Projective Plane

and

Homogeneous Coordinates

Definition 1. *Given a set X, a relation $R \subset X \times X$ is called* **equivalence** *(and denoted with $x \sim y$ for $(x, y) \in R$) if it is reflexive ($x \sim x$), symmetric ($x \sim y \Rightarrow y \sim x$), and transitive ($x \sim y,\ y \sim z\ \Rightarrow x \sim z$).*
An **equivalence class** *with representative $a \in X$ is the subset of all elements of X which are equivalent to a.*

Definition 2. *Two elements of $\mathbb{R}^3 \backslash \{(0,0,0)\}$ are* **projectively equivalent**

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} \quad \textit{if} \quad \exists \lambda \in \mathbb{R} \setminus \{0\} : \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \lambda \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}$$

Definition 3. *The* **projective plane** $\mathbb{P}^2$ *is the set of all projective equivalence classes of* $\mathbb{R}^3 \setminus \{(0,0,0)\}$.

Example: Points in $\mathbb{R}^3$ lying in the same line through the origin are projectively equivalent. A line through the origin is then an equivalence class. Its representative can be for example the pair of antipodal intersections with the unit sphere.

Injection of $\mathbb{R}^2$ in $\mathbb{P}^2$:

- A point $(x', y') \in \mathbb{R}^2$ is injected in $\mathbb{P}^2$ with the addition of the homogeneous coordinate 1: $(x', y', 1) \in \mathbb{P}^2$.

- Only the points (x, y, w) of $\mathbb{P}^2$ with $w \neq 0$ can be mapped to $\mathbb{R}^2$ as $(x/w, y/w)$. If projective equivalence can be visualized as lines through the origin in $\mathbb{R}^3$, then $\mathbb{R}^2$ is injected as the plane $w = 1$.

Projective Transformation

Definition 4. *A* **projective transformation** *is any invertible matrix transformation* $\mathbb{P}^2 \rightarrow \mathbb{P}^2$*.*

- A projective transformation A maps p to $p' \sim Ap$. This means that $\det(A) \neq 0$ and that there exists $\lambda \neq 0$ such that $\lambda p' = Ap$. Observe that we will write either $p' \sim Ap$ or $\lambda p' = Ap$.

- If A maps a point to Ap, then A maps a line l to $l' = A^{-T}l$. This can be shown starting from the line equation $l^T p = 0$ and replacing p with $A^{-1}p'$.

- A projective transformation λA is the same as A since they map to projectively equivalent points. Hence, we will be able to determine a projective transformation only up to a scale factor.

- A projective transformation preserves incidence.
- Three collinear points are mapped to three collinear points and three concurrent lines are mapped to three concurrent lines.
- The latter might involve the case that the point of intersection is mapped to a point at infinity.
- Because of the incidence preservation, projective transformations are also called **collineations**.

How many points suffice to determine a projective transformation?

Assume that a mapping A maps the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ to the non-collinear points a, b and c. Since $a \sim \alpha a$, $b \sim \beta b$, and $c \sim \gamma c$ we can write:

$$\begin{pmatrix} a & b & c \end{pmatrix} \sim \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is obvious that for each choice of $\alpha, \beta, \gamma \neq 0$ we can build a matrix A mapping the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ to a, b and c. Let us assume that the same A maps $(1,1,1)$ to the point d. Then, the following should hold:

$$\lambda d = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

hence

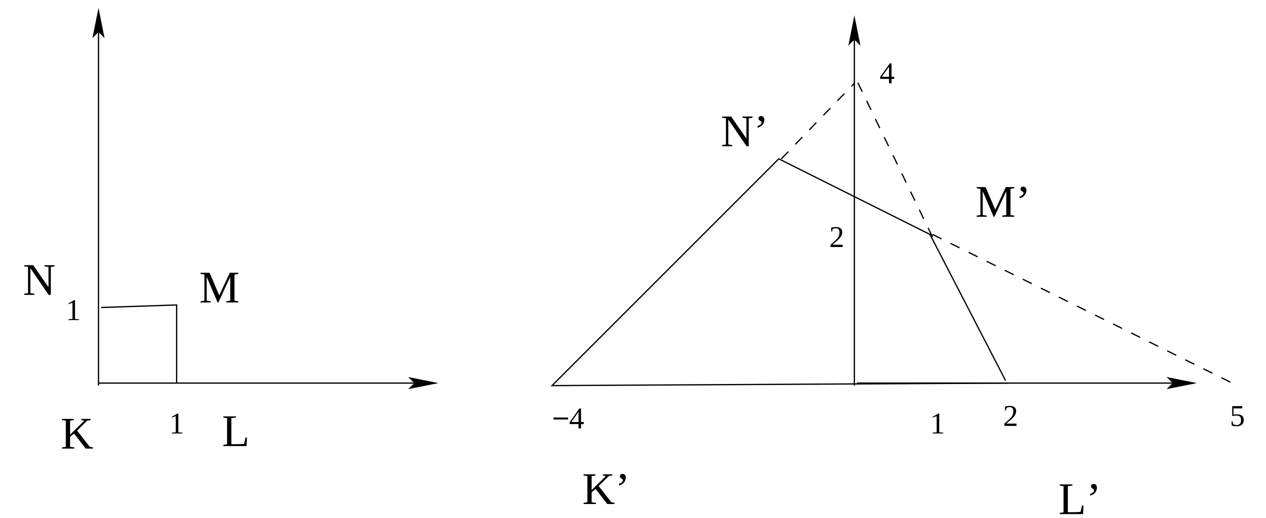
$$\lambda d = \alpha a + \beta b + \gamma c.$$

There always exist such $\lambda, \alpha, \beta, \gamma$ because four elements of $\mathbb{R}^3 \backslash \{(0,0,0)\}$ are always linearly dependent. Because a, b, c are not collinear, there exist unique $\alpha/\lambda, \beta/\lambda, \gamma/\lambda$ for writing this linear combination. Since A is the same as A/λ we solve for α, β, γ such that $d = \alpha a + \beta b + \gamma c$, which can be written as a linear system

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = d.$$

Since a, b, c are not collinear we can always find a unique triple α, β, γ. The resulting projective transformation is $A = \left(\alpha a \quad \beta b \quad \gamma c \right)$.

Proposition 1. *Four points not three of them collinear suffice to recover unambiguously a projective transformation.*



Find projective transformation mapping $(a, b, c, d) \rightarrow (a', b', c', d')$:

To determine this mapping we go through the four points used in the paragraph above. We find the mapping from $(1, 0, 0), etc$ to (a, b, c, d) and we call it T: $a \sim T(1, 0, 0)^T, etc$. We find the mapping from $(1, 0, 0), etc$ to (a', b', c', d') and we call it T': $a' \sim T'(1, 0, 0)^T, etc$. Then, back-substituing $(1, 0, 0)^T \sim T^{-1}a, etc$ we obtain that $a' = T'T^{-1}a, etc$. Thus, the required mapping is $T'T^{-1}$.