Homework - 7

7.3)

a) $P\{W>0\} = \frac{1}{2}$ because there is $\frac{1}{2}$ chance the player will win on their first gamble & stop with a net profit.

b) $P\{W<0\} = 1 - P\{W>0\} - P\{W=0\}$

$P\{W=0\} = \frac{1}{4}$ because there is $\frac{1}{2}$ chance the player will lose on their first gamble, then also a $\frac{1}{2}$ chance they will win on their second gamble and stop with a profit of 0.

hence, $P\{W<0\} = 1 - \frac{1}{2} - \frac{1}{4}$

$$\Rightarrow P\{W<0\} = \frac{1}{4}$$

c) Net winnings $(W) = 2 - X$

$$E[X] = \frac{1}{p} = \frac{1}{1/2} = 2$$

$$E[W] = E[2-X]$$

$$= 2 - E[X]$$

$$\Rightarrow E[W] = 0$$

7.4) given,

$$f_{X,Y}(x,y) = \begin{cases} 1/y, & \text{if } 0<y<1,\ 0<x<y \\ 0, & \text{otherwise.} \end{cases}$$

a) $E[XY] = \iint xy\, f(x,y)\, dx\, dy$

$$= \int_0^1 \int_0^y xy \frac{1}{y}\, dx\, dy = \int_0^1 \int_0^y x\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2}\right]_0^y dy = \int_0^1 \frac{y^2}{2}\, dy$$

$$= \left[\frac{y^3}{6}\right]_0^1 = \frac{1}{6}$$

$\Rightarrow E[XY] = \frac{1}{6}$

b) $E[X] = \int_0^1 \int_0^y x.f(x,y)\, dx\, dy = \int_0^1 \int_0^y x \cdot \frac{1}{y}\, dx\, dy$

$$= \int_0^1 \frac{1}{y}\left[\frac{x^2}{2}\right]_0^y dy = \int_0^1 \frac{1}{y} \cdot \frac{y^2}{2}\, dy$$

$$= \frac{1}{2}\int_0^1 y\, dy = \frac{1}{2}\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{4}$$

$\Rightarrow E[X] = \frac{1}{4}$

c) $E[Y] = \int_0^1 \int_0^y y f(x,y)\, dx\, dy = \int_0^1 \int_0^y y \frac{1}{y}\, dx\, dy$

$= \int_0^1 \int_0^y dx\, dy = \int_0^1 [x]_0^y\, dy = \int_0^1 y\, dy$

$= \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$

$\Rightarrow E[Y] = \frac{1}{2}$

7.7) Probability that item is chosen $P(A) = P(B) = \frac{3}{10}$

$P(A') = 1 - P(A) = 1 - \frac{3}{10} = \frac{7}{10} = P(B')$

a) $P(A \cap B) = \frac{3}{10} \times \frac{3}{10} = \frac{9}{100} = 0.09$

Expected no. of objects chosen by both $= P(A \cap B) \times 10$

$= 0.09 \times 10$

$\Rightarrow E(A \cap B) = 0.9$

b) $P(A' \cap B') = \frac{7}{10} \times \frac{7}{10} = \frac{49}{100} = 0.49$

Expected no. of objects chosen by none $= P(A' \cap B') \times 10$

$= 0.49 \times 10$

$\Rightarrow E(A' \cap B') = 4.9$

c) $P(A \cap B') + P(A' \cap B) = \left(\frac{3}{10} \times \frac{7}{10}\right) + \left(\frac{7}{10} \times \frac{3}{10}\right) = \frac{42}{100} = 0.42$

Expected no. of objects chosen by exactly one $= 0.42 \times 10$

$= 4.2$

7.23) Let $X_i = 1$, if the i^{th} white ball in urn 1 is ultimately one of the 3 Selected & '0' otherwise.

Let $Y_i = 1$, if the i^{th} white ball initially in urn 2 is ultimately selected & '0' otherwise

Prob. the i^{th} white ball is transferred from urn 1 to urn 2 is $= \frac{2}{11}$ (as there are 11 balls in urn 1 & we are picking 2)

Prob. i^{th} white ball from urn 1 is then picked from urn 2 is $= \frac{3}{20}$ (as there are 20 balls in urn 2 & we are picking 3)

So, $E[X_i] = \frac{2}{11} \times \frac{3}{20} = \frac{6}{220}$

Prob that the i^{th} white ball from urn 2 is ultimately picked is $= \frac{3}{20}$ (as there are 20 balls in urn 2 & we are picking 3)

