Math 234

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October 16, 2020

Contents

1 Riemann integral

A **partition** P of $[a, b]$:

Lastly in this section we define (general) Riemann sums. If $P = \{x_0, \ldots, x_N\}$ is a partition of $[a, b]$ and t_j is <u>any</u> point in $[x_{j-1}, x_j]$,

$$\sum_{j=1}^{N} f(t_j)(x_j - x_{j-1})$$

is called a **Riemann sum** for f associated to the partition P. Clearly

$$m_j \leq f(t_j) \leq M_j$$

and so

$$s_P(f) \leq \sum_{j=1}^{N} f(t_j)(x_j - x_{j-1}) \leq S_P(f).$$

So, if $s_P(f)$ and $S_P(f)$ are close to $\int_a^b f(x)dx$, then any Riemann sum is close to $\int_a^b f(x)dx$. We will make this more qualitative in following sections.

2 Some criterias for Riemann integrability

Theorem 2.1 *If f is bounded and monotone on $[a, b]$, then f is integrable on $[a, b]$*

Proof: Suppose that f is monotone increasing, the other case is similar. Let $\epsilon > 0$. Let $P_k = \{x_0, x_1, \ldots, x_k\}$ where $x_0 = a$, $x_k = b$ and $x_j = a + j\left(\frac{b-a}{k}\right)$, $j = 1, \ldots, k-1$. Since f is increasing $m_j = f(x_{j-1})$ and $M_j = f(x_j)$ for $j = 1, \ldots, k$. So

$$s_{P_k}(f) = \sum_{j=0}^{k-1} \frac{b-a}{k} f(x_j) \quad \text{and} \quad S_{P_k}(f) = \sum_{j=1}^{k} \frac{b-a}{k} f(x_j).$$

Then

$$S_{P_k}(f) - s_{P_k}(f) = \frac{b-a}{k}(f(x_k) - f(x_0)) = \frac{b-a}{k}(f(b) - f(a)) < \epsilon$$

for large enough k, concluding the proof. $\square$

Theorem 2.2 *If f is continuous on $[a, b]$, then f is integrable on $[a, b]$*

Proof: Since f is continuous on $[a, b]$, it is uniformly continuous on the same interval. Letting $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{b-a}$ whenever $x, y \in [a, b]$ and $|x - y| < \delta$.

Let P be a partition of $[a, b]$ such that $\max_{1 \leq j \leq N} |x_j - x_{j-1}| < \delta$. Then

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}$$

whenever x, y are in same subinterval of the partition. Then

$$\begin{aligned} S_P(f) - s_P(f) &= \sum_{j=1}^{N} M_j(x_j - x_{j-1}) - \sum_{j=1}^{N} m_j(x_j - x_{j-1}) \\ &= \sum_{j=1}^{N} (M_j - m_j)(x_j - x_{j-1}) \\ &< \frac{\epsilon}{b-a} \sum_{j=1}^{N} (x_j - x_{j-1}) \\ &= \epsilon. \end{aligned}$$

Therefore, $f \in R[a, b]$ as asserted. □

Our next goal is to prove that functions on $[a, b]$ with "few" discontinuous are still integrable. For this purpose we need the concept of Jordan measurability.

Definition 2.1 *A set $Z \subset \mathbb{R}$ is said to have* ***zero content*** *if for any $\epsilon > 0$, there exist a finite collection of intervals $I_1, \ldots, I_N$ such that*

(i) $Z \subset \bigcup_{i=1}^{N} I_i$, *and*

(ii) $\sum_{i=1}^{N} |I_i| < \epsilon$, *where $|I_i|$ is the length of I_i.*

Example 2.1 *Any finite set has zero content.*

Example 2.2 *We claim that any convergent sequence x_n has zero content. To see this let $\epsilon > 0$, and assume that $x_n \to x \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x| < \frac{\epsilon}{4}$. Let $I_{N+1} = (x - \epsilon/4, x + \epsilon/4)$ and $I_j = (x_j - \epsilon/2^{j+1}, x_j + \epsilon/2^{j+1})$. Then it is easily seen that $\sum_{j=1}^{N+1} |I_j| < \epsilon$ and $\{x_n : n \geq 1\} \subset \bigcup_{j=1}^{N+1} I_j$. This proves our claim.*

Theorem 2.3 *If f is a bounded function on $[a, b]$ and if the set of points at which f is discontinuous has zero content, then $f \in R[a, b]$.*

Proof: As usual letting $\epsilon > 0$ we will find a partition P of $[a, b]$ such that $S_P(f) - s_P(f) < \epsilon$. Let

$$m = \inf\{f(x) : x \in [a, b]\} \quad \text{and} \quad M = \sup\{f(x) : x \in [a, b]\}$$

- which exist since f is bounded. Recall that the set of discontinuities of f has zero content and so we may choose a covering $\bigcup_{i=1}^{L} I_i$ for the set of discontinuities where I_i's are intervals and where $\sum_{i=1}^{L} |I_i| < \frac{\epsilon}{2(M-m)}$.

Let now

$$U = \bigcup_{i=1}^{L} I_i = \bigcup_{i=1}^{L} [\alpha_i, \beta_i],$$

and

$$V = [a, b] - U^{(int)} = [a, b] - \bigcup_{i=1}^{L} (\alpha_i, \beta_i).$$

So $|U| \leq \frac{\epsilon}{2(M-m)}$ and also V is a finite union of closed intervals on each of which f is continuous.

Next we let P be any partition of $[a,b]$ including endpoints of I_m's, $m = 1, \ldots, L$. Then

$$S_P(f) = S_P^U(f) + S_P^V(f)$$

and

$$s_P(f) = s_P^U(f) + s_P^V(f)$$

where $S_P^U(f)$ (resp. $S_P^V(f)$) is the sum of terms $M_j(x_j - x_{j-1})$ in $S_P(f)$ for which the interval $[x_{j-1}, x_j]$ is contained in U (resp., V), and likewise for $s_P^U(f)$ and $s_P^V(f)$.

Then since f is continuous on each closed interval forming V, we have

$$S_P^V(f) - s_P^V(f) < \frac{\epsilon}{2}$$

by choosing P sufficiently large. Also

$$S_P^U(f) - s_P^U(f) = \sum_{[x_{j-1},x_j]\subset U} (M_j - m_j)(x_j - x_{j-1}) \leq (M-m)|U| < (M-m)\frac{\epsilon}{2(M-m)} < \frac{\epsilon}{2}.$$

Thus

$$S_P(f) - s_P(f) = S_P^U(f) - s_P^U(f) + S_P^V(f) - s_P^V(f) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

The following corollary is now immediate.

Corollary 2.1 *If is a bounded function on $[a,b]$ that has finitely many discontinuities, then $f \in R[a,b]$.*

Exercise 2.1 *(Thomae's function) Is the function $f : [0,1] \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{q}, & \text{if } x \text{ is rational with } x = p/q,\ gcd(p,q) = 1 \end{cases}$$

Riemann integrable?

We conclude this section by a result related to modification of a function on a small set.

Proposition 2.1 *Suppose $f, g \in R[a,b]$ and $f(x) = g(x)$ for all but finitely many $x \in [a,b]$. Then*

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

Proof: Assume we have proved the result when one of the functions equals 0 for all $x \in [a,b]$. Then $f(x) - g(x) = (f-g)(x) = 0$ for all but finitely many x. Then via the previous corollary

$$0 = \int_a^b (f-g)(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

implying

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

Thus the general case follows if we show the case where one of the functions equals 0 for all $x \in [a, b]$.

So let's prove the case where g is identically 0. Then $f(x) = 0$ except some finitely many points $\{x_1, \ldots, x_N\}$. Let now P_k be the partition of $[a, b]$ into k equal subintervals, and take k large enough so that each x_j is in a distinct subinterval. Then

$$S_{P_k}(f) = \frac{b-a}{k} \sum_{j=1}^{n} \max\{f(x_j), 0\}$$

and

$$s_{P_k}(f) = \frac{b-a}{k} \sum_{j=1}^{n} \min\{f(x_j), 0\}.$$

So

$$S_{P_k}(f) - s_{P_k}(f) = \frac{b-a}{k} \sum_{j=1}^{n} (\max\{f(x_j), 0\} - \min\{f(x_j), 0\}) \to 0$$

as $k \to \infty$. Result follows. □

3 Fundamental theorem of calculus and related

Theorem 3.1 *(FTC form I) Let $f \in R[a, b]$. For $x \in [a, b]$, let $F(x) = \int_a^x f(t)dt$. Then (i) $F \in C[a, b]$. Also; (ii) if f is continuous at $x \in (a, b)$, then $F'(x)$ exists and equals $f(x)$.*

Proof: (i) If $x, y \in [a, b]$, then

$$F(y) - F(x) = \int_a^y f(t)dt - \int_a^x f(t)dt = \int_x^y f(t)dt.$$

Letting $C = \sup\{|f(t)| : t \in [a, b]\}$, we have

$$|F(y) - F(x)| \leq \int_x^y |f(t)|dt \leq C|y - x|.$$

So F is continuous (indeed it is uniformly continuous).

(ii) Assume that f is continuous at $x \in (a, b)$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ whenever $|y - x| < \delta$.

$$\begin{aligned} \frac{F(y) - F(x)}{y - x} - f(x) &= \frac{F(y) - F(x)}{y - x} - \frac{f(x)}{y - x}(y - x) \\ &= \frac{1}{y - x}\left(\int_x^y f(t)dt - \int_x^y f(x)dt\right) \\ &= \frac{1}{y - x}\left(\int_x^y (f(t) - f(x))dt\right). \end{aligned}$$

Let now $\epsilon > 0$i and choose δ as above. Then for any $0 < |y - x| < \delta$, we have

$$\left|\frac{F(y) - F(x)}{y - x} - f(x)\right| \leq \frac{1}{|y - x|} \int_{\min\{x,y\}}^{\max\{x,y\}} |f(t) - f(x)| dt < \frac{1}{|y - x|} \int_{\min\{x,y\}}^{\max\{x,y\}} \epsilon dt = \epsilon.$$

Thus

$$\lim_{y \to x} \frac{F(y) - F(x)}{y - x} = f(x),$$

that is $F'(x) = f(x)$. $\square$

Theorem 3.2 *(FTC form I) Let $F \in C[a, b]$ and assume it is differentiable except perhaps at finitely many points in (a, b), and let f be a function on $[a, b]$ that agrees with F' at all points where the latter is defined. If $f \in R[a, b]$, then*

$$\int_a^b f(t) dt = F(b) - F(a).$$

In other words, F is an antiderivative of f.

Proof: The idea is to find a partition P of $[a, b]$ such that

(1) $s_P(f) \leq F(b) - F(a) \leq S_P(f)$,

(2) $\int_a^b f(t) dt - S_P(f) < \frac{\epsilon}{3}$, and

(3) $S_P(f) - s_P(f) < \frac{\epsilon}{3}$.

Once such a partition is found, we have

$$\begin{aligned} \left|\int_a^b f(t) dt - (F(b) - F(a))\right| &\leq \left|\int_a^b f(t) dt - S_P(f)\right| + |S_P(f) - s_P(f)| + |s_P(f) - (F(b) - F(a))| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

and we are done.

So let's find a partition P satisfying (1)-(3). First, let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$ in which the points where f is not differentiable are among x_j's. Then for each j, F is continous on $[x_{j-1}, x_j]$ and is differentiable on (x_{j-1}, x_j), an so MVT says that there exists $t_j \in (x_{j-1}, x_j)$ such that

$$F(x_j) - F(x_{j-1}) = F'(t_j)(x_j - x_{j-1}) = f(t_j)(x_j - x_{j-1}).$$

Adding up these for varying j, we obtain

$$\begin{aligned} F(b) - F(a) &= F(x_N) - F(x_0) \\ &= \sum_{j=1}^{N} F(x_j) - F(x_{j-1}) \\ &= \sum_{j=1}^{N} f(t_j)(x_j - x_{j-1}) \end{aligned}$$

implying

$$s_P(f) \le F(b) - F(a) \le S_P(f).$$

So (1) is satisfied for P. We are now done since (2) and (3) can be satisfied as well by adding points to P properly since $f \in R[a,b]$. $\square$

Note 3.1 *(N1) Under regularity conditions we have seen that*

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Next we may ask

$$\frac{d}{dx}\int_a^{\phi(x)} f(t)dt$$

when $\phi(x)$ is a nice enough function. To answer this letting $F(x) = \int_a^x f(t)dt$, we are trying to find $\frac{d}{dx}F(\phi(x))$. But using the chain rule we then immediately obtain (under smoothness assumptions)

$$\frac{d}{dx}\int_a^{\phi(x)} f(t)dt = \frac{d}{dx}F(\phi(x)) = F'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x).$$

(N2) How about the derivative of

$$F(x) = \int_a^{\phi(x)} f(x,y)dy?$$

Later in these notes we will prove that

$$F'(x) = f(x,\phi(x))\phi'(x) - \int_a^{\phi(x)} \frac{\partial f}{\partial x}(x,y)dy.$$

4 Some further results in 1-D

Proposition 4.1 *Suppose $f \in R[a,b]$ and $|f(x)| \le C$ for each $x \in [a,b]$. Then given $\epsilon > 0$, there exists $\delta > 0$ such that if $P = \{x_0, \ldots, x_N\}$ is any partition of $[a,b]$ satisfying*

$$mesh(P) = \max_{1\le j\le N}(x_j - x_{j-1}) < \delta,$$

the sums $s_P(f)$ and $S_P(f)$ differ from $\int_a^b f(x)dx$ by at most ϵ. The same holds for any Riemann sum.[1]

Proof of this proposition will require the following lemma.

Lemma 4.1 *Suppose $f \in R[a,b]$ and $|f(x)| \le C$ for each $x \in [a,b]$. Let $P = \{x_0, \ldots, x_N\}$ be a partition of $[a,b]$ such that $mesh(P) < \delta$, and let P' be another partition of $[a,b]$ by adding L extra points to P. Then*

$$S_P(f) < S_{P'}(f) + 2CL\delta$$

and

$$s_P(f) > s_{P'}(f) + 2CL\delta.$$

[1] General Riemann sums were defined at the end of Section 1.

Proof: We prove the first inequality, the second one is similar. There are two cases for the summand corresponding to (x_{j-1}, x_j).

Case 1: If no extra point is added in (x_{j-1}, x_j), both sums contain the term $M_j(x_j - x_{j-1})$ where $M_j = \sup_{x \in [x_{j-1}, x_j]} f(x)$.

Case 2: If extra points are added, the term $M_j(x_j - x_{j-1})$ in $S_P(f)$ is replaced by a sum - call it R_j - of similar terms corresponding to the subinterval $[x_{j-1}, x_j]$. Now

$$|M_j(x_j - x_{j-1})| < C\delta$$

and similarly

$$|R_j| = \left| \sum M'_\ell (x_\ell - x_{\ell-1}) \right| \leq \sum |M'_\ell| (x_\ell - x_{\ell-1} < C\delta.$$

So

$$0 < M_j(x_j - x_{j-1}) - R_j < C\delta + C\delta = 2C\delta.$$

The total change from $S_P(f)$ to $S_{P'}(f)$ is the sum of these differences, of which there are at most L many, so it is less than $2CL\delta$. $\square$

Proof of Proposition 4.1. Note that it is enough to prove the result for $s_P(f)$ and $S_P(f)$, as all Riemann sums lie in between. We begin by considering a partition Q of $[a, b]$ satisfying

$$S_Q(f) < \int_a^b f(x)dx + \frac{\epsilon}{2}$$

and

$$s_Q(f) > \int_a^b f(x)dx - \frac{\epsilon}{2}.$$

Recall that $|f(x)| \leq C$ for each x, and let K be the number of subdivision points in Q. Suppose P is another partition satisfying $mesh(P) < \delta$ where $\delta < \frac{\epsilon}{4KC}$.

Then the partition $P \cup Q$ is obtained by adding at most K points to P. Previous lemma now yields

$$S_P(f) < S_{P\cup Q}(f) + 2KC\delta < S_{P\cup Q}(f) + \frac{\epsilon}{2} \leq S_Q(f) + \frac{\epsilon}{2} \leq \int_a^b f(x)dx + \epsilon.$$

Likewise

$$s_P(f) > S_{P\cup Q}(f) - 2KC\delta < s_{P\cup Q}(f) + \frac{\epsilon}{2} \geq s_Q(f) + \frac{\epsilon}{2} \geq \int_a^b f(x)dx - \epsilon.$$

Now,

$$S_P(f) < \int_a^b f(x)dx + \epsilon \quad \text{and} \quad s_P(f) > \int_a^b f(x)dx - \epsilon$$

imply

$$\epsilon > S_P(f) - \int_a^b f(x)dx > 0 \quad \text{and} \quad \epsilon > \int_a^b f(x)dx - S_P(f) > 0.$$

So both $S_P(f)$ and $s_P(f)$ are at most ϵ distance from $\int_a^b f(x)dx$. $\square$

5 Double integrals

A **rectangle** is a set of the form $R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], y \in [c, d]\}$. A partition of R is a subdivision of R into subrectangles

$$P = \{R_{jk} : [x_{j-1}, x_j] \times [y_{k-1}, y_k], j = 1, \ldots, N_1, k = 1, \ldots, N_2\}.$$

Here, R_{jk} has area ΔA_{jk}. (We may alternatively represent P as $P = \{x_0, \ldots, x_{N_1}; y_0, \ldots, y_{N_2}\}$ with $a = x_0 < \cdots < x_{N_1} = b$, $c = y_0 < \cdots < y_{N_2} = d$.)

Now given a partition P of R, set

$$m_{jk} = \inf\{f(x, y) : (x, y) \in R_{j,k}\}$$

and

$$m_{jk} = \sup\{f(x, y) : (x, y) \in R_{j,k}\}$$

and define lower and upper sums by

$$s_P(f) = \sum_{j=1}^{N_1} \sum_{j=1}^{N_2} m_{jk} \Delta A_{jk}$$

and

$$S_P(f) = \sum_{j=1}^{N_1} \sum_{j=1}^{N_2} M_{jk} \Delta A_{jk}.$$

Lower and upper integrals are then

$$\underline{I}_R(f) = \sup_P s_P(f)$$

and

$$\overline{I}_R(f) = \inf_P S_P(f).$$

When $\underline{I}_R(f) = \overline{I}_R(f)$, f is called **(Riemann) integrable** and the common value is the integral of f over R, denoted by

$$\int\int_R f dA \quad \text{or} \quad \int\int_R f(x, y) dx dy.$$

What if the region of integration is not a rectangle? To approach this problem, first for $S \subset \mathbb{R}^2$, let's define the characteristic function of S to be the function

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Now let $S \subset \mathbb{R}^2$ be a bounded set and $f : \mathbb{R}^2 \to \mathbb{R}$ be bounded as well. Let R be any rectangle with $S \subset R$. Then f is **integrable on** S if $f\chi_S$ is integrable on R, and in this case we define the integral of S to be

$$\int\int_S f dA = \int\int_R f \chi_S dA$$

I leave it as an exercise for you to show that the current definition satisfies standard properties of Riemann integration in one dimensional setting such as linearity, domination, etc.

