Exercises

1. Find the generating function for the finite sequence 2, 2, 2, 2, 2, 2.

2. Find the generating function for the finite sequence 1, 4, 16, 64, 256.

In Exercises 3–8, by a **closed form** we mean an algebraic expression not involving a summation over a range of values or the use of ellipses.

3. Find a closed form for the generating function for each of these sequences. (For each sequence, use the most obvious choice of a sequence that follows the pattern of the initial terms listed.)

a) 0, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, ...
b) 0, 0, 0, 1, 1, 1, 1, 1, 1, ...
c) 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, ...
d) 2, 4, 8, 16, 32, 64, 128, 256, ...
e) $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \ldots, \binom{7}{7}, 0, 0, 0, 0, 0, \ldots$
f) 2, −2, 2, −2, 2, −2, 2, −2, ...
g) 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, ...
h) 0, 0, 0, 1, 2, 3, 4, ...

4. Find a closed form for the generating function for each of these sequences. (Assume a general form for the terms of the sequence, using the most obvious choice of such a sequence.)

a) −1, −1, −1, −1, −1, −1, −1, 0, 0, 0, 0, 0, 0, ...
b) 1, 3, 9, 27, 81, 243, 729, ...
c) 0, 0, 3, −3, 3, −3, 3, −3, ...
d) 1, 2, 1, 1, 1, 1, 1, 1, 1, ...
e) $\binom{7}{0}, 2\binom{7}{1}, 2^2\binom{7}{2}, \ldots, 2^7\binom{7}{7}, 0, 0, 0, 0, \ldots$
f) −3, 3, −3, 3, −3, 3, ...
g) 0, 1, −2, 4, −8, 16, −32, 64, ...
h) 1, 0, 1, 0, 1, 0, 1, 0, ...

5. Find a closed form for the generating function for the sequence $\{a_n\}$, where

a) $a_n = 5$ for all $n = 0, 1, 2, \ldots$.
b) $a_n = 3^n$ for all $n = 0, 1, 2, \ldots$.
c) $a_n = 2$ for $n = 3, 4, 5, \ldots$ and $a_0 = a_1 = a_2 = 0$.
d) $a_n = 2n + 3$ for all $n = 0, 1, 2, \ldots$.
e) $a_n = \binom{8}{n}$ for all $n = 0, 1, 2, \ldots$.
f) $a_n = \binom{n+4}{n}$ for all $n = 0, 1, 2, \ldots$.

6. Find a closed form for the generating function for the sequence $\{a_n\}$, where

a) $a_n = -1$ for all $n = 0, 1, 2, \ldots$.
b) $a_n = 2^n$ for $n = 1, 2, 3, 4, \ldots$ and $a_0 = 0$.
c) $a_n = n - 1$ for $n = 0, 1, 2, \ldots$.
d) $a_n = 1/(n+1)!$ for $n = 0, 1, 2, \ldots$.
e) $a_n = \binom{n}{2}$ for $n = 0, 1, 2, \ldots$.
f) $a_n = \binom{10}{n+1}$ for $n = 0, 1, 2, \ldots$.

7. For each of these generating functions, provide a closed formula for the sequence it determines.

a) $(3x - 4)^3$ **b)** $(x^3 + 1)^3$
c) $1/(1 - 5x)$ **d)** $x^3/(1 + 3x)$
e) $x^2 + 3x + 7 + (1/(1 - x^2))$
f) $(x^4/(1 - x^4)) - x^3 - x^2 - x - 1$
g) $x^2/(1 - x)^2$ **h)** $2e^{2x}$

8. For each of these generating functions, provide a closed formula for the sequence it determines.

a) $(x^2 + 1)^3$ **b)** $(3x - 1)^3$
c) $1/(1 - 2x^2)$ **d)** $x^2/(1 - x)^3$
e) $x - 1 + (1/(1 - 3x))$ **f)** $(1 + x^3)/(1 + x)^3$
***g)** $x/(1 + x + x^2)$ **h)** $e^{3x^2} - 1$

