MATH 48A - Midterm I

1. Consider $x^3 + y^3 = c$. When is this curve non-singular? Put the curve in Weierstrass form.

2. Let E be an elliptic curve over a field of characteristic $\neq 2, 3$ given by
$$y^2 = x^3 + ax + b.$$
   (a) Show that there is E is isomorphic to an elliptic form in the form
   $$y^2 = x(x-1)(x-\lambda)$$
   for some $\lambda \neq 0, 1$.

   (b) Find the relation between a, b and λ (how do you obtain one from the other?).

   (c) Find j in terms of λ. How many possibilities are there for λ for each value of j? Is the number always the same?

3. Show that there exist positive rational numbers r_1 and r_2 s.t. $r_1^3 + r_2^3 = 7$ by starting with one point on the elliptic curve $x^3 - y^3 = 7$ and using the group operations.

4. Consider the cubic curves
$$x^3 + 2y^3 - x - 2y = 0, \qquad 2x^3 - y^3 + 2x + y = 0.$$
Find all their intersection points and prove that any cubic going through 8 of the intersection points $\neq (0, 0)$ must go through $(0, 0)$ by explicit calculation.

5. Given an elliptic curve $y^2 = x^3 + ax + b$ and a point $P = (x_0, y_0)$ on it, find the x coordinate of $3P$. Find all 3 torsion points on the curve $y^2 = x^3 + 1$.

6. Characterize when the line $y = mx + n$ is an inflectional tangent to the curve $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$. Characterize when such a curve has a 3-torsion point with coordinates in $\mathbb{Q}$.

7. Let P, P_1, P_2, P_3 be points in $\mathbb{P}^2$, and let L be a line in $\mathbb{P}^2$.

   (a) If P_1, P_2 and P_3 do not lie on aline, prove that there is a projective transformation of $\mathbb{P}^2$ so that
   $$P_1 \mapsto (0:0:1), P_2 \mapsto (0:1:0), P_3 \mapsto (1:0:0).$$

(b) If no three of P_1, P_2, P_3 and P lie on a line, prove that there is a unique projective transformation as in (a) that also sends P to $(1:1:1)$.

(c) Prove that there is a projective transformation of $\mathbb{P}^2$ so that L is sent to the line $z=0$.

(d) If P does not lie on L, prove that there is a projective transformation of $\mathbb{P}^2$ so that L is sent to the line $z=0$ and P is sent to the point $(0:0:1)$.

8. Assume char$k \neq 2$ and $x^2+1=0$ has a solution i in k. Let E be

$$y^2 = x^3 - x.$$

(a) Show that $\varphi(x,y) = (-x+iy)$ is an endomorphism of F and it satisfies $\varphi^2+[1]=0$ in $\mathrm{End}(E)$.

(b) For $a,b \in \mathbb{Z}$ show that the degree of the endomorphism $[a]+[b]\varphi$ is a^2+b^2.

(c) Explicitly find $[1]+\varphi$.

(d) Find $\ker([1]+\varphi)$.

9. Find as many solutions to the following equation in $\mathbb{Q}$ as possible:

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} = 4.$$