6 Integrability in two dimensions

Let S be a bounded subset of $\mathbb{R}^2$, and R be a rectangle containing R. When we look at the double integral $\int\int_S = \int\int_R f\chi_S dA$, we should be concerned with the fact that $f\chi_S$ is not continuous on the boundary of S for integrability of f on S.

Definition 6.1 *$Z \subset \mathbb{R}^2$ is said to have zero content if for all $\epsilon > 0$, there exist a finite collection of rectangles $R_1, \ldots, R_M$ in mathbbR^2 such that*

(i) $Z \subset \bigcup_{i=1}^M R_i$, and

(ii) $\sum_{i=1}^M |R_i| < \epsilon$.

Now, for rectangular regions we have

Theorem 6.1 *Suppose f is a bounded function on a rectangle R. If the set of points in R at which f is discontinuous has zero content, then f is integrable on R.*

Proof of this theorem follows the same steps in case of $\mathbb{R}$, we just replace intervals by rectangles.

The more important issue here is finding an integrability condition on an arbitrary set $S \subset \mathbb{R}^2$. First a few more notes on zero content.

Proposition 6.1 *(i) If $Z \subset \mathbb{R}^2$ has zero content and $U \subset Z$, then U has zero content.*

(ii) If $Z_1, \ldots, Z_k$ have zero content, then so does $\bigcup_{j=1}^k Z_j$.

(iii) If $f : (a_0, b_0) \to \mathbb{R}^2$ is of class C^1, then $f([a, b])$ has zero content whenever $a_0 < a < b < b_0$.

Proof: (i) Just note that if $I_1, \ldots, I_M$ is a cover for Z, then it is also a cover for U.

(ii) Let $\epsilon > 0$. Let $I_{j_i}^{(i)}$, $i = 1, \ldots, k$, $j_i = 1, \ldots, N_i$ be such that

$$\bigcup_{\ell=1}^{N_i} I_\ell^{(i)} \supset Z_i$$

and

$$\sum_{\ell=1}^{N_i} |I_\ell^{(i)}| < \frac{\epsilon}{k}.$$

Then

$$\bigcup_{i=1}^{k} \bigcup_{\ell=1}^{N_i} I_\ell^{(i)} \supset \bigcup_{i=1}^{k} Z_i$$

and

$$\sum_{i=1}^{k} \sum_{\ell=1}^{N_i} |I_\ell^{(i)}| < \epsilon.$$

So $\{I_1^{(1)}, \ldots, I_{N_1}^{(1)}, \ldots, I_1^{(k)}, I_{N_k}^{(k)}\}$ satisfies the required condition for having zero content.

(iii) Let $P_k : a = t_0 < t_1 < \cdots < t_k = b$ be a partition of $[a,b]$ into k equal parts of length $\delta = \frac{b-a}{k}$. Set $C = \sup_{t\in[a,b]} \|f'(t)\|$. Be the mean value theorem applied to components of $f(t) = (x(t), y(t))$, we have

$$|x(t) - x(t_j)| \le C\delta \quad \text{and} \quad |y(t) - y(t_j)| \le C\delta$$

whenever $t \in [t_{j-1}, t_j]$. So

$$f([t_{j-1}, t_j]) \subset A,$$

where A is the square with the center at $f(t_j)$, and whose edge length iis given by $2C\delta$. Then $f([a,b])$ is contained in a union of these squares and the sum of their areas is

$$k(2C\delta)^2 = k4C^2\frac{(b-a)^2}{k^2} = \frac{4C^2(b-a)^2}{k}.$$

This can be made arbitrarily small by choosing k large enough. Therefore $f[a,b]$ has zero content as asserted. □

Exercise 6.1 *Let $f : [a,b] \to \mathbb{R}$ be in $R[a,b]$. Show that the graph of f in $\mathbb{R}^2$ has zero content.*

Recall that when f is bounded on a rectangle, f is integrable if the set of discontinuities of f has zero content. How about $\int\int_S f dA$ for an arbirary bounded set $S \subset \mathbb{R}^2$? Since

$$\int\int_S f dA = \int\int_R f\chi_S dA,$$

we will need to understand χ_S a bit more.

Lemma 6.1 *χ_S is discontinuous at x if and only if x is on boundary of S*

Proof: If $x \in S^{int}$, then χ_S equals 1 on some ball containing x, so it is continuous at x. Likewise, if $x \in (S^c)^{int}$, then χ_S is 0 near x, and so is continuous at x.

But if $x \in \partial S$, then there are points x_1, x_2 in any ball $B_r(x)$ with $\chi_S(x_1) = 0$ and $\chi_S(x_2) = 1$. So χ_S is not continuous at x. □

Based on our discussions so far, the integrability of f requires boundary of S to have zero content, and the set of discontinuities of f on S have zero content.

For a more precise treatment, we say that $S \subset \mathbb{R}^2$ is **Jordan measurable** if it is bounded and its boundary has zero content.

Theorem 6.2 *Let S be a Jordan measurable subset of $\mathbb{R}^2$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded and set of points in S at which f is discontinuous has zero content. Then f is integrable on S.*

Proof: Let R be a rectangle containing S. The set of points where $f\chi_S$ can be discontinuous are the points Z_1 which is boundary of S, and the set of points Z_2 where f is discontinuous on S. Now Z_1 has zero content since S is Jordan measurable and Z_2 has zero content by assumption. Then we know that $Z_1 \cup Z_2$, being a finite union, has zero content as well. □

Exercise 6.2 *Suppose $Z \subset \mathbb{R}^2$ has zero content. If $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded, then f is integrable on Z and $\int_Z f dA = 0$.*

Proposition 6.2 *Suppose f is integrable on the set $S \subset \mathbb{R}^2$. If $g(x) = f(x)$ except for x on a set Z of zero content, then*

$$\int\int_S g dA = \int\int_S f dA.$$

Proof: We have

$$\begin{aligned}
\int\int_S (g-f) dA &= \int\int_S g dA - \int\int_S f dA \\
&= \int\int_{S-Z} g dA + \int\int_Z g dA - \int\int_{S-Z} f dA - \int\int_Z f dA \\
&= \int\int_S (g-f) dA \\
&= 0.
\end{aligned}$$

where we used the last exercise in last step. We conclude

$$\int\int_S g dA = \int\int_S f dA = \int\int_S (g-f) dA = 0.$$

□

7 Multiple integrals

The discussion of integrals in dimensions higher than two is similar. We just replace rectangles by n-dimensional boxes:

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n].$$

The notation for n-dimensional integrals over a region $S \subset \mathbb{R}^n$ is

$$\int \cdots \int_S f dV^n = \int \cdots \int_S f(x) d^n x = \int \cdots \int_S f(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

Theorem 7.1 *(Mean value theorem for integrals) Let S be a compact, connected, measurable subset of $\mathbb{R}^n$. Let f, g be continuous functions on S with $g \geq 0$. Then there exists a $a \in S$ such that*

$$\int \cdots \int_S f(x) g(x) d^n x = f(a) \int \cdots \int_S g(x) d^n x.$$

Proof: Let $m = \inf\{f(x) : x \in S\}$ and $M = \sup\{f(x) : x \in S\}$. Now, $g \geq 0$ implies

$$mg \leq fg \leq Mg$$

on S. So

$$m \int \cdots \int_S g(x) d^n x \leq \int \cdots \int_S f(x) g(x) d^n x \leq M \int \cdots \int_S g(x) d^n x.$$

This says

$$\inf\{f(x) : x \in S\} = m \leq \frac{\int \cdots \int_S f(x)g(x)d^n x}{\int \cdots \int_S g(x)d^n} \leq M = \sup\{f(x) : x \in S\}.$$

Calling the ratio in middle C, we have $C \in [m, M]$. Noting that f is continuous, intermediate value theorem then yields the existence of $a \in S$ such that $f(a) = C$. That is

$$\frac{\int \cdots \int_S f(x)g(x)d^n x}{\int \cdots \int_S g(x)d^n} = f(a),$$

concluding the proof. $\square$

Taking $g = 1$, in previous theorem gives:

Corollary 7.1 *Let S be a compact, connected, measurable subset of $\mathbb{R}^n$. Let f be a continuous function on S Then there exists $a \in S$ such that*

$$\int \int_S f(x)d^n x = f(a)|S|$$

where $|S|$ is the n-dimensional volume of S.

Note 7.1 $\frac{\int \int_S f(x)d^n x}{|S|}$ *is called the mean value (or, average) of f on S. The corollary says that when f is continuous and S is compact and connected, there exists $a \in S$ at which the actual value of f is the average value.*

8 Iterated integrals and Fubini's theorem

We focus on $\mathbb{R}^2$. Consider the integral of f on a rectangle R. Given a partition $P = \{x_0, \ldots, x_J; y_0, \ldots, y_K\}$ of R, pick points $\tilde{x}_j \in [x_{j-1}, x_j]$ and $\tilde{y}_k \in [y_{k-1}, y_k]$ ($1 \leq j \leq J$, $1 \leq k \leq K$) and form the Riemann sum

$$\sum_{j=1}^{J} \sum_{k=1}^{K} f(\tilde{x}_j, \tilde{y}_k)\Delta x_j \Delta y_k,$$

where $\Delta x_j = x_j - x_{j-1}$ and $\Delta y_k = y_k - y_{k-1}$. If f is integrable on R, then we have

$$\int \int_R f(x, y)dA \approx \sum_{j=1}^{J} \sum_{k=1}^{K} f(\tilde{x}_j, \tilde{y}_k)\Delta x_j \Delta y_k,$$

that is, the integral is approximated by the sum. Also, for fixed y, $\sum_{j=1}^{J} f(\tilde{x}_j, y)\Delta x_j$ is a Riemann sum for $\int_a^b f(x,y)dx$ and $\sum_{k=1}^{K} g(\tilde{y}_k)\Delta y_k$ is a Riemann sum for $\int_c^d g(y)dy$. So

$$\begin{aligned}
\int\int_R f(x,y)dA &\approx \sum_{j=1}^{J}\sum_{k=1}^{K} f(\tilde{x}_j, \tilde{y}_k)\Delta x_j \Delta y_k \\
&= \sum_{k=1}^{K}\sum_{j=1}^{J} f(\tilde{x}_j, \tilde{y}_k)\Delta x_j \Delta y_k \\
&\approx \sum_{k=1}^{K} \int_a^b f(x, \tilde{y}_k)dx\Delta y_k \\
&\approx \int_c \int_d f(x,y)dydx.
\end{aligned}$$

Theorem 8.1 *(Fubini's theorem) Let $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$ and let f be integrable on R. Suppose that for each $y \in [c,d]$, the function f_y defined by $f_y(x)$ is integrable on $[a,b]$ and the function $g(y) = \int_a^b f(x,y)dx$ is integrable on $[c,d]$. Then*

$$\int\int_R f(x,y)dA = \int_c^d \left(\int_a^b f(x,y)dx\right) dy.$$

Similarly, if for each $x \in [a,b]$, $f^x(y) = f(x,y)$ is integrable on $[a,b]$, and $h(x) = \int_c^d f(x,y)dy$ is integrable on $[a,b]$, then

$$\int\int_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy\right) dx.$$

In particular, the iterated integrals coincide.

Proof requires two lemmas. We already proved the following in Section 4:

Lemma 8.1 *Suppose $f \in R[a,b]$. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $P = \{x_0, \ldots, x_N\}$ is any partition of $[a,b]$ satisfying*

$$\max_{1\leq j\leq N}\{x_j - x_{j-1}\} < \delta,$$

then any Riemann sum $\sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$ differs from $\int_a^b f(x)dx$ by at most ϵ.

Second lemma has a similar proof, adapt it yourself.

Lemma 8.2 *Suppose f is integrable on the rectangle $R = [a_1, b_1] \times [a_2, b_2]$. Given $\epsilon > 0$, there exists $\delta > 0$ such that if*

$$P = \{x_{10}, \ldots, x_{1N}; x_{20}, \ldots, x_{2N}\}$$

is any partition of R satisfying

$$\max\{1 \leq i \leq 2\} \max_{1\leq j\leq N_i}\{x_{ij} - x_{i(j-1)}\} < \delta,$$

then any Riemann sum for f associated to R differs from $\int\int_R f dA$ by at most ϵ.

Proof of Fubini's theorem. Let's prove $\int\int_R f(x,y)dA = \int_c^d \left(\int_a^b f(x,y)dx\right) dy$, the other one is similar. Let $P_{JK} = \{x_0, x_1, \ldots, x_J; y_0, y_1, \ldots, y_K\}$ be the partition of $[a,b]$ and $[c,d]$ respectively into J and K equal subintervals of length $\Delta x = \frac{b-a}{J}$ and $\Delta y = \frac{d-c}{K}$.

Since f is integrable, letting $\epsilon > 0$, there exists $N' \in \mathbb{N}$ such that

$$\left|\int\int_R f(x,y)dA - \sum_{j=1}^{J}\sum_{k=1}^{K} f(x_j, y_k)\Delta x_j \Delta y_k\right| < \frac{\epsilon}{3}, \tag{1}$$

provide that $\min\{J, K\} \geq N'$ by Lemma 8.2.

Also, using Lemma 8.1 on the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(y) = \int_a^b f(x,y)dx$ (which we assumed to be integrable on $[c,d]$)

$$\left|\int_c^d g(y)dy - \sum_{k=1}^{K} g(y_k)\Delta y\right| < \frac{\epsilon}{3},$$

or equivalently

$$\left|\int_c^d \left(\int_a^b f(x,y)dx\right) dy - \sum_{k=1}^{K}\left(\int_a^b f(x,y_k)dx\right)\Delta y\right| < \frac{\epsilon}{3}, \tag{2}$$

provided $K \geq N''$ for some $N'' \in \mathbb{N}$.

Let $N = \max\{N', N''\}$, and fix $K = N$ so that y_k's are also fixed. Now again by Lemma 8.1, we can choose J large enough so that

$$\left|\int_a^b f(x,y_k)dx - \sum_{j=1}^{J} f(x_j, y_k)\Delta x\right| < \frac{\epsilon}{3(d-c)} \tag{3}$$

for all $k = 1, \ldots, K$ (Here, actually $J = \max\{J_1, \ldots, J_K\}$ where J_i depends on y_i.) Then

$$\begin{aligned}
\left|\sum_{j=1}^{J}\sum_{k=1}^{K} f(x_j,y_k)\Delta x\Delta y - \sum_{k=1}^{K}\int_a^b f(x,y_k)dx\Delta y\right| &\leq \sum_{k=1}^{K}\left|\sum_{j=1}^{J} f(x_j,y_k)\Delta x - \int_a^b f(x,y_k)dx\right|\Delta y \\
&\underbrace{<}_{(3)} \sum_{k=1}^{K}\frac{\epsilon}{3(d-c)}\Delta y \\
&= \frac{K\epsilon}{3(d-c)}\frac{d-c}{K} \\
&= \frac{\epsilon}{3}.
\end{aligned} \tag{4}$$

Therefore

$$\begin{aligned}
\left|\int\int_R f(x,y)dA - \int_c^d \int_a^b f(x,y)dxdy\right| &\leq \left|\int\int_R f(x,y)dA - \sum_{j=1}^J \sum_{k=1}^K f(x_j,y_k)\Delta x_j \Delta y_k\right| \\
&\quad + \left|\sum_{j=1}^J \sum_{k=1}^K f(x_j,y_k)\Delta x \Delta y - \sum_{k=1}^K \int_a^b f(x,y_k)dx\Delta y\right| \\
&\quad + \left|\sum_{k=1}^K \left(\int_a^b f(x,y_k)dx\right)\Delta y - \int_c^d \left(\int_a^b f(x,y)dx\right)dy\right| \\
&< \underbrace{\frac{\epsilon}{3}}_{(1)} + \underbrace{\frac{\epsilon}{3}}_{(4)} + \underbrace{\frac{\epsilon}{3}}_{(2)} \\
&= \epsilon.
\end{aligned}$$

$\square$

For arbitrary $S \subset \mathbb{R}^2$, the limits of integration need to be adjusted. For example, if

$$S = \{(x,y) : a \leq x \leq b, \phi(x) \leq y \leq \psi(x)\},$$

then

$$\int\int_S f(x,y)dA = \int_a^b \left(\int_{\phi(x)}^{\psi(x)} f(x,y)dy\right)dx.$$

Example 8.1 *Let S be the region between the parabolas $x = 4 - y^2$ and $x = 4 - y^2$. Then there are two ways two interpret $\int\int_S f(x,y)dA$:*

$$\int\int_S f(x,y)dA = \int_{-2}^2 \int_{y^2-4}^{4-y^2} f(x,y)dxdy,$$

or

$$\int\int_S f(x,y)dA = \int_{-4}^0 \int_{-\sqrt{4+x}}^{\sqrt{4+x}} f(x,y)dydx + \int_0^4 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} f(x,y)dydx.$$

The ideas in higher dimensions are entirely similar. Here is a typical situation in three dimensions. Region of integration S is between two graphs:

$$S = \{(x,y,z) : (x,y) \in U, \phi(x,y) \leq\leq z \leq \psi(x,y)\},$$

based on some region U in $x - y$ plane

$$U = \{(x,y) : a \leq x \leq b : \sigma(x) \leq y \leq \tau(x)\}.$$

We then have

$$\int\int\int_S f dV = \int_a^b \int_{\sigma(x)}^{\tau(x)} \int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)dzdydx.$$

Example 8.2 *Find the mass of the tetrahedron T formed by the three coordinate planes and the plane $x + y + 2z = 2$ if the density is $\rho(x,y,z) = e^{-z}$.*

Solution. The mass is given by

$$\int\int\int_T e^{-z}dV = \int_0^2\int_0^{2-x}\int_0^{1-\frac{x+y}{2}} e^{-z}dzdydx = \cdot = 2-4e^{-1}.$$

□

Example 8.3 *(The order which you try integrate may matter a lot) Find*

$$\int\int_R \frac{\sin x}{x}dA,$$

where R is the triangle with corners at $(0,0),(1,0)$ and $(1,1)$.

Solution. We have

$$\int\int_R \frac{\sin x}{x}dA = \int_0^1\int_0^x \frac{\sin x}{x}dydx = \int_0^1 \sin x dx = 1-\cos 1.$$

□

Example 8.4 *Find $\int_0^2\int_{y/2}^1 ye^{-x^3}dxdy$.*

Solution. Note that e^{-x^3} has no anti-derivative, so we use Fubini's theorem to interchange the integrals:

$$\int_0^2\int_{y/2}^1 ye^{-x^3}dxdy = \int_0^1\int_0^{2x} = \int_0^1 2x^2e^{-x^3}dx = \cdots = \frac{2}{3}\left(1-\frac{1}{e}\right).$$

□

9 Change of variables for multiple integrals

Recall that if g is a one-to-one C^1 function on $[a,b]$, then for any $f\in C[a,b]$,

$$\int_a^b f(g(u))g'(u)du = \int_{g(a)}^{g(b)} f(x)dx. \tag{5}$$

That is, we do the substitution $x=g(u)$.

