Because loops are not present at all the vertices of the directed graph of S, this relation is not reflexive. It is symmetric and not antisymmetric, because every edge between distinct vertices is accompanied by an edge in the opposite direction. It is also not hard to see from the directed graph that S is not transitive, because (c, a) and (a, b) belong to S, but (c, b) does not belong to S. ◀

Exercises

1. Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 1), (1, 2), (1, 3)\}$
b) $\{(1, 2), (2, 1), (2, 2), (3, 3)\}$
c) $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
d) $\{(1, 3), (3, 1)\}$

2. Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
b) $\{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)\}$
c) $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$
d) $\{(2, 4), (3, 1), (3, 2), (3, 4)\}$

3. List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

5. How can the matrix representing a relation R on a set A be used to determine whether the relation is irreflexive?

6. How can the matrix representing a relation R on a set A be used to determine whether the relation is asymmetric?

7. Determine whether the relations represented by the matrices in Exercise 3 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

8. Determine whether the relations represented by the matrices in Exercise 4 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

9. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 100\}$ consisting of the first 100 positive integers have if R is

a) $\{(a, b) \mid a > b\}$? **b)** $\{(a, b) \mid a \neq b\}$?
c) $\{(a, b) \mid a = b + 1\}$? **d)** $\{(a, b) \mid a = 1\}$?
e) $\{(a, b) \mid ab = 1\}$?

10. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 1000\}$ consisting of the first 1000 positive integers have if R is

a) $\{(a, b) \mid a \leq b\}$?
b) $\{(a, b) \mid a = b \pm 1\}$?
c) $\{(a, b) \mid a + b = 1000\}$?
d) $\{(a, b) \mid a + b \leq 1001\}$?
e) $\{(a, b) \mid a \neq 0\}$?

11. How can the matrix for $\overline{R}$, the complement of the relation R, be found from the matrix representing R, when R is a relation on a finite set A?

12. How can the matrix for R^{-1}, the inverse of the relation R, be found from the matrix representing R, when R is a relation on a finite set A?

13. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing

a) R^{-1}. **b)** $\overline{R}$. **c)** R^2.

14. Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$. **c)** $R_2 \circ R_1$.
d) $R_1 \circ R_1$. **e)** $R_1 \oplus R_2$.

15. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find the matrices that represent

a) R^2. **b)** R^3. **c)** R^4.

16. Let R be a relation on a set A with n elements. If there are k nonzero entries in $\mathbf{M}_R$, the matrix representing R, how many nonzero entries are there in $\mathbf{M}_{R^{-1}}$, the matrix representing R^{-1}, the inverse of R?

17. Let R be a relation on a set A with n elements. If there are k nonzero entries in $\mathbf{M}_R$, the matrix representing R, how many nonzero entries are there in $\mathbf{M}_{\overline{R}}$, the matrix representing $\overline{R}$, the complement of R?

18. Draw the directed graphs representing each of the relations from Exercise 1.

19. Draw the directed graphs representing each of the relations from Exercise 2.

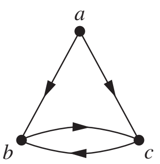
20. Draw the directed graph representing each of the relations from Exercise 3.

21. Draw the directed graph representing each of the relations from Exercise 4.

22. Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$.

In Exercises 23–28 list the ordered pairs in the relations represented by the directed graphs.

23.



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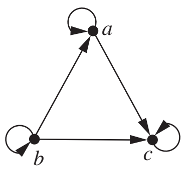

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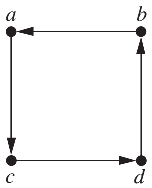

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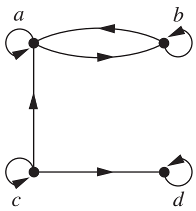

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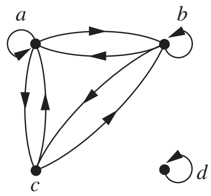

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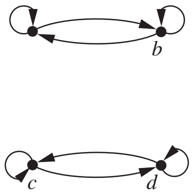


29. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is asymmetric?

30. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is irreflexive?

31. Determine whether the relations represented by the directed graphs shown in Exercises 23–25 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

32. Determine whether the relations represented by the directed graphs shown in Exercises 26–28 are reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive.

33. Let R be a relation on a set A. Explain how to use the directed graph representing R to obtain the directed graph representing the inverse relation R^{-1}.

34. Let R be a relation on a set A. Explain how to use the directed graph representing R to obtain the directed graph representing the complementary relation $\overline{R}$.

