PS I

Q1.1 Metric defn : d is a metric iff $\forall a,b,c \in X$, we've
(i) $d(a,b) \geq 0$ (ii) $d(a,b)=0 \Leftrightarrow a=b$ (iii) $d(a,b)=d(b,a)$
(iv) $d(a,b) \leq d(a,c)+d(c,b)$

In our case X - set & $d: X \times X \to \mathbb{R}$. If d is a metric in X, by the definition of d, the two conditions are already satisfied. Thus, assume that $\forall a,b,z \in X$
1) $d(a,b)=0 \Leftrightarrow a=b$ 2) $d(a,b) \leq d(z,a)+d(z,b)$ are satisfied.
We need to verify (i) & (iii) parts of requirements of a metric:

(iii) Take $z=b$ in 2) to get $d(a,b) \leq d(b,a) + \underbrace{d(b,b)}_{\overset{(1)}{=}0} \Rightarrow$
$d(a,b) \leq d(b,a)$. Similarly taking $z=a$ and with the roles of a & b interchanged we arrive at
$d(b,a) \leq d(z,b)+d(z,a) = d(a,b) + \underbrace{d(a,a)}_{\overset{(1)}{=}0} \Rightarrow d(b,a) \leq d(a,b)$
Therefore $d(a,b)=d(b,a)$.

(i) Here take $a=b$ in 2) to have $\underbrace{d(b,b)}_{=0 \text{ by } (1)} \leq d(z,b)+d(z,b)$
$\Rightarrow 0 \leq 2d(z,b) \Rightarrow 0 \leq d(z,b)$, let $z=a$
to get $d(a,b) \geq 0$.

Q 1.2 Suppose d is a metric on a set X.
WTS : $|d(x,y)-d(z,w)| \leq d(x,z)+d(y,w) \quad \forall x,y,z,w \in X$.
$d(x,y) \leq d(x,z)+d(z,y) \leq d(x,z)+d(z,w)+d(w,y) \Rightarrow$
$d(x,y)-d(z,w) \leq d(x,z)+d(w,y)$ (1). Also
$d(z,w) \leq d(z,y)+d(y,w) \leq d(z,x)+d(x,y)+d(y,w) \Rightarrow$
$d(z,w)-d(x,y) \leq d(z,x)+d(y,w)$ (2) By (1) & (2), done.

Q 1.3 $X \neq \emptyset$ set and $d(a,a)=0 \ \forall a \in X$ & $d(a,b)=1 \ \forall a,b \in X$ with $a \neq b$. WTS : d is a metric.

$d : X \times X \to \{0,1\}$ or $d(a,b) = \begin{cases} 0 & \text{if } a=b \\ 1 & \text{o.w.} \end{cases}$ From this, clearly $d(a,b) \geq 0$ and $d(a,b) = 0 \iff a = b$. It is also immediate that $d(a,b) = d(b,a)$. Left to show the triangle inequality : $d(a,b) \leq d(a,c) + d(c,b)$:

Case 1 : $a = b$. $d(a,b) = 0$ & $d(a,c) = d(c,a) \geq 0$.

Case 2 : $a \neq b$. $1 \leq d(a,c) + d(c,b)$ either $c \neq a$ or $c \neq b$ in a worst situation since $a \neq b$, it follows, either $d(a,c) = 1$ or $d(c,b) = 1$ at least.

Q 1.6 : Suppose d & e are metrics on a set X. $g : X \times X \to \mathbb{R}^+ \cup \{0\}$ by $(x,y) \longmapsto \min\{d(x,y), e(x,y)\}$. WTS : g need not be a metric on X and find a condition under which it is a metric.

Let us construct a counter example : Set $X = \mathbb{R}^2$ and

$d(x,y) = d((x_1,x_2),(y_1,y_2)) = \left[(x_1-y_1)^2 + \frac{1}{4}(x_2-y_2)^2\right]^{1/2}$ &

$e(x,y) = e((x_1,x_2),(y_1,y_2)) = \left[\frac{1}{4}(x_1-y_1)^2 + (x_2-y_2)^2\right]^{1/2}$. (which are clearly metrics) and let $x = (1,0)$, $y = (0,1)$, $z = (0,0)$

