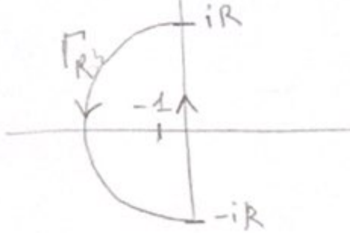
PS IX (338)

Chapter 12

1. a) $I \; : \; z(t) = it \;, \; -\infty < t < \infty \;, \quad f(z) = \dfrac{e^z}{(z+1)^4}$



$$\int_{-iR}^{iR} f(z)\,dz + \int_{\Gamma_R} f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z); -1)$$

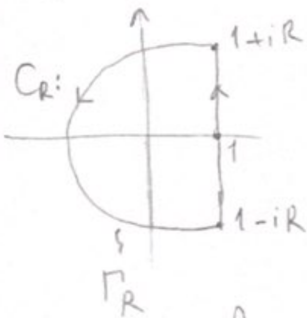
$$e^z = e^{-1} e^{z+1} = e^{-1}\left(1 + (z+1) + \frac{(z+1)^2}{2} + \cdots\right)$$

$$\Rightarrow \mathrm{Res}(f(z);-1) = \frac{1}{6e} \quad \text{also}$$

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \leq \int_{\Gamma_R} |f(z)||dz| = \int_{\Gamma_R} \frac{e^{\mathrm{Re}\,z}}{|z^4+1|}|dz| \leq \frac{C}{R^3} \to 0 \quad \text{as } R \to \infty$$

Thus $\displaystyle\int_{-i\infty}^{i\infty} f(z)\,dz = \frac{\pi i}{3e}$

b) $I \; : \; z(t) = 1 + it \;, \; -\infty < t < \infty \;, \quad f(z) = \dfrac{a^z}{z^2} \;, \quad 0 < a < \infty$



C_R is as drawn on the left side (containing 0)

$$\int_{1-iR}^{1+iR} f(z)\,dz + \int_{\Gamma_R} f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z); 0)$$

$$a^z = e^{z \log a} = \sum_0^{\infty} \frac{[(\log a) z]^n}{n!} \Rightarrow \mathrm{Res}(f(z);0) = \log a \quad \text{and}$$

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \leq \int_{\Gamma_R} \frac{a^{\mathrm{Re}\,z}}{|z|^2}|dz| \underset{\downarrow}{\leq} \frac{C}{R} \to 0 \quad \text{as } R \to \infty.$$

$\mathrm{Re}\,z \leq 1$ for $a > 1$, $0 < a < 1$ part is left as an ex.

So $\displaystyle\int_{1-i\infty}^{1+i\infty} f(z)\,dz = 2\pi i \log a$

2) $\displaystyle\int_{|z|=2} \frac{dz}{\sqrt{4z^2-8z+3}} =: I$, the zeros of $4z^2-8z+3$ occur at $r_{\pm} = 1 \pm \dfrac{\sqrt{3}}{4}$ encircled by the circle $|z| = 2$

$$I = 2\pi i\left[\mathrm{Res}\left(\frac{1}{\sqrt{4z^2-8z+3}}; r_+\right) + \mathrm{Res}\left(\frac{1}{\sqrt{4z^2-8z+3}}; r_-\right)\right] = \int_{|z|=R} \frac{dz}{\sqrt{4z^2-8z+3}} =: I_R$$

where $R > 2$. For large $R > 0$, $\sqrt{4z^2-8z+3} \sim 2z$, $|z| = R$

Since $\frac{1}{2}\int_{|z|=R} \frac{dz}{z} = \pi i$, as $R \to \infty$ $I_R \to \pi i$. To see

