PS II (331)

Q2.6) (X, d) m.s. $F \subseteq X$ finite. WTS : $\mathrm{acc}(F) = \emptyset$

remember : for $z \in X$ & $S \subseteq X$; z accumulation pt of S in X $\Leftrightarrow$ $\mathrm{dist}(z, S \setminus \{z\}) = 0$.

For $z \in \mathrm{acc}(F)$, $d(z,y) \neq 0$ $\forall y \in F \setminus \{z\}$ $\rightsquigarrow$

$$\mathrm{dist}(z, F\setminus\{z\}) = \inf_{y \in F\setminus\{z\}} \{d(z,y)\} \underset{F\text{-fin}}{=} \min_{y\in F\setminus\{z\}} \{d(z,y)\} > 0$$

Q2.7) C : collection of subsets of a m.s. X, and $S \in C$.

WTS :

(i) x isolated pt of S & $x \in \bigcap C \Rightarrow x$ isolated pt of $\bigcap C$

(ii) x isolated pt of $\bigcup C$ & $x \in S \Rightarrow x$ isolated pt of S

To see (i) note that $x \in \mathrm{iso}(S) \Leftrightarrow x \in S$ & $\mathrm{dist}(x, S\setminus\{x\}) > 0$ $\Leftrightarrow x \in S$ & $\inf_{y\in S\setminus\{x\}} \{d(x,y)\} > 0$. So, as $\bigcap C \setminus \{x\} \subset S\setminus\{x\}$,

$$\inf_{z \in \bigcap C\setminus\{x\}} \{d(x,z)\} \geq \inf_{y\in S\setminus\{x\}} \{d(x,y)\} > 0 \rightsquigarrow x \in \mathrm{iso}(\bigcap C).$$

And to establish (ii), we note : $x \in \mathrm{iso}(\bigcup C) \Leftrightarrow$ $x \in \bigcup C$ & $\mathrm{dist}(x, \bigcup C \setminus\{x\}) > 0 \Leftrightarrow x \in \bigcup C$ & $\inf_{y \in \bigcup C\setminus\{x\}} \{d(x,y)\} > 0$

Using this along with the fact that $S\setminus\{x\} \subseteq \bigcup C \setminus \{x\}$, we obtain :

$$\underbrace{\inf_{z \in S\setminus\{x\}} \{d(x,z)\}}_{= \mathrm{dist}(x, S\setminus\{x\})} > \inf_{y\in \bigcup C\setminus\{x\}} \{d(x,y)\} > 0$$

$\rightsquigarrow x \in \mathrm{iso}(S)$ as $x \in S$.

Q2.8) C : nonempty collection of subsets of a m.s. X

WTS :

1) $\mathrm{acc}(\bigcap S) \subseteq \bigcap \{\mathrm{acc}(S) \mid S \in C\}$

2) $\bigcup \{\mathrm{acc}(S) \mid S \in C\} \subseteq \mathrm{acc}(\bigcup C)$

also we'll show inclusions in (1) & (2) may be proper.

For (1), let $x \in acc(\cap C)$ then $dis(x, \cap C \setminus \{x\}) = 0$, and need to show that $x \in acc(S)$ $\forall S \in C$.

$$0 \leq dist(x, S \setminus \{x\}) \leq dist(x, \cap C \setminus \{x\}) = 0$$

$$S \setminus \{x\} \supset \cap C \setminus \{x\}$$

$\Rightarrow dist(x, S \setminus \{x\}) = 0 \quad \therefore x \in acc(S)$.

To see that this inclusion may be proper, set $(X, d) = (\mathbb{R}, euc)$ & $C = \{ \underbrace{(0, 1/n)}_{=: S_n} : n \in \mathbb{N} \}$. Recalling the notation:

$\mathcal{C} = \{S : S \subset X\}$, $\cap C := \bigcap_{S \in C} S$, we arrive at:

$acc(S_n) = [0, 1/n] \Rightarrow \bigcap acc(S_n) = \{0\}$ yet $\cap C = \emptyset \Rightarrow acc(\cap C) = \emptyset$. As for (2), let $x \in \bigcup\{acc(S) : S \in C\}$, then $\exists\ S \in C$ s.t. $x \in acc(S)$, hence as $S \subset \cup C$, $x \in acc(\cup C)$; and for the proper inclusion part we set: $(X, d) = (\mathbb{R}, euc)$, $C = \{ \underbrace{\{x\}}_{=S} : x \in [0,1] \}$

$\cup C = [0,1]$, $acc(\cup C) = [0,1]$, S: singleton $\Rightarrow acc(S) = \emptyset$ hence $\bigcup \{acc(S)\} = \emptyset$.

