Math 162

Ch 4 - Supp. Ex.

Q 29 Adopt the proof that there are infinitely many primes to show that there are infinitely many primes in the arithmetic progression $6k+5$, $k=0,1,2,\ldots$

Suppose for a contradiction that there are finitely many primes of the form $6k+5$, say $p_0, p_1, p_2, \ldots, p_n$ are all of them.

Let $\theta = 6p_1p_2 \cdots p_n + 5$

Since we've listed all primes of the form $6k+5$, and θ is greater than each of them θ can not be prime. Then θ can be written as a product of odd prime numbers

All possibilities for the prime factors of θ are of the form : $6k+1$, $\cancel{6k+2}$, $\cancel{6k+3}$, $\cancel{6k+4}$, $6k+5$, $\cancel{6k}$

even, 3, not prime, even, even

$3 \nmid \theta$

Note that product of two primes of the form $6k+1$

$(6k+1)(6\ell+1) = 36k\ell + 6k + 6\ell + 1 = 6(6k\ell + k + \ell) + 1$

Hence one of the prime divisors of θ must have the form : $6k+5$

Case 1) It is equal to $p_0 = 5$ ie $5 \mid \theta$

Then $5 \mid \theta - 5$ Then $5 \mid 6p_1p_2 \cdots p_n$ since 5 is prime

$5 \mid 6$ or $5 \mid p_1$ or $\cdots$ $5 \mid p_n$ X since $p_1, \ldots p_n$ prime and $\neq 5$

Case 2.) It is equal to p_j for some $j \in \{1,2,\ldots n\}$

ie $p_j \mid \theta \to p_j \mid \theta - 6p_1p_2\cdots p_j \to p_j \mid 5$ for some $j \in \{1,\ldots n\}$

Contradiction since $p_1 = 11$, $p_2 = 17, \ldots$

Hence there are infinitely many prime numbers of the form $6k+5$

Q30 Explain why you cannot directly adopt the proof that there are infinitely many primes of the form $3k+1$

$k = 1, 2, \ldots$

Suppose that there are finitely many primes of the form $3k+1$ which are $p_0, p_1, \ldots p_n$

Let $\theta = 3p_0p_1p_2\cdots p_n + 1$

θ is of the form $3k+1$ and $\geq p_i$ $\forall i \in \{0,1,\ldots n\}$

θ can not be prime since we've listed all the primes of the form $3k+1$. Then θ can be written as a product of primes. All possibilities $\cancel{3k}$, $3k+1$, $3k+2$

3 or not prime

$3 \nmid \theta$

Assume that some prime divisor of θ is of the form $3k+1$

Then $p_j \mid \theta$ for some $j \in \{0,\ldots n\} \to p_j \mid \theta - 3p_0p_1\cdots p_n$

$\to p_j \mid 1$ ✗ (since p_j is prime). So this is not possible

Hence θ must be a product of primes of the form

$3k+2$

$$(3k+2)(3l+2) = 9kl + 6k + 6l + 4 = 3(3kl + 2k + 2l + 1) + 1$$

θ can be obtained by this way. Hence we can not get a contradiction ch5 – Supp

Q37 (24) Use M.I to prove that if n people stand in a line and if the first person in the line is a woman, and the last person in line is a man, then somewhere in the line there is a woman directly in front of a man

Let $P(n)$ denote the given statement.

Basis step: Shortest line should consist of 2 people

$P(2)$: $\underset{1}{\underline{W}}\ \underset{2}{\underline{M}}$ the statement is true since the woman is directly in front of the man

Inductive step: Assume that $p(k)$ is true for some $k \geq 2$. lets consider a line of $k+1$ people : $\underbrace{W \ldots X}_{k \text{ people}} M$

→ If the person in front of the last man is woman $(X=W)$ then the statement is true

→ If X is a man then look at the shorter line of length k obtained by removing the last man We have line starting with a woman ending with a man of length k. Therefore by induction hypothesis, there is a woman directly in front of man in the shorter line

Hence in our line of length $k+1$, there is a woman directly in front of man. By M.I. we have proved that the statement is true for all integers $n \geq 2$

