MATH 162 First Midterm Examination

1	/15 points
2	/15 points
3	/10 points
4	/20 points
5	/20 points
6	/20 points

March 20, 2018
Name:
Surname:
Signature:

During the exams, you should have no book, no tablet, no computer, no calculator, no phone, no electronic device at all.

You should explain your reasoning; this is a necessary condition for getting any credit.

You have to make some progress to the correct answer in order to deserve partial credit.

Good luck.

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(1) Let A be any subset of $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ having 6 elements. Does there necessarily exist in A two integers whose sum is equal to 15? (15 points)

We consider 5 boxes, labelled III-XII, IV-XI, V-X, VI-IX, VII-VIII. We put the six elements of A to these boxes, placing the number a in A to the box labelled by a Roman numeral representing a. Since there are 6 elements of A and 5 boxes, the pigeonhole principle yields that one of the boxes has more elements than one. The numbers in that box add to 15.

Thus there necessarily exist two integers in A with sum 15.

(2) A class has 40 students. The students are from 17 to 34 years of age. You claim that there are at least x students of the same age. How large can you make x and be sure that your claim is true. (15 points)

The years of age, which we think of as boxes, are $34 - 16 = 18$ in number. There are 40 students, whom we think of as balls. By the generalized piegonhole principle, $\left\lceil \frac{40}{18} \right\rceil = \left\lceil 2 + \frac{4}{18} \right\rceil = 3$ students will be of the same age for sure, whereas there do not have to be 4 students of the same age.

Thus $x = 3$.

(3) What is the coefficient of $x^2y^4z^5$ in the expansion of $(x+y+z)^{11}$? (10 points)

In expanding the product $(x+y+z)^{11}$, we will write a product of 11 factors, each of which is x or y or z. Each factor comes from one and only one parenthesis. We have eleven parentheses to choose x, y,z from. Designating a parenthesis from which x is taken by x, and likewise for y and z, we understand that we are to count the number of 11-letter words consisting of letters x, y, z; more precisely, consisting of 2 x's, 4 y's, 5 z's. This is the number of words that can be formed by using the letters of the word $xxyyyyzzzzz$, and it equals

$$\frac{11!}{2!\,4!\,5!}.$$

(4) A battery shop has 8 different kinds of batteries; one of these kinds is $A76$. Assume there are at least 30 batteries of each type.

(i) In how many ways can we form an inventory of 30 batteries? (5 points)

We are to choose 30 objects from 8 kinds, the answer is $\binom{8-1+30}{30} = \binom{37}{30}$.

(ii) In how many ways can we form an inventory 30 batteries, if at least 4 pieces of $A76$ batteries must be included? (5 points)

First we put 4 pieces of $A76$ batteries. Then we are to choose $30 - 4 = 26$ objects of 8 kinds, and this can be done in $\binom{8-1+26}{26} = \binom{33}{26}$ ways.

(iii) In how many ways can we form an inventory 30 batteries, if at most 3 pieces of $A76$ batteries must be included? (10 points)

From parts (i) and (ii), we deduce that this can be done in

$$\binom{37}{30} - \binom{33}{26}$$

ways.

[Alternatively, in $\binom{36}{30} + \binom{35}{29} + \binom{34}{28} + \binom{33}{27}$ ways.]

(5) Let E and F be the events that a family of n children has children of both sexes and has at most one boy, respectively. (Here the age order of the children is important; for example when $n = 3$, BBG is to be considered distinct from BGB.) Find all positive integers n such that E and F are independent. (20 points)

There are 2^n ways of having n children. They are represented by a sequence of n letters, each letter being B or G.

The event E of both sexes B and G occurs in the case all of the 2^n sequences of B's and G's except $BB \dots B$ and $GG \dots G$, so $p(E) = \dfrac{2^n - 2}{2^n}$.

The event F that there is at most one boy is represented by the sequence $GG \dots G$, and the sequences of $n-1$ G's and one B. The place occupied by B can be chosen in $\dbinom{n}{1} = n$ ways. Thus $p(F) = \dfrac{1+n}{2^n}$.

The event $E \cap F$ that there is at exactly one boy is represented by the sequences of $n-1$ G's and one B. As we have computed above, this can happen in $\dbinom{n}{1} = n$ ways. Thus $p(E \cap F) = \dfrac{n}{2^n}$.

The events E and F are independent if and only if $p(E \cap F) = p(E)p(F)$, so if and only if

$$\begin{aligned}
\frac{n}{2^n} &= \frac{2^n - 2}{2^n} \cdot \frac{1+n}{2^n} \\
\frac{n}{n+1} &= \frac{2^n - 2}{2^n} \\
1 - \frac{1}{n+1} &= 1 - \frac{1}{2^{n-1}} \\
n + 1 &= 2^{n-1} \\
n &= 3.
\end{aligned}$$

(6) How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

(20 points)

Let the signals be a, b, c, where a takes 1 microsecond, b and c takes 2 microseconds for transmittal.

Let u_n denote the number of different messages that can be transmitted in n microseconds.

Then $u_1 = 1$ since only the signal a can be transmitted in 1 second. The signals taking 2 microseconds are aa, b, c, so $u_2 = 3$.

Next we will find a recurrence relation of u_n. A signal requiring n microseconds may end in a, or in b, or in c. In the first case, it is a signal requiring $n-1$ microseconds followed by a; and there are u_{n-1} signals of this type. In the second case, the signal is a signal requiring $n-2$ microseconds followed by b; the number of such signals equals u_{n-2}. In the third case, too, we have a signal that takes $n-2$ microseconds followed by c and there are u_{n-2} signals in the third case. Thus $u_n = u_{n-1} + 2u_{n-2}$.

We now find the general solution of this recurrence relation. The characteristic polynomial is $x^2 - x - 2 = (x+1)(x-2)$ with roots -1 and 2. So the general solution is $\alpha(-1)^n + \beta 2^n$. So $u_n = \alpha(-1)^n + \beta 2^n$ for some constants α, β.

In order to find these constants, we use the initial conditions $u_1 = 1$ and $u_2 = 3$. We have

$$\begin{aligned}(-1)\alpha + 2\beta &= u_1 = 1\\ \alpha + 4\beta &= u_2 = 3,\end{aligned}$$

so $6\beta = 4$ and $\beta = \dfrac{2}{3}$, and $\alpha = 2\beta - 1 = \dfrac{4}{3} - 1 = \dfrac{1}{3}$.

This yields the formula

$$u_n = \frac{1}{3}(-1)^n + \frac{2}{3}2^n = \frac{2^{n+1} + (-1)^n}{3}$$

for the number of different messages that can be transmitted in n microseconds.