HW-6

① given, $f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)$ $0 < x < 1,\ 0 < y < 2$

a) $\iint \cancel{f_{x}} f_{X,Y}\, f(x,y)\, dx\, dy = 1.$

So as to be a valid joint density function.

$$= \frac{6}{7}\int_0^2\int_0^1 \left(x^2 + \frac{xy}{2}\right) dx\, dy$$

$$= \frac{6}{7}\int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{4}\right]_0^1 dy$$

$$= \frac{6}{7}\int_0^2 \left(\frac{1}{3} + \frac{y}{4}\right) dy$$

$$= \frac{6}{7}\left[\frac{y}{3} + \frac{y^2}{8}\right]_0^2$$

$$= \frac{6}{7}\left[\frac{2}{3} + \frac{4}{8}\right]$$

$$= 1$$

Hence, this is indeed a joint density function.

b) to find the density function of X.

$$f_X(x) = \int_0^x f_{X,Y}(x,y)\, dy \quad , \quad 0 < x < 1.$$

$$= \frac{6}{7}\left[x^2y + \frac{xy^2}{4}\right]_0^2$$

$$= \frac{6}{7}\left[2x^2 + x\right]$$

Hence,

$$\Rightarrow \quad f_X(x) = \begin{cases} \frac{6}{7}\left[2x^2+x\right] & ; 0 < x < 1 \\ 0 & ; \text{otherwise.} \end{cases}$$

c) find $P\{X > Y\} = \int_0^1 \int_0^x \frac{6}{7}\left[x^2 + \frac{xy}{2}\right] dy\, dx$

$$= \frac{6}{7}\int_0^1 \left[x^2y + \frac{xy^2}{4}\right]_0^x dx$$

$$= \frac{6}{7}\int_0^1 \left[x^3 + \frac{x^3}{4}\right] dx$$

$$= \frac{6}{7} \times \frac{5}{4}\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{15}{56}.$$

d) find $P\{Y > 1/2 \mid X < 1/2\}$

$$= \frac{P(Y > 1/2 \text{ and } X < 1/2)}{P(X < 1/2)}$$

$$= \frac{\frac{6}{7}\int_{1/2}^{2}\int_{0}^{1/2}\left[x^2 + \frac{xy}{2}\right] dx\, dy}{\frac{6}{7}\int_{0}^{1/2}\left[2x^2 + x\right] dx}$$

$$= \frac{\frac{6}{7}\int_{1/2}^{2}\left[\frac{x^3}{3} + \frac{x^2y}{4}\right]_0^{1/2} dy}{\frac{6}{7}\left[\frac{2}{3}x^3 + \frac{x^2}{2}\right]_0^{1/2}}$$

$$= \frac{\int_{1/2}^{2}\left(\frac{1}{24} + \frac{y}{16}\right) dy}{\frac{1}{12} + \frac{1}{8}}$$

$$= \frac{\left[\frac{y}{24} + \frac{y^2}{32}\right]_{1/2}^{2}}{\frac{5}{24}}$$

$$= \frac{69}{80}$$

e) $E(X) = \int x f_X[x]\,dx$

$= \frac{6}{7}\int_0^1 [2x^3 + x^2]\,dx$

$= \frac{6}{7}\left[\frac{2x^4}{4} + \frac{x^3}{3}\right]_0^1$

$\Rightarrow E[X] = \frac{5}{7}$

f) $E[Y] =$

$f_Y(y) = \int_0^1 f_{X,Y}(x,y)\,dx \,,\ 0 < y < 2$

$= \frac{6}{7}\left[\frac{x^3}{3} + \frac{x^2 y}{4}\right]_0^1$

$= \frac{6}{7}\left[\frac{1}{3} + \frac{y}{4}\right] \;;\ 0 < y < 2$

$E[Y] = \int y f_Y(y)\,dy$

$= \frac{6}{7}\int_0^1 \left[\frac{y}{3} + \frac{y^2}{4}\right] dy$

$= \frac{6}{7}\left[\frac{y^2}{6} + \frac{y^3}{12}\right]_0^2$

$\Rightarrow E[Y] = \frac{8}{7}$

6.13) Let 'x' be the arrival time of the man.
'y' be the arrival time of the woman.

hence, $X \sim Unif(15, 45)$

$Y \sim Unif(0, 60)$.

