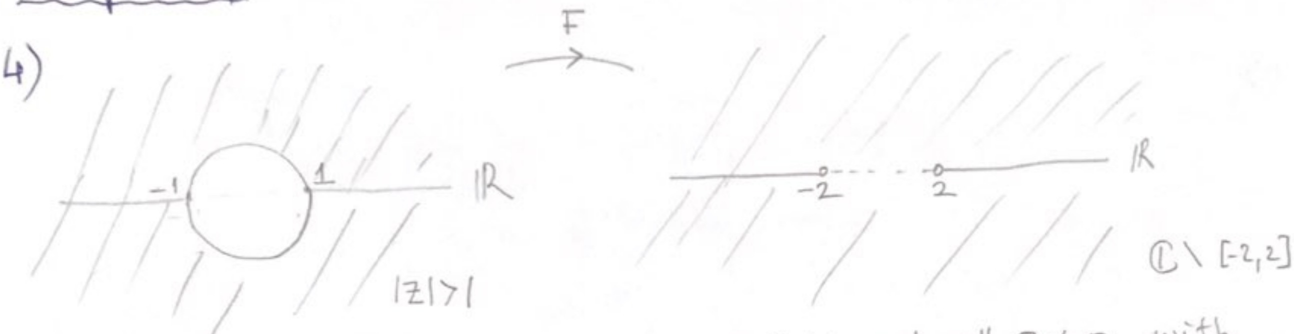
PS X (338)

Chapter 14

4)

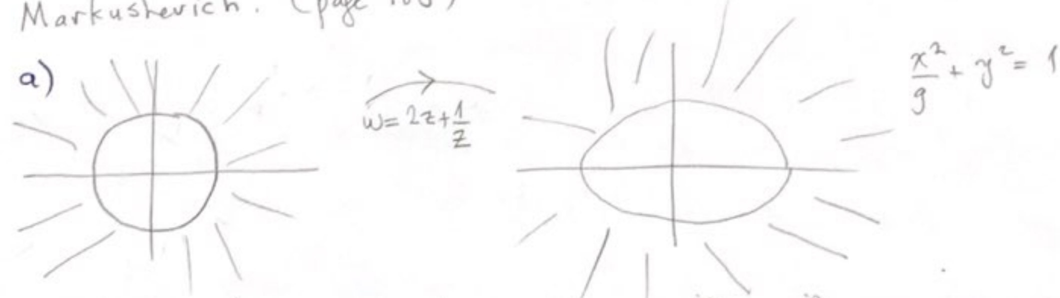


Firstly note that $F(z) = z + \frac{1}{z}$ is analytic at all $z \neq 0$ with $F'(z) = \frac{z^2-1}{z^2} \Rightarrow$ « $F'(z) = 0 \Rightarrow z = \pm 1$ », so $F(z)$ is conformal at $z \neq 0, \pm 1$. Clearly upper and lower semicircles are mapped to $[-2, 2]$ as $-2 \leq \underbrace{e^{i\theta} + e^{-i\theta}}_{= F(e^{i\theta})} \leq 2$ for $0 \leq \theta < 2\pi$

Also one observation is that $UHP \cap |z| > 1$ is mapped to UHP because letting $w = z + \frac{1}{z}$ gives $w = \operatorname{Re} z\left(1 + \frac{1}{|z|^2}\right) + i \operatorname{Im} z\left(1 - \frac{1}{|z|^2}\right)$ so that if $z \in UHP \cap |z|>1$ then $\operatorname{Im} w > 0$ as $\underbrace{\operatorname{Im} z}_{>0}\underbrace{\left(1 - \frac{1}{|z|^2}\right)}_{>0 \; |z|>1} > 0$