Note 9.1 *If g is decreasing, then the endpoints on right are in "wrong" order, and we can make it in "right" order by introducing a minus sign: $\int_{g(a)}^{g(b)} = -\int_{g(b)}^{g(a)}$. Since g is increasing or decreasing according as $g'>0$ or $g'<0$, we may rewrite (5) as*

$$\int_{[a,b]} f(g(u))|g'(u)|du = \int_{g([a,b])} f(x)dx. \tag{6}$$

Integrations here are from left-end point to the right-end point of the domain of integration.

Setting $I = g([a,b])$, we have $[a,b] = g^{-1}(I)$ and so (6) is saying

$$\int_I f(x)dx = \int_{g^{-1}(I)} f(g(u))|g'(u)|du.$$

More generally, in given dimension n, let G be a one-to-one transform from $R \subset \mathbb{R}^n$ to $S \subset G(R) \subset \mathbb{R}^n$. Then $R = G^{-1}(S)$, and the formula we are looking for should be something like

$$\int \cdots \int_S f(x)d^n x = \int \cdots \int_{G^{-1}(S)} f(G(u))[???]d^n u.$$

Theorem 9.1

Proof:

$\square$

A more general result which we are not able to prove at this level is:

Theorem 9.2 *Given open sets U and V in $\mathbb{R}^n$, let $G : U \to V$ be a one-to-one map of class C^1 whose derivative $DG(u)$ is invertible for all $u \in U$. Suppose $T \subset U$ and $S \subset V$ are measurable sets such that $G(T) = S$. If f is integrable on S, then $f \circ G$ is measurable on T, and*

$$\int \cdots \int_S f(x)d^n x = \int \cdots \int_T f(G(u))|\det(DG(u))||d^n u.$$

Example 9.1 *When $G(u) = Au$ with some invertible A, $DG(u) = A$ for each u. Then*

$$|\det DG(u)| = |\det A|$$

and so we recover our previous theorem - the linear transformations case.

Example 9.2 *(polar coordinates) Here, $G(r,\theta) = (x,y) = (x(r,\theta), y(r,\theta))$ where*

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Then

$$DG(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial x}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta, \end{pmatrix}$$

giving $|\det DG(r,\theta)| = r$. So

$$\int\int_S f(x,y)dxdy = \int\int_{G^{-1}(S)} f(r\cos\theta, r\sin\theta)rdtd\theta.$$

Here, $G^{-1}(S)$ is the representation of S in polar coordinates.

Example 9.3 *Find volume of the region S above the surface $z = x^2+y^2$ and below the plane $z = 4$.*

Solution. The volume is given by

$$V = \int\int_R (4 - x^2 - y^2 dxdy),$$

where $R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$. Changing to polar coordinates

$$V = \int_0^2 \int_0^{2\pi} (4 - r^2) r d\theta dr = \cdots = 8\pi.$$

□

Example 9.4 *(Cylindirical coordinates in 3-D) This time* $G(r, \theta, z) = (r\cos\theta, r\sin\theta, z) = (x(r, \theta, z), y(r, \theta, z), z(r, \theta, z))$. *An easy computation then shows that* $DG(r, \theta, z) = r$ *and so*

$$\int\int\int_S f(x, y, z) dxdydz = \int\int\int_{G^{-1}(S)} f(r\cos\theta, r\sin\theta, z) r dt d\theta dz.$$

Example 9.5 *(Spherical coordinates in 3-D) Letting* $G(r, \phi, \theta) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta)$, *we have* $\det DG(r, \phi, \theta) = r^2 \sin\phi$. *So in this case*

$$\int\int\int_S f(x, y, z) dxdydz = \int\int\int_{G^{-1}(S)} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\theta) r^2 \sin\phi dr d\phi d\theta.$$

Example 9.6 *Let* P *be the parellogram bounded by the lines* $x - y = 0$, $x + 2y = 0$, $x - y = 1$, $x + 2y = 6$. *Find* $\int\int_P xy dA$.

Solution. Let $u = x - y$, $v = x + 2y$. Then $x = \frac{1}{3}(2u+)v$ and $y = \frac{1}{3}(v - u)$ so that

$$G(u, v) = \begin{pmatrix} \frac{1}{3}(2u + v) & \frac{1}{3}(v - u) \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We conclude

$$DG(u, v) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

and

$$|\det DG(u, v)| = \frac{2}{9} + \frac{1}{9} = \frac{1}{3}.$$

Therefore,

$$\int\int_P xy dA = \frac{1}{3} \int_0^1 \int_0^6 \left(\frac{2u + v}{3} \frac{v - u}{3} \right) dv du = \cdots = \frac{77}{27}.$$

□

10 Functions defined by integrals

Let $f(x, y)$ be a real valued function defined on $\mathbb{R}^m \times \mathbb{R}^n$ (So, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$). Assume that $f(x, y)$ is integrable on $S \subset \mathbb{R}^n$ as a function of y for each x. Define

$$F(x) = \int \cdots \int_S f(x, y) d^n y.$$

Assuming that $\lim_{x \to a} f(x, y) = g(y)$ for all $y \in S$, does the following hold

$$\lim_{x \to a} F(x) \lim_{x \to a} \int \cdots \int_S f(x, y) d^n y =_? \int \cdots \int_S g(y) d^n y.$$

The answer in general is no. Let's see an example.

Example 10.1 *Let*

$$f(x, y) = \begin{cases} \frac{x^2 y}{(x^2+y^2)^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if if } (x, y) = (0, 0). \end{cases}$$

First, let's show that $\lim_{x \to 0} f(x, y) = 0$ *for any* y*. When* $y = 0$*, we have*

$$\lim_{x \to 0} f(x, y) = \lim_{x \to 0, y=0} \frac{0}{x^4} = 0.$$

Also, when $y \neq 0$*, we have*

$$0 \leq \left| \frac{x^2 y}{(x^2 + y^2)^2} \right| \leq \left| \frac{x^2 y}{y^4} \right| = \left| \frac{x^2}{y^3} \right| \to 0, \quad \text{as } x \to 0.$$

So, $\lim_{x \to 0} f(x, y) = 0$ *in either case.*

Also

$$\int_0^1 \frac{x^2 y}{(x^2 + y^2)^2} dy = \cdots = -\frac{x^2}{2(x^2 + y^2)} \Big|_{y=0}^{y=1} = \frac{1}{2(1 + x^2)} \to \frac{1}{2}, \quad \text{as } x \to 0.$$

We conclude that

$$\frac{1}{2} = \lim_{x \to 0} \int_0^1 \frac{x^2 y}{(x^2 + y^2)^2} dy \neq \int_0^1 \lim_{x \to 0} \frac{x^2 y}{(x^2 + y^2)^2} dy = 0.$$

Now, it is time to see a positive result.

Theorem 10.1 *Let* S, T *be compact subsets of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively, and assume that* S *is measurable. If* $f(x, y)$ *is continuous on the set* $T \times S = \{(x, y) : x \in T, y \in S\}$*, then*

$$F(x) = \int \cdots \int_S f(x, y) d^n y$$

is continuous on T*.*

Proof: Letting $\epsilon > 0$, we would like to find $\delta > 0$ such that $|F(x) - F(x')| < \epsilon$ whenever $\|x - x'\| < \delta$. Now, since S and T are compact, $S \times T$ is compact. So f, being continuous on the compact set $S \times T$, is uniformly continuous. So there exists $\delta > 0$ such that $|f(x, y) - f(x', y)| < \frac{\epsilon}{|S|}$, where $y \in S$, $x, x' \in T$, $\|x - x'\| < \delta$ and where $|S|$ is the volume of S. Then

$$|F(x) - F(x')| \leq \int \cdots \int_S |f(x, y) - f(x', y)| d^n y < \int \cdots \int_S \frac{\epsilon}{|S|} d^n y = \epsilon.$$

$\square$

Next question is the validity of the equation

$$\frac{\partial}{\partial x} \int \int_S f(x, y) dy = \int \int_S \frac{\partial}{\partial x} f(x, y) dy$$

under certain assumptions.

Theorem 10.2 *Suppose $S \subset \mathbb{R}^n$ is compact and measurable, $T \subset \mathbb{R}^m$ is open. If f is a continuous function on $T \times S$ that is of class C^1 as a function of $x \in T$ and $y \in S$, then the function*

$$F(x) = \int \cdots \int_S f(x, y) d^n y$$

is C^1 on T and

$$\frac{\partial F}{\partial x_j}(x) = \int \cdots \int_S \frac{f}{\partial x_j}(x, y) d^n y, \quad x \in T, \ j = 1, \ldots, m.$$

Proof: We prove the case $n = m = 1$ and write $x_1 = x$, $y_1 = y$ (More general case is similar with more notation).

Let $x_0 \in T$. Let $r > 0$ such that $x \in T$ whenever $|x - x_0| \leq 2r$. We claim that F is C^1 on $B(r, x_0)$ (and, in particular, F is C^1 on T since $x_0 \in T$ is arbitarary). For this purpose, consider $x \in T$, $h \in \mathbb{R}$ such that $|x - x_0| \leq r$ and $0 < |h| \leq r$. By the mean value theorem we know that

$$f(x + h, y) - f(x, y) = h f_x(x + th, y),$$

for some $t \in [0, 1]$ (depending on x, h, y). Then

$$\begin{aligned}
\frac{F(x+h) - F(x)}{h} - \int \int_S f_x(x, y) dy &= \int \int_S \frac{f(x+h) - f(x)}{h} dy - \int \int_S f_x(x, y) dy \\
&= \int \int_S f_x(x + th, y) dy - \int \int_S f_x(x, y) dy \\
&= \int \int_S (f_x(x + th, y) f_x(x, y)) dy.
\end{aligned}$$

Now, since $f \in C^1$ on $T \times S$, f_x is continuous on the compact set $\overline{B(2r, x_0)} \times S$, and therefore f_x is uniformly continuous on the same set. Then, letting $\epsilon < 0$, we may choose $0 < \delta < r$ such that

$$|f_x(x + th, y) - f_x(x, y)| < \frac{\epsilon}{|S|},$$

where again $|S|$ is the volume of S. Combining this with our observation above we obtain

$$\left|\frac{F(x+h)-F(x)}{h} - \int\int_S f_x(x,y)dy\right| \leq \int\int_S |f_x(x+th,y)-f_x(x,y)|dy < \frac{\epsilon}{|S|}\int\int_S dy = \epsilon.$$

We conclude that

$$\lim_{h\to} \frac{F(x+h)-F(x)}{h} = \int\int_S f_x(x,y)dy$$

as required.

The claim that the derivative is continuous follows from the previous theorem. $\square$

Next we discuss interchange of limits and integration. First a definition.

Definition 10.1 *A sequence of functions $\{f_n(x) : n \geq 1\}$ defined on $S \subset \mathbb{R}^n$ is said to **converge uniformly** on S to the function $f(x)$ if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|f_n(x) - f(x))| < \epsilon, \quad \textit{whenever } n \geq N,\ x \in S.$$

This is denoted by $f_n \to_u f$.

Theorem 10.3 *Suppose S is a measurable subset of $\mathbb{R}^n$, and f_n is a sequence of integrable functions on S converging uniformly to an integrable function f on S. Then*

$$\int\cdots\int_S f(x)d^nx = \lim_{k\to\infty}\int\cdots\int_S f_k(x)d^nx.$$

Proof: Let $\epsilon > 0$. Choose N large enough so that $|f_n(x) - f(x)| < \frac{\epsilon}{|S|}$ for all $n \geq N$ and $x \in S$. Then

$$\left|\int\cdots\int_S f_k(x)d^nx - \int\cdots\int_S f(x)d^nx\right| \leq \int\cdots\int_S |f_k(x)-f(x)|\,d^nx < \frac{\epsilon}{|S|} = \epsilon.$$

$\square$

Theorem 10.4 *(Bounded convergence theorem) Suppose S is a measurable subset of $\mathbb{R}^n$ and f_j be a sequence of integrable functions on S. Assume that $f_j(x) \to f(x)$ for each $x \in S$, where f is an integrable function, and that there exists some $C \in \mathbb{R}$ such that $|f_j(x)| \leq C$ for each $x \in S$ and $j \in \mathbb{N}$. Then*

$$\int\cdots\int_S f(x)d^nx = \lim_{j\to\infty}\int\cdots\int_S f_j(x)d^nx.$$

We are not able to prove this result at this level. Let's just note this is a fundamental result in analysis, and let's see an application of it. Recall from our discussion on fundamental theorem of calculus that, under smoothness assumptions we had

$$\frac{d}{dx}\int_a^{\phi(x)} f(y)dy = f(\phi(x))\phi'(x).$$

Our goal now is to understand

$$\frac{d}{dx}\int_a^{\phi(x)} f(x,y)dy.$$

Theorem 10.5 *Under smoothness assumptions*

$$\frac{d}{dx}\int_a^{\phi(x)} f(x,y)dy = f(x,\phi(x))\phi'(x) + \int_a^{\phi(x)} \frac{\partial f}{\partial x}(x,y)dy.$$

Proof:

□

11 Series: basics

An infinite series is an expression of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots .$$

In this course, $a_n \in \mathbb{R}$ for each n. Given a_n, $n \geq 0$, setting

$$s_0 = a_0, \quad s_1 = a_0 + a_1, \quad \cdots \quad s_k = a_0 + a_1 + \cdots + a_k,\ k \geq 1,$$

s_k is called the k^{th} partial sum of the series. We say $\sum_{n=0}^{\infty} a_n$ converges to S if s_k converges to S. Otherwise we say $\sum_{n=0}^{\infty} a_n$ diverges.

Theorem 11.1 *If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent with respective sums S and T, then $\sum_{n=0}^{\infty}(a_n + b_n)$ converges to $S+T$.*

Proof: Let $s_n = \sum_{k=0}^{n} a_k$ and $t_n = \sum_{k=0}^{n} b_k$ so that $s_n \to S$ and $t_n \to T$ as $n \to \infty$. Letting $\epsilon > 0$, there exist then $N_1, N_2 \in \mathbb{N}$ such that

$$|s_n - S| < \epsilon/2, \qquad n \geq N_1,$$

and

$$|t_n - T| < \epsilon/2, \qquad n \geq N_2.$$

Choosing $N = \max\{N_1, N_2\}$, for $n \geq N$, we then have

$$|s_n + t_n - (S+T)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Result follows. □

Exercise 11.1 *If $\sum_{n=0}^{\infty} a_n$ converges to S, then for $c \in \mathbb{R}$, $\sum_{n=0}^{\infty} ca_n \to cS$.*

Theorem 11.2 *(n^{th} term test) (i) If $\sum_{n=0}^{\infty} a_n$ is convergent (say, to S), then $\lim_{n\to\infty} a_n = 0$.*
(ii) If a_n does not converge to 0, as $n \to \infty$, then $\sum_{n=0}^{\infty} a_n$ is divergent.

Proof: (i) Letting $s_n \sum_{k=0}^{n} a_k$, we have $a_n = s_n - s_{n-1}$. Sending $n \to \infty$ on both sides, we obtain $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = S - S = 0$.
(ii) Contrapositive of (i) . □

Example 11.1 $\sum_{n=0}^{\infty} \left(1 - \frac{1}{n}\right)^n$ *is divergent since* $\left(1 - \frac{1}{n}\right)^n \to e^{-1} \neq 0$.

We conclude this section with two reminders from calculus:

Note 11.1 *(N1) A series of the form* $\sum_{k=0}^{\infty} x^k$ *is called a* ***geometric series****. Letting*

$$s_n = 1 + x + \cdots + x^n,$$

$$xs_n = x + x^2 + \cdots + x^n + x^{n+1}.$$

Subtracting the second line of equations from the first one, we obtain $(1-x)s_n = 1 - x^{n+1}$*, or*

$$s_n = \frac{1 - x^{n+1}}{1-x}.$$

Observe that when $|x| < 1$*,* $s_n \to \frac{1}{1-x}$*, and when* $|x| \geq 1$*,* s_n *diverges by the* n^{th} *term test.*

(N2) Suppose $a_0 = b_0$*,* $a_n = b_n - b_{n-1}$*,* $n \geq 1$*. In this case* $\sum a_n$ *is called a telescoping series. Here, observe that*

$$\begin{aligned} s_n &= a_0 + a_1 + \cdots + a_n \\ &= b_0 + (b_1 - b_0) + \cdots + (b_n - b_1) \\ &= b_n. \end{aligned}$$

So, $\sum_{n=0}^{\infty} a_n$ *converges if and only if* b_n *converges. When it does* $\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} b_k$*.*

Example 11.2 *We have*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = 1.$$

12 A brief discussion on infinite products

Let a_n be a sequence of real numbers. The infinite product $\prod_{n=1}^{\infty} a_n$ converges to $p \in \mathbb{R}$ if $\lim_{k\to\infty} \prod_{n=1}^{k} a_k = p$.

See Piazza for more challenging problems on infinite products

Example 12.1 *(E1)* $\prod_{n=1}^{\infty} \frac{1}{n} = 0$

(E2) $\prod_{n=1}^{\infty} \frac{n^2-1}{n^2} = \frac{1}{2}$*.*

Exercise 12.1 *Prove the claims in previous example.*

I would just like to show one result on infinite products here. Before the statement just remember that $e^x \geq 1 + x$ for any $x \in \mathbb{R}$; proof just uses basic calculus.

Theorem 12.1 *Let* b_n*,* $n \geq 1$ *be a sequence of real numbers such that* $b_n \geq 1$ *for each* n*. Let* $a_n = b_n - 1$ *so that* $a_n \geq 0$ *for all* n*. If* $\sum_{n=1}^{\infty} a_n$ *i s convergent, then* $\prod_{n=1}^{\infty}(1 + a_n)$ *is convergent.*

Proof: Let $p_N = \prod_{i=1}^N b_i$ and recall that $1 + x \leq \exp(x)$. We have

$$\begin{aligned} p_N = \prod_{i=1}^N b_i = p_N = \prod_{i=1}^N (1+a_i) &\leq p_N = \prod_{i=1}^N \exp(a_i) \\ &= \exp\left(\sum_{i=1}^N a_i\right) \\ &\leq \exp\left(\sum_{i=1}^\infty a_i\right) \\ &< \infty. \end{aligned}$$

So, the sequence P_N is bounded above. Also noting that $p_N > 0$ and that $b_N \geq 1$ for each N, we have

$$p_N = b_N p_{N-1} \geq p_{N-1},$$

that is, p_N is non-decreasing. Therefore, $p_N = \prod_{i=1}^N b_i$ converges by monotone sequence theorem. $\square$

13 Use of Taylor's theorem

Recall that when $f \in C^k(-c, c)$, $c > 0$,

$$f(x) = \underbrace{f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \cdots + f^{(k)}(0)\frac{x^k}{k!}}_{k\text{th order Taylor polynomial}} + R_k(x), \qquad |x| < c$$

Now if $R_k(x) \to 0$ as $k \to \infty$, we get the Taylor series

$$f(x) = \sum_{n=0}^\infty f^{(n)}(0)\frac{x^n}{n!}.$$

One more recall: assuming that $f \in C^{k+1}(I)$ where I is some open interval, for $a \in I$,

$$R_{a,k}(h) = \frac{h^{k+1}}{k!}\int_0^1 (1-t)^k f^{(k+1)}(a+th)dt.$$

Taking $I = (-c, c)$, we have

$$|R_{0,k}(x)| \leq \sup_{|t|\leq|x|} |f^{(k+1)}(t)|\frac{|x|^{k+1}}{(k+1)!}, \qquad |x| < c.$$

Theorem 13.1 *Let $f \in C^\infty(-c, c)$, $c \in (0, \infty]$.*

(i) If there exist positive a, b such that $|f^{(k)}(x)| \leq ab^k k!$ from each x with $|x| < c$ and $k \geq 0$, then

$$f(x) = \sum_{n=0}^\infty f^{(n)}(0)\frac{x^n}{n!}$$

for all x with $|x| < \min(c, 1/b)$.

