Math 338 PS I

1) c) $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4 = \left(e^{i\frac{2\pi}{3}}\right)^4 = e^{i\frac{8\pi}{3}} = \underbrace{e^{i2\pi}}_{=1} \cdot e^{i\frac{2\pi}{3}}$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

2) $\sqrt{-8+6i} = a+bi \Rightarrow a^2-b^2=-8 \;\&\; 2ab=6 \Rightarrow (b=3/a)$

$a^4+8a^2-9=0 \Rightarrow (a^2+9)(a^2-1)=0$

$\Rightarrow a^2=-9,\; a^2=1 \Rightarrow a=3i, \pm 1 \Rightarrow b=-i, \pm 3$

respectively. Therefore, $\sqrt{-8+6i} = \pm(1+3i)$

12) b) $z^4=-1 \Rightarrow z=\sqrt[4]{-1} = \sqrt[4]{e^{\pi i}\cdot \underbrace{e^{2\pi m i}}_{=1}} = e^{\frac{\pi i}{4}+\frac{2\pi m i}{4}}$ for

$m=0,1,2,3$, $z_0 = e^{\pi i/4} = \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$, $z_1 = e^{\frac{3\pi}{4}i} = -\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$

$z_2 = e^{\frac{5\pi i}{4}} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$, $z_3 = e^{\frac{7\pi i}{4}} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

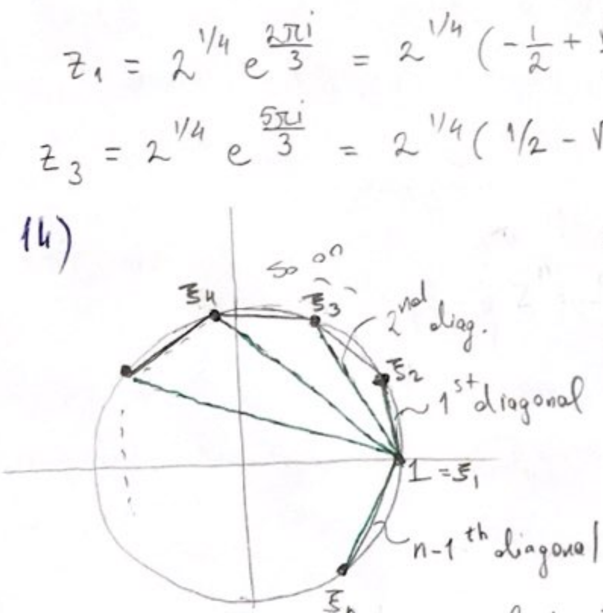
c) $z^4 = -1+\sqrt{3}i \Rightarrow z^4 = 2e^{\frac{2\pi i}{3}+2m\pi i} \Rightarrow z = 2^{1/4}e^{\frac{\pi i}{6}+\frac{2m\pi i}{4}}$

for $m=0,1,2,3$ so $z_0 = 2^{1/4}e^{\pi i/6} = 2^{1/4}\left(\frac{\sqrt{3}}{2}+i/2\right)$,

$z_1 = 2^{1/4}e^{\frac{2\pi i}{3}} = 2^{1/4}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$, $z_2 = 2^{1/4}e^{\frac{7\pi i}{6}} = 2^{1/4}\left(-\frac{\sqrt{3}}{2}-i/2\right)$

$z_3 = 2^{1/4}e^{\frac{5\pi i}{3}} = 2^{1/4}\left(1/2 - \sqrt{3}/2\, i\right)$

14)



we want to show that the product of the lengths of the diagonals is equal to n.

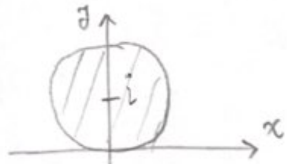
n^{th} roots of the unit are the roots of the eqn $z^n=1$ or

$z^n-1 = (z-1)\underbrace{(z^{n-1}+z^{n-2}+\dots+z+1)}_{=0}$

let $\underset{=1}{\xi_1}, \xi_2, \dots, \xi_n$ be the n^{th} roots of unity, so then $z^{n-1}+\dots+z+1 = (z-\xi_2)\dots(z-\xi_n)$. Clearly the length of j^{th} diagonal is $|\xi_j - 1|$, thus substituting

$z=1$ into above equation and taking absolute value of both sides we have $n = |1-\bar{\xi}_2||1-\bar{\xi}_3| \cdots |1-\bar{\xi}_n| = \prod_{j=2}^{n} |1-\bar{\xi}_j|$ which is the product of the length of $n-1$ - diagonals.

