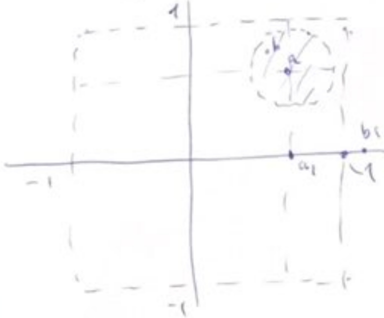
PS VI

Q5.6) Let $a = (a_1, a_2) \in S = \{ a \in \mathbb{R}^2 : a_1, a_2 \in (-1,1) \}$



and also let $r = \min\{ |1-a_1|, |1+a_1|, |1-a_2|, |1+a_2| \}$. WTS: $b[a; r/2) \subseteq S$

Take $b \in b[a; r/2)$ and show that $b \in S$.

Thus $d(a,b) < r/2 \leftrightarrow (|a_1-b_1|^2 + |a_2-b_2|^2)^{1/2} < r/2$

need to show: $|b_j| < 1 \quad j = 1,2$.

If $b_1 > 1$ then

$d(a,b) = (\underbrace{|a_1-b_1|^2}_{\geq |a_1-1|^2} + |a_2-b_2|^2)^{1/2} \geq (|a_1-1|^2 + |a_2-b_2|^2)^{1/2} \geq |a_1-1|$

$\geq r > r/2$ ↯. or if $b_1 < -1$ then contradiction

$d(a,b) = (\underbrace{|a_1-b_1|^2}_{\geq |a_1+1|^2} + |a_2-b_2|^2)^{1/2} \geq |a_1+1| \geq r > r/2$ ↯.

Similarly, if $b_2 > 1$ then $|a_2 - b_2| \geq |a_2 - 1| \Rightarrow d(a,b) \geq |a_2-1| > r/2$ ↯

& if $b_2 < -1$ then $|a_2-b_2| \geq |a_2+1| \Rightarrow d(a,b) \geq |a_2+1| > r/2$ ↯

Q5.7) Defn: The collection of open subsets of m.s. (X,d) is called the topology determined by d.

Suppose X - m.s. Z is a metric subspace of X, the topology of Z is $\{ U \cap Z : U \text{ - open in } X \}$

$\mu_1(a,b) = \sum_{i=1}^{n} \tau_i(a_i,b_i)$, $\mu_2(a,b) = \left(\sum_{i=1}^{n} (\tau_i(a_i,b_i))^2 \right)^{1/2}$

$\mu_\infty(a,b) = \max\{ \tau_i(a_i,b_i) : i \in \mathbb{N}_n \}$

$\mu_\infty(a,b) \leq \mu_2(a,b) \leq \mu_1(a,b)$

↳ conserving

Thm 5.1: $n \in \mathbb{N}$, $\forall i \in \mathbb{N}_n$, (X_i, τ_i) is a m.s. Endow

$P = \prod_{i=1}^{n} X_i$ with a conserving metric d. The topology on P is the collection of all unions of members of the set :

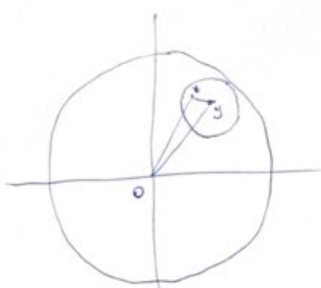
$\{ \prod_{i=1}^{n} U_i : U_i - \text{open in } X_i \}$

In our situation, $\mathbb{R}^2$ is a m.s. with Euclidean metric clearly which is conserving. $\mathbb{R} \times \{0\}$ is a metric subspace of $\mathbb{R}^2$. So the topology of $\mathbb{R} \times \{0\}$, by the above facts, consists of unions of $\{ (U \times V) \cap \mathbb{R} \times \{0\} : U, V$ open in $\mathbb{R} \} = \{ U \times \{0\} : U - \text{open in } \mathbb{R} \}$ as $\{0\}$ is open in $\{0\}$.

But clearly $U \times \{0\}$ is not open in $\mathbb{R}^2$ as by Theorem 4.5.1 open sets are those unions of $\prod_{i=1}^{2} U_i$ U_i open in $\mathbb{R}$. $\{0\}$ - not open in $\mathbb{R}$. Yet taking $U = \phi$., $U \times \{0\} = \phi$ - open

5.8) Let $t = 1 - d(0, y) > 0$ for y satisfying $d(0,y) < 1$.

WTS : $b[y;t) \subset S = \{ a \in \mathbb{R}^2 : a_1^2 + a_2^2 < 1 \}$ i.e. we want $z \in S \ \forall z \in b[y;t)$.



Thus $d(y,z) < t \rightsquigarrow$

$$d(0,z) \leq d(0,y) + d(y,z) < d(0,y) + t$$

$$= \cancel{d(0,y)} + 1 - \cancel{d(0,y)} = 1 . \implies z \in S.$$

5.11) A metric on a set X that satisfies $d(a,c) \leq \max \{ d(a,b), d(b,c) \} \ \forall a,b,c \in X$ is called an ultrametric on X. Suppose B is an open ball of (X, d)

WTS : every point of B is a center for B.

