Chapter 9

PS VII (338)

1) Assume that $f(z) \to \infty$ as $z \to z_0$, an isolated singularity.

WTS : f has a pole at z_0

By the assumption, defining $h(z) := \frac{1}{f(z)}$ which is analytic in some punctured neighborhood of z_0 so that we have $h(z) \to 0$ as $z \to z_0 \Rightarrow h(z)$ is bounded near z_0. By the Corollary 9.4 of Riemann's theorem of removable singularities, $h(z)$ has a removable singularity at z_0 and can be extended to an analytic function by defining $h(z_0) = 0$. So $h(z)$ has a zero of order N say, so that $f(z)$ has a pole of order N.

Pf of Corollary 9.4 (not given in the text)

f is analytic in the punctured disc $D = \{z : 0 < |z - z_0| < r\}$ (by the defn of isolated singularity), by Corollary 9.10,

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k \quad \text{where} \quad a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} dz$$

$C = C(z_0; s)$, $0 < s < r$. Since f is bounded in D, $\exists M > 0$ s.t. $|f(z)| < M \ \forall z \in D$. Thus,

$$|a_n| \leq \frac{1}{2\pi} \frac{M}{s^{n+1}} 2\pi s = \frac{M}{s^n} \to 0 \text{ as } s \to 0 \text{ for } n < 0 \Rightarrow$$

$a_n = 0$ for $n < 0$.

2) As $|f(z)| \sim \exp(1/|z|) \to \infty$ as $|z| \to 0$, by question above f would have a pole at $z = 0$, say of order N then $|f(z)| \sim \frac{C}{|z|^N}$.

3) Suppose that f is entire 1-1 function. WTS : $f(z) = az + b$

Since entire functions can be given by power series, we shall consider $g(z) := f(1/z)$ which has 0 as isolated singularity. Clearly this singularity can't be removable, so 0 is either a pole or an essential singularity. If it were an essential singularity, then considering deleted neighbourhood $D = \{z : 0 < |z| < r\}$ of 0, we would deduce by Casorati-Weierstrass Theorem that the range $g(D)$ be dense in $\mathbb{C}$, however this contradicts with the 1-1 ness of f because g is analytic at points other than 0, hence maps open set $A = \{z : r+1 < |z| < r+2\}$ to an open set $g(A)$ by the Open mapping theorem, for which $g(A) \cap g(D) \neq \emptyset$ by the density of $g(D)$. Thus g is not 1-1 $\Rightarrow$ f is not 1-1. So 0 is the pole of g $\Rightarrow$ $\exists N \in \mathbb{N}$ s.t. $g(z) = \sum_{j=0}^{N} a_j \frac{1}{z^j}$ $\Rightarrow$ $f(z) = \sum_{j=0}^{N} a_j z^j$

with power series expansion of f

by fund. thm. of algebra

when $N \geq 2$, f is not injective. To see this, assume $z_1, \dots, z_N$ are s.t. $f(z_j) = 0$, so if $\exists\, j \neq k$ s.t. $z_j \neq z_k$ then f is not injective. Thus for $z_1 = \dots = z_N$, we've $f(z) = c(z - z_1)^N$ yielding that $f(z_1 + 1) = c = c\underbrace{e^{2\pi i}}_{=1} = c\left(e^{2\pi i/N}\right)^N = f(z_1 + e^{2\pi i/N})$ $\Rightarrow$ f is not injective.

Thus $N = 1$ $\Rightarrow$ $f(z) = a_0 + a_1 z$.

4) f is analytic in $\mathbb{C} \setminus \{0\}$ and $|f(z)| \leq \sqrt{|z|} + 1/\sqrt{|z|}$ $\Rightarrow$

$$\lim_{|z| \to 0} |z||f(z)| \leq \lim_{|z| \to 0} |z|^{3/2} + |z|^{1/2} = 0 \Rightarrow \lim_{|z| \to 0} |zf(z)| = 0$$

Then by Riemann's principle of removable singularities, 0 is the removable singularity so there is entire function g s.t. $g = f$ on $\mathbb{C} \setminus \{0\}$. Then for $z \neq 0$, $\exists\, C > 0$ s.t. $|g(z)| = |f(z)| \leq C|z|$ which along with the extended Liouville's thm g is a degree-1 polynomial, but from the given bound for f, g must be constant and hence f is constant.

