Quiz 2 - Solutions

① Let f be an entire function such that $f(z)=f(z+\sqrt{2})=f(z+i\sqrt{5})$ for every $z\in\mathbb{C}$. Show that f is constant.

1pt. First note that $f(z+n\sqrt{2})=f(z)$ and $f(z+im\sqrt{5})$ for all $z\in\mathbb{C}$ and $m,n\in\mathbb{Z}$.

2pts. Note also that for each $z\in\mathbb{C}$, there is w in the rectangle

$$R=\{z\in\mathbb{C} : \mathrm{Re}(z)\in[0,\sqrt{2}],\ \mathrm{Im}(z)\in[0,\sqrt{5}]\}$$

such that $z=w+n\sqrt{2}+im\sqrt{5}$. In particular, $f(z)=f(w)$.
So f is determined by $f\restriction_R$.

1pt. Since R is compact, and f continuous, f is bounded on R.
Therefore f is bounded.

1pt. So f is constant by Liouville Theorem.

② Let $f(z)=\dfrac{1}{z^2-1}$ for $z\in\mathbb{C}\setminus\{-1,1\}$. Does f have an anti-derivative on $\mathbb{C}\setminus\{-1,1\}$?

1pt. Clearly, f is continuous on the open set $S=\mathbb{C}\setminus\{-1,1\}$.
So we know that f has an antiderivative on S if and only if $\int_C f=0$ for every closed curve $C\subseteq S$.

We'll show that $\int_C f\neq 0$, where C is the circle centered at

1 and has radius 1.

1pt. Write $f(z) = \frac{1}{z^2-1} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$.

So $\int_C f = \frac{1}{2}\int_C \frac{dz}{z-1} - \frac{1}{2}\int_C \frac{dz}{z+1}$; let $I_1 = \int_C \frac{dz}{z-1}$, $I_2 = \int_C \frac{dz}{z+1}$.

1pt. Note that $\frac{1}{z+1}$ is analytic on $D = D(1, \frac{3}{2})$. So $I_2 = 0$.

2pts. Let's calculate I_1:

$$\int_C \frac{dz}{z-1} = \int_0^{2\pi} \frac{1}{1+e^{i\theta}-1}\, i e^{i\theta} d\theta = i\int_0^{2\pi} d\theta = i\theta \Big|_0^{2\pi} = 2\pi i \neq 0.$$

(• Last step is done in class; could be directly referred.

• The second step is not necessary, but makes it much easier.

• Another curve can be taken.)