Lecture 6 - Outline

Sunday, September 15, 2019 1:06 PM

Linear Equation Systems

- Elementary examples
- Existence + uniqueness
- Augmented matrix
- Row echelon form

Graph and graph theory

Markov chains

Linear Equation Systems

Sunday, September 15, 2019 4:29 PM

$\underline{A}\,\underline{x} = \underline{b}$

- System of linear algebraic equations
- Transformation for $\underline{x}$ to $\underline{b}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

system (usually known)

usually unknowns

usually known

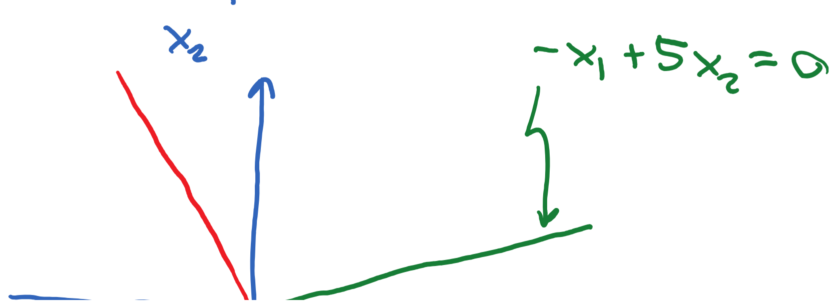
Elementary examples

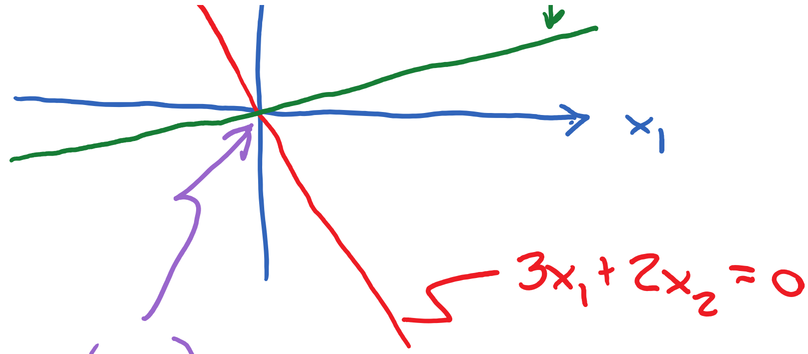
$\boxed{I}$ $\underline{A}\,\underline{x} = \underline{0}$

$$\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Homogeneous set

Examine equations in x_1–x_2 plane





$(0,0)$ ~ only intersection

For homogeneous set, always have the trivial solution $x_1 = 0,\ x_2 = 0$

Here, $\det(\underline{A}) = 3(5) - (-1)(2) = 17 \neq 0$

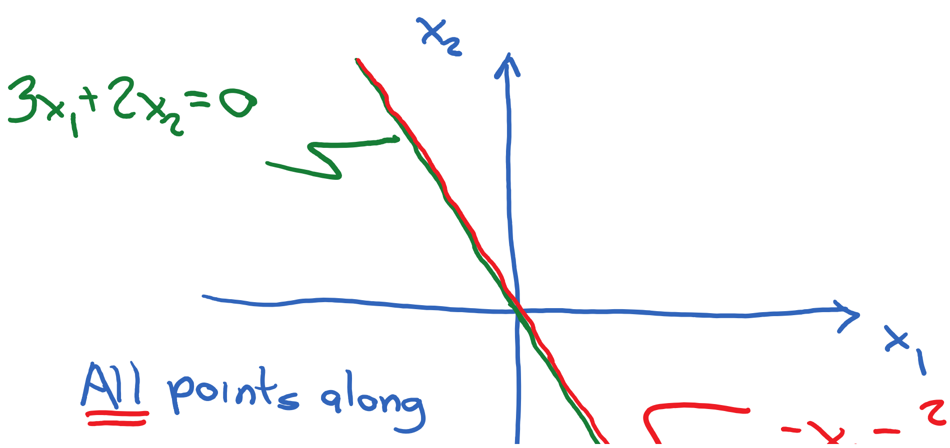
and thus this solution is $\underline{\underline{\text{unique}}}$