Then $g(x,y) = \min\{d(x,y), e(x,y)\} = \frac{\sqrt{5}}{2} > \min\{\underbrace{d(x,z)}_{=1}, \underbrace{e(x,z)}_{=1/2}\}$
$+ \min\{\underbrace{d(z,y)}_{=1/2}, \underbrace{e(z,y)}_{=1}\} = \underbrace{g(x,z)}_{=1/2} + \underbrace{g(z,y)}_{=1/2} = 1$

Also, we may construct another examle using example 1.4.4 of our book : Set $Z = \{x, y, z\}$. $(\mathbb{R}, |\cdot|)$ – metric space and $f, h : Z \to \mathbb{R}$ injective functions defined by

$f(x) = 3$, $f(y) = 0$, $f(z) = 1$; $h(x) = 3$, $h(y) = 0$, $h(z) = 2$

Let $d(x,y) = |f(x) - f(y)|$, $e(x,y) = |h(x) - h(y)|$. Then

$g(x,y) = \min\{\underbrace{d(x,y)}_{=3}, \underbrace{e(x,y)}_{=3}\} = 3 > \min\{\underbrace{d(x,z)}_{=2}, \underbrace{e(x,z)}_{=1}\} + \min\{\underbrace{d(z,y)}_{=1},$
$\underbrace{e(z,y)}_{=2}\} = \underbrace{g(x,z)}_{=1} + \underbrace{g(z,y)}_{=1} = 2$. Clearly, all requirements of being a metric, except the triangle ineq., are satisfied

for g. So we must put a condition on this requirement. Thus if we have $d(a,b) \leq e(a,b)$ $\forall a,b \in X$ then it as well is satisfied:

$$g(x,y) = d(x,y) \underset{\text{d metric}}{\leq} d(x,z) + d(z,y) = g(x,z) + g(z,y).$$

Q 1.8 $\mathcal{F}(S)$ - set of all finite subsets of a set S. $\forall A,B \in \mathcal{F}(S)$, $\Delta(A,B) = (A \setminus B) \cup (B \setminus A)$. Let $d(A,B) = \text{card}(\Delta(A,B))$

Is d a metric?

(i) $d(A,B) = \text{card}(\Delta(A,B)) \geq 0$ $\forall A,B \in \mathcal{F}(S)$ ✓ clear.

(ii) $d(A,B) = 0$ iff $A = B$ because $d(A,B) = 0 \Leftrightarrow \text{card}(\Delta(A,B)) = 0 \Leftrightarrow (A \setminus B) \cup (B \setminus A) = \phi \Leftrightarrow A \setminus B = \phi$ & $B \setminus A = \phi \Leftrightarrow A \subseteq B$ & $B \subseteq A \Leftrightarrow A = B$.

(iii) $d(A,B) = d(B,A)$ ✓ clear.

(iv) $d(A,B) \leq d(A,C) + d(C,B)$

Observe that $\Delta(A,B) \subseteq \Delta(A,C) \cup \Delta(C,B)$ $\forall A,B,C \in \mathcal{F}(S)$.

because if $x \in A \setminus B$ then $x \in A$ & $x \notin B \rightarrow$ if $x \in C$ then $x \in \Delta(C,B)$ as $x \notin B$, but if $x \notin C$ then $x \in \Delta(A,C)$ thus in any case $x \in \Delta(A,C) \cup \Delta(C,B)$. The case $x \in B \setminus A$ is treated similarly. Using this observation, we proceed as

$$d(A,B) = \text{card}(\Delta(A,B)) \underset{\text{obser.}}{\leq} \text{card}(\Delta(A,C) \cup \Delta(C,B)) \leq \text{card}(\Delta(A,C))$$
$$+ \text{card}(\Delta(C,B)) = d(A,C) + d(C,B).$$

Q 1.10 $(\text{Poly}(\mathbb{R}, d))$, $p, q \in \text{Poly}(\mathbb{R})$; $p = \sum_{i=0}^{\infty} \alpha_i x^i$ & $q = \sum_{i=0}^{\infty} \beta_i x^i$, $\alpha_i, \beta_i \in \mathbb{R}$ $\leadsto$ all except a finite number $= 0$.

$$d(p,q) = \sup \{ |a_i - b_i| : i \in \mathbb{N} \cup \{0\} \}$$
$$= \max \{ |a_i - b_i| \; i \in N \} \quad \text{where } N = \{ i \in \mathbb{N} \cup \{0\} :$$