9. Find the coefficient of x^{10} in the power series of each of these functions.

a) $(1 + x^5 + x^{10} + x^{15} + \cdots)^3$
b) $(x^3 + x^4 + x^5 + x^6 + x^7 + \cdots)^3$
c) $(x^4 + x^5 + x^6)(x^3 + x^4 + x^5 + x^6 + x^7)(1 + x + x^2 + x^3 + x^4 + \cdots)$
d) $(x^2 + x^4 + x^6 + x^8 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x^4 + x^8 + x^{12} + \cdots)$
e) $(1 + x^2 + x^4 + x^6 + x^8 + \cdots)(1 + x^4 + x^8 + x^{12} + \cdots)(1 + x^6 + x^{12} + x^{18} + \cdots)$

10. Find the coefficient of x^9 in the power series of each of these functions.

a) $(1 + x^3 + x^6 + x^9 + \cdots)^3$
b) $(x^2 + x^3 + x^4 + x^5 + x^6 + \cdots)^3$
c) $(x^3 + x^5 + x^6)(x^3 + x^4)(x + x^2 + x^3 + x^4 + \cdots)$
d) $(x + x^4 + x^7 + x^{10} + \cdots)(x^2 + x^4 + x^6 + x^8 + \cdots)$
e) $(1 + x + x^2)^3$

11. Find the coefficient of x^{10} in the power series of each of these functions.

a) $1/(1 - 2x)$ **b)** $1/(1 + x)^2$
c) $1/(1 - x)^3$ **d)** $1/(1 + 2x)^4$
e) $x^4/(1 - 3x)^3$

12. Find the coefficient of x^{12} in the power series of each of these functions.

a) $1/(1 + 3x)$ **b)** $1/(1 - 2x)^2$
c) $1/(1 + x)^8$ **d)** $1/(1 - 4x)^3$
e) $x^3/(1 + 4x)^2$

13. Use generating functions to determine the number of different ways 10 identical balloons can be given to four children if each child receives at least two balloons.

14. Use generating functions to determine the number of different ways 12 identical action figures can be given to five children so that each child receives at most three action figures.

15. Use generating functions to determine the number of different ways 15 identical stuffed animals can be given to six children so that each child receives at least one but no more than three stuffed animals.

16. Use generating functions to find the number of ways to choose a dozen bagels from three varieties—egg, salty, and plain—if at least two bagels of each kind but no more than three salty bagels are chosen.

17. In how many ways can 25 identical donuts be distributed to four police officers so that each officer gets at least three but no more than seven donuts?

18. Use generating functions to find the number of ways to select 14 balls from a jar containing 100 red balls, 100 blue balls, and 100 green balls so that no fewer than 3 and no more than 10 blue balls are selected. Assume that the order in which the balls are drawn does not matter.

19. What is the generating function for the sequence $\{c_k\}$, where c_k is the number of ways to make change for k dollars using \$1 bills, \$2 bills, \$5 bills, and \$10 bills?

20. What is the generating function for the sequence $\{c_k\}$, where c_k represents the number of ways to make change for k pesos using bills worth 10 pesos, 20 pesos, 50 pesos, and 100 pesos?

21. Give a combinatorial interpretation of the coefficient of x^4 in the expansion $(1 + x + x^2 + x^3 + \cdots)^3$. Use this interpretation to find this number.

22. Give a combinatorial interpretation of the coefficient of x^6 in the expansion $(1 + x + x^2 + x^3 + \cdots)^n$. Use this interpretation to find this number.

23. a) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1 + x_2 + x_3 = k$ when x_1, x_2, and x_3 are integers with $x_1 \geq 2$, $0 \leq x_2 \leq 3$, and $2 \leq x_3 \leq 5$?

b) Use your answer to part (a) to find a_6.

24. a) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1 + x_2 + x_3 + x_4 = k$ when x_1, x_2, x_3, and x_4 are integers with $x_1 \geq 3$, $1 \leq x_2 \leq 5$, $0 \leq x_3 \leq 4$, and $x_4 \geq 1$?

b) Use your answer to part (a) to find a_7.