$\boxed{2}$ $\quad \underline{B}\,\underline{x} = \underline{0}$

$$\begin{bmatrix} 3 & 2 \\ -1 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

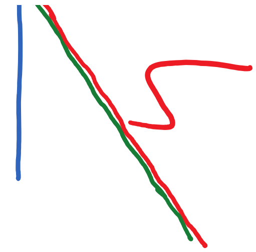
Again, homogeneous set

Always have trivial solution $x_1 = 0,\ x_2 = 0$



All points along

All points along line are solutions



$-x_1 - \frac{2}{3}x_2 = 0$

Here, $\det(\underline{B}) = 3\left(-\frac{2}{3}\right) - (-1)(2) = 0$

Equations (rows) are linearly dependent

Non-uniqueness of the solution; an infinity of solutions

$\boxed{3}$

$$\underline{A}\,\underline{x} = \underline{b}$$

$\underline{b} \neq \underline{0}$

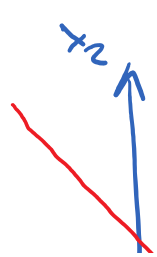
$$\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

Nonhomogeneous set

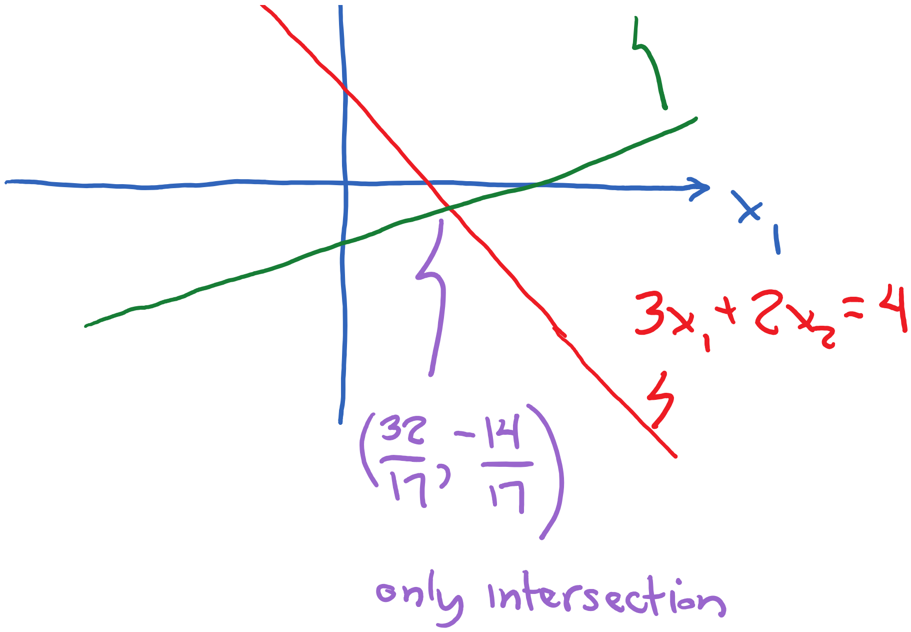
Note: $\underline{x} = \underline{0}$ is not a solution

