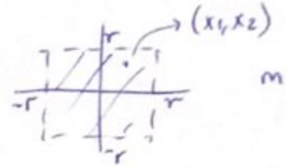
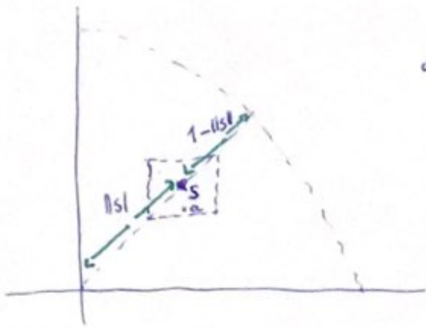
PS VII

Q 5.8 revisited) Ball $B(0;r)$:



$\max\{|x_1|,|x_2|\}<r$



define $\underset{(s_1,s_2)}{\|s\|} = \sqrt{s_1^2+s_2^2}$

$S=\{a\in\mathbb{R}^2 : a_1^2+a_2^2<1\}$

WTS: S is open in $\mathbb{R}^2$ with max metric

Given $s\in S$, find $r\in\mathbb{R}^+$ s.t. $\forall a\in B(s,r)$ we should have $a_1^2+a_2^2<1$:

$a\in B(s,r) \Rightarrow \quad d(a,s) = \max\{|a_1-s_1|, |a_2-s_2|\} < r \quad \Rightarrow$

$$a_1^2+a_2^2 \leq (|a_1-s_1|+|s_1|)^2 + (|a_2-s_2|+|s_2|)^2 < (|s_1|+r)^2+(|s_2|+r)^2$$

$$= |s_1|^2+|s_2|^2+2r^2+2|s_1|r+2|s_2|r = \|s\|^2+2r^2+2r\underbrace{(|s_1|+|s_2|)}_{\leq 2\|s\|}$$

$$\underset{\substack{\|s\|<1\\ r<1}}{\leq} \|s\|+2r+4r \underset{\text{want}}{<} 1 \quad\Rightarrow\quad r<\frac{1-\|s\|}{6}$$

So choose $r = \dfrac{1-\|s\|}{12}$.

Q 6.2) Define a real sequence recursively by :

$x_1 = 0$

$x_{2n} = x_{2n-1}/2$

$x_{2n+1} = x_{2n}+1/2, \quad \forall n\in\mathbb{N}.$

remember : $\text{tail}_m(x) = \{x_n : n\in\mathbb{N}, n\geq m\}$ - mth tail of $x=(x_n)$.

& $\limsup x_n := \inf\{\sup \text{tail}_n(x) : n\in\mathbb{N}\}$

$\liminf x_n := \sup\{\inf \text{tail}_n(x) : n\in\mathbb{N}\}$

$x_1=0, \; x_2=0, \; x_3=1/2, \; x_4=1/4, \; x_5=1/4+1/2, \; x_6=1/8+1/4$

$x_7=\frac{1}{8}+\frac{1}{4}+\frac{1}{2}, \; x_8=\frac{1}{16}+\frac{1}{8}+\frac{1}{4}, \; x_9=\frac{1}{16}+\frac{1}{8}+\frac{1}{4}+\frac{1}{2}$

Thus, for $n \geq 2$, $x_{2n} = \sum_{k=2}^{n} \frac{1}{2^k}$; $x_{2n+1} = \sum_{k=1}^{n} \frac{1}{2^k}$

$\sup tail_1(x) = \sup tail_2(x) = \sup tail_3(x) = \dots = \sup tail_m(x) = \sum_{k=1}^{\infty} \frac{1}{2^k}$

$= \frac{1}{1-\frac{1}{2}} - 1 = 1$, $\forall m$. $\Rightarrow \inf \{\sup tail_n(x) : n \in \mathbb{N}\} = 1$.

$\inf tail_1(x) = \inf tail_2(x) = 0$

$\inf tail_3(x) = \underbrace{\frac{1}{4}}_{= x_4} = \inf tail_4(x) = x_4$

$\inf tail_5(x) = \underbrace{\frac{1}{4} + \frac{1}{8}}_{= x_6} = \inf tail_6(x) = x_6$

$\inf tail_7(x) = \underbrace{\frac{1}{4} + \frac{1}{8} + \frac{1}{16}}_{= x_8} = \inf tail_8(x) = x_8$

$\vdots \qquad\qquad \vdots$

$\Rightarrow \sup \{\inf tail_n(x) : n \in \mathbb{N}\} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \frac{1}{4} \underbrace{\sum_{k=0}^{\infty} \frac{1}{2^k}}_{\frac{1}{1-1/2}} = \frac{1}{2}$

Q 6.3) Suppose (x_n) is a seqn. in $\mathbb{R}$ & $k \in \mathbb{N}$ WTS:

(i) $\limsup x_n = \inf \{\sup tail_n(x) : n \in \mathbb{N}, n \geq k\}$

(ii) $\liminf x_n = \sup \{\inf tail_n(x) : n \in \mathbb{N}, n \geq k\}$

(iii) $\liminf x_n \leq \limsup x_n$

(i) for any k, $tail_1(x) \supseteq tail_2(x) \supseteq \dots \supseteq tail_k(x) \supseteq \dots$ (1)

$\Rightarrow \underbrace{\sup tail_1(x)}_{=: a_1} \geq \underbrace{\sup tail_2(x)}_{=: a_2} \geq \dots \geq \underbrace{\sup tail_k(x)}_{=: a_k} \geq \dots$

Thus $(a_n)_{n \in \mathbb{N}}$ is a nonincreasing sequence which implies that $\inf \{(a_n)_{n \in \mathbb{N}}\} = \inf \{(a_n)_{n \in \mathbb{N}} : n \geq k\}$ $\Rightarrow$