35. Show that if $\mathbf{M}_R$ is the matrix representing the relation R, then $\mathbf{M}_R^{[n]}$ is the matrix representing the relation R^n.

36. Given the directed graphs representing two relations, how can the directed graph of the union, intersection, symmetric difference, difference, and composition of these relations be found?

9.4 Closures of Relations

Introduction

A computer network has data centers in Boston, Chicago, Denver, Detroit, New York, and San Diego. There are direct, one-way telephone lines from Boston to Chicago, from Boston to Detroit, from Chicago to Detroit, from Detroit to Denver, and from New York to San Diego. Let R be the relation containing (a, b) if there is a telephone line from the data center in a to that in b. How can we determine if there is some (possibly indirect) link composed of one or more telephone lines from one center to another? Because not all links are direct, such as the link from Boston to Denver that goes through Detroit, R cannot be used directly to answer this. In the language of relations, R is not transitive, so it does not contain all the pairs that can be linked. As we will show in this section, we can find all pairs of data centers that have a link by constructing a transitive relation S containing R such that S is a subset of every transitive relation containing R. Here, S is the smallest transitive relation that contains R. This relation is called the **transitive closure** of R.

In general, let R be a relation on a set A. R may or may not have some property **P**, such as reflexivity, symmetry, or transitivity. If there is a relation S with property **P** containing R such that S is a subset of every relation with property **P** containing R, then S is called the **closure**

Exercises

1. Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

2. Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (2, 2), (3, 3)\}$

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)\}$

3. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is taller than b?

b) a is not taller than b?

c) $a = b$ or a is an ancestor of b?

d) a and b have a common friend?

4. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is no shorter than b?

b) a weighs more than b?

c) $a = b$ or a is a descendant of b?

d) a and b do not have a common friend?

5. Which of these are posets?

a) $(\mathbf{Z}, =)$ **b)** $(\mathbf{Z}, \neq)$ **c)** $(\mathbf{Z}, \geq)$ **d)** $(\mathbf{Z}, \nmid)$

6. Which of these are posets?

a) $(\mathbf{R}, =)$ **b)** $(\mathbf{R}, <)$ **c)** $(\mathbf{R}, \leq)$ **d)** $(\mathbf{R}, \neq)$

7. Determine whether the relations represented by these zero–one matrices are partial orders.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

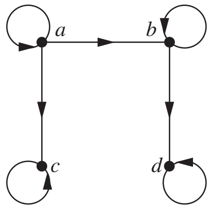
8. Determine whether the relations represented by these zero–one matrices are partial orders.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

In Exercises 9–11 determine whether the relation with the directed graph shown is a partial order.

9.



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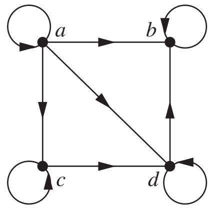


11.

12. Let (S, R) be a poset. Show that (S, R^{-1}) is also a poset, where R^{-1} is the inverse of R. The poset (S, R^{-1}) is called the **dual** of (S, R).

13. Find the duals of these posets.

a) $(\{0, 1, 2\}, \leq)$ **b)** $(\mathbf{Z}, \geq)$

c) $(P(\mathbf{Z}), \supseteq)$ **d)** $(\mathbf{Z}^+, |)$

14. Which of these pairs of elements are comparable in the poset $(\mathbf{Z}^+, |)$?

a) 5, 15 **b)** 6, 9 **c)** 8, 16 **d)** 7, 7

15. Find two incomparable elements in these posets.

a) $(P(\{0, 1, 2\}), \subseteq)$ **b)** $(\{1, 2, 4, 6, 8\}, |)$

16. Let $S = \{1, 2, 3, 4\}$. With respect to the lexicographic order based on the usual "less than" relation,

a) find all pairs in $S \times S$ less than $(2, 3)$.

b) find all pairs in $S \times S$ greater than $(3, 1)$.

c) draw the Hasse diagram of the poset $(S \times S, \preccurlyeq)$.

17. Find the lexicographic ordering of these n-tuples:

a) $(1, 1, 2), (1, 2, 1)$ **b)** $(0, 1, 2, 3), (0, 1, 3, 2)$

c) $(1, 0, 1, 0, 1), (0, 1, 1, 1, 0)$

18. Find the lexicographic ordering of these strings of lowercase English letters:

a) *quack, quick, quicksilver, quicksand, quacking*

b) *open, opener, opera, operand, opened*

c) *zoo, zero, zoom, zoology, zoological*

19. Find the lexicographic ordering of the bit strings 0, 01, 11, 001, 010, 011, 0001, and 0101 based on the ordering $0 < 1$.

