MATH 48A - Homework I

Due: 14.04.2021

1. Let $F(x, y, z) \in \daleth[x, y, z]$ be a homogeneous polynomial of degree d.

    (a) Prove that the three partial derivatives of F are homogeneous polynomials of degree $d - 1$.

    (b) Prove that

    $$X\frac{\partial F}{\partial X} + Y\frac{\partial F}{\partial Y} + Z\frac{\partial F}{\partial Z} = d \cdot F(x, y, z).$$

    (Hint: differentiate $F(tx, ty, tz) = t^d F(x, y, z)$ with respect to t)

2. Let C be the projective curve given by the equation

    $$y^2 z - x^3 - z^3 = 0.$$

    (a) Show that C has only one point at infinity, namely the point $(0 : 1 : 0$ corresponding to the vertical direction $x = 0$.

    (b) Let C_0 given by $y^2 - x^3 - 1 = 0$ be the affine part of C in the chart $z \neq 0$, and let (r_i, s_i) be a sequence of points on C_0 with $r_i \to \infty$. Let L_i be the tangent line to C_0 at the point (r_i, s_i). Prove that as $i \to \infty$, the slopes of the lines L_i approach infinity, i.e., they approach the slope of the line $x = 0$.

3. Let P, P_1, P_2, P_3 be points in $\mathbb{P}^2$, and let L be a line in $\mathbb{P}^2$.

    (a) If P_1, P_2 and P_3 do not lie on aline, prove that ther is a porijective transformation of $\mathbb{P}^2$ so that

    $$P_1 \mapsto (0 : 0 : 1), P_2 \mapsto (0 : 1 : 0), P_3 \mapsto (1 : 0 : 0).$$

    (b) If no three of P_1, P_2, P_3 and P lie on a line, prove that there is a unique projective transformation as in (a) that also sends P to $(1 : 1 : 1)$.

    (c) Prove that there is a projective transformation of $\mathbb{P}^2$ so that L is sent to the line $z = 0$.

    (d) If P does not lie on L, prove that there is a projective transformation of $\mathbb{P}^2$ so that L is sent to the line $z = 0$ and P is sent to the point $(0 : 0 : 1)$.

4. Consider the affine curve C given by $y^4 - xy - x^3 = 0$. Show that at the origin $(x, y) = (0, 0)$, the curve C meets the y-axis four times, the x-axis three times, and every other line through the origin twice.

5. Find all singular points on each of the following curves. How does the picture look locally (union of lines) in each case?

   (a) $y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy = 0$

   (b) $x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1 = 0$

   (c) $y^2 + (x^2 - 5)(4x^4 - 20x^2 + 25) = 0.$

6. Let P be a point of order 2 on the curve C given by $F = 0$. Show that the curve looks locally like the intersection of two separate lines (a double node) if $F_{xy}(P)^2 \neq F_{xx}(P) \cdot F_{yy}(P)$.