Fall Math 48X Homework 1, due August 28, 23:59

Topics covered: More probability, number theory basics, first connections

Problem 0.1 *Assume that each year there are* $1,000,000$ *French people who receive scam emails promising them a huge amount of money. Assume also that each of these recipient of the scam emails, independently, becomes a victim with probability* $1/500,000$.

(i) Write down the exact probability that exactly 2 Americans will be victims by scam emails next year.

(ii) Compute the expected number of such scam cases in the next 10 years.

Problem 0.2 *Let* $X_1, \ldots, X_n$ *be independent random variables with finite mean* μ *and finite variance* σ^2. *Let* $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ *be the sample mean of* $X_1, \ldots, X_n$.

i. *Find the expectation of* $\overline{X}$.

ii. *Find the variance and standard deviation of* $\overline{X}$.

iii. *Show that* $Cov(X_i - \overline{X}, \overline{X}) = 0$ *for each* $i = 1, 2, \ldots, n$.

Problem 0.3 *Let* g *be strictly increasing and nonnegative. Show that*

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}[g(|X|)]}{g(a)}, \qquad for \qquad a > 0.$$

Problem 0.4 *A sequence of random variables* $X_1, X_2, \ldots$ *is said to* ***converge to*** b ***in quadratic mean*** *if*

$$\lim_{n\to\infty} \mathbb{E}[(X_n - b)^2] = 0. \tag{1}$$

Show that (1) is satisfied if and only if

$$\lim_{n\to\infty} \mathbb{E}[X_n] = b \qquad and \qquad \lim_{n\to\infty} Var(X_n) = 0.$$

Problem 0.5 *Prove that if a sequence of random variables* $X_1, X_2, \ldots$ *converges to a constant* b *in quadratic mean, then the sequence also converges to* b *in probability.*

Problem 0.6 *Let* $X_1, X_2, \ldots$ *be a sequence of random variables, and suppose that for* $n = 1, 2, 3, \ldots,$ *the distribution of* X_n *is as follows:*

$$\mathbb{P}\left(X_n = \frac{1}{n}\right) = 1 - \frac{1}{n^2} \qquad and \qquad \mathbb{P}(X_n = n) = \frac{1}{n^2}.$$

a. Does there exist a constant c *to which the sequence converges in probability? b. Does there exist a constant* c *to which the sequence converges in quadratic mean?*

Problem 0.7 *A sequence of mean-zero random variables* $(X_n)_{n\in\mathbb{N}}$ *is called weakly stationary if there is a function* ϕ *such that*

$$\mathbb{E}[X_i X_j] = \phi(j - i) < \infty \qquad \forall i, j.$$

Suppose that for some such sequence, we have $\phi(k) \to 0$ *as* $k \to \infty$. *Show that the weak law of large numbers is valid, that is,* $\frac{X_1 + \cdots + X_n}{n}$ *converges to* 0 *in probability.*

Problem 0.8 *Read hypergeometric distribution*

Problem 0.9 *Find the Dirichlet series of*

(i) $a_n = n^\alpha$, $n = 1, 2, \ldots$

(ii) $b_n = \ln n$, $n = 1, 2, \ldots$

(iii) the Von Mangoldt function.

Problem 0.10 *(i) What is the natural asymptotic density of the even numbers in* $\mathbb{N}$*?*

(ii) What is the natural asymptotic density of prime numbers among natural numbers?

Problem 0.11 *Give an example of a subset of* $\mathbb{N}$ *which does not have an natural asymptotic density.*

Problem 0.12 *Let* $f : \mathbb{N} \to \mathbb{N}$ *be a multiplicative function. Show that the function* g *defined by*

$$g(n) = \sum_{d|n} f(d), \qquad n = 1, 2, \ldots$$

is multiplicative as well.

Problem 0.13 *Prove that for any* $n \in \mathbb{N}$

$$\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0.$$

Problem 0.14 *Let* f *be a multiplicative function so that*

$$\lim_{m\to\infty} f(p^m) = 0$$

for any prime p*. Prove then that*

$$\lim_{n\to\infty} f(n) = 0.$$

Problem 0.15 *An integer valued discrete random variable* X *is said to be* ***log-concave*** *if its pmf* f *satisfies*

$$f^2(i) \geq f(i-1)f(i+1), \qquad \text{for all } i \in \mathbb{Z}.$$

(i) Prove that a log-concave pmf is unimodal.

(ii) Prove that the binomial distribution is log-concave, and is therefore unimodal.

Problem 0.16 *Suppose that the random variable* X *with support* $\mathbb{N}$ *satisfies the relation*

$$\mathbb{P}(mn \text{ divides } X) = \mathbb{P}(m \text{ divides } X)\mathbb{P}(n \text{ divides } X)$$

whenever $gcd(m, n) = 1$*. Assuming* $\mathbb{P}(X = 1) > 0$*, prove that the function*

$$f(n) = \frac{\mathbb{P}(X = n)}{\mathbb{P}(X = 1)}$$

is multiplicative.

Problem 0.17 *(i) Let X and Y be independent and each have G_p distribution. Let H be the greatest common divisor of X and Y. Then*

$$\mathbb{P}_s(H = n) = \frac{1}{n^{2s}\zeta(2s)}.$$

(ii) Conclude from the first part that the probability of X and Y being coprime is $\frac{1}{\zeta(2s)}$.

Problem 0.18 *($\star$) Prove the following statement: Let X be a random variable from some distribution $\mathbb{P}$ on natural numbers whose (random) prime factorization is given by*

$$X = \prod_{i=1}^{\infty} p_i^{N_i}$$

where p_i is the i-th prime number.

(a) $\mathbb{P}$ has the factorization property if and only if the prime powers $\{N_i : i \geq 1\}$ are independent.

(b) $\mathbb{P}$ has the factorization property if and only if

$$\mathbb{P}(X = n) = \frac{f(n)}{n^s F(s)}$$

where $F(s)$ is the Dirichlet series of some non-negative arithmetic function, i.e. $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$.