20. Draw the Hasse diagram for the "greater than or equal to" relation on $\{0, 1, 2, 3, 4, 5\}$.

21. Draw the Hasse diagram for the "less than or equal to" relation on $\{0, 2, 5, 10, 11, 15\}$.

22. Draw the Hasse diagram for divisibility on the set

a) $\{1, 2, 3, 4, 5, 6\}$. **b)** $\{3, 5, 7, 11, 13, 16, 17\}$.

c) $\{2, 3, 5, 10, 11, 15, 25\}$. **d)** $\{1, 3, 9, 27, 81, 243\}$.

23. Draw the Hasse diagram for divisibility on the set

a) $\{1, 2, 3, 4, 5, 6, 7, 8\}$. **b)** $\{1, 2, 3, 5, 7, 11, 13\}$.

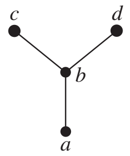
c) $\{1, 2, 3, 6, 12, 24, 36, 48\}$.

d) $\{1, 2, 4, 8, 16, 32, 64\}$.

24. Draw the Hasse diagram for inclusion on the set $P(S)$, where $S = \{a, b, c, d\}$.

In Exercises 25–27 list all ordered pairs in the partial ordering with the accompanying Hasse diagram.

25.



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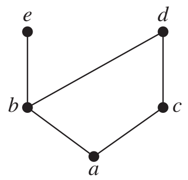

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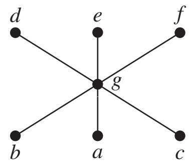


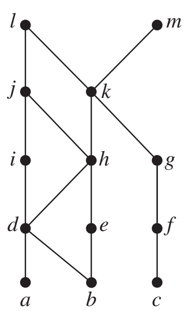
28. What is the covering relation of the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 12\}$?

29. What is the covering relation of the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set of S, where $S = \{a, b, c\}$?

30. What is the covering relation of the partial ordering for the poset of security classes defined in Example 25?

31. Show that a finite poset can be reconstructed from its covering relation. [*Hint:* Show that the poset is the reflexive transitive closure of its covering relation.]

32. Answer these questions for the partial order represented by this Hasse diagram.



a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{a, b, c\}$.

f) Find the least upper bound of $\{a, b, c\}$, if it exists.

g) Find all lower bounds of $\{f, g, h\}$.

h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

33. Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{3, 5\}$.

f) Find the least upper bound of $\{3, 5\}$, if it exists.

g) Find all lower bounds of $\{15, 45\}$.

h) Find the greatest lower bound of $\{15, 45\}$, if it exists.

34. Answer these questions for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{2, 9\}$.

f) Find the least upper bound of $\{2, 9\}$, if it exists.

g) Find all lower bounds of $\{60, 72\}$.

h) Find the greatest lower bound of $\{60, 72\}$, if it exists.

35. Answer these questions for the poset $(\{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \subseteq)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{\{2\}, \{4\}\}$.

f) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists.

g) Find all lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$.

h) Find the greatest lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$, if it exists.

36. Give a poset that has

a) a minimal element but no maximal element.

b) a maximal element but no minimal element.

c) neither a maximal nor a minimal element.

37. Show that lexicographic order is a partial ordering on the Cartesian product of two posets.

38. Show that lexicographic order is a partial ordering on the set of strings from a poset.

39. Suppose that $(S, \preccurlyeq_1)$ and $(T, \preccurlyeq_2)$ are posets. Show that $(S \times T, \preccurlyeq)$ is a poset where $(s, t) \preccurlyeq (u, v)$ if and only if $s \preccurlyeq_1 u$ and $t \preccurlyeq_2 v$.

40. **a)** Show that there is exactly one greatest element of a poset, if such an element exists.

b) Show that there is exactly one least element of a poset, if such an element exists.

41. **a)** Show that there is exactly one maximal element in a poset with a greatest element.

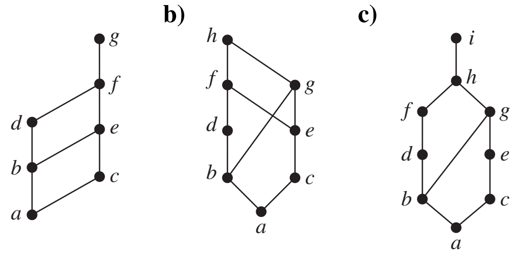
b) Show that there is exactly one minimal element in a poset with a least element.

42. **a)** Show that the least upper bound of a set in a poset is unique if it exists.

b) Show that the greatest lower bound of a set in a poset is unique if it exists.