(ii) If there exist positive A, B such that $|f^{(k)}(x)| \leq AB^k$ for all x with $|x| < c$ and $k \geq 0$, then

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

for all x with $|x| < c$.

Proof: As already noted above, idea of the proof is to show that $R_{0,k}(x) \to 0$ as $k \to \infty$.

(i) Let $|x| < \min(c, 1/b)$. We have

$$\begin{aligned}
|R_{0,k-1}(x)| &\leq \sup_{|t| \leq |x|} |f^{(k)}(t)| \frac{|x|^k}{k!} \\
&\leq \sup_{|t| \leq |x|} \frac{abk!}{k!} |x|^k \\
&= a|bx|^k \\
&\leq ac_x^k \qquad (\text{where } 0 < c_x = b|x| < 1 \text{ since } |x| < 1/b) \\
&\to 0
\end{aligned}$$

as $k \to \infty$.

(ii) Recall first that earlier we proved that $\frac{M^k}{k!} \to 0$ as $k \to \infty$ for any $M > 0$. This implies that $\frac{A(B/b)^k}{k!} \to 0$ as $k \to \infty$ for any $A, B, b > 0$. Letting

$$a = \max_{k \geq 1} \{ \frac{A(B/b)^k}{k!} \},$$

we have

$$AB^k = \left(\frac{A(B/b)^k}{k!} \right) \leq ab^k k!.$$

So the assumption that $|f^{(k)}(x)| \leq AB^k$ implies $|f^{(k)}(x)| \leq ab^k k!$ for any $b > 0$ and $a = a(b) > 0$. Result now follows from (i). $\square$

Example 13.1 *Let $f(x) = \cos x$. Then $f^{(k)}(x) = \pm \cos x$ or $f^{(k)}(x) = \pm \sin x$. So $\|f^{(k)}(x)\|_\infty \leq 1$ for any k. Using (ii) of previous theorem with $A = B = 1$, we conclude*

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Example 13.2 *Let $f(x) = e^x$ so that $f^{(k)}(x) = e^x$ for any k. Observe that we do not have a sufficient estimate on $f^{(k)}(x) =$ working for all x at once. However, letting $c > 0$, for $|x| < c$, we have $|f^{(k)}(x)| < e^c$. Using (ii) of previous theorem with $A = e^c$, $B = 1$, we obtain*

What is the major difference between these two examples?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < c.$$

Since $c > 0$ is arbitrary, we indeed conclude that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for any $x \in \mathbb{R}$.

14 Integral test

Idea is pretty simple: Integrals are sometimes easier to compute than the corresponding sums. e.g. compare $\int_1^\infty 1/x^2 dx$ and $\sum_1^\infty 1/n^2$.

Theorem 14.1 *Suppose f is a positive, non-increasing function on $[a, \infty)$, $a \in \mathbb{R}$. Then for any $j, k \in \mathbb{Z}$ with $a \leq j < k$,*

$$\sum_{n=j}^{k-1} f(n) \geq \int_j^k f(x)dx \geq \sum_{n=j+1}^{k} f(n).$$

Proof: f is non-increasing implies for $n \leq x \leq n+1$,

$$f(n) \geq f(x) \geq f(n+1)$$

and so

$$f(n) = \int_n^{n+1} f(n)dx \geq \int_n^{n+1} f(x)dx \geq \int_n^{n+1} f(n+1)dx = f(n+1).$$

Adding these up from $n = j$ to $n = k - 1$, result follows. □

Corollary 14.1 *(Integral test) Suppose f is a positive, non-increasing function on $[1, \infty)$. Then $\sum_{n=1}^\infty f(n)$ converges if and only if $\int_1^\infty f(x)dx$ converges.*

Proof: ($\Leftarrow$) Let $s_k = \sum_{n=1}^k f(n)$. If $\int_1^\infty f(x)dx < \infty$, for any k, we have

$$s_k = f(1) + \sum_{n=2}^{k} f(n) \leq f(1) + \int_1^k f(x)dx \leq f(1) + \int_1^\infty f(x)dx < \infty.$$

Result follows using the monotone sequence theorem.

($\Rightarrow$) Proof uses contrapositive. Assume $\int_1 \infty f(x)dx = \infty$. Then

$$s_k = \sum_{n=1}^{k-1} f(n) + f(k) \geq \int_1^k f(x)dx + f(k) \to \infty.$$

□

Note 14.1 *Of course we may instead compare $\sum_{n=m}^\infty f(n)$ with $\int_m^\infty f(x)dx$ for any $m \in \mathbb{Z}$.*

Corollary 14.2 $\sum_{n=1}^\infty n^{-p}$ *converges if $p > 1$ and diverges if $p \leq 1$.*

Proof: Just do a comparison with $\int_1^\infty x^{-p}dx$. □

Discussions similar to we just had also helps to understand the difference

$$\sum_{n=k}^{\infty} f(n) - \int_k^\infty f(x)dx$$

in same setting.

Proposition 14.1 *Suppose f is a positive, non-increasing function on $[1, \infty)$ and $\int_1^\infty f(x)dx < \infty$. Then*

(i) For $k \in \mathbb{N}$,

$$\int_k^\infty f(x)dx \leq \sum_{n=k}^\infty f(n) \leq f(k) + \int_k^\infty f(x)dx.$$

(ii) In particular,

$$0 \leq \sum_{n=k}^\infty f(n) - \int_k^\infty f(x)dx \leq f(k).$$

Proof: Recall that for $j \geq k+1$

$$\sum_{n=k}^{j-1} f(n) \geq \int_k^j f(x)dx \geq \sum_{n=k+1}^{j} f(n).$$

Send $j \to \infty$ to get

$$\sum_{n=k}^\infty f(n) \geq \int_k^\infty f(x)dx \geq \sum_{n=k+1}^\infty f(n).$$

So

$$\sum_{n=k}^\infty f(n) \geq \int_k^\infty f(x)dx \tag{7}$$

Also $\int_k^\infty f(x)dx \geq \sum_{n=k+1}^\infty f(n)$ implies that

$$f(k) + \int_k^\infty f(x)dx \geq \sum_{n=k}^\infty f(n) \tag{8}$$

Result follows by combining (7) and (8).

(ii) is immediate from (i). □

Note 14.2 *(ii) of previous proposition further implies that*

$$\sum_{n=0}^\infty f(n) - \sum_{n=0}^{k-1} f(n) = \int_k^\infty f(x)dx + E, \tag{9}$$

where $0 \leq E \leq f(k)$. This is saying that if we do the finite sum approximation $\sum_{n=0}^{k-1} f(n)$ to the infinite series $\sum_{n=0}^\infty f(n)$, we have an error at most $\int_k^\infty f(x)dx + f(k)$.

Example 14.1 *Evaluate $\sum_{n=1}^\infty \frac{1}{n^4}$ with an error of at most 0.0001.*

Solution 14.1 *Take $k = 10$ in (9):*

$$\sum_{n=1}^\infty \frac{1}{n^4} = \sum_{n=1}^{9} \frac{1}{n^4} + \int_{10}^\infty + E,$$

where $E \leq f(10) = 0.0001$. Now a calculator show that

$$\sum_{n=1}^{9} \frac{1}{n^4} + \int_{10}^\infty = \sum_{n=1}^{9} \frac{1}{n^4} + \frac{1}{3}10^{-3} \approx 1.08266$$

is an approximation satisfying the restriction.

15 More convergence tests I

We continue with analysis of series with non-negative terms.

Theorem 15.1 *(Comparison I) If $0 \le a_n \le b_n$, $n \ge 0$, and (i) if $\sum b_n$ converges, then so does $\sum a_n$; (ii) if $\sum a_n$ diverges, then so does $\sum b_n$.*

Proof: Second claim is just the contrapositive of the first one, so let's focus on (i). Let $s_k = \sum_{n=0}^{k} a_n$ and $t_k = \sum_{n=0}^{k} b_n$. Then, clearly, $0 \le s_k \le t_k$. Since $\sum b_n < \infty$, t_k is a bounded sequence, and therefore s_k is bounded as well. Also, since $a_n \ge 0$ for each n, s_k is also monotone non-decreasing. Hence s_k converges by monotone sequence theorem. $\square$

Note 15.1 *$0 \le a_n \le b_n$ can be replaced by $0 \le a_n \le cb_n$ for some $c > 0$ (Why?).*

Example 15.1 *$\sum \frac{1}{2n-1}$ diverges because $\frac{1}{2n-1} > \frac{1}{2n} = \frac{1}{2}\frac{1}{n}$ for $n \ge 1$.*

Example 15.2 *$\sum \frac{1}{n^2-6n+10}$ converges because*

Also remember Eylül's nice completing the squares solution in lecture.

$$\frac{1}{n^2 - 6n + 10} < \frac{1}{n^2/2 + 10} < \frac{1}{n^2/2} = \frac{2}{n^2}, \qquad n \ge 13.$$

Theorem 15.2 *(Comparison II) Suppose a_n, b_n are sequences of positive real numbers and $a_n/b_n \to L \in (0, \infty)$ as $n \to \infty$. Then $\sum a_n$ and $\sum b_n$ are either both convergent, or both divergent.*

Proof: We have $L/2 < a_n/b_n < 2L$ for large n since $a_n/b_n \to L \in (0, \infty)$. This in particular says $a_n < 2Lb_n$ and $b_n < \frac{2a_n}{L}$ for large n. Result now follows from the comparison theorem I. $\square$

Theorem 15.3 *(Ratio test) Suppose a_n is a sequence of positive numbers.*
(i) If $\frac{a_{n+1}}{a_n} < r < 1$ for all sufficiently large n, then $\sum a_n$ converges.
(ii) If $\frac{a_{n+1}}{a_n} \ge 1$ for all sufficiently large n, then $\sum a_n$ diverges.
(iii) Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ exists. Then

$$\sum a_n \begin{cases} \textit{converges}, & \textit{if } L < 1 \\ \textit{diverges}, & \textit{if } L > 1 \\ \textit{no conclusion}, & \textit{if } L = 1. \end{cases}$$

Proof: (i) Suppose $\frac{a_{n+1}}{a_n} < r < 1$ for $n \ge N$. Then

$$a_{N+1} < ra_N \Rightarrow a_{N+2} < ra_{N+1} < r^2 a_N,$$

and more generally,

$$a_{N+m} < r^m a_N, \qquad m > 0.$$

Then

$$\sum a_n < a_0 + a_1 + \cdots + a_{N-1} + a_N(1 + r + r^2 + \cdots).$$

Since the infinite series on right-hand side is geometric with $|r| < 1$, the result follows from the comparison test I.

(ii) In this case $a_{n+1} \geq a_n$ for each $n \geq N$ for some $N \in \mathbb{N}$. Noting that a_n's are all positive, this says that a_n does not converge to 0, and therefore the result follows from the nth term test.

(iii) Take $a_n = \frac{1}{n^p}$. Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ for any $p > 0$. But $\sum a_n$ converges when $p > 1$ and diverges when $p \leq 1$. $\square$

Theorem 15.4 *(Root test) Let a_n be a sequence of positive real numbers.*

(i) If $a_n^{1/n} < r$ for large n, with $r < 1$, then $\sum a_n$ converges.

(ii) If $a_n^{1/n} \geq 1$ for all large enough n, then $\sum a_n$ diverges.

(iii) Suppose $\lim_{n\to\infty} a_n^{1/n} = L$ exists. Then

$$\sum a_n \begin{cases} \textit{converges}, & \textit{if } L < 1 \\ \textit{diverges}, & \textit{if } L > 1 \\ \textit{no conclusion}, & \textit{if } L = 1. \end{cases}$$

Proof: (i) If $a_n^{1/n} < r$ for $n \geq N$, $N \in \mathbb{N}$, then $a_n < r^n$ for $n \geq N$. Result now follows from the comparison test (via the comparison with geometric series).

(ii) If $a_n^{1/n} \geq 1$ for $n \geq N$, then $a_n \geq 1$ for $n \geq N$, and so a_n does not converge to 0. So $\sum a_n$ does not converge by the nth term test.

(iii) Left for you. Just note that you can still use $\sum \frac{1}{n^p}$ to obtain the result. $\square$

Example 15.3 *Does $\sum \frac{n^2}{2^n}$ converge?*

Solution 15.1 *We have*

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1.$$

Convergence follows from the ratio test.

Also

$$\lim_{n\to\infty} \left(\frac{n^2}{2^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{2} n^{2/n} = \frac{1}{2}.$$

So convergence follows from the root test as well.

ex: Show that $\lim_{n\to\infty} n^{1/n} = 1$.

Example 15.4 *Let $a_n = \frac{1\cdot 4\cdot 7\cdots(3n+1)}{2^n n!}$. Does $\sum a_n$ converge?*

Solution 15.2 *We use the ratio test. We have*

$$\frac{a_{n+1}}{a_n} = \frac{(1\cdot 4\cdot 7\cdots(3n+1)(3n+4))/(2^{n+1}(n+1)!)}{(1\cdot 4\cdot 7\cdots(3n+1))/(2^n n!)} = \frac{3n+4}{2(n+1)} \longrightarrow \frac{3}{2} > 1.$$

So $\sum a_n$ diverges.

16 Absolute convergence

Definition 16.1 *$\sum a_n$ is **absolute convergent** if $\sum |a_n|$ is convergent.*

Theorem 16.1 *Every absolutely convergent series is convergent.*

Proof: Suppose $\sum |a_n|$ converges. Let

$$S_k = \sum_{n=0}^{k} |a_n| \qquad \text{and} \qquad R_k = \sum_{n=0}^{k} a_n.$$

By assumption, S_k is convergent, and therefore it is Cauchy. Being Cauchy implies for any $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$S_k - S_j = |a_{j+1}| + \cdots + |a_k| < \epsilon$$

whenever $K \leq j \leq k$. Then

$$|R_k - R_j| = |a_{j+1} + \cdots + a_k| \leq \sum_{\ell=j+1}^{k} |a_\ell| < \epsilon$$

for any $K \leq j \leq k$. This says that R_k is Cauchy, and hence convergent as well.

Note 16.1 *(N1) The converse of this result is not true - we will see an example right away. If a series converges, but does not converge absolutely, it is said to be **conditionally convergent**.*

(N2) ex: Do a comparison to improper integrals.

Example 16.1 *Let*

$$a_n = \begin{cases} \frac{1}{n+1}, & \text{if } n = 2k \\ -\frac{1}{n}, & \text{if } n = 2k+1, \end{cases}$$

for $k \in \mathbb{N}_0$. Then

$$\sum a_n = 1 - 1 + \frac{1}{3} - \frac{1}{3} + \frac{1}{5} - \frac{1}{5} + \cdots.$$

So

$$\sum_{n=0}^{k} a_n = \begin{cases} 0, & \text{if } k \text{ is odd} \\ \frac{1}{k+1}, & \text{if } k \text{ is even,} \end{cases}$$

clearly yields $\sum a_n = 0$. However,

$$\sum |a_n| = 1 + 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \cdots = 2 \sum \frac{1}{2n+1} = \infty.$$

Note 16.2 *As in case of functions, for a given sequence a_n of real numbers, we define*

$$a_n^+ = \max\{a_n, 0\}, \qquad \text{and} \qquad a_n^- = \max\{-a_n, 0\},$$

and observe that

(O1) $a_n = a_n^+ - a_n^-$*;*

(O2) $|a_n| = a_n^+ + a_n^-$*;*

(O3) $a_n^+, a_n^- \geq 0$*;*

(O4) $a_n^+ \leq |a_n|$ *and* $a_n^- \leq |a_n|$*.*

Theorem 16.2 *(i) If* $\sum a_n$ *is absolutely convergent, then the series* $\sum a_n^+$ *and* $\sum a_n^-$ *are both convergent.*

(ii) (!!!) If $\sum a_n$ *is conditionally convergent, the series* $\sum a_n^+$ *and* $\sum a_n^-$ *are both divergent.*

Proof: (i) Note that $a_n^+ \leq |a_n|$ and $a_n^- \leq |a_n|$ for each n. So both $\sum a_n^+$ and $\sum a_n^-$ converge by the comparison test.

(ii) (Proof via contrapositive) Suppose at least one of $\sum a_n^+$ or $\sum a_n^-$ is convergent.

If both are convergent, clearly $\sum a_n$ is convergent.

Assume that just one of them is divergent; say $\sum a_n^+ = \infty$ and $\sum a_n^- = S < \infty$. Then for any $C > 0$, and for $k \geq K$ for some $K \in \mathbb{N}$, we have

$$\sum_{n=1}^{k} a_n^+ > C + S, \qquad \text{and} \qquad \sum_{n=1}^{k} a_n^- \leq S.$$

So for $k \geq K$,

$$S_k = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k} (a_n^+ - a_n^-) > C + S - S = C.$$

Since this is true for any $k \geq K$, $C > 0$, $\sum a_n$ diverges. $\square$

17 Raabe's test

Motivation is that we were not even able to handle the series $\sum \frac{1}{n^p}$, $p > 0$ using ratio or root tests.

Theorem 17.1 *(Raabe's test) Let* x_n *be a sequence of non-zero real numbers.*

(i) If there exists $a > 1$ *and* $K \in \mathbb{N}$ *such that*

$$\left| \frac{x_{n+1}}{x_n} \right| \leq 1 - \frac{a}{n}, \qquad n \geq K,$$

then $\sum x_n$ *is absolutely convergent.*

(ii) If there exists $a \leq 1$ *and* $K \in \mathbb{N}$ *such that*

$$\left| \frac{x_{n+1}}{x_n} \right| \geq 1 - \frac{a}{n}, \qquad n \geq K,$$

then $\sum x_n$ *is not absolutely convergent.*

Proof: (i) We know that $\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n}, \qquad n \geq K$, for each $n \geq K$. This implies

$$n|x_{n+1}| \leq (n-a)|x_n| = (n-1)|x_n| - (a-1)|x_n|, \qquad n \geq K,$$

and in particular

$$(n-1)|x_n| - n|x_{n+1}| \geq (a-1)|x_n| > 0, \qquad n \geq K. \tag{10}$$

So for $n \geq K$, $n|x_{n+1}|$ is a decreasing sequence. Also, adding the left-hand side of (10) for $n = K, \ldots, N$, $N > K$, we get

$$(K-1)|x_K| - N|x_{N+1}| \geq (a-1)(|x_K| + \cdots + |x_N|).$$

Thus, $\sum |x_n|$ is bounded, and we conclude that $\sum x_n$ is absolutely convergent by monotone sequence theorem.