5) Suppose f & g are entire with $|f(z)| \leq |g(z)|$ $\forall z$.
<u>WTS</u> : $f(z) = c\,g(z)$, for some constant c.
Write $h = \frac{f}{g}$, as $|h| \leq 1$ $(*)$, if we show that h is entire then by Liouville's theorem $h = c$, for some constant so the result will follow. As f & g are entire, to see that so is h, we must verify that the possible roots of g are removable singularities of h. So assume that z_0 is the zero of g of order n, i.e. $g^{(j)}(z_0) = 0$ $0 \leq j \leq n-1$, $g^{(n)}(z_0) \neq 0$. If $f(z_0) \neq 0$ then we could have $h(z) \to \infty$ as $z \to z_0$, contradicting $(*)$. Then if f has zero of order $m < n$ at z_0, again $h(z) \to \infty$ as $z \to z_0$. So must take $m \geq n$. Therefore we write $f(z) = A(z)(z-z_0)^m$ and $g(z) = B(z)(z-z_0)^n$ where A & B are entire functions s.t. $A(z_0) \neq 0$, $B(z_0) \neq 0$. This yields $h(z) = \frac{A(z)(z-z_0)^{m-n}}{B(z)} \Rightarrow$
$\lim_{z \to z_0} (z-z_0)h(z) = 0$ which, by Riemann's thm, shows that the singularity is removable.
<u>Note</u> : Indeed, since $|h(z)| \leq 1$ $\forall z$, by the Corollary 9.4, the result immediately follows.

8) This question is a content of Casorati-Weierstrass Thm
However we just prove :
if z_0 is an essential singularity of $f(z)$, then for all $w_0 \in \mathbb{C}$ $\exists$ seqn. $z_n \to z_0$ s.t. $f(z_n) \to w_0$. (This is stronger than the statement of "$\Rightarrow$" part of the question)
Argue contrapositive. Suppose there is some $w_0 \in \mathbb{C}$ that is not a limit of values of $f(z)$ as above. Then there is some small $\varepsilon > 0$ s.t. $|f(z) - w_0| > \varepsilon$ for all z near z_0. Hence $h(z) = \frac{1}{f(z) - w_0}$ is bounded near z_0. By Corollary 9.4, $h(z)$ has removable singularity at z_0. Hence $h(z) = (z-z_0)^N g(z)$ for some $N \geq 0$ and some analytic fnc. $g(z)$ with $g(z_0) \neq 0$.
Thus $f(z) - w_0 = \frac{1}{h(z)} = (z-z_0)^{-N}\left(\frac{1}{g(z)}\right)$, where $\frac{1}{g(z)}$ is

is analytic at z_0. If $N=0$, $f(z)$ extends to be analytic at z_0, while if $N>0$, $f(z)$ has a pole of order N at z_0. This establishes what we wanted to show.

9) a) $\frac{1}{z^4+z^2} = \frac{1}{z^2(z^2+1)}$, double pole at 0, simple pole at $\pm i$

b) $\cot z = \frac{\cos z}{\sin z}$, simple pole at $z = k\pi$, $k \in \mathbb{Z}$.

c) $\csc z = \frac{1}{\sin z}$, simple pole at $z = k\pi$, $k \in \mathbb{Z}$.

d) $\frac{\exp(1/z^2)}{z-1}$, simple pole at $z=1$, essential singularity at 0.

also, $\exp(z) = 1 + z + z^2/2! + \cdots$ has an essential singularity at ∞.

10) $f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2} = \frac{1}{(z-i)^2[(z-i)+2i]^2} = \frac{1}{(z-i)^2}\frac{1}{(2i)^2\left[1+\frac{z-i}{2i}\right]^2}$

$= \frac{-1/4}{(z-i)^2}\frac{1}{\left[1+\frac{z-i}{2i}\right]^2}$ $(*)$. Note that $\frac{1}{(1+w)^2} = \frac{d}{dw}\left(\frac{-1}{1+w}\right) =$

$\frac{d}{dw}\left(\sum_0^\infty (-1)^{n+1} w^n\right) \underset{|w|<1}{=} \sum_1^\infty (-1)^{n+1} n w^{n-1} = 1 - 2w + 3w^2 - \cdots$

So for $|z-i|<2$, $(*) = \frac{-1/4}{(z-i)^2}\left[1 - 2\frac{(z-i)}{2i} + 3\frac{(z-i)^2}{-4} - \cdots\right]$

$= \frac{-1/4}{(z-i)^2} - \frac{i/4}{(z-i)} + 3/16 - \cdots$

So the principal part of f is $\frac{-1/4}{(z-i)^2} - \frac{i/4}{(z-i)}$.