43. Determine whether the posets with these Hasse diagrams are lattices.

a) **b)** **c)**



44. Determine whether these posets are lattices.

a) $(\{1, 3, 6, 9, 12\}, |)$ **b)** $(\{1, 5, 25, 125\}, |)$

c) $(\mathbf{Z}, \geq)$

d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S

45. Show that every nonempty finite subset of a lattice has a least upper bound and a greatest lower bound.

46. Show that if the poset (S, R) is a lattice then the dual poset (S, R^{-1}) is also a lattice.

47. In a company, the lattice model of information flow is used to control sensitive information with security classes represented by ordered pairs (A, C). Here A is an authority level, which may be nonproprietary (0), proprietary (1), restricted (2), or registered (3). A category C is a subset of the set of all projects {*Cheetah, Impala, Puma*}. (Names of animals are often used as code names for projects in companies.)

a) Is information permitted to flow from (*Proprietary*, {*Cheetah, Puma*}) into (*Restricted*, {*Puma*})?

b) Is information permitted to flow from (*Restricted*, {*Cheetah*}) into (*Registered*, {*Cheetah, Impala*})?

c) Into which classes is information from (*Proprietary*, {*Cheetah*, *Puma*}) permitted to flow?

d) From which classes is information permitted to flow into the security class (*Restricted*, {*Impala, Puma*})?

48. Show that the set S of security classes (A, C) is a lattice, where A is a positive integer representing an authority class and C is a subset of a finite set of compartments, with $(A_1, C_1) \preccurlyeq (A_2, C_2)$ if and only if $A_1 \leq A_2$ and $C_1 \subseteq C_2$. [*Hint:* First show that $(S, \preccurlyeq)$ is a poset and then show that the least upper bound and greatest lower bound of (A_1, C_1) and (A_2, C_2) are $(\max(A_1, A_2), C_1 \cup C_2)$ and $(\min(A_1, A_2), C_1 \cap C_2)$, respectively.]

***49.** Show that the set of all partitions of a set S with the relation $P_1 \preccurlyeq P_2$ if the partition P_1 is a refinement of the partition P_2 is a lattice. (See the preamble to Exercise 49 of Section 9.5.)

50. Show that every totally ordered set is a lattice.

51. Show that every finite lattice has a least element and a greatest element.

52. Give an example of an infinite lattice with

a) neither a least nor a greatest element.

b) a least but not a greatest element.

c) a greatest but not a least element.

d) both a least and a greatest element.

53. Verify that $(\mathbf{Z}^+ \times \mathbf{Z}^+, \preccurlyeq)$ is a well-ordered set, where $\preccurlyeq$ is lexicographic order, as claimed in Example 8.

54. Determine whether each of these posets is well-ordered.

a) $(S, \leq)$, where $S = \{10, 11, 12, \ldots\}$

b) $(\mathbf{Q} \cap [0, 1], \leq)$ (the set of rational numbers between 0 and 1 inclusive)

c) $(S, \leq)$, where S is the set of positive rational numbers with denominators not exceeding 3

d) $(\mathbf{Z}^-, \geq)$, where $\mathbf{Z}^-$ is the set of negative integers

A poset $(R, \preccurlyeq)$ is **well-founded** if there is no infinite decreasing sequence of elements in the poset, that is, elements $x_1, x_2, \ldots, x_n$ such that $\cdots \prec x_n \prec \cdots \prec x_2 \prec x_1$. A poset $(R, \preccurlyeq)$ is **dense** if for all $x \in S$ and $y \in S$ with $x \prec y$, there is an element $z \in R$ such that $x \prec z \prec y$.

55. Show that the poset $(\mathbf{Z}, \preccurlyeq)$, where $x \prec y$ if and only if $|x| < |y|$ is well-founded but is not a totally ordered set.

56. Show that a dense poset with at least two elements that are comparable is not well-founded.

57. Show that the poset of rational numbers with the usual "less than or equal to" relation, $(\mathbf{Q}, \leq)$, is a dense poset.

***58.** Show that the set of strings of lowercase English letters with lexicographic order is neither well-founded nor dense.

59. Show that a poset is well-ordered if and only if it is totally ordered and well-founded.

60. Show that a finite nonempty poset has a maximal element.

61. Find a compatible total order for the poset with the Hasse diagram shown in Exercise 32.

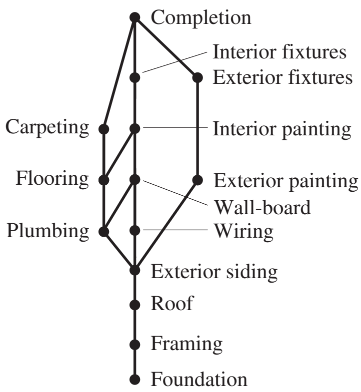
62. Find a compatible total order for the divisibility relation on the set $\{1, 2, 3, 6, 8, 12, 24, 36\}$.