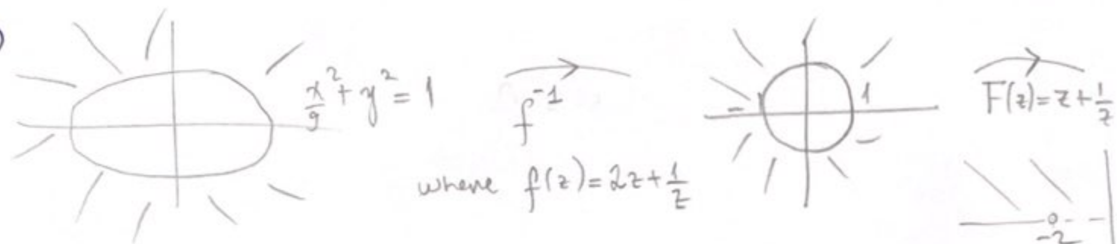
Similarly, $LHP \cap |z|>1 \longrightarrow LHP$. Uniqueness part follows from the arguments of solutions part of the text and the book: « Theory of Functions of a complex variable Volume II » of Markushevich. (page 189)

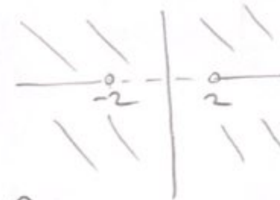
5) a)



let $f(z) = 2z + \frac{1}{z}$ so that $f(e^{i\theta}) = 2e^{i\theta} + e^{-i\theta} = 2\cos\theta + 2i\sin\theta + \cos\theta - i\sin\theta = 3\cos\theta + i\sin\theta = w = u + iv \Rightarrow \frac{u}{3} = \cos\theta$ & $v = \sin\theta \Rightarrow \frac{u^2}{9} + v^2 = 1$. We see that $\partial_{circle} \longrightarrow \partial_{ellipse}$. $\operatorname{Ext}(\text{Circle}) \ni 2i \mapsto \frac{7i}{2} \in \operatorname{Ext}(\text{Ellipse})$. By the OMT, Ext(Circle) maps conformally onto Ext(Ellipse) as f is conformal at $z \neq 0, \pm\frac{1}{\sqrt{2}}$. Also ontoness follows by analyzing the quadratic equation $2z^2 - zw + 1 = 0$ by means of its roots in w.

b)



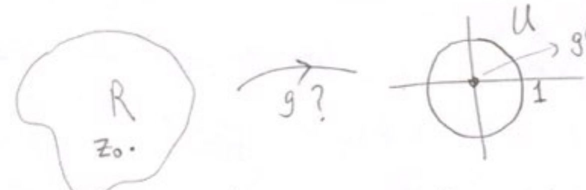


the map : $F \circ f^{-1}(z)$

to determine f^{-1} : $w = 2z + \frac{1}{z} \Rightarrow 2z^2 - zw + 1 = 0$

$$\Rightarrow f^{-1}(w) = \frac{w + \sqrt{w^2-8}}{2}.$$

6)



$g(z_0)=0$, $g'(z_0) > 0$

To find a such a mapping g, consider $f: R \longrightarrow \mathcal{U}$ and the automorphisms of the unit disc $\mathcal{U}$: $e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$, $|\alpha|<1$.

So as $|f(z)|<1 \ \forall z \in R$, we can take $e^{i\theta}\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}$ as a mapping $R \longrightarrow \mathcal{U}$ so that set $g(z) = e^{i\theta}\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}$ and evaluate $g'(z_0) = e^{i\theta}\frac{f'(z_0)}{1-|f(z_0)|^2}$. Since we want $g'(z_0)>0$, let $\theta = -\mathrm{Arg}\, f'(z_0)$, to have $e^{i\theta}f'(z_0) = e^{-i\mathrm{Arg} f'(z_0)}|f'(z_0)|e^{i\mathrm{Arg} f'(z_0)}$
$= |f'(z_0)| > 0$ also due to $|f(z_0)|<1$, we infer that $g'(z_0)>0$.

<u>Recall</u> : (Riemann Mapping thm) For any simply connected domain $R\ (\neq \mathbb{C})$ and $z_0 \in R$, $\exists$ unique conformal mapping g of R onto $\mathcal{U}$ s.t. $g(z_0)=0$ & $g'(z_0)>0$.

