HW-5

5.7) given, $f(x) = \begin{cases} a + bx^2 & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$

$E(X) = \frac{3}{5}$

from property of density function,

* $\int_{-a}^{a} f(x)\,dx = 1$
* $f(x) \ge 0$ for all x

therefore, $\int_{-a}^{a} (a + bx^2)\,dx = 1$

$$\int_{0}^{1} (a + bx^2)\,dx = 1$$

$$\left[ax + \frac{bx^3}{3} \right]_0^1 = 1$$

$$a + \frac{b}{3} = 1 \rightarrow 1$$

from expectation we get,

$$E(x) = \int_0^1 x(a+bx^2)\,dx = \frac{3}{5}$$

$$\left[\frac{ax^2}{2} + \frac{bx^4}{4}\right]_0^1 = \frac{3}{5}$$

$$\frac{a}{2} + \frac{b}{4} = \frac{3}{5}$$

$$a + \frac{b}{2} = \frac{6}{5}. \Rightarrow 2$$

Now, eq 1 - eq 2 ; we get

$$\left(\frac{b}{3} - \frac{b}{2}\right) = 1 - \frac{6}{5}$$

$$\frac{-b}{6} = \frac{-1}{5}$$

$$\boxed{b = \frac{6}{5}}$$

Substitute $b = 6/5$ in 1

$$a = 1 - \frac{\frac{\cancel{6}\,2}{5}}{\cancel{3}} \Rightarrow a = 1 - \frac{2}{5}. = \frac{3}{5}$$

Therefore, $a = \frac{6}{5}$ & $b = \frac{3}{5}$

5.13) We know that, A random variable has a uniform density if and only if its PMF is given by:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{, otherwise.} \end{cases}$$

here, $a = 0$ to $b = 30$

$$f(x) = \frac{1}{b-a} = \frac{1}{30}$$

So, $f(x) = \frac{1}{30}$ where $0 < x < 30$

where x is the waiting time.

a) P(waiting longer than 10min) $= P(X > 10)$

$$P(x > 10) = \int_{10}^{30} \frac{1}{30}\,dx = \frac{1}{30}[x]_{10}^{30}$$

$$P(x > 10) = \frac{20}{30} = \frac{2}{3}$$

b) P(waiting 15 min to 25 min) $= P(15 < x < 25)$

$$P(15 < x < 25) = \int_{15}^{25} \frac{1}{30}\,dx = \frac{1}{30}[x]_{15}^{25}$$

$$P(15 < x < 25) = \frac{10}{30} = \frac{1}{3}$$

5.16) Let $P(X>50)$ be the probability of having a rainfall greater than 50 inches in a year.

So, it can be written in standard normal variable as:

$$P\left(Z > \frac{50-\mu}{\sigma}\right)$$

given, $\mu = 40$ & $\sigma = 4$

$$\text{so, } P\left(Z > \frac{50-40}{4}\right)$$

$$P(Z > 2.5) \Rightarrow 1 - P(Z \leq 2.5)$$

from the Z-table, $= 1 - 0.9938$.

$$\cancel{P(Z>2.5)} = 0.00621$$

The probability that it will take over 10 yrs before a year occurs having a rainfall over 50 inches is given as:

= probability that there is no rainfall of over 50 inches over next 10 years.

$$= (1 - 0.00621)^{10}$$

$$= \cancel{0.936}$$

$$= 0.9396.$$

Here, the assumption made is that all the years have independent rainfall.

5.21) given, $\mu = 71$, $\sigma^2 = 6.25$

$\sigma = 2.5$

$6'2''$ tall $= (6 \times 12 + 2)$

$= 74$ inches.

Probability of 25 year old men are over $6'2''$ tall is given by $P(X > 6'2'') = P(X > 7\cancel{4})$.

writting it in Standard normal variable,

$$\cancel{P(Z \geq 174} = P\left(Z > \frac{X-\mu}{\sigma}\right) = P\left(Z > \frac{74-71}{2.5}\right)$$

$$\Rightarrow P\left(Z > \frac{6}{5}\right) = P(Z > 1.2)$$

$$= 1 - P(Z \leq 1.2)$$

$$= 1 - 0.88493$$

$$= 0.11507$$

Hence, Percentage of 25 year old men are over $6'2''$ tall is 11.5%.

The probability of men in the 6 footer club are over $6'5''$ is given by $P(X > 6'5'') = P(X > 77 \text{ inches})$

$$P\left(Z > \frac{X-\mu}{\sigma}\right) = P\left(Z > \frac{77-71}{2.5}\right)$$

$$= P(Z > 2.4)$$

$$= 1 - P(Z \leq 2.4)$$

$$= 1 - 0.99180$$

$$= 0.0082$$

Hence, the percentage of men in the 6-footer club are taller than 6 feet 5 inches is $\underline{\underline{0.82\%}}$

5.32) given, $\lambda = \frac{1}{2}$.

from the property of memoryless property of exponential function. we know.

$$f(x) = e^{-\lambda x}.$$

a) Probability that a repair exceeds 2 hours is given by $P(X > 2) = 1 - P(X \leq 2)$

$$= 1 - (1 - e^{-1})$$

$$= e^{-1}$$

$$= 0.3679.$$

b) The conditional probability that a repair takes at least 10hrs, given that its duration exceeds 9 hours.

$$P(X \geq 10 \mid X \geq 9) = \frac{P(X \geq 10)}{P(X \geq 9)}$$

from Memory less property,

$$P(X \geq 10 \mid X \geq 9) = P(X \geq 1)$$

$$P(X \geq 1) = e^{-1/2 \times 1}$$

$$= e^{-0.5}$$

$$= 0.6065$$

5.40) given, X is a random variable following uniform distribution with parameter 0 and 1
let 'Y' is the random variable & it is defined as

$Y = e^x$

The CDF of 'Y' is $F_Y(y) = P(Y \leq y)$

$= P(e^X \leq y)$

$= P(\ln(e^X) \leq \ln(y))$

$= P(X \ln(e) \leq \ln(y))$

$= P(X \leq \ln(y))$

Limits of 'X' is: $0 < X < 1$

$X = 0 \Rightarrow Y = e^0 \Rightarrow Y = 1$

$X = 1 \Rightarrow Y = e^1 \Rightarrow Y = e.$

Limits of 'Y' is $1 < Y < e$.

Now, differentiate CDF to get PDF w.r.t 'y'

$f_X(x) = F'_X(\ln(y))$

$= f_X(\ln(y)) \frac{d}{dy}(\ln(y))$

$= f_X(\ln(y)) \left(\frac{1}{y}\right)$

$= \frac{1}{y} f_X(\ln(y))$

But $f_X(\ln(y)) = 1$ because $X \sim U(0,1)$

So, the PDF ~~of~~ of Y is

$$\Rightarrow f_Y(y) = \begin{cases} 1/y & ; 1 < y < e \\ 0 & ; \text{otherwise} \end{cases}$$