PS IV

Q3.2) Suppose X is a m.s. & $A, B \subseteq X$ for which $\partial B \subseteq A \subseteq B$. WTS : $\partial B \subseteq \partial A$

recall : $x \in \partial B$ iff $dist(x, B) = dist(x, B^c) = 0$

Let $x \in \partial B$. As $\partial B \subseteq A$, $x \in A$. Then $dist(x, A) \overset{(1)}{=} 0$

$A \subseteq B \Rightarrow B^c \subseteq A^c$. Then $\{d(x,y) \mid y \in B^c\} \subseteq \{d(x,z) \mid z \in A^c\}$

$\Rightarrow dist(x, A^c) \leq dist(x, B^c)$. Using the assumption that $x \in \partial B$ we have $dist(x, B^c) = 0$. Hence $dist(x, A^c) \overset{(2)}{=} 0$

By (1) & (2), $x \in \partial A$.

Q3.3) X m.s. $S \subseteq X$, $a \in iso(S)$. WTS : $a \in \partial S \Leftrightarrow a \notin iso(X)$

« $a \in iso(S) \Leftrightarrow dist(a, S \setminus \{a\}) \neq 0$ & $a \in S$ »

($\Rightarrow$) Assume $a \in \partial S$. Hence $dist(a, S) = dist(a, S^c) \overset{(*)}{=} 0$.

$a \in iso(S)$ given, so $a \in S \rightsquigarrow a \notin S^c \Rightarrow S^c = S^c \setminus \{a\}$

$\Rightarrow dist(a, S^c \setminus \{a\}) = 0$ by $(*)$. Also $S^c \setminus \{a\} \subseteq X \setminus \{a\}$ yields that $dist(a, X \setminus \{a\}) \leq dist(a, S^c \setminus \{a\}) = 0 \Rightarrow$

$dist(a, X \setminus \{a\}) = 0$. This implies that $a \notin iso(X)$.

($\Leftarrow$) Suppose $a \notin iso(X)$. Thus $dist(a, X \setminus \{a\}) = 0$

also given that $a \in iso(S)$, $dist(a, S \setminus \{a\}) =: r > 0$.

WTS : $a \in \partial S$, i.e. $dist(a, S) = dist(a, S^c) = 0$.

Clearly, $a \in iso(S) \Rightarrow a \in S \Rightarrow dist(a, S) \overset{(1)}{=} 0$. So we shall show that $dist(a, S^c) = 0$. Let $\varepsilon < r/2$. By the defn of $dist(\cdot, *) = \inf_{\circ \in *} \{d(\cdot, \circ)\}$ $\exists y \in X \setminus \{a\}$

s.t. $d(y, a) \leq \underbrace{dist(a, X \setminus \{a\})}_{=0} + \varepsilon$. Notice that if $y \in S \setminus \{a\}$

$\subseteq X \setminus \{a\}$ then $d(y, a) \leq \varepsilon \leq r/2 < r = dist(a, S \setminus \{a\})$ a contradiction, so y must lie in $(X \setminus \{a\}) \setminus (S \setminus \{a\}) = X \setminus S$

$= S^c$. Hence as ε can be taken arbitrary, we deduce that $\inf_{y \in S^c} \{d(y,a)\} = 0 \iff \operatorname{dist}(a, S^c) \overset{(2)}{=} 0$. By (1) & (2), $a \in \partial S$.

Q3.5) Recall: $\mathcal{C}$ is a non-empty finite collection of subsets of a m.s. X. Then

(i) $\partial(\bigcup \mathcal{C}) \subseteq \bigcup \{\partial A \mid A \in \mathcal{C}\}$ & (ii) $\partial(\bigcap \mathcal{C}) \subseteq \bigcup \{\partial A \mid A \in \mathcal{C}\}$

With reference to these we show that: $\bigcap_{j=1}^{n} \partial F_j \not\subset \partial \bigcap_{j=1}^{n} F_j$ or $\bigcap_{j=1}^{n} \partial F_j \not\subset \partial \bigcup_{j=1}^{n} F_j$ where $F_j \subset X$ $j = 1, 2, \dots, n$.

For instance let $F_1 = (0,1)$ & $F_2 = (1,2)$ with $X = \mathbb{R}$ equiped with Euclidean metric. Thus $\partial F_1 = \{0,1\}$ & $\partial F_2 = \{1,2\}$ in which case $\underbrace{\bigcap_{i=1}^{2} \partial F_i}_{= \{1\}} \not\subset \overset{\emptyset}{\overset{\shortparallel}{\partial \bigcap_{i=1}^{2} F_i}}$. Moreover, for $F_1 = (0,1]$

& $F_2 = (1,2) \rightsquigarrow \underbrace{\bigcap_{i=1}^{2} \partial F_i}_{= \{1\}} \not\subset \underbrace{\partial \bigcup_{i=1}^{2} F_i}_{= \{0, 2\}}$

Q3.6) Consider $A = \{1/n : n \in \mathbb{N}\}$. Then $\partial_{\mathbb{R}} A = A \cup \{0\}$.

