Chapter 5 PS IV (338)

7) An alternate proof of Liouville's thm: If f is entire and if for some integer $k \geq 0$, $\exists\ A, B \in \mathbb{R}^+$ s.t. $|f(z)| \leq A + B|z|^k$ then f is a polynomial of degree at most k.

As f is entire, for any $a \in \mathbb{C}$, it has a power series representation $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$, $\forall z$. Now consider $z \in \mathbb{C}$ with $|z| = R$ for which $|f(z)| \leq A + BR^k$. Taking $a = 0$ we have by a) part of question 6), $\left|\frac{f^{(j)}(0)}{j!}\right| = |c_j| \leq \frac{A + BR^k}{R^j} \longrightarrow 0$ as $R \to \infty$ for $j > k$. (we can let R go to ∞ because f has a power series representation for all z). Thus $c_j \equiv 0$ $\forall j > k$ so that $f(z) = c_k z^k + \dots + c_0$, $c_i = \frac{f^{(i)}(0)}{i!}$, $i = 1, 2, \dots, k$.

9) Suppose f is entire & $|f'(z)| \leq |z|$, $\forall z$. WTS: $f(z) = a + bz^2$ with $|b| \leq 1/2$.

Since f' is entire and given that $|f'(z)| \leq |z|$, $\forall z$, by Liouville's theorem then $f'(z) = \tilde{a} + \tilde{b}z$, $\tilde{a}, \tilde{b} \in \mathbb{C}$. It is easy to see that $\tilde{a} = 0$ since $|f'(0)| = |\tilde{a}| \leq 0$. Upon integrating: $f(z) = a + bz^2$ for some a, b, but as f is entire, by its power series representation at 0, $a = f(0)$, $b = \frac{f''(0)}{2!}$. By question 4) with $g = f'$, which is entire, we have

$$f''(0) = g'(0) \underset{g \text{ entire}}{=} \frac{1}{2\pi i} \int_{|\eta|=1} \frac{g(\eta)}{\eta^2}\, d\eta = \frac{1}{2\pi i} \int_{|\eta|=1} \frac{f'(\eta)}{\eta^2}\, d\eta \Rightarrow$$

$$|b| = \frac{1}{2}|f''(0)| \leq \frac{1}{4\pi} \int_{|\eta|=1} \frac{|f'(\eta)|}{|\eta|^2}\, d\eta \underset{|f'(z)| \leq |z|}{\leq} \frac{1}{4\pi} \int_{|\eta|=1} \frac{1}{\underbrace{|\eta|}_{=1}}\, d\eta = \frac{1}{4\pi} 2\pi = \frac{1}{2}.$$

16) First note that if $z_1, \dots, z_n$ roots of the polynomial $P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ then $\sum_{k=1}^{n} z_k = -a_{n-1}/a_n$, to see this divide $P_n(z) = a_n(z - z_1)\cdots(z - z_n)$ by a_n to get

$$z^n + \frac{a_{n-1}}{a_n} z^{n-1} + \dots + a_0 = (z - z_1)\cdots(z - z_n) = z^n - \left(\sum_{k=1}^{n} z_k\right) z^{n-1} + \dots$$

comparing the coefficients, done ✓. Centroid of the complex roots $z_1, \dots, z_n$ of $P_n(z)$ is $\frac{1}{n}(z_1 + \dots + z_n)$. WTS: centroid of the zeros of P_n is the same as the centroid of zeros of P_n'.

So then, by application of the induction argument to Fund. thm. of algebra, there are $n-1$ roots, say $\xi_1, \xi_2, \dots, \xi_{n-1}$ of P_n'.

