Homework - 10

① To find the Method of moments estimators for the parameters θ_1 & θ_2 based on the given uniform distributions.

Given that $Y_1, Y_2, \ldots, Y_m$ are i.i.d uniform $(0, \theta_1]$, the first moment (mean) of Y is given by:

$$E[Y] = \frac{\theta_1}{2}$$

Similarly, $Z_1, Z_2, \ldots, Z_n$ are i.i.d uniform $[\theta_1, \theta_2]$, the first moment (mean) of Z is given by:

$$E[Z] = \frac{\theta_1 + \theta_2}{2}$$

<u>Method of Moments for θ_1:</u>

Sample mean for Y is given by: $\bar{Y} = \frac{1}{m} \sum_{i=1}^{m} Y_i$

By the method of moments, we equate the sample mean to the population mean, we have $\frac{\theta_1}{2} = \bar{Y}$

$$\Rightarrow \theta_1 = 2\bar{Y}$$

<u>Method of Moments for θ_2:</u>

Sample mean for Z is given by $\bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i$

from method of moments, $\frac{\theta_1 + \theta_2}{2} = \bar{Z}$

we know $\theta_1 = 2\bar{Y}$

$$2\bar{Y} + \theta_2 = 2\bar{Z}$$

$$\theta_2 = 2\bar{Z} - 2\bar{Y}$$

$$\theta_2 = 2(\bar{Z} - \bar{Y})$$

Therefore, the method of moments estimator for θ_1 is

$$\Rightarrow \boxed{\hat{\theta}_{1MM} = 2\bar{Y}}$$

The method of moments estimator for θ_2 is

$$\Rightarrow \boxed{\hat{\theta}_{2MM} = 2(\bar{Z} - \bar{Y})}$$

(2) Given PDF for a Rayleigh distributed random Variable X is :

$$f(x|\sigma^2) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$$

To find Maximum likelihood Estimator (MLE) of σ^2

A function $f(x)$ is a valid PDF if $\int f(x)dx = 1$

Let $x_1, x_2, \ldots, x_n$ be a random sample. The likelihood function 'L' is the product of the induvidual PDFs of the observations:
$$L = \prod_{i=1}^{n} f(x_i)$$

$$L(\sigma^2) = \prod_{i=1}^{n} \left(\frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \right)$$

Taking log on both sides

$$\log L(\sigma^2) = \sum_{i=1}^{n} \log \left(\frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \right)$$

$$\log L(\sigma^2) = \sum_{i=1}^{n} \left(\log x_i - \log \sigma^2 - \frac{x_i^2}{2\sigma^2} \right)$$

$$\log L(\sigma^2) = \sum_{i=1}^{n} \log x_i - n \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2$$

Now, differentiating with respect to σ^2 & setting it to zero to find the MLE

$$\frac{d}{d\sigma^2} \log L(\sigma^2) = 0$$

$$0 - \frac{n}{\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^{n} x_i^2 = 0$$

$$\sigma^2 = \frac{1}{2n} \sum_{i=1}^{n} x_i^2$$

Hence, the MLE of σ^2 for a Rayleigh Distribution is

$$\Rightarrow \boxed{\hat{\sigma}^2 = \frac{1}{2n} \sum_{i=1}^{n} x_i^2}$$