7) R - simply connected region $\neq \mathbb{C}$ & symmetric wrt $\mathbb{R}$

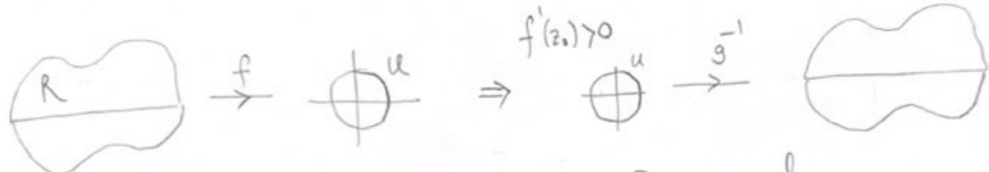
$f: R \longrightarrow \mathcal{U}$ (unit disc) Riemann mapping with $f(z_0)=0$ & $f'(z_0)>0$, $z_0 \in R \cap \mathbb{R}$. WTS : $f(\bar{z}) = \overline{f(z)} \ \forall z \in R$.

To see so, recall

Schwarz' Lemma : f is analytic in $\mathcal{U}$, $f \ll 1$ and $f(0)=0$ then (i) $|f(z)| \le |z|$ (ii) $|f'(0)| \le 1$ with equality in (i) or (ii) iff $f(z) = e^{i\theta} z$

To begin let $g(z) = \overline{f(\bar{z})}$. Clearly, $g(z_0) = \overline{f(\bar{z}_0)} \underset{z_0 \in \mathbb{R}}{=} \overline{f(z_0)} = 0$

and $g'(z_0) = \bar{f}'(\bar{z}_0) = \overline{f'(z_0)} \underset{,}{=} f'(z_0)$. Now set $h = f \circ g^{-1} : \mathcal{U} \to \mathcal{U}$



hence $h(0) = f \circ g^{-1}(0) = f(z_0) = 0$ and

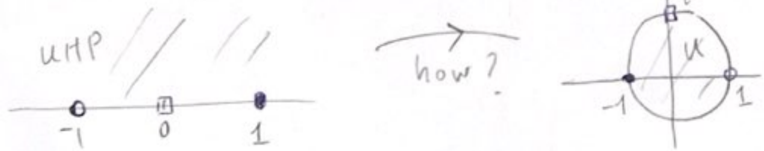
$$h'(0) = f'(z_0)(g^{-1})'(0) = \frac{f'(z_0)}{g'(z_0)} = 1$$

Since h is clearly analytic, by Schwarz's Lemma we easily deduce that $h(z) = z$ so that $f(z) = g(z) \Rightarrow \overline{f(z)} = f(\bar{z})$.

8) a) $\dfrac{(w-i)(-2)}{(w-1)(-1-i)} = \dfrac{z \cdot 2}{(z+1)} \Rightarrow \dfrac{w-i}{w-1} = \dfrac{z}{z+1}(1+i) \Rightarrow$

$\dfrac{1-i}{w-1} = \dfrac{z(1+i)}{z+1} - 1 \Rightarrow (1-i)\dfrac{(z+1)}{iz-1} = w-1 \Rightarrow \boxed{w = \dfrac{z-i}{iz-1}}$

unique by Thm 13.23



Answer:

$|w| = \left|\dfrac{z-i}{z+i}\right| < 1$ because $z \in \text{UHP} \to \operatorname{Im} z > 0$ which follows $(\operatorname{Im} z - 1)^2 < (\operatorname{Im} z + 1)^2 \Rightarrow (\operatorname{Re} z)^2 + (\operatorname{Im} z - 1)^2 = |z-i|^2 < (\operatorname{Re} z)^2 + (\operatorname{Im} z + 1)^2 = |z+i|^2$