this, for a sufficiently large $R > 0$, we can find $\varepsilon(z)$

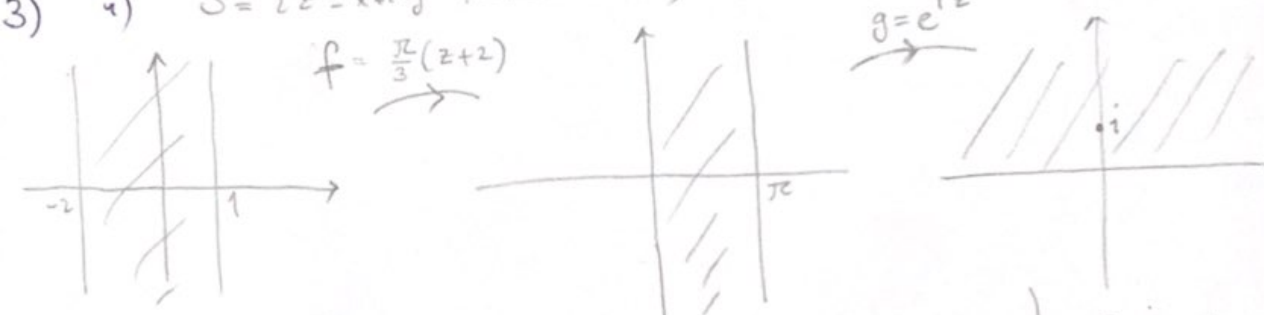
s.t. $\varepsilon(z)/z \to 0$ as $z \to \infty$ and $\sqrt{4z^2-8z+3} = 2z + \varepsilon(z)$

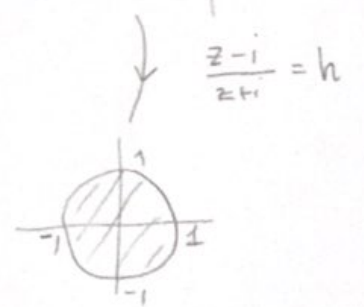
Thus $\left| I_R - \int_{|z|=R} \frac{dz}{2z} \right| \leq \int_{|z|=R} \left| \frac{|\varepsilon(z)|}{2z(2z+\varepsilon(z))} \right| |dz| \leq \frac{2\pi R}{2R} \max_{|z|=R} \left| \frac{\varepsilon(z)}{2z+\varepsilon(z)} \right|$

$\to 0$ as $R \to \infty$. So $I = \pi i$.

Chapter 13

3) i) $S = \{z = x+iy : -2 < x < 1\}$; $T = D(0,1)$





So the desired map is $h \circ g \circ f$

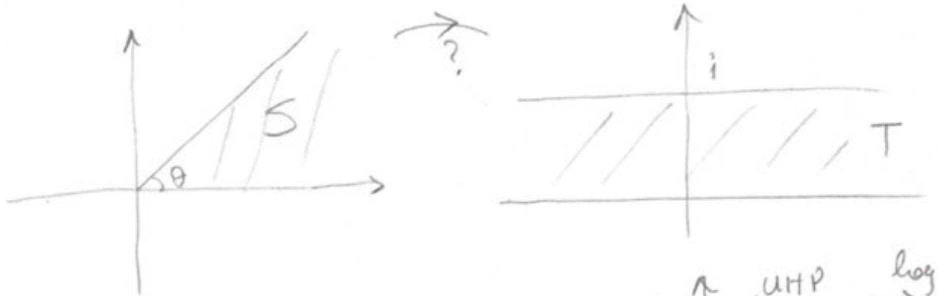
ii) $S = T$ upper half plane, $f(-2) = -1$
$f(0) = 0$ and $f(2) = 2$.

thm 13.23: $w = f(z)$ mapping z_1, z_2, z_3 into w_1, w_2, w_3 given by $\frac{(w-w_2)(w_3-w_1)}{(w-w_1)(w_3-w_2)} = \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ $\Rightarrow z_1 = -2$, $z_2 = 0$, $z_3 = 2$ & $w_1 = -1$, $w_2 = 0$, $w_3 = 2$

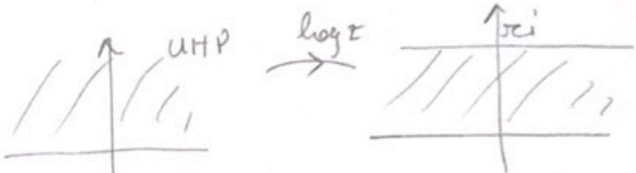
$\frac{w \cdot 3}{(w+1)\cancel{2}} = \frac{z \cdot 4}{(z+2)\cancel{2}} \Rightarrow 1 - \frac{1}{w+1} = \frac{4z}{3(z+2)} \Rightarrow \frac{6-z}{3(z+2)} = \frac{1}{w+1}$

