MATH 338 – Complex Analysis I

FINAL EXAM - 25.06.2021

We use the following conventions:

- *Given $S \subseteq \mathbb{C}$, $\overline{S}$ denotes its closure.*
- $\mathbb{D} := D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

Questions

1. Let f be an entire function such that $f(z) = f(x+iy) = u(x) + i\,v(y)$ for real valued functions u, v of variables x and y respectively. Show that f is a linear function of z.

2. Write down the isolated singularities of $f(z) = \frac{1}{\sin(\pi z)}$ and express their kinds. Also calculate the residues of f at each singularity.

3. Let $a \in \mathbb{C}$ be such that $|a| > e$ and let $n > 0$ be an integer. How many solutions does the equation $\exp(z) = az^n$ have in $\mathbb{D}$. Explain your answer.

4. Find the Laurent series of $\exp(z + \frac{1}{z})$ around the origin and determine its annulus of convergence.

5. Write down **all** conformal mappings between $\mathbb{D} \cap \mathbb{H}$ and $\mathbb{D}$. Explain your answer.

6. Let C be the boundary of $\mathbb{D}$. Let f be a non-constant continuous map on $\overline{\mathbb{D}}$ that is analytic on $\mathbb{D}$ and $f(C) \subseteq C$.

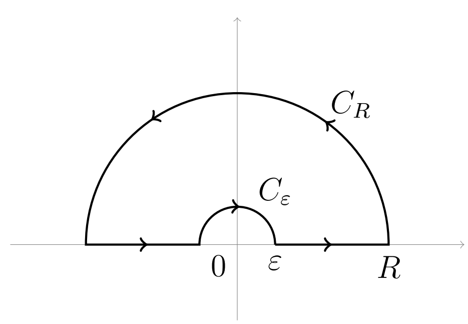
   (a) Show that $f(\mathbb{D}) \subseteq \mathbb{D}$.

   (b) Show that f has a zero in $\mathbb{D}$.

   (c) Show that f maps $\mathbb{D}$ onto itself; this means $f(\mathbb{D}) = \mathbb{D}$. (Hint: Modify f by a Möbius transformation.)

7. Show that
$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx = -\frac{\pi}{4}.$$

   [*Hints: 1. Define an appropriate branch of the complex logarithm and consider the integral of an appropriate function along the following path:*



   2. You don't have to calculate the exact value of any real integrals.]