(ii) Now assume $\left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n}, \qquad n \geq K$ for some $a \leq 1$ and for $n \geq K$. Then

$$n|x_{n+1}| \geq (n-a)|x_n| \geq (n-1)|x_n|, \qquad n \geq K$$

since $a \leq 1$. That is $n|x_{n+1}| \geq (n-1)|x_n|, \qquad n \geq K$, which clearly implies

$$n|x_{n+1}| \geq (K-1)|x_K|, \qquad n \geq K,$$

implying

$$|x_{n+1}| \geq \frac{(K-1)|x_K|}{n}, \qquad n \geq K.$$

Comparison to $\sum \frac{1}{n}$ yields $\sum |x_n| = \infty$. □

Corollary 17.1 *Let x_n be a non-zero sequence and let*

$$L = \lim_{n\to\infty} \left(n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right) \right),$$

assuming the limit exists. Then

$$x_n \begin{cases} \textit{absolutely converges when } L > 1 \\ \textit{does not converge absolutely when } L < 1 \\ \textit{no conclusion when } L = 1. \end{cases}$$

Example 17.1 *Let $x_n = \frac{1\cdot 4\cdot 7\cdots(3n+1)}{n^2 3^n n!}$. Does $\sum a_n$ converge?*

Solution 17.1 *We use Raabe's test. We have*

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{x_{n+1}}{x_n} = \frac{(1\cdot 4\cdot 7\cdots(3n+1)(3n+4))/((n+1)^2 3^{n+1}(n+1)!)}{(1\cdot 4\cdot 7\cdots(3n+1))/(n^2 3^n n!)} = \frac{(3n+4)n^2}{3(n+1)^3}.$$

(Observe at this point that the ratio test does not apply for this series.) Then

$$n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right) = n\left(1 - \frac{(3n+4)n^2}{3(n+1)^3}\right) = \frac{5n^3 < 9n^2 + 3}{3(n+1)^3} \longrightarrow \frac{5}{3} > 1.$$

So $\sum x_n$ converges.

Exercise 17.1 *What does Raabe's test say for $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$?*

18 Rearrangements

Definition 18.1 *A series $\sum y_k$ is a rearrangement of $\sum x_n$ is there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $y_k = x_{f(k)}$, for each $k \in \mathbb{N}$.*

Theorem 18.1 *(Rearrangement theorem I) Let $\sum x_n$ be absolutely convergent. Then any rearrangement $\sum y_k$ of $\sum x_n = L$ converges to L.*

Proof: Let $S_n = \sum_{j=1}^{n} x_j$, and $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $n, q > N$, then

$$|L - S_n| < \epsilon, \qquad \text{and} \qquad \sum_{k=N+1}^{q} |x_k| < \epsilon.$$

Let $M = M(N) \in \mathbb{N}$ be such that all terms $x_1, \ldots, x_N$ are contained as summands in $T_M = \sum_{j=1}^{M} y_j$.

Then for $m \geq M$, $T_m - S_n$ is the sum of a finite number of terms x_k with index $k > N$. Hence, for some $q > N$,

$$|T_m - S_n| \leq \sum_{k=N+1}^{q} |x_k| < \epsilon.$$

Thus, if $m \geq M$

$$|T_m - L| \leq |T_m - S_n| + |S_n - L| < \epsilon + \epsilon = 2\epsilon.$$

So $\sum y_k \to L$ as required. □

Theorem 18.2 *(Rearrangement theorem II) Suppose $\sum x_n$ is conditionally convergent. Given any real number S, there is a rearrangement $\sum x_{f(n)}$ of $\sum x_n$ that converges to S.*

Amazing result attributed to Riemann. Indeed, he proves a slightly more general version with lim sup and lim inf.

Sketch of proof. (Recall that since $\sum x_n$ is conditionally convergent, $\sum x_n^+ \to \infty$ and $\sum x_n^- \to \infty$) Now, the construction of the required rearrangement is as follows:

Suppose $S \geq 0$ (other case is similar);

1. Add up positive terms from $\sum x_n$ (in their original order) until the sum exceeds S. Stop as soon as sum exceeds S.
2. Start adding negative terms from $\sum x_n$ (in their original order) until the sum becomes less than S.
3. Repeat the first and the second steps ad infinitum.

Resulting arrangement then converges to S. □

Exercise 18.1 *Prove Theorem 18.2. (Hint: Keep mind that given $\epsilon > 0$, $|a_n| < \epsilon$ for sufficiently large n.)*

19 More convergence tests II

The tests we developed for series with non-negative summands yield the following analogues in absolutely convergent setting.

Theorem 19.1 *(i) (Comparison) If $|a_n| \le Cn^{-1-\epsilon}$ for some $C > 0, \epsilon > 0$, then $\sum a_n$ convergens absolutely. On the other hand, if $|a_n| \ge C/n$ for some $C > 0$, then $\sum a_n$ either conditionally converges or diverges.*

(ii) (Ratio) If $\left|\frac{a_{n+1}}{a_n}\right| \to L$ as $n \to \infty$, then $\sum a_n$ absolutely converges if $L < 1$, and diverges if $L > 1$.

(iii) (Root) If $|a_n|^{1/n} \to L$ as $n \to \infty$, then $\sum a_n$ absolutely converges if $L < 1$, and diverges if $L > 1$.

Another example about ratio test.

Recall: A **power series** is a function of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$.

Example 19.1 *Consider the power series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n(x-3)^n}{(n+1)2^{2n+1}}.$$

For which $x \in \mathbb{R}$, does this series converge absolutely? conditionally?

Solution 19.1 *We have*

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((-1)^{n+1}(x-3)^{n+1})/((n+2)2^{2n+3})}{((-1)^n(x-3)^n)/((n+1)2^{2n+1})}\right| = \frac{n+1}{n+2}\frac{|x-3|}{4} \to \frac{|x-3|}{4}$$

as $n \to \infty$. So the series converges absolutely when $|x-3| < 4$, and diverges when $|x-3| > 4$.

We are left with understanding the behavior of the series at the endpoints of the interval $|x-3| < 4$: $x = 7$ and $x = -1$.

(a.) When $x = 7$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(n+1)2^{2n+1}} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

which clearly converges conditionally.

(b.) When $x = -1$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-4)^n}{(n+1)2^{2n+1}} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n+1}$$

which diverges.

Next, alternating series test.

Theorem 19.2 *(Alternating series test) Suppose a_n a decreasing sequence for which $\lim_{n\to\infty} a_n = 0$ (so, in particular, $a_n > 0$ for each n.). Then* *(i)* $\sum_{n=0}^{\infty}(-1)^n a_n$ *is convergent.*

(ii) Further, if $s_k = \sum_{n=0}^{k}(-1)^n a_n$ *and* $S = \sum_{n=0}^{\infty}(-1)^n a_n$*, then the following three hold:*

a. $s_k > S$ *for* k *even;*

b. $s_k < S$ *for* k *odd;*

c. $|s_k - S| < a_{k+1}$ *for each* k.

Proof: (i) Since $a_k \geq a_{k+1}$ for each k, we have

$$s_{2m+1} = s_{2m-1} + \underbrace{a_{2m} - a_{2m-1}}_{\geq 0} \geq s_{2m-1}$$

and

$$s_{2m+2} = s_{2m} + \underbrace{-a_{2m} + a_{2m-1}}_{\leq 0} \leq s_{2m}.$$

So s_{2m+1} and s_{2m} are increasing and decreasing, respectively.

Also

$$s_{2m-1} = s_{2m-2} - a_{2m-1} \leq s_{2m-2} \leq s_0 < \infty, \text{ for all } m$$

and

$$s_{2m} = s_{2m-1} + a_{2m} \geq s_{2m-1} \geq s_1 > -\infty, \text{ for all } m.$$

Via monotone sequence theorem, we may then conclude that s_{2m} and s_{2m-1} both converge; say, $s_{2m} \to L_1$ and $s_{2m-1} \to L_2$.

Since $s_{2m} - s_{2m-1} = a_{2m} \to 0$, we conclude that $L_1 = L_2 = S := \sum_{n=0}^{\infty}(-1)^n a_n$. This in particular says that $s_k \to S$ as well.

(ii) We have

$$S < s_{2m} \quad \text{and} \quad S > s_{2m-1}, \text{ for all } m.$$

So,

$$0 < S - s_{2m-1} < s_{2m} - s_{2m-1} = a_{2m}$$

and

$$0 < s_{2m} - S < s_{2m} - s_{2m+1} = a_{2m+1}.$$

These yield

$$|S - s_k| < a_{k+1}$$

for each k. $\square$

Example 19.2 $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ *clearly does not converge absolutely. But* $1/n$ *is a decreasing sequence converging to zero. Therefore, alternating series test gives the conditional convergence of* $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$.

20 Review Problems I

Problem 20.1 *Comment: Previously we proved that if f_n's are uniformly continuous on a compact set, and if f_n's converge uniformly to some function f, then f is continuous as well. Dini's theorem provides a partial converse to this result.*

Dini's theorem. *Assume that f_n is a monotone sequence of continuous functions on $[a, b]$ converging to a continous function f. Then the convergence is uniform.*

We did not prove Dini's theorem in lecture, but here are some related exercises:

i. *Let $f_n(x) = 1$ for $x \in (0, 1/n)$ and $f_n(x) = 0$ elsewhere in $[0, 1]$. Show that f_n is a decreasing sequence of discontinous functions that converges to a continuous limit function, but the convergence is not uniform on $[0, 1]$.*

ii. *Let $f_n(x) = x^n$, $x \in [0, 1]$, $n \in \mathbb{N}$. Show that f_n is a decreasing sequence of continuous functions converging to a function that is not continous, but the convergence is not uniform on $[0, 1]$.*

iii. *Let $f_n(x) = x/n$, for $x \in [0, \infty)$, $n \in \mathbb{N}$. Show that f_n is a decreasing sequence of continuous functions that converges to a continuous limit function, but the convergence is not uniform on $[0, \infty)$*

iv. *Give an example of a decreasing sequence f_n of continuous functions on $[0, 1)$ that converges to a continuous limit function, but the convergence is not uniform on $[0, 1)$.*

v. *Explain in detail what the purpose of this problem is.*

Problem 20.2 *(Cesaro means) Let a_n be a sequence convering to a. Letting $b_n = \frac{1}{n}\sum_{i=1}^{n} a_i$, prove that $\lim_{n\to\infty} b_n = a$.*

Problem 20.3 *(Cauchy condensation test) (i) Let $\sum_{n=1}^{\infty} a_n$ be such that a_n is a decreasing sequence of strictly positive numbers. If S_n denotes the nth partial sum, show (by grouping the terms in S_{2^n} in two different ways) that*

$$\frac{1}{2}(a_1 + 2a_2 + \cdots + 2^n a_{2^n}) \leq S_{2^n} \leq (a_1 + 2a_2 + \cdots + 2^{n-1} a_{2^{n-1}}) + a_{2^n}.$$

Use these inequalities to show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Problem 20.4 *Use Cauchy condensation test to establish the divergence of*

$$\sum \frac{1}{n(\ln n)(\ln \ln n)}.$$

Problem 20.5 *By using partial fractions, show that*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}.$$

Problem 20.6 *Assume that $\sum a_n$ with $a_n > 0$ is convergent.*

(i) Is $\sum a_n^2$ always convergent?

(ii) Is $\sum \sqrt{a_n}$ always convergent?

(iii) Is $\sum \sqrt{a_n a_{n+1}}$ always convergent?

Problem 20.7 *If $f \in R[a,b]$ and $c \in \mathbb{R}$, we define g on $[a+c, b+c]$ by $g(y) = f(y-c)$. Prove that $g \in R[a+c, b+c]$ and that $\int_{a+c}^{b+c} g = \int_a^b f$. (The function g is called the c-**translate** of f.*

Problem 20.8 *(i) (Sequeeze theorem for integrals) Let $f : [a,b] \to \mathbb{R}$. Then $f \in R[a,b]$ if and only if for every $\epsilon > 0$, there exist functions α_ϵ and w_ϵ in $R[a,b]$ with*

$$\alpha_\epsilon \le f(x) \le w_\epsilon, \quad \text{for all } x \in [a,b]$$

and such that

$$\int_a^b (w_\epsilon - \alpha_\epsilon) < \epsilon.$$

(ii) If $\alpha(x) = -x$ and $w(x) = x$, and if $\alpha(x) \le f(x) \le w_\epsilon$, for all $x \in [0,1]$, does it follow from the sequeeze theorem that $f \in R[0,1]$?

Problem 20.9 *(i) Show that the Thomae's function $t : [0,1] \to \mathbb{R}$ Riemann integrable.*

(ii) Show that composite function $sgn \circ t$ is not Riemann integrable.

Problem 20.10 *(Cauchy-Schwarz-Bunyakowski inequality) Let $f, g \in R[a,b]$.*

(a) If $t \in \mathbb{R}$, show that $\int_a^b (tf+g)^2 \ge 0$.

(b) Use (a) to show that $2\left|\int_a^b fg\right| \le t\int_a^b f^2 + (1/t)\int_a^b g^2$ for $t > 0$.

(c) If $\int_a^b f^2 = 0$, show that $\int_a^b fg = 0$.

(d) Now prove that

$$\left|\int_a^b fg\right|^2 \le \left(\int_a^b |fg|^2\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$$

Problem 20.11 *(L^2 distance) For $f \in R[a,b]$, define*

$$\|f\|_2 \left(\int_a^b |f|^2 dx\right)^{1/2}.$$

Suppose $f, g, h \in R[a,b]$. Prove that the triangle inequality

$$\|f-h\|_2 \le \|f-g\|_2 + \|g-h\|_2$$

holds. (Hint: Cauchy-Schwarz-Bunyakowski inequality)

Problem 20.12 *(Preserving uniform continuity) (i) Show that if f_n, g_n converge uniformly on the set A to f, g, respectively, then $(f_n + g_n)$ converges uniformly on A to $f+g$.*

(ii) Show that if $f_n(x) = x + 1/n$ and $f(x) = x$ for all $x \in \mathbb{R}$, then f_n converges uniformly on $\mathbb{R}$ to f, but the sequence f_n^2 does not converge uniformly on $\mathbb{R}$. (Thus the product of uniformly convergent sequences of functions may not converge uniformly.)

(iii) Let f_n, g_n be sequences of bounded functions on A that converge uniformly on A to f, g, respectively. Show that $f_n g_n$ converges uniformly on A to fg.

Problem 20.13 *Show that the sequence* $f(x) = x^2 e^{-nx}$ *converges uniformly on* $[0, \infty)$.

Problem 20.14 *Show that the sequence* $\frac{x^n}{1+x^n}$ *does not converge uniformly on* $[0, 2]$ *by showing that the limit function is not continuous on* $[0, 2]$.

Problem 20.15 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be uniformly continuous on* $\mathbb{R}$, *and let* $f_n(x) = f(x + 1/n)$ *for* $x \in \mathbb{R}$. *Show that* f_n *converges uniformly to* f *on* $\mathbb{R}$.

Problem 20.16 *Suppose the sequence* f_n *converges uniformly to* f *on* A, *and suppose that each* f_n *is bounded on* A. *Show that the function* f *is bounded on* A.

Problem 20.17 *If* $a > 0$, *show that*

$$\lim_{n\to\infty} \int_a^\pi (\sin nx)/(nx) dx = 0.$$

What happens if $a = 0$?

Problem 20.18 *Let* $g_n(x) = nx(1 - x)^n$ *for* $x \in [0, 1]$, $n \in \mathbb{N}$. *Discuss the convergence of* g_n *and* $\int_0^1 g_n dx$.

Problem 20.19 *Let* $0 < a < 1$ *and consider the series*

$$a^2 + a + a^4 + a^3 + \cdots + a^{2n} + a^{2n-1} + \cdots .$$

Show that the root test applies, but that the ratio test does not apply.

Problem 20.20 *Find a series expansion for* $\int_0^x e^{-t^2} dt$ *for* $x \in \mathbb{R}$. *(Try to) Justify your answer.*

Problem 20.21 *Prove that* $\sum_p 1/p$ *diverges where the sum extends over all primes.*

Problem 20.22 *Suppose that* f *is a bounded function on* $[a, b]$, *and* $f^2 \in R[a, b]$. *Does it follow that* $f \in R[a, b]$? *Does the answer change if we assume that* $f^3 \in R[a, b]$?

Problem 20.23 *Folland, 4.1: 7*

Problem 20.24 *Folland, 4.2: 2*

Problem 20.25 *Folland, 4.3: 13*

Problem 20.26 *Folland, 4.5: 3, 4, 7*

Problem 20.27 *Folland, 4.6: 1, 2, 3,*

Problem 20.28 *Folland, 6.1: 1, 2*

Problem 20.29 *Folland, 6.2: All exercises from 1 to 14, Also 20, 21*

Problem 20.30 *Folland, 7.1: 1, 8*

21 Dirichlet and Abel tests

Proof of Dirichlet's test will require summation by parts as a prerequisite.

Question: How does summation by parts compare with integration by parts?

Lemma 21.1 *(Summation by parts) Given $a_n, b_n, n \geq 0$, let $a'_n = a_n - a_{n-1}$ and $B_n = b_0 + \cdots + b_n$. Then*

$$\sum_{n=0}^{k} a_n b_n = a_k B_k - \sum_{n=1}^{k} a'_n B_{n-1}, \quad n \geq 1.$$

Proof: We have $b_0 = B_0$ and $b_n = -B_{n-1} + B_n$, $n \geq 1$. Then

$$\begin{aligned}
\sum_{n=0}^{k} a_n b_n &= a_0 b_0 + a_1 b_1 + \cdots + a_k b_k \\
&= a_0 B_0 - a_1 B_0 + a_1 B_1 - a_2 B_1 + a_2 B_2 - \cdots - a_k B_{k-1} + a_k B_k \\
&= -a'_1 B_0 - a'_2 B_1 - \cdots - a'_k B_{k-1} + a_k B_k \\
&= a_k B_k - \sum_{n=1}^{k} a'_n B_{n-1}, \quad n \geq 1.
\end{aligned}$$

□

Theorem 21.1 *(Dirichlet' test) Let a_n, b_n be sequences such that a_n is decreasing and $a_n \to 0$ as $n \to \infty$. Also suppose that the sums $B_n = b_0 + b_1 + \cdots + b_n$ are bounded by $C > 0$ for all n. Then $\sum_{n=0}^{\infty} a_n b_n$ converges.*

Proof: As before, let $a'_n = a_n - a_{n-1}$. Summation by parts says:

$$\sum_{n=0}^{k} a_n b_n = a_k B_k - \sum_{n=1}^{k} a'_n B_{n-1}.$$

So let's show: (a) $a_k B_k$ converges and (b) $\sum_{n=1}^{k} a'_n B_{n-1}$ (so that then their difference will converge as well.).

Proof of (a) is clear as $|a_k B_k| \leq C|a_k| \to 0$ as $k \to \infty$.

