PS V

Q 4.1) Defn: A is open iff $A \cap \partial A = \emptyset$

A is closed iff $\partial A \subset A$

$$(\Rightarrow) \quad \partial A = \emptyset \quad \begin{cases} A \cap \partial A = \emptyset \rightsquigarrow A \text{ is open} \\ \partial A = \emptyset \subset A \rightsquigarrow A \text{ is closed} \end{cases}$$

$(\Leftarrow)$ $\partial A \subset A$ & $A \cap \partial A = \emptyset$ both imply that $\partial A = \emptyset$.

Q 4.2) X m.s. WTS: $S^\circ = \emptyset$ iff $\underbrace{S^c \text{ is dense in } X}_{\Updownarrow \; \overline{S^c} = X}$.

$$S^\circ = S \setminus \partial S \;\Downarrow\; S \subseteq \partial S$$

$(\Rightarrow)$ $S \subseteq \partial S \Rightarrow \overline{S^c} = S^c \cup \partial S^c = S^c \cup \partial S \supseteq S^c \cup S = X$,

other inclusion is trivial.

$(\Leftarrow)$ $\underset{S^c \cup S}{X} = \overline{S^c} = S^c \cup \underbrace{\partial S^c}_{= \partial S} \rightsquigarrow S \subseteq \partial S$.

Q 4.3) $A = \{2, 3, 4, \dots\}$ & $B = \{2 + 1/2, 3 + 1/3, 4 + 1/4, \dots\}$

$\operatorname{dist}(A, B) = \inf\{d(a,b) \mid a \in A, b \in B\} \leq \inf\{d(n, n + 1/n) : n \in \mathbb{N} \setminus \{1\}\} = \inf\{1/n : n \in \mathbb{N} \setminus \{1\}\} = 0$

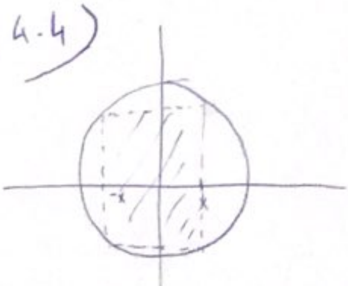
Criteria for being closed: X m.s. $S \subseteq X$. $\partial S \subseteq S \Leftrightarrow S = \overline{S}$

$\Leftrightarrow \operatorname{acc}(S) \subseteq S$.

$\operatorname{acc}(A) = \operatorname{acc}(B) = \emptyset$.

Q 4.4)

$$\mathcal{U} = \bigcup_{x \in (0,1)} \{(-x, x) \times (-\sqrt{1-x^2}, \sqrt{1-x^2})\}$$



4.8) $S = \{x \in \mathbb{Q} \mid x \in [0,1]\}$ WTS : $S \subset \mathbb{Q}$ is closed, with its usual metric. I.e. $\partial S \subset S$?

$S^c = ((-\infty, 0) \cup (1, \infty)) \cap \mathbb{Q}$, $\partial S = \{a \in \mathbb{Q} : dist(a,S) = dist(a,S^c) = 0\}$. If $a \in S$ then $dist(a, S^c) = 0$ for $a = 0$ or $a = 1$.

and If $a \in S^c$ then $dist(a, S) = 0$ never holds. Indeed such $a \in \partial S^c$ but $\partial S^c \not\subset S^c$. Thus $\partial S = \{0, 1\} \subset S$.

4.9) Defn: A metric space (X, d) is called a discrete m.s. $\Leftrightarrow$ all its subsets are open (and therefore closed) in X .

WTS : the points of a discrete m.s. X are all isolated.

Assume otherwise, i.e., assume that there is $x \in X$ which is not isolated. Hence $dist(x, X \setminus \{x\}) = 0$

Also as $x \in \{x\} = (X \setminus \{x\})^c$, $dist(x, (X \setminus \{x\})^c) = 0$.

These require that $x \in \partial(X \setminus \{x\})$. Since X is a discrete m.s. , $X \setminus \{x\} \subset X$ is closed so that we get $\partial(X \setminus \{x\}) \subseteq X \setminus \{x\}$, $\therefore$ $x \in X \setminus \{x\}$, a contradiction.

