HW-3

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1)

If we say, the sequence is monotone and bounded, automatically it converges since it is a sequence in $\mathbb{R}$. So;

we have $a_{n+2} = (n.a_{n+1} + a_n)/n + 1$ where $a_1 = 0, a_2 = 1$.

Clearly, $a_n > 0, \forall n > 0$.

I have a strong intuation that it is bounded above by 1. But I could not prove it algebraically. Only idea that comes up my mind is that $a_{n+2} < a_2, \forall n > 0$. Because $a_{n+2} = [(n/n+1).a_{n+1} + (1/n+1).a_n] < a_{n+1} + a_n$. For the recursive relation; since $a_1 = 0$, we only get a_2,which is 1. ($a_3 < a_2 + a_1 = a_2 = 1$. Continuing repeatedly, yields the above intuiton.)

Also we have;

$n + 1 = (n.a_{n+1}/a_{n+2}) + (a_n/a_{n+2}) \implies n.(1 - a_{n+1}/a_{n+2}) = (a_n/a_{n+2} - 1) \implies 0 < n = (a_n/a_{n+2} - 1)/(1 - a_{n+1}/a_{n+2})$. So the nominator and denominator have the same sign. Either case we reach that a_n is monotone.

Thus, a monotone and bounded sequence in $\mathbb{R}$ is convergent.

2)

a)

$sup(S)$ is the least upper bound of S; say r s.t. $s \leq r \;\; \forall s \in S$. Also, $inf(S)$ is the greatest lower bound of S; say t, s.t. $t \leq s \, \forall s \in S$. So, $t \leq s \leq r \implies inf(S) \leq s \leq sup(S) \Longrightarrow inf(S) \leq sup(S)$.

$\square$

b)

Since $S \subset \mathbb{R}$ bounded. We have a bounded interval and $sup(S) = inf(S)$. S consists of only one element (a singleton) since otherwise; say $a, b \in S$ where $a \neq b$. Hence it is either $a < b$ or $b < a$. In either case we have $inf(S) \leq a < b \leq sup(S)$ or $inf(S) \leq b < a \leq sup(S)$ which implies $sup(S) \neq inf(S)$ which is a contradiciton.

$\square$

3)

Let $A \in \mathbb{R}^n$ be arbitrary. A is bounded means that A is contained in a ball $\iff A \subset B(\vec{a}; \vec{r}) \subset \mathbb{R}$ where $r > 0$. $\iff \forall x, y \in A \implies x, y \in B(\vec{a}, \vec{r})$. We can say that $diam(B) = sup\{|x-y|, x, y \in B(\vec{a}, \vec{r})\}$ is an upper bound for $\{|c-d|; c, d \in A\}$ all the differences for arbitrary $c, d \in A$. So, $\{|c-d|; c, d \in A\}$ is a bounded subset of $\mathbb{R} \implies diam(A) = sup\{|x-y|, x, y \in A\}$ exists.

$\square$