Here $\det(\underline{A}) = 3(5) - (-1)(2) = 17 \neq 0$

$\therefore$ Unique non-trivial solution



$-x_1 + 5x_2 = -6$



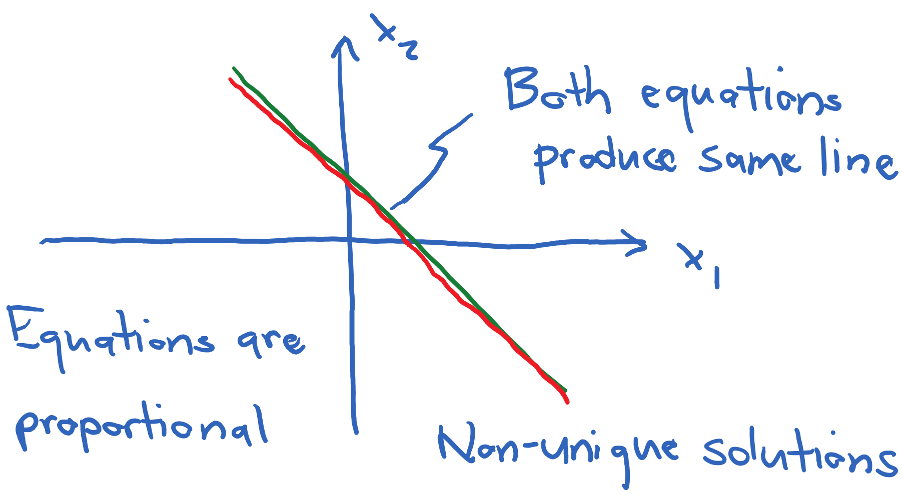
$\boxed{4}$

$$\underline{B}\,\underline{x} = \underline{c}$$

$$\begin{bmatrix} 3 & 2 \\ -1 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -\frac{4}{3} \end{bmatrix}$$

$$\det(\underline{B}) = 3\left(-\frac{2}{3}\right) - (-1)(2) = 0$$

$\therefore$ $\underline{B}$ is singular

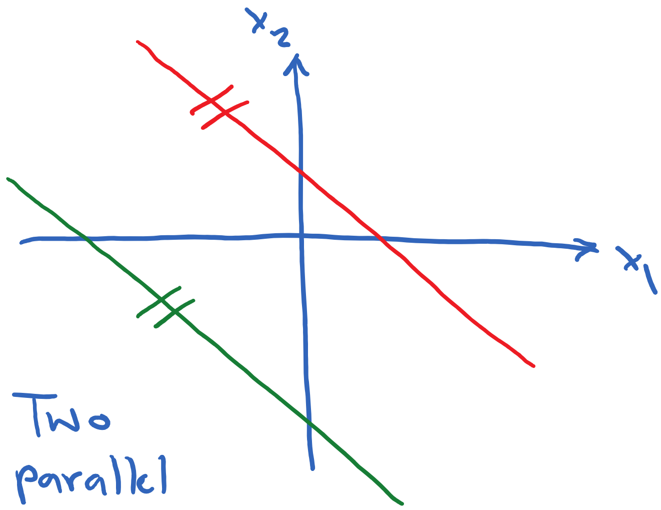


How many solutions?

$\boxed{5}$

$$\underline{B}\,\underline{x} = \underline{d}$$

$$\begin{bmatrix} 3 & 2 \\ -1 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



Two parallel lines

$\det(\underline{B}) = 0$, but eqns are not proportional (due to $\underline{d}$ vector)

No intersections → No solution exists

Equations are inconsistent

Overall, three possibilities: No solution; one solution; an infinity of solutions

What happens if the lines are nearly parallel?

$\det(\underline{B}) \approx 0$

Terminology $\underline{A}\,\underline{x} = \underline{b}$

Homogeneous eqns $\underline{b} = \underline{0}$

Non-homogeneous eqns $\underline{b} \neq \underline{0}$

Trivial solutions $\underline{x} = \underline{0}$

Non-trivial solutions $\underline{x} \neq \underline{0}$

Linear dependence

Try to write one equation as a linear combination of the others

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n - b_i = 0$$

$$\begin{aligned}
&= c_1(a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n - b_1) \\
&+ c_2(\qquad\qquad) \\
&+ \dots \\
&+ c_{i-1}(a_{i-1\,1}x_1 + \dots \qquad -b_{i-1}) \\
&+ c_{i+1}(a_{i+1\,1}x_1 + \dots \qquad -b_{i+1}) \\
&+ \dots \\
&+ c_n(a_{n1}x_1 + a_{n2}x_2 + \dots \qquad -b_n) \qquad (1)
\end{aligned}$$

If we can find $c_1, c_2, \dots, c_n$, such that (1) is true, then equation i is linearly dependent

