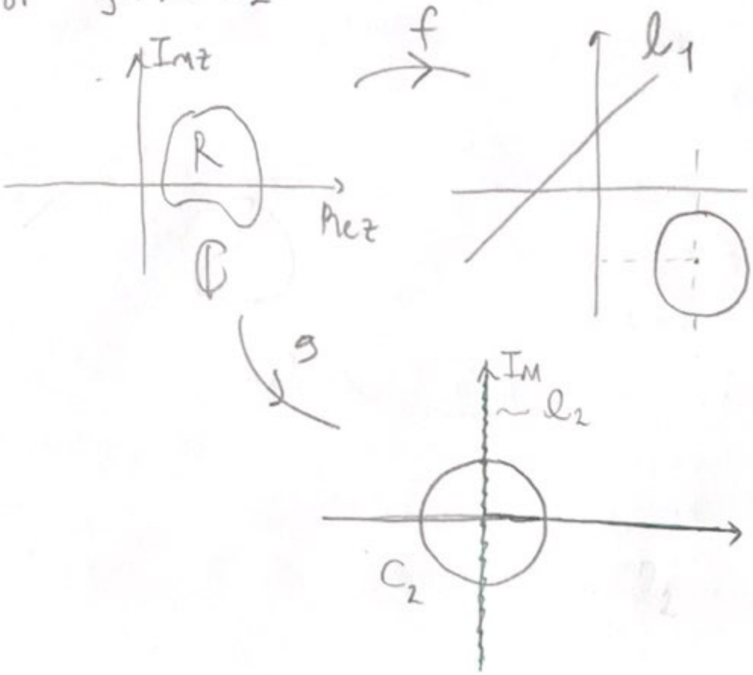
PS II (338)

Chapter 3

7) Suppose on the contrary that f is a nonconstant analytic fnc that maps a region R into a straight line or into a circular arc. Then one can find suitable constants c_1 & c_2 so that we have $g(z) = c_1 f(z) + c_2$, mapping $g(R) \subseteq$ Imaginary axis or $g(R) \subseteq C_2$.



What proposition 3.7 says is that given an analytic fnc f on a region R with modulus $|f|$ is constant there, f is constant as well.

Clearly since for any $z \in R$, $g(z) \in C_2$ – circle centered at zero, $|g(z)| = r$ – radius of C_2 $\forall z \in R$ $\Rightarrow$ $|g|$ is constant on R $\overset{\text{prop 3.7}}{\Rightarrow}$ g is constant $\Rightarrow f$ "

Moreover, using prop 3.6 which states that if $f = u + iv$ is analytic in a region D and u is constant, then f is constant (Similarly taking v constant leads to the same conclusion)

In our case, $g(R) \subseteq$ Imaginary axis, so $\forall z \in R$, $\exists w \in \mathbb{R}$ s.t $g(z) = iw$. By prop 3.6, g is contant, say $g(z) = c$ $\forall z \in \mathbb{C}$

$\Rightarrow f(z) = \dfrac{c - c_2}{c_1}$ – a fixed complex number, $z \in \mathbb{C}$. So f is contant.

8) Finding all analytic fncs $f = u + iv$ with $u(x,y) = x^2 - y^2$

Cauchy-Rieman eqns : $\dfrac{\partial u}{\partial x} = 2x = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y} = -2y = -\dfrac{\partial v}{\partial x}$

$2xy + c(x) \equiv v(x,y) \Rightarrow v_x = 2y + c'(x) \overset{?}{=} 2y$

$\Rightarrow c(x) = c$ – constant. So, $f = u + iv = x^2 - y^2 + i(2xy + c)$

$= x^2 - y^2 + 2ixy + iC \underset{z = x+iy}{=} z^2 + iC$

9) No analytic fnc $f = u + iv$ with $u(x,y) = x^2 + y^2$.
By the computations of question 8, we'd have $v_x = 2y + c'(x) = -2y \Rightarrow c(x) = -4xy + c$ which is absurd as $c(x)$ is a function of x only. So Cauchy-Riemann eqns cannot be satisfied in this case.

