Homework-8

7.64) a) let 'x' be the lifetime of the light bulb.

X_i be the lifetime of type 'i' bulb.

given, $P(x=1) = P$ & $P(x=2) = 1-P$

$E(x) = E(x|i=1) \cdot P(i=1) + E(x|i=2) \cdot P(i=2)$

$\Rightarrow \boxed{E(x) = Pu_1 + (1-P)u_2}$

b) $Var(x) = E(x^2) - [E(x)]^2$

$Var(x|i=1) = \sigma_1^2$ & $Var(x|i=2) = \sigma_2^2$

$Var(x|i=1) = E(x^2|i=1) - E(x|i=1)^2$

$Var(x|i=2) = E(x^2|i=1) - E(x|i=2)^2$

$E(x^2|i=1) = Var(x|i=1) + E(x/i=1)^2$

$= \sigma_1^2 + \mu_1^2$

$E(x^2/i=2) = Var(x/i=2) + E(x/i=2)^2$

$= \sigma_2^2 + \mu_2^2$

$E(x^2) = E(x^2|i=1).P(i=1) + E(x^2/i=2)\,P(i=2)$

$= (\sigma_1^2 + \mu_1^2)P + (\sigma_2^2 + \mu_2^2)(1-P)$

Hence,

$\Rightarrow \boxed{Var(x) = (\sigma_1^2 + \mu_1^2)P + (\sigma_2^2 + \mu_2^2)(1-P) - [\mu_1 P + \mu_2(1-P)]}$

7.65)

a) Let X_1 be a R.V denoting no. of storms in a good year with mean $= 3$

Let P_1 be the prob. of good year $= 0.4$ $\{P(X_1) = 0.4\}$

Let X_2 be a R.V denoting no. of storms in a bad year with mean $= 5$

Let P_2 be the prob of bad year $= 0.6$ $\{P(X_2) = 0.6\}$

The expected value is given as:

$$E(X) = E(X_1) \cdot P(X_1) + E(X_2) \cdot P(X_2)$$

$$= 3(0.4) + 5(0.6)$$

$$= 1.2 + 3$$

$$\Rightarrow \boxed{E(X) = 4.2}$$

The variance is given by: $Var(X) = E(X^2) - E(X)^2$

$$E(X^2) = E(X_1^2) P(X_1) + E(X_2^2) P(X_2)$$

we know, $Var(X_1) = E(X_1^2) - E(X_1)^2$

In poisson distribution, variance $=$ mean $\{\mu = E(X)\}$

So, $E(X_1^2) = Var(X_1) + E(X_1)^2 = 3 + 3^2$

$$= 12$$

Similarly, $E(X_2^2) = Var(X_2) + E(X_2)^2 = 5 + 5^2$

$$= 30$$

$E(X^2) = 12(0.4) + 30(0.6)$

$= 4.8 + 18$

$= 22.8$

Hence, $Var(X) = 22.8 - (4.2)^2$

$\Rightarrow \boxed{Var(X) = 5.16}$

7.79) given, the weekly sales follow bivariate normal distribution.

Mean = 40, S.D = 6, Correlation = 0.6

$E(X_1 + X_2) = E(X_1) + E(X_2)$

$= 40 + 40$

$= 80$

$Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2\,Cov(X_1 X_2)$

we know, $Cor(X_1, X_2) = \dfrac{Cov(X_1, X_2)}{\sqrt{Var(X_1)\,Var(X_2)}}$

$Cov(X_1, X_2) = Cor(X_1, X_2)\sqrt{Var(X_1)\,Var(X_2)}$

$Var(X_1 + X_2) = 6^2 + 6^2 + 2\left(0.6\sqrt{6^2(6^2)}\right)$

$= 72 + 43.2$

$= 115.2$

a) P(Sales exceeds 90) $\Rightarrow P(x>90) = P\left(z > \frac{x-\mu}{\sigma}\right)$

$$P(x>90) = 1 - P\left(\frac{90-80}{\sqrt{115.2}}\right)$$

$$= 1 - \phi(0.9316)$$

$$\Rightarrow \underline{\underline{P(x>90) = 0.1762}} \quad \{\text{from z-table}\}$$

b) If correlation were 0.2 instead of 0.6 then the probability would decrease.
This is because the mean of the sales is less than 90 and the probability that it exceeds 90 is increased as the variance of sales

c) given correlation $= 0.2$

$$Var(x_1 + x_2) = 6^2 + 6^2 + 2\left(0.2\sqrt{6^2(6^2)}\right)$$

$$= 72 + 14.4$$

$$= \underline{\underline{86.4}}$$

P(Sales exceeds 90) $\Rightarrow P(x>90) = P\left(z > \frac{x-\mu}{\sigma}\right)$

$$P(x>90) = 1 - P\left(\frac{90-80}{\sqrt{86.4}}\right)$$

$$= 1 - \phi(1.075)$$

$$\Rightarrow \underline{\underline{P(x>90) = 0.1423}} \quad \{\text{from z-table}\}$$

Theoretical problems

7.40)

The conditional variance of a random variable 'x' given another random variable 'y' is defined as

$$Var(x/Y) = E[Var(X/Y)] + Var[E(X|Y)]$$

Consider, X is a geometric R.V with parameter p.

the probability mass function (PMF) of a geometric distribution is given by $P(X=k) = (1-P)^{k-1}P$

Mean of geometric distribution with parameter P is $E(x) = 1/p$ & the variance is $Var(X) = (1-P)/p^2$

So, $Var(x|Y) = E(Var(X|Y)) + Var(E(X|Y))$

$$Var(X|Y) = E\left(\frac{1-P}{p^2}\right) + Var\left(\frac{1}{P}\right)$$

$$Var(x|Y) = \frac{1-P}{p^2} + Var\left(\frac{1}{P}\right)$$

Variance of constant times a R.V is the constant squared times the variance of the R.V $E(X|Y) = \frac{1}{p}$

$$\therefore var(E(X|Y)) = Var\left(\frac{1}{P}\right) = 0$$

Therefore, Conditional Variance is:

$$\Rightarrow Var(x/Y) = \frac{1-P}{p^2}$$

7.51) Given, $\sum_{i=1}^{n} X_i$; $X_1, X_2 \ldots X_n$ are independent & follows exponential R.V with mean $\frac{1}{\lambda}$

we know,

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1} \quad \forall\, i = 1, 2, \ldots n$$

$$M_{\sum_{i=1}^{n} X_i}(t) = E\left(e^{t\left(\sum_{i=1}^{n} X_i\right)}\right)$$

$$= E\left(e^{tX_1 + tX_2 + \cdots + tX_n}\right) = E\cdot\left(e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}\right)$$

$$= M_X(t) \ldots M_{X_n}(t) = \left[M_X(t)\right]^n$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-n} = \left(\frac{\lambda}{\lambda - t}\right)^n$$

this is possible because X_i are independent & identically distributed

So we have, $M_Y(t) = \left(\frac{\lambda}{\lambda - t}\right)^n$

here, Y has a gamma distribution with parameters n & λ & $M_Y(t)$ is of the form $\left(\frac{\lambda}{\lambda - t}\right)^s$ with

$$\text{mean} = \frac{s}{\lambda} = \frac{n}{\lambda} \quad \& \quad \text{Variance} = \frac{s}{\lambda^2} = \frac{n}{\lambda^2}$$

Hence, the given function $\sum_{i=1}^{n} X_i$ follows gamma distribution.