Q42 (28) Use M.I to show that if a, b and c are the length of the sides of a right triangle, where c is the length of the hypotenuse, then $a^n + b^n < c^n$ for all integers n with $n \geq 3$

Let $P(n)$ denote the given statement

Basis step: $P(3)$:

$a^3+b^3 = a^2a + b^2b < a^2c+b^2c$ (since c is the hypothenus)

$= c(a^2+b^2) = c.c^2 = c^3$

Hence $a^3+b^3 < c^3$

Inductive step: Assume $a^k+b^k < c^k$ for some integer $k \geq 3$

Then $a^{k+1}+b^{k+1} = a^k a + b^k . b < a^k . c + b^k . c = (a^k+b^k).c$

$\underset{\text{by I.H}}{<} c^k . c = c^{k+1}$ Hence $a^{k+1}+b^{k+1} < c^{k+1}$

by M.I $a^n+b^n < c^n$ for all integers $n \geq 3$

Q41 (27) Show that if n is a positive integer, then

$$\sum_{j=1}^{n}\left((2j-1)\left(\sum_{k=j}^{n}\frac{1}{k}\right)\right) = \frac{n.(n+1)}{2}$$

Let $P(n)$ denote the given equality

Basis step $P(1)$

$$\sum_{j=1}^{1}(2j-1)\left(\sum_{k=j}^{1}\frac{1}{k}\right) = (2.1-1).\frac{1}{1} = 1 = \frac{1.(1+1)}{2}$$ is true

Inductive step: Assume $P(m)$ is true for some integer $m \geq 1$

Then $$\sum_{j=1}^{m+1}(2j-1)\left(\sum_{k=j}^{m+1}\frac{1}{k}\right) = \left[\sum_{j=1}^{m}(2j-1)\left(\sum_{k=j}^{m}\frac{1}{k}\right)\right] + (2.(m+1)-1).\frac{1}{m+1}$$

$$= \sum_{j=1}^{m}(2j-1)\left(\sum_{k=j}^{m}\frac{1}{k} + \frac{1}{m+1}\right) + \frac{2m+1}{m+1}$$

$$= \sum_{j=1}^{m} \left(\left((2j-1) \sum_{k=j}^{m} \frac{1}{k} \right) + \frac{(2j-1)}{m+1} \right) + \frac{2m+1}{m+1}$$

$$= \underbrace{\sum_{j=1}^{m} (2j-1) \sum_{k=j} \frac{1}{k}}_{\downarrow \text{by I.H}} + \sum_{j=1}^{m} \frac{2j-1}{m+1} + \frac{2m+1}{m+1}$$

$$= \frac{m.(m+1)}{2} + \frac{2}{m+1} \sum_{j=1}^{m} j - \frac{1}{m+1} \sum_{j=1}^{m} 1 + \frac{2m+1}{m+1}$$

$$= \frac{m.(m+1)}{2} + \frac{2}{\cancel{m+1}} \frac{m.(\cancel{m+1})}{2} - \frac{m}{m+1} + \frac{2m+1}{m+1}$$

$$= \frac{m^2+m+2m}{2} + \frac{m+1}{m+1} = \frac{m^2+3m+2}{2} = \frac{(m+2)(m+1)}{2}$$

We've proved $P(m+1)$ is true wherever $P(m)$ is true.

By M-I the equality is true for all positive integers n

5.2

Q30 Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, wherever a is a nonzero real number

Basis step: $a^0 = 1$ is true by definition of a^0

Inductive step: Assume that $a^j = 1$ for all nonnegative integers j with $j \leq k$ Then note that

$$a^{k+1} = \frac{a^k . a^k}{a^{k-1}} = \frac{1.1}{1} = 1$$

Answer $a^{k-1} = 1$ does not fit the inductive hypothesis since $k-1$ is not necessarily a nonnegative number since k may equal to zero

Q7 Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction

2, 4, 5, 6, 7, 8, 9, -

Claim $n=2$ or $\forall n \geq 4$ n can be formed using just two dollar bills and five dollar bills.

$n=2$ is clear.

Denote $P(n)$ that the statement n dollars can be formed using just 2-dollar bills and five-dollar bills.

