Midterm - Solutions

① Suppose that f is analytic on a region D and that it satisfies $f(x+iy) = x^2y^2 + i\,v(x,y)$ for some real valued $v(x,y)$.

Then $\boxed{2xy^2 = v_y}$ 0.15 and $\boxed{-2x^2y = v_x}$. 0.15 From the first one, we get $\boxed{v(x,y) = \frac{2}{3}xy^3 + c(x)}$ 0.2 for some function $c(x)$ of x. We also need $\boxed{v_x = \frac{2}{3}y^3 + c'(x) = -2x^2y}$. 0.2 So $\boxed{c'(x) = -2x^2y - \frac{2}{3}y^3}$ 0.05 and hence $c(x)$ needs to be $\boxed{-\frac{2}{3}x^3y - \frac{2}{3}xy^3 + d}$ 0.15 for some constant c. But <u>then $c(x)$ is not a fuction of x.</u> Hence there are no such analytic functions.

0.1

② (a) Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. So for all $\varepsilon > 0$ there is $N > 0$ such that for all $n > N$ we have $\left|\frac{a_{n+1}}{a_n} - L\right| < \varepsilon$. So $L - \varepsilon < \frac{a_{n+1}}{a_n} < L + \varepsilon$.

Now take $m \geq N$ and write $a_m = a_N \frac{a_{N+1}}{a_N} \cdots \frac{a_m}{a_{m-1}}$.

Then $(L-\varepsilon)^{m-N} < \frac{a_m}{a_N} < (L+\varepsilon)^{m-N}$, and hence

$$\underbrace{a_N^{1/m}}_{\to 1}(L-\varepsilon)\underbrace{(L-\varepsilon)^{-N/m}}_{\to 1} < a_m^{1/m} < \underbrace{a_N^{1/m}}_{\to 1}(L-\varepsilon)\underbrace{(L-\varepsilon)^{-N/m}}_{\to 1}$$

as $m\to\infty$ as $m\to\infty$

Therefore $\lim_{m\to\infty} a_m^{1/m} = L$.

(b) We know that the radius of convergence is $R = \dfrac{1}{\limsup_n a_n^{1/n}}$.

Also, if $\lim_{n\to\infty} a_n^{1/n}$ exists, then it equals $\limsup_n a_n^{1/n}$.

And by the previous part, it suffices to show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists.

In this case $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\,/(n+1)^{n+1}}{n!\,/\,n^n} = \lim_{n\to\infty} \frac{\cancel{(n+1)}!\; n^n}{\cancel{n!}\,(n+1)^n\cdot\cancel{(n+1)}} = \big)$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}.$$

Hence $R = \frac{1}{L} = e$ is the radius of convergence.

(c) First note that $f(z) = \frac{z^2}{2+z^2}$ is defined anywhere except $\pm\sqrt{2}$.

$$f(z) = \frac{z^2}{2+z^2} = 1 - \frac{2}{2+z^2} = 1 - \frac{1}{1+\frac{z^2}{2}} = 1 - \sum_{n=0}^{\infty} (-1)^n \left(\frac{z^2}{2}\right)^n$$

as long as $|z^2/2| < 1$ ($\Leftrightarrow |z| < \sqrt{2}$)

So $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} z^{2n}$ for $|z| < \sqrt{2}$.

③ Since f' is also entire, we apply the generalized Liouville theorem to conclude that $f'(z)$ is a polynomial of order at most 1.

Say $f'(z) = A_1 z + A_2$. We know that $|f'(0)| \le |0|$. So $f'(0) = 0$, and hence $A_2 = 0$. Also $|f'(1)| \le |1|$.

Now $f(z) = \frac{1}{2} A_1 z^2 + B$ for some $B \in \mathbb{C}$. Then we get the desired result by taking $A = \frac{A_1}{2}$.

④ Suppose $f(x+2ix) = |x|$ for all $x \in [-3,1]$. Let $r(z) = e^{-i\theta} \cdot z$ be rotation by θ where $\theta = \arctan 2$. Also let $S^* = r(S)$.

Note that S^* is included in the upper half plane and its closure intersects $\mathbb{R}$ on a line segment (namely, $[-3,1]$.)

Note that $f^* := f \circ r^{-1}$ defines a continuous function on $\overline{S^*}$ that is analytic on S^* and moreover f^* gets real values on $\mathbb{R}$, indeed $f^*(x) = |x|$.

So by Schwarz Reflection principle, we get an analytic function on $\overline{S^*} \cup \{z : \bar{z} \in S^*\}$. However $\frac{df^*}{dz}(0)$ doesn't exist, because

$$\frac{f(h)-f(0)}{h} = \begin{cases} 1 & : h \in \mathbb{R}_{>0} \\ -1 & : h \in \mathbb{R}_{<0} \end{cases}.$$ So such an f can't exist.

(5) Let $r>0$. Note that $\dfrac{1}{|z^2+1|} \leq \dfrac{1}{|z|^2+1}$ (just triangle inequality.)

Then for $z \in C_r$ we have $\dfrac{1}{|z^2+1|} \leq \dfrac{1}{r^2+1}$; let $M := \dfrac{1}{r^2+1}$.

Now $$|I(r)| = \left| \int_{C_r} \frac{dz}{z^2+1} \right| \leq \int_{C_r} \frac{dz}{|z^2+1|} \underset{\text{ML-inequality}}{\leq} M \cdot \pi r = \frac{\pi r}{r^2+1}.$$

Therefore $\lim\limits_{r\to\infty} |I(r)| = 0$, and hence $\lim\limits_{r\to\infty} I(r) = 0$.