PSV (338)

Chapter 6

4) Assume that f is analytic in $|z| \leq 1$. WTS: $\exists\, n \in \mathbb{Z}^+$ s.t. $f(1/n) \neq \frac{1}{n+1}$.

recall: Corollary 6.10
If two functions f and g, analytic in a region D, agree at a set of points with an accumulation point in D, then $f \equiv g$ throughout D.

To obtain a contradiction suppose that $\forall n \in \mathbb{Z}^+$, $f(1/n) = \frac{1}{n+1}$.

So setting $z_n = \frac{1}{n}$, $n = 1, 2, \ldots$ we have a sequence $\{z_n\}$ of distinct pts s.t. $z_n \to 0 \in \{z \in \mathbb{C} : |z| \leq 1\} =: D$. Also

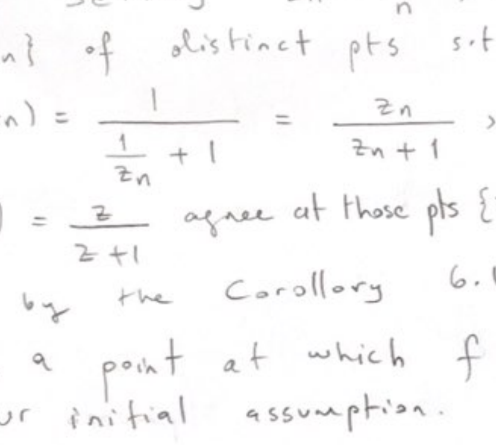
$$f(z_n) = \frac{1}{\frac{1}{z_n} + 1} = \frac{z_n}{z_n + 1}, \quad n = 1, 2, \ldots$$

In other words, $f(z)$ and $g(z) = \frac{z}{z+1}$ agree at those pts $\{z_n\}$ with accumulation point $0 \in D$

So by the Corollary 6.10, $f(z) = \frac{z}{z+1}$ on D but $-1 \in D$ is a point at which f is not analytic contradicting our initial assumption.

7) Suppose f is entire and that $|f(z)| \geq |z|^N$ for sufficiently large z. thus $f(z) \to \infty$ as $z \to \infty$ which at once implies by Theorem 6.11 that f is a polynomial, and the assumption $|f(z)| \geq |z|^N$ for large z yields that f is atleast of degree N.

10), 9)



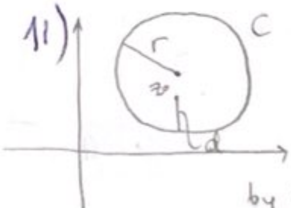
Let (x_0, y_0) & $(x_1, y_1) \in \partial D$, then
$\quad\;\; \overset{\shortparallel}{p_0} \qquad\quad \overset{\shortparallel}{p_1}$

$$\max_{z \in D} |e^z| \underset{z = x+iy}{=} \max_{x \in \text{proj}_{\mathbb{R}} D} e^x = e^{x_1} \quad \text{since}$$

x_1 is the largest elt. of $\text{proj}_{\mathbb{R}} D$ - projection of D onto real axis and e^x is an increasing function. Similarly, $\min_{z \in D} |e^z| = e^{x_0}$

So the max. & min. modulus of e^z are always attained on the boundary. Next consider $z^2 - z$ in the disk $|z| \leq 1$. It is apperant that $\max_{|z| \leq 1} |z^2 - z| = 2$ which is assumed at $z = -1$ and $\min_{|z| \leq 1} |z^2 - z| = 0$ assumed at $z = 0, 1$. We see that for

z^2-z, maximum and minimum modulus occur at the boundary points as well as at the interior point. This is no suprise because min. modulus thm asserts that minimum modulus can be attained at interior point, say z, provided that $f(z)=0$. Notice here that both e^z & z^2-z are non-constant analytic functions.