Thus $p_n'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \dots + a_1 = n a_n (z - \xi_1) \cdots (z - \xi_{n-1})$

By the note above, $\sum_{k=1}^{n-1} \xi_k = \frac{(1-n) a_{n-1}}{n a_n}$, so the centroid of the roots of p' is $\frac{\xi_1 + \dots + \xi_{n-1}}{n-1} = \frac{\cancel{(1-n)} a_{n-1}}{\cancel{(n-1)} n a_n} = -\frac{a_{n-1}}{n a_n}$

$= \frac{1}{n} \sum_{k=1}^{n} z_k$

17) Let's do it by induction. As the set is convex, the statement is true for $n = 1, 2$ trivially. Now assume that the claim is true for $n = k$ and try to show it for $n = k+1$: $z_1, \dots, z_{k+1}$ belong to a convex set, say S,

WTS: $\sum_{j=1}^{k+1} a_j z_j \in S$; $a_i \geq 0$ $\forall i$, $\sum_{j=1}^{k+1} a_j = 1$.

So set $A = \sum_{j=1}^{k} a_j$, clearly $1 - A = \sum_{j=1}^{k+1} a_j - A = a_{k+1}$ $\Rightarrow$

$$\sum_{j=1}^{k+1} a_j z_j = \sum_{j=1}^{k} a_j z_j + a_{k+1} z_{k+1} = A \underbrace{\sum_{j=1} \left(\frac{a_j}{A}\right) z_j}_{\in S} + (1-A) \underbrace{z_{k+1}}_{\in S} \in S$$

since induction hypothesis $\Rightarrow$ $\sum_{j=1}^{k} \left(\frac{a_j}{A}\right) z_j \in S$ as $\sum_{j=1}^{k} \frac{a_j}{A} = \frac{1}{A} \underbrace{\sum_{j=1}^{k} a_j}_{=A} = 1$.

& $z_1, \dots, z_k \in S$.

18) $P_k(z) = 1 + z + z^2/2! + \dots + z^k/k!$,

a) let $z_1, z_2, \dots, z_k$ be k roots of $P_k(z)$, then

$P_k(z) = \frac{1}{k!}(z - z_1) \cdots (z - z_k)$ $\Rightarrow$ $\sum_{j=1}^{k} z_j = -\frac{1/(k-1)!}{1/k!} = -k$. Therefore

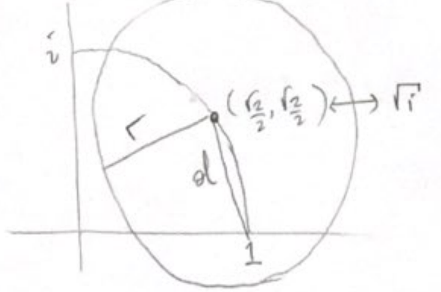
centroid $\frac{\sum_{k=1}^{k} z_j}{k} = -\frac{k}{k} = -1$ for all $k \geq 1$.

b) Exercise!

19) $P(z) = \frac{d}{dz}\left(z^n + z^{n-1} + \cdots + z + 1\right) =: \frac{d}{dz} Q(z)$. The roots of $Q(z)$ are the n roots of $z^{n+1} - 1 = (z-1)Q(z) = 0$ other than 1, both of which clearly are on the unit circle. Let $z_1, \ldots, z_n$ be these roots, hence $|z_j| = 1$ for $j = 1, 2, \ldots, n$. By Gauss-Lucas theorem, if $w \in \mathbb{C}$ s.t. $P(w) = 0$, then $\exists\ a_1, \ldots, a_n$ all positive with $\sum_{i=1}^{n} a_i = 1$ s.t. $w = \sum_{i=1}^{n} a_i z_i \Rightarrow |w| \leq \sum_{i=1}^{n} |a_i z_i| = \sum_{i=1}^{n} a_i \underbrace{|z_i|}_{=1} = \sum_{i=1}^{n} a_i$

$$= 1$$

20) First we shall make some observations to run the useful theorems / lemmas in the text. Set $f(z) = z^2 - i = 0$. Apparently $\sqrt{i}$ is a simple zero since $f(z) = (z - \sqrt{i})(z + \sqrt{i}) = 0$. Also define $g(z) = z - \frac{z^2 - i}{2z}$. By Lemma 5.18, for f is analytic and has a zero of order 1 at $z = \sqrt{i}$, g is analytic at $\sqrt{i}$ and $g'(\sqrt{i}) = 0$ (direct calculation also immediately shows this by noting that $\sqrt{i} = e^{i\pi/4}$). Let $z_0 = 1$.