$\Rightarrow w+1 = \frac{3z+6}{6-z} \Rightarrow w = \frac{4z}{6-z}$

iii) $S = \{re^{i\theta} : r > 0 \text{ and } 0 < \theta < \pi/4\}$, $T = \{x+iy : 0 < y < 1\}$

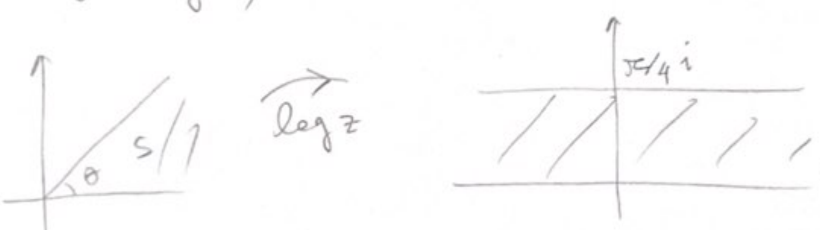


remember how $\log z$ maps :



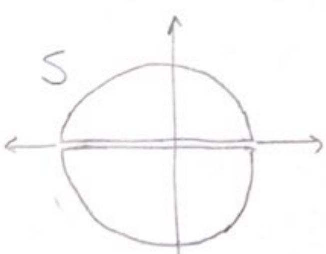
i.e. UHP is mapped conformally onto the strip $0 < \operatorname{Im} z < \pi$

To determine the above map, use defn : $\log z = \log|z| + i\theta$ where $\theta = \operatorname{Arg} z$, so as $0 < \theta < \pi/4$ we see that

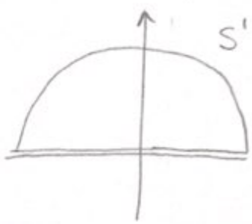


thus the desired map is $f(z) = \frac{4}{\pi} \log z$

iv) $S = D(0,1) \setminus [0,1]$; $T = D(0,1)$.