11) 

f is analytic inside and on a circle C with $|f(z)| \le M$ on C. Let z_0 be a point inside C. So denoting $f^n = \underbrace{f \cdot f \cdot \cdots \cdot f}_{n\text{-times}}$, we've the estimate by Cauchy's integral formula $f^n(z_0) = \frac{1}{2\pi i}\int_C \frac{f^n(z)}{z-z_0}\,dz$:

$$|f(z_0)|^n = |f^n(z_0)| \le \frac{1}{2\pi}\int_C \frac{|f(z)|^n}{|z-z_0|}\,dz \underset{\substack{|f|\le M \\ \text{on } C}}{\le} \frac{M^n}{2\pi d}\,\text{length}(C)$$
$$= \frac{M^n}{\cancel{2\pi} d}\,\cancel{2\pi} r = \frac{r}{d}M^n$$

where d is the distance of z_0 to the circle C, and r is the radius of the circle both of which are independent of n. Therefore,

$$|f(z_0)| \le \left(\frac{r}{d}\right)^{1/n} M \underset{n\to\infty}{\Longrightarrow} |f(z_0)| \le M.$$

13) <u>recall</u>

<u>Fund. thm. of Algebra</u> : Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$.

<u>Min. mod. thm</u> : If f is a non-constant analytic function in a region D, then <u>no</u> point $z \in D$ can be relative minimum of f unless $f(z)=0$.

Assume $p(z) = a_n z^n + \cdots + a_0$ is a non-constant (i.e. at least one coeff. $a_j \ne 0$ for $j \ne 0$) polynomial s.t. $p(z) \ne 0$ for all $z \in \mathbb{C}$. So application of min. mod. thm with region $|z| \le R$ guarantees that $p(z)$ assumes its minimum on $|z|=R$, that is on the boundary, since $p(z)$ is analytic. But $p(z)$ is non-constant, so when $z \to \infty$, $p(z) \to \infty$. This ensures that if we chose sufficiently large R, then we could find $z_0 \in \mathbb{C}$ with $|z_0| \lneq R$ s.t. $|P(z_0)| < |P(z)|$ for each z with $|z|=R$ since $|P(z_0)|$ is a fixed positive constant, contradiction to min.

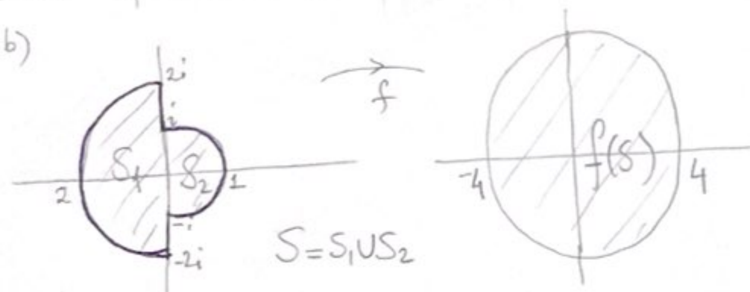
modulus theorem. So such a polynomial has a zero in $\mathbb{C}$ that establishes the fund. thm of algebra.

Chapter 7

3) Recall (open mapping Thm). Let f be a non-constant analytic function and S be an open set, then $f(S)$ is an open set.

a) Given that f is a nonconstant analytic function on S and that $f(S) = T$. Then if $f(z) \in \partial T$ then by the Open mapping Thm $z \notin \operatorname{Int} S$, so there is one choice left: $z \in \partial S$, as $S = \operatorname{Int} S \cup \partial S$. Indeed $\underbrace{\text{Open mapping Thm}}_{\text{(OMT)}}$ certifies that $f(\operatorname{Int} S) \subseteq \operatorname{Int} T$.

b)



$f(z) = z^2$, $z = re^{i\theta} \Rightarrow$

$f(re^{i\theta}) = r^2 e^{2i\theta}$

Consider points z on ∂S_1

$-\pi/2 \leq \operatorname{Arg} z \leq \pi/2 \overset{f}{\Rightarrow}$

$-\pi \leq \operatorname{Arg} f(z) \leq \pi$

and $|z| = 2 \overset{f}{\Rightarrow} |f(z)| = 4$. By the OMT, $f(S) = \{z \in \mathbb{C} : |z| \leq 4\}$

In the same way, $f(S_2) = \{z \in \mathbb{C} : |z| \leq 1\} \subseteq \operatorname{Int} f(S)$, hence $f(\partial S_2) \subseteq \operatorname{Int} f(S)$. Also $\{iy : 1 \leq y < 2 \text{ or } -2 < y \leq 1\} \subseteq \operatorname{Int} f(S)$.

remark: To sketch an image of an analytic function, we use OMT as well, just looking to where boundary points mapped. Also this question inquires that boundary points are mapped by boundary points by virtue of OMT, and that not all boundary points are mapped to boundary points.

