Math 48X PS 1

Ümit Işlak

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Problem 0.1 *A box contains 30 red balls, 30 white balls and 30 blue balls. If 10 balls are selected at random, without replacement, what is the probability that at least one color will be missing from the selection?*

Solution: Let E_r, E_w, E_b be the events that a red, a white, a blue ball is missing in the selection, respectively. By using the inclusion-exclusion principle with $n = 3$, and the underlying symmetry we have

$$\begin{aligned}\mathbb{P}(E_r \cup E_w \cup E_b) &= 3\mathbb{P}(E_r) - 3\mathbb{P}(E_r \cap E_w) + \mathbb{P}(E_r \cap E_w \cap E_b)\\ &= 3\frac{\binom{60}{10}}{\binom{90}{10}} - 3\frac{\binom{30}{10}}{\binom{90}{10}}.\end{aligned}$$

$\square$

Problem 0.2 *Suppose that A, B, and C are three events such that A and B are disjoint, A and C are independent, and B and C are independent. Suppose also that $4\mathbb{P}(A) = 2\mathbb{P}(B) = \mathbb{P}(C) > 0$ and $\mathbb{P}(A \cup B \cup C) = 5\mathbb{P}(A)$. Determine the value of $\mathbb{P}(A)$.*

Solution: Let $p = \mathbb{P}(A)$. Then we have

$$\begin{aligned}p = \mathbb{P}(A) &= \frac{\mathbb{P}(A \cup B \cup C)}{5}\\ &= \frac{\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - (\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)) + \mathbb{P}(A \cap B \cap C)}{5}\\ &= \frac{p + 2p + 4p - 4p^2 - 8p^2}{5}.\end{aligned}$$

Solving this equation, we obtain $\mathbb{P}(A) = p = 1/6$. $\square$

Problem 0.3 *An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn c additional balls of the same color are put in with it. Now suppose that we draw another ball. Find the probability that the first ball drawn was black given that the second ball drawn was red.*

Solution: Let B be the event that the first chosen ball is black and R be the event that the second chosen ball is red. We should find $P(B|R)$. We have

$$P(B|R) = \frac{\mathbb{P}(R|B)\mathbb{P}(B)}{\mathbb{P}(R)} = \frac{\frac{b}{b+r}\frac{r}{b+r+c}}{\frac{b}{b+r}\frac{r}{b+r+c} + \frac{r}{b+r}\frac{r+c}{b+r+c}} = \cdots = \frac{b}{b+r+c},$$

where $\cdots$ is some elementary manipulations. □

Problem 0.4 *A fair coin is continually flipped. What is the probability that the pattern T, H, H, H occurs before the pattern H, H, H, H? (Hint : How can the pattern H, H, H, H occur first?)*

Solution: The only way H, H, H, H can occur first is when the first four trials are each heads. So the required probability is $1-(1/2)^4 = 15/16$. □

Problem 0.5 *Let $A_1, A_2, \ldots, A_n$ be independent events. Show that probability that none of the $A_1, A_2, \ldots, A_n$ occur is less than or equal to* $\exp(-\sum_{i=1}^n \mathbb{P}(A_i))$.

Solution: First note that

$$\mathbb{P}(A_1^c \cap \cdots \cap A_n^c) = \Pi_{i=1}^n \mathbb{P}(A_i^c) = \Pi_{i=1}^n (1-\mathbb{P}(A_i)).$$

Next, using the inequality $e^x \geq 1+x$, we get

$$\mathbb{P}(A_1^c \cap \cdots \cap A_n^c) == \Pi_{i=1}^n (1-\mathbb{P}(A_i)) \leq \Pi_{i=1}^n e^{-\mathbb{P}(A_i)} = \exp(-\sum_{i=1}^n \mathbb{P}(A_i)),$$

concluding the proof. □

Problem 0.6 *You ask your neighbor to water a sickly plant while you are on vacation. Without water it will die with probability* .8*; with water it will die with probability* .15*. You are 90 percent certain that your neighbor will remember to water the plant.*

a. What is the probability that the plant will be alive when you return?

b. If it is dead, what is the probability your neighbor forgot to water it?

Solution: Let D be the event that the plant is dead and W be the event that your neighbor waters the plant.

(a.) The question is asking for $\mathbb{P}(D^c)$. Using law of total probability, we have

$$\mathbb{P}(D^c) = \mathbb{P}(D^c|W^c)\mathbb{P}(W^c) + \mathbb{P}(D^c|W)\mathbb{P}(W) = (0.2)(0.1) + (0.85)(0.9).$$

(b.) This time we would like to find $\mathbb{P}(W^c|D)$. Using Bayes' theorem

$$\mathbb{P}(W^c|D) = \frac{\mathbb{P}(D|W^c)\mathbb{P}(W^c)}{\mathbb{P}(D|W^c)\mathbb{P}(W^c) + \mathbb{P}(D|W)\mathbb{P}(W)}.$$

Probabilities in last expression can now be found easily by following the statement of problem. $\square$

Problem 0.7 *An urn contains* 10 *balls, numbered 1 through 10. If 5 balls are independently withdrawn in sequence, each time replacing the ball selected previously, find* $\mathbb{P}(X = k), k = 1, ..., 10$, *where* X *is the maximum of the* 5 *chosen numbers. Hint : First find* $\mathbb{P}(X \leq k)$.