$f(z) = \sqrt{z}$

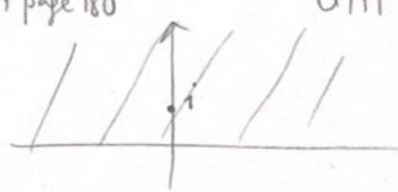


$S' = \{z : |z| < 1, \operatorname{Im} z > 0\}$

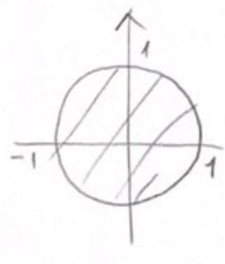
$g(z) = -\dfrac{(z-1)^2}{4(z+1)^2}$

done in text page 180

UHP

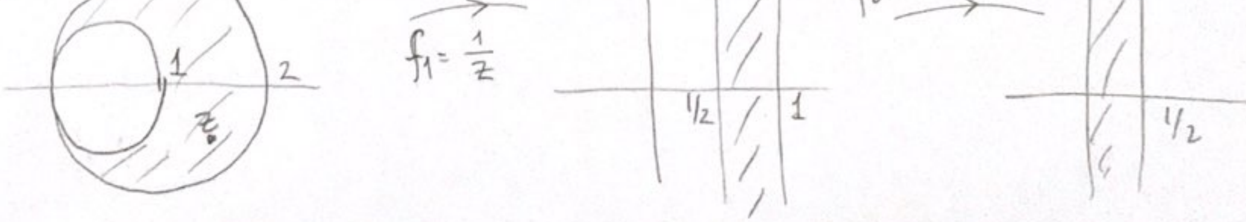


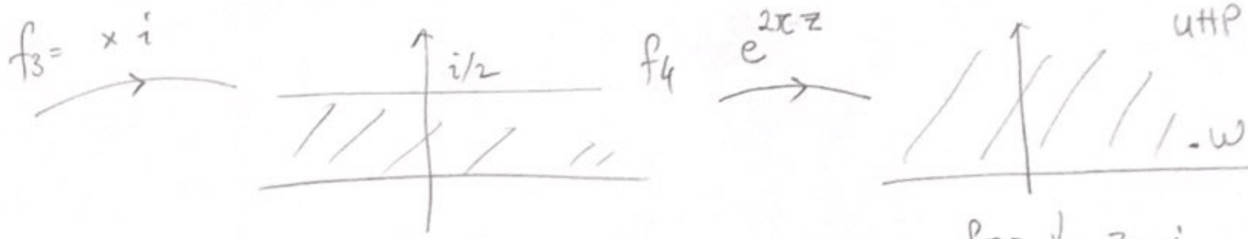
our map : $h \circ g \circ f$

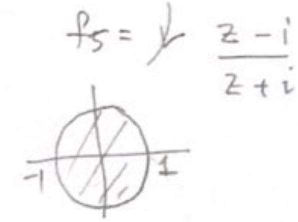


$h(z) = \dfrac{z-i}{z+i}$

4)







the desired mapping is

$$f_5\circ f_4\circ f_3\circ f_2\circ f_1(z) = \frac{e^{2\pi i(\frac{1}{z}-\frac{1}{2})}-i}{e^{2\pi i(1/z-1/2)}+i}$$

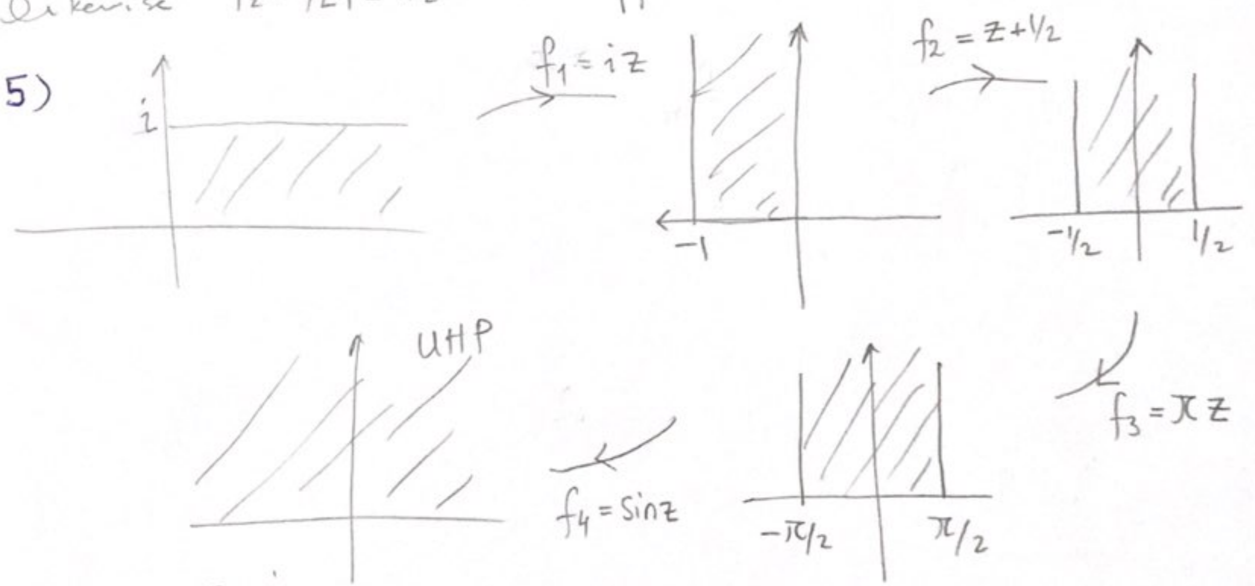
note:

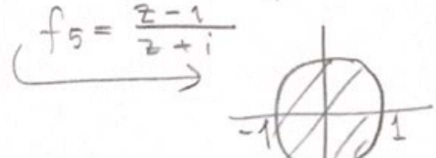
$w = \frac{1}{z}$ maps the circle $|z-1| = 1$ onto the line $\operatorname{Re} w = \frac{1}{2}$ because for $z \neq 0$, $|z-1| = 1 \underset{w=1/z}{\Leftrightarrow} |\frac{1}{w}-1| = 1 \Leftrightarrow |w-1|^2 = |w|^2$