5) Recall from textbook: $B_\alpha(z) = \dfrac{z - \alpha}{1 - \bar{\alpha} z}$ is analytic throughout $|z| \leq 1$ where $|\alpha| < 1$ as $\left|\frac{1}{\bar{\alpha}}\right| > 1$ (notice $\frac{1}{\bar{\alpha}}$ is a zero of the denominator) Also on $|z| = 1$, $|B_\alpha| \equiv 1$ (see the textbook)

Start by assuming that $\alpha_1, \dots, \alpha_n$ are the zeros of f with $|\alpha_j| < 1$ for $j = 1, 2, \dots, n$. Then as $|B_{\alpha_j}(z)| = 1$, $|z| = 1$, $\forall j$

and $|f|=1$ on $|z|=1$, setting $g(z)=\dfrac{f(z)}{\prod_{j=1}^{n} B_{\alpha_j}(z)}$ we have $|g|\equiv 1$ on the unit circle. So by max. modulus theorem $|g(z)|\leq 1, |z|\leq 1$. Since the only zeros of $g(z)=\prod_{j=1}^{n}\left[(1-\bar{\alpha}_j z)\dfrac{f(z)}{z-\alpha_j}\right]$ are $1/\bar{\alpha}_j$, $j=1,\dots,n$ where $\left|\frac{1}{\bar{\alpha}_j}\right|>1$, we've $g(z)\neq 0$ for all $z\in\{z:|z|\leq 1\}$

So then min. modulus thm may be invoked to assert that g has to be constant since, by min. mod thm, there is no $z\in\{z:|z|<1\}$ (as $g(z)\neq 0, z\in\{z:|z|<1\}$) such that a non-constant analytic g $\underbrace{\text{has relative minimum at } z}_{\text{in } \{z:|z|<1\}}$.

Indeed so the minimum is achived at boundary $|z|=1$ for which we have $|g|\equiv 1$, thus $|g|\equiv 1$ over $\{z:|z|\leq 1\}$. As a consequence, $f(z)=C\prod_{j=1}^{n}B_{\alpha_j}(z)=C\prod_{j=1}^{n}\dfrac{z-\alpha_j}{1-\bar{\alpha}_j z}$ however we notice that $1/\bar{\alpha}_1,\dots,1/\bar{\alpha}_n$ are the zeros of the denominator which forces that each $\alpha_j\equiv 0$, otherwise f would not be entire! Thus $f(z)=Cz^n$.

<u>note</u>: If we had a point $z\in\{z:|z|<1\}$ s.t. $g(z)=0$, this z would be the relative min. of g by min. mod. thm, so having $g(z)\neq 0\ \forall z\in\{z:|z|<1\}$ is crucial above.

6) By the pole, textbook means that here f being a rational function i.e. $\exists$ polynomials p & q s.t. $f(z)=\dfrac{p(z)}{q(z)}$ where q has finitely many zeros. So then assuming $\alpha_1,\dots,\alpha_n$ are the poles of $f(z)$ with $|\alpha_j|<1$, by fund. thm algebra if $f=\frac{p}{q}$ then $f(z)=\dfrac{p(z)}{C\prod_{j=1}^{n}(z-\alpha_j)}$, p - some polynomial which does not take any α_j as a root and C - constant.

Consider $g(z)=\left(\prod_{j=1}^{n}\dfrac{z-\alpha_j}{1-\bar{\alpha}_j z}\right)f(z)=\dfrac{1}{C}p(z)\prod_{j=1}^{n}\dfrac{1}{1-\bar{\alpha}_j z}$, at worst this has poles at $z=1/\bar{\alpha}_j$, $j=1,\dots,n$ whenever $p(1/\bar{\alpha}_j)\neq 0$, $j=1,\dots,n$ and as $|1/\bar{\alpha}_j|>1$ this g satisfies the criterion of the question since $|g(z)| = \underbrace{\left|\prod_{j=1}^{n} \frac{z-\alpha_j}{1-\bar{\alpha}_j z}\right|}_{=1} |f(z)|$ for $|z|=1$.