15) a) $|z-i| \leq 1$



Closed disk centered at i with radius 1 not a region

b) $\left|\frac{z-1}{z+1}\right| = 1 \underset{z=x+iy}{\Rightarrow} (x-1)^2 + y^2 = (x+1)^2 + y^2$
$\Rightarrow x=0$ i.e. imaginary axis which is not a region.

c) $|z-2| > |z-3| \underset{z=x+iy}{\Rightarrow} (x-2)^2 + y^2 > (x-3)^2 + y^2$
$\Rightarrow x > 5/2$ or. $\operatorname{Re} z > 5/2$ which is a region as it is open & connected.

f) $|z|^2 = \operatorname{Im} z \underset{z=x+iy}{\Rightarrow} x^2+y^2 = y \Rightarrow x^2+y^2-y+1/4 = 1/4$
$\Rightarrow x^2 + (y-1/2)^2 = 1/4$

which is the circle centered at $i/2$ with radius $1/2$, not a region.

16) b) $|z-1| + |z+1| = 4$

First note that $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$ by ex 10. (this can easily be obtained by using $|z|^2 = z\bar{z}$). Now

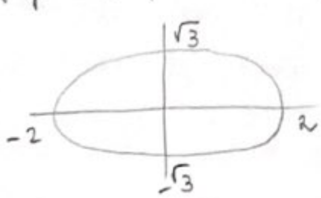
$|z-1|^2 + |z+1|^2 + 2|z-1||z+1| = 16 \underset{\text{using note}}{\Rightarrow}$

$\cancel{2}(|z|^2+1 + |z-1||z+1|) = \cancel{16}\ 8 \Rightarrow$

$|z-1||z+1| = 7-|z|^2 \Rightarrow |z^2-1| = 7-|z|^2 \underset{z=x+iy}{\overset{\text{squaring both sides}}{\Rightarrow}}$

$(x^2-y^2-1)^2 + 4x^2y^2 = (7-x^2-y^2)^2 \Rightarrow 3x^2+4y^2 = 12$ or

$\frac{x^2}{4} + \frac{y^2}{3} = 1$ which is the eqn of ellipse



c) $z^{n-1} = \bar{z}$, multiply both sides by z to obtain

$z^n = |z|^2 \Rightarrow z^n \in \mathbb{R}^+$ if we write $z = re^{i\theta}$ then

$z^n = r^n(\cos n\theta + i\sin n\theta) \in \mathbb{R}^+ \Rightarrow n\theta = 2k\pi,\ k \in \mathbb{Z}$. Thus

$\theta = \operatorname{Arg} z = \dfrac{2k\pi}{n}$, $k \in \mathbb{Z}$. As for the magnitude $|z|$, take $|\cdot|$ of both sides of the given eqn:

$|z|^{n-1} = |z| \Rightarrow |z|(|z|^{n-2} - 1) = 0 \Rightarrow |z| = 0$ or $|z| = 1$

20) a) $P(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$. By considering $(1-z)P(z)$ we will show that all the zeros of $P(z)$ are inside the unit disk, that is if $z_0 \in \mathbb{C}$ with $P(z_0) = 0$ then $|z_0| \le 1$. For a contradiction, let $z_0 \in \mathbb{C}$ be a root with $|z_0| > 1$. Then $(1-z_0)P(z_0) = 0$ as well, by a simple algebra, $(1-z_0)P(z_0) = 1 + z_0 + z_0^2 + \cdots + z_0^{n-1} - nz_0^n = 0$

$$\Rightarrow n|z_0|^n = |1 + z_0 + z_0^2 + \cdots + z_0^{n-1}| \underset{\text{triangle ineq.}}{\le} 1 + |z_0| + |z_0|^2 + \cdots + |z_0|^{n-1}$$

$\underset{|z_0|>1}{<} n|z_0|^n$ which is a contradiction!