63. Find all compatible total orderings for the poset $(\{1, 2, 4, 5, 12, 20\}, |\}$ from Example 26.

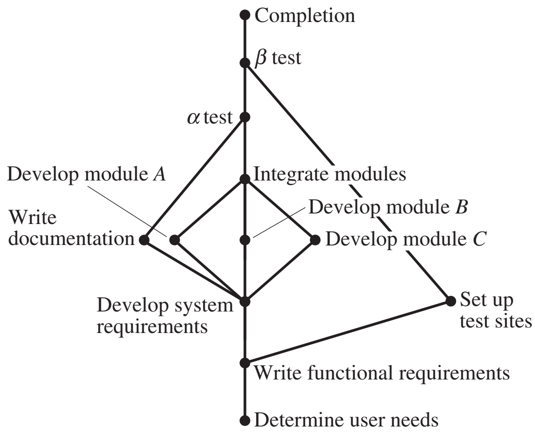
64. Find all compatible total orderings for the poset with the Hasse diagram in Exercise 27.

65. Find all possible orders for completing the tasks in the development project in Example 27.

66. Schedule the tasks needed to build a house, by specifying their order, if the Hasse diagram representing these tasks is as shown in the figure.



67. Find an ordering of the tasks of a software project if the Hasse diagram for the tasks of the project is as shown.



Key Terms and Results

TERMS

binary relation from A to B: a subset of $A \times B$

relation on A: a binary relation from A to itself (i.e., a subset of $A \times A$)

$S \circ R$: composite of R and S

R^{-1}: inverse relation of R

R^n: nth power of R

reflexive: a relation R on A is reflexive if $(a, a) \in R$ for all $a \in A$

symmetric: a relation R on A is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$

antisymmetric: a relation R on A is antisymmetric if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$

transitive: a relation R on A is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$

n-ary relation on $A_1, A_2, \ldots, A_n$: a subset of $A_1 \times A_2 \times \cdots \times A_n$

relational data model: a model for representing databases using n-ary relations

primary key: a domain of an n-ary relation such that an n-tuple is uniquely determined by its value for this domain

composite key: the Cartesian product of domains of an n-ary relation such that an n-tuple is uniquely determined by its values in these domains

selection operator: a function that selects the n-tuples in an n-ary relation that satisfy a specified condition

projection: a function that produces relations of smaller degree from an n-ary relation by deleting fields

join: a function that combines n-ary relations that agree on certain fields

directed graph or digraph: a set of elements called vertices and ordered pairs of these elements, called edges

loop: an edge of the form (a, a)

closure of a relation R with respect to a property P: the relation S (if it exists) that contains R, has property **P**, and is contained within any relation that contains R and has property **P**

path in a digraph: a sequence of edges $(a, x_1), (x_1, x_2), \ldots, (x_{n-2}, x_{n-1}), (x_{n-1}, b)$ such that the terminal vertex of each edge is the initial vertex of the succeeding edge in the sequence

circuit (or cycle) in a digraph: a path that begins and ends at the same vertex

R^* (connectivity relation): the relation consisting of those ordered pairs (a, b) such that there is a path from a to b

equivalence relation: a reflexive, symmetric, and transitive relation

equivalent: if R is an equivalence relation, a is equivalent to b if aRb

$[a]_R$ (equivalence class of a with respect to R): the set of all elements of A that are equivalent to a

$[a]_m$ (congruence class modulo m): the set of integers congruent to a modulo m

partition of a set S: a collection of pairwise disjoint nonempty subsets that have S as their union

partial ordering: a relation that is reflexive, antisymmetric, and transitive

poset (S, R): a set S and a partial ordering R on this set

comparable: the elements a and b in the poset $(A, \preccurlyeq)$ are comparable if $a \preccurlyeq b$ or $b \preccurlyeq a$

incomparable: elements in a poset that are not comparable

total (or linear) ordering: a partial ordering for which every pair of elements are comparable

totally (or linearly) ordered set: a poset with a total (or linear) ordering

well-ordered set: a poset $(S, \preccurlyeq)$, where $\preccurlyeq$ is a total order and every nonempty subset of S has a least element

Exercises

1. Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
c) $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
d) $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

2. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(a, b) \mid a$ and b are the same age$\}$
b) $\{(a, b) \mid a$ and b have the same parents$\}$
c) $\{(a, b) \mid a$ and b share a common parent$\}$
d) $\{(a, b) \mid a$ and b have met$\}$
e) $\{(a, b) \mid a$ and b speak a common language$\}$

