PS VIII (338)

Chapter 10

1) d) $\underbrace{\frac{\exp(1/z^2)}{z-1}}^{=: f(z)}$, simple pole at $z=1$ where

$$\operatorname{Res}(f(z);1) = \frac{\exp(1/z^2)}{\frac{d}{dz}(z-1)}\Big|_{z=1} = \frac{\exp(1/z^2)}{1}\Big|_{z=1} = e.$$ From

the previous PS, Laurent expansion of f about $z=0$ is $-e\sum_{m=0}^{\infty} z^m - \sum_{m=-\infty}^{-1}\left(e - \sum_{k=0}^{j-1} 1/k!\right) z^m$ where, for the second sum, $m=-2j$ or $m=-2j+1$, $j=1,2,\ldots$ In other words, if say $\sum_{k=-\infty}^{\infty} a_k z^k$ is the Laurent expansion representation for f about $z=0$ then

$$a_k = \begin{cases} -e & k=0,1,\ldots \\ -e+\sum_{n=0}^{j-1} 1/n! , & k=-2j \text{ or } k=-2j+1 \\ & j=1,2,\ldots \end{cases}$$

0 is the essential singularity, $\operatorname{Res}(f;0) = -e+1$ (search a_{-1})

f) $\sin\frac{1}{z}$, 0 is the essential singularity. Laurent series about 0 is $\frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} - \cdots$, $\operatorname{Res}(\sin\frac{1}{z}, 0) = 1$

4) Firstly recall that $\int_{|z|=1} z^m\, dz = \begin{cases} 2\pi i & m=-1 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{|z|=1} (z+1/z)^{2m+1} dz = \int_{|z|=1} \frac{(1+z^2)^{2m+1}}{z^{2m+1}} dz = \int_{|z|=1} \frac{\sum_{j=0}^{2m+1}\binom{2m+1}{j} z^{2j}}{z^{2m+1}} dz =$$

$$\sum_{j=0}^{2m+1}\binom{2m+1}{j}\int_{|z|=1} z^{2j-2m-1} dz \underset{j=m}{\underset{\downarrow}{=}} 2\pi i\binom{2m+1}{m}, \quad m\in\mathbb{Z}^+\cup\{0\}.$$

5) Let $g(w) := \frac{f(w)}{p(w)}\cdot\frac{p(w)-p(z)}{w-z}$. By Corollary 10.6,

$P(z) = \sum_{j=1}^{n}\operatorname{Res}(g;w_j) + \operatorname{Res}(g;z)$. Start by evaluating the residue at z:

$\lim_{w\to z}(w-z)\,g(w) = 0$, So the residue at w_j :

$$\lim_{w\to w_j}(w-w_j)\frac{f(w)}{\prod_{k=1}^{n}(w-w_k)}\,\frac{p(w)-p(z)}{w-z} = \frac{f(w_j)}{\prod_{k\neq j}^{n}(w_j-w_k)}\,\frac{\overbrace{p(w_j)}^{=0}-p(z)}{w_j-z}$$

$$= C_j\frac{p(z)}{z-w_j} = C_j\prod_{k\neq j}^{n}(z-w_k)$$ where C_j is the constant $f(w_j)\Big/\prod_{k\neq j}^{n}(w_j-w_k)$. Therefore, $P(z)=\sum_{j=1}^{n}C_j\prod_{k\neq j}^{n}(z-w_k)$ is the polynomial of degree $n-1$ satisfying $P(w_i)=f(w_i)$, $i=1,2,\dots,n$.

Since $$P(w_i) = \sum_{j=1}^{n} C_j\prod_{k\neq j}^{n}(w_i-w_k) = \sum_{j=1}^{n}\frac{f(w_j)}{\prod_{k\neq j}^{n}(w_j-w_k)}\prod_{k\neq j}^{n}(w_i-w_k)$$

when $i\neq j$ this product is zero

$$= \frac{f(w_i)}{\cancel{\prod_{k\neq i}^{n}(w_i-w_k)}}\cancel{\prod_{k\neq i}^{n}(w_i-w_k)} = f(w_i)$$

8) $|f|\geq|g|$ and $f+g\neq 0$ on γ. ($f\neq 0$) $\quad f(z)+g(z) = f(z)\left[1+\frac{g(z)}{f(z)}\right]$

$\Rightarrow \operatorname{Arg}(f(z)+g(z)) = \operatorname{Arg}f(z) + \operatorname{Arg}\left(1+\frac{g(z)}{f(z)}\right)$