$a_i \neq 0$ or $b_i \neq 0 \}$ which explains this sup must be real.

i) $d(p,q) \geq 0$ $\forall p,q \in \text{poly}(\mathbb{R})$ ✓

ii) $d(p,q) = 0 \Leftrightarrow \sup\{ |\alpha_i - \beta_i| : i \in \mathbb{N} \cup \{0\} \} = 0 \Leftrightarrow$

$|\alpha_i - \beta_i| = 0 \quad \forall i \in \mathbb{N}\cup\{0\} \iff \alpha_i = \beta_i \quad \forall i \in \mathbb{N}\cup\{0\} \iff p = q$.

iii) $d(p,q) = d(q,p)$ ✓

iv) $|a_i - b_i| \leq |a_i - c_i| + |c_i - b_i| \leq \sup_i \{|a_i - c_i| + |b_i - c_i|\}$

$\leq \sup_i \{|a_i - c_i|\} + \sup_i \{|b_i - c_i|\} = d(a,c) + d(c,b), \forall i$.

Taking sup : $\underbrace{\sup_{i \in \mathbb{N}\cup\{0\}} |a_i - b_i|}_{= d(a,b)} \leq d(a,c) + d(c,b)$.

Q 1.17 (X,d), $k \in \mathbb{R}^+$. WTS : $v : x \mapsto \delta_z(x) + k$ is pointlike

v is pointlike iff $\forall x, y \in X \quad v(x) - v(y) \leq d(x,y) \leq v(x) + v(y)$

thus $v(x) - v(y) = \delta_z(x) + k - (\delta_z(y) + k) = d(z,x) - d(z,y) \leq$

$d(x,y) \leq d(x,z) + d(z,y) \underset{k \in \mathbb{Z}^+}{\leq} d(z,x) + d(z,y) + 2k =$

$(\delta_z(x) + k) + (\delta_z(y) + k) = v(x) + v(y)$.

Q 1.18 Suppose (X,d) is a nonempty metric sp. and $u : X \to \mathbb{R}^{\oplus}$. We call u a pointlike fnc. on X iff $u(a) - u(b) \leq d(a,b) \leq u(a) + u(b) \quad \forall a, b \in X$.

$(X,d) = (\mathbb{R}\setminus\{0\}, |\cdot|)$, $v(x) = \delta_0(x) = d(0,x)$

$v(x) - v(y) = d(x,0) - d(0,y) \leq d(x,y) \leq d(x,0) + d(0,y) =$

$v(x) + v(y)$

0 is not the min of v : If it would $v(x) = 0$ for some $x \in \mathbb{R}\setminus\{0\}$ then $d(0,x) = 0 \Rightarrow x = 0 \notin X$ so a contradiction.

Suppose min of v is achieved at some $x \in \mathbb{R}\setminus\{0\}$, say $v(x) = c > 0$ then $v(x) = |x| = c$ i.e. $x = \pm c$ but choosing $y = c/2$ gives that $v(y) = c/2 < c$, a contradiction

Note here that clearly such a choice of pointlike fnc differs from that in the previous question.

Q 1.21. (X, d) & (Y, e) are metric spaces & $\phi : X \to Y$ then ϕ is called an isometry $\Leftrightarrow$ $e(\phi(a), \phi(b)) = d(a,b)$ $\forall a, b \in X$.

WTS : Every isometry is injective.

If not, then $\exists\, x, y \in X$ with $x \neq y$ s.t. $\phi(x) = \phi(y)$. In this case, since $x \neq y$ $0 < d(x,y) \underset{\phi \text{ is iso.}}{=} e(\phi(x), \phi(y)) =$

$e(\phi(x), \phi(x)) \underset{e \text{ is a metric}}{=} 0$, a contradiction. Thus ϕ has be injective.

But isometry need not be surjective onto Y but onto $\phi(X)$ ✓

Q 1.22 $\|\phi(x)\|_Y = \|x\|_X$ $\forall x \in X$. Need ϕ be an isometry i.e. $\|x - y\|_X = \|\phi(x) - \phi(y)\|_Y$. If ϕ were linear then the statement would be true : $\|x - y\|_X = \|\phi(x-y)\|_Y = \|\phi(x) - \phi(y)\|_Y$. So ϕ not necessarily be an isom.