HW-4

4.27 let X = event that product of two dice.
when the two dices are rolled, the sample space is $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

$$S = \{1,2,3,4,5,6,8,9,10,12,15,16,18,20,24,25,30,36\}$$

when, $X=1 \Rightarrow$ outcome is $(1,1) \Rightarrow P(X=1) = \frac{1}{36}$

$X=2 \Rightarrow$ outcomes are $(1,2),(2,1) \Rightarrow P(X=2) = 2/36$

$X=3 \Rightarrow$ outcomes are $(1,3)(3,1) \Rightarrow P(X=3) = 2/36$

$X=4 \Rightarrow$ outcomes are $(4,1),(1,4),(2,2) \Rightarrow P(X=4) = 3/36$.

$X=5 \Rightarrow$ outcomes are $(5,1),(1,5) \Rightarrow P(X=5) = 2/36$

$X=6 \Rightarrow$ outcomes are $(6,1)(1,6)(2,3)(3,2) \Rightarrow P(X=6) = 4/36$.

$X=8 \Rightarrow$ outcomes are $(2,4),(4,2) \Rightarrow P(X=8) = 2/36$.

$X=9 \Rightarrow$ outcomes ~~are~~ is $(3,3) \Rightarrow P(X=9) = 1/36$

$X=10 \Rightarrow$ outcomes are $(5,2),(2,5) \Rightarrow P(X=10) = 2/36$.

$X=12 \Rightarrow$ outcomes are $(2,6)(6,2)(3,4)(4,3) \Rightarrow P(X=12) = 4/36$

$X=15 \Rightarrow$ outcomes are $(3,5),(5,3) \Rightarrow P(X=15) = 2/36$

$X=18 \Rightarrow$ outcomes are $(3,6),(6,3) \Rightarrow P(x=16) = 2/36$.

$X=16 \Rightarrow$ outcomes ~~are~~ is $(4,4) = 1/36$.

$X=20 \Rightarrow$ outcomes are $(4,5),(5,4) \Rightarrow P(X=20) = 2/36$

$X = 24 \Rightarrow$ outcomes are $(4,6), (6,4) \Rightarrow P(X=24) = 2/36$

$X = 25 \Rightarrow$ outcomes ~~are~~ $(5,5) \Rightarrow P(X=25) = 1/36$

$X = 30 \Rightarrow$ Outcomes are $(6,5), (5,6) \Rightarrow P(X=30) = 2/36$

$X = 36 \Rightarrow$ outcome is $(6,6) \Rightarrow P(X=36) = 1/36$

Hence, the total probability $= 1$.

4.13) Let $P(A)$ is the probability of a sale in 1^{st} appointment
$P(B)$ is the probability of a sale in 2^{nd} appointment
$P(D)$ is the probability of selling the deluxe model
$P(S)$ is the probability of selling the standard model.

given, $P(A) = 0.3 \Rightarrow 1 - P(A) = 0.7$
$P(B) = 0.6 \Rightarrow 1 - P(B) = 0.4$
$P(D) = 0.5$
$P(S) = 0.5$.

'X' is the total dollar value of all sales, which is either 0, 500, 1000, 1500 or 2000.

when $X = 0$, i.e no sale

$P(X=0) = \cancel{P(A) \times P(}(1 - P(A)) \times (1 - P(B))$
$= 0.7 \times 0.4$

$P(X=0) = 0.28$

when $x = 500$,

$$P(X=500) = \left[\left[1 - P(A)\right] \times P(B) \times P(S)\right] + \left[P(A) \times (1 - P(B)) \times P(S)\right]$$

$$= (0.7 \times 0.6 \times 0.5) + (0.3 \times 0.4 \times 0.5)$$

$$= 0.21 + 0.06$$

$$\Rightarrow P(X=500) = \cancel{0.06}\; 0.27$$

when $x = 1000$,

$$P(X=1000) = \left[P(A) \times P(B) \times 2 P(S)\right] + \left[P(A) \times (1 - P(B)) \times P(D)\right] + \left[(1 - P(A)) \times P(B) \times P(D)\right]$$

$$= (0.3 \times 0.6 \times 2(0.5)) + (0.3 \times 0.4 \times 0.5) + (0.7 \times 0.6 \times 0.5)$$

$$= 0.045 + 0.06 + 0.21$$

$$P(X=1000) = 0.315$$

when $X = 1500$,

$$P(X=1500) = \left[P(A) \times P(B) \times P(S) \times P(D)\right] + \left[P(A) \times P(B) \times P(D) \times P(S)\right]$$

$$= (0.3 \times 0.6 \times 0.5 \times 0.5) + (0.3 \times 0.6 \times 0.5 \times 0.5)$$

$$= 0.045 + 0.045$$

$$P(X=1500) = 0.09$$

When $X = 2000$,

$$P(X=2000) = P(A) \times P(B) \times P(D) \times P(D)$$

$$= 0.3 \times 0.6 \times 0.5 \times 0.5$$

$$= \cancel{0.045}\ 0.045$$

$$P(X=2000) = 0.045$$

hence, $P(X=0) = 0.28$

$P(X=500) = 0.27$

$P(X=1000) = 0.315$

$P(X=1500) = 0.09$

$P(X=2000) = 0.045$.

