1. Let $\{x_k\}$ be a convergent sequence in $\mathbb{R}$. Show that the set $\{x_1, x_2, \dots\}$ has zero content.

   Defn. *A set $Z \subset \mathbb{R}$ has zero content* if for any $\epsilon > 0$ there is a finite collection of intervals $\{I_1, \dots, I_N\}$ such that

   (a) $Z \subset \bigcup_{n=1}^{N} I_n$, and

   (b) the sum of the lengths of the I_n's is less than ϵ.

   Proof. Let $\epsilon > 0$ be given. Since $x_k \to L$ for some $L \in \mathbb{R}$, there exists a $K \in \mathbb{N}$ such that for all $k > K$ $|x_k - L| < \epsilon/6$. Let

   $$I_{K+1} = (L - \frac{\epsilon}{6}, L + \frac{\epsilon}{6}) \quad \text{and} \quad I_k = (x_k - \frac{\epsilon}{4K}, x_k + \frac{\epsilon}{4K}) \ \text{ for all } \ k \leq K.$$

   Note that we have K many intervals of length $|I_k| = \epsilon/2K$ and one interval of length $|I_{K+1}| = \epsilon/3$. We get

   $$|I_1| + \cdots + |I_K| + |I_{K+1}| = K\frac{\epsilon}{2K} + \frac{\epsilon}{3} = \frac{5}{6}\epsilon < \epsilon,$$

   thus the set $\{x_1, x_2, \dots\}$ has zero content.

   □

2. Suppose that $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are both uniformly continuous on S. Show that $f + g$ is uniformly continuous on S. (Exercise. Try to do it under 4 minutes!)

3. A function g on $\mathbb{R}$ to $\mathbb{R}^q$ is periodic if there exists a positive number p such that $g(x+p) = g(x)$ for every $x \in \mathbb{R}$. Show that a continuous periodic function is bounded and uniformly continuous on all of $\mathbb{R}$.

   Proof: Let g be continuous and periodic on $\mathbb{R}$, so that $g(x+p) = g(x)$ for every $x \in \mathbb{R}$. Fix $x_0 \in \mathbb{R}$, and consider the interval $I = [x_0 - p, x_0 + p]$.

   g is bounded on $\mathbb{R}$: Since I is compact g is bounded on I, so there exists $M \in \mathbb{R}$ such that

   $$|g(x)| \leq M \text{ for every } x \in I.$$

   Let $y \in \mathbb{R}$, then there exists $k \in \mathbb{Z}$ such that $y + kp \in I$. [1] So we have

   $$|g(y)| = |g(y + kp)| \leq M,$$

   therefore g is a bounded function on $\mathbb{R}$.

   g is uniformly continuous on $\mathbb{R}$: Since I is compact g is uniformly continuous on I. Let $\epsilon > 0$, then there exists $\delta > 0$ such that

   $$|g(x) - g(y)| \leq \epsilon \ \text{ whenever } \ |x - y| \leq \delta \text{ and } x, y \in I.$$

   Take $\tilde{\delta} = \min\{\delta, p\}$. Let $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$ wlog and $|x_1 - x_2| = x_2 - x_1 \leq \tilde{\delta}$.

   There exists $y_1, y_2 \in I$ such that $g(x_1) = g(y_1)$ and $g(x_2) = g(y_2)$ with $|y_1 - y_2| = y_2 - y_1 \leq \tilde{\delta}$. [2]

   Then

   $$|g(x_1) - g(x_2)| = |g(y_1) - g(y_2)| \leq \epsilon,$$

   therefore g is a uniformly continuous function on $\mathbb{R}$.

   □

[1] Take $k = \frac{\lfloor x_0 - y \rfloor}{p}$, so that $x_0 - p \leq y + kp \leq x_0$; or $k = \frac{\lceil x_0 - y \rceil}{p}$, so that $x_0 \leq y + kp \leq x_0 + p$.

[2] Take $k = \frac{\lfloor x_0 - x_1 \rfloor}{p}$, so that $x_0 - p \leq y_1 \leq y_2 \leq x_0 + p$ where $y_1 = x_1 + kp$ and $y_2 = x_2 + kp$