Math 162 PS 2

4.3 Q33 Use the Euclidean Algorithm to find

a. gcd(12,18) b. gcd(111,201)

c. gcd(1001,1331) d. gcd(12345,54321)

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a. $18 = 12 \cdot 1 + 6$

$12 = 6 \cdot 2 + 0$

gcd(12,18) = 6; last non-zero remainder

b. $201 = 111 \cdot 1 + 90$

$111 = 90 \cdot 1 + 21$

$90 = 21 \cdot 4 + 6$

$21 = 6 \cdot 3 + \boxed{3}$

$6 = 3 \cdot 2 + 0$

gcd(201,111) = 3

c. $1331 = 1 \cdot 1001 + 330$

$1001 = 3 \cdot 330 + \boxed{11}$

$330 = 30 \cdot 11 + 0$

gcd(1331,1001) = 11

d. $54321 = 12345 \cdot 4 + 4941$

$12345 = 4941 \cdot 2 + 2463$

$4941 = 2463 \cdot 2 + 15$

$2463 = 15 \cdot 164 + \boxed{3}$

$15 = 3 \cdot 5 + 0$

gcd(54321, 12345) = 3

(31) Q43 Use the extended Euclidean Algorithm to express gcd(144, 89) as a linear combination of 144 and 89

aim to find c, d s.t

$144c + 89d = \gcd(89,144) = 1.$

$144 = 89 \cdot 1 + 55$

$89 = 55 \cdot 1 + 34$

$55 = 34 \cdot 1 + 21$

$34 = 21 \cdot 1 + 13$

$21 = 13 \cdot 1 + 8$

$13 = 8 \cdot 1 + 5$

$8 = 5 \cdot 1 + 3$

$5 = 3 \cdot 1 + 2$

$3 = 2 \cdot 1 + \boxed{1}$

Numbers in paranthesis corresponds to the question numbers in the global 7nd edition

$1 = 3 - 1.2$

$= 3 - 1(5-3)$

$= 2.3 - 1.5$

$= 2(8-5) - 1.5$

$= 2.8 - 3.5$

$= 2.8 - 3.(13-8)$

$= 5.8 - 3.13$

$= 5.(21-13) - 3.13$

$= 5.21 - 8.13$

$= 5.21 - 8.(34-21)$

$= 13.21 - 8.34$

$= 13(55-34) - 8.34$

$= 13.55 - 21.34$

$= 13.55 - 21(89-55)$

$= 34.55 - 21.89$

$= 34(144-89) - 21.89$

$= 34\ 144 - 55.89$

So $1 = \gcd(89, 144) = 34.144 - 55.89$

Chapter 4- Supplementary Ex.

25. Use the Euclidean Algorithm to find gcd (10, 223) and (33, 341)

$223 = 10.22 + 3$

$10 = 3.3 + \boxed{1}$

$\gcd(10, 223) = 1$

$341 = 33.10 + 11$

$33 = 11.3 + 0$

$\gcd(341, 33) = 11$

Q27 Find $\gcd(2n+1, 3n+2)$ where n is a positive integer

$3n+2 = (2n+1).1 + n+1$

$2n+1 = (n+1).1 + n$

$n+1 = n\,1 + 1 \longrightarrow \gcd(2n+1, 3n+2) = 1$

$n = 1.n + 0$

Chapter 5
review questions

(1) Q1.c. Find a formula for the sum of the first n even positive integers and prove it using mathematical induction.

