PS II

2.1. (X, d) m.s with more than one point. Find subsets A & B of X s.t. $\text{diam}(A \cup B) \geq \text{diam}(A) + \text{diam}(B)$

Take $(X, d) = (\mathbb{R}, \text{Eucl})$, $A = (0,1)$ & $B = \{2\}$ then $\text{diam}(A \cup B) = 2$, $\text{diam}(A) = 1$, $\text{diam}(B) = 0$. Here $\text{diam}(S) = \sup\{d(r,t) : r,t \in S\}$.

2.2 Remember : Thm 2.13 : X m.s. $A, B \subseteq X$ for which $A \subseteq B$. Then $\text{diam}(A) \leq \text{diam}(B)$.
What condition on (X, d) can be put to ensure that $\exists\ A, B \subseteq X$ with $A \subset B$ s.t. $\text{diam}(A) = \text{diam}(B)$.
Take $A = \{x_1, x_2\}$ and let $d(x_1, x_2) = \sup\{d(x_i, x_j) : x_i, x_j \in B\}$.

2.3 $X = I \cup J$ m.s. where $I = (0,1)$, $J = [4, 7)$.
WTS : $\text{dist}(\sup_X I, I) \neq 0$.

recall : $\text{dist}(x, A) = \inf\{d(x,a) : a \in A\}$
$\sup_X I = \inf J = 4$, as J is the upper bound for I in X.
therefore $\text{dist}(\sup_X I, I) = \text{dist}(4, I) = \inf\{d(4,a) : a \in I\}$
$= 3$

2.4 Recall : X m.s. $\overset{\neq\phi}{A}, \overset{\neq\phi}{B} \subseteq X$ for which $A \subseteq B$. Then
$\text{dist}(x, B) \leq \text{dist}(x, A) \leq \text{dist}(x, B) + \text{diam}(B)$.
want to find a m.s. X, $x \in X$, nonempty $A, B \subseteq X$ with $A \subseteq B$ s.t. $\text{dist}(x, A) > \text{dist}(x, B) + \text{diam}(B \setminus A)$

$\text{dist}(x, A) = \inf\{d(x,a) : a \in A\}$
$\text{diam}(A) = \sup\{d(r,s) : r,s \in A\}$ Take $A = [0,5]$
$B = [0,5] \cup [6,7]$, also pick $x = 7$. Then

$\mathrm{dist}(x, A) = 2$, $\mathrm{dist}(x, B) = 0$ and

$\mathrm{diam}(B \setminus A) = \mathrm{diam}([6,7]) = 1$. Thus

$$\underbrace{\mathrm{dist}(x, A)}_{=2} > \underbrace{\mathrm{diam}(B \setminus A)}_{=1} + \underbrace{\mathrm{dist}(x, B)}_{=0} .$$