Write $B = b[x;r) = \{ y \in X : d(x,y) < r \}$ then for

2,VI

any $y, z \in B$, $d(z,y) \leq \max\{d(y,x), d(x,z)\} < r$ $\forall y \in B$, z.

$$\{y \in X : d(z,y) < r\} = b[z;r)$$

5.12) $d(a,b) = \begin{cases} |b-a| & a,b \in \mathbb{R}^- \\ b^2 - a & a \in \mathbb{R}^-, b \in \mathbb{R}^{\oplus} \\ a^2 - b & a \in \mathbb{R}^{\oplus}, b \in \mathbb{R}^- \\ |b^2 - a^2| & a,b \in \mathbb{R}^{\oplus} \end{cases}$

For $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases} \Rightarrow$

$d(a,b) = |f(a) - f(b)|$ $\forall a,b \in \mathbb{R}$. $d_{euc}(a,b) = |a-b|$

If $0 \leq a < b$ then

$$(a,b) = b\left[\frac{f(\sqrt{a}) + f(\sqrt{b})}{2} ; \frac{f(\sqrt{b}) - f(\sqrt{a})}{2}\right)$$

or if $b \geq 0$ & $a < 0$ then

$$(a,b) = b\left[\frac{f(a) + f(\sqrt{b})}{2} ; \frac{f(\sqrt{b}) - f(a)}{2}\right)$$

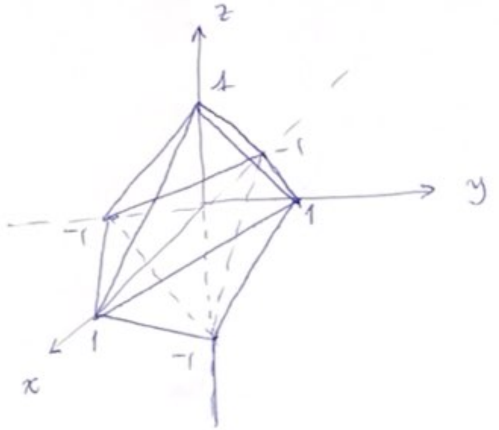
5.15) $x \longmapsto \sqrt{x_1^2 + x_2^2 + x_3^2} = \|x\|_2$

$(a,b) \longmapsto |a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| = \|a-b\|_1$.

$\|x\|_\infty = \max\{|x_i| : i \in \mathbb{N}_3\}$

Description of the shape of the open unit ball of $\mathbb{R}^3$ endowed with $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$:

$\|\cdot\|_1$: $\|\underset{(x_1, x_2, x_3)}{\underset{\shortparallel}{x}}\|_1 = 1$ $\Leftrightarrow$ $|x_1| + |x_2| + |x_3| = 1$



unit $\|\cdot\|_2$: $\|x\|_2 = 1 \Leftrightarrow (x_1^2 + x_2^2 + x_3^2)^{1/2} = 1 \Leftrightarrow x_1^2 + x_2^2 + x_3^2 = 1$

unit sphere :

$\|\cdot\|_\infty$: $\|x\|_\infty = 1 \Leftrightarrow$

$\max\{|x_i| : 1 \le i \le 3\} = 1$

unit cube

5.17) X - normed linear space & C is a convex subset of X

<u>WTS</u> : C^0 is convex. <u>Def</u> : Suppose V is a normed linear space $C \subseteq V$ then C is said to be convex iff $\forall a, b \in C$ the line segment $\{(1-t)a + tb : t \in [0,1]\}$ joining a & b is included in C.

Let $x_1, x_2 \in C^\circ$. So $\exists r_1, r_2 > 0$ s.t. $b[x_1; r_1) \subseteq C$ & $b[x_2; r_2) \subseteq C$. Let

$x = \alpha x_1 + (1-\alpha) x_2$ for $\alpha \in [0,1]$, and set

$r = \alpha r_1 - (1-\alpha) r_2 > 0$ for sufficiently small r_2.

<u>WTS</u> : $b[x; r) \subseteq C$

For $y \in b[x; r)$, $z \in b[x_2; r_2)$

& $\|y - x\| < r \Rightarrow \|y - \alpha x_1 - (1-\alpha) x_2\| = \|y - (\alpha x_1 + (1-\alpha) z)$

$+ (1-\alpha)(z - x_2)\| \quad \Rightarrow$

$$\| y - (\alpha x_1 + (1-\alpha) z) \| - \| (1-\alpha)(z - x_2) \| < r \Rightarrow$$

$$\| y - (\alpha x_1 + (1-\alpha) z) \| < r + (1-\alpha) \underbrace{\| z - x_2 \|}_{< r_2}$$

$$= \alpha r_1 - \cancel{(1-\alpha) r_2} + \cancel{(1-\alpha) r_2} = \alpha r_1$$

$$\| [y - (1-\alpha) z] - \alpha x_1 \| \leq \alpha r_1 \Rightarrow$$

$$y - (1-\alpha) z \in b[\alpha x_1 ; \alpha r_1) \Rightarrow$$

$$y \in (1-\alpha) z + b[\alpha x_1 ; \alpha r_1) \Rightarrow$$

$\exists\, \alpha \tilde{x} \in b[\alpha x_1 ; \alpha r_1)$ s.t. $y = \alpha \tilde{x} + (1-\alpha) z$

$$\Updownarrow$$

$$\alpha \tilde{x} \in \alpha x_1 + \alpha r_1 b[0;1) \Rightarrow$$

$$\tilde{x} \in x_1 + r_1 b[0;1) = b[x_1 ; r_1).$$

$\therefore\ y = \alpha \tilde{x} + (1-\alpha) z$, $\tilde{x} \in b[x_1 ; r_1) \subseteq C$ & $z \in b[x_2 ; r_2) \subseteq C$. Since C is convex $y \in C$. This proves that C° is convex.

∂C need not be convex:

Take $C = (0,1)$ which is convex but $\partial C = \{0, 1\}$ clearly not convex.

$\overline{C}$ is convex whenever C is convex: exercise!