25. Explain how generating functions can be used to find the number of ways in which postage of r cents can be pasted on an envelope using 3-cent, 4-cent, and 20-cent stamps.

a) Assume that the order the stamps are pasted on does not matter.

b) Assume that the stamps are pasted in a row and the order in which they are pasted on matters.

c) Use your answer to part (a) to determine the number of ways 46 cents of postage can be pasted on an envelope using 3-cent, 4-cent, and 20-cent stamps when the order the stamps are pasted on does not matter. (Use of a computer algebra program is advised.)

d) Use your answer to part (b) to determine the number of ways 46 cents of postage can be pasted in a row on an envelope using 3-cent, 4-cent, and 20-cent stamps when the order in which the stamps are pasted on matters. (Use of a computer algebra program is advised.)

26. a) Show that $1/(1 - x - x^2 - x^3 - x^4 - x^5 - x^6)$ is the generating function for the number of ways that the sum n can be obtained when a die is rolled repeatedly and the order of the rolls matters.

b) Use part (a) to find the number of ways to roll a total of 8 when a die is rolled repeatedly, and the order of the rolls matters. (Use of a computer algebra package is advised.)

27. Use generating functions (and a computer algebra package, if available) to find the number of ways to make change for \$1 using

a) dimes and quarters.

b) nickels, dimes, and quarters.

c) pennies, dimes, and quarters.

d) pennies, nickels, dimes, and quarters.

28. Use generating functions (and a computer algebra package, if available) to find the number of ways to make change for \$1 using pennies, nickels, dimes, and quarters with

a) no more than 10 pennies.

b) no more than 10 pennies and no more than 10 nickels.

***c)** no more than 10 coins.

29. Use generating functions to find the number of ways to make change for \$100 using

a) \$10, \$20, and \$50 bills.

b) \$5, \$10, \$20, and \$50 bills.

c) \$5, \$10, \$20, and \$50 bills if at least one bill of each denomination is used.

d) \$5, \$10, and \$20 bills if at least one and no more than four of each denomination is used.

30. If $G(x)$ is the generating function for the sequence $\{a_k\}$, what is the generating function for each of these sequences?

a) $2a_0, 2a_1, 2a_2, 2a_3, \ldots$

b) $0, a_0, a_1, a_2, a_3, \ldots$ (assuming that terms follow the pattern of all but the first term)

c) $0, 0, 0, 0, a_2, a_3, \ldots$ (assuming that terms follow the pattern of all but the first four terms)

d) $a_2, a_3, a_4, \ldots$

e) $a_1, 2a_2, 3a_3, 4a_4, \ldots$ [*Hint: Calculus required* here.]

f) $a_0^2,\ 2a_0a_1,\ a_1^2 + 2a_0a_2,\ 2a_0a_3 + 2a_1a_2,\ 2a_0a_4 + 2a_1a_3 + a_2^2, \ldots$

31. If $G(x)$ is the generating function for the sequence $\{a_k\}$, what is the generating function for each of these sequences?

a) $0, 0, 0, a_3, a_4, a_5, \ldots$ (assuming that terms follow the pattern of all but the first three terms)

b) $a_0, 0, a_1, 0, a_2, 0, \ldots$

c) $0, 0, 0, 0, a_0, a_1, a_2, \ldots$ (assuming that terms follow the pattern of all but the first four terms)

d) $a_0, 2a_1, 4a_2, 8a_3, 16a_4, \ldots$

e) $0, a_0, a_1/2, a_2/3, a_3/4, \ldots$ [*Hint: Calculus required* here.]

f) $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \ldots$

32. Use generating functions to solve the recurrence relation $a_k = 7a_{k-1}$ with the initial condition $a_0 = 5$.

33. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.

34. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 4^{k-1}$ with the initial condition $a_0 = 1$.

35. Use generating functions to solve the recurrence relation $a_k = 5a_{k-1} - 6a_{k-2}$ with initial conditions $a_0 = 6$ and $a_1 = 30$.

36. Use generating functions to solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2} + 2^k$ with initial conditions $a_0 = 4$ and $a_1 = 12$.

