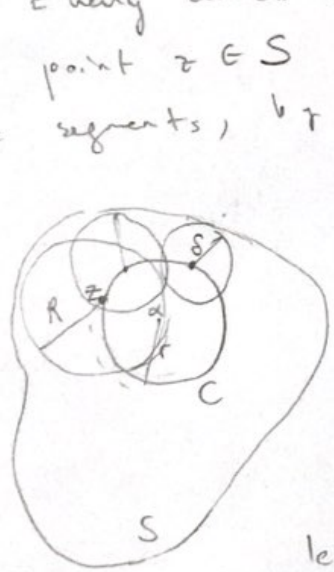
PS VI (338)

1) WTS: A star like region S is simply connected.
Let α be the point in S connected to every point in S via line segments and let $z \in S^c$. Consider
γ: $\gamma(t) = tz + (1-t)\alpha$, $t \geq 1$ which is the portion of the line connecting α, through z, to ∞. If there were some point

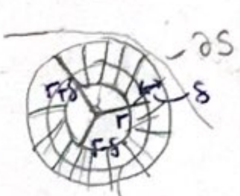
$z_0 \in \gamma \cap S$, then, as S is star-like, the line segment l connecting z_0 to α would lie in S. In this case, z being the initial point of γ would have to be in S, a contradiction to initial assumption.

2) Every convex region S is a star-like region because any point $z \in S$ is connected to any other $w \in S$ through line segments, by question above S is simply connected.

3)



$C = \{z : |z-\alpha| = r\} \subset S$, for each $z \in C$, define $\delta(z) = \max\{R : D(z;R) \subseteq S\}$ which is a continuous fnc of $z \in C$. for, if $z_n \in C$ s.t. $z_n \to z \in C$, we would have $\delta(z_n) \to \delta(z)$ (S is open). Since C is cpt, this $\delta(z)$ assumes its minimum on C, so let $\delta = \min_{z \in C} \delta(z)$. Thus we form an annulus



$A = \{z : r - \delta \leq |z-\alpha| \leq r+\delta\} \subset S$ that is contained in S. Our aim has been to show that the disc $D(\alpha;r) = \{z : |z-\alpha| \leq r\} \subset S$.
Suppose, for a while, that there is some $z_0 \in D(\alpha;r)$ that belongs to S^c, but there is no continuous curve $\gamma(t)$, $0 \leq t < \infty$ with $\gamma(0) = z_0$ and $d(\gamma(t), S^c) < \varepsilon \leq \delta$ for all $t \geq 0$, s.t. $\gamma(t) \to \infty$ as $t \to \infty$ because $\exists\, t_0 \geq 0$ s.t. $\gamma(t_0) \in C$ so that $d(\gamma(t_0), S^c) > \delta$ by the construction of annulus A. So this yields a contradiction to simply connectedness of S, consequently $D(\alpha, r) \subset S$.

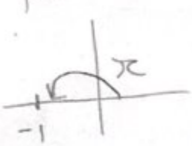
8) <u>recall</u>

Thm 8.8 : Suppose D is simply connected and that $0 \notin D$. Choose $z_0 \in D$, fix a value of $\log z_0$ and set

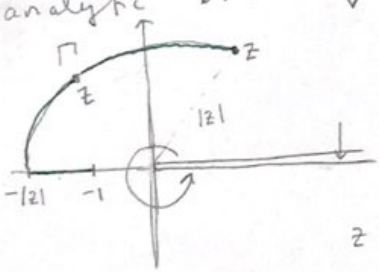
$$f(z) = \int_{z_0}^{z} \frac{d\zeta}{\zeta} + \log z_0$$

then f is analytic branch of $\log z$.

remember $\log z = \log|z| + i \operatorname{Arg} z$ where $\operatorname{Arg} z = \theta + 2k\pi,\ k \in \mathbb{Z}$ for $z = Re^{i\theta}$. Thus letting $z_0 = -1$, $\log(-1) = \underbrace{\log|-1|}_{=0} + \pi i$



so that, by Thm 8.8, $\displaystyle\int_{-1}^{z} \frac{d\zeta}{\zeta} + \pi i$ is the analytic branch of $\log z$ in $D = \{ z \in \mathbb{C} : z \notin [0,\infty) \}$ - simply connected.



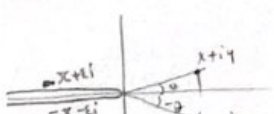
$0 < \operatorname{Arg} z < 2\pi$. Here $-1 \in D$, and we set $\operatorname{Arg}(-1) = \pi$ $(k = 0)$.

integration : from -1 to $-|z|$ and then $-|z|$ to z by a circular arc Γ : $\boxed{\displaystyle\int_{-1}^{-|z|} + \int_{\Gamma} = \int_{-1}^{z}}$

9) Consider the analytic branch of $\log z = \log|z| + i \operatorname{Arg} z$ where $-\pi < \operatorname{Arg} z < \pi$, with this branch of logarithm define $f(z) = e^{z \log z}$ which is analytic in $D = \mathbb{C} \setminus (-\infty, 0]$. Clearly for $x \in (0,\infty)$, $f(x) = e^{x \log x} = e^{x(\ln|x| + i \operatorname{Arg} x)} = e^{x(\ln x + i \cdot 0)}$ $= e^{x \ln x} = e^{\ln x^x} = x^x$. Next observe that for $z = x+iy$

$\operatorname{Arg}(\bar{z}) = \operatorname{Arg}(x - iy) = -\operatorname{Arg}(x+iy) = -\operatorname{Arg}(z)$ (1) ,

$\log \bar{z} = \ln|\bar{z}| + i \operatorname{Arg}(\bar{z}) \overset{(1)}{=} \ln|z| - i \operatorname{Arg}(z) =$



$\overline{\ln|z| + i \operatorname{Arg} z} = \overline{\log z}$ (2) $\Rightarrow$

$e^{\bar{z}} = e^{x-iy} = e^{x}(\cos(-y) + i\sin(-y)) = \overline{e^{x}(\cos y - i \sin y)}$ $= \overline{e^{x}(\cos y + i \sin y)} = \overline{e^{z}}$ (3) $\Rightarrow f(\bar{z}) = e^{\bar{z}\log\bar{z}} \overset{(2)}{=} e^{\overline{z \log z}} \overset{(3)}{=} \overline{e^{z \log z}}$ $= \overline{f(z)}$, $f(i) = e^{i \log i} = e^{i(\underbrace{\ln|i|}_{=0} + i\underbrace{\operatorname{Arg}(i)}_{\pi/2})} = e^{-\pi/2}$ and using $f(\bar{z}) = \overline{f(z)}$ we see $f(-i) = f(\bar{i}) = \overline{f(i)} = \overline{e^{-\pi/2}} = e^{-\pi/2}$.