Let f have an isolated singularity at z_0. Show that if $\lim_{z \to z_0} |f(z)| = \infty$, then the singularity cannot be essential.

0.3 Recall Casorati-Weierstrass Theorem which states that for any deleted neighbourhood D of z_0, the image $f(D)$ is dense in $\mathbb{C}$.

0.3 If $\lim_{z \to z_0} |f(z)| = \infty$, then there is $\varepsilon > 0$ such that for every $z \in D(z_0, \varepsilon) \setminus \{z_0\}$ $|f(z)| > 1$.

0.4 But then $f(D(z_0,\varepsilon) \setminus \{z_0\}) \subseteq \mathbb{C} \setminus D(0,1)$, hence is not dense.

② Let $f(z) = \dfrac{1}{z^5 + 2z^3 + z}$. Calculate $\mathrm{Res}(f, i)$.

0.1 First note that $z^5 + 2z^3 + z = z(z+i)^2 (z-i)^2$. So $z = i$ is a pole of f of order 2.

0.5 Then $\mathrm{Res}(f, i) = \left((z-i)^2 f(z)\right)'(i) = \left(\dfrac{1}{z(z+i)^2}\right)'(i)$

0.4

$$= \left(\left(z(z+i)^2\right)^{-1}\right)'(i)$$

$$= \left[-1 \left(z(z+i)^2\right)^{-2} \left((z+i)^2 + 2z(z+i)\right)\right](i)$$

$$= -1 \left(i(2i)^2\right)^{-2} \left((2i)^2 + 2i(2i)\right)$$

$$= -1 \, \frac{1}{-16} \cdot -8 = \frac{1}{2}.$$

$$= \frac{2i-4}{16}.$$