Homework - 9

8.2) Given, $\mu = 75 = E(x)$

a) by markov's inequality,

we know, $P(x \geq a) \leq \frac{E(x)}{a}$

An upperbound for the prob. that a student's test score will exceed 85 is

$\Rightarrow P(x \geq 85) \leq \frac{75}{85} = \frac{15}{17}$

b) From Chebyshev's inequality,

we know, $P\{|X-\mu| \geq a\} = \frac{\sigma^2}{a^2}$, when $a > 0$

given, Variance $(\sigma^2) = 25 \Rightarrow \sigma = 5$

$P\{65 < X < 85\} = P\{|X-75| \geq 10\} \geq \frac{25}{100}$

$P\{|x-75| \leq 10\} \geq 1 - \frac{1}{4}$

$\Rightarrow P\{65 < x < 85\} \geq \frac{3}{4}$

c) Let n represent required no. of Students.

$$\bar{x} = \frac{\sum x_i}{n}$$

Where, x_i is the test score of the i^{th} student.

Then, $E[x_i] = E[x] = \mu$

$Var(x_i) = Var(x) = \sigma^2$

& $E(\bar{x}) = \mu$ & $Var(\bar{x}) = \frac{\sigma^2}{n}$

From Chebyshev's inequality,

$$P\{|\bar{x}-75| \geq 5\} \leq \frac{\sigma^2}{5^2(n)} = \frac{25}{25(n)} = \frac{1}{n}$$

$$P\{|\bar{x}-75| < 5\} \geq 1-\frac{1}{n}$$

Given, $P\{|\bar{x}-75|\} \geq 0.9$.

Now, $1-\frac{1}{n} \geq 0.9 \rightarrow \frac{1}{n} \geq 0.1$

$\Rightarrow$ Hence, $\boxed{n \geq 10}$

8.3) $\bar{X} = \frac{\sum x_i}{n}$, $\mu = 75$, $\sigma^2 = Var(\bar{X}) = \frac{25}{n}$

From Central limit theorem,

$$P\left\{ \frac{\sum_{i=1}^{n} (x_i - \mu_i)}{\sqrt{\sum_{i=1}^{n} \sigma^2}} < a \right\} \rightarrow \phi(a)$$

$$P\{|\bar{x} - 75| \leq 5\} = P\{70 \leq \bar{x} \leq 80\} = 0.9$$

$$\phi\left(\frac{80-75}{\sqrt{25/n}}\right) - \phi\left(\frac{70-75}{\sqrt{25/n}}\right) = 0.9$$

$$\phi(\sqrt{n}) - \phi(-\sqrt{n}) = 0.9$$

$$2\phi(\sqrt{n}) - 1 = 0.9$$

$$\phi(\sqrt{n}) = 0.95$$

$$\sqrt{n} = 1.65 \quad \{\text{from z-table}\}$$

$$\Rightarrow \boxed{n \approx 2.72}$$

8.4) Let X_i be independent Poisson R.V.; $i = 1, 2, \cdots 20$

$\mu = 1$

a) Markov inequality : $P\left\{\sum_1^{20} X_i > 15\right\}$

from markov's inequality,

we know, $P(X \geq a) \leq \frac{E(X)}{a}$

$$\Rightarrow P\left(\sum_1^{20} X_i > 15\right) \leq \frac{E\left(\sum_1^{20} X_i\right)}{15}$$

Since, X_i are independent with mean 1,

$$P\left(\sum_1^{20} X_i > 15\right) = \frac{\sum_1^{20} (1)}{15} = \frac{20}{15} = \frac{4}{3}$$

Hence, $\boxed{P\left\{\sum_1^{20} X_i > 15\right\} = \frac{4}{3}}$

b) Central limit theorem: $P\left\{\sum_1^{20} X_i > 15\right\}$

$$P\left(\sum_1^{20} X_i \geq 15\right) = P\left(\frac{\sum_1^{20} X_i - E\left(\sum_1^{20} X_i\right)}{\sqrt{Var\left(\sum_1^{20} X_i\right)}} \geq \frac{15 - E\left(\sum_1^{20} X_i\right)}{\sqrt{Var\left(\sum_1^{20} X_i\right)}}\right)$$

$$= P\left(\frac{\sum X_i - 20}{\sqrt{20}} \geq \frac{15.5 - 20}{\sqrt{20}}\right)$$

$$= P\left(Z \geq \frac{-4.5}{2\sqrt{5}}\right)$$

$$= P(Z \geq -1.006)$$

$$= 1 - 0.1587$$

$$\Rightarrow \boxed{P\left\{\sum_{1}^{20} X_i > 15\right\} = 0.8413}$$

8.15) Let X_i be the yearly income for i^{th} policy holder.
Let 'x' be the yearly claim per policy holder.
the Sum of claims for 10,000 policy holders, denoted as 'S', can be expressed as $S = X_1 + X_2 + \cdots + X_{10000}$

given, $\mu_x = 240$, $\sigma_x = 800$, $n = 10,000$

Mean of the Sum, $\mu_s = n \cdot \mu_x$

$$= 10000(240)$$

$$\mu_s = 2400000$$

S.D of the Sum, $\sigma_s = \sqrt{n}\,(\sigma_x)$

$$= \sqrt{10000}\,(800)$$

$$\sigma_s = 8000$$

the probability that the total yearly claim (S) exceeds \$2.7 million is given by:

$$P\left(\sum_{1}^{10000} X_i > 2.7\right) = P\left(Z > \frac{2700000 - 2400000}{8000}\right)$$

$$= P(Z > 3.75)$$

$$\Rightarrow \boxed{P\left(\sum_{1}^{10000} X_i > 2.7\right) \simeq 0}$$

Theoretical problems:

8.1) Let 'X' be a random variable with mean 'μ' and standard deviation 'σ'.

From Chebyshev's inequality,

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2} \quad ; \; k > 0$$

Consider a R.V $Y = (X-\mu)^2$ which represents the squared devation from the mean.

Hence, $E[Y] = E[(X-\mu)^2] = Var(X) = \sigma^2$

Now, Apply Markov's inequality to 'Y'!

$$P(Y \geq k^2\sigma^2) \leq \frac{E[Y]}{k^2\sigma^2}$$

Now, $P((X-\mu)^2 \geq k^2\sigma^2) \leq \frac{\sigma^2}{k^2\sigma^2}$

$$\Rightarrow \text{Hence, } \boxed{P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}}$$

8. 9) The Strong law of large numbers suggests that as the no. of coin tosses increases, the proportion of heads should approach the expected probability, which is 0.5 for a fair coin. However, this convergence is not immediate or compensatory for short-term deviations. For instance, if the first 100 tosses all results in heads, it doesn't guarantee an immediate correction in the subsequent tosses.

In the scenario where the first 100 tosses are all heads, the expected value of the next 900 tosses is not influenced by the initial outcome. i.e.,

$$E[900 \text{ tosses} \mid 100 \text{ tosses are heads}] = E[900 \text{ tosses}].$$

The law of large numbers implies that over a large no. of tosses, the impact of the initial 100 heads becomes insignificant.

The Statement emphasizes that the law doesn't componesate for or adjust probabilities based on short-term results. While expectation is that, the proportion of heads will converge to 0.5, this doesn't necessarily mean an immediate balancing of outcomes in next 900 tosses.