37. Use generating functions to solve the recurrence relation $a_k = 4a_{k-1} - 4a_{k-2} + k^2$ with initial conditions $a_0 = 2$ and $a_1 = 5$.

38. Use generating functions to solve the recurrence relation $a_k = 2a_{k-1} + 3a_{k-2} + 4^k + 6$ with initial conditions $a_0 = 20$, $a_1 = 60$.

39. Use generating functions to find an explicit formula for the Fibonacci numbers.

***40.** **a)** Show that if n is a positive integer, then

$$\binom{-1/2}{n} = \frac{\binom{2n}{n}}{(-4)^n}.$$

b) Use the extended binomial theorem and part (a) to show that the coefficient of x^n in the expansion of $(1-4x)^{-1/2}$ is $\binom{2n}{n}$ for all nonnegative integers n.

***41.** (*Calculus required*) Let $\{C_n\}$ be the sequence of Catalan numbers, that is, the solution to the recurrence relation $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$ with $C_0 = C_1 = 1$ (see Example 5 in Section 8.1).

a) Show that if $G(x)$ is the generating function for the sequence of Catalan numbers, then $xG(x)^2 - G(x) + 1 = 0$. Conclude (using the initial conditions) that $G(x) = (1 - \sqrt{1-4x})/(2x)$.

b) Use Exercise 40 to conclude that

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n,$$

so that

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

c) Show that $C_n \geq 2^{n-1}$ for all positive integers n.

42. Use generating functions to prove Pascal's identity: $C(n,r) = C(n-1,r) + C(n-1,r-1)$ when n and r are positive integers with $r < n$. [*Hint:* Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

43. Use generating functions to prove Vandermonde's identity: $C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)C(n,k)$, whenever m, n, and r are nonnegative integers with r not exceeding either m or n. [*Hint:* Look at the coefficient of x^r in both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.]

44. This exercise shows how to use generating functions to derive a formula for the sum of the first n squares.

a) Show that $(x^2+x)/(1-x)^4$ is the generating function for the sequence $\{a_n\}$, where $a_n = 1^2 + 2^2 + \cdots + n^2$.

b) Use part (a) to find an explicit formula for the sum $1^2 + 2^2 + \cdots + n^2$.

The **exponential generating function** for the sequence $\{a_n\}$ is the series

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

For example, the exponential generating function for the sequence 1, 1, 1, ... is the function $\sum_{n=0}^{\infty} x^n/n! = e^x$. (You will find this particular series useful in these exercises.) Note that e^x is the (ordinary) generating function for the sequence $1, 1, 1/2!, 1/3!, 1/4!, \ldots$.

45. Find a closed form for the exponential generating function for the sequence $\{a_n\}$, where

a) $a_n = 2$. **b)** $a_n = (-1)^n$.
c) $a_n = 3^n$. **d)** $a_n = n+1$.
e) $a_n = 1/(n+1)$.

46. Find a closed form for the exponential generating function for the sequence $\{a_n\}$, where

a) $a_n = (-2)^n$. **b)** $a_n = -1$.
c) $a_n = n$. **d)** $a_n = n(n-1)$.
e) $a_n = 1/((n+1)(n+2))$.

47. Find the sequence with each of these functions as its exponential generating function.

a) $f(x) = e^{-x}$ **b)** $f(x) = 3x^{2x}$
c) $f(x) = e^{3x} - 3e^{2x}$ **d)** $f(x) = (1-x) + e^{-2x}$
e) $f(x) = e^{-2x} - (1/(1-x))$
f) $f(x) = e^{-3x} - (1+x) + (1/(1-2x))$
g) $f(x) = e^{x^2}$

48. Find the sequence with each of these functions as its exponential generating function.

a) $f(x) = e^{3x}$ **b)** $f(x) = 2e^{-3x+1}$
c) $f(x) = e^{4x} + e^{-4x}$ **d)** $f(x) = (1+2x) + e^{3x}$
e) $f(x) = e^x - (1/(1+x))$
f) $f(x) = xe^x$ **g)** $f(x) = e^{x^3}$

49. A coding system encodes messages using strings of octal (base 8) digits. A codeword is considered valid if and only if it contains an even number of 7s.

a) Find a linear nonhomogeneous recurrence relation for the number of valid codewords of length n. What are the initial conditions?

b) Solve this recurrence relation using Theorem 6 in Section 8.2.

c) Solve this recurrence relation using generating functions.

