① Riemann Sum of a function of two variables Constructed over a rectangular region is given by:

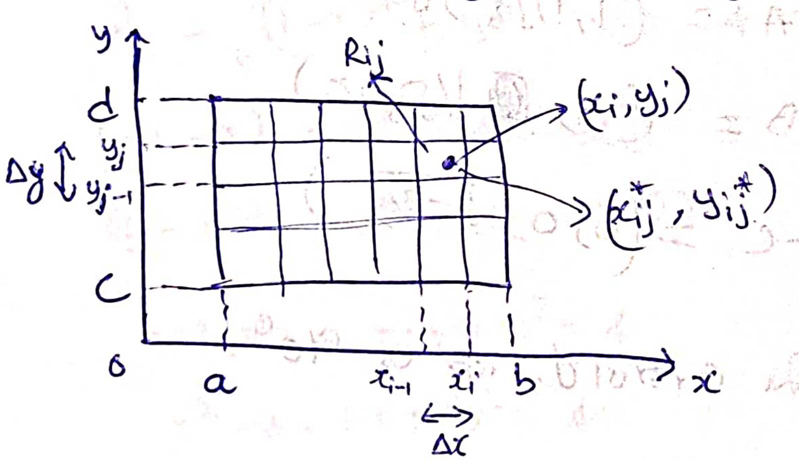
$$\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*)\,\Delta x\,\Delta y.$$

Let $f(x, y)$ be defined on rectangular domain $R = [a,b] \times [c, d]$. If we partition the interval $[a,b]$ in 'm' equal sub intervals & partitions $[c,d]$ in n equal sub intervals.

~~the~~ So, $\Delta x = \frac{b-a}{m}$ and $\Delta y = \frac{d-c}{n}$.

where, $a = x_0 < x_1 < \dots < x_m = b$ and

$b = y_0 < y_1 < \dots < y_n = d$.



Now, area of each rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$

Therefore, Area of each sub rectangle is $R_{ij} = f(x_i^*, y_j^*)\Delta x \Delta$

where, x_i^*, y_j^* are the points between $[x_{i-1}, x_i]$ & $[y_{j-1}, y_j]$ respectively.

Thus, Approximate Volume under the graph of $f(x,y)$ is

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*)\,\Delta x\,\Delta y.$$

This sum is known as Riemann sum and can be used to approximate the value of the volume of the solid.

* Here, the double sum means that for each sub-rectangle we evaluate the function at the chosen point, multiply by the area of each rectangle and then add all the results.

Since, multiplication is a commutative function, Therefore $\Delta x \Delta y = \Delta y \Delta x$ thus we can to have:

$$\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta x \Delta y = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta y \Delta x.$$

Hence, The above ~~equa~~ Riemann Sum is true.

(2) From, Fundamental theorem of Calculus for function of one variable we know that.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where, $f(x)$ is a continuous function defined on closed interval $[a,b]$.

$$F'(x) = f(x)$$

Similarly, let $f(x,y)$ is a function defined over a region in the xy-plane, let's consider a Rectangle 'R' defined by $[a,b]$ in the x-direction and $[c,d]$ in the y-direction. To find the double integral of $f(x,y)$ over this Region R,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$\Rightarrow \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

$$\Rightarrow \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy.$$

* the notation $\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$ means that we integrate $f(x,y)$ with respect to 'y' while holding 'x' constant.

* The notation $\int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$ means that we integrate $f(x,y)$ with respect to 'x' while holding 'y' constant.

③ Fubini's Theorem :-

Fubini's theorem deals with the interchangeability of multiple integral. It means that double integral can be split into interated integrals. This theorem is an essential tool for evaluating double integrals.

Let $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$ then

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy.$$

$$= \int_a^b\int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Fubini's theorem is true if 'f' is bounded on R and f is discontinuous only on a finite number of continous curves which means 'f' has to be integrable over 'R'!