Q 2.9 (X_i, τ_i) - m.s. $i \in \underbrace{\{1, \dots, n\}}_{=: \mathbb{N}_n}$, $n \in \mathbb{N}$. Suppose d is conserving on $P = \prod_{i=1}^{n} X_i$

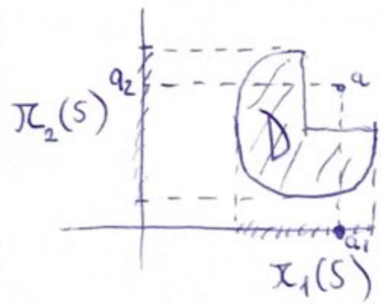
Recall: $\mu_1(a, b) = \sum_{i=1}^{n} \tau_i(a_i, b_i)$, $\mu_2(a, b) = \left(\sum_{i=1}^{n} (\tau_i(a_i, b_i))^2 \right)^{1/2}$

$\mu_\infty(a, b) = \max\{\tau_i(a_i, b_i) : i \in \mathbb{N}_n\}$; $a, b \in P$. It had been shown in Thm 1.6.1 that $\mu_\infty(a, b) \leq \mu_2(a, b) \leq \mu_1(a, b)$.

defn: e is a metric on P then it is called conserving on P wrt τ_i iff $\forall a, b \in P$, we've $\mu_\infty(a, b) \leq e(a, b) \leq \mu_1(a, b)$.

Suppose $S \subseteq P$ & $a \in S$. denoting $\pi_i : P \to X_i$ by a a projection of P onto X_i, we wonder whether it is true that $a \in iso(S) \Leftrightarrow a_i \in iso(\pi_i(S))\ \forall i \in \mathbb{N}_n$.

($\Rightarrow$) part is not necessarily true :



set $S = D \cup \{a\} \subset \mathbb{R}^2$, as the figure suggests, $a \in iso(S)$ but neither a_1 nor a_2 (where $a = (a_1, a_2)$) is isolated in projections of $\pi_j : S \to \pi_j(S)$.

But it is true whenever $S = P = \prod_{i=1}^{n} X_i$. To see so, we use contrapositive argument : let $a_j \notin iso(\pi_j(S)) = iso(\pi_j(P)) = iso(X_j)$ for some $j \in \mathbb{N}_n$. Thus for any $\varepsilon > 0$, we can find $n^j \in X_j \setminus \{a_j\}$ s.t. $\tau_j(a_j, n^j) < \varepsilon$ (remember τ_j is a metric on X_j). Then since $d : P \to [0, \infty]$ is conserving on P, we've the following ineq. : $d(n, a) \leq \sum_{i=1}^{n} \tau_i(a_i, n^i) = \tau_j(a_j, n^j) < \varepsilon$ for $n = (n^1, n^2, \dots, n^j, \dots, n^n) = (a_1, a_2, \dots, n^j, \dots a_n)$ i.e. $n^i = a_i$ for $i \neq j$. Therefore since ε is arbitrary, $dist(a, P \setminus \{a\}) = 0$ (clearly $n \neq a = (a_1, \dots, a_n)$)

$$\overset{''}{\inf}\{d(a,y) : y \in P \setminus \{a\}\}$$

$\Rightarrow a \notin iso(P)$.

For the ($\Leftarrow$) part, suppose that $a_i \in iso(\pi_i(S))$ for all $i \in \mathbb{N}_n$, and also further suppose that $a = (a_1, \dots, a_n) \notin iso(S)$. Since given that $a \in S$, by

Theorem 2.6.4 ($z \in acc(S)$ iff $z \notin iso(S)$, whenever $z \in S$), $a \in acc(S)$. Hence $\forall \varepsilon > 0\ \exists x \in S$ s.t. $d(a,x) < \varepsilon$. Since $a_i \in iso(\pi_i(S))$ for all i, $dist(a_i, \pi_i(S) \setminus \{a_i\}) \neq 0\ \forall i$. So set

$c = \min_i \{ dist(a_i, \pi_i(S) \setminus \{a_i\}) \}$ and let $\varepsilon = c/2$ then

$$0 \leq \max_{i \in \mathbb{N}_n} \{ \tau_i(a_i, x_i) \} \underset{d \text{ conserving}}{\leq} d(a,x) < \varepsilon = c/2 < c < \tau_i(a_i, x_i)$$

for each $x_i \in \pi_i(S) \setminus \{a_i\}$, a contradiction.

Q 2.12 $S \subseteq \mathbb{R}$, $x \in \mathbb{R}$. WTS: $\exists$ at most two nearest pts of S to x.

« $s \in S$ is nearest pt of S to x iff $d(x,s) = dist(x,S)$ »

$$dist(x,S) = \inf_{y \in S} \{ d(x,y) \}$$

Note that the stament is not true for an arbitrary metric. Consider, for instance, $\mathbb{R}$ with the discrete metric. Let $S \subset \mathbb{R}$ with $|S| \geq 3$ and $x \in \mathbb{R} \setminus S$ then $dist(x,S) = 1 = d(x,s)$, $\forall s \in S$. I.e. all points of S are nearest points of S to x. Thus we take d as Euclidean metric.

Suppose on the contrary that $\exists$ 3 distinct pts in $\mathbb{R}$ say a,b,c s.t. $|x-a| = |x-b| = |x-c|$ & wlog $a<b<c$.

Then
$$\left.\begin{array}{l} |x-a| = |x-b| \rightsquigarrow x = \frac{a+b}{2} \\ |x-b| = |x-c| \rightsquigarrow x = \frac{b+c}{2} \end{array}\right\} \Rightarrow a+b = b+c \rightarrow a=c$$

a contradiction