***50.** A coding system encodes messages using strings of base 4 digits (that is, digits from the set $\{0, 1, 2, 3\}$). A codeword is valid if and only if it contains an even number of 0s and an even number of 1s. Let a_n equal the number of valid codewords of length n. Furthermore, let b_n, c_n, and d_n equal the number of strings of base 4 digits of length n with an even number of 0s and an odd number of 1s, with an odd number of 0s and an even number of 1s, and with an odd number of 0s and an odd number of 1s, respectively.

a) Show that $d_n = 4^n - a_n - b_n - c_n$. Use this to show that $a_{n+1} = 2a_n + b_n + c_n$, $b_{n+1} = b_n - c_n + 4^n$, and $c_{n+1} = c_n - b_n + 4^n$.

b) What are a_1, b_1, c_1, and d_1?

c) Use parts (a) and (b) to find a_3, b_3, c_3, and d_3.

d) Use the recurrence relations in part (a), together with the initial conditions in part (b), to set up three equations relating the generating functions $A(x)$, $B(x)$, and $C(x)$ for the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, respectively.

e) Solve the system of equations from part (d) to get explicit formulae for $A(x)$, $B(x)$, and $C(x)$ and use these to get explicit formulae for a_n, b_n, c_n, and d_n.

Generating functions are useful in studying the number of different types of partitions of an integer n. A **partition** of a positive integer is a way to write this integer as the sum of positive integers where repetition is allowed and the order of the integers in the sum does not matter. For example, the partitions of 5 (with no restrictions) are $1+1+1+1+1$, $1+1+1+2$, $1+1+3$, $1+2+2$, $1+4$, $2+3$, and 5. Exercises 51–56 illustrate some of these uses.

51. Show that the coefficient $p(n)$ of x^n in the formal power series expansion of $1/((1-x)(1-x^2)(1-x^3)\cdots)$ equals the number of partitions of n.

52. Show that the coefficient $p_o(n)$ of x^n in the formal power series expansion of $1/((1-x)(1-x^3)(1-x^5)\cdots)$ equals the number of partitions of n into odd integers, that is, the number of ways to write n as the sum of odd positive integers, where the order does not matter and repetitions are allowed.

53. Show that the coefficient $p_d(n)$ of x^n in the formal power series expansion of $(1+x)(1+x^2)(1+x^3)\cdots$ equals the number of partitions of n into distinct parts, that is, the number of ways to write n as the sum of positive integers, where the order does not matter but no repetitions are allowed.

54. Find $p_o(n)$, the number of partitions of n into odd parts with repetitions allowed, and $p_d(n)$, the number of partitions of n into distinct parts, for $1 \le n \le 8$, by writing each partition of each type for each integer.

55. Show that if n is a positive integer, then the number of partitions of n into distinct parts equals the number of partitions of n into odd parts with repetitions allowed; that is, $p_o(n) = p_d(n)$. [*Hint:* Show that the generating functions for $p_o(n)$ and $p_d(n)$ are equal.]

****56.** (*Requires calculus*) Use the generating function of $p(n)$ to show that $p(n) \le e^{C\sqrt{n}}$ for some constant C. [Hardy and Ramanujan showed that $p(n) \sim e^{\pi\sqrt{2/3}\sqrt{n}}/(4\sqrt{3}n)$, which means that the ratio of $p(n)$ and the right-hand side approaches 1 as n approaches infinity.]

Suppose that X is a random variable on a sample space S such that $X(s)$ is a nonnegative integer for all $s \in S$. The **probability generating function** for X is

$$G_X(x) = \sum_{k=0}^{\infty} p(X(s) = k)x^k.$$

57. (*Requires calculus*) Show that if G_X is the probability generating function for a random variable X such that $X(s)$ is a nonnegative integer for all $s \in S$, then

a) $G_X(1) = 1$. **b)** $E(X) = G'_X(1)$.

c) $V(X) = G''_X(1) + G'_X(1) - G'_X(1)^2$.