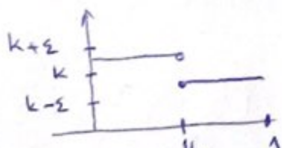
Q.3.7) $\tilde{\mathcal{F}}$: set of fncs from $[0,1]$ to $[0,1]$ with metric

$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$

$\mathcal{C}$: collection of constant functions in $\tilde{\mathcal{F}}$. WTS: $\partial \mathcal{C} = \mathcal{C}$.

First, $\mathcal{C} \subset \partial \mathcal{C}$: Let $k \in \mathcal{C}$, we will show i) $\operatorname{dist}(k, \mathcal{C}) = 0$ which immediately follows as $k \in \mathcal{C}$ & ii) $\operatorname{dist}(k, \mathcal{C}^c) = 0$

This also follows because for any $\varepsilon > 0$ we can find $f \in \mathcal{C}^c$ s.t. $d(k, f) < \varepsilon$: for ex take $f(x) = \begin{cases} k + \varepsilon/2, & x \in [0, 1/2) \\ k - \varepsilon/2 & \text{o.w} \end{cases}$



. To show $\partial \mathcal{C} \subset \mathcal{C}$, let $f \in \partial \mathcal{C}$.

Then $\operatorname{dist}(f, \mathcal{C}) = \operatorname{dist}(f, \mathcal{C}^c) = 0$

WTS: $f \in \mathcal{C}$. So suppose $f \notin \mathcal{C}$, then $\exists\, x, y \in [0,1]$ s.t. $f(x) \neq f(y)$. Let $|f(x) - f(y)| = c$. Since

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$dist(f, \mathcal{C}) = 0$, given $\varepsilon > 0$ $\exists\, k \in \mathcal{C}$ s.t. $d(f,k) < \varepsilon$. Choose $\varepsilon = c/2$ and suppose $d(f,k) < \varepsilon$ for some $k \in \mathcal{C}$. By definition of d,

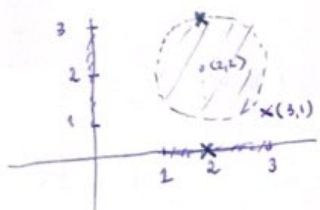
$$\frac{c}{2} = \frac{|f(x) - f(y)|}{2} \leq \frac{|f(x) - k|}{2} + \frac{|k - f(y)|}{2} \overset{\text{defn of } d}{\leq} d(f,k) < \varepsilon = \frac{c}{2}$$

$\rightsquigarrow c < c$, a contradiction. Thus $f \in \mathcal{C}$.

Q.3.11) $i \in \mathbb{N}_n$, (X_i, τ_i) non-empty m.s., d is a conserving metric on $P = \prod_{i=1}^{n} X_i$. $S \subseteq P$, we explore the relationship between $\partial_P S$ & $\partial_{X_i} \pi_i(S)$, $i \in \mathbb{N}_n$ through an example:

Take $X_j = \mathbb{R}$, $\tau_j = d_{euc}$, $j = 1,2$. I.e., $P = \mathbb{R}^2$ and let

$S = \{ (x,y) : (x-2)^2 + (y-2)^2 < 1 \quad x, y \in \mathbb{R} \}$



$\partial_P S = \{ (x,y) \in \mathbb{R}^2 : (x-2)^2 + (y-2)^2 = 1 \}$

take $(2,3) \in \partial_P S$, $\pi_1(S) = (1,3) \rightsquigarrow \partial_{X_1} \pi_1(S) = \{1,3\}$

yet $\pi_1((2,3)) = 2 \notin \partial_{X_1} \pi_1(S)$.

Similarly, $\pi_2(S) = (1,3) \rightarrow \partial_{X_2} \pi_2(S) = \{1,3\}$

and $3 \in \partial_{X_1} \pi_1(S)$ & $1 \in \partial_{X_2} \pi_2(S)$ yet $(3,1) \notin \partial_P(S)$ since $dist((3,1), S) > 0$. So no inclusion conclusion can be drawn.

Note: Since $d(a,b) = \left((a_1 - b_1)^2 + (a_2 - b_2)^2 \right)^{1/2}$ is a conserving metric

$\because \max\{ |x_i - y_i| : i \in \mathbb{N}_2 \} = |x_j - y_j| = \sqrt{|x_j - y_j|^2} \leq \left((x_1 - y_1)^2 + (x_2 - y_2)^2 \right)^{1/2}$

$\leq |x_1 - y_1| + |x_2 - y_2|$, done.

