrix}$$

$$ABC = \begin{bmatrix} \frac{\sqrt{3}-1}{4} & \frac{-\sqrt{3}+1}{8} \\ \frac{-(\sqrt{3}+1)}{4} & \frac{\sqrt{3}+1}{8} \end{bmatrix}$$ here, $m=2$

$n=2$

Now, RREF

$$R_2 \to R_2 + \left(\frac{1+\sqrt{3}}{-1+\sqrt{3}}\right) R_1$$

$$\begin{bmatrix} \frac{-1+\sqrt{3}}{4} & \frac{-\sqrt{3}+1}{8} \\ 0 & 0 \end{bmatrix}$$

the rank of matrix $(r) = 1$

$\text{Dim}(\text{column space}) = \text{Dim}(C(ABC)) = r = 1$

$\text{Dim}(\text{row space}) = \text{Dim}(C((ABC)^T)) = r = 1$

$\text{Dim}(\text{Null space}) = \text{Dim}(N(ABC)) = n - r = 2-1 = 1$

$\text{Dim}(\text{left Null space}) = \text{Dim}(N(ABC^T)) = m - r = 2-1 = \underline{\underline{1}}$

② given,

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & \alpha & 2 & 2 \\ 0 & 0 & 0 & \beta & 2 \end{bmatrix}$$ & rank = 2

$R_3 \Rightarrow R_3 - R_2$

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & \alpha & 2 & 2 \\ 0 & 0 & 0-\alpha & \beta-2 & 0 \end{bmatrix}$$

In order to have rank 2, the matrix must have all the last row elements equal to zeros.

hence, $0-\alpha=0$. & $\beta-2=0$

$\alpha=0$ & $\beta=2$.

therefore, for $\alpha=0$, $\beta=2$. the given matrix has rank = 2.

④ given,

$$H = \text{span}\left\{ \begin{bmatrix} 5 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 14 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 38 \\ 8 \\ 6 \\ 24 \end{bmatrix}, \begin{bmatrix} 47 \\ 10 \\ 7 \\ 28 \end{bmatrix}, \begin{bmatrix} 10 \\ 2 \\ 3 \\ 12 \end{bmatrix} \right\}$$

the matrix is

$$H = \begin{bmatrix} 5 & 14 & 38 & 47 & 10 \\ 1 & 3 & 8 & 10 & 2 \\ 1 & 2 & 6 & 7 & 3 \\ 4 & 8 & 24 & 28 & 12 \end{bmatrix}$$

$R_1 \to R_1 - 4R_2$ & $R_4 \to R_4 - 4R_3$

$$H = \begin{bmatrix} 1 & 2 & 6 & 7 & 2 \\ 1 & 3 & 8 & 10 & 2 \\ 1 & 2 & 6 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_3$ and $R_3 \to R_3 - R_1$

$$H = \begin{bmatrix} 1 & 2 & 6 & 7 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\uparrow \quad \uparrow \qquad\qquad\qquad \uparrow$$

the rank of Matrix $H = 3$

The pivot Column are C_1, C_2 & C_5

$$\text{Basis} = \left\{ \begin{bmatrix} 5 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 14 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 10 \\ 2 \\ 3 \\ 12 \end{bmatrix} \right\}$$

Hence, the Dimension $= 3$ as it has 3 linearly independent vectors.

(1) A) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = y\}$

* the zero vector is in S as where $x = 0, y = 0$
* Under vector addition, if (x, y, z) & (x', y', z') are in S, then $(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$

Also, since $x = y \Rightarrow x' = y' \Rightarrow x + x' = y + y'$

this implies $(x + x', y + y', z + z') \in S$.

Thus S is closed under vector addition.