4.22) Let 'X' be the random variable that marks the no. of games that are played.

for $i=2$,

the No. of chances that both teams, (A & B) win two games is given by the sample space.

$\{AA, BB, ABB, ABA, BAA, BAB\}$

It is given that, if any of team wins two games then the game ends.

Let the probability that A wins be 'p'

the probability that B wins be '1-p'

the expected no. of games played when $i=2$ is:

$$E(x) = 2\left[p.p + (1-p)(1-p)\right] + 3\left[p(1-p)(1-p) + p(1-p)p + (1-p)p.p + (1-p)p(1-p)\right]$$

$$= 2\left[p^2 + 1 + p^2 - 2p\right] + 3\left[p(1+p^2-2p) + 2p^2(1-p) + p(1+p^2-2p)\right]$$

$$E(x) = -2p^2 + 2p + 2$$

for max. value of $E(x)$ at $p = 1/2$

$$\frac{\partial E(x)}{\partial p} = 0 \quad \text{and} \quad \frac{\partial^2 E(x)}{\partial p^2} \leq 0$$

$$\frac{\partial(E(x))}{\partial p} = 0$$

$$\left.\frac{\partial(-2p^2+2p+2)}{\partial p}\right|_{p=\frac{1}{2}} = 0$$

$$\left[-4p+2\right]_{p=\frac{1}{2}}$$

$$= -4\left(\frac{1}{2}\right)+2$$

$$= \underline{\underline{0}}$$ {this is maximum, as 2^{nd} derivative is $-ve$}

Also, $$\left.\frac{\partial^2 E(x)}{\partial p^2}\right|_{p=\frac{1}{2}} = 0$$

$$= \frac{\partial(-4p+2)}{\partial p}$$

$$= -4 < 0.$$

So, $E(x)$ is maximum at $p=\frac{1}{2}$.

for $i=3$;

the Sample Space for the no. of times that both teams win 3 games is given by:

{AAA, BBB, AABA, ABAA, BAAA, BBAB, BABB, ABBB, AABBA, ABBAA, BBAAA, BBAAB, BAABB, AABBB, BABAB, ABABA}

It is given that, if any team wins 3 games then the game ends.

let the probability of team A winning is 'p'

the probability of team B winning is '1-p'

the expected no. of games played when $i=3$ is.

$$
\begin{aligned}
E(X) = {} & 3\left[p.p.p + (1-p)(1-p)(1-p)\right] + 4\left[p.p(1-p)p + \right. \\
& p(1-p).p.p + (1-p)p.p.p + (1-p)(1-p)p(1-p) + \\
& \left.(1-p)\,p(1-p)(1-p) + p(1-p)(1-p)(1-p)\right] + \\
& 5\left[p.p(1-p)(1-p)p + p(1-p)(1-p)pp + (1-p)(1-p).p.p.p \right. \\
& + (1-p)(1-p)\,pp(1-p) + (1-p)pp(1-p)(1-p) + pp(1-p)(1-p) \\
& \left.(1-p) + (1-p)p(1-p)p(1-p) + p(1-p)p(1-p)p\right]
\end{aligned}
$$

$$= -4p^4 + 8p^3 - 7p^2 + 3p + 3$$

$$\frac{\partial E(x)}{\partial p}\Big|_{p=\frac{1}{2}} = 0$$

$$= -16p^3 + 24p^2 - 14p + 3$$

$$= -\cancel{16}^{2}\left(\frac{1}{\cancel{8}}\right) + \cancel{24}^{6}\left(\frac{1}{\cancel{4}}\right) - \cancel{14}^{7}\left(\frac{1}{\cancel{2}}\right) + 3$$

$$= -2 + 6 - 7 + 3$$

$$= \underline{\underline{0}}$$ { this is maximum, as 2nd derivative is −ve }

Also, $$\frac{\partial^2 E(x)}{\partial p^2}\Big|_{p=\frac{1}{2}} = 0$$

$$= \frac{\partial}{\partial p}\left(-16p^3 + 24p^2 - 14p + 3\right)$$

$$= -48p^2 + 48p - 14$$

$$= -\cancel{48}^{12}\left(\frac{1}{\cancel{4}}\right) + 48\left(\frac{1}{2}\right) - 14$$

$$= -12 + 24 - 14$$

$$= -2 < 0$$

Hence, $E(x)$ is maximum at $p = \frac{1}{2}$ for $i = 3$

4.45) given that,

probability of passing the student independently if he has an on day = 0.8

probability of passing the student independently if he has an off day = 0.4.

probability of off day = 2/3

probability of on day = 1/3

→ If the student requests 3 examiners then.