So, $E[Y_i] = \frac{3}{20}$

The no. of white balls in the trio $= \sum_1^5 X_i + \sum_1^8 Y_i$

$$= 5\left(\frac{6}{220}\right) + 8\left(\frac{3}{20}\right)$$

$$= \underline{\underline{\frac{147}{110}}}$$

7.45) Given, mean $= 0$ & variance $= 1$

a) $Cov[X_1 + X_2$ and $X_2 + X_3]$

$$Cov[(x_1+x_2),(x_2+x_3)] = E[(x_1+x_2)(x_2+x_3)] - E[(x_1+x_2)]E[(x_2+x_3)]$$

$$= E[x_1x_2 + x_1x_3 + x_2^2 + x_2x_3] - 0(0)$$

$$= E[x_2^2]$$

$$Var[x_2] = 1$$

$$E[x_2^2] - E[x_2]^2 = 1$$

$$E[x_2^2] - 0 = 1 \Rightarrow E[x_2^2] = 1$$

$$\Rightarrow Cov[(x_1+x_2),(x_2+x_3)] = 1$$

$$Var[x_i + x_j] = Var[x_i] + Var[x_j] = 1 + 1 = 2$$

$$Corr[(x_1+x_2),(x_2+x_3)] = \frac{Cov[(x_1+x_2),(x_2+x_3)]}{\sqrt{Var[x_1+x_2]}\sqrt{Var[x_2+x_3]}}$$

$$= \frac{1}{\sqrt{2}\sqrt{2}}$$

$$\Rightarrow Corr[(x_1+x_2),(x_2+x_3)] = \frac{1}{2}$$

b) $Cov[(x_1+x_2),(x_3+x_4)]$

$$= E[x_1x_2 + x_1x_3 + x_2x_3 + x_2x_4] - E[x_1+x_2]E[x_3+x_4]$$

$$= 0 - 0(0)$$

$$= 0$$

Hence, $Corr[(x_1+x_2),(x_3+x_4)] = 0$

7.49) Let N_i be the no. of heads in final i flips.

Let C_1 be the event when coin 1 is flipped

Let C_2 be the event when coin 2 is flipped.

Let H be the event that 2 of the first 3 flips lands on head.

$P(C_1 \text{ landing on head}) = 0.4 \Rightarrow P(C_1) = 0.6$

$P(C_2 \text{ landing on head}) = 0.7 \Rightarrow P(C_2) = 0.3$

$$P(C_1|H) = \frac{P(H/C_1)\,P(C_1)}{P(H|C_1)\,P(C_1) + P(H|C_2)\,P(C_2)}$$

$$= \frac{3(0.4)^2(0.6)}{3(0.4)^2(0.6) + 3(0.7)^2(0.3)}$$

$$P(C_1/H) = \frac{32}{81}$$

$$P(C_2/H) = \frac{P(H/C_2)\,P(C_2)}{P(H/C_2)\,P(C_2) + P(H|C_1)\,P(C_1)} = \frac{3(0.7)^2(0.3)}{3(0.7)^2(0.3) + 3(0.4)^2(0.6)}$$

$$\Rightarrow P(C_2/H) = \frac{49}{81}$$

$$E[N_{10}|H] = 2 + E[N_7|H]$$

$$E[N_7|H] = E[N_7|HC_1]\,P[C_1H] + E[N_7|HC_2]\,P[C_2H]$$

$$= 7(0.4)\left(\frac{32}{81}\right) + 7(0.7)\left(\frac{49}{81}\right)$$

$$E[N_7|H] = 4.07$$

$$\Rightarrow E[N_{10}|H] = 2 + 4.07 = \underline{\underline{6.07}}$$

Theoretical problems

7.13)

a) $X_i = \begin{cases} 1, & \text{if record occurs at time } i \\ 0, & \text{otherwise} \end{cases}$

$X = \sum_{i=1}^{n} X_i$ $\{X_i \to iid\}$

$E[X_i] = P[X_i = 1] = \frac{(i-1)!}{i!} = \frac{1}{i}$

$E[X] = E\left[\sum_{i=1}^{n} X_i\right]$

$\Rightarrow E(X) = \sum_{i=1}^{n} \frac{1}{i}$

b) $Var[X_i] = E[X_i^2] - E[X_i]^2$

$E[X_i] = E[X_i^2] = \frac{1}{i}$

$Var[X_i] = \frac{1}{i} - \frac{1}{i^2}$

$Var(X) = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i^2}\right)$

$= \sum_{i=1}^{n} \frac{i-1}{i^2}$

7.17) Estimate of mean (μ) of weighted average is

$$\hat{\mu} = \lambda x_1 + (1-\lambda) x_2$$

The value that λ yields the estimate having lowest variance is given by:

$$Var(\hat{\mu}) = Var(\lambda x_1 + (1-\lambda) x_2)$$

$$= \lambda^2 var(x_1) + (1-\lambda)^2 var(x_2) + 2\lambda(1-\lambda) Cov(x_1, x_2)$$

$$Var(\hat{\mu}) = \lambda^2 \sigma_1^2 + (1-\lambda)^2 \sigma_2^2$$

diff. with respect to λ.

$$\frac{\partial}{\partial \lambda}\left(\lambda^2 \sigma_1^2 + (1-\lambda)^2 \sigma_2^2\right) = 2\lambda\sigma_1^2 - 2(1-\lambda)^2\sigma_2^2$$

when $\frac{d}{d\lambda}\left(Var(\hat{\mu})\right) = 0$,

$$2\lambda\sigma_1^2 - 2(1-\lambda)^2\sigma_2^2 = 0$$

$$\lambda\sigma_1^2 = (1-\lambda)\sigma_2^2$$

$$\lambda\sigma_1^2 = \sigma_2^2 - \lambda\sigma_2^2$$

$$\lambda(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$

$$\Rightarrow \boxed{\lambda = \frac{\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)}}$$