7) a) $\left|\frac{R(z-\alpha)}{R^2-\bar{\alpha}z}\right| = \left|\frac{\cancel{R}(z-\alpha)}{R^{\cancel{2}}\left(1-\frac{\bar{\alpha}}{R}\frac{z}{R}\right)}\right| = \left|\frac{z/R-\alpha/R}{1-\bar{\alpha}/R\; z/R}\right| \underset{\text{set } z/R=w,\ \alpha/R=a}{=} \left|\frac{w-a}{1-\bar{a}w}\right|$

we know that $\frac{w-a}{1-\bar{a}w}$, $|a|<1$ is analytic throughout $|w|\leq 1$. and that $\left|\frac{w-a}{1-\bar{a}w}\right| = 1$ on $|w|=1$. This shows that $\frac{R(z-\alpha)}{R^2-\bar{\alpha}z}$ is analytic for $|z|\leq R$ and hence $\left|\frac{R(z-\alpha)}{R^2-\bar{\alpha}z}\right|$ is analytic for $|z|\leq R$, also $\left|\frac{R(z-\alpha)}{R^2-\bar{\alpha}z}\right| = 1$ on $|z|=R$.

b) write $\sqrt[n]{|z-\alpha_1|\cdots|z-\alpha_n|} = \prod_{j=1}^{n} |R^2-\bar{\alpha}_j z|^{1/n} \times \frac{1}{R}\prod_{j=1}^{n}\left|\frac{R(z-\alpha_j)}{R^2-\bar{\alpha}_j z}\right|^{1/n}$

$= R\left(\prod_{j=1}^{n} \left|1-\frac{\bar{\alpha}_j}{R}\cdot\frac{z}{R}\right| \left|\frac{R(z-\alpha_j)}{R^2-\bar{\alpha}_j z}\right|\right)^{1/n} =: RI$

On $|z|=R$, by a) above, $\left|\frac{R(z-\alpha)}{R^2-\bar{\alpha}_j z}\right| = 1$ and given that $\exists$ at least one $j\in\{1,\dots,n\}$ s.t. $|\alpha_j|\neq 0$ otherwise $\prod_1^n \left|1-\frac{\bar{\alpha}_j z}{R^2}\right| = 1$ always!

$1-\left|\frac{\bar{\alpha}_j z}{R^2}\right| \leq \left|1-\frac{\bar{\alpha}_j}{R}\cdot\frac{z}{R}\right| \leq 1+\left|\frac{\bar{\alpha}_j\cdot z}{R^2}\right|$

on $|z|=R$ $\Rightarrow$ $\underbrace{1-\underbrace{\frac{|\alpha_j|}{R}}_{>0 \text{ by assump.}}}_{<1} \leq \left|1-\frac{\bar{\alpha}_j\cdot z}{R^2}\right| \leq \underbrace{1+\underbrace{\frac{|\alpha_j|}{R}}_{>0 \text{ by assump.}}}_{>1}$

(as $|\alpha_j|<R$ of course $1-\frac{|\alpha_j|}{R}>0$ & $1+\frac{|\alpha_j|}{R}<2$). So on $|z|=R$ by max mod thm $\exists$ point z s.t. $I=\prod_{j=1}^{n}\left|1-\frac{\bar{\alpha}_j}{R}\frac{z}{R}\right|^{1/n}>1$ so then $RI>R$. Similarly by min mod thm, $\exists$ point z on $|z|=R$ s.t. $I<1$ so that $RI<R$.

9) f analytic in $|z|<2$ & $|f(z)|<10$ for $|z|<2$ and $f(1)=0$ $\overset{\text{consider}}{\Rightarrow}$ $g(z)=\frac{1}{10}f(2z)$ so that $|g(z)|<1$ for $|z|<1$ and hence using $f(1)=0 \Rightarrow g(1/2)=0$, we could define $h(z)=\begin{cases} g(z)/B_{1/2}(z) & z\neq 1/2 \\ \frac{3}{4}g'(1/2) & z=1/2 \end{cases}$

where $B_{1/2}(z) = \frac{z-1/2}{1-\frac{1}{2}z}$ $\left(\text{because } \frac{g(z)}{\frac{z-1/2}{1-\frac{1}{2}z}} \underset{g(1/2)=0}{=} \frac{1}{1} \underbrace{(1-\tfrac{1}{2}z)}_{\downarrow \, 3/4} \underbrace{\frac{g(z)-g(1/2)}{z-1/2}}_{\downarrow \, g'(1/2)} \text{ as } z \to 1/2 \right)$

Since $|g|<1$ for $|z|<1$ and $|B_{1/2}(z)| = 1$ for $|z|=1$, we deduce that as $|z| \to 1$, $|h(z)| \le 1 \Rightarrow |g(z)| \le |B_{1/2}(z)|$ for $|z| \le 1$.