If no such relations exist for the entire set of equations, then set is linearly independent

Augmented matrix

Place $\underline{b}$ in the $n+1^{st}$ column

$$\left[\underline{A} \ \vdots \ \underline{b} \right]$$

$$\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 3 & 2 & 4 \\ -1 & 5 & -6 \end{array}\right]$$ Augmented matrix

Solve $\underline{A}\,\underline{x} = \underline{b}$ by applying elementary row operations on augmented matrix

1. Multiply row (equation) by a non-zero scalar
2. Add an equation to another equation
3. Interchange two rows (equations)

Each of these EROs has a well-defined effect on the determinant

1. For scalar f, determinant equals $f\det(\underline{A})$
2. No effect on determinant
3. Multiplies determinant by −1

Example: $\underline{A}\,\underline{x} = \underline{b}$, augmented matrix

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 2 & 2 \end{array}\right] \Leftarrow *2$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ -2 & 4 & 4 \end{array}\right] +$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 3 & 5 \end{array}\right] \Leftarrow *1/3$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 1 & 5/3 \end{array}\right] +$$

$$\left[\begin{array}{cc|c} 2 & 0 & 8/3 \\ 0 & 1 & 5/3 \end{array}\right] \Leftarrow *1/2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4/3 \\ 0 & 1 & 5/3 \end{array}\right] \Longrightarrow \begin{array}{l} x_1 = 4/3 \\ x_2 = 5/3 \end{array}$$

↳ Reduced row echelon form (rref)

Note: If

$$\begin{bmatrix} a & b & c \\ 0 & d & e \end{bmatrix}$$ Row echelon form

Example: $\underline{A}\,\underline{x} = \underline{b}$, augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 3 \end{array}\right]$$

$\downarrow$ EROs

Try to obtain this rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{array}{c} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{array}$$

$\det(\underline{A}) = 1(1)(0) = 0 \Rightarrow \underline{A}^{-1}$ does not exist

$\therefore$ infinite number of solutions (x_3 is a free variable)

Example: $\underline{A}\,\underline{x} = \underline{b}$, augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 3 \end{array}\right]$$

$$\Big\downarrow \text{ EROs}$$

Try to obtain this rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Longrightarrow \begin{array}{c} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 1 \end{array}$$