10) $f(x,y) = u(x) + iv(y)$ - entire $\overset{CR}{\Rightarrow} \underbrace{u_x}_{\text{fnc of } x} = \underbrace{v_y}_{\text{fnc of } y} \Rightarrow$
this must be constant, $u_x = v_y = C$

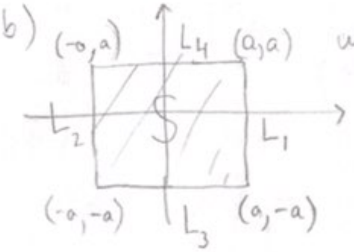
$\Rightarrow u(x) = Cx + b_1$ & $v(y) = Cy + b_2$

$\Rightarrow f(x,y) = C(x+iy) + b_1 + b_2 \underset{\substack{x+iy=z \\ b_1+b_2=b}}{=} Cz + b = f(z)$

14) d) $e^z = 1 + i \underset{z=x+iy}{\Rightarrow} e^x e^{iy} = \sqrt{2}\, \text{cis}\, \pi/4 \Rightarrow x = \log\sqrt{2}$
and $y = \frac{\pi}{4} + 2\pi k$, $k \in \mathbb{Z}$.

16) a) $\sin(\frac{\pi}{2} + iy) = \frac{1}{2i}(e^{i\pi/2 - y} - e^{-i\pi/2 + y}) = \frac{1}{2i}(ie^{-y} + ie^{y})$

$= \frac{1}{2}(e^{-y} + e^{y}) = \cosh y$

b) 

where $a = (N + \frac{1}{2})\pi$, $N \in \mathbb{Z}^+$. WTS: $|\sin z| \geq \frac{1}{4}$ $\forall z \in \partial S = \bigcup_{j=1}^{4} L_j$

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$, using original definition does not help much (try & see) instead lets set $z = x + iy$ and observe

$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$

(here we $\cosh y = \frac{1}{2}(e^{y} + e^{-y})$ & $\sinh y = \frac{1}{2}(e^{y} - e^{-y})$ to see)

then $|\sin z|^2 = |\sin(x+iy)|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$
$= (1-\cos^2 x)\cosh^2 y + \cos^2 x \sinh^2 y = \cosh^2 y + \cos^2 x(\sinh^2 y - \cosh^2 y) = \cosh^2 y - \cos^2 x \Rightarrow$
$|\sin z| = (\cosh^2 y - \cos^2 x)^{1/2}$, At points $z \in L_1 \cup L_2$ we have $x = \operatorname{Re} z = \pm(N+1/2)\pi \Rightarrow \cos x = 0$ thus
$|\sin z| = |\cosh y| \geq 1$ as $\cosh^2 y = 1 + \underbrace{\sinh^2 y}_{\geq 0} \geq 1$. Next consider the points $z \in L_3 \cup L_4$ at which we have
$y = \operatorname{Im} z = \pm(N+\frac{1}{2})\pi$. $\cosh y = \frac{1}{2}(e^y + e^{-y}) = \cosh(-y)$
$\Rightarrow$ so suffices to consider $y = (N+\frac{1}{2})\pi$ alone. Here $N \in \mathbb{Z}^+$. Also $\frac{d}{dy}\cosh y = \frac{1}{2}(e^y - e^{-y}) \geq 0$, $\forall y \geq 0$. Hence $\cosh y$ is an increasing fnc for $y \geq 0$. It follows, for $N \in \mathbb{Z}^+$, $\cosh(N+1/2)\pi \geq \cosh\frac{\pi}{2} = \frac{1}{2}(e^{\pi/2} + e^{-\pi/2}) \geq \frac{3}{2}$
$\Rightarrow |\sin z|^2 = |\cosh^2 y - \cos^2 x| \geq |\cosh^2 y| - |\cos x|^2 \geq \frac{5}{4}$

$z = x + i(N+\frac{1}{2})\pi$ $\qquad$ $|\cos x| \leq 1$

$\Rightarrow |\sin z| \geq \frac{\sqrt{5}}{2} > 1.$ $\qquad$ $\left(\begin{array}{l} e > 2.5,\ \pi/2 > 3/2 \\ \Rightarrow e^{\pi/2} > \sqrt{15} > 3 \end{array}\right)$

c) By b), $\sin z = \sin x \cosh y + i \cos x \sinh y \Rightarrow$
$|\sin z| = (\cosh^2 y - \cos^2 x)^{1/2}$, as $\cos x$ is a bounded function and $\cosh y = \frac{1}{2}(e^y + e^{-y})$, when $y \to \pm\infty$ clearly $\cosh y \to \infty \Rightarrow |\sin z| \to \infty$.