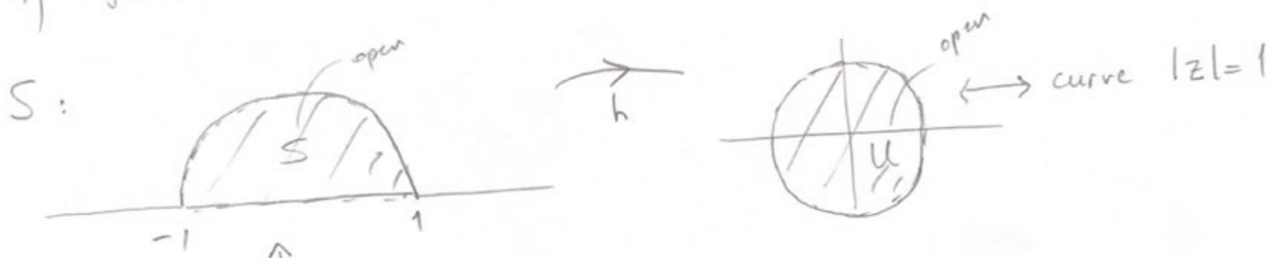
So w maps the UHP into unit disc.

b) From part a), $w = \dfrac{z-i}{1-iz}$ works well.

12) a) From the question 6) of previous PS we know that this conformal map could be taken of the form

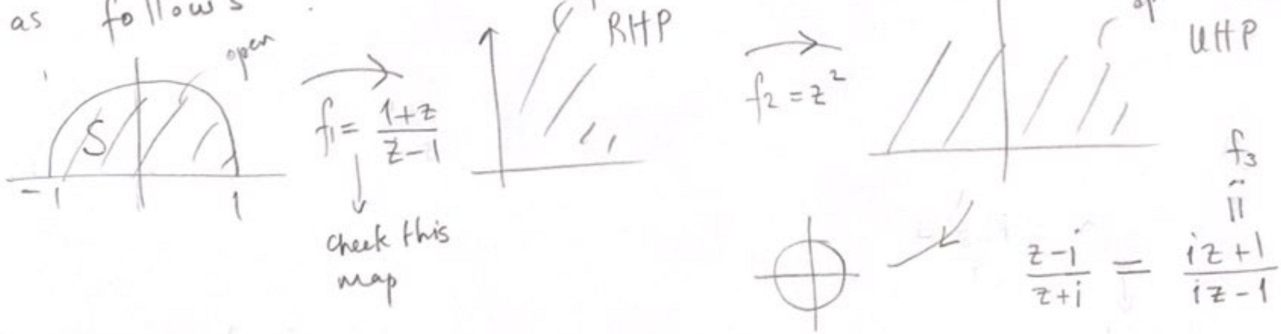
$$h(z) = \frac{-(z-1)^2 - 4i(z+1)^2}{-(z-1)^2 + 4i(z+1)^2}$$

this mapping extends to a homeomorphism between $\bar{S}$ & $\bar{U}$ due to Carathéodory – Osgood Theorem :
Any conformal mapping between two Jordan regions can be extended to a homeomorphism between the closures of the two regions.
Recall <u>defn</u> : R is called a Jordan region if it is the interior of Jordan curve (simple closed curve)



curve $\{z \in \mathbb{C} : [-1,1] \cup$ upper unit semi circle$\}$, both curves are closed simple so Jordon curves, Carathéodory – Osgood Theorem applies.

b) Tedious Check!
In part a) instead of that h, we would rather construct the desired map as follows :



the map :

$$f_1 \circ f_2 \circ f_3(z) = \frac{i\left(\frac{z+1}{z-1}\right)^2 + 1}{i\left(\frac{z+1}{z-1}\right)^2 - 1} = \frac{(i+1)(z^2+1) + 2(i-1)z}{(i-1)(z^2+1) + 2(i+1)z}$$

$$= \frac{(i+1)^2(z^2+1) + 2(i^2-1)z}{(i^2-1)(z^2+1) + 2(i+1)^2 z} = -i\,\frac{z^2+2iz+1}{z^2-2iz+1}$$

, this is easier to check in terms of analyticity.