$\lightning$

$$0 = 1$$

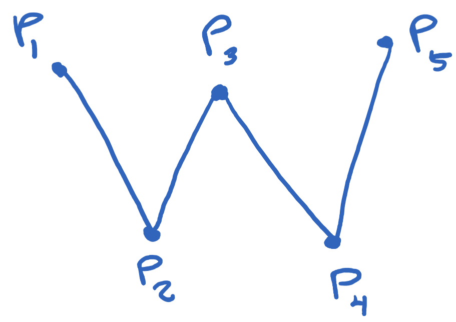
$\therefore$ No solution exists

Note: $\det(\underline{A}) = 1(1)(0) = 0$

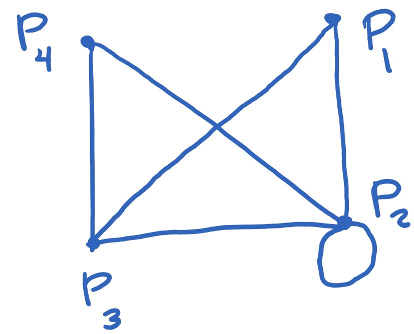
Graphs & Graph Theory

Sunday, September 15, 2019 8:56 PM

Graph: A finite collection of vertices and edges

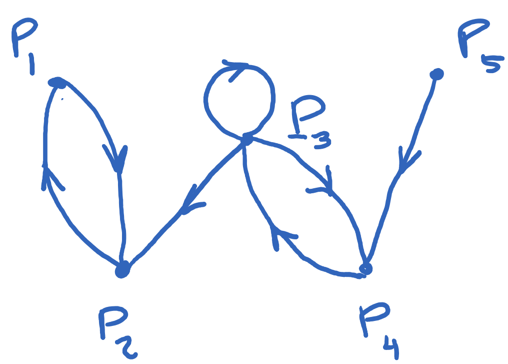


G_1

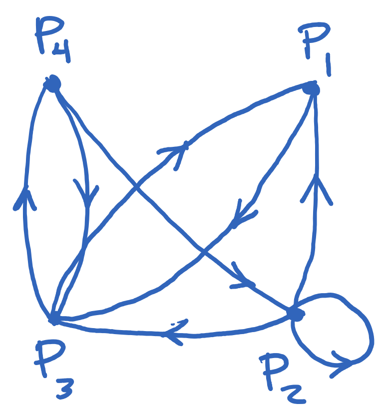


G_2

Directed Graph: A graph that indicates direction



D_1



D_2

Digraphs

Applications:

Biology/computer science – Neural networks

Linguistics – language inter-relationships

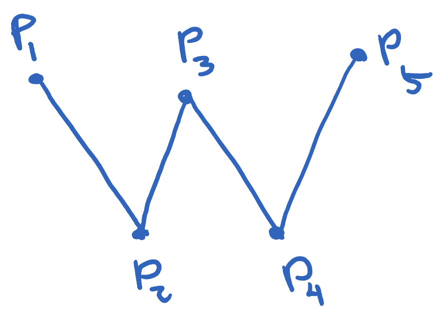
Chemistry – molecules

Computer science – websites

Scientific computing – meshing

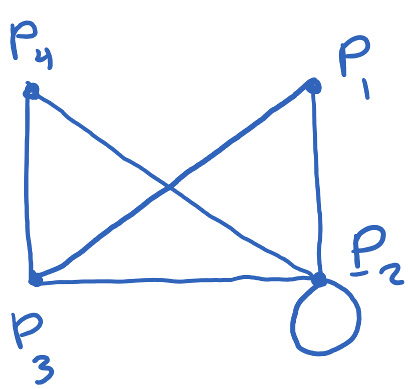
Adjacency Matrix

Matrix entry (i,j) is 1 if there is an edge between P_i and P_j; 0 otherwise

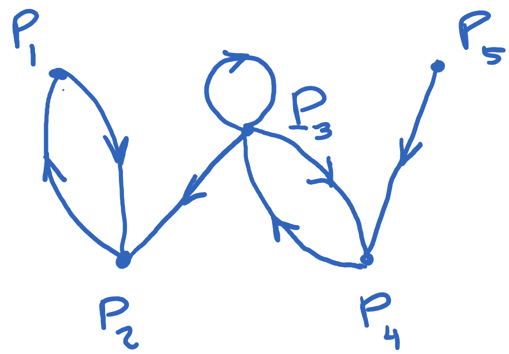


$$G_1: \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

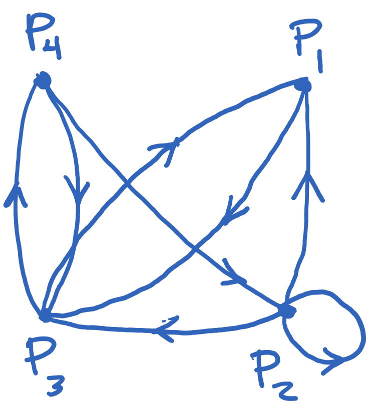
What type of matrix is G_1?



$$G_2: \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



$$D_1: \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



$$D_2: \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Matrix representation of a graph or digraph

How is it useful?