2
$2+4 = 6 = 2.3$
$2+4+6 = 12 = 3.4$
$2+4+6+8 = 20 = 4.5$

Let $P(n)$ denote the statement $\sum_{i=1}^{n} 2i = n(n+1)$

Basis step $n=1$ $\quad P(1) \quad 2 = 1(1+1)$ is true

Inductive step: Assume that $P(k)$ is true for some positive integer k. • We have $\sum_{i=1}^{k} 2i = k(k+1)$ (I.H)

• We will prove $\sum_{i=1}^{k+1} 2i = (k+1)(k+2)$

$$\sum_{i=1}^{k+1} 2i = \left(\sum_{i=1}^{k} 2i\right) + 2.(k+1) \underset{\text{By I.H}}{=} k(k+1) + 2.(k+1) = (k+1)(k+2).$$

Hence $P(k+1)$ is true. By Mathematical Induction we've showed that $P(n)$ is true for all positive integers

(4)

Q4 Give two examples of proofs that use the strong induction

1. If n is an integer greater than 1, then n can be written as the products of primes

Denote $P(n)$ the statement that n can be written as the products of primes

Basis step: $n=2$ $P(2)$ is true, since 2 is prime itself

Inductive Step: Assume $P(j)$ is true for all integers $2 \leq j \leq k$

Now if $k+1$ is prime then $P(k+1)$ is true, We're done

if $k+1$ is not prime then $k+1$ has a divisor m

s.t $2 \leq m < k+1$ Then $k+1 = m.l$ for some $m, l \in \mathbb{Z}$

s.t $2 \leq m, l < k+1$. By inductive hypothesis

m and l can be written as the product of primes Thus

$n = m.l$ " " " " " "

By M.I every integer greater than 1, can be written as the products of primes.

2. Every positive integer can be written as a sum of distinct powers of 2.

Let $P(n)$ denote the statement n can be written as a sum of distinct powers of 2.

Basis step $P(1)$ is true since $1 = 2^0$

Inductive Step: Assume that $P(j)$ is true for all integers

$1 \le j \le k$

Case 1 $k+1$ is odd then k is even

By I.H k can be written as a sum of distinct powers of 2. We add 2^0 to k to get $k+1$. Thus $k+1$ can be written as a sum of distinct powers of 2 (since k is even, it does not include 2^0)

Case 2 $k+1$ is even then $\frac{k+1}{2}$ is a positive integer

and by I.H can be written as a sum of distinct powers of 2 i.e $\frac{k+1}{2} = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_m}$

Multiplying by 2 $\rightarrow$ $k+1 = 2^{a_1+1} + 2^{a_2+1} + \cdots + 2^{a_m+1}$

$a_1+1, a_2+1, \cdots a_m+1$ is distinct since $a_1, a_2, \cdots a_m$ distinct.

Hence by M.I the statement is true for all positive integers

Supplementary Exercises

(3) 5 Show that $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$ whenever n is a positive integer

Basis step $P(1)$:

$$\frac{1}{(3.1-2)(3.1+1)} = \frac{1}{1.4} = \frac{1}{4} = \frac{1}{3.1+1} \quad \text{is true}$$

Inductive step: Assume that $P(k)$ is true for some integer $k \geq 1$. Then

$$\underbrace{\frac{1}{1.4} + \frac{1}{4.7} + \dots + \frac{1}{(3k-2)(3k+1)}}_{\text{by I.H} \ \frac{k}{3k+1}} + \frac{1}{(3(k+1)-2)(3(k+1)+1)} = $$

$$+ \frac{1}{(3k+1)(3k+4)} =$$

$$= \frac{k.(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)} = \frac{(3k+1)(k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4} = \frac{k+1}{3(k+1)+1}$$

Hence $P(k+1)$ is true, wherever $P(k)$ is true. By M.I we've proved that $P(n)$ is true for all positive integers

(5) 9) Use mathematical induction to prove that $a-b$ is a factor of $a^n - b^n$ wherever n is a positive integer.

Let $P(n)$ denote the statement given above

Basis step: $P(1)$ is true since $a-b$ is a factor of $a^1 - b^1$.

Inductive step: Assume that $a-b$ is a factor of $a^k - b^k$ for some positive integer k

ie $a^k - b^k = (a-b)M$ for some $M \in \mathbb{Z}^+$

Then $a^{k+1} - b^{k+1} = a.a^k - b.b^k$

$= a.a^k - b.b^k + b.a^k - b.a^k$

$= a^k(a-b) + b(a^k - b^k)$ } by I.H

$= a^k(a-b) + b.(a-b)M$

$= (a-b).(a^k + bM)$

Hence $(a-b)$ is a factor of $a^{k+1} - b^{k+1}$ and we've proved that $P(k+1)$ is true.