Solution. Let $X_i, i = 1, \ldots, 5$ be the number we choose in i^{th} trial. Then

$$\mathbb{P}(X \leq k) = \mathbb{P}(\max\{X_1, X_2, X_3, X_4, X_5\} \leq k) = (\mathbb{P}(X_1 \leq k))^5 = \left(\frac{k}{10}\right)^5, \quad k = 1, 2, \ldots, 10.$$

So

$$\mathbb{P}(X = k) = \mathbb{P}(X \leq k) - \mathbb{P}(X \leq k-1) = \left(\frac{k}{10}\right)^5 - \left(\frac{k-1}{10}\right)^5.$$

$\square$

Problem 0.8 *Suppose that a school has 20 classes: 16 with 25 students in each, three with 100 students in each, and one with 300 students for a total of 1000 students.*

a. *What is the average class size?*

b. *Select a student randomly out of the 1000 students. Let the random variable X equal the size of the class to which this student belongs and define the pmf of X.*

c. *Find the mean of X. Does this answer surprise you?*

Solution. (i.) The average class size is given by

$$\frac{1000}{20} = 50.$$

(ii.) The pmf of X is

$$f(x) = \begin{cases} \frac{4}{10}, & \text{if } x = 25 \\ \frac{3}{10}, & \text{if } x = 100 \\ \frac{3}{10}, & \text{if } x = 300 \\ 0, & \text{otherwise} \end{cases}.$$

(iii.) The expected value of X is given by

$$\mathbb{E}[X] = \frac{4}{10}25 + \frac{3}{10}100 + \frac{3}{10}300 = 130.$$

Note that this expected value is much bigger than the answer we found in part i. I am not sure if this was surprising for you, but it definitely is for several other people. This phenomenon is called **size bias in sampling**. In case you are interested, check the paper "*Size bias for one and all*" by Arratia, Goldstein and Kochman available here http://arxiv.org/abs/1308.2729.

$\square$

Problem 0.9 *(Method of indicators) There are n types of coupons. Each newly obtained coupon is, independently, type i with probability p_i, $i = 1, ..., n$. Let X be the number of distinct types obtained in a collection of k coupons. Find the expected value of X. (Hint: Use method of indicators.)*

Solution. Let $X_i = 1$ if i^{th} kind of coupon is in sample, and let $X_i = 0$. Then $X = \sum_{i=1}^n X_i$. We have

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n \mathbb{P}(X_i = 1) = \sum_{i=1}^n (1 - \mathbb{P}(X_i = 0)) = \sum_{i=1}^n (1 - (1 - p_i)^k).$$

$\square$

Problem 0.10 *Suppose that there are N different types of coupons and each time one obtains a coupon it is equally likely to be any one of the N types.*

(a) Find the expected number of different types of coupons that are contained in a set of n coupons. (Hint: Use method of indicators.)

(b) Let Y_i, $i = 0, 1, \ldots, N-1$, be the number of additional coupons that need be obtained after i distinct types have been collected in order to obtain another distinct type. Find $\mathbb{P}(Y_i = k)$ for $k \geq 1$.

(c) Find the expected number of coupons one needs for obtaining a complete set of at least one of each type. (Hint: Use part b.) (Warning: Just writing the answer from Math 361 is not enough!, The proof from Math 361 is okay, but I would prefer a Math 345 proof.)

(Bonus) (5 points) Find the variance of different types of coupons that are contained in a set of n coupons.

Solution. (a) For $i = 1, \ldots, N$, let

$$X_i = \begin{cases} 1, & \text{if } i \text{ is in our sample} \\ 0, & \text{otherwise.} \end{cases}$$

Then we are looking for $\sum_{i=1}^{N} \mathbb{P}(X_i = 1) = \sum_{i=1}^{N}(1 - \mathbb{P}(X_i = 0)) = \sum_{i=1}^{N}(\frac{N-1}{N})^n = N(\frac{N-1}{N})^n$.

(b) We have

$$\mathbb{P}(Y_i = k) = \left(\frac{i}{N}\right)^{k-1} \frac{N-i}{N}, \qquad k \geq 1.$$

(c) Just note that the time required will be $1 + Y_2 + Y_3 + \cdots + Y_N$. You can complete the problem by taking expectations. $\square$