P(pass if 2 or more examiners pass him)

$$= \frac{2}{3}\left[\binom{3}{2}(0.4)^2(0.6) + \binom{3}{3}(0.4)^3\right] + \frac{1}{3}\left[\binom{3}{2}(0.8)^2(0.2) + \binom{3}{3}(0.8)^3\right]$$

{from binomial theorem}

$$= \frac{2}{3}(0.288 + 0.064) + \frac{1}{3}(0.384 + 0.512)$$

$$= 0.2347 + 0.2987$$

$$= 0.533$$

→ If the student request 5 examiners then,

from binomial theorem,

p

P(pass if 3 or more examiners pass the Student)

$$= \frac{2}{3}\left[\binom{5}{3}(0.4)^3(0.6)^2 + \binom{5}{4}(0.4)^4(0.6) + \binom{5}{5}(0.4)^5\right] +$$

$$\frac{1}{3}\left[\binom{5}{3}(0.8)^3(0.2)^2 + \binom{5}{4}(0.8)^4(0.2) + \binom{5}{5}(0.8)^5\right]$$

$$= \frac{2}{3}[0.2304 + 0.0768 + 0.0102] + \frac{1}{3}[0.2048 + 0.4096 + 0.3276]$$

$$= 0.2117 + 0.314$$

$$= 0.526$$

The Student should prefer 3 examiners as the probability of passing is more with 3 examiners.

4.55) given that,

there are 2 typist and the average number of errors per article by 1^{st} typist is 3

the avg. no. of error per article by 2^{nd} typist is 4.2

from, the probability mass function of poisson distribution we have.

$$P(X=x) = \frac{e^{-\lambda}(\lambda)^x}{x!} \quad \text{where } x = 0, 1, 2, \ldots$$

here, $\lambda_1 = 3$ & $\lambda_2 = 4.2$

for zero error by 1^{st} typist,

$$P(X=0) = \frac{e^{-3}(3^0)}{0!} = e^{-3} = 0.0497$$

for zero error by 2^{nd} typist,

$$P(X=0) = \frac{e^{-4.2}(4.2)^0}{0!} = e^{-4.2} = 0.0149$$

P(Selecting a typist for today) = 0.5

so, = $0.5(0.0497) + 0.5(0.0149)$

$= 0.02485 + 0.00745$

$= 0.0323$

4.52) Let 'x' be the random variable that denotes the no. of airplane crashes.
Since, event of a airplane crash is a rare event it can be modelled using poisson distribution.

given, $\lambda = 3.5$

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

a) P(at least 2 such accidents)

$$= P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[\frac{e^{-3.5}(3.5^{0})}{0!} + \frac{e^{-3.5}(3.5)}{1!}\right]$$

$$= 1 - [0.0301 + 0.1056]$$

$$= 0.8643.$$

b) P(at most 1 accident)

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \frac{e^{-3.5}(3.5^{0})}{0!} + \frac{e^{-3.5}(3.5)}{1}$$

$$= 0.0301 + 0.1056$$

$$= 0.1357$$

As the probability ~~that~~ of at least 2 such accidents happening in the next month is 0.8643 which is high, whereas, the probability of at most 1 accident happening in the next month is 0.1357 which is low.

4.78) Probability of selecting 2 white & 2 black ball is

$$P = \frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{\frac{4!}{2!2!} \times \frac{4!}{2!2!}}{\frac{8!}{4!4!}}$$

$$= \frac{\frac{\cancel{4}\times 3\times 2\times 1 \times \overset{2}{\cancel{4}}\times 3\times \cancel{2}\times 1}{\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2}}}{\frac{2\cancel{8}\times 7\times \cancel{6}\times 5\times \cancel{4!}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times 1\times\cancel{4!}}} = \frac{36}{70}$$

$$P = \frac{18}{35}.$$

a) The repetitions are independent, so the number of selections needed is a geometric random variable with success probability $p = \frac{18}{35}$

Hence, P(Making exactly 'n' selections) $= P(1-P)^{n-1}$

$$= \frac{18}{35}\left(1-\frac{18}{35}\right)^{n-1} = \frac{18}{35}\left(\frac{17}{35}\right)^{n-1}$$

b) the expected number of selections is given by $= \frac{1}{p}$

$= \frac{1}{\frac{18}{35}}$

$= \frac{35}{18}$

Theoretical Problem.

4.11) Consider a particular arrangement of 'k' successes and (n-k) failures.

Let the arrangement be {SSS...SSFFF....FF}

first 'k' trials result in success and (n-k) last trails result in failure.

So, the probability of this event is

$= p \times p \times p \times \cdots \times p \times (1-p)(1-p)(1-p) \cdots (1-p)$

here, 'p' is multiplied 'k' times for the 'k' successes

'1-p' is multiplied 'n-k' times for 'n-k' failures.

hence, probability of this event $= p^k (1-p)^{n-k}$

If we have a sightly different arrangement like { FSS..SSSFFF...F} i.e, firstly a failure, followed by 'k' Success & then $(n-k-1)$ failures again.

So, the probabability of the event will be.

$$= (1-p) \times p \times p \times p \times \cdots \times p (1-p)(1-p)(1-p) \cdots (1-p)$$

$$= p^k (1-p)^{n-k}$$

Each of the arrangements will have equal probability because, the bernouli variable is independent.

Hence, the total probability of getting 'k' Success & $(n-k)$ failures = sum of probability of each possible arrangement

$$= \underline{\underline{\binom{n}{k} p^k (1-p)^{n-k}}}$$