We will prove $P(n)$ is true for all $n \geq 4$

Basis step: $4=2+2$ so $P(4)$ is true.

Inductive step: Let $P(j)$ is true for all integers $4 \leq j \leq k$

Then $k+1 = (k-1)+2$

Given $4 \leq j \leq k < k+1 \rightarrow 2 < k-1 \rightarrow 3 \leq k-1$

Case 1: If $k-1=3$ then $k+1=5$ then $P(k+1)$ is true

Case 2 If $k-1 \geq 4$ then by I.H. $k-1$ dollars can be formed just using 2-dollar bills and 5-dollar bills.

Then $k+1 = (k-1)+2$ can be formed just using 2-dollar bills and 5-dollar bills. Then $P(k+1)$ is true

Thus by strong induction $P(n)$ is true for all $n \geq 4$

4.1

Q 38 Show that if n is an integer then $n^2 \equiv 0$ or $1 \pmod 4$

Case 1) n is even Then $n=2k$ for some $k \in \mathbb{Z}$

Then $n^2 = 4k^2 \equiv 0 \pmod 4$

Case 2) n is odd. Then $n=2k+1$ for some $k \in \mathbb{Z}$.

Then $n^2 = 4k^2+4k+1$ then $4 \mid n^2-1$ ie $n^2 \equiv 1 \pmod 4$

We've considered all cases Thus $n^2 \equiv 0$ or $1 \pmod 4$

4.3

Q44 Use the extended Euclidean algorithm to express gcd(100001, 1001) as a linear combination of 1001 and 100001

$100001 = 1001 \cdot 99 + 902$

$1001 = 902 \cdot 1 + 99$ Thus gcd (1001, 100001) = 11

$902 = 99 \cdot 9 + 11$

$99 = 11 \cdot 9 + 0$

$11 = 902 - 99 \cdot 9$

$= 902 - (1001 - 902) \cdot 9$

$= 10 \cdot 902 - 9 \cdot 1001$

$= 10 \cdot (100001 - 1001 \cdot 99) - 9 \cdot 1001$

$= 10 \cdot 100001 - 999 \cdot 1001$

Q49 Prove that the product of any three consecutive integers is divisible by 6.

Let $x = n \cdot (n+1) \cdot (n+2)$ $n \in \mathbb{Z}$.

Then either n or $n+1$ is even thus $2 | n$ or $2 | n+1$

Then $2 | x$

If $n \bmod 3 = 0$ then $3 | n$

If $n \bmod 3 = 1$ then $3 | n+2$

if $n \bmod 3 = 2$ then $3 | n+1$

Hence $3 | n$ or $3 | n+1$ or $3 | n+2$

Then $3 | x$.

We have $2 | x$ and $3 | x$ since 2 and 3 are relatively prime $2 \cdot 3 = 6 | x$

Q54 Adapt the proof in the text that there are infinitely many primes to prove that there are infinitely many primes of the form $3k+2$, where k is a nonnegative integer (Hint: Suppose that there are only finitely many such primes $q_1, q_2 \ldots q_n$, and consider the number $3q_1q_2\ldots q_n - 1$)

Suppose that there are only finitely many primes of the form $3k+2$; call them $q_1, q_2, \ldots q_n$

Let $\theta = 3q_1q_2\ldots q_n - 1$ (or $\theta' = 3q_2\ldots q_n + 2$)

since θ (or θ') is of the form $3k+2$ and $> p_i$ $\forall i \in \{1, \ldots n\}$

θ can not be a prime number. Then θ (or θ') has a prime factor of the form $\cancel{3k}$, $3k+1$ or $3k+2$

3 or not prime

$3 \nmid \theta$

Note that product of two primes of the form $3k+1$

$(3k+1)(3l+1) = 9kl + 3k + 3l + 1 = 3(3kl + k + l) + 1$

Hence one of the prime divisors of θ must have the form $3k+2$ ie for some $j \in \{1, \ldots n\}$ $q_j \mid \theta$

Then $q_j \mid \theta - 3q_1q_2\ldots q_n$ then $q_j \mid -1$ X since q_j is prime

Hence there are infinitely many primes of the form $3k+2$