* If c (a scalar) is multiplied i.e $c(x, y, z) = (cx, cy, cz)$

here, $x = y \Rightarrow cx = cy$ hence $(cx, cy, cz) \in S$

Thus S is closed under scalar multiplication

Hence, S is a vector subspace in $\mathbb{R}^3$

B) $T = \{(x, y, z) \in R^3 \mid y = 1\}$

* The zero vector is not in T as for zero vector $y = 0$ & $y \neq 1$
* Under vector addition, if $(x, y, z) + (x', y', z') = (x+x', y+y', z+z')$. Since $y = 1$ (given), $y + y' = 1 + 1 = 2 \neq 1$.

Thus 'T' is not closed under vector addition.

* Under scalar multiplication if (x, y, z) is multiplied by c (a scalar) then $c(x, y, z) = (cx, cy, cz)$. Since $y = 1$, $cy = c(1) = c$
$\Rightarrow$ cy is not necessarily equal to 1. Hence T is not closed under scalar multiplication

Hence, T is not a vector subspace in R^3.

C) $V = \{(x, y, z) \in R^3 \mid x \cdot y \cdot z = 0\}$

* The zero vector is in U as $xyz = 0$ $(x = 0, y = 0, z = 0)$
* under vector addition, if $(x, y, z) + (x', y', z') = (x+x', y+y', z+z')$

Since $xyz = 0$ & $x' \cdot y' \cdot z' = 0$

$$(x+x')(y+y')(z+z') = (xy + x'y' + 0)(z+z')$$
$$= 0(z+z') = 0$$

Hence, U is closed under vector addition.

* Under scalar multiplication, if (x, y, z) is multiplied by c (a scalar), then $c(x,y,z) = (cx, cy, cz)$
Since $xyz = 0$, $c(xy)(cz) = (cxy)(cz) = 0$
hence, U is closed under scalar multiplication
therefore, U is a vector subspace of $\mathbb{R}^3$

D) $V = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$
* the zero vector is in V as $x+y+z=0$
$\{0+0+0\}$
* Under vector addition $(x,y,z) + (x',y',z') =$
$(x+x', y+y', z+z')$
Since, $x+y+z=0$, $x'+y'+z'=0$
$\therefore (x+x') + (y+y') + (z+z') = (x+y+z) + (x'+y'+z')$
$= 0 + 0$
$= 0$
Thus 'V' is closed under vector addition.
* Under scalar multiplication, if c (a scalar) is multiplied with V, then $c(x,y,z) = (cx, cy, cz)$
Since, $x+y+z=0$, $c(x+y+z) = c(0) = 0$
* Thus, V is closed under scalar multiplication
therefore, V is a vector subspace of $\mathbb{R}^3$

(5) $A = \begin{pmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 3 & 9 & 1 & -7 & 23 & 8 \\ 1 & 3 & 1 & -3 & 9 & 2 \\ 1 & 3 & -1 & -1 & 5 & 4 \end{pmatrix}$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$

$$A = \begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_4$; $R_4 \to R_4 + R_2$

$$A = \begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 0 & 0 & 1 & -1 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns are $\Rightarrow C_1 , C_3$

$C(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

column space

Null Space -

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \begin{bmatrix} -3(x_2 - 2x_4 + 7x_5 + 3x_6 \\ x_2 \\ -(-x_4 + 2x_5 - x_6) \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$= \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -7 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$A^T = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 3 & 9 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ -2 & -7 & -3 & -1 \\ 7 & 23 & 9 & 5 \\ 3 & 8 & 2 & 4 \end{bmatrix} \xrightarrow[\substack{R_5 \to R_5 - 7R_1 \\ R_6 \to R_6 - 3R_1}]{\substack{R_2 \to R_2 - 3R_1 \\ R_4 \to R_4 + 2R_1}} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_4$;

$R_6 \to R_6 - R_4$

$R_5 \to R_5 + 2R_4$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row space $= R(A) = C(A^T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 1 \\ -7 \\ 23 \\ 8 \end{bmatrix} \right\}$

Left Null space $= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -(-2x_3 + 4x_4) \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix}$

$$= \text{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$