P1

(i) It is obvious that we have a binomial distribution. Let p be the prob. that being a victim. And X be the number of victims. So we are asked to find $P(X=2)$.

Chose 2 people with prob. $1/500\,000$ among $1\,000\,000$ people.

$$\binom{1\,000\,000}{2}\left(\frac{1}{500\,000}\right)^2\left(1-\frac{1}{500\,000}\right)^{999\,998}$$

(ii)

$$X=\sum_{i=0}^{n} X_i \Rightarrow E(X)=\sum_{i=0}^{n} E(X_i)=\sum_{i=0}^{1\,000\,000}\binom{1\,000\,000}{i}\left(\frac{1}{500\,000}\right)^{i}\left(1-\frac{1}{500\,000}\right)$$

where

$$X_i=\begin{cases}1 & \text{if } i \text{ th person is victim}\\ 0 & \text{otherwise}\end{cases}$$

(P13) $\mu(n)\cdot\mu(n+1)\cdot\mu(n+2)\,\mu(n+3)=0$ since any consecutive 4 numbers contain some product of 4. WLOG say $n=4k$ for some $k\in\mathbb{N}$. Then $\mu(4.k)=0$ by the definition of Möbius function as $4.k$ is not square free.

(P14) $\lim_{m\to\infty} f(p^m)=0 \Rightarrow \lim_{n\to\infty} f(n)=0$

&

f is multiplicative

proof: By Contradiction assume $\lim_{n\to\infty} f(n)=L$ for some L.

I worked on this question with Berk. After we thought that the question is not solvable, we couldn't manage to fix the problem. I tried to find an ε, yet I guess it is impossible.

P11 Consider the set of integers which have an odd # of digits. We have $\frac{d(9)}{9} = 1$, $\frac{d(99)}{99} = \frac{9}{99}$, $\frac{d(999)}{999} = \frac{909}{999}$ ...

One can see that for $x = 10^k - 1$, if $k =$ is even $\frac{d(x)}{x} = 1/11$

whereas if k is odd, then $\frac{d(x)}{x} > \frac{9}{10}$. So limit does not exists. Hence the set has no natural density

P12 f is multiplicative $\iff f(mn) = f(m) \cdot f(n)$, $\gcd(m,n) = 1$

We have

$$g(n) = \sum_{d \mid n} f(d) \implies g(n) = f(d_1) + f(d_2) + \dots + f(d_k)$$

where n has k many divisors d_i.

for n and z;

$$g(n \cdot z) = \sum_{t,d \mid n \cdot z} f(n \cdot z) \; \dots \; = \sum f(n) \cdot \sum f(z)$$

can be shown by some algebraic manipulation.

P10

(i) $d(A) = \lim\limits_{n \to \infty} \dfrac{n}{a_n}$ if the limit exists.

we have $A = \{2n : n \in N\}$ so $a_n = 2n$ and $d(A) = 1/2$

(ii) we have $\lim\limits_{x \to \infty} \dfrac{\pi(x)}{\left[\dfrac{x}{\log x}\right]} = 1$ wher $\pi(x)$ denotest the number of primes less than or equal to x.

So as $x \to \infty$, $\dfrac{x}{\log x} \to 0$. we can say prime numbers have 0 density among natural numbers

(iii) we already showed in the lecture notes that (Möbius fnc)

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

Using this result and (ii), we see that $\dfrac{-\zeta'(s)}{\zeta(s)}$ is the Dirichlet series of $\ln * \mu$. But $\ln * \mu = \Lambda$, and thus

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \frac{-\zeta'(s)}{\zeta(s)} \qquad \blacksquare$$

(ii) $b_n = \ln n$. Look at the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

- Note that the derivative of $\frac{1}{n^s}$ wrt s is $\frac{-\ln n}{n^s}$.

Also we know $\zeta(s)$ is absolutely convergent for each $s>1$, we can differentiate the series term by term to get

$$-\zeta'(s) = \sum_{n=1}^{\infty} \frac{\ln n}{n^s}$$

(P9) $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$

(i) $a_n = n^{\alpha}$, $\alpha = 1, 2, 3 \ldots$ Let $f(n) = n^{\alpha}$. Clearly $f(n)$ is multiplicative function. So we have

$$D(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \text{ prime}} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right)$$

P5 we want to show $X_n \xrightarrow[L^2]{} b \Longrightarrow X_n \xrightarrow[P]{} b$.

$$P(|X_n - b| > \varepsilon) = P(|X_n - b|^2 > \varepsilon^2) \leq \frac{E|X_n - b|^2}{\varepsilon^2} \to 0$$

P6

* $X_n \xrightarrow{P} b \iff P(|X_n - b| > \varepsilon) \to 0$ as $n \to \infty$, where $\varepsilon > 0$

* $X_n \xrightarrow[L^2]{} b \iff E(X_n - b)^2 \to 0$

P

(P4)

we know that $Var(X_n - b) \geq 0 \implies$

$E(X_n - b)^2 \geq (E|X_n - b|)^2$, taking the limit as $n \to \infty$

we get $0 \geq \limsup_{n\to\infty} E|X_n - b|$ and $\liminf_{n\to\infty} E|X_n - b| \geq 0$

Then $\lim_{n\to\infty} E|X_n - b| = 0$ so $\lim_{n\to\infty} E(X_n) = b$.

And since b is a constant, it has 0 variance so,

$\lim_{n\to\infty} var(X_n) = 0$.

P3

Take a look at $P(|X| \geq a)$. Since g is strictly increasing and non-negative, we have

$$X \geq a \iff g(X) \geq g(a)$$

So $P(|X| \geq a) = P(g(x) \geq g(a))$. At this stage, we have Markov's Inequality. $P(X \geq t) \leq \frac{E[X]}{t}$, where t, X are non-

So

$$P(|X| \geq a) = P(g(X) \geq g(a)) \leq \frac{E[g(|X|)]}{g(a)} \quad \text{for } a > 0$$

as desired.