$\Leftrightarrow (w-1)\overline{(w-1)} = |w|^2 \Leftrightarrow |w|^2 - 2\operatorname{Re} w + 1 = |w|^2 \Leftrightarrow \operatorname{Re} w = 1/2$.

likewise $|z - 1/2| = 1/2$ is mapped onto $\operatorname{Re} w = 1$.

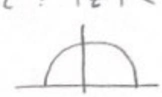
5)



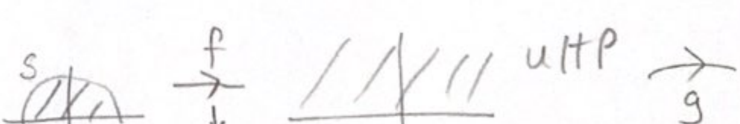


$$f_5\circ f_4\circ f_3\circ f_2\circ f_1(z) = \frac{\sin(\pi(iz+1/2)) - i}{\sin(\pi(iz+1/2)) + i}$$

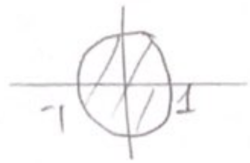
6) $S = \{z : |z|<1, \operatorname{Im} z > 0\}$ semi disc $\xrightarrow{?}$ unit disc



known from text :



$$f(z) = \frac{-(z-1)^2}{4(z+1)^2}$$

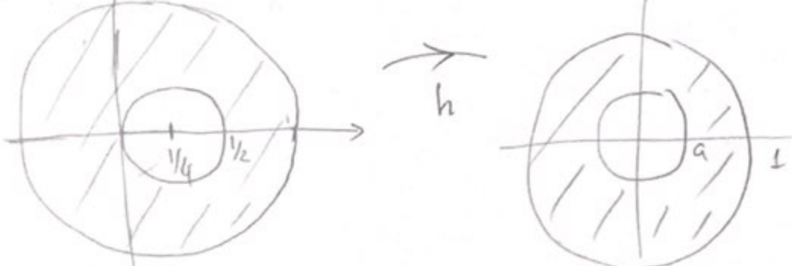


where $g(z) = \frac{z-i}{z+i}$, so $g \circ f(z) = \dfrac{\frac{-(z-1)^2}{4(z+1)^2} - i}{\frac{-(z-1)^2}{4(z+1)^2} + i}$

evaluation is left as an ex.

20) For $|\alpha|<1$, $f(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ maps unit disc $|z|<1$ onto itself

Similarly, for the circle $|z - 1/4| = 1/4$, set $4z-1=w$ to have $|w| = 1$ so that $g(w) = a\,\frac{w-\beta}{1-\bar{\beta}w}$ maps the disc $|w|<1$ onto $|z|<a<1$ for $|\beta|<1$.

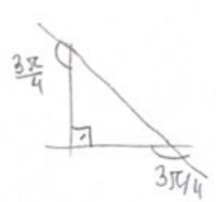


thus $h(z) = f(z)$ & $h(z) = g(4z-1)$

$$\Rightarrow \frac{z-\alpha}{1-\bar{\alpha}z} = a\,\frac{4z-\beta}{1-\bar{\beta}(4z-1)}$$

find α to determine this map.

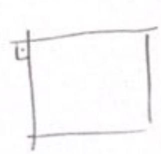
22) $f(z) = \int_0^z (\zeta^2-1)^{-3/4}\, d\zeta$



$\alpha_j \pi = \frac{3\pi}{4} \Rightarrow \alpha_j = 3/4$

$j = 1, 2.$

23) $f(z) = \int_0^z \zeta^{-1/2}(\zeta^2-1)^{-1/2}\, d\zeta$



$\alpha_j \pi = \frac{\pi}{2} \Rightarrow \alpha_j = \frac{1}{2}$

$j = 1, 2, 3.$