PS VII

Q6.10) Set $(X, d) = (\mathbb{R}^+, euc)$ take the sequence $(x_n)_n$ with $x_n = \frac{1}{n}$, $n = 1, 2, \ldots$ Then clearly $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ and $1/N \overset{\text{Archimedean prop.}}{\leq} \varepsilon$

$$d(x_n, x_{n+1}) = \left| \frac{1}{n} - \frac{1}{n+1} \right| = \frac{1}{n(n+1)} < \frac{1}{N} \leq \varepsilon$$ shows that $\inf\{ d(x_n, x_m) : m, n \in \mathbb{N}, n \neq m \} = 0$. But no subsequence of $(x_n)_n$ converges in X.

Q6.11) Suppose X is a m.s., $z \in X$ and $(x_n)_n \subseteq X$.

WTS: If (x_n) has a subseq. converging to z then $\mathrm{dist}(z, \{x_n : n \in \mathbb{N}\}) = 0$. Also we show that converse need not true.

Let (x_{n_k}) be a convergent subsequence of (x_n) s.t. $x_{n_k} \to z$ as $k \to \infty$. Then $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n_k > N$ we have $d(x_{n_k}, z) < \varepsilon$ so that $\mathrm{dist}(z, \{x_n : n \in \mathbb{N}\}) = \inf\{ d(z, x_n) : n \in \mathbb{N}\} \leqslant d(z, x_{n_k}) < \varepsilon$ gives the result as ε is arbitrary.

For the second assertion, consider $(x_n)_n$ with $x_n = n$ in $\mathbb{R}$, clearly taking $z = x_1 (= 1)$ we see that $\mathrm{dist}(z, \{x_n ; n \in \mathbb{N}\}) = 0$ but no subseq of $(x_n)_n$ converges in $X = \mathbb{R}$ with $d = euc$.

Q 6.13) Suppose X is a m.s. $S \subseteq X$.

WTS : S is dense in X $\Leftrightarrow$ $\forall x \in X$, $\exists$ a seq. (x_n) in S s.t. $x_n \to x$.

remember Corollary 6.6.2 : X m.s. $z \in X$ and $S \subseteq X$
TFAE (i) $z \in \overline{S}$ (ii) $\exists$ seq. $(x_n) \subset S$ s.t. $x_n \to z$

($\Rightarrow$) If $\overline{S} = X$, then $z \in X = \overline{S}$, so by Corollary $\exists$ a seq. $(x_n) \subseteq S$ s.t. $x_n \to z$.

($\Leftarrow$) If for all $x \in X$ $\exists$ a seq. $(x_n) \subseteq S$ s.t. $x_n \to x$, again by Corollary $x \in \overline{S}$. Thus $\overline{S} = X$.

Extra Questions :

Q1) Prove that a subspace of a complete m.s. X is complete iff it is closed.

Defn : (Completeness) (X, d) is a complete m.s. iff every Cauchy seq. in X converges in X.

X complete m.s., $S \subseteq X$

WTS : S is complete $\Leftrightarrow$ S is closed in X.

($\Rightarrow$) We need to show that $acc(S) \subset S$ (which implies S is closed). Let $y \in acc(S)$ i.e. $dist(y, S \setminus \{y\}) = 0$ $\rightsquigarrow$ $\forall n \in \mathbb{N}$, $\exists y_n \in S \setminus \{y\}$ s.t. $d(y, y_n) < 1/n$ which implies that $y_n \to y$. As $(y_n) \subset S$ is convergent and hence is Cauchy in a complete subset S, $y \in S$.

($\Leftarrow$) Let $(y_n) \subseteq S$ be a Cauchy sequence. We need to show that (y_n) converges in S. Yet (y_n) is Cauchy seq in X, which is complete, so it converges in X, say to $y \in X$. Hence $y \in acc(S)$. Since S is closed, $acc(S) \subset S$ so that $y \in S$. Thus S is complete.

Q2) Suppose (x_n) is a sequence in $\mathbb{R}$ which converges to $x \in \mathbb{R}$. Define a new sequence (y_n) by $y_n = \dfrac{\sum_{i=1}^{n} x_i}{n}$. Show that (y_n) converges to the same point.

<u>Notice</u>! Converse is not true!

Take, for instance, $x_n = (-1)^n$ which apparently do not converge but $y_n = \begin{cases} 0 & n \text{ even} \\ -1/n & n \text{ odd} \end{cases}$ converges to 0. I.e. convergence of y_n does not necessarily imply the convergence of x_n in general.

For the Q2), we need to show that, given $\varepsilon > 0$; $\exists N \in \mathbb{N}$ s.t. $\forall n \geqslant N$ $|y_n - x| < \varepsilon$

As $x_n \to x$, given $\varepsilon > 0$ $\exists M \in \mathbb{N}$ s.t. $\forall m > M$, we have $|x_m - x| < \varepsilon$. So choose $N > M$, $n \geqslant N$

$$|y_n - x| = \left| \frac{\sum_{i=1}^{n} x_i - nx}{n} \right| = \left| \frac{\sum_{i=1}^{n} (x_i - x)}{n} \right| \leq 1/n \sum_{i=1}^{n} |x_i - x|$$

$$= \frac{1}{n}\sum_{i=1}^{M}|x_i-x| + \underbrace{\frac{1}{n}\sum_{i=M+1}^{N}|x_i-x|}_{\substack{\leqslant \frac{n-(M+1)}{n}\varepsilon \\ \leqslant \varepsilon}}$$

Since M does not depend on the choice of N (as M is a fixed number) we can take N sufficiently large so that $\frac{1}{n}\sum_{i=1}^{M}|x_i-x| < \varepsilon$

i.e. choose $N > \max\left\{ \frac{\sum_{i=1}^{M}|x_i-x|}{\varepsilon}, M \right\}$ to conclude that $y_n \to x$.