HW-3

3.38) from the given data,

$P(\text{tails}) = P(T) = \frac{1}{2}$ & $P(\text{Heads}) = P(H) = \frac{1}{2}$

$P(\text{white ball} \mid \text{tails}) = P(\text{white ball from Urn B})$

$P(W/T) = \frac{3}{15}$

$P(\text{white ball} \mid \text{heads}) = P(\text{white ball from Urn A})$

$P(W/H) = \frac{5}{12}$

from Bayes' theorem,

$$P(T/W) = \frac{P(W/T)\,P(T)}{P(W/T)\,P(T) + P(W/H)\,P(H)}$$

$$= \frac{\frac{3}{15} \times \frac{1}{2}}{\left(\frac{3}{15} \times \frac{1}{2}\right) + \left(\frac{5}{12} \times \frac{1}{2}\right)} = \frac{\frac{3}{30}}{\frac{37}{120}}$$

$$\Rightarrow P(T/W) = \frac{36}{111} = \frac{12}{37}$$

3.47) let "A" ~~dento~~ denote choosing of 'n' balls that are white.

E_i be the result of rolling the die.

where $i = \{1, 2, 3, 4, 5, 6\}$

Since it is a fair die, all the outcomes have an equal probability to appear.

$$\Rightarrow P(E_i) = \frac{1}{6} \text{ where } i = 1, 2, 3, 4, 5, 6$$

When the die is rolled, choosing a combination of 'i' balls among black and white balls is given by:

$$P(A|E_1) = \frac{\binom{5}{1}}{\binom{15}{1}} = \frac{5}{15} = \frac{1}{3}$$

$$P(A|E_2) = \frac{\binom{5}{2}}{\binom{15}{2}} = \frac{10}{105} = \frac{2}{21}$$

$$P(A|E_3) = \frac{\binom{5}{3}}{\binom{15}{3}} = \frac{10}{455} = \frac{2}{91}$$

$$P(A|E_4) = \frac{\binom{5}{4}}{\binom{15}{4}} = \frac{1}{273}$$

$$P(A|E_5) = \frac{\binom{5}{5}}{\binom{15}{5}} = \frac{1}{3003}$$

$$P(A|E_6) = \frac{\binom{5}{6}}{\binom{15}{6}} = 0.$$

the probability that all of the balls ~~sett~~ selected are white is

$$P(A) = \sum_{i=1}^{6} P(A|E_i)\, P(E_i)$$

$$= \frac{1}{6}\left(\frac{1}{3} + \frac{2}{21} + \frac{2}{91} + \frac{1}{273} + \frac{1}{3003} + 0\right)$$

$$= \frac{1}{6}\left(\frac{1001 + 286 + 66 + 11 + 1}{3003}\right) = \frac{1}{6}\left(\frac{1365}{3003}\right)$$

we get,

$$P(A) = \frac{1365}{18018}.$$

$$\Rightarrow \boxed{P(A) = \frac{5}{66}}$$

The conditional probability that the die landed on 3 if all the balls ~~are~~ selected are white is given by:

$$P(E_3/A) = \frac{P(A|E_3)\, P(E_3)}{P(A)} = \frac{(2/91)(1/6)}{(5/66)}.$$

$$\Rightarrow \boxed{P(E_3/A) = \frac{22}{455}}$$

3.90) Let 'E_1' be the event where Judge 1 Casts a guilty vote
'E_2' be the event where Judge 2 Casts a guilty vote
'E_3' be the event where Judge 3 Casts a guilty vote.
'G' be the defendant is guilty
'I' be the defendant is innocent.

given, $P(G) = 0.7 \Rightarrow P(I) = 0.3$

probability of voting guilty when the defendant is actually guilty

$$P(E_1|G) = P(E_2|G) = P(E_3|G) = 0.7$$

probability of voting guilty when the defendant is innocent

$$P(E_1/I) = P(E_2/I) = P(E_3/I) = 0.2.$$

a) $$P(E3|E1,E2,G) = \frac{[(0.7\times0.7\times0.7)0.7] + ([(0.3\times0.3\times0.3)]0.2)}{(0.7\times0.7\times0.7)+[(0.2\times0.2\times\cancel{0.2})0.3]}$$

$$= \frac{0.2401 + 0.0054}{0.343 + 0.012}$$

$$= 0.683$$

b) $P(E_3 \mid E_1, \sim E_2, G) = \dfrac{0.147 + 0.7 + 0.032 \times 0.3}{0.21 \times 0.7 + 0.16 \times 0.3}$

$= 0.5769$

c) $P(E_3 \mid \sim E_1, \sim E_2, I) = \cancel{0.3 \times 0.3 \times 0.7 \times 0.7}$

$= \dfrac{(0.3 \times 0.3 \times 0.7 \times 0.7) + (0.8 \times 0.8 \times 0.2 \times 0.3)}{(0.3 \times 0.3 \times 0.7) + (0.8 \times 0.8 \times 0.3)}$

$= \dfrac{0.0825}{0.255}$

$= 0.3235$

~~Since,~~ the event E_1, E_2 & E_3 are dependent events, as the outcome of one Judge's vote is affecting the probability of another judge's vote. as it does not satisfy

$$P(E_1, E_2, E_3) = P(E_1) \cdot P(E_2) \cdot P(E_3)$$

Also, $E_1, E_2 E_3$ are not conditionally idependent Since, it does not Satisfy the below conditions.

$$P(E_1, E_2, E_3 \mid G) = P(E_1 \mid G) \cdot P(E_2 \mid G) \cdot P(E_3 / G)$$

$$P(E_1, E_2, E_3 \mid I) = P(E_1 \mid I) \cdot P(E_2 / I) \cdot P(E_3 / I)$$

Theoretical problems.