$|a|^2 + |b|^2 \leq (|a| + |b|)^2$

Q 3.12) WTS: every countable subset S of $\mathbb{R} \sim (\mathbb{R}, euc)$ has $\text{int}_{\mathbb{R}} S = \phi$

$\therefore S \subseteq \partial_{\mathbb{R}} S$.

defn: $\text{int}(S) = S \setminus \partial S$

The statement is true for Euclidean metric: $(\mathbb{R}$, discrete metric$)$

$\rightsquigarrow \text{int}(\mathbb{Q}) = \mathbb{Q} \setminus \underbrace{\partial \mathbb{Q}}_{= \phi} = \mathbb{Q} \neq \phi$ as there is no

$q \in \mathbb{R}$ with $dist(q, \mathbb{Q}) = dist(q, \mathbb{Q}^c) = 0$ in the discrete metric (no $q \in \mathbb{Q}$ & $q \in \mathbb{Q}^c$ at the same time).

To see the statement is valid for the Euclidean metric case, let $S \subset \mathbb{R}$ be countable and assume $int(S) \neq \phi$. Then $\exists\, a \in S$ with $a \notin \partial S \Rightarrow dist(a, S^c) \neq 0$. Thus write $dist(a, S^c) = c > 0$. This gives that $(a - c/2, a + c/2) \cap S^c = \phi$ $\Rightarrow (a - c/2, a + c/2) \subset S$. Since the interval $(a - c/2, a + c/2)$ contains an uncountably many elements for $c > 0$, this contradicts the fact that S is countable.

3.15) X m.s. $A \subset X$. Is $\partial A = \partial \bar{A}$ always?

$\bar{A} = A \cup \partial A$. NonExample : $(\mathbb{R}, euc)$, $A = \mathbb{Q}$, $\partial \mathbb{Q} = \mathbb{R} \neq \phi$ $= \partial \mathbb{R} = \partial \bar{\mathbb{Q}}$.

3.16) X m.s. $S \subseteq X$. WTS : $diam(S^o)$ need not be the same as $diam(S)$.

$diam(S) = \sup\{d(s_1, s_2) : s_1, s_2 \in S\}$. Take $S = [1,2] \cup \{5\}$ & $X = \mathbb{R}$ then $S^o = (1,2) \rightsquigarrow diam\, S^o = 1$ but $diam\, S = d(1,5)$ $= 4$.

3.17) X m.s. $S \subseteq X$ WTS : $\bar{S} = acc(S) \cup iso(S)$

$(\supseteq)$: Let $x \in iso(S)$ then $x \in S \subset \bar{S} = S \cup \partial S$, and when $x \in acc(S)$, two cases possible : if $x \in S$ then done, otherwise, i.e., if $x \notin S$, that is $x \in S^c$ we have $dist(x, S^c) = 0$. Also since $x \in acc(S)$, $dist(x, \underset{S \setminus \{x\}}{\underset{\|}{S}}) = 0$.

So $x \in \partial S \subseteq \bar{X}$.

$(\subseteq)$: when $x \in S$ $<$ $dist(x, S \setminus \{x\}) = 0 \Rightarrow x \in acc(S)$; $dist(x, S \setminus \{x\}) \neq 0 \Rightarrow x \in iso(S)$

when $x \in \partial S \setminus S$, then $dist(x, S \setminus \{x\}) = 0 \Rightarrow x \in acc(S) \Rightarrow$ $x \in RHS$.

3.18) (X, d) m.s. $A, B \subseteq X$. WTS: $\mathrm{dist}(\bar{A}, \bar{B}) = \mathrm{dist}(A, B)$

i.e. ~~wts~~ $\inf\{d(a,b) : a \in A, b \in B\} = \inf\{d(\bar{a}, \bar{b}) : \bar{a} \in \bar{A}, \bar{b} \in \bar{B}\}$

Since $\bar{A} \supseteq A$ & $\bar{B} \supseteq B$, $\mathrm{dist}(\bar{A}, \bar{B}) \leq \mathrm{dist}(A, B)$ follows at once. (Since we take inf over a larger set). For the $\geq$ part it suffices to show the following:

Given $\varepsilon > 0$ $\exists\, a \in A, b \in B$ s.t. $d(a,b) < \mathrm{dist}(\bar{A}, \bar{B}) + \varepsilon$ (taking inf of things $\mathrm{dist}(A,B) \leq \mathrm{dist}(\bar{A}, \bar{B})$ follows when $\varepsilon \to 0$)

Clearly, for $\varepsilon_1 > 0$ $\exists\, \bar{a} \in \bar{A}, \bar{b} \in \bar{B}$ s.t. $d(\bar{a}, \bar{b}) < \mathrm{dist}(\bar{A}, \bar{B}) + \varepsilon_1$

& for $\varepsilon_2 > 0$ $\exists\, a \in A, b \in B$ s.t. $d(a, \bar{a}) < \varepsilon_2$ & $d(b, \bar{b}) < \varepsilon_2$

Let $\varepsilon_1 = \varepsilon/2$ & $\varepsilon_2 = \varepsilon/4$ to get

$$d(a,b) \leq d(a, \bar{a}) + d(\bar{a}, \bar{b}) + d(\bar{b}, b) < \mathrm{dist}(\bar{A}, \bar{B}) + \varepsilon.$$