$\left|\frac{g(z)}{f(z)}\right| \leq 1$ on γ

$1+\frac{g(z)}{f(z)}$

$\operatorname{Arg}\left(1+\frac{g(z)}{f(z)}\right)$ stays in the right of the imaginary axis as z moves along γ, so change in $\operatorname{Arg}\left(1+\frac{g(z)}{f(z)}\right)=0$. Thus $\operatorname{Arg}(f(z)+g(z))$ & $\operatorname{Arg}(f(z))$ have the same change around γ

$$\mathbb{Z}(f) = \frac{1}{2\pi i}\int_\gamma\frac{f'}{f} = \frac{1}{2\pi}\Delta\operatorname{Arg}f(z) = \frac{1}{2\pi}\Delta\operatorname{Arg}(f(z)+g(z)) = \mathbb{Z}(f+g).$$

b) $p(z) = z^5+2z^4+1$, $|z|<1$. Let $z^5+1=g(z)$, $2z^4=f(z)$

On $|z|=1$, $|f(z)| = 2 \geq |g(z)|$. As $\mathbb{Z}(f) = 4$

$f+g\neq 0$ on $|z|=1$ $\quad$ # of zeros of f in $|z|<1$

$\Rightarrow \mathbb{Z}(P) = 4$.

Rouche's thm

9) c) $f_3(z) = z^4 - 5z + 1$ in $1 \leq |z| \leq 2$,

for $|z| = 2$, $|5z - 1| \leq 1 + 5|z| = 11 < 16 = |z|^4$, 4 zeros in $|z| \leq 2$. For $|z| = 1$, $|z^4 + 1| \leq 2$ whereas $|5z| = 5$,

$\underset{\text{Rouche's}}{\Longrightarrow}$ 1 zero in $|z| \leq 1$. Thus 3 zeros in $1 \leq |z| \leq 2$.

10) $\lambda > 1$, WTS: $\lambda - z - e^{-z} = 0$ has exactly one root in the right half-plane.

Let $f(z) = z + e^{-z} - \lambda$. Consider the circle centered at 0 with radius R, cut it into two with the y-axis and let the right semicircle denoted by C_R. It suffices to show that for every $R > \lambda + 1$, f has a single zero inside C_R (because the interiors of such semicircles cover the entire right half plane) For $z \in C_R$,

$|f(z) - (z - \lambda)| = |e^{-z}| = e^{Re(-z)} = e^{-Re z} \leq 1 < |R - \lambda| = ||z| - |\lambda||$

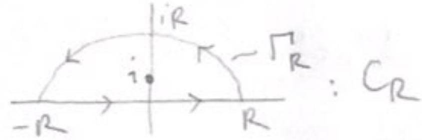
$\leq |z - \lambda|$. $\Rightarrow$ Rouche's thm implies that $f(z)$ & $z - \lambda$ have the same number of zeros inside C_R. The only root λ of $z - \lambda$ lies inside C_R (as $R > \lambda > 0$), therefore f also has a single root inside C_R

14) Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ (assume $a_n \neq 0$). Choose R s.t. $|a_n| R > \sum_{j=0}^{n-1} |a_j|$ and $R > 1$. For $|z| = R$,

$$\begin{aligned} |f(z) - a_n z^n| &\leq |a_{n-1} z^{n-1}| + \dots + |a_1 z| + |a_0| \\ &= |a_{n-1}| R^{n-1} + |a_{n-2}| R^{n-2} + \dots + |a_1| R + |a_0| \\ &\leq (|a_{n-1}| + |a_{n-2}| + \dots + |a_0|) R^{n-1} < |a_n| R^n = |a_n z^n| \end{aligned}$$

By Rouche's thm, in $|z| < R$, f & $a_n z^n$ have the same number of zero which is n.

Chapter 11



4) a) $\displaystyle\int_0^\infty \frac{\cos(ax)}{(x^2+1)^2}\,dx = I, \quad a \geq 0.$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2+1)^2}\,dx = \frac{1}{2}\operatorname{Re}\Big(\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+1)^2}\,dx\Big)$$

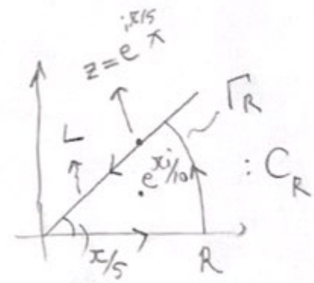
$$= \frac{1}{2}\operatorname{Re}\Big(2\pi i \operatorname{Res}\Big(\frac{e^{iaz}}{(z^2+1)^2}; i\Big)\Big).$$ Double pole at i, so

by recalling the identity: if f has a pole of order k at z_0, $\displaystyle \operatorname{Res}(f; z_0) = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right]\Big|_{z=z_0}$, we have

$$\operatorname{Res}\Big(\frac{e^{iaz}}{(z^2+1)^2}; i\Big) = \frac{d}{dz}\Big[\frac{e^{iaz}}{(z+i)^2}\Big]\Big|_{z=i} = -\frac{i}{4}e^{-a}(a+1) \Rightarrow$$

$$I = \frac{\pi}{4}e^{-a}(a+1).$$

Note: Since $a \geq 0$, still $\displaystyle\int_{\Gamma_R} R(z)e^{iaz}\,dz \to 0$ holds, see page 145 of the textbook.