3. Which of these relations on the set of all functions from **Z** to **Z** are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(f, g) \mid f(1) = g(1)\}$
b) $\{(f, g) \mid f(0) = g(0)$ or $f(1) = g(1)\}$
c) $\{(f, g) \mid f(x) - g(x) = 1$ for all $x \in \mathbf{Z}\}$
d) $\{(f, g) \mid$ for some $C \in \mathbf{Z}$, for all $x \in \mathbf{Z}$, $f(x) - g(x) = C\}$
e) $\{(f, g) \mid f(0) = g(1)$ and $f(1) = g(0)\}$

4. Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

5. Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

6. Define three equivalence relations on the set of classes offered at your school. Determine the equivalence classes for each of these equivalence relations.

7. Show that the relation of logical equivalence on the set of all compound propositions is an equivalence relation. What are the equivalence classes of **F** and of **T**?

8. Let R be the relation on the set of all sets of real numbers such that $S\,R\,T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0, 1, 2\}$ and **Z**?

9. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.

a) Show that R is an equivalence relation on A.
b) What are the equivalence classes of R?

10. Suppose that A is a nonempty set and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$.

11. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

12. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree except perhaps in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

13. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings that agree in their first and third bits is an equivalence relation on the set of all bit strings of length three or more.

14. Let R be the relation consisting of all pairs (x, y) such that x and y are strings of uppercase and lowercase English letters with the property that for every positive integer n, the nth characters in x and y are the same letter, either uppercase or lowercase. Show that R is an equivalence relation.

15. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

16. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

17. (*Requires calculus*)

a) Show that the relation R on the set of all differentiable functions from **R** to **R** consisting of all pairs (f, g) such that $f'(x) = g'(x)$ for all real numbers x is an equivalence relation.

b) Which functions are in the same equivalence class as the function $f(x) = x^2$?

18. (*Requires calculus*)

a) Let n be a positive integer. Show that the relation R on the set of all polynomials with real-valued coefficients consisting of all pairs (f, g) such that $f^{(n)}(x) = g^{(n)}(x)$ is an equivalence relation. [Here $f^{(n)}(x)$ is the nth derivative of $f(x)$.]

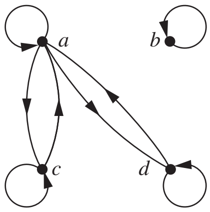
b) Which functions are in the same equivalence class as the function $f(x) = x^4$, where $n = 3$?

19. Let R be the relation on the set of all URLs (or Web addresses) such that $x\,R\,y$ if and only if the Web page at x is the same as the Web page at y. Show that R is an equivalence relation.

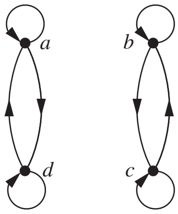
20. Let R be the relation on the set of all people who have visited a particular Web page such that $x\,R\,y$ if and only if person x and person y have followed the same set of links starting at this Web page (going from Web page to Web page until they stop using the Web). Show that R is an equivalence relation.

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.

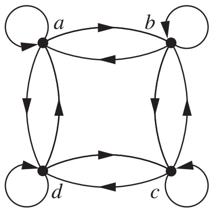
21.



22.



23.



24. Determine whether the relations represented by these zero–one matrices are equivalence relations.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

25. Show that the relation R on the set of all bit strings such that $s\,R\,t$ if and only if s and t contain the same number of 1s is an equivalence relation.

26. What are the equivalence classes of the equivalence relations in Exercise 1?

27. What are the equivalence classes of the equivalence relations in Exercise 2?

28. What are the equivalence classes of the equivalence relations in Exercise 3?

29. What is the equivalence class of the bit string 011 for the equivalence relation in Exercise 25?

30. What are the equivalence classes of these bit strings for the equivalence relation in Exercise 11?

a) 010 **b)** 1011 **c)** 11111 **d)** 01010101

31. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 12?

32. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 13?

33. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_4 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_4 if and only if they are equal or they are both at least four bits long and agree in their first four bits.)

34. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_5 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_5 if and only if they are equal or they are both at least five bits long and agree in their first five bits.)

35. What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2? **b)** 3? **c)** 6? **d)** -3?

36. What is the congruence class $[4]_m$ when m is

a) 2? **b)** 3? **c)** 6? **d)** 8?