b) Now let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $a_i \in \mathbb{R}$ with $0 \le a_0 \le a_1 \le \cdots \le a_n$. Following the same idea of the proof above, assume $z_0 \in \mathbb{C}$ with $|z_0| > 1$ s.t. $P(z_0) = 0$, then $0 = (1-z_0)P(z_0) = a_0 + (a_1 - a_0)z_0 + (a_2 - a_1)z_0^2 + \cdots + (a_n - a_{n-1})z_0^n - a_n z_0^{n+1}$

$$\Rightarrow a_n |z_0|^{n+1} = \left| a_0 + \sum_{j=1}^{n} (a_j - a_{j-1}) z_0^j \right| \underset{\text{tr. ineq}}{\le} a_0 + \sum_{j=1}^{n} (a_j - a_{j-1}) |z_0|^j \underset{|z_0|>1}{<} a_0|z_0|^n + (a_1 - a_0)|z_0|^n + \cdots + (a_n - a_{n-1})|z_0|^n$$

$$= a_n |z_0|^n \Rightarrow |z_0| < 1 \text{ a contradiction}$$

27) a)

$$\xi = \frac{x}{x^2+y^2+1}, \quad \eta = \frac{y}{x^2+y^2+1}, \quad \zeta = \frac{x^2+y^2}{x^2+y^2+1}$$

$1/z$ is the stereographic proj. of (ξ', η', ζ')

$$1/z = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$$

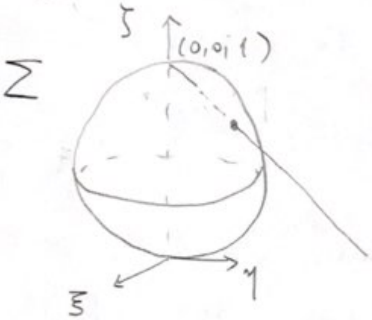
$$\xi' = \frac{\frac{x}{x^2+y^2}}{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{-y}{x^2+y^2}\right)^2 + 1} = \frac{\frac{x}{x^2+y^2}}{\frac{\cancel{x^2+y^2}}{(x^2+y^2)^{\cancel{2}}} + 1} = \frac{\frac{x}{\cancel{x^2+y^2}}}{\frac{1 + x^2+y^2}{x^2+y^2}} = \xi$$

$$\eta' = \frac{\frac{-y}{x^2+y^2}}{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{-y}{x^2+y^2}\right)^2 + 1} = \underset{\text{as above}}{\cdots} = \frac{-y}{x^2+y^2+1} = -\eta$$

$$\zeta' = \frac{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{-y}{x^2+y^2}\right)^2}{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{-y}{x^2+y^2}\right)^2 + 1} = \frac{\frac{1}{x^2+y^2}}{\frac{1}{x^2+y^2}+1} = \frac{1}{x^2+y^2+1}$$

$$= 1 - \frac{x^2+y^2}{x^2+y^2+1} = 1 - \zeta$$

28) $f(z) = 1/z$ maps circles & lines in $\mathbb{C}$ onto other circles & lines.



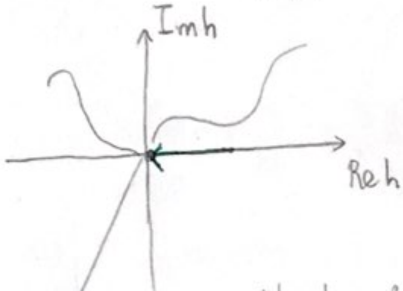
By a circle on Σ it is meant the intersection of Σ with a plane of the form $A\xi + B\eta + C\zeta \overset{*}{=} D$. By question 27, $(\xi', \eta', \zeta') = (\xi, -\eta, 1-\zeta)$ hence $\xi = \xi'$, $\eta = -\eta'$, $\zeta = 1-\zeta'$ $\Rightarrow$ $*$ is equivalent to $A\xi' - B\eta' + C(1-\zeta') = D$ $\Rightarrow$ $A\xi' - B\eta' - C\zeta' = D - C$. Here note that to understand the behaviour of the map $z \longmapsto \frac{1}{z}$ by means of mapping circles / lines onto other circles / lines, we use the associated (Riemann) spheres Σ with coordinates $(\xi, \eta, \zeta) \longleftrightarrow (\xi', \eta', \zeta')$.

Chapter 2

2) a) $f(z) \in \mathbb{R}$, $\forall z \in \mathbb{R}$. WTS: $f'(z) \in \mathbb{R}$, for $z \in \mathbb{R}$.