b) $\displaystyle\int_0^\infty \frac{x^2}{x^{10}+1}\,dx$, Consider the contour:

$$\int_{\Gamma_R} \frac{z^2}{z^{10}+1}\,dz << \frac{\pi}{5}R \max_{\Gamma_R}\frac{|z|^2}{|z^{10}+1|} < \frac{\pi}{5}\frac{R^3}{R^{10}} \to 0 \text{ as } R\to\infty.$$

hence, letting $R\to\infty$ $\displaystyle\lim_{R\to\infty}\int_{C_R}\frac{z^2}{z^{10}+1}\,dz = \int_0^\infty \frac{x^2}{x^{10}+1}\,dx - e^{\frac{3\pi i}{5}}\int_0^\infty \frac{x^2}{x^{10}+1}\,dx = (1-e^{3\pi i/5})\int_0^\infty \frac{x^2}{x^{10}+1}\,dx$. Moreover,

$$\int_{C_R}\frac{z^2}{z^{10}+1}\,dz = 2\pi i\Big(\operatorname{Res}\Big(\frac{z^2}{z^{10}+1}; e^{\pi i/10}\Big)\Big) = 2\pi i\,\frac{z^2}{10z^9}\Big|_{e^{\pi i/10}}$$

$$= 2\pi i\,\frac{z^3}{10z^{10}}\Big|_{e^{\pi i/10}} = 2\pi i\,\frac{e^{3\pi i/10}}{10e^{\pi i}} = -\frac{\pi}{5}i\,e^{3\pi i/10} \Rightarrow$$

$$\int_0^\infty \frac{x^2}{x^{10}+1}\,dx = -\frac{\pi i}{5} e^{3\pi i/10}\left(1 - e^{3\pi i/5}\right)^{-1}.$$

c) $\int_0^{2\pi} e^{e^{i\theta}}\,d\theta =: I$, will use : $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ by setting $z = e^{i\theta}$. Notice that $e^{i\theta} = \cos\theta + i\sin\theta = z \rightsquigarrow I = \frac{1}{i}\int_{|z|=1} \frac{e^z}{z}\,dz = \frac{1}{i}\,2\pi i\,\mathrm{Res}\left(\frac{e^z}{z}; 0\right)$

$= 2\pi$

5) WTS : $\underbrace{\int_0^{2\pi} (\cos x)^{2m}\,dx}_{=:I} = \frac{2\pi}{4^m}\binom{2m}{m}$

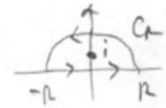
As above letting $z = e^{ix}$, we have $dx = \frac{dz}{iz}$ and $\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right)$. $I = \int_{|z|=1} \left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^{2m}\frac{dz}{iz} = \frac{1}{4^m i}\int_{|z|=1} \frac{(z^2+1)^{2m}}{z^{2m+1}}\,dz$

$$\underset{z^2=w}{=} \frac{2\pi}{4^m}\left[\frac{1}{2\pi i}\int_{|w|=1} \frac{(1+w)^{2m}}{w^{m+1}}\,dw\right] = \frac{2\pi}{4^m}\binom{2m}{m}$$

wraps the circle twice!

page 154 (6) $\rightarrow$ indeed this comes from Laurent series expansion (a formula given by Corollary 9.10) and the observation that $\binom{2n}{n}$ = coefficient of z^n in $(1+z)^{2n}$.

6) WTS : $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)\pi}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n}$

Using the contour :  , the integral is equal to $2\pi i\,\mathrm{Res}\left(\frac{1}{(1+z^2)^{n+1}}; i\right)$, pole of order $n+1$ at i for which the residue is evaluated as $\frac{1}{n!}\frac{d^n}{dz^n}\left(\frac{1}{(z+i)^{n+1}}\right)\Big|_{z=i}$

$$= \frac{1}{n!}(-1)^n (n+1)(n+2)\cdots 2n\,(2i)^{-2n-1}$$

$$= \frac{(-1)^n}{2i}(-1)^{-n}\frac{(n+1)(n+2)\cdots 2n}{n!\,2^{2n}} = \frac{1}{2i}\frac{(n+1)(n+2)\cdots 2n}{(2\cdot 4\cdot \ldots 2n)\,2^n} =$$

$$\frac{1}{2i} \frac{n!\,(n+1)(n+2)\cdots(2n)}{(2\cdot 4 \cdots 2n)^2} = \frac{1}{2i} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}$$

so the result follows