Thus $|f(1/2)| = 10|g(1/4)| \le 10 \left| \frac{1/4 - 1/2}{1 - 1/8} \right| = \frac{20}{7}$.

13) Recap: (Morera's thm) f - cont. on D - open. If $\int_\Gamma f(z)dz = 0$, $\Gamma = \partial R$ where R is the closed rectangle in D then f is analytic.

a) $f(z) = \int_0^1 \frac{\sin zt}{t} dt$

$\sin z$ is an entire function, so for any $R>0$ with $|z| \le R$ we can estimate: $\left|\frac{\sin z}{z}\right| = \left| \sum_{n \ge 0} \frac{(-1)^n}{(2n+1)!} z^{2n} \right| \le \sum_{n \ge 0} \frac{|z|^{2n}}{(2n+1)!} \le \frac{1}{R} \sum_{n\ge 0} \frac{R^{2n+1}}{(2n+1)!} =$

$$\frac{1}{2R}\left[\left(1 + R + \frac{R^2}{2!} + \frac{R^3}{3!} + \dots\right) - \left(1 - R + \frac{R^2}{2!} - \frac{R^3}{3!} + \dots\right) \right] = \frac{1}{2}\left[\sum_{n \ge 0} \frac{R^n}{n!} - \sum_{n\ge 0} \frac{(-1)^n R^n}{n!} \right]$$

$$= \frac{1}{2R}\left[e^R - e^{-R}\right] = \frac{\sinh R}{R}.$$

Let Γ be the boundary of some closed rectangle, so $\exists R>0$ s.t. $\Gamma \subseteq D(0,R)$, $(z \in \Gamma \subseteq D(0,R) \Rightarrow zt \in D(0,R)$ as $t \in [0,1])$

$$\int_\Gamma \int_0^1 \left| \frac{\sin zt}{t} \right| dt\, dz = \int_\Gamma \int_0^1 |z| \left| \frac{\sin zt}{zt} \right| dt\, dz \underset{|z|,|zt| \le R}{\le} \cancel{R} \frac{\sinh R}{\cancel{R}} \int_\Gamma 1 dz$$

So for any $R>0$ this is convergent, that is $\int_\Gamma \int_0^1 \frac{\sin zt}{t} dt\, dz$ is absolutely convergent, $\quad = 0$, 1 is analytic everywhere (rectangle thm)

So we can change the order of integration by Fubini's thm to get

$$\int_\Gamma \int_0^1 \frac{\sin zt}{t} dt\, dz = \underbrace{\int_0^1 \int_\Gamma \underbrace{\frac{\sin zt}{t}}_{\text{entire}} dz\, dt}_{=0 \text{ (rectangle thm)}} = 0$$

and clearly f is continuous. So by Morera's thm f is analytic on $\mathbb{C}$.

recall Fubini: if f is integrable on $X \times Y$, say $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$

$$\int_X \int_Y |f(x,y)| dy\, dx < \infty \text{ then } \int_X \int_Y f(x,y) dy\, dx = \int_Y \int_X f(x,y) dx\, dy.$$

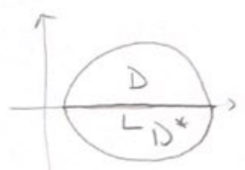
b) The series expansion : $\frac{\sin zt}{t} = \frac{1}{t}\left(zt - \frac{(zt)^3}{3!} + \frac{(zt)^5}{5!} - \dots \right) =$ $\left(z - \frac{z^3t^2}{3!} + \frac{z^5t^4}{5!} - \dots\right)$, this is a power series with radius of convergence ∞, so we can integrate term by term :

$$f(z) = \int_0^1 \frac{\sin zt}{t}\, dt = \left[z - \frac{z^3}{3\cdot 3!} + \frac{z^5}{5\cdot 5!} - \dots \right] = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{(2n+1)(2n+1)!}}_{=:a_n} z^{2n+1}$$

look at $\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n} = 0$. To see this utilize Stirling's formula : $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ so $(n!)^{1/n} \approx \underset{\infty}{\underset{\downarrow}{\frac{n}{e}}} \underset{1}{\underset{\downarrow}{\underbrace{(2\pi n)^{1/2n}}}}$ $\to \infty$ as $n \to \infty$, so $\left|\frac{(-1)^n}{(2n+1)(2n+1)!}\right|^{1/n} \to 0$ as $n \to \infty$.