By M.I $(a-b)$ is a factor of $a^n - b^n$ for all positive integers n.

(10) Q16 For which positive integers n, is $n+6 < \frac{(n^2-8n)}{16}$?

Prove your answer using mathematical induction

$n+6 < \frac{n^2-8n}{16} \rightarrow 16n+96 < n^2-8n \rightarrow n(n-24) > 96 \rightarrow n \geq 28$

Let $P(n)$ denote the statement $n+6 < \frac{n^2-8n}{16}$

Basis step $P(28)$ $28+6 < \frac{28.20}{16}$

$34 < 35$ is true

Inductive step: Assume that $P(k)$ is true for some integer $k \geq 28$. Then

$(k+1)+6 = (k+6)+1 \underset{\text{by I.H}}{<} \frac{k^2-8k}{16} + 1 = \frac{k^2-8k+16}{16} = \frac{k^2-6k-2k+16}{16} < \frac{k^2-6k-7}{16}$

Since $k \geq 28$
$-2k+16 < -7$

$= \frac{k^2+2k+1-8k-8}{16} = \frac{(k+1)^2-8(k+1)}{16}$

Hence $P(k+1)$ is true whenever $P(k)$ is true. By M.I we have proved that $P(n)$ is true for all integers $n \geq 28$.

(18) 28 Suppose that the sequence $x_1, x_2, \ldots x_n, \ldots$ is recursively defined by $x_1 = 0$ & $x_{n+1} = \sqrt{x_n + 6}$

a. Use M.I to show that $x_1 < x_2 < \ldots < x_n < \ldots$ that is the sequence $\{x_n\}$ is monotonically increasing

Let $P(n)$ denote the statement $x_n < x_{n+1}$

Basis step: $x_1 < x_2$ since $x_1 = 0$ and $x_2 = \sqrt{0+6} = \sqrt{6}$

$P(1)$ is true

Inductive step: Assume that $P(k)$ is true for some positive integer k. That is $x_k < x_{k+1}$. Then

$x_{k+1} = \sqrt{x_k + 6} \underset{\text{By I.H}}{<} \sqrt{x_{k+1} + 6} = x_{k+2}$. Hence by M.I we have proved that $P(n)$ is true for all positive integers n

b. Use M.I to prove that $x_n < 3$ for $n = 1, 2, \ldots$

Let $P(n)$ denote the statement $x_n < 3$.

Basis step: $x_1 = 0 < 3$, $P(1)$ is true

Inductive step: Assume that $P(k)$: $x_k < 3$ is true for some positive integer k

$x_{k+1} = \sqrt{x_k + 6} < \sqrt{3+6} = \sqrt{9} = 3 \rightarrow x_{k+1} < 3$

Therefore, $P(k+1)$ is true whenever $P(k)$ is true

By M.I, $P(n)$ is true for all positive integers n

($x_n < 3$ for all n)

c. Show that $\lim_{n\to\infty} x_n = 3$

Since $\{x_n\}$ is monotonically increasing and banded

$\{x_n\}$ converges. Say $\lim_{n\to\infty} x_n = L$, $L \in \mathbb{R}$

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{x_n+6} = \sqrt{\lim_{n\to\infty}(x_n+6)} = \sqrt{\lim_{n\to\infty} x_n + 6}$$

$$= \sqrt{L+6}$$

Then $L = \sqrt{L+6}$

$\rightarrow L^2 = L+6$

$\rightarrow L^2 - L - 6$

-3

$+2$

$\rightarrow (L-3)(L+2) = 0$

Since $\{x_n\}$ is increasing and > 0 $\lim_{n\to\infty} x_n = 3$ //