58. Let X be the random variable whose value is n if the first success occurs on the nth trial when independent Bernoulli trials are performed, each with probability of success p.

a) Find a closed formula for the probability generating function G_X.

b) Find the expected value and the variance of X using Exercise 57 and the closed form for the probability generating function found in part (a).

59. Let m be a positive integer. Let X_m be the random variable whose value is n if the mth success occurs on the $(n+m)$th trial when independent Bernoulli trials are performed, each with probability of success p.

a) Using Exercise 32 in the Supplementary Exercises of Chapter 7, show that the probability generating function G_{X_m} is given by $G_{X_m}(x) = p^m/(1-qx)^m$, where $q = 1-p$.

b) Find the expected value and the variance of X_m using Exercise 57 and the closed form for the probability generating function in part (a).

60. Show that if X and Y are independent random variables on a sample space S such that $X(s)$ and $Y(s)$ are nonnegative integers for all $s \in S$, then $G_{X+Y}(x) = G_X(x)G_Y(x)$.

8.5 Inclusion–Exclusion

Introduction

A discrete mathematics class contains 30 women and 50 sophomores. How many students in the class are either women or sophomores? This question cannot be answered unless more information is provided. Adding the number of women in the class and the number of sophomores probably does not give the correct answer, because women sophomores are counted twice. This observation shows that the number of students in the class that are either sophomores or women is the sum of the number of women and the number of sophomores in the class minus the number of women sophomores. A technique for solving such counting problems was introduced in

Therefore, each element in the union is counted exactly once by the expression on the right-hand side of the equation. This proves the principle of inclusion–exclusion. ◁

The inclusion–exclusion principle gives a formula for the number of elements in the union of n sets for every positive integer n. There are terms in this formula for the number of elements in the intersection of every nonempty subset of the collection of the n sets. Hence, there are $2^n - 1$ terms in this formula.

EXAMPLE 5 Give a formula for the number of elements in the union of four sets.

Extra Examples

Solution: The inclusion–exclusion principle shows that

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| \\ & - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| \\ & + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|.\end{aligned}$$

Note that this formula contains 15 different terms, one for each nonempty subset of $\{A_1, A_2, A_3, A_4\}$. ◀

Exercises

1. How many elements are in $A_1 \cup A_2$ if there are 12 elements in A_1, 18 elements in A_2, and
   a) $A_1 \cap A_2 = \emptyset$?
   b) $|A_1 \cap A_2| = 1$?
   c) $|A_1 \cap A_2| = 6$?
   d) $A_1 \subseteq A_2$?
2. There are 345 students at a college who have taken a course in calculus, 212 who have taken a course in discrete mathematics, and 188 who have taken courses in both calculus and discrete mathematics. How many students have taken a course in either calculus or discrete mathematics?
3. A survey of households in the United States reveals that 96% have at least one television set, 98% have telephone service, and 95% have telephone service and at least one television set. What percentage of households in the United States have neither telephone service nor a television set?
4. A marketing report concerning personal computers states that 650,000 owners will buy a printer for their machines next year and 1,250,000 will buy at least one software package. If the report states that 1,450,000 owners will buy either a printer or at least one software package, how many will buy both a printer and at least one software package?
5. Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and if
   a) the sets are pairwise disjoint.
   b) there are 50 common elements in each pair of sets and no elements in all three sets.
   c) there are 50 common elements in each pair of sets and 25 elements in all three sets.
   d) the sets are equal.
6. Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in A_1, 1000 in A_2, and 10,000 in A_3 if
   a) $A_1 \subseteq A_2$ and $A_2 \subseteq A_3$.
   b) the sets are pairwise disjoint.
   c) there are two elements common to each pair of sets and one element in all three sets.
7. There are 2504 computer science students at a school. Of these, 1876 have taken a course in Java, 999 have taken a course in Linux, and 345 have taken a course in C. Further, 876 have taken courses in both Java and Linux, 231 have taken courses in both Linux and C, and 290 have taken courses in both Java and C. If 189 of these students have taken courses in Linux, Java, and C, how many of these 2504 students have not taken a course in any of these three programming languages?
8. In a survey of 270 college students, it is found that 64 like brussels sprouts, 94 like broccoli, 58 like cauliflower, 26 like both brussels sprouts and broccoli, 28 like both brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables?
9. How many students are enrolled in a course either in calculus, discrete mathematics, data structures, or programming languages at a school if there are 507, 292, 312, and 344 students in these courses, respectively; 14 in both calculus and data structures; 213 in both calculus and programming languages; 211 in both discrete mathematics

