PS III (338)

Chapter 4

10) i) $\int_0^i e^z dz$, what we know from the previous chapter is that $e^z = \frac{d}{dz} e^z$ $\forall z$, thus e^z is analytic everywhere let Γ be given by $\Gamma : z(t) = it$, $0 \le t \le 1$ then by Prop. 4.12 $\int_\Gamma e^z dz = e^{z(1)} - e^{z(0)} = e^i - 1$. Also by a direct calculation, $\int_\Gamma e^z dz = \int_0^1 e^{it} i \, dt = e^{it}\big|_0^1 = e^i - 1$.

ii) $\int_{\pi/2}^{\pi/2+i} \cos 2z \, dz$, now consider $\Gamma : z(t) = \pi/2 + it$, $0 \le t \le 1$ again by $\cos z = (\sin z)'$ which is given in the previous chapter (write $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ and use $e^z = \frac{d}{dz} e^z$ to see that $\frac{d}{dz}\sin z = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$) we see by prop 4.12 that $\int_\Gamma \cos 2z = \frac{1}{2}[\sin 2z(1) - \sin 2z(0)] = \frac{1}{2}[\sin(\pi + 2i) - \underbrace{\sin \pi}_{=0}] = \frac{1}{4i}(\underbrace{e^{i\pi}}_{=-1} e^{-2} - \underbrace{e^{-i\pi}}_{=-1} e^{2}) = \frac{1}{4i}(-e^{-2} + e^2)$. Also directly

$$\int_\Gamma \cos 2z \, dz = \int_0^1 \cos(\pi + 2it) \cdot i \, dt = \frac{i}{2}\int_0^1 (\underbrace{e^{\pi i}}_{-1} e^{-2t} + \underbrace{e^{-\pi i}}_{=-1} e^{2t}) \, dt$$

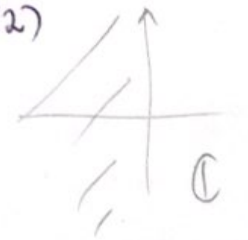
$$= \frac{i}{2}\left[+\frac{1}{2} e^{-2t}\Big|_0^1 - \frac{1}{2} e^{2t}\Big|_0^1 \right] = \frac{i}{2}\left[\frac{1}{2} e^{-2} - \cancel{\frac{1}{2}} - \frac{1}{2} e^2 + \cancel{\frac{1}{2}} \right]$$

$$= \frac{1}{4i}(-e^{-2} + e^2)$$

11) Since f is analytic in a convex region D, we consider the line segment ℓ joining a and b, which is given by $\ell : z(t)$, $0 \le t \le 1$

and $\int_\ell f'(z) dz = f(z(1)) - f(z(0)) = f(b) - f(a)$

$$\Rightarrow |f(b)-f(a)| \leq \int_{\ell} |f'(z)|\, dz \underset{|f'|\leq 1}{\leq} \text{length}(\ell) = |b-a|$$

12) 

left half plane, i.e., $\operatorname{Re} z < 0$ is a convex region $\subseteq \mathbb{C}$, and e^z is everywhere analytic (entire), so by the previous question $|e^a - e^b| < |a-b|$, for a, b with $\operatorname{Re} a, \operatorname{Re} b < 0$. since $\left|\frac{d}{dz} e^z\right| = |e^z| \underset{\text{say } z = x+iy,\ x<0}{=} \underbrace{|e^{iy}|}_{=1} |e^x| \underset{x<0}{\leq} 1$.

Chapter 5

1, 2) For an entire function f, and $a \in \mathbb{C}$, we have $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$ by Corollary 5.7. So when $f(z) = z^2$ and $a = 2$, we write $f^{(n)}(z) = 0$, $n \geq 3$. Thus

$$z^2 = f(2) + \frac{f^{(1)}(2)}{1!}(z-2) + \frac{f^{(2)}(2)}{2!}(z-2)^2 = 4 + 4(z-2) + (z-2)^2$$

also power series expansion for $f(z) = e^z$ about any point a is $e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \underset{\frac{d^n}{dz^n} e^z = e^z\ \forall n \in \mathbb{N}.}{=} e^a \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!}$

3) f - odd, that is, $f(z) = -f(-z) \Rightarrow f(0) = 0$, $f'(z) = f'(-z) \Rightarrow$ f' is even, one more differentiation gives $f''(z) = -f''(-z) \Rightarrow$ f'' is odd $\Rightarrow f''(0) = 0$ and so on. In general, $f^{(n)}(0) = 0$ for even n. Thus for entire odd f, $f(z) = \sum_{k=1,3,5,\dots} \frac{f^{(k)}(0)}{k!} z^k$.
Analyze the situation for entire even f as exercise.