18) Finding $\sin^{-1}(2)$: set $w = e^{iz} = \cos z + i\underbrace{\sin z}_{=2}$ for as $\sin^2 z + \cos^2 z = 1 \Rightarrow \cos z = \pm\sqrt{3}\,i$. So
$w = i(2 \pm \sqrt{3})$. Then $e^x e^{iy} = (2 \pm \sqrt{3}) \operatorname{cis} \pi/2$ as w lies on the imaginary axis thus $x = \ln(2 \pm \sqrt{3})$ and

$y = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$.

19) solns of $e^{e^z} = 1$: To find the solution set, first we shall solve : $e^W = 1$, for W : using the definition $e^W \underset{W=x+iy}{=} e^x(\cos y + i\sin y) = 1$ we have $e^x\cos y = 1$ & $e^x \sin y = 0$. The latter implies that $y = n\pi$, $n \in \mathbb{Z}$ as $e^x \neq 0$ $\forall x$. Plugging these y values into former equation we see that $1 = \begin{cases} e^x, & \text{for } n \text{ even} \\ -e^x, & \text{for } n \text{ odd} \end{cases}$

Since $e^x > 0$ for each $x \in \mathbb{R}$, we omit the second case where n is odd. So we get $x = 0$ for n even so that $\boxed{W = 2n\pi i,\ n \in \mathbb{Z}.}$ Now setting $W = e^z$, we have

$e^z = 2n\pi i$, $n \in \mathbb{Z}$. First observe that for $n = 0$, this has no solution since $|e^z| = e^x > 0$. So then we consider two cases where $n > 0$ or $n < 0$. For the first one, $2n\pi i = e^{\ln(2n\pi)}e^{\pi i/2} \Rightarrow e^z = e^{\ln(2n\pi) + i(\pi/2 + 2k\pi)}$ $n = 1, 2, \ldots$; $k \in \mathbb{Z}$. For $n < 0$ write $2n\pi i = (-2n\pi)(-i) = e^{\ln(-2n\pi)} \times e^{-\pi i/2} \Rightarrow e^z = e^{\ln(-2n\pi) + i(-\pi/2 + 2k\pi)}$, $\ldots, -2, -1 = n$; $k \in \mathbb{Z}$.

Therefore $z = \begin{cases} \ln(2n\pi) + i(\pi/2 + 2k\pi), & n > 0, k \in \mathbb{Z} \\ \ln(-2n\pi) + i(-\pi/2 + 2k\pi), & n < 0, k \in \mathbb{Z}. \end{cases}$

Chapter 4

5) Assume $F' \equiv 0$, C - smooth curve with initial point a & terminal point b, $z : [t_0, t_1] \longrightarrow \mathbb{C}$, $z(t_0) = a$ & $z(t_1) = b$

$$0 = \int_C F'(z)\,dz \underset{\text{prop } 4.12}{=} F(z(t_1)) - F(z(t_0)) = F(b) - F(a) \Rightarrow$$

$F(b) = F(a)$

6) $f(z) \in \mathbb{R}$, $f \ll 1$ that is $|f| \leq 1$,

WTS : $\int_{|z|=1} f \ll 4$

Using hint , need to show $\int_{|z|=1} f \ll \int_0^{2\pi} |\sin t| dt$, because

$\int_0^{2\pi} |\sin t| dt = 2\int_0^{\pi} \sin t dt = -2\cos t \Big|_0^{\pi} = -2[-1-1] = 4$. Then

as $\int_{|z|=1} f(z)dz \in \mathbb{C}$, we write $\int_{|z|=1} f(z)dz \underset{*}{=} Re^{i\theta}$, $0 < R = \left|\int_{|z|=1} f(z)dz\right|$

& $\theta = \text{Arg}\left(\int_{|z|=1} f(z)dz\right)$. Taking $z = e^{it}$ (recall $|z|=1$) , $dz =$

$ie^{it}dt$ and using $*$, we have

$$R = i\int_0^{2\pi} e^{i(t-\theta)} f(e^{it})dt = \text{Re}\left[i\int_0^{2\pi} \cos(t-\theta) f(e^{it})dt - \int_0^{2\pi} \sin(t-\theta) f(e^{it})dt\right]$$

$$\Rightarrow \left|\int_{|z|=1} f(z)dz\right| = R = \int_0^{2\pi} \sin(\theta - t) f(e^{it})dt \underset{|f|\leq 1}{\leq} \int_0^{2\pi} |\sin(\theta-t)| \cdot 1 \, dt$$

$$= \int_0^{2\pi} |\sin t| dt .$$