For (b), recalling a_n is decreasing, we have $a'_n = a_n - a_{n-1} \leq 0$ for all n. Then

$$\begin{aligned}
\sum_{n=1}^{k} |a'_n B_{n-1}| &\leq C \sum_{n=1}^{k} |a'_n| \\
&= C[(a_0 - a_1) + (a_1 - a_2) + \cdots + (a_{k-1} - a_k)] \\
&= C(a_0 - a_k) \\
&\leq C a_0.
\end{aligned}$$

So $\sum_{n=1}^{k} |a'_n B_{n-1}|$ is increasing and bounded above, and therefore converges by monotone sequence theorem. Since $\sum_{n=1}^{k} a'_n B_{n-1}$ absolutely converges, it converges as well yielding (b), and the proof is over. □

Note 21.1 *Taking $b_n = (-1)^n$ in Dirichlet test, for which $B_n = 1$ or $B_n = 0$ depending on whether n is even or odd, we obtain the alternating series test.*

Of course Dirichlet test is a lot more general than the alternating one. Another instance where Dirichlet's test should be kept in mind is when b_n involves trigonometric functions. In particular, here we deal with the case where $b_n = \sin n\theta$ and $b_n = \cos n\theta$. First, an elementary preliminary lemma

Lemma 21.2 *If θ is not an integer multiple of 2π, then*

$$\sum_{n=1}^{k} \sin n\theta = \frac{\sin\left(\frac{1}{2}(k+1)\theta\right)\sin\left(\frac{k\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

and

$$\sum_{n=1}^{k} \cos n\theta = \frac{\cos\left(\frac{1}{2}(k+1)\theta\right)\sin\left(\frac{k\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

Proof is omitted. Now let's see the promised application of Dirichlet's test.

Corollary 21.1 *Suppose a_n decreases to 0 as $n \to \infty$.*
(i) $\sum_{n=1}^{\infty} a_n \cos n\theta$ converges for all θ except perhaps integer multiples of 2π.
(ii) $\sum_{n=1}^{\infty} a_n \sin n\theta$ converges for all θ.

Proof: Using previous lemma, if $b_n = \cos n\theta$ or $b_n = \sin n\theta$, and $B_n = b_0 + \cdots + b_n$, we have $|B_n| \leq |\csc(\theta/2)|$ since both since and cosine functions are bounded by 1 – note that the upper bound is independent of n. So, hypotheses of Dirichlet test are satisfied for $\theta \neq 2\pi j$.

When $\theta = 2\pi j$, the series $\sum a_n \sin n\theta$ converges trivially since $\sin n\theta = 0$ for all n. $\square$

There is also Abel's test, a conclusion of Dirichlet's, which we mention without proof. I urge the interested reader to prove it.

Theorem 21.2 *(Abel's test) If a_n is a convergent monotone sequence and $\sum b_n$ is convergent, then the series $\sum a_n b_n$ is convergent as well.*

22 Double series

A double series is a series of the form $\sum_{m,n=0}^{\infty} a_{mn}$. Of course, this is not rigorous, and it is on clear how to interpret it:

$$\lim_{k\to\infty} \sum_{m,n=0}^{k} a_{mn}, \quad \text{OR} \quad \lim_{k\to\infty} \sum_{m+n\leq} a_{mn} \quad \text{OR} \quad \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{mn}\right) \text{OR} \quad \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{mn}\right).$$

There are obviously infinitely many different ways to approach the double sums (or more generally multiple sums) problem. We treat the double sums problem similar to (improper) double integrals.

Definition 22.1 *Given any one-to-one correspondence $j \leftrightarrow (m,n)$ between $\mathbb{N}_0$ and $\mathbb{N}_0 \times \mathbb{N}_0$, we can set $b_j = a_{mn}$, and form the ordinary infinite series $\sum_{j=0}^{\infty} b_j$; such a series is called an* ***ordering*** *of $\sum a_{mn}$.*

The main idea in the following is that any ordering of $\sum a_{mn}$ can be considered as a rearrangement of one other, and therefore our previously developed theory on rearrangements can still be used.

The theory is summarized in following two items:

- If $a_{mn} \geq 0$ for all m, n, then either all orderings of $\sum a_{mn}$ or all orderings covergence (to the same limit). So the sum $\sum a_{mn}$ is well-defined (though possibly equals $\pm\infty$).

- If a_{mn}'s are not necessarily non-negative, if $\sum |b_j|$ converges for one ordering of $\sum a_{mn}$, then the same is true for any ordering. We then say $\sum a_{mn}$ is absolutely convergent, and by rearrangement theorem we know that any ordering of $\sum a_{mn}$ converges to the limit.

An important exercise for you is the following iterated sums result.

Exercise 22.1 *$\sum a_{mn}$ is absolutely convergent if and only if $\sum_m (\sum_n a_{mn})$ is absolutely convergent, and in this case*

$$\sum a_{mn} = \sum_m \left(\sum_n a_{mn} \right) = \sum_m \left(\sum_n a_{mn} \right) = \sum_n \left(\sum_m a_{mn} \right).$$

Note 22.1 *(N1) What the exercise says is the following: Evaluating $\sum |a_{mn}|$ by some ordering and finding it a finite number, we can evaluate $\sum a_{mn}$ treating it as an iterated series (or, by ordering it any way we like.)*

(N2) If $\sum a_{mn}$ is not absolutely convergent, then the theory moves in lines of the theory for improper double integrals.

23 How to multiply series

First way to interpret multiplication of two infinite series is to consider the product as

$$\sum_{m=0}^{\infty} a_m \sum_{n=0}^{\infty} b_n = \lim_{k\to\infty} \left(\sum_{m=0}^{k} a_m \sum_{n=0}^{k} b_n \right) = \lim_{k\to\infty} \sum_{m,n=0}^{k} a_m b_n = \sum_{m,n=0}^{\infty} a_m b_n.$$

The following result gives a sufficient criteria for absolute convergence of product of two series.

Theorem 23.1 *Suppose $\sum_{m=0}^{\infty} a_m$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent with sums A and B. Then the double series $\sum_{m,n=0}^{\infty} a_m b_n$ is absolutely convergent with sum AB.*

Proof: Consider

$$\sum_{m,n=0}^{k} a_m b_n = \left(\sum_{m=0}^{k} a_m \sum_{n=0}^{k} b_n \right). \tag{11}$$

Observe that

$$\sum_{m,n=0}^{k} |a_m b_n| = \sum_{m=0}^{k} |a_m| \sum_{n=0}^{k} |b_n| \leq A, B$$

for all k. Then $\sum_{m,n=0}^{k} a_m b_n$ is absolutely convergent, and therefore convergent. Letting now $k \to \infty$ in (11), we obtain

$$\sum_{m,n=0}^{\infty} a_m b_n = AB.$$

□

We conclude this section by defining the Cauchy product of two series.

Definition 23.1 *Given two convergent series $\sum_{m=0}^{\infty} a_m$ and $\sum_{n=0}^{\infty} b_n$, their* ***Cauchy product*** *is the series*

$$\sum_{j=0}^{\infty}\left(\sum_{m+n=j} a_n b_m\right) = \sum_{j=0}^{\infty}(a_0 b_j + a_1 b_{j-1} + \cdots + a_{j-1} b_1 + a_j b_1).$$

Cauchy product of series will be useful when we discuss power series later.

24 Back to uniform convergence

Recall that f_n **converges to** f **pointwise** on $S \subset \mathbb{R}^n$, denoted $f_n \to_p f$, if $f_n(x) \to f(x)$ for all $x \in S$ as $n \to \infty$. Also we say that f_n **converges to** f **uniformly** on $S \subset \mathbb{R}^n$, denoted $f_n \to_u f$ if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon, \quad \text{whenever } n > N \text{ and } x \in S$$

which is equivalent to

$$\sup_{x \in S} |f_n(x) - f(x)| < \epsilon, \quad \text{whenever } n > N.$$

So far, we proved the following three results related to uniform convergence.

Theorem 24.1 *If $f_n \to_u f$ on S and if each f_n is continuous on S, then f is continuous as well.*

Theorem 24.2 *Suppose $S \subset \mathbb{R}^k$ is measurable and f_n is a sequence of functions on S such that $f_n \to_u f$ on S. Then*

$$\int \cdots \int_S f_n(x) d^k x \longrightarrow \int \cdots \int_S f(x) d^k x.$$

Theorem 24.3 *Let $f \in C^1[a,b]$. Suppose $f_n \to_p f$ and $f_n' \to_u g$ on $[a,b]$. Then $f \in C^1$ and $g = f'$.*

In this section we will learn a little bit more on uniform convergence. First we discuss a result that will help us to the understand the relation between the uniform convergence and uniformly Cauchy concepts.

Theorem 24.4 *$f_n \to_u f$ on S if and only if there exists a sequence C_n of positive numbers such that $|f_k(x) - f(x)| \leq C_k$ for all $x \in S$ and $\lim_{n\to\infty} C_k = 0$.*

Proof: ($\Rightarrow$) When $f_n \to_u f$, we can choose $C_k = \sup_{x \in S} |f_k(x) - f(x)|$.

($\Leftarrow$) Assume that $\sup_{x \in S} |f_k(x) - f(x)| \le C_k$ where $C_k \to 0$. Then, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k > N$, we have $C_k < \epsilon$. This gives

$$\sup_{x \in S} |f_k(x) - f(x)| < \epsilon, \quad \text{for } k > N.$$

Therefore, $f_n \to_u f$ on S. □

Now, let's introduce the concept of uniform Cauchy.

Definition 24.1 *The sequence of functions f_n on S is* ***uniformly Cauchy*** *if for all ϵ, there exists $N \in \mathbb{N}$ such that*

$$|f_j(x) - f_k(x)| < \epsilon, \quad \textit{whenever } j, k > N \textit{ and } x \in S$$

or

$$\sup_{x \in S} |f_j(x) - f_k(x)| < \epsilon, \quad \textit{whenever } j, k > N.$$

Theorem 24.5 *f_n is uniformly Cauchy on S if and only if there is a function f on S such that $f_n \to_u f$ on S.*

Proof: ($\Rightarrow$) Assume that f_n is uniformly Cauchy on S. Then for each $x \in S$, the sequence $f_n(x)$ is Cauchy, and therefore $f_n(x)$ converges to a limit which we call $f(x)$.

Now, for given $\epsilon > 0$, since f_n is uniformly Cauchy on S, we know that there exists $N \in \mathbb{N}$ such that

$$|f_j(x) - f_k(x)| < \frac{\epsilon}{2}, \quad \text{whenever } j, k > N \text{ and } x \in S.$$

Letting $k \to \infty$, we obtain

$$|f_j(x) - f(x)| < \epsilon, \quad \text{whenever } j > N \text{ and } x \in S.$$

So, $f_n \to_u f$ on S.

($\Leftarrow$) Suppose that there exists f on S such that $f_n \to f$. Then by previous theorem, we know that $|f_n(x) - f(x)| \le C_n$ for all $x \in S$, $n \in \mathbb{N}$, where $C_n \to 0$ as $n \to \infty$. Then

$$|f_j(x) - f_k(x)| \le |f_j(x) - f(x)| + |f_k(x) - f(x)| \le C_j + C_k.$$

Given $\epsilon > 0$, choosing N large enough so that $C_j + C_k < \epsilon$ for $j, k > N$, we are done. □

25 Uniform convergence and infinite series

Definition 25.1 *(i) $\sum_{n=0}^{\infty} f_n(x)$, $x \in S$ is said to* ***converge pointwise*** *if $\sum_{n=0}^{\infty} f_n(x)$ converges for each $x \in S$.*

(ii) $\sum_{n=0}^{\infty} f_n(x)$, $x \in S$ is said to ***converge uniformly*** *if the partial sums $\sum_{n=0}^{k} f_n(x)$* ***converges uniformly*** *on S.*

Example 25.1 *We know that the geometric series* $\sum_{n=0}^{\infty} x^n$ *converges pointwise to* $\frac{1}{1-x}$ *on* $S = (-1, 1)$*. In this case, the* k*th partial sum is*

$$s_k(x) = \frac{1 - x^{k+1}}{1 - x}$$

and so

$$\left| s_k(x) - \frac{1}{1-x} \right| = \left| \frac{1 - x^{k+1}}{1 - x} - \frac{1}{1-x} \right| = \frac{|x|^{k+1}}{1-x}.$$

Noting that $\frac{|x|^{k+1}}{1-x} \to \infty$ *as* $x \to 1$*, the convergence is not uniform on* $(-1, 1)$*.*

However, for $r \in (0, 1)$*, convergence is uniform over* $[-r, r]$ *since*

$$\frac{|x|^{k+1}}{1-x} \leq \frac{r^{k+1}}{1-r}, \quad \textit{for } |x| \leq r,$$

and this can be made arbitrarily small by choosing k *large enough.*

Next, we discuss a fundamental result on uniform convergence of series.

Theorem 25.1 *(Weierstrass M-test) Let* f_n *be a sequence of functions on* S*. Assume there exists a sequence* M_n*,* $n \geq 0$*, of positive numbers such that*

(i) $|f_n(x)| \leq M_n$*, for all* $x \in S$*,*

(ii) $\sum_{n=0}^{\infty} M_n < \infty$*.*

Then $\sum_{n=0}^{\infty} f_n(x)$ *is absolutely and uniformly convergent.*

Proof: The claim that $\sum_{n=0}^{\infty} f_n(x)$ is absolutely convergent is clear for each x is clear since

$$\sum_{n=0}^{\infty} |f_n(x)| \leq \sum_{n=0}^{\infty} M_n < \infty.$$

Let's next focus on uniform convergence. Let $s(x) = \sum_{n=0}^{\infty} f_n(x)$, $s_k(x) = \sum_{n=0}^{k} f_n(x)$ and $C_k = \sum_{n=k+1}^{\infty} M_n$. Then for $x \in S$,

$$|s(x) - s_k(x)| = \left| \sum_{n=k+1}^{\infty} f_n(x) \right| \leq \sum_{n=k+1}^{\infty} |f_n(x)| \leq \sum_{n=k+1}^{\infty} M_n = C_k.$$

Now since $\sum_{n=0}^{\infty} M_n < \infty$, $C_k \to 0$ as $k \to \infty$. Result follows from Theorem 24.4. □

Example 25.2

Theorem 25.2

Proof:

□

Theorem 25.3 *Suppose that f_n is a sequence of functions on the interval $[a,b]$ and that the series $\sum_{n=0}^{\infty} f_n$ converges pointwise on $[a,b]$.*

(i) If $\sum_{n=0}^{\infty} f_n$ converges uniformly on $[a,b]$, then

$$\int_a^b \sum_{n=0}^{\infty} f_n(x)dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)dx.$$

(ii) If the f_n's are of class C^1 and the series $\sum_{n=0}^{\infty} f_n'$ converges uniformly on $[a,b]$, then the sum $\sum_{n=0}^{\infty} f_n$ is of class C^1 on $[a,b]$ and

$$\frac{d}{dx}\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} f_n'(x).$$

Proof:

□

26 Power series: basics

Definition 26.1 *A power series is a series of the form*

$$\sum_{n=0}^{\infty} a_n(x-b)^n = a_0 + a_1(x-b) + a_2(x-b)^2 + \cdots$$

Note 26.1 *Study of power series can be reduced to the special case $b=0$ by using the change of variable $x \mapsto x+b$.*

Lemma 26.1 *If $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_0$, then it converges absolutely for all x such that $|x| < |x_0|$.*

Proof: $\sum_{n=0}^{\infty} a_n x_0^n$ converges implies that $a_n x_0^n \to 0$ as $n \to \infty$ by the nth term test. In particular, $|a_n x_0^n| \leq C$ for some $C > 0$ and for all $n \geq 0$. Since

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x^n}{x_0^n}\right| \leq C \left|\frac{x}{x_0}\right|^n,$$

when $|x| < |x_0|$, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely via the comparison with the geometric series $\sum_{n=0}^{\infty} \left|\frac{x}{x_0}\right|^n$. □

Theorem 26.1 *For any power series $\sum_{n=0}^{\infty} a_n x^n$, there exists $R \in (0, \infty]$, called the **radius of convergence** the series, such that the series converges absolutely for $|x| < R$ and diverges for $|x| > R$.*

Proof: Let

$$R = \sup\{|x_0| : \sum_{n=0}^{\infty} a_n x_0^n \text{ converges}\}.$$

Then $R \geq 0$ since any power series converges at $x_0 = 0$. By definition $\sum_{n=0}^{\infty} a_n x^n$ diverges if $|x| > R$.

On the other hand, if $|x| < R$, there exists x_0 such that $|x_0| > |x|$ and $\sum_{n=0}^{\infty} a_n x_0^n$ converges, and now $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely by the previous lemma. □

Note 26.2 *$R = 0$ means that the series converges only for $x = 0$. $R = \infty$ says that the series converges absolutely for any x.*

Example 26.1 *As examples to previous note,*

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges for all x and equals x, and

$$\sum_{n=0}^{\infty} x^n n!$$

does not converge for any $x \neq 0$.

The theorem we have regarding the radius of converges gives only an open interval. One needs to check the endpoints of this open interval separately.

Example 26.2 *Find where the following power series converges:*

(i) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$*;*

(ii) $\sum_{n=1}^{\infty} x^n$*;*

(iii) $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Solution 26.1 *By an application of the ratio test it is to see that $R = 1$ for each of these three. So we just need to check the endpoints.*

(i) When $x = \pm 1$, $\sum_{n=1}^{\infty} \left| \frac{(\pm 1)^n}{n^2} \right|$ is finite by comparison test. So, the interval of convergence in this case is $[-1, 1]$.

(ii) For $x = \pm 1$, x^n does not converge to 0, so the series diverges by nth term test. That is, the interval of convergence in this case is $(-1, 1)$.

(iii) When $x = 1$, the series reduces to $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. When $x = 1$, this time the series reduces to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test. Therefore, the interval of convergence is $[-1, 1)$.

We proved that $\sum_{n=1}^{\infty} a_n x^n$ converges when $|x| < R$. Next, we prove that this convergence is indeed uniform on any compact subset of $|x| < R$.

Theorem 26.2 *Let R be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$. Then (i) for any $0 \leq r < R$, $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $\{x : |x| \leq r\}$, and (ii) its sum is a continuous function on the set $\{x : |x| < R\}$.*

Proof: (i) We will use the M-test. Choose $M_n = |a_n r^n|$. Then

$$f_n(x) \leq M_n, \qquad x \in [-r, r],$$

and

$$\sum_{n=0}^{\infty} M_n \leq \sum_{n=0}^{\infty} |a_n r^n| < \infty$$

since $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent at $x = r$. So $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $\{x : |x| \leq r\}$ by the M-test.

(ii) This follows from the fact that uniform limits of continuous functions is continuous.

□

Theorem 26.3 *Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$.*

(i) If $-R < a < b < R$, then

$$\int_a^b f(x)dx = \int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} a_n \frac{b^{n+1} - a^{n+1}}{n+1}.$$

(ii) If F is any anti-derivative of f, then

$$F(x) = F(0) + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \qquad |x| < R.$$

Proof: (i) We need to show that we may interchange integration and summation. From the previous theorem $\sum_{n=0}^{\infty} a_n x^n$ uniformly converges on $[a, b]$ for any $-R < a < b < R$, and then the result follows from part (i) of Theorem 25.3.

