Quiz 4 - Solutions

① Let $f(z) = \frac{1}{1+z^2}$. The poles of of f are $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, $-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, $-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. The first two lie on the upper half plane.

Clearly, $|f(z)| \leq \frac{1}{|z|^2}$ for all $z \neq 0$.

Hence $\displaystyle\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \left(\mathrm{Res}\left(f, \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) + \mathrm{Res}\left(f, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \right)$

$$\mathrm{Res}\left(f, \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{1}{4\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^3} = \frac{1}{\sqrt{2}\,(-2+2i)}$$

$$\mathrm{Res}\left(f, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{1}{4\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^3} = \frac{1}{\sqrt{2}\,(2+2i)}$$

So $$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \left(\frac{1}{2\sqrt{2}} \left(\frac{1}{(-1+i)} + \frac{1}{(1+i)} \right) \right) = \frac{\pi i}{\sqrt{2}} \left(\frac{1+i-1+i}{-2} \right) = \frac{\pi}{\sqrt{2}}$$

Since f is even, we have $\displaystyle\int_0^{\infty} f(x)\,dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2\sqrt{2}}$.

② Let $f(z) = z^9 + z^5 - 8z^3 + 2z + 1$. And let C_1 and C_2 be circles centered at 0 of radii 1 and 2.

First note that if we take $g_1(z) = -8z^3$, then $|g_1(z)| > |z^9 + z^5 + 2z + 1|$ for $z \in C_1$. Hence $f(z)$ has the same number of zeros as $g_1(z)$ inside C_1. This means that f has 3 zeros in the unit disc.

Now let $g_2(z) = z^9$. Then $|g_2(z)| > |z^5 - 8z^3 + 2z + 1|$ for $z \in C_2$.

So f has as many zeros as g_2 in the disc centered at 0 & of radius 2.

Therefore f has 9 zeros in that disc.

Thus f has $9 - 3 = 6$ zeros in the given annulus.