37. Give a description of each of the congruence classes modulo 6.

38. What is the equivalence class of each of these strings with respect to the equivalence relation in Exercise 14?

a) *No* **b)** *Yes* **c)** *Help*

39. **a)** What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 15?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 15. [*Hint:* Look at the difference $a - b$ corresponding to (a, b).]

40. **a)** What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 16?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 16. [*Hint:* Look at the ratio a/b corresponding to (a, b).]

41. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$ **b)** $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$
c) $\{2, 4, 6\}, \{1, 3, 5\}$ **d)** $\{1, 4, 5\}, \{2, 6\}$

42. Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$?

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$
b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$
c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$
d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

43. Which of these collections of subsets are partitions of the set of bit strings of length 8?

a) the set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01

b) the set of bit strings that contain the string 00, the set of bit strings that contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11

c) the set of bit strings that end with 00, the set of bit strings that end with 01, the set of bit strings that end with 10, and the set of bit strings that end with 11

d) the set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 00

e) the set of bit strings that contain $3k$ ones for some nonnegative integer k; the set of bit strings that contain $3k + 1$ ones for some nonnegative integer k; and the set of bit strings that contain $3k + 2$ ones for some nonnegative integer k.

44. Which of these collections of subsets are partitions of the set of integers?

a) the set of even integers and the set of odd integers

b) the set of positive integers and the set of negative integers

c) the set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3

d) the set of integers less than -100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100

e) the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6

45. Which of these are partitions of the set $\mathbf{Z} \times \mathbf{Z}$ of ordered pairs of integers?

a) the set of pairs (x, y), where x or y is odd; the set of pairs (x, y), where x is even; and the set of pairs (x, y), where y is even

b) the set of pairs (x, y), where both x and y are odd; the set of pairs (x, y), where exactly one of x and y is odd; and the set of pairs (x, y), where both x and y are even

c) the set of pairs (x, y), where x is positive; the set of pairs (x, y), where y is positive; and the set of pairs (x, y), where both x and y are negative

d) the set of pairs (x, y), where $3 \mid x$ and $3 \mid y$; the set of pairs (x, y), where $3 \mid x$ and $3 \nmid y$; the set of pairs (x, y), where $3 \nmid x$ and $3 \mid y$; and the set of pairs (x, y), where $3 \nmid x$ and $3 \nmid y$

e) the set of pairs (x, y), where $x > 0$ and $y > 0$; the set of pairs (x, y), where $x > 0$ and $y \leq 0$; the set of pairs (x, y), where $x \leq 0$ and $y > 0$; and the set of pairs (x, y), where $x \leq 0$ and $y \leq 0$

f) the set of pairs (x, y), where $x \neq 0$ and $y \neq 0$; the set of pairs (x, y), where $x = 0$ and $y \neq 0$; and the set of pairs (x, y), where $x \neq 0$ and $y = 0$

46. Which of these are partitions of the set of real numbers?

a) the negative real numbers, $\{0\}$, the positive real numbers

b) the set of irrational numbers, the set of rational numbers

c) the set of intervals $[k, k+1]$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

d) the set of intervals $(k, k+1)$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

e) the set of intervals $(k, k+1]$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

f) the sets $\{x + n \mid n \in \mathbf{Z}\}$ for all $x \in [0, 1)$

47. List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

a) $\{0\}, \{1, 2\}, \{3, 4, 5\}$

b) $\{0, 1\}, \{2, 3\}, \{4, 5\}$

c) $\{0, 1, 2\}, \{3, 4, 5\}$

d) $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

48. List the ordered pairs in the equivalence relations produced by these partitions of $\{a, b, c, d, e, f, g\}$.

a) $\{a, b\}, \{c, d\}, \{e, f, g\}$

b) $\{a\}, \{b\}, \{c, d\}, \{e, f\}, \{g\}$

c) $\{a, b, c, d\}, \{e, f, g\}$

d) $\{a, c, e, g\}, \{b, d\}, \{f\}$

A partition P_1 is called a **refinement** of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2.

49. Show that the partition formed from congruence classes modulo 6 is a refinement of the partition formed from congruence classes modulo 3.

50. Show that the partition of the set of people living in the United States consisting of subsets of people living in the same county (or parish) and same state is a refinement of the partition consisting of subsets of people living in the same state.

51. Show that the partition of the set of bit strings of length 16 formed by equivalence classes of bit strings that agree on the last eight bits is a refinement of the partition formed from the equivalence classes of bit strings that agree on the last four bits.

In Exercises 52 and 53, R_n refers to the family of equivalence relations defined in Example 5. Recall that $s\, R_n\, t$, where s and t are two strings if $s = t$ or s and t are strings with at least n characters that agree in their first n characters.