(ii) First part shows that $G(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is an antiderivative of f on $(-R, R)$ for which $G(0) = 0$. Then any other antiderivative F of f will differ from G by $F(0)$. So

$$F(x) = F(0) + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \qquad |x| < R$$

as required. □

Example 26.3 *We know that*

$$\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x}, \quad \textit{for } |x| < 1.$$

Then for $|x| < 1$, we have (with $a = 0, b = x$ in previous theorem)

$$\begin{aligned}
\ln(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x \sum_{n=0}^{\infty} (-t)^n dt &= \sum_{n=0}^{\infty} \int_0^x (-t)^n dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n}.
\end{aligned}$$

Exercise 26.1 *Show that*

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| < 1.$$

Example 26.4 *The function $f(x) = \frac{\sin x}{x}$ has no elementary antiderivative. However,*

$$\begin{aligned}
\int_0^x \frac{\sin t}{t} &= \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^x t^{2n} dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)}.
\end{aligned}$$

Next we discuss differentiation of power series.

Definition 26.2 $\sum_{n=0}^{\infty} n a_n x^{n-1}$ *is called the* ***derived series*** *of* $\sum_{n=0}^{\infty} a_n x^n$.

Theorem 26.4 *The radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is equal to the radius of convergence of $\sum_{n=0}^{\infty} n a_n x^{n-1}$.*

Proof: Let R and R' be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} n a_n x^{n-1}$. We have two claims from which the result will follow: (C1) $R' \le R$ and (C2) $R \le R'$.

Proof of C1. Suppose $|x| < R'$ so that $\sum_{n=0}^{\infty} n a_n x^{n-1}$ is absolutely convergent, and

$$|a_n x^n| = \frac{|x|}{n} |n a_n x^{n-1}| \le |n a_n x^{n-1}| \quad \text{for large } n.$$

So $\sum_{n=0}^{\infty} a_n x^n$ is absolute convergent via the comparison test. Thus, $|x| < R'$ implies that $|x| < R$ which is to say that $R' \le R$.

Proof of C2. Assume $|x| < R$. Pick r such that $|x| < r < R$. Then $\sum_{n=0}^{\infty} a_n r^n$ is absolutely convergent and

$$|n a_n x^{n-1}| = \frac{1}{|x|} \left(n \left| \frac{x}{r} \right|^n \right) |a_n| r^n.$$

Since $|x/r| < 1$, $n|x/r|^n \to 0$ as $n \to \infty$ and so $n \left| \frac{x}{r} \right|^n < 1$ for large n. We then deduce

$$|n a_n x^{n-1}| \le |a_n| r^n, \quad \text{for large } n.$$

So $\sum_{n=0}^{\infty} n a_n x^{n-1}$ converges absolutely by comparison to $\sum_{n=0}^{\infty} a_n |a_n r^n|$. Therefore, $|x| < R$ implies $|x| \le R'$ yielding $R \le R'$.

Combining (C1) and (C2)i we conclude that $R = R'$ as asserted. □

Theorem 26.5 *Suppose radius of convergence of $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is $R > 0$. Then $f \in C^{\infty}(-R, R)$, and its kth derivative may be computed on $(-R, R)$ by differentiating the series $\sum_{n=0}^{\infty} a_n x^n$ termwise k-times.*

Proof: For $|x| < R$, we know that $\sum_{n=0}^{\infty} n a_n r^{n-1}$ converges uniformly by the previous theorem and Theorem 25.3 now allows termwise differentiation.

General k case can be completed using induction. $\square$

Corollary 26.1 *Every power series $\sum_{n=0}^{\infty} a_n x^n$ with a positive radius of convergence is the Taylor series of its sum; that is, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$, $(R > 0)$, then*

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Proof: Noting that $\left(\frac{d}{dx}\right)^n x^k = 0$ when $k < n$ and $\left(\frac{d}{dx}\right)^n x^n = n!$, we have

$$f^{(n)}(x) = \frac{d^n}{dx^n}(a_0 + a_1 x + \cdots + a_n x^n + \cdots = n! a_n + \text{terms containing } x^m, m \geq 1).$$

So

$$f^{(n)}(0) = n! a_n, \quad \text{giving} \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

$\square$

Corollary 26.2 *If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$, $|x| < R$, $(R > 0)$, then $a_n = b_n$ for all n.*

Proof: We have $a_n = \frac{f^{(n)}(0)}{n!} = b_n$ for all n. $\square$

Example 26.5 *Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Show that*

$$f(x) = -\int_0^t \frac{\ln(1-t)}{t} dt.$$

Solution 26.2 *Observe that*

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}, \quad x f'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad \text{and} \quad (x f'(x))' = \sum_{n=1}^{\infty} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

So $x f'(x) = -\ln(1-x)$ or

$$f'(x) = -\frac{\ln(1-x)}{x}.$$

Integration yields the result.

27 Review Problems II

Problem 27.1 *Suppose $a_n > 0$ for each n, and $\sum_{n=1}^{\infty} a_n < \infty$. Is $\sum_{n=1}^{\infty} \frac{\sin a_n}{\sqrt{n} + n a_n}$ convergent?*

Problem 27.2 *For which values of α, does the series $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^{\alpha}}$ converge?*

Problem 27.3 *Does $\sum_{n=1}^{\infty} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n}$ converge?*

Problem 27.4 *If the partial sums s_n os $\sum_{n=1}^\infty$ are bounded show that $\sum_{n=1}^\infty a_n/n$ converges $\sum_{n=1}^\infty s_n/(n(n+1))$.*

Problem 27.5 *Is the series*

$$\frac{1}{1}+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{4n-3}+\frac{1}{4n-2}-\frac{1}{4n-1}-\frac{1}{4n}$$

absolutely convergent. If not, is it conditionally convergent?

Problem 27.6 *Prove Abel's test in lecture notes.*

Problem 27.7 *Let $a_n = \frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n+1)}$. Does $\sum_{n=1}^\infty a_n$ converge?*

Problem 27.8 *Let $a_{ij} = 1$ if $i-j=1$, $a_{ij}=-1$ if $i-j=-1$ and $a_{ij}=0$ elsewhere. Compute the corresponding double sums. How do your answers entie to the theory developed in lecture?*

Problem 27.9 *Find an explicit expression for the n^{th}partial sum of $\sum_{n=2}^\infty \ln(1-1/n^2)$ to show that this series converges to $-\ln 2$. Is this convergence absolute?*

Problem 27.10 *If $\sum a_n$ is an absolutely convergent series, prove that the series $\sum a_n \sin nx$ is absolutely and uniformly convergent.*

Problem 27.11 *If $\alpha\in\mathbb{R}$ and $|k|<1$, the integral*

$$F(\alpha,k)=\int_0^\alpha (1-k^2(\sin x)^2)^{-1/2}$$

is called an elliptic integral of the first kind. Show that

$$F(\pi/2,k)=\frac{\pi}{2}\sum_{n=0}^\infty\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdot 2n}\right)^2 k^{2n},$$

for $|k|<1$.

Problem 27.12 *Show by integrating the series for $1/(1+x)$ that if $|x|<1$, then*

$$\ln(1+x)=\sum_{n=1}^\infty\frac{(-1)^{n+1}}{n}x^n.$$

28 Abel's theorem

We know that if the radius of convergence of $f(x)=\sum_{n=0}^\infty a_nx^n$ is $R>0$, then the series converges to $f(x)$ uniformly on any compact subset of $(-R,R)$, and there f is continuous on $(-R,R)$.

Now assume that the series converges at one endpoint, say $x=R$. The question to follow is whether the uniformity of convergence and the continuity extend to this point. If the series converges absolutely at $x=R$, then the M-test (with $M_n=|a_n|R^n$) shows that the series converges absolutely and uniformly on $[-R,R]$, so its sum is continuous there.

When the series conditionally converges at the endpoint, the question is more interesting.

Theorem 28.1 *(Abel's theorem) If the series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = R$ (resp. $x = -R$), then it converges uniformly on the interval $[0, R]$ (resp. $[-R, 0]$) and hence defines a continuous function.*

Proof: Convergence at f $x = -R$ (and uniform convergence on $[-R, 0]$) of $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is equivalent to the convergence at $x = R$ (and uniform convergence on $[0, R]$) of $f(x) = \sum_{n=0}^{\infty} (-1)^n a_n x^n$. So it is enough to consider the convergence at $x = R$.

Also, convergence at $x = R$ (and uniform convergence on $[0, R]$) of $\sum_{n=0}^{\infty} a_n x^n$ is the same as convergence at $x = 1$ (and uniform convergence on $[0, 1]$) of $g(x) = \sum_{n=0}^{\infty} a_n R^n x^n$. So it is enough to assume that $\sum_{n=0}^{\infty} a_n$ converges and to prove that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, 1]$.

For $k \geq 1$, let $A_k = \sum_{n=k}^{\infty} a_n$ so that $a_k = A_k - A_{k+1}$. For $\ell > k$ and $x \in [0, 1]$, we have

$$\begin{aligned} a_k x^k + \cdots + a_\ell x^\ell &= (A_k - A_{k+1}) x^k + \cdots + (A_\ell - A_{\ell+1}) x^\ell \\ &= A_k x^k + A_{k+1}(x^{k+1} - x^k) + \cdots + A_\ell (x^\ell - \ell - 1) - A_{\ell+1} x^\ell. \end{aligned}$$

Now sending $\ell \to \infty$, (i) $A_{\ell+1} \to 0$ since $\sum_{n=0}^{\infty} a_n$ converges, and (ii) x^ℓ remains bounded. So the last term $A_{\ell+1} x^\ell$ in above equation tends to zero as $\ell \to \infty$. We obtain

$$\sum_{n=k}^{\infty} a_n x^n = A_k x^k + \sum_{n=k}^{\infty} A_{n+1}(x^{n+1} - x^n). \tag{12}$$

Letting $\epsilon > 0$, we may choose $K \in \mathbb{N}$ such that $|A_n| < \epsilon/2$ whenever $n \geq K$. Since $x \in [0, 1]$, $x^{n+1} - x^n \leq 0$, so (12) gives

$$\left| \sum_{n=K}^{\infty} a_n x^n \right| \leq |A_K| x^K + \sum_{n=K}^{\infty} |A_{n+1}| (x^n - x^{n+1}) \leq \frac{\epsilon}{2} x^K + \frac{\epsilon}{2} \sum_{n=K}^{\infty} (x^n - x^{n+1}).$$

When $x = 1$, $\sum_{n=K}^{\infty} (x^n - x^{n+1})$ vanishes; and if $0 \leq x < 1$, it equals to x. In either case, we obtain

$$\left| \sum_{n=K}^{\infty} a_n x^n \right| \leq \epsilon x^K \leq \epsilon, \quad \text{for all } x \in [0, 1]$$

when K is sufficiently large. Result follows. $\square$

29 Functions defined by improper integrals

In this section we are interested in examples of the form

$$F(x) = \int_c^{\infty} f(x, t) dt.$$

Definition 29.1 *$F(x) = \int_c^{\infty} f(x, t) dt$, $c \in \mathbb{R}$, is said to* ***converge uniformly*** *on an interval I if $\int_c^d f(x, t)$ converges to $\int_c^{\infty} f(x, t)$ uniformly for $x \in I$ as $d \to \infty$; i.e.*

$$\sup_{x \in I} \left| \int_d^{\infty} f(x, t) dt \right| \to 0$$

as $d \to 0$.

The next result is the analogue of the M-test for improper integrals.

Theorem 29.1 *Let g be a non-negative function on $[c, \infty)$ such that*

(i) $|f(x,t)| \leq g(t)$ for all $x \in I$ and $t \geq c$;

(ii) $\int_c^\infty g(t)dt < \infty$.

Then $\int_c^\infty f(x,t)dt$ converges absolutely and uniformly.

Proof: For the absolute convergence note that $\int_c^d |f(x,t)|dt$ is non-decreasing in d and that it is bounded by $\int_c^\infty g(t)dt < \infty$. So result follows from the monotone convergence theorem.

For the uniform convergence observe that

$$\sup_{x \in I} \left| \int_d^\infty f(x,t)dt \right| \leq \sup_{x \in I} \int_d^\infty g(t)dt \to 0$$

as $d \to \infty$. □

The consequences (and relevant proofs) of uniform convergence on continuity, differentiation and integration are pretty much the same as the case for sums. Therefore we will be having a brief discussion on them.

Theorem 29.2 *Suppose that $f(x,t)$ is a continous function on the set $\{(x,t) : x \in I, t \geq c\}$ and that the integral $\int_c^\infty f(x,t)dt$ is uniformly convergent for $x \in I$. Then the function $F(x) = \int_c^\infty f(x,t)dt$ is continuous on I.*

Proof: We already know from our discussion on proper integrals that the function $\int_c^d f(x,t)dt$ is continuous on I since f is assumed to be so. Then

$$\int_c^\infty f(x,t)dt = \lim_{d \to \infty} \int_c^d f(x,t)dt$$

is continuous as well since the uniform limits of continuous functions is continuous. □

The next result concerns the interchange of an improper and a proper integral.

Theorem 29.3 *Suppose that $f(x,t)$ is a continous function on the set $\{(x,t) : x \in I, t \geq c\}$ and that the integral $\int_c^\infty f(x,t)dt$ is uniformly convergent for $x \in I$. If $[a,b] \subset I$, then*

$$\int_a^b \int_c^\infty f(x,t)dtdx = \int_c^\infty \int_a^b f(x,t)dxdt.$$

Proof: Using uniform convergence for a sequence tending to ∞, we have

$$\int_a^b \int_c^\infty f(x,t)dtdx = \int_a^b \lim_{n \to \infty} \int_c^{d_n} f(x,t)dtdx = \lim_{n \to \infty} \int_c^{d_n} \int_a^b f(x,t)dxdt = \int_c^\infty \int_a^b f(x,t)dxdt$$

□

We do not go into details of the proof of the following result which are similar to our discussions above.

Theorem 29.4 *Suppose that $f(x,t)$ and its partial derivatives $\partial_x f(x,t)$ are continuous on the set $\{(x,t) : x \in I, t \geq c\}$. Assuming that $\int_c^\infty f(x,t)dt$ converges for all $x \in I$ and $|\partial_x f(x,t)| \leq g(t)$ for all $x \in I$, where $\int_c^\infty g(t)dt < \infty$, we gave*

$$\frac{d}{dx}\int_c^\infty f(x,t)dt = \int_c^\infty \frac{\partial f}{\partial x}(x,t)dt.$$

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30 Some examples

Example 30.1 *Show that for $0 < a < b$*

$$\int_0^\infty \frac{\arctan(bt) - \arctan(at)}{t}dt = \frac{\pi}{2}\ln\left(\frac{b}{a}\right).$$

Solution. First observe that

$$\frac{\arctan(bt) - \arctan(at)}{t} = \int_a^b \frac{1}{x^2t^2+1}dx.$$

Here, for $x \geq a$, $\frac{1}{x^2t^2+1} \leq \frac{1}{a^2t^2+1}$ and also $\int_0^\infty \frac{1}{a^2t^2+1}dt < \infty$. So $\int_0^\infty \frac{1}{x^2t^2+1}dt$ is uniformly convergent for $x \geq a$. Then

$$\begin{aligned}
\int_0^\infty \frac{\arctan(bt) - \arctan(at)}{t}dt &= \int_0^\infty \int_a^b \frac{1}{x^2t^2+1}dxdt \\
&= \int_a^b \int_0^\infty \frac{1}{x^2t^2+1}dtdx \\
&= \int_a^b \left(\frac{\arctan(xt)}{x}\Big|_0^\infty\right)dx \\
&= \int_a^b \frac{\pi}{2x} \\
&= \frac{\pi}{2}\ln\left(\frac{b}{a}\right).
\end{aligned}$$

□

Example 30.2 *Prove that*

$$\int_0^\infty t^{2k}e^{-xt^2}dt = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(k-\frac{1}{2}\right)\frac{\sqrt{\pi}}{2x^{k+1/2}}.$$

Solution. Let $F(x) = \int_0^\infty e^{-xt^2}dt$ for $x > 0$. We will investigate $F^{(k)}(x)$ in two distinct ways.

Approach 1. By doing a substitution in Gauss integral, we have $F(x) = \frac{\sqrt{\pi}}{2}x^{-1/2}$. Differentiating this k many times we have

$$F^{(k)}(x) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(k-\frac{1}{2}\right)\frac{\sqrt{\pi}}{2x^{k+1/2}}.$$

Approach 2. Note that $\frac{\partial^k}{\partial x^k}(e^{-xt^2}) = (-t^2)^k e^{-xt^2}$. It then follows that $(-1)^k \int_0^\infty t^{2k} e^{-xt^2} dt$ converges uniformly for $0 < \delta \leq x < \infty$ since $t^{2k}e^{-xt^2} \leq t^{2k}e^{-\delta t^2}$ for $x \geq \delta$ (and $\int_0^\infty t^{2k}e^{-\delta t^2} dt < \infty$). Therefore

$$F^{(k)}(x) = \frac{d^k}{dx^k} \int_0^\infty e^{-xt^2} dt = \frac{\partial^k}{\partial x^k} \int_0^\infty e^{-xt^2} dt = \int_0^\infty t^{2k} e^{-xt^2} dt.$$

Comparing the two expressions for $F^{(k)}(x)$, result follows. □

31 Gamma function: Basic properties

Definition 31.1 *The* ***gamma function*** $\Gamma(x)$ *is defined for* $x > 0$ *by*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Theorem 31.1

Proof:

□

Note 31.1 *(N1) It can also be shown that* $\Gamma(x)$ *is uniformly convergent on* $[\delta, C]$ *for any* $0 < \delta < C$*. By Theorem XX,* $\Gamma(x)$ *is then continuous.*

(N2) $\Gamma(x)$ *is actually* C^∞*, and its derivatives can be calculated by differentiating under the integral sign:*

$$\Gamma^{(k)}(x) = \int_0^\infty (\ln t)^k t^{x-1} e^{-t} dt.$$

Proposition 31.1 $\Gamma(x)$ *satisfies the functional equation*

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0.$$

Proof: We have

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty t^x e^{-t} \\ -t^x e^{-t}\Big|_0^\infty + \int_0^\infty x t^{x-1} e^{-t} dt \\ &= x\Gamma(x). \end{aligned}$$

□

The proof of the following theorem is now immediate, and is left for you.

Proposition 31.2 *(i)* $\Gamma(1) = 1$*;*

(ii) $\Gamma(n) = (n-1)!$*;*

(iii) $\Gamma(1/2) = \sqrt{\pi}$*;*

(iv) $\Gamma(n + 1/2) = (n - 1/2) \cdots 3/2 \cdot 1/2 \cdot \sqrt{\pi}$.

Note that (ii) says that the gamma function is an extension of the factorial function.