4) For a power series $f(z) = \sum_0^{\infty} a_n z^n$ with a nonzero radius of convergence we know, by differentiating the series term by term, that $f^{(n)}(z) = n!\, a_n + (n+1)!\, a_{n+1} z + \frac{(n+2)!}{2!} a_{n+2} z^2 + \dots \Rightarrow \boxed{\frac{f^{(n)}(0)}{n!} = a_n}$ (1). Consider now an entire f and

circle C centered at 0 with radius $R = |a| + 1$, $a \neq 0$
hence as $R > |a|$, $C : Re^{i\theta}$, $0 \leq \theta \leq 2\pi$ $\hat{\mathbb{C}}$

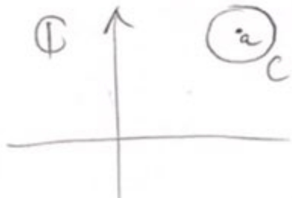
by the Cauchy Integral formula, for z with $|z| \leq |a|$, we write $f(z) = \frac{1}{2\pi i} \int_C \frac{f(\beta)\, d\beta}{\beta - z}$. Also $\frac{1}{\beta - z} = \frac{1}{\beta(1 - \frac{z}{\beta})} =$

$$\frac{1}{\beta} \sum_{k=0}^{\infty} \left(\frac{z}{\beta}\right)^k \Rightarrow f(z) = \frac{1}{2\pi i} \int_C \frac{f(\beta)}{\beta} \sum_{k=0}^{\infty} \left(\frac{z}{\beta}\right)^k d\beta$$

$$\underset{\substack{\text{convergence}\\ \text{is uniform throughout}\\ C}}{=} \sum_{k=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i} \int_C \frac{f(\beta)}{\beta^{k+1}}\, d\beta\right]}_{= a_k \ (2)} z^k$$

$$\Rightarrow \text{By (1) \& (2)},\ f^{(k)}(0) = \frac{k!}{2\pi i} \int_C \frac{f(\beta)}{\beta^{k+1}}\, d\beta, \quad k = 0, 1, 2, \ldots$$

5) WTS : $f^{(k)}(a) = \frac{k!}{2\pi i} \int_C \frac{f(w)}{(w-a)^{k+1}}\, dw$, $k = 1, 2, \ldots$ where C surrounds the point a & f is entire



writing $g(z) = f(z+a)$ we see that $g^{(n)}(z) = f^{(n)}(z+a)$
$\Rightarrow g^{(n)}(0) = f^{(n)}(a)$. As g is entire, implementing previous exercise to g, we obtain

$$f^{(k)}(a) = g^{(k)}(0) = \frac{k!}{2\pi i} \int_{|w|=R} \frac{g(w)}{w^{k+1}}\, dw = \frac{k!}{2\pi i} \int_{|w|=R} \frac{f(w+a)}{w^{k+1}}\, dw$$

$$\underset{w \mapsto w-a}{=} \frac{k!}{2\pi i} \int_{|w-a|=R} \frac{f(w)}{(w-a)^{k+1}}\, dw \quad \text{where } C : |w-a| = R$$

6) a) Suppose an entire f is bounded by M along $|z| = R$ that is $|f(z)| \leq M$, $|z| = R$. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, c_k, by ex 4, is given by $c_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}}\, dw$, $k = 0, 1, 2, \ldots$ Hence

$$|c_k| \leq \frac{1}{2\pi} \int_{|w|=R} \frac{|f(w)|}{|w|^{k+1}}\, dw \underset{|f| \leq M \text{ on } |z|=R}{\leq} \frac{M}{2\pi} \frac{2\pi R}{R^{k+1}} = \frac{M}{R^k}$$

b) A polynomial is entire, write for instance,

$p(z) = a_k z^k + a_{k-1} z^{k-1} + \dots + a_1 z + a_0$, $a_j \in \mathbb{C}$, $j = 0, 1, \dots, k$

Clearly, power series expansion of $p(z)$ about 0 is exactly itself, then assuming $|p(z)| \leq 1$, $|z| = 1$, we have by part a), that $|a_j| \leq \frac{1}{1} = 1$, $j = 0, 1, \dots, k$.