52. Show that the partition of the set of all bit strings formed by equivalence classes of bit strings with respect to the equivalence relation R_4 is a refinement of the partition formed by equivalence classes of bit strings with respect to the equivalence relation R_3.

53. Show that the partition of the set of all identifiers in C formed by the equivalence classes of identifiers with respect to the equivalence relation R_{31} is a refinement of the partition formed by equivalence classes of identifiers with respect to the equivalence relation R_8. (Compilers for "old" C consider identifiers the same when their names agree in their first eight characters, while compilers in standard C consider identifiers the same when their names agree in their first 31 characters.)

54. Suppose that R_1 and R_2 are equivalence relations on a set A. Let P_1 and P_2 be the partitions that correspond to R_1 and R_2, respectively. Show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2.

55. Find the smallest equivalence relation on the set $\{a, b, c, d, e\}$ containing the relation $\{(a, b), (a, c), (d, e)\}$.

56. Suppose that R_1 and R_2 are equivalence relations on the set S. Determine whether each of these combinations of R_1 and R_2 must be an equivalence relation.

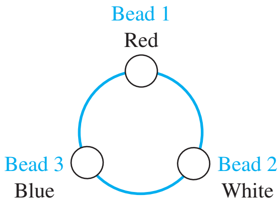
a) $R_1 \cup R_2$ **b)** $R_1 \cap R_2$ **c)** $R_1 \oplus R_2$

57. Consider the equivalence relation from Example 2, namely, $R = \{(x, y) \mid x - y \text{ is an integer}\}$.

a) What is the equivalence class of 1 for this equivalence relation?

b) What is the equivalence class of 1/2 for this equivalence relation?

***58.** Each bead on a bracelet with three beads is either red, white, or blue, as illustrated in the figure shown.



Define the relation R between bracelets as: (B_1, B_2), where B_1 and B_2 are bracelets, belongs to R if and only if B_2 can be obtained from B_1 by rotating it or rotating it and then reflecting it.

a) Show that R is an equivalence relation.

b) What are the equivalence classes of R?

***59.** Let R be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either red or blue so that (C_1, C_2), where C_1 and C_2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C_2 can be obtained from C_1 either by rotating the checkerboard or by rotating it and then reflecting it.

a) Show that R is an equivalence relation.

b) What are the equivalence classes of R?

60. a) Let R be the relation on the set of functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ such that (f, g) belongs to R if and only if f is $\Theta(g)$ (see Section 3.2). Show that R is an equivalence relation.

b) Describe the equivalence class containing $f(n) = n^2$ for the equivalence relation of part (a).

61. Determine the number of different equivalence relations on a set with three elements by listing them.

62. Determine the number of different equivalence relations on a set with four elements by listing them.

***63.** Do we necessarily get an equivalence relation when we form the transitive closure of the symmetric closure of the reflexive closure of a relation?

***64.** Do we necessarily get an equivalence relation when we form the symmetric closure of the reflexive closure of the transitive closure of a relation?

65. Suppose we use Theorem 2 to form a partition P from an equivalence relation R. What is the equivalence relation R' that results if we use Theorem 2 again to form an equivalence relation from P?

66. Suppose we use Theorem 2 to form an equivalence relation R from a partition P. What is the partition P' that results if we use Theorem 2 again to form a partition from R?

67. Devise an algorithm to find the smallest equivalence relation containing a given relation.

***68.** Let $p(n)$ denote the number of different equivalence relations on a set with n elements (and by Theorem 2 the number of partitions of a set with n elements). Show that $p(n)$ satisfies the recurrence relation $p(n) = \sum_{j=0}^{n-1} C(n-1, j)p(n-j-1)$ and the initial condition $p(0) = 1$. (*Note:* The numbers $p(n)$ are called **Bell numbers** after the American mathematician E. T. Bell.)

69. Use Exercise 68 to find the number of different equivalence relations on a set with n elements, where n is a positive integer not exceeding 10.

9.6 Partial Orderings

Introduction

We often use relations to order some or all of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y), where x comes before y in the dictionary. We schedule projects using the relation consisting of pairs (x, y), where x and y are tasks in a project such that x must be completed before y begins. We order the set of integers using the relation containing the pairs (x, y), where x is less than y. When we add all of the pairs of the form (x, x) to these relations, we obtain a relation that is reflexive, antisymmetric, and transitive. These are properties that characterize relations used to order the elements of sets.

Links

DEFINITION 1 A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

We give examples of posets in Examples 1–3.

EXAMPLE 1 Show that the "greater than or equal" relation $(\geq)$ is a partial ordering on the set of integers.