The joint pdf is given by

$$f(x, y) = \begin{cases} \frac{1}{30} \cdot \frac{1}{60} & , 15 \leq x \leq 45 \\ & \quad 0 \leq y \leq 60. \\ 0 & , \text{otherwise} \end{cases}$$

$$f(x, y) = \begin{cases} \frac{1}{1800} & ; 15 \leq x \leq 45 \;\&\; 0 \leq y \leq 60 \\ 0 & ; \text{otherwise} \end{cases}$$

i) P[first to arrive waits no longer than 5 min]

when woman arrives first,

$$P[X-5 < Y < X] = \int_{15}^{45} \int_{x-5}^{x} \frac{1}{1800} \, dy \, dx.$$

$$= \int_{15}^{45} \frac{1}{1800} [\cancel{x} - \cancel{x} + 5] \, dx$$

$$= \frac{1}{360} \int_{15}^{45} dx.$$

$$= \frac{1}{360} \{45 - 15\}$$

$$= \frac{1}{12}.$$

when man arrives first,

$$P[X < Y < X+5] = \int_{15}^{45}\int_{x}^{x+5} \frac{1}{1800}\, dy\, dx$$

$$= \frac{1}{360}[45-15]$$

$$= \frac{1}{12}$$

Hence, P[first to arrive waits no longer than 5 min] $= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$

ii) P[Man arrives first]

$$P[X < Y] = \int_{15}^{45}\int_{x}^{60} \frac{1}{1800}\, dy\, dx.$$

$$= \frac{1}{1800}\int_{15}^{45} (60-x)\, dx$$

$$= \frac{1}{1800} \times \left[60x - \frac{x^2}{2}\right]_{15}^{45}$$

$$\Rightarrow P[X < Y] = \frac{1}{2}.$$

6.22) given, $f(x,y) = \begin{cases} x+y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$

a) for x & y to be independent

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

$$f_X(x) = \int_y f(x,y)\, dy \quad \cancel{\text{marginal}}$$

$$= \int_0^1 (x+y)\, dy.$$

$$f_X(x) = \left[xy + \frac{y^2}{2} \right]_0^1 = \left[x + \frac{1}{2} \right]$$

Similarly, $f_Y(y) = \left[y + \frac{1}{2} \right]$

Now, $f_X(x) \cdot f_Y(y) = \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right)$

$$= xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{4}$$

Since, $f(x,y) \neq f_X(x) \cdot f_Y(y)$

x and y are not independent.

b) density function of x

$$f_X(x) = \int_y f(x,y)\,dy$$

$$= \int_0^1 (x+y)\,dy$$

$$= \left[xy + \frac{y^2}{2}\right]_0^1$$

$$\Rightarrow f_X(x) = x + \frac{1}{2} \quad ; \quad 0 < x < 1$$

c) $$P[X+Y<1] = \int_0^1 \int_0^{1-y} (x+y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + xy\right]_0^{1-y} dy$$

$$= \int_0^1 \frac{(1-y)^2}{2} + y(1-y)\,dy$$

$$= \int_0^1 \left(\frac{1^2 + y^2 - 2y}{2} + y - y^2\right) dy$$

$$= \frac{1}{2}\left[y + \frac{y^3}{6} - \frac{2y^2}{2}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 + \left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\cancel{1} + \frac{1}{6} - \cancel{1}\right] + \frac{1}{2} + \frac{1}{3} = \frac{1}{12} + \frac{1}{2} + \frac{1}{3} = \frac{1+6+4}{12}$$

$$= \int_0^1 \frac{1 + y^2 - 2y + 2y - 2y^2}{2} \, dy$$

$$= \frac{1}{2}\int_0^1 1 - y^2 \, dy \qquad = \frac{1}{2}\left[y - \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

6.23) given, $f(x,y) = \begin{cases} 12xy(1-x) & ; 0<x<1, 0<y<1 \\ 0 & ; \text{otherwise.} \end{cases}$

a) for x & y to be independent

$$f(x,y) = f_x(x) \cdot f_y(y)$$

$$f_x(x) = \int_0^1 12xy(1-x) \, dy$$

$$= 12x(1-x) \int_0^1 y \, dy$$

$$= 12x(1-x)\left[\frac{y^2}{2}\right]_0^1$$

$$\Rightarrow f_x(x) = 6x(1-x)$$

$$f_y(y) = \int_0^1 12xy(1-x) \, dx$$

$$= \cancel{\int} 12y \int_0^1 (x - x^2) \, dx$$

$$= 12y\left(\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right)$$

$$= 12y\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 12y \cdot \frac{1}{6}.$$

$$\Rightarrow f_Y(y) = 2y$$

Now, $f_X(x) \cdot f_Y(y) = 6x(1-x) \cdot 2y$

$$= 12xy(1-x)$$

$$= f(x,y)$$

Hence, X & Y are independent.