Proposition 31.3 *For $a, b > 0$, we have*

$$a(a+b)\cdots(a+nb) = b^{n+1}\frac{\Gamma(a/b+n+1)}{\Gamma(a/b)}.$$

Proof: First observe that for $c > 0$,

$$\Gamma(c+n+1) = (c+n)\Gamma(c+n) = \cdots = c\Gamma(c).$$

Using this for $c = a/b$, we obtain

$$\begin{aligned} a(a+b)\cdots(a+nb) &= b^{n+1}(a/b)(a/b+1)\cdots(a/b+n) \\ &= b^{n+1}\frac{\Gamma(a/b+n+1)}{\Gamma(a/b)} \end{aligned}$$

□

Exercise 31.1 *Can you describe how $\Gamma(x)$ looks like for $x > 0$.*

Next, we will see that some useful integrals can be transformed into expressions involving the gamma function. For example,

$$\int_0^\infty t^{x-1}e^{-bt}dt \underbrace{=}_{u=bt} \int_0^\infty \left(\frac{u}{b}\right)^{x-1} e^{-u}\frac{du}{b} = b^{-x}\Gamma(x), \quad (b > 0),$$

and

$$\int_0^\infty t^{2x-1}e^{-t}dt \underbrace{=}_{v=t^2} v^{\frac{2x-1}{2}}e^{-v}\frac{dv}{2v^{1/2}} = \frac{1}{2}\Gamma(x).$$

One other important function related to the gamma function is the beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x, y < 0.$$

By using the substitution $t = \sin^2\theta$ so that $1-t = \cos^2\theta$, $dt = 2\sin\theta\cos\theta d\theta$, we have

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}dt \\ &= \int_0^{\pi/2} (\sin^2\theta)^{x-1}(\cos^2\theta)^{y-1}2\sin\theta\cos\theta d\theta \\ &= 2\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta d\theta. \end{aligned}$$

32 Relation between Beta and Gamma functions

Theorem 32.1 *For $x, y > 0$*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof: We have

Note that the proof here is very similar to the argument we used for computing $\int_0^\infty e^{-x^2}dx$.

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= 4\int_0^\infty t^{2x-1}e^{-t^2}dt\int_0^\infty s^{2y-1}e^{-s^2}ds \\
&= 4\int_0^\infty\int_0^\infty s^{2y-1}t^{2x-1}e^{-s^2-t^2}dsdt \\
&= \int_0^{\pi/2}\int_0^\infty (r\cos\theta)^{2y-1}(r\sin\theta)^{2x-1}e^{-r^2}rdrd\theta \\
&= \int_0^{\pi/2}(\cos\theta)^{2y-1}(\sin\theta)^{2x-1}d\theta\int_0^\infty r^{2x+2y-1}e^{-r^2}rdr \\
&= B(x,y)\Gamma(x,y).
\end{aligned}$$

□

The relation we just proved will help us to prove two other useful results.

Theorem 32.2 *(Duplication formula) For $x > 0$, we have*

$$\Gamma(2x) = \pi^{-1/2}2^{2x-1}\Gamma(x)\Gamma(x+1/2).$$

Proof: Recall that

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{13}$$

Choosing $y = x$, this gives

$$\frac{(\Gamma(x))^2}{\Gamma(2x)} = \int_0^1 (t(1-t))^{x-1}dt,$$

and noting that $(t(1-t))^{x-1}$ is symmetric around $t = 1/2$, this yields

$$\frac{(\Gamma(x))^2}{\Gamma(2x)} = 2\int_0^{1/2}(t(1-t))^{x-1}dt.$$

Now, using the substitution $t = \frac{1}{2}(1-s^{1/2})$ so that $dt = -\frac{1}{4}s^{-1/2}ds$ and $t(1-t) = \frac{1}{4}(1-s)$, we obtain

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2^{1-2x}\int_0^1 s^{-1/2}(1-s)^{x-1}ds = 2^{1-2x}\frac{\Gamma(1/2)\Gamma(x)}{\Gamma(x+1/2)},$$

where in the last step we used the relation in (13). □

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Next, we prove:

Theorem 32.3 *For $a > 0$,*

$$\lim_{x\to\infty}\frac{\Gamma(x+a)}{x^a\Gamma(x)} = 1.$$

Proof:

□

33 Stirling's formula: a preliminary version

34 Stirling's formula

35 Path integrals

Given a "smooth" curve C parametrized by $g(t)$, $a \leq t \leq b$, we know that the arclength of C is given by

$$\int_a^b \|g'(t)\| dt.$$

Since $g'(t)$ is the velocity vector at time t, the arc length is the integral of the speed from time a to time b.

More generally, we may define integrals with respect to its arclength. We focus on 3-D setting although the theory for other dimensions is the same. For a given function f on $\mathbb{R}^3$ and for some curve C, let (x_k, y_k, z_k) form a partition on C. Letting

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k$$

where Δs_k is the distance between successive parts of the partition, $\int_C f(x, y, z)ds$ is defined as the "limit" of S_n - fill in the details yourself.

Such integrals are, for example, used in mass and moment computations.

For computing path integrals, we first parametrize C, say

$$r(t) = g(t)i + h(t)j + k(t)k, \qquad a \leq t \leq b.$$

Noting that

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k = \sum_{k=1}^{n} f(x_k, y_k, z_k)\frac{\Delta s_k}{\Delta t_k}\Delta t_k,$$

we then compute the integral as

$$\int_C f(x, y, z)ds = \int_a^b f(x(t), y(t), z(t))\|r'(t)\| dt :$$

(Here and below C is still assumed to be smooth. Why?)

Example 35.1 *Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1, 1, 1)$.*

Solution 35.1 *Parametrize the curve by $r(t) = ti + tj + tk$, $0 \leq t \leq 1$. An easy computation shows that $r'(t) = \sqrt{3}$. Then*

$$\int_C f(x, y, z) = \int_0^1 f(t, t, t)\sqrt{3}dt = \cdots = 0.$$

Note 35.1 *Path integrals are additive over curves. If $C_1, C_2, \ldots, C_n$ are disjoint curves, then (with some abuse of notation)*

$$\int_{C_1 \cup \cdots \cup C_n} f ds = \sum_{i=1}^{n} \int_{C_i} f ds.$$

36 Line integrals of vector fields

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field defined on some neighborhood of a smooth curve C in $\mathbb{R}^n$. Then the **line integral** of F over C is

$$\int_C F \cdot dx = \int_C (F_1 dx_1 + \cdots + F_n dx_n).$$

If C is parametrized by $x = g(t)$, $a \le t \le b$, then

$$\int_C F \cdot dx = \int_a^b F(g(t)) \cdot g'(t) dt.$$

Example 36.1 *Let C be the ellipse formed by the intersection of the circular cylinder $x^2 + y^2 = 1$ and the plane $z = 2y + 1$, oriented counterclockwise as viewed from above, and let $F(x, y, z) = (y, z, x)$. Compute $\int_C F \cdot d\mathbf{x}$.*

Solution. First we will parametrize C. Clearly we may let $x = \cos t, y = \sin t$, $0 \le t \le 2\pi$ since (x, y) should lie on the unit circle - keep orientation in mind. Then $z = 2y + 1$ says that z should be $z = 2 \sin t + 1$. Summary: the parametrization is given by

$$x = \cos t, y = \sin t, z = 2 \sin t + 1, \qquad 0 \le t \le 2\pi.$$

From this we see that $d\mathbf{x} = (-\sin t, \cos t, 2 \cos t)$ and

$$F \cdot d\mathbf{x} = (\cos 2t + \sin 2t + \cos^2 t) dt.$$

Then

$$\int_C F \cdot d\mathbf{x} = \int_0^{2\pi} (\cos 2t + \sin 2t + \cos^2 t) dt = \pi.$$

□

Note 36.1 *(N1) (Representing the line integral as a path integral) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, C be a curve with parametrization $x = g(t) : \mathbb{R} \to \mathbb{R}^n$, $a \le t \le b$. Define the unit tangent function $t = t(x)$*

$$t(g(u)) = \frac{g'(u)}{\|g'(u)\|}$$

and the component $F_{tang} = F_{tang}(x)$ of F in direction t

$$F_{tang}(x) = F(x) \cdot t(x).$$

Then

$$\begin{aligned} F(g(u)) \cdot g'(u) du &= F(g(u)) \cdot t(g(u)) \|g'(u)\| du \\ &= F_{tang}(g(u)) ds. \end{aligned}$$

So,

$$\int_C F \cdot dx = \int_C F_{tang} ds.$$

Conclusion: *The line integral $\int_C F \cdot dx$ is the path integral of the tangential component of F with respect to arc length.*

(N2) If F is a force field, then $\int_C F \cdot dx$ is the work done by the force field F on a particle that traverses the curve C.

There are other physical interpretations such as flux, flow, etc. (Maybe I include a discussion in a lter version of these notes.)

(N3) $F \cdot dx = F_1 dx_1 + \cdots + F_n x_n$ is called a differential form - third year.

(N4) When $n = 1$, a vector field turns into a scalar function. There is just a small twist off:when we write $\int_{[a,b]} f(x)dx$, we mean $\int_a^b f(x)dx$, and not $\int_b^a f(x)dx$.

37 Green's theorem

Green's theorem is a special case of a general theorem which basically says that (quoted from Folland) "the integral of something over the boundary of a region equals the integral of something else over the region itself". Of course we will need some assumptions on the region and the integrand.

Definition 37.1 *A **simple closed curve** is a curve that can be parametrized by a continuous map $x = g(t), a \leq t \leq b$ such that $g(a) = g(b)$, but $g(t) \neq g(s)$ unless $\{s, t\} = \{a, b\}$.*

Definition 37.2 *A **regular region** in $\mathbb{R}^n$ is a compact set in $\mathbb{R}^n$ that is the closure of its interior.*

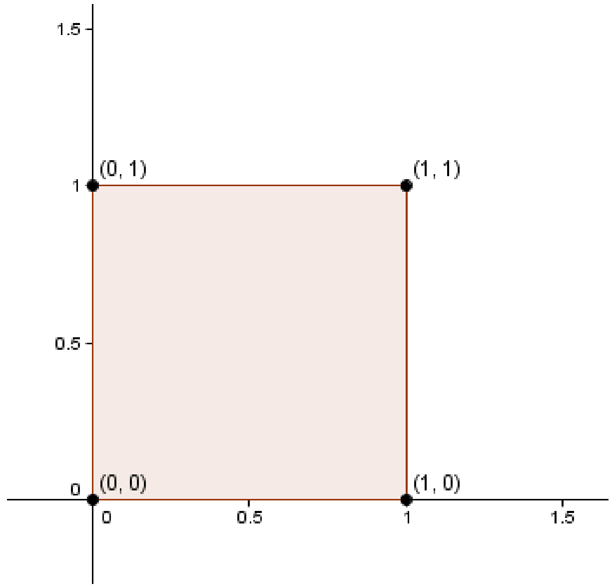
Example 37.1 *(E1) A closed ball in $\mathbb{R}^n$ is a regular region.*

(E2) A closed line in $\mathbb{R}^n$, $n > 1$, is not a regular region since its interior is the empty set.

Below in this section we now <u>assume</u> that $n = 2$.

Definition 37.3 *A regular region S is said to have a **piecewise smooth boundary** if the boundary consists of a finite union of disjoint, piecewise smooth simple closed curves.*

Example 37.2 *A square has a piecewise smooth boundary:*



Example 37.3 *Note that we allow S to contain "hole(s)" like a simit:*

That is, the boundary of S can be disconnected. In this case, the positive orientation on ∂S is the orientation on each of the closed curves that make up the boundary such that the region S is on the left with respect to the positive direction on the curve.

We need one last definition.

Definition 37.4 *We say that a region S is x-**simple** if it has the following form*

$$S = \{(x, y) : a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}, \tag{14}$$

*where ϕ_1 and ϕ_2 are continuous, piecewise smooth functions on $[a, b]$. S is said to be y-**simple** if it has the following form*

$$S = \{(x, y) : c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}, \tag{15}$$

where ψ_1 and ψ_2 are continuous, piecewise smooth functions on $[c,d]$.

Example 37.4 *Consider the region S bounded by the curve $y = \frac{1}{8}x^3 - 1$ and the line $x+2y = 2$:*

FIGURE

Then, it is easily seen that S is both x-simple and y-simple by respective choices (in (14) and (14))

$$a = 0, \quad b = 2, \quad \phi_1(x) = \frac{1}{8}x^3 - 1, \quad \phi_2(x) = 1 - \frac{1}{2}x,$$

and

$$c = -1, \quad d = 1, \quad \psi_1(y) = 0, \quad \psi_2(y) = \begin{cases} 2(y+1)^{1/3}, & \text{if } -1 \le y \le 0 \\ 2 - 2y, & \text{if } 0 \le y \le 1. \end{cases}$$

Before stating Green's theorem, we lastly note that for a given continuous vector field $F = (F_1, F_2)$ on $\mathbb{R}^2$, we denote by $\int_{\partial S} F \cdot dx$ or $\int_{\partial S} F_1 dx_1 + \cdots + F_n dx_n$ the sum of the line integrals of F over the positively oriented closed curve that makes up ∂S.

The following discussion is pretty much the same as Folland.

Theorem 37.1 *(Green's theorem) Suppose S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S. Suppose that F is a vector field of class C^1 on $\overline{S}$. Then, denoting $F = (P,Q)$ we have*

$$\int_{\partial S} Pdx + Qdy = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Proof of Green's theorem for x-simple and y-simple regions. Assume that S is x-simple and y-simple. Since S is x-simple we may consider ∂S as the union of (i) the curve $y = \phi_1(x)$, oriented from left to right, (ii) the curve $y = \phi_2(x)$, oriented from right to left, (iii) vertical lines segments at $x = a$ and $x = b$ (which can be single points). In particular, the line integral $\int_{\partial S} Pdx$ will be the sum of these four separate integrals.

On the vertical line segments, x is constant and so $dx = 0$, so no contribution is done through these pieces. On the curves $\phi_1(x)$ and $\phi_2(x)$, we take x as the parameter, noting that the orientation is reversed for ϕ_2, and we obtain

$$\int_{\partial S} Pdx = \int_a^b P(x, \phi_1(x))dx - \int_a^b P(x, \phi_2(x))dx.$$

Using the fundamental theorem of calculus, we then have

$$\int\int_S \frac{\partial P}{\partial y} dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y} dydx = \int_a^b (P(x, \phi_2(x)) - P(x, \phi_1(x)))dx.$$

These two yield

$$\int_{\partial S} Pdx = -\int\int_S \frac{\partial P}{\partial y} dA.$$

The same approach also gives

$$\int_{\partial S} Qdy = \int\int_S \frac{\partial Q}{\partial x} dA.$$

(Why no minus sign here?)

Adding the last two equalities result follows for x-simple and y-simple regions. $\square$

Proof of Green's theorem for a slightly more general setting. This time we assume that S can be up into finitely many subregions, say $S = S_1 \cup \cdots \cup S_k$ under the assumptions

(i) the S_j's may intersect along common edges but have disjoint interiors;

(ii) each S_j has piecewise smooth boundary and is both x-simple and y-simple. See the following example

FIGURE

Here the S_j's overlap only in a set of <u>zero content</u>. Using additivity, we then have

$$\int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \sum_{j=1}^{k} \int\int_{S_j} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Also, we know that

$$\int_{\partial S} (Pdx + Qdy) = \sum_{j=1}^{k} \int_{\partial S_j} (Pdx + Qdy).$$

Here, note that the integrals over the parts of the boundaries of the S_j's that are not parts of the boundary of S all cancel out - due to opposite directions.

The proof of Green's theorem under this setting is now proven. $\square$

Proof of Green's theorem in even more general setting. See appendix of Folland - it is complicated though.

Example 37.5 *Let C be the unit circle $x^2 + y^2 = 1$, oriented counterclockwise. find*

$$\int_C (\sqrt{1+x^2} - ye^{xy} + 3y)dx + (x^2 - xe^{xy} + \ln(1+y^4))dy.$$

Solution. Letting D be the unit disc and using Green's theorem, we have

$$\int_C (\sqrt{1+x^2} - ye^{xy} + 3y)dx + (x^2 - xe^{xy} + \ln(1+y^4))dy$$

equals

$$\int\int_D \left(\frac{\partial}{\partial x}\left((x^2 - xe^{xy} + \ln(1+y^4)\right) - \frac{\partial}{\partial y}\left(\sqrt{1+x^2} - ye^{xy} + 3y)dx\right)\right) = \int\int_D (2x-3)dA = -6\pi.$$

$\square$

Example 37.6 *This is an example to keep in mind. Suppose that we are interested in computing the area of a rectangular region S in the plane. This can be done using Green's theorem, and turning the problem into a line integral. Indeed,*

$$\text{Area of } S = \int_{\partial S} xdy = -\int_{\partial S} ydx = \int_{\partial S} \frac{1}{2}(xdy - ydx).$$

All these integrals are equal to $\int\int_S 1dA = $ Area of S. $\square$

Example 37.7 *Use Green's Theorem to calculate the area of the disk S of radius r defined by $x^2 + y^2 \leq \alpha^2$.*

Solution. Parametrize the boundary C of the disk counterclockwise by

$$r(t) = (r\cos t, r\sin t), \qquad 0 \leq t \leq 2\pi$$

so that

$$r'(t) = (-r\sin t, r\cos t).$$

Then, by Green's theorem

$$\begin{aligned}
\text{Area of } S &= \int\int_S dA \\
&= \frac{1}{2}\int_C xdy - ydx \\
&= \frac{1}{2}\int_0^{2\pi} ((r\cos t)(r\cos t) - (r\sin t)(-r\sin t))dt \\
&= \cdots \\
&= \pi r^2.
\end{aligned}$$

An important point next.

Recall: The line integral $\int_{\partial S} F \cdot dx$ is the integral of the tangential component F over ∂S.

Besides this recall, Green's theorem can also be interpreted as a statement about the integral of the normal component of the vector field. This later becomes important in both theory and applications.

Corollary 37.1 *Suppose S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S, and let $n(x)$ be the unit outward normal vector to ∂S at $x \in \partial S$. Suppose further that F is a vector field of class C^1 on $\overline{S}$. Then*

$$\int_{\partial S} F \cdot nds = \int\int_S \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right) dA.$$

Proof. In order to proceed, quoting Folland, recall that counterclockwise and clockwise rotations by 90 degrees in the plane are given by $R_+(x,y) = (-y,x)$ and $R_-(x,y) = (y,-x)$, respectively. So, if $t = (t_1, t_2)$ is the unit tangent vector to ∂S at a point on ∂S, point in the forward direction. Then $n = R_-(t) = (t_2, -t_1)$ is the unit normal vector to ∂S pointing out of S. Given a vector field $F = (F_1, F_2)$, let $\overline{F} = R_+(F) = (-F_2, F_1)$ be the vector obtained by rotating the values of F by 90 degrees counterclockwise. Then the normal component of F is the tangential component of $\overline{F}$,

$$F \cdot n = F_1 t_2 - F_2 t_1 = \overline{F} \cdot t.$$

Result follow by Green's theorem. □

Note that $\nabla f \cdot n$ is the directional derivative of f in outward normal direction to ∂S, denoted by $\partial f/\partial n$. Then the corollary says that

$$\int_{\partial S} \frac{\partial f}{\partial n} ds = \int \int_S \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} \right) dA.$$

The integrand here is the Laplacian of f, which you will see at several other places.

Lastly, a corollary to the corollary:

Corollary 37.2 *Assume that F is the gradient of a C^2 function f. Then*

$$\int_{\partial S} \nabla f \cdot n dx = 0.$$

Proof. We have $F_1 = \partial_1 f$ and $F_2 = \partial_2 f$. Then we obtain

$$\int_{\partial S} \nabla f \cdot n dx = \int \int_S (\partial_1 \partial_2 f - \partial_2 \partial_1 f) dA = \int \int_S 0 dA = 0.$$

$\square$

38 Surface area

39 Surface integrals

40 Vector derivatives

41 Divergence theorem

42 Stoke's theorem

43 Further topics