So the radius of convergence is ∞ which shows that f is entire.

16) Schwarz Reflection principle : f is cont on ∂D & analytic in Int$D$ $\acute{C}$-analytic in a region D which is contained in either upper or lower half plane and whose boundary contains a segment L on the real axis, and suppose $f(z) \in \mathbb{R}$ for $z \in \mathbb{R}$.



Then we can define an analytic "extension" g of f to the region $D \cup L \cup D^*$ that symmetric wrt real axis by setting $g(z) = \begin{cases} f(z) & z \in D \cup L \\ \overline{f(\bar{z})} & z \in D^* \end{cases}$

For the question suppose f is bounded and analytic in $\mathrm{Im}\, z \geq 0$ and is real on the real axis. So by Schwarz reflection principle there is an analytic extension g of f to the $\mathbb{C}$ defined as

$g(z) = \begin{cases} f(z) & \mathrm{Im}\, z \geq 0 \\ \overline{f(\bar{z})} & \mathrm{Im}\, z < 0 \end{cases}$ so that g is bounded, therefore g being bounded and entire, has to be constant by Liouville's thm. So f is constant as well.

20) Suppose, on the contrary that there is f which is non-constant analytic in the unit disc $|z| \leq 1$ and $f(z) \in \mathbb{R}$ $\forall z$ with $|z| = 1$. Set (as in question 19)

$g(z) = \begin{cases} f(z) & |z| \leq 1 \\ \overline{f\left(\frac{1}{\bar{z}}\right)} & |z| > 1 \end{cases}$ which is entire as f is analytic in $|z| \leq 1$ and $\left|\frac{1}{\bar{z}}\right| < 1$ for $|z| > 1$ $\Rightarrow g'(z) = \overline{h'(\bar{z})}$ where $h(z) = f(1/z)$ for $|z| > 1$. Since f is continuous, it is bounded on compact domain $|z| \le 1$, likewise $f(1/\bar{z})$ is bounded on $|z| \geq 1$ ($\frac{1}{|\bar{z}|} \le 1$) thus g is bounded then by Liouville's thm g is constant so that f is constant.

22) Let $z = x+iy$, $f(z) = u+iv$. Assume f maps the lines $y=y_1$ & $y=y_2$ onto $v=v_1$ and $v=v_2$ with $y_2-y_1=c$ and $v_2-v_1=d$.

WTS : $f(z+2ci) = f(z)+2di \quad \forall z$.

Since f is entire, to make use of Corollary 7.9 :
If f is analytic in a region symmetric wrt the real axis and if f is real for real z, then $f(z) = \overline{f(\bar{z})}$.

we introduce the functions $g_1(z) = f(z+iy_1) - iv_1$ and $g_2(z) = f(z+iy_2) - iv_2$ both of which map $\mathbb{R}$ to $\mathbb{R}$ by our assumption, i.e. for $z \in \mathbb{R}$ $f(z+iy_j) = iv_j$, $j=1,2$.

Thus for all $z \in \mathbb{C}$, by Corollary 7.9

$$f(z+iy_j) - iv_j = g_j(z) = \overline{g_j(\bar{z})} = \overline{f(\bar{z}+iy_j)} + iv_j, \quad j=1,2 \Rightarrow$$

$$f(z+iy_j) \overset{(*)}{=} \overline{f(\bar{z}+iy_j)} + 2iv_j, \quad j=1,2.$$ So setting $z=x+iy$ we have

$$f(z) = f(x+iy) = f(x+i(y-y_1)+iy_1) \underset{\text{by }(*)}{=} \overline{f(x-i(y-y_1)+iy_1)} + 2iv_1$$

$$= \overline{f(x-i(y-2y_1+y_2)+iy_2)} + 2iv_1$$

$$\underset{\text{by }(*)}{=} f(x+i(y-2y_1+y_2)+iy_2) + 2iv_1 - 2iv_2$$

$$= f(z+2i(y_2-y_1)) + 2i(v_1-v_2)$$

$$= f(z+2ci) - 2di \quad \Rightarrow \quad f(z+2ci) = f(z)+2di \quad \forall z$$

$$\Rightarrow f'(z) = f'(z+2ci),$$ so f' is periodic with period $2ci$.

23) try to do it by question 22).