and data structures; 43 in both discrete mathematics and programming languages; and no student may take calculus and discrete mathematics, or data structures and programming languages, concurrently?

10. Find the number of positive integers not exceeding 100 that are not divisible by 5 or by 7.

11. Find the number of positive integers not exceeding 100 that are either odd or the square of an integer.

12. Find the number of positive integers not exceeding 1000 that are either the square or the cube of an integer.

13. How many bit strings of length eight do not contain six consecutive 0s?

***14.** How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish*, *rat* or *bird*?

15. How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123?

16. How many elements are in the union of four sets if each of the sets has 100 elements, each pair of the sets shares 50 elements, each three of the sets share 25 elements, and there are 5 elements in all four sets?

17. How many elements are in the union of four sets if the sets have 50, 60, 70, and 80 elements, respectively, each pair of the sets has 5 elements in common, each triple of the sets has 1 common element, and no element is in all four sets?

18. How many terms are there in the formula for the number of elements in the union of 10 sets given by the principle of inclusion–exclusion?

19. Write out the explicit formula given by the principle of inclusion–exclusion for the number of elements in the union of five sets.

20. How many elements are in the union of five sets if the sets contain 10,000 elements each, each pair of sets has 1000 common elements, each triple of sets has 100 common elements, every four of the sets have 10 common elements, and there is 1 element in all five sets?

21. Write out the explicit formula given by the principle of inclusion–exclusion for the number of elements in the union of six sets when it is known that no three of these sets have a common intersection.

***22.** Prove the principle of inclusion–exclusion using mathematical induction.

23. Let E_1, E_2, and E_3 be three events from a sample space S. Find a formula for the probability of $E_1 \cup E_2 \cup E_3$.

24. Find the probability that when a fair coin is flipped five times tails comes up exactly three times, the first and last flips come up tails, or the second and fourth flips come up heads.

25. Find the probability that when four numbers from 1 to 100, inclusive, are picked at random with no repetitions allowed, either all are odd, all are divisible by 3, or all are divisible by 5.

26. Find a formula for the probability of the union of four events in a sample space if no three of them can occur at the same time.

27. Find a formula for the probability of the union of five events in a sample space if no four of them can occur at the same time.

28. Find a formula for the probability of the union of n events in a sample space when no two of these events can occur at the same time.

29. Find a formula for the probability of the union of n events in a sample space.

8.6 Applications of Inclusion–Exclusion

Introduction

Many counting problems can be solved using the principle of inclusion–exclusion. For instance, we can use this principle to find the number of primes less than a positive integer. Many problems can be solved by counting the number of onto functions from one finite set to another. The inclusion–exclusion principle can be used to find the number of such functions. The famous hatcheck problem can be solved using the principle of inclusion–exclusion. This problem asks for the probability that no person is given the correct hat back by a hatcheck person who gives the hats back randomly.

An Alternative Form of Inclusion–Exclusion

There is an alternative form of the principle of inclusion–exclusion that is useful in counting problems. In particular, this form can be used to solve problems that ask for the number of elements in a set that have none of n properties $P_1, P_2, \ldots, P_n$.

Let A_i be the subset containing the elements that have property P_i. The number of elements with all the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ will be denoted by $N(P_{i_1} P_{i_2} \ldots P_{i_k})$.