Path: Path between P_i and P_j in a graph is a finite sequence of edges, where

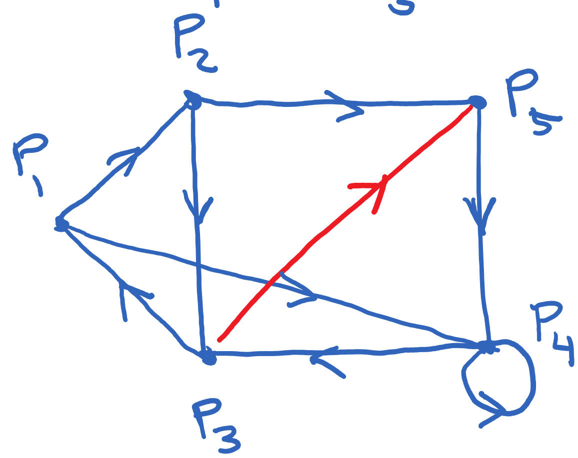
a) First edge starts at P_i

b) Last edge ends at P_j

c) Each edge after the first begins at the vertex where the prior edge ends

Length (of path): Number of edges to go from P_i to P_j

Example: G_3



$P_1 \to P_2 \to P_5$ (length = 2)

$P_1 \to P_2 \to P_3 \to P_5$ (length = 3)

2

Counting paths: How many paths of a given length exist from P_i to P_j?

Theorem

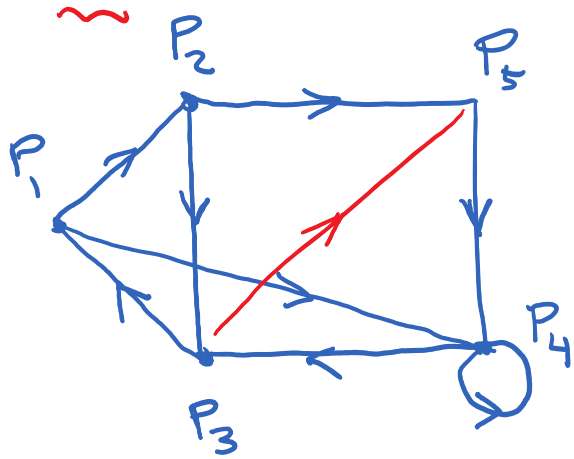
Let $\underline{A}$ be the adjacency matrix for vertices $P_1, P_2, \ldots, P_n$

The number of paths of length k from P_i to P_j is given by the i,j value of $\underline{A}^k$

Corollary

The number of paths with length $\leq k$ is given by the sum of the i,j values from $\underline{A}, \underline{A}^2, \ldots, \underline{A}^k$

G_3



$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \boxed{0} & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & \boxed{0} & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Check $\left.\begin{array}{l} P_2 \to P_4 \\ P_4 \to P_2 \end{array}\right\}$ zero paths of length 1

From Matlab

$$\underline{A}^2 = \begin{bmatrix} 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & \boxed{1} & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Minimum length from $P_2 \to P_4$? Length 2

$P_2 \to P_5 \to P_4$

$$\underline{A}^3 = \begin{bmatrix} 2 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ 1 & \boxed{1} & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Minimum length from $P_4 \to P_2$? Length 3

$P_4 \to P_3 \to P_1 \to P$

Three paths of length 3 from $P_2 \to P_4$

What are these?

$P_2 \to P_3 \to P_1 \to P_4$

$P_2 \to P_3 \to P_5 \to P_4$

$P_2 \to P_5 \to P_4 \to P_4$

EROs - Effect on Determinant

Monday, September 16, 2019 11:36 AM

$$\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(\underline{A}) = ad - bc$$

$$\underline{B} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} \qquad \det(\underline{B}) = (a+c)d - (c)(b+d)$$

$$= ad - bc + cd - cd$$

$$= \det(\underline{A})$$

$$\underline{C} = \begin{bmatrix} fa & fb \\ c & d \end{bmatrix} \qquad \det(\underline{C}) = fad - fcb$$

$$= f \det(\underline{A})$$

$$\underline{D} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \qquad \det(\underline{D}) = cb - ad$$

$$= -\det(A)$$

$$= -\det(\underline{A})$$

$$\begin{vmatrix} a & & & & & \\ 0 & b & & & & \\ 0 & 0 & c & & & \\ 0 & 0 & 0 & d & & \\ 0 & 0 & 0 & 0 & \ddots & \\ 0 & 0 & 0 & 0 & 0 & z \end{vmatrix} = a \cdot b \cdot c \cdot d \cdot \ldots z$$