3.8) given that $P(A|C) > P(B|C)$ and $P(A|C^c) > P(B|C^c)$

let A, B, & C be events relating to the experiment of rolling a pair of dice.

The events A, B and C for which that relationship is not true.

a) prove that $P(A) > P(B)$

By the condition on whether or not event 'C' occurs

i.e., $P(A) = P(A|C) \cdot P(C) + P(A|C^c) \cdot P(C^c)$

& $P(B) = P(B|C) \cdot P(C) + P(B|C^c) \cdot P(C^c)$

Subtracting P(B) from P(A) to get the required conditions:

$$P(A) - P(B) = P(A|C) \cdot P(C) + P(A|C^c) \cdot P(C^c) - (P(B|C) \cdot P(C) + P(B|C^c) \cdot P(C^c))$$

$$= P(C)(P(A|C) - P(B|C)) + P(C^c)(P(A|C^c) - P(B|C^c))$$

Now,

$$P(A|C) - P(B|C) > 0$$

$$P(A|C^c) - P(B|C^c) > 0.$$

hence, $P(A) - P(B) > 0$

$\Rightarrow P(A) > P(B)$

b) given that $P(A|C) > P(A|C^c)$ and $P(B|C) > P(B|C^c)$
prove that. $P(AB|C) > P(AB|C^c)$

~~As, the event~~

As, it is not mentioned that events A, B & C are independent or not,

Hence, $P(AB) = P(A|B) \cdot P(B)$

By the condition on whether or not event 'C' occurs

$P(B) = P(B|C) \cdot P(C) + P(B|C^c) \cdot P(C^c)$

$P(AB) = P(AB|C) \cdot P(C) + P(AB|C^c) \cdot P(C^c)$

Combining the above conditions,

$P(AB) = P(AB|C) \cdot P(B|C) \cdot P(C) + P(AB|C^c) \cdot P(B|C^c) P(C^c)$

$P(AB) = P(A|B) \cdot P(B|C) \cdot P(C) + P(A|B) \cdot P(B|C^c) \cdot P(C^c)$.

Equate the coefficients in two equations for $P(AB)$:

$P(AB|C) = P(A|B) \cdot P(B|C)$ &

$P(AB|C^c) = P(A|B) \cdot P(B|C^c)$

from the given condition.

$P(AB|C) - P(AB|C^c) > 0$

$P(B|C) - P(B|C^c) > 0$.

hence,

$P(AB|C) = P(A|B) \cdot P(B|C)$

$P(AB|C^c) = P(A|B) \cdot P(B|C^c)$

$P(AB|C) - P(AB|C^c) = P(A|B)(P(B|C) - P(B|C^c))$.

$$\Rightarrow P(AB|C) - P(AB|C^c) > 0$$

$$\underline{\underline{P(AB|C) > P(AB|C^c)}}.$$

3.9) Given, 'A' be the event that the first toss results in heads.

'B' be the event that the second toss results in heads.

'C' be the event that the both tosses the coin lands on the same side.

A, B, C are pairwise independent. {AB, BC, AC}.

to prove A, B, C are not independent.

$S = \{HH, HT, TH, TT\}$

$A = \{HH, HT\} \Rightarrow P(A) = \frac{2}{4} = \frac{1}{2}$

$B = \{HH, TH\} \Rightarrow P(B) = \frac{2}{4} = \frac{1}{2}$

$C = \{HH, TT\} = P(C) = \frac{2}{4} = \frac{1}{2}$.

$P(AB) = \cancel{P(A \cap B)} = P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(BC) = \cancel{P(B \cap C)} = P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(AC) = \cancel{P(A \cap C)} = P(A) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{4} \rightarrow P(AB)$

$P(B \cap C) = \frac{n(B \cap C)}{n(S)} = \frac{1}{4} = P(BC)$

$P(A \cap C) = \frac{n(A \cap C)}{n(S)} = \frac{1}{4} = P(AC)$

$P(A \cap B \cap C) = \frac{n(A \cap B \cap C)}{n(S)} = \frac{1}{4}$

$P(ABC) = P(A) \, P(B) \, P(C)$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{1}{8}.$

here, $P(A \cap B \cap C) \neq P(A) \, P(B) \, P(C)$

$\frac{1}{4} \neq \frac{1}{8}$

Hence, A, B and C are not independent.

3.10) Let B = women of age 45 yrs having breast cancer.

$P(B) = 0.02 \Rightarrow P(B^c) = 0.98$

Let A = Women have positive mammograms.

$P(A/B) = 0.9$ & $P(A|B^c) = 0.08$

to calculate the probability that women has breast cancer given that she has a positive mammogram is given by:

$$P(B/A) = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B^c)\,P(B^c)}$$

{from Bayes' theorem}

$$P(B/A) = \frac{(0.9)(0.02)}{(0.9)(0.02) + (0.08)(0.98)}$$

$$P(B|A) = \frac{0.018}{0.018 + 0.0784}$$

$$P(B|A) = 0.1867$$

Hence, $P(B|A) = 0.1867$