* when a function is continuous, we can ~~break~~ divide the region of integration into infinitely small pieces and accurately calculate the integral over each small piece and finally add up all these small peices then we get the integral over the entire region.

Exercises

① A) $\int_0^1 6x^2y^3\,dy.$

Integrating with respect to 'y' and keeping 'x' constant.

$$\int_0^1 6x^2y^3\,dy = \left[6x^2 \times \frac{y^4}{4}\right]_0^1$$

$$= 6x^2\left(\frac{1}{4}\right)$$

$$= \frac{3x^2}{2}$$

B) $\int_0^2 \int_0^{\pi/2} x \sin y \, dy\, dx.$

from fubini's theorem.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

so,

$$\int_0^2 \int_0^{\pi/2} x \sin y\,dy\,dx = \int_0^2 \left[\int_0^{\pi/2} x \sin y\,dy\right] dx$$

$$= \int_0^2 \left[x(-\cos y)_0^{\pi/2}\right] dx$$

$$= \int_0^2 \left[-x(\cos y)_0^{\pi/2}\right] dx$$

$$= \int_0^2 \left[-x\left(\cos \frac{\pi}{2} - \cos 0^\circ\right) dx\right.$$

$$= \int_0^2 -x(0-1)\,dx$$

$$= \int_0^2 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{4}{2} - 0$$

$$= \underline{\underline{2}}$$

Hence, $\displaystyle\int_0^2 \int_0^{\pi/2} x \sin y \, dy\, dx = \underline{\underline{2}}$

(2) $f(x, y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the Region $R = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, the double integral can be written as

$$\int_a^b \int_c^d f(x)\, g(y)\, dx\, dy = \int_a^b f(x)\, dx \cdot \int_c^d g(y)\, dy.$$

The above formula holds when:

* the region of integration is a rectangle defined by the limits $[a, b]$ in the x-direction and $[c, d]$ in y-direction. The region must be a simple rectangular shape.

* the functions $f(x)$ & $g(y)$ are continuous over this rectangular region.
* The order of integration (first 'x' then 'y' or vice versa) does not affect the result, this can be true when both $f(x)$ & $g(y)$ are continuous over the entire region of integration.

If any of the conditions are not met, then the formula does not hold.

(3) $\iint_S e^{xy} dA > 3$

where $0 \leq x \leq 1$ and $0 \leq y \leq 1$

and S is the unit square.

* Here, e^{xy} is a +ve exponential function. So, $e^{xy} \geq 1$. Also, the unit Square 'S' has an area of 1 sq. units as both its sides are of length 1.
* Let $z = e^{xy}$, and here, the double integral represents the volume under the surface over this region. Since the function is always greater than or equal to 1.
* This volume is greater than or equal to the volume of a unit cube with side length 1.
* the volume of a unit cube with length = 1 is '1' cc. Since the integral represents a volume that is greater than or equal to '1' cc unit, it cannot be greater than 3.

Therefore, we should not believe ~~that~~ the claim that $\iint_S e^{xy} dA > 3$. whereas it's value can be less than or equal to 3, based on the above discussed points.