4.14) Suppose $\bigcap \mathcal{F}$ contains two distinct elt. a & b. Then $a, b \in A$ $\forall A \in \mathcal{F}$ s.t. $d(a,b) = t > 0$. Thus $diam(A) \geq t$ $\forall A \in \mathcal{F}$, a contradiction to the assumption that $\inf\{diam(A) \mid A \in \mathcal{F}\} = 0$.

4.15) Want to find ~~X with~~ countble m.s. X which is not a discrete m.s. i.e. $iso(X) \neq X$.

Take $\mathbb{Q}$ with euc. metric.

Q.4.16)

remark: A is closed $\Leftrightarrow acc(A) \subseteq A$.

$\uparrow$ defn

$\partial A \subseteq A$

Pf) $(\Rightarrow)$ Suppose $\partial A \subseteq A$. Let $x \in acc(A)$ but $x \notin A$ then $A \setminus \{x\} = A$. $x \in acc(A) \Rightarrow dist(x, A \setminus \{x\}) = 0 \Rightarrow dist(x, A) = 0$ If $dist(x, A^c) = 0$ then $x \in \partial A \subseteq A$. So $dist(x, A^c) \neq 0$ but this also implies that $x \notin A^c \leftrightarrow x \in A$. So must have: $x \in A$ for a contradiction i.e. $acc(A) \subseteq A$.

$(\Leftarrow)$ Suppose $acc(A) \subseteq A$. WTS: $\partial A \subseteq A$. so let $x \in \partial A$ s.t. $x \notin A$. As $x \in \partial A$, $dist(x, A) = dist(x, A^c) = 0$. Since $A = A \setminus \{x\}$ $dist(x, A \setminus \{x\}) = 0 \leadsto x \in acc(A) \subseteq A$, a contradiction. Thus $x \in A \leadsto \partial A \subseteq A$.

We use this remark in the following: Let X m.s.

WTS: Sets A with $A = acc(A)$ are closed and having no isolated pts, $iso(A) = \phi$. $\rightarrow$ perfect sets criterion

$(\Rightarrow)$ $A = acc(A)$. By the remark A is closed. Also $iso(A) = A \setminus acc(A) = \phi$, by question 2.5 (Chapter 2).

$(\Leftarrow)$ Assume A is a closed set which has no isolated pts. Again by remark, $acc(A) \subseteq A$ as A is closed. Since no $a \in A$ satisfies $a \in iso(A)$, it follows that $a \in acc(A)$ $\forall a \in A$. Thus $A \subseteq acc(A)$. $\therefore$ $A = acc(A) \leftrightarrow A$ is perfect.

Q 4.18) X m.s. $\mathcal{C}$:= collection of all dense subsets of X. WTS: $\bigcap \mathcal{C} = iso(X)$.

$(\supseteq)$: Let a $iso(X)$, want to show $a \in \bigcap \mathcal{C}$. Suppose the converse, then $\exists A \in \mathcal{C}$ s.t. $a \notin A$. We want to obtain a contradiction by showing that $\bar{A} \neq X$. Therefore as $a \notin A$ & $a \in iso(X)$, $dist(a, A) \neq 0$. This implies that $\bar{A} \neq X$ because $a \notin A \Rightarrow dist(a, X \setminus \{a\}) \leq dist(a, A \setminus \{a\}) = dist(a, A) \underset{\uparrow}{=} 0$. ~~...~~

when $\bar{A} = X$, $a \in \partial A$.

$\therefore$ ~~...~~

($\subseteq$) Let $a \in \bigcap \mathcal{C}$ need to show that $a \in iso(X)$.
If $a \in acc(X)$ then $X \setminus \{a\}$ is a dense subset of X.
Hence $X \setminus \{a\} \in \mathcal{C}$ $\Rightarrow$ $a \notin \bigcap \mathcal{C}$. Thus $a \in iso(X)$.