$d = \sqrt{\left(\frac{\sqrt{2}}{2} - 1\right)^2 + \left(\frac{\sqrt{2}}{2} - 0\right)^2} = \sqrt{2 - \sqrt{2}} < 1$, let $r = \sqrt{2 - \sqrt{2}} + \delta$ for $\delta > 0$ such small that $0 \notin D(\sqrt{i}, r)$ and $z_0 = 1 \in D(\sqrt{i}, r)$. So then we evaluate

$$g'(z) = \frac{1}{2} - \frac{i}{2z^2} \Rightarrow g''(z) = \frac{i}{z^3} \Rightarrow \exists M > 0$$

s.t. $|g''(z)| \leq M \quad \forall z \in D(\sqrt{i}, r)$ since $0 \notin D(\sqrt{i}, r)$. Next define the sequence $\{z_n\}$ recursively as $z_{n+1} = g(z_n) = z_n - \frac{z_n^2 - i}{2z_n}$ $n = 0, 1, 2, \ldots$. So by Lemma 5.20, $|z_1 - \sqrt{i}| \leq \frac{M}{2} |\sqrt{i} - 1|^2$ and by Theorem 5.22, the sequence $\{z_n\} \to \sqrt{i}$ quadratically, that is $\varepsilon_n = |z_n - \sqrt{i}| \leq K \varepsilon_{n-1}^2 = K|z_{n-1} - \sqrt{i}|^2 \leq \cdots \leq K^n |1 - \sqrt{i}|^{2n}$

$$= K^n (\underbrace{2 - \sqrt{2}}_{<1})^n$$

Chapter 6

2) $f(z) = \dfrac{1}{1-z-2z^2} = \dfrac{A}{(1-2z)} + \dfrac{B}{z+1} \Rightarrow B = 1/3 \ , \ A = 2/3$

$\dfrac{1}{z+1} = 1 - z + z^2 - z^3 + \cdots = \sum_{k=0}^{\infty} (-1)^k z^k \quad ; \quad \dfrac{1}{1-2z} = 1 + 2z + 4z^2 + \cdots = \sum_{k=0}^{\infty} 2^k z^k$

$\Rightarrow f(z) = \dfrac{1}{3} \sum_{k=0}^{\infty} \left[(-1)^k + 2^{k+1} \right] z^k$

3) $\sum_{n=0}^{\infty} z^n = \underset{|z|<1}{\dfrac{1}{1-z}} \Rightarrow \dfrac{1}{(1-z)^2} = \dfrac{d}{dz} \sum_{n=0}^{\infty} z^n = \sum_{n=1}^{\infty} n z^{n-1} \xrightarrow[\text{by } z]{\text{multiplying}} \dfrac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n$

$\dfrac{2}{(1-z)^3} = \dfrac{d}{dz}\left(\dfrac{1}{(1-z)^2}\right) = \sum_{n=2}^{\infty} n(n-1) z^{n-2} \xrightarrow[\text{by } z^2]{\text{mult.}} \dfrac{2z^2}{(1-z)^3} = \sum_{n=1}^{\infty} n^2 z^n - \underbrace{\sum_{n=1} n z^n}_{= \frac{z}{(1-z)^2}}$

$\Rightarrow \dfrac{2z^2}{(1-z)^3} + \dfrac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n^2 z^n \Rightarrow \dfrac{z^2+z}{(1-z)^3} = \sum_{n=1}^{\infty} n^2 z^n$