b) $E[X] = \cancel{\int_0^1\int_0^1 x(12xy(1-x))\,dx\,dy}.$

$$= \int_0^1 x(6x(1-x))\,dx.$$

$$= 6\int_0^1 (x^2 - x^3)\,dx$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= \frac{6}{12} \Rightarrow E[X] = \frac{1}{2}$$

c) $E[Y] = \int_0^1 y \cdot 2y \, dy$

$= 2\left[\frac{y^3}{3}\right]_0^1$

$= \frac{2}{3}$

d) $E[X^2] = \int_0^1 x^2 (6x(1-x)) \, dx$

$= 6\int_0^1 (x^3 - x^4) \, dx$

$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$

$= 6\left[\frac{1}{4} - \frac{1}{5}\right] = 6\left(\frac{5-4}{20}\right) = \frac{6}{20}.$

$E[X^2] = \frac{3}{10}$

$Var(X) = E[X^2] - (E[X])^2$

$= \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{3}{10} - \frac{1}{4} = \frac{6-5}{20}$

$= \frac{1}{20}$

e) $E[y^2] = \int_0^1 y^2 \, 2y \, dy$

$= 2\int_0^1 y^3 \, dy$

$= 2\left[\frac{y^4}{4}\right]_0^1$

$= \frac{1}{2}$

$Var(y) = E[y^2] - (E[y])^2$

$= \frac{1}{2} - \left(\frac{2}{3}\right)^2$

$= \frac{1}{2} - \frac{4}{9}$

$\Rightarrow Var(y) = \frac{1}{18}$

6.32) given, Mean = 100 , Standard deviation = 5

a) Let 'x' is the no. of months having sales greater than 100.

P[Exactly 3 of next 6 months have greater sales than 100]

~~P[X = 3]~~

As, Monthly sales are normally distributed, probability of having sales greater than 100 irrespective of months it is 1/2.

~~P[X = 3]~~

from, Binomial distribution for a PMF is

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

here, $n = 6$, $k = 3$, $p = \frac{1}{2}$

$$P(X=3) = \binom{6}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3$$

$$= 20 \times \frac{1}{8} \times \frac{1}{8}$$

$$\Rightarrow P(X=3) = \frac{5}{16}.$$

b) P[Total of the Sales in the next 4 months is greater than 420]

Mean of month Sales for next four months

$= 4 \times 100$

$= 400$

Similarly, Standard Deviation $= 4 \times 25$

$= 100$

$$P[x > 420] = P\left[\frac{X-400}{10} > \frac{420-400}{10}\right]$$

$$\left\{P\left(Z > \frac{X-\mu}{\sigma}\right)\right\}$$

$= P[Z > 2]$

$= 1 - P(Z < 2)$ {from Z-table}

$= 1 - 0.9772.$

$\Rightarrow P(x > 420) = 0.0228$

6.40) given, $P(1,1) = \frac{1}{8}$ $\quad P(1,2) = \frac{1}{4}$

$P(2,1) = \frac{1}{8}$ $\quad P(2,2) = \frac{1}{2}$

~~x\y~~	~~1~~	~~2~~	~~P(y)~~
~~1~~	~~1/8~~	~~1/4~~	
~~2~~	~~1/8~~	~~1/2~~	
~~P(x)~~			

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$P(x=1) = P(1,1) + P(1,2) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

$P(y=1) = P(1,1) + P(2,1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

a) conditional mass function of X given $Y=i$, $i = 1, 2$.

$P_{X|Y}(1|1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/8}{1/4} = \frac{1}{2}$ //

$P_{X|Y}(2|1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/8}{1/4} = \frac{1}{2}$ //

$P(y=2) = P(1,2) + P(2,2) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$P_{X|Y}(1|2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{1/4}{3/4} = \frac{1}{3}$ //

$$P_{X|Y}(2,2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{1/2}{3/4} = \frac{2}{3}$$

b) for Independence

$$p(X=1) \cdot P(Y=1) = p(1,1)$$

$$p(X=1) \cdot p(Y=1) = \frac{3}{8} \times \frac{1}{4} = \frac{3}{32}$$

$$P(1,1) = \frac{1}{8} \neq \frac{3}{32}$$

Hence, X & Y are not independent.

c) $P(XY \leq 3) = P(1,1) + P(1,2) + P(2,1)$

$$= \frac{1}{8} + \frac{1}{4} + \frac{1}{8}$$

$$= \frac{2}{8} + \frac{1}{4} = \frac{4}{8}$$

$$P(XY \leq 3) = \frac{1}{2}$$

$P(X+Y > 2) = P(1,2) + P(2,1) + P(2,2)$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{2}$$

$$= \frac{2+1+4}{8}$$

$$\Rightarrow P(X+Y > 2) = 7/8$$

$$p(x/y>1) = p(2,1)$$
$$= \frac{1}{8}$$