11) a) $\frac{1}{z^2(z^2+1)} = \frac{1}{z^2}\frac{1}{1+z^2} = \frac{1}{z^2}\sum_0^\infty (-1)^k z^{2k} = \frac{1}{z^2} - 1 + z^2 - z^4$
$+ \cdots = \sum_{k=-1}^\infty (-1)^{k+1} z^{2k}$

b) $\frac{\exp(1/z^2)}{z-1} =: f(z)$, $\exp(1/z^2) = \sum_{k=0}^\infty \frac{1}{k!\, z^{2k}} = 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \cdots$

$\frac{1}{z-1} = -\sum_{k=0}^\infty z^k = -1 - z - z^2 - \cdots$

$f(z) = -\sum_{k=0}^\infty \frac{z^{-2k}}{k!}\sum_{n=0}^\infty z^n = -\sum_{k=0}^\infty\sum_{n=0}^\infty \frac{z^{n-2k}}{k!}$ $(*)$, two cases to consider : $n-2k = m$, $m = 0,1,2,\dots$ and $n-2k=-m$, $m=1,2,\dots$; ... (we exclude $m=0$ in the second case in order to avoid summing the associated term two times). In the first case since $n=2k+m$ where $n=0,1,2,\dots$, we take $k=0,1,2,\dots$. In the second, $-m=-2k$ or $-m=-2k+1$ (*) for $k=1,2,\dots$

$$(*) = -\left[\sum_{m=0}^{\infty}\underbrace{\left(\sum_{k=0}^{\infty}\frac{1}{k!}\right)}_{=e} z^m + \sum_{m=1}^{\infty}\underbrace{\left(\sum_{k=j}^{\infty}\frac{1}{k!}\right)}_{=\left(e-\sum_{k=0}^{j-1}\frac{1}{k!}\right)} z^{-m}\right] \quad \text{where}$$

$m = 2j$ or $m = 2j-1$, $j=1,2,\dots$ with(*)

$$= -e\sum_{m=0}^{\infty} z^m - \sum_{m=-\infty}^{1}\left(e-\sum_{k=0}^{j-1} 1/k!\right) z^m$$

now where $m=-2j$ or $m=-2j+1$, $j=1,2,\dots$

12) $f(z) = \dfrac{1}{z(z-1)(z-2)}$

a) $0<|z|<1$

$f(z) = \dfrac{1}{z}\left(\dfrac{1}{z-2}-\dfrac{1}{z-1}\right)$, $\dfrac{1}{z-2} = \dfrac{-1}{2\left(1-\frac{z}{2}\right)}$, in this region $0<\left|\frac{z}{2}\right|<1$, thus $\dfrac{1}{z-2} = -\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{2^n}$, also $\dfrac{1}{z-1} \underset{|z|<1}{=} -\displaystyle\sum_{n=0}^{\infty} z^n$

therefore $f(z) = -\displaystyle\sum_{n=0}^{\infty}\left(\frac{z^{n-1}}{2^{n+1}} + z^{n-1}\right)$

b) $1<|z|<2$

In this region since $0<\left|\frac{z}{2}\right|<1$, we take the same series for $\dfrac{1}{z-2}$. But since we nomore have $|z|<1$, we rewrite $\dfrac{1}{z-1} = \dfrac{1}{z}\cdot\dfrac{1}{\left(1-\frac{1}{z}\right)}$ because $\left|\frac{1}{z}\right|<1$, so

$\dfrac{1}{z-1} = \dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$. Thus $f(z) = \dfrac{1}{z}\left(-\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right)$

$$= -\left(\sum_{n=0}^{\infty}\frac{z^{n-1}}{2^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{z^{n+2}}\right)$$

c) $|z|>2$

In this case, $$\frac{1}{z-2} = \frac{1}{z}\,\frac{1}{\left(1-\frac{2}{z}\right)} \underset{|z|>2}{=} \frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

likewise $$\frac{1}{z-1} = \frac{1}{z}\,\frac{1}{1-\frac{1}{z}} \underset{|z|>2}{=} \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} \Longrightarrow$$

$$f(z) = \frac{1}{z}\left(\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{2^n-1}{z^{n+2}}$$