Consult on the definition of a derivative directly:

$f'(z) = \lim\limits_{h \to 0} \frac{f(z+h) - f(z)}{h}$, As we suppose that the derivative exists at $z \in \mathbb{R}$, the value for the derivative does not change at z regardless of which path taken for h, as $h \to 0$. So for $\mathbb{R} \ni h \to 0$, and $z \in \mathbb{R}$, we've $f(z+h)$, $f(z) \in \mathbb{R}$ so that $f'(z) \in \mathbb{R}$.



b) Let $z = iw$, $w \in \mathbb{R}$, given that f is diff'ble at all such z,

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$
$$= \lim_{\substack{ih \to 0 \\ h \in \mathbb{R}}} \frac{f(z+ih) - f(z)}{ih}$$
$$= \underbrace{-i \lim_{\substack{ih \to 0 \\ h \in \mathbb{R}}} \underbrace{\frac{f(z+ih) - f(z)}{h}}_{\in \mathbb{R}}}_{\in \mathbb{R}}$$

3) a) $P(x+iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y)$

$P_y = -6xy + i(3x^2 - 3y^2 - 1)$, $P_x = 3x^2 - 3y^2 - 1 + 6ixy$

$\Rightarrow iP_x = P_y$, So P is analytic. Indeed,

$$P(x+iy) = x^3 + i3x^2y - 3xy^2 - iy^3 - x - iy$$
$$= (x+iy)^3 - (x+iy) = z^3 - z$$

b) $P(x+iy) = x^2 + iy^2$

$x^2 + iy^2 = \sum\limits_{j=0}^{N} a_j (x+iy)^j$, set $y = 0$ to get $x^2 = \sum\limits_{j=0}^{N} a_j x^j \Rightarrow$

$\sum\limits_{\substack{j=0 \\ j \neq 2}}^{N} a_j x^j + (a_2 - 1)x^2 = 0 \Rightarrow a_j = 0, j \neq 2$, $a_2 = 1$.

Thus $x^2 + iy^2 = (x+iy)^2$ which is obviously false!

so not analytic.

5) $P(x+iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y) = z^3 - z$

$\frac{dP}{dz} = P'(z) = 3z^2 - 1 = 3x^2 - 3y^2 - 1 + i\,6xy = P_x \; (= -iP_y)$

P is analytic polynomial

The reason is that for a diffble fnc f, say at $z \in \mathbb{C}$, $f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}$

$$\underset{\text{exists}}{\text{limit}} = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z+h) - f(z)}{h} = \frac{\partial}{\partial x} f(\underset{x+iy}{\underset{\|}{z}})$$

by diffbility of f at z

9) a) $\sum_{n=0}^{\infty} z^{n!} = z + z^2 + z^6 + z^{24} + \cdots$ has radius of convergence 1 as $\overline{\lim_{n \to \infty}} |C_n|^{1/n} = 1$ where $C_n = \begin{cases} 1 & n = k!,\ k \geq 0 \\ 0 & \text{o.w.} \end{cases}$

$$\underset{\text{radius of conv}}{R} = \frac{1}{\overline{\lim_{n \to \infty}} |C_n|^{1/n}} = 1$$

b) $\sum_{n=0}^{\infty} (n + 2^n) z^n$, $\overline{\lim_{n \to \infty}} |C_n|^{1/n} = \lim_{n \to \infty} \left(\sup_{k \geq n} (k + 2^k)^{1/k} \right)$

$= 2$

as $2 = (2^k)^{1/k} < (k + 2^k)^{1/k} < (2^k + 2^k)^{1/k} = 2^{\frac{k+1}{k}} \to 2$ as $k \to \infty$.

thus $R = 1/2$.

10) $\sum c_n z^n$ has radius of convergence R, hence $\overline{\lim_{n \to \infty}} |c_n|^{1/n} = \frac{1}{R}$

a) $\sum n^p c_n z^n$, $\overline{\lim_{n \to \infty}} |n^p c_n|^{1/n} = \overline{\lim_{n \to \infty}} (n^{1/n})^p |C_n|^{1/n}$