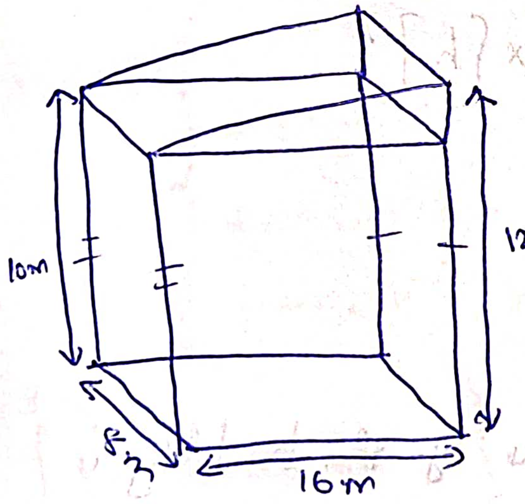
④ given,

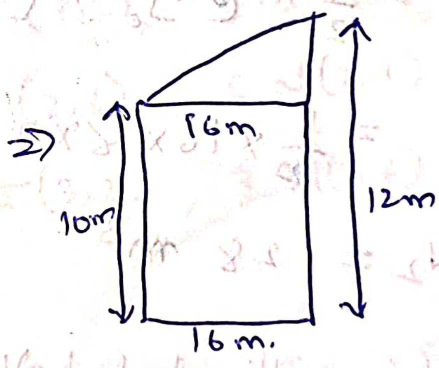
width of the building $(w) = 8m$

length of the building $(l) = 16m$.

height of the roof at one corner $(h_1) = 12m$.

height of the roof at other corner $(h_2) = 10m$.





Since there are two geometric figures in the above figure, we will find the volume of each figure separately and finally add it.

To calculate volume (V_1) of rectangular prism,

$$V_1 = \iiint dV_1 = \int_0^{10}\int_0^{8}\int_0^{16} dl\, dw\, dh_2$$

$$V_1 = [l]_0^{16} \times [w]_0^{8} \times [h_2]_0^{10}$$

$$V_1 = 16 \times 8 \times 10$$

$V_1 = 1280$ cubic metre.

To calculate the volume (V_2) of ~~rectangular~~ triangular prism,

$$V_2 = \iiint dV_2 = \int_0^2 \int_0^8 \int_0^{16} \frac{1}{2}\, dl.\, dw.\, dh_1$$

As the total height is ranging from 0 to 12, but the height of the triangular height has to be considered from 10 to 12 (or) 0 to 2.

$$V_2 = \frac{1}{2} \times [l]_0^{16} \times [w]_0^{8} \times [h_1]_0^{2}$$

$$= \frac{1}{\cancel{2}} \times 16 \times 8 \times \cancel{2}$$

$$V_2 = 128 \text{ m}^3$$

therefore, the total volume of the building is

$$V = V_1 + V_2$$

$$= 1280 + 128$$

$$\Rightarrow V = \underline{\underline{1408 \text{ m}^3}}$$

(5) given,

$0 \leq x \leq 3$ & $0 \leq y \leq 6$

6	1	2	0	0
4	3	5	6	2
2	2	4	5	1
0	0	1	4	1
y/x	0	1	2	3

to find the population of the town.

let the function of population be $P(x)$

$$\int_0^3 \int_0^6 P(x)\, dA = \int_0^3 \int_0^6 P(x,y)\, dy\, dx$$

$$P(x,y) = \begin{cases} 0 \text{ at } (0,0),\ 2 \text{ at } (0,2),\ 3 \text{ at } (0,4),\ 1 \text{ at } (0,6) \\ 1 \text{ at } (1,0),\ 4 \text{ at } (1,2),\ 5 \text{ at } (1,4),\ 2 \text{ at } (1,6) \\ 4 \text{ at } (2,0),\ 5 \text{ at } (2,2),\ 6 \text{ at } (2,4),\ 0 \text{ at } (2,6) \\ 1 \text{ at } (3,0),\ 1 \text{ at } (3,2),\ 2 \text{ at } (3,4),\ 0 \text{ at } (3,6) \end{cases}$$

$$\int_0^3 \int_0^6 P(x,y)\, dy\, dx = \int_0^3 \int_0^6 (0+2+3+1+1+4+5+2+4+5+6+0+1+1+2+0)\, dy\, dx$$

$$= \int_0^3 \int_0^6 37\, dy\, dx$$

$$= 37 \int_0^3 [y]_0^6\, dx$$

$$= 37 \int_0^3 6\, dx$$

$$= 37 \times 6 \times [x]_0^3$$

$$= 37 \times 6 \times 3$$
$$= 666.$$

Therefore, the population of the town is 666,000 people.