# MATH 323
# RINGS, FIELDS AND GALOIS THEORY

(FALL TERM 2020-21)

# FINAL EXAM

In what follows, $K$ denotes a field.

(1) Let $E$ be an extension field of $K$. An element $\varepsilon \in E$ is called separable over $F$, if $\varepsilon$ is algebraic over $K$ with separable irreducible polynomial $m_{\varepsilon,K}(X) \in K[X]$ over $K$.
Now assume that $E/K$ is a finite extension such that $E = K(\varepsilon_1, \cdots, \varepsilon_s)$ where $\varepsilon_1, \cdots, \varepsilon_s \in E$ are all separable over $K$. Prove that, there exists $\varepsilon \in E$ such that
$$E = K(\varepsilon).$$

(2) Recall that, a finite extension $E/K$ is respectively called :
– a separable extension, if every $\varepsilon \in E$ is separable over $K$;
– a normal extension , if every irreducible polynomial $p(X) \in K[X]$ that has a zero in $E$ actually splits in $E[X]$;
– a Galois extension, if $E/K$ is both normal and separable.
Prove: For a finite extension $E/K$, T.F.A.E.:
(i) $E/K$ is Galois.
(ii) $E = \mathrm{Split}_K(p_1(X), \cdots, p_s(X))$ for some separable polynomials $p_1(X), \cdots, p_s(X) \in K[X]$ over $K$.
(iii) $E = \mathrm{Split}_K(p(X))$ of a separable polynomial $p(X) \in K[X]$ over $K$.

(3) Let $E/K$ be a finite extension. Let $F$ be an intermediate field of $E/K$; which means that, we have a chain of field extensions $K \subseteq F \subseteq E$. Prove that:
$$E/K : \text{Galois} \Rightarrow E/F : \text{Galois}.$$

(4) Let $E$ be an (not necessarily finite) extension of $K$. Introduce
$$\mathrm{Aut}(E/K) = \{\varphi : E \xrightarrow{\sim} E \mid \ \varphi(\kappa) = \kappa, \ \forall \kappa \in K\}.$$
Prove that $\mathrm{Aut}(E/K)$ is a group under composition of mappings.
In case $E/K$ is Galois, $\mathrm{Aut}(E/K)$ is denoted by $\mathrm{Gal}(E/K)$, and called the Galois group of $E/K$.

(5) Prove that $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not a Galois extension but $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ is Galois. Determine the Galois group of $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$.

(6) Let $E/K$ be a finite Galois extension. To simplify the discussion, introduce the collection of all intermediate fields of $E/K$
$$\mathrm{I}(E/K) := \{F \mid K \subseteq F \subseteq E\},$$
and the collection of all subgroups of $\mathrm{Gal}(E/K)$
$$\mathrm{S}(E/K) := \{H \mid H \leq \mathrm{Gal}(E/K)\}.$$

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Define two mappings

$$i_{E/K} : \mathrm{I}(E/K) \to \mathrm{S}(E/K)$$

and

$$j_{E/K} : \mathrm{S}(E/K) \to \mathrm{I}(E/K)$$

by the rules

$$i_{E/K}(F) = \mathrm{Gal}(E/F),\ \forall F \in \mathrm{I}(E/K)$$

and

$$j_{E/K}(H) = \{\varepsilon \in E \mid \sigma(\varepsilon) = \varepsilon,\ \forall \sigma \in H\},\ \forall H \in \mathrm{S}(E/K)$$

respectively.

(i) Prove that $i$ and $j$ are inverses of each other, and both define "inclusion-reversing" bijections.
(ii) Prove that there are finitely many intermediate fields of $E/K$.
(iii) Prove that for $F \in \mathrm{I}(E/K)$, $\mid \mathrm{Gal}(E/F) \mid = [E : F]$.
(iv) For $\varphi \in \mathrm{Gal}(E/K)$ and $F \in \mathrm{I}(E/K)$, prove that $\mathrm{Gal}(E/\varphi(F)) = \varphi \mathrm{Gal}(E/F) \varphi^{-1}$.
(v) Prove that, for $F \in \mathrm{I}(E/K)$, the extension $F/K$ is Galois iff $i_{E/K}(F) \trianglelefteq \mathrm{Gal}(E/K)$. If this is the case,

$$\mathrm{Gal}(F/K) \simeq \mathrm{Gal}(E/K)/i_{E/K}(F).$$