$= \lim (n^{1/n})^p \, \overline{\lim} |C_n|^{1/n} = 1 \cdot \overline{\lim} |C_n|^{1/n} = \frac{1}{R}$

since $\lim (n^{1/n})^p = 1 \geq 0$ (this is possible since if $\{a_n\}$ & $\{b_n\}$ two bounded sequences with $b_n \to b \geq 0$ then $\overline{\lim_{n \to \infty}} (a_n b_n) = (\overline{\lim} a_n) \times b$)

c) $\sum c_n^2 z^n$, given that radius of convergence of $\sum c_n z^n$ is $R = (\overline{\lim} |c_n|^{1/n})^{-1}$. Let $\tilde{R}$ be the radius of convergence of $\sum c_n^2 z^n$. Then from the elementary property of $\overline{\lim}$ or limsup we know that if $\{a_n\}$ & $\{b_n\}$ are arbitrary sequences of nonnegative real numbers, then $\overline{\lim}_{n\to\infty} (a_n b_n) \leq (\overline{\lim}\, a_n) \times (\overline{\lim}\, b_n)$. Implementing this to our case, $\tilde{R} = (\overline{\lim} |c_n^2|^{1/n})^{-1} \geq (\overline{\lim} |c_n|^{1/n})^{-2} = R^2$. So the radius of convergence of this series is at least R^2.

11) $\underbrace{\sum (a_n+b_n) z^n}_{\text{(say } R \text{ as rad. of conv.)}} = \underbrace{\sum a_n z^n}_{R_1} + \underbrace{\sum b_n z^n}_{R_2}$, assume first $R_1 \neq R_2$ then this series is obviously convergent for $|z| < \min(R_1, R_2)$ so we have to take $R = \min(R_1, R_2)$ only. Otherwise whenever $R_1 = R_2$, we can find examples so that $R \geq R_1 = R_2$, for instance let $a_n = -b_n$ so that $R = \infty$.

12) $$\sum_{n=1}^{\infty} \frac{z^n}{n} \underset{z = \text{cis}\,\theta}{=} \sum_{n=1}^{\infty} \frac{\cos n\theta + i \sin n\theta}{n} = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} + i \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}$$

recall Dirichlet's test : $\{a_n\} \in \mathbb{R}$, $\{b_n\} \in \mathbb{C}$ satisfying

- $\{a_n\}$ is monotone
- $\lim a_n = 0$
- $\left|\sum_{n=1}^{N} b_n\right| \leq \underset{\mathbb{R}^+}{M} \quad \forall N \in \mathbb{Z}^+$

then $\sum a_n b_n$ converges

also if $\theta \neq 2\pi j$, $j \in \mathbb{Z}$ then

$$B_k^1 = \sum_1^k \cos n\theta = \frac{\cos 1/2 (k+1)\theta \cdot \sin 1/2 k\theta}{\sin 1/2 \theta}$$

$$B_k^2 = \sum_1^k \sin n\theta = \frac{\sin 1/2 (k+1)\theta \cdot \sin 1/2 k\theta}{\sin 1/2\theta}$$

$\Rightarrow |B_k^j| \leq |\csc 1/2 \theta| \quad \forall n$

so by Dirichlet's test above series converge.

15) a) For $x \in [\pi/6 + 2k\pi, \frac{5\pi}{6} + 2k\pi] =: I_k$, $k \in \mathbb{Z}^+$

$|\sin x| \geq \frac{1}{2}$, $\left| [\pi/6 + 2k\pi, \frac{5\pi}{6} + 2k\pi] \right| = \frac{2\pi}{3} > 2$, so there are infinitely many such intervals of length > 2 so that $\sin n \geq \frac{1}{2}$, $n \in I_k \cap \mathbb{Z}^+$, $k \in \mathbb{Z}^+$. It follows

$1 = \lim \left(\frac{1}{2}\right)^{1/n} \leq \overline{\lim} |\sin n|^{1/n} \leq 1$, so that the radius of convergence of $\sum \sin n z^n$ equals to 1.

b) $\sum_{n=0}^{\infty} e^{-n^2} z^n$, $a_n = e^{-n^2} > 0$, $\lim \frac{a_{n+1}}{a_n} = \lim \frac{e^{-(n+1)^2}}{e^{-n^2}}$

$= \lim e^{-2n-1} = 0$, so the radius of conv. ∞. by question 13 a).

16) $\overline{\lim_{k \to \infty}} \left| (1 + \frac{1}{k})^{k^2} \right|^{1/k} = \lim_{k \to \infty} (1 + \frac{1}{k})^k = e$

$\overline{\lim} |2^k|^{1/k} = 2 \Rightarrow e > 2$, so $\overline{\lim} |c_n|^{1/n} = e$

$\Rightarrow R = 1/e$.