$\inf \{\sup tail_n(x) : n \in \mathbb{N}, n \geq k\} = \inf \{\sup tail_n(x) : n \in \mathbb{N}\} = \limsup x_n$

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(ii) From (1), $\inf \text{tail}_1(x) \leq \inf \text{tail}_2(x) \leq \cdots \leq \inf \text{tail}_k(x) \leq \cdots$

$$\underbrace{\inf \text{tail}_1(x)}_{= b_1} \leq \underbrace{\inf \text{tail}_2(x)}_{= b_2} \leq \cdots \leq \underbrace{\inf \text{tail}_k(x)}_{= b_k} \leq \cdots$$

Since (b_n) is a nondecreasing sequence, we have the equality:
$\sup\{b_n : n \in \mathbb{N}\} = \sup\{b_n : n \in \mathbb{N}, n \geq k\}$ for any k.
Hence,
$\liminf x_n = \sup\{\inf \text{tail}_n(x) : n \in \mathbb{N}\} = \sup\{\inf \text{tail}_n(x) : n \in \mathbb{N}, n \geq k\}$

(iii) For any n, what we know is the comparison:

$$\underbrace{\inf \text{tail}_n(x)}_{= b_n} \leq \underbrace{\sup \text{tail}_n(x)}_{= a_n}$$

By the arguments of (i) & (ii), b_n is increasing & a_n is decreasing. So the following assertion is not obvious in a sense:

<u>Claim</u>: $b_n \leq a_n \ \forall n$ where $b_n \uparrow$ & $a_n \searrow \Rightarrow b_n \leq \inf a_n \ \forall n$.
Assume that there is $k \in \mathbb{N}$ s.t. $b_k > \inf a_n$. Hence taking $\varepsilon := \left| \frac{b_k - \inf a_n}{2} \right| > 0$, there is $k_0 \in \mathbb{N}$ s.t.
$a_{k_0} < \inf a_n + \varepsilon \underset{\text{assumption}}{<} b_k$ (2). It follows that for all $n \geq \max\{k_0, k\}$

we have: $a_n \underset{a_n \searrow}{\leq} a_{k_0^*} \underset{(2)}{<} b_k \underset{b_n \uparrow}{\leq} b_n$ which is equivalent to our statement:

$\inf \text{tail}_n(x) \leq \sup \text{tail}_n(x) \Rightarrow$
$\inf \text{tail}_n(x) \leq \inf\{\sup \text{tail}_n(x) : n \in \mathbb{N}\}$ true for any n
$\Rightarrow \sup\{\inf \text{tail}_n(x)\} \leq \inf\{\sup \text{tail}_n(x)\}$
i.e. $\liminf x_n \leq \limsup x_n$.

(notation :
$\liminf_n x_n := \sup_{n \in \mathbb{N}} \inf_{m \geq n} x_m = \sup \{ \inf \{ x_m : m \geq n \} : n \in \mathbb{N} \}$
$= \sup \{ \inf \mathrm{tail}_n (x) : n \in \mathbb{N} \}$, $\limsup_n x_n$ – likewise)

Q 6.9)
recall : Thm 6.14 : (X, d) m.s. , $z \in X$ & (x_n) is a sequence in X. Then $x_n \to z$ in X iff $(d(x_n, z))_{n \in \mathbb{N}} \to 0$ in $\mathbb{R}$.

(z^n) converges in $\mathbb{C}$ for those $z \in \mathbb{C}$ with $|z| \leq 1$.

1) For $|z| < 1$, $d(z^n, 0) = |z^n| = |z|^n \to 0$ in $\mathbb{R}$. Thus for such $z \in \mathbb{C}$, $z^n \to 0$.

2) For $|z| = 1$, if $z = e^{\frac{2\pi i k}{n}}$ $z^n \to 1$. But if $z = e^{i\theta}$, $\theta \neq \frac{2\pi k}{n}, k \in \mathbb{Z}$, no convergence !

Q. 6.14) $P = \prod_{i=1}^{\infty} X_i$, $X_i = [0,1]$ for each $i \in \mathbb{N}$.
endow P with the supremum metric $d(x,y) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$. Assume that $(a_m)_{m \in \mathbb{N}} \subseteq P$ converges in P.

Claim : $(\pi_i(a_m))_{m \in \mathbb{N}}$ converges in $[0,1]$ $\forall i \in \mathbb{N}$.

Suppose $(a_m)_{m \in \mathbb{N}} \to x = (x_1, x_2, \dots)$. Then $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ we have

$\sup\{ |x_j - \pi_j(a_m)| : j \in \mathbb{N} \} < \varepsilon \Rightarrow$

$|x_j - \pi_j(a_m)| < \varepsilon$ $\forall j$. Since $\varepsilon > 0$ is arbitrary,

$(\pi_j(a_m))_{m \in \mathbb{N}} \to x_j$ in X_j

Q6.15) Construct a metric on $\mathbb{R}$ in which the sequence $(1/n)$ of inverses of natural numbers converges to a limit other than 0.

Define d as follows:

For $x, y \neq 0, 1$, $d(x,y) = |x-y|$. For $x \neq 0, 1$
$d(x,1) = d(1,x) = |x|$, $d(x,0) = d(0,x) = |x-1|$
$d(0,1) = d(1,0) = 1$. ~~Check that~~ Also $d(0,0) =$
$d(1,1) = 0$. Check that triangle ineq. is satisfied in each case. Thus,

$$d(1/n, 1) = \left|\frac{1}{n}\right| = \frac{1}{n} \longrightarrow 0 \text{ in } \mathbb{R}. \quad \text{Thus}$$

$$\frac{1}{n} \longrightarrow 1 \text{ as } n \to \infty, \text{ in } \mathbb{R}.$$