Midterm II (Answer Key)

Let α, β, γ be ordinals.

1. Show that $\alpha \cap \beta$ is also an ordinal.

Clearly since α and β well ordered $\alpha \cap \beta$ is also well ordered.
Let $a \in \alpha \cap \beta$ then $a = \alpha_a$ $a = \beta_a$ so all elements of a belong to both α and β hence $a = (\alpha \cap \beta)_a$.

2. Show that if $\gamma \subset \alpha$ then γ is equal to $\alpha_a = a$ for some $a \in \alpha$.

We first need to find for which $a = \alpha_a$ γ is equal to.
Note that $\alpha \setminus \gamma \subseteq \alpha$ is a non empty subset of α, since α is well ordered, $\alpha \setminus \gamma$ has a minimal element, say $a \in \alpha$.
We claim that $a = \alpha_a = \gamma$. Since $\gamma \subset \alpha$ every element of γ is less then a, hence $\gamma \subseteq \alpha_a$. (1)
Conversely let $b \in \alpha_a$, i.e $b \in \alpha$ and $b < a$
if $b \notin \gamma$ then $b \in \alpha \setminus \gamma$ and $b < a$
Contradicting the minimality of a; hence
$\alpha_a \subseteq \gamma$ (2) By (1) and (2) $a = \alpha_a = \gamma$. $\blacksquare$

3. Conclude that for any ordinals α and β either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

By Q1. $\alpha \cap \beta$ is an ordinal, suppose $\alpha \setminus \beta$ and $\beta \setminus \alpha$ are both non empty,

since $\alpha \cap \beta \subseteq \alpha$, By Q.2. $\alpha \cap \beta = \alpha_a = a$ for some $a \in \alpha \setminus \beta$.

Similarly $\alpha \cap \beta \subseteq \beta$ Again by Q2 $\alpha \cap \beta = \beta_b = b$ for some $b \in \beta \setminus \alpha$.

But $a = \alpha_a = \alpha \cap \beta = \beta_b = b$

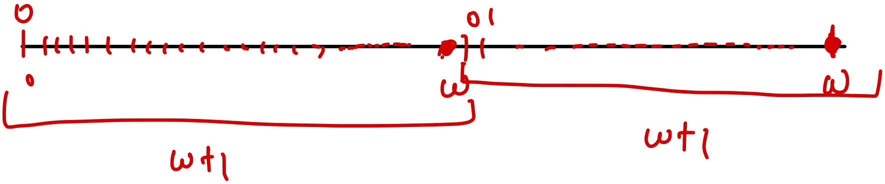
is in both α and β. hence $\alpha \cap \beta$ ⨳.

either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

4. $\alpha = (\omega+1)\cdot 2$ $\qquad$ $\beta = 2\cdot(\omega+1)$

$\gamma = (\omega+1)\cdot\omega$ $\qquad$ $\delta = \omega\cdot(\omega+1)$

$$\alpha = (\omega+1)\, S(1) = (\omega+1) + (\omega+1)$$



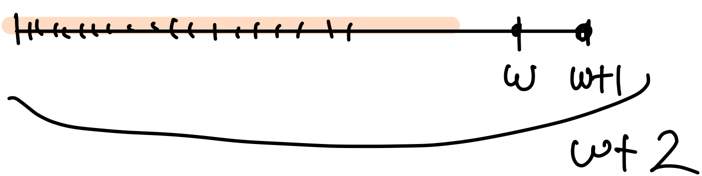
$$\underbrace{\omega+\omega+1} = 2\omega+1 = \alpha$$

$$\beta = 2\cdot(\omega+1) = 2\cdot S(\omega) = 2\cdot\omega + 2$$

$$= \bigcup_{n\in\omega} 2\cdot n + 2$$

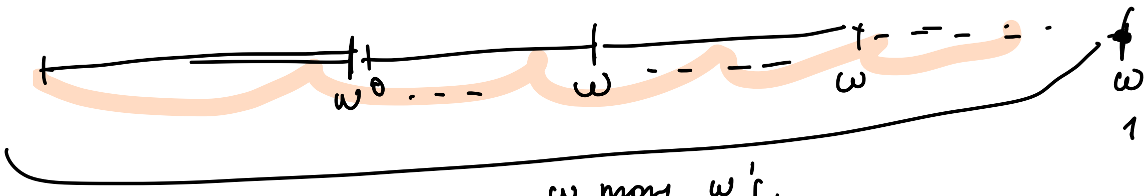
$$= \underbrace{\bigcup_{m\in\omega} m}_{\omega} + 2$$

finite things end to end



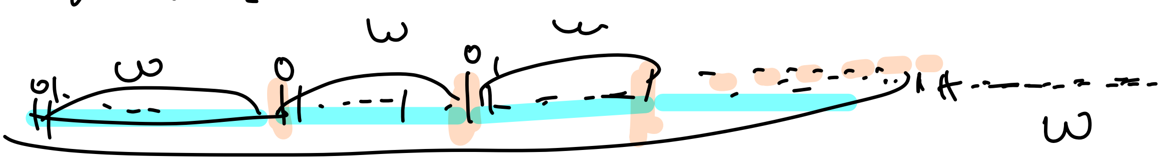
$$\beta = \omega+2 < 2\omega+1 = \alpha$$

$$\gamma = (\omega+1)\cdot\omega = \bigcup_{n\in\omega} (\omega+1)\cdot n$$



$\omega\cdot\omega+1$

$$\gamma = \omega\cdot(\omega+1) = \omega\cdot S(\omega) = \omega\cdot\omega+\omega$$



$\omega\cdot\omega$

$$\gamma = \omega\cdot\omega+1 < \omega\cdot\omega+\omega = \delta$$

$$\beta < \alpha < \gamma < \delta.$$