Problem Section

1. PS-I

1.1. Part I

Exercise 1. Suppose that d is a metric on a set X. Prove that the inequality $|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$ holds for all $w, x, y, z \in X$.

Solution. Starting with $d(x,y)$, we have

$$d(x,y) \leq d(x,z) + d(z,y) \leq d(x,z) + d(z,w) + d(y,w).$$

Hence we have

$$d(x,y) - d(z,w) \leq d(x,z) + d(y,w).$$

Now, if we start with $d(z,w)$, we would get with the same reasoning:

$$d(z,w) - d(x,y) \leq d(x,z) + d(y,w).$$

This two together yields the result.

Exercise 2. Suppose (X,d) is a metric space and $e(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$. Show that e is a metric on X.

Solution. Positiveness and symmetry is inherited by d. We only need to check the triangle inequality:

$$e(x,y) \leq e(x,z) + e(z,y).$$

We know that

$$d(x,y) \leq d(x,z) + d(z,y).$$

Now dividing by $1 + d(x,y)$, we have

$$e(x,y) \leq \frac{d(x,z)}{1+d(x,y)} + \frac{d(z,y)}{1+d(x,y)}.$$

Now, if we have $d(x,y) \geq d(x,z), d(z,y)$, we have

$$e(x,y) \leq \frac{d(x,z)}{1+d(x,y)} + \frac{d(z,y)}{1+d(x,y)} \leq e(x,z) + e(z,y),$$

and we are done.

Now, for the other cases, let us observe that the function $f(x) = \dfrac{x}{1+x}$ is an increasing function. This provides us with the fact that if $d(x,y) \leq d(x,z)$ (or $d(z,y)$) we have $e(x,y) \leq e(x,z)$ (or $e(z,y)$ respectively.) This with the fact that $e \geq 0$ finishes the proof.

Exercise 3. Suppose Z is a set, (X,d) is a metric space and $f : Z \to X$ is an injective function. Show that $(a,b) \to d(f(a), f(b))$ is a metric on Z.

Solution. Let us call the map defined as $\zeta(a,b)$. Since d is a metric and is greater than or equal to 0, we observe that ζ inherits this property as well. Now, suppose $\zeta(a,b) = 0$. Then we have

$$d(f(a), f(b)) = 0,$$

hence $f(a) = f(b)$. By injectivity of f, we see that $a = b$.

Next, we observe that

$$\zeta(a,b) = d(f(a), f(b)) = d(f(b), f(a)) = \zeta(b,a).$$

Hence ζ is symmetric.

For the triangle inequality, we use the triangle inequality to see

$$\zeta(a,b) = d(f(a), f(b)) \leq d(f(a), f(c)) + d(f(c), f(b)) = \zeta(a,c) + \zeta(c,b).$$

Thus, ζ is a metric.

Exercise 4. Show that every isometry is injective.

Solution. Let (X,d) and (Y,m) be two metric spaces and suppose $f : X \to Y$ is an isometry. Suppose $f(a) = f(b)$. Then we see that

$$0 = m(f(a), f(b)) = d(a,b),$$

and since d is a metric on X, this yields $a = b$. Hence, f is injective.

1.2. Part II

Exercise 5. With reference to 2.3.1, find a metric space X, an element x of X and non-empty subsets of A and B of X with $A \subset B$ such that $dist(x, A) > dist(x, B) + diam(B/A)$.

Solution. In 2.3.1, we prove that with the above conditions

$$dist(x, B) \leq dist(x, A) \leq dist(x, B) + diam(B).$$

Now, if we can construct sets such that $diam(B/A) = 0$, and $dist(x, A) > dist(x, B)$, with this theorem we are done. Let $\mathbb{R}$ be our metric space with Euclidean metric. Now, let $A = (0, 1)$, and $B = (0, 1) \cup \{10\}$. Clearly we have $A \subset B$, and $diam(B/A) = diam(\{10\}) = 0$. If we let $x = 11$, we see that

$$dist(x, A) = 10 > 1 = dist(x, B) = dist(x, B) + diam(B/A).$$

Exercise 6. Suppose that X is a metric space and that $S \subset X$. Show that $iso(S) = S/acc(S)$.

Solution. First, let us recall the definitions.

A point $a \in S$ is an isolated point iff $dist(a, S/\{a\}) \neq 0$.

A point $a \in S$ is an accumulation point iff $dist(a, S/\{a\}) = 0$.

Let $a \in iso(S)$, then by definition, $dist(a, S/\{a\}) \neq 0$. Hence $a \notin acc(S)$. Thus, $a \in S/acc(S)$. This proves that $iso(S) \subset S/acc(S)$.

To prove the other inclusion, let $a \in S/acc(S)$. Hence, $dist(a, S/acc(S)) \neq 0$. Thus $a \in iso(S)$.

Exercise 7. Suppose C is a non-empty collection of subsets of a metric space X. Show that there are inclusions $acc(\cap C) \subset \cap\{acc(S) : S \in C\}$, and $\cup\{acc(S) : S \in C\} \subset acc(\cup C)$ and that each of them may be proper.

Solution. We will only show the intersection. Same type of reasoning may be used for the other one. Let $a \in acc(\cap C)$. Hence by definition we have $dist(a, \cap C/\{a\}) = 0$. Now, we know by theorem 2.4.3 that

$$\sup(dist(a, S/\{a\}) : S \in C) \leq dist(a, \cap C/\{a\}),$$

hence we have $dist(a, S/\{a\}) = 0$ for all $S \in C$, which yields the inclusion.

To show that the inclusion may be proper, we need to find a point that is in the intersection of the accumulation points, but not in the accumulation points of the intersection. Let $\mathbb{R}$ be the metric space and $C = \{\mathbb{Q}, \mathbb{R}/\mathbb{Q}\}$. Then each set is dense, so the accumulation points are all of $\mathbb{R}$. However, their intersection is empty. Hence, the inclusion is proper, i.e $\emptyset \subset \mathbb{R}$.

2. PS-II

Exercise 8. Verify that the graph $\Gamma = \{(x, \sin(\frac{1}{x})) : x \in \mathbb{R}^+\}$ has boundary $\Gamma \cup \{(0, y) : y \in [-1, 1]\}$.

Solution. Γ is a curve in $\mathbb{R}^2$, hence we know that it is itself in the boundary. Since any neighborhood of a point in Γ will be an open disc which contains points other that the curve. Then, we have that if a point $x \notin \Gamma$ with $x \in \partial\Gamma$, we need to have $x \in acc(\Gamma)$. First, we observe that a point $(x, y) \notin \Gamma \cup \{(0, y) : y \in [-1, 1]\}$ cannot be in the boundary. Suppose there is a point (x, y) that is in the boundary. We know that $x \neq 0$, and $f(x) \neq y$. However, the boundary condition means that there is sequence $(x_n, y_n) \to (x, y)$. But with $f(x_n) = y_n$ which contradicts the continuity.

Now, let $(0, y)$ be a point. Then for any neighborhood of radius ϵ, we have that $\sin(\frac{1}{\epsilon}) \to \infty$ with oscillation. Hence, we can place an element of the graph around the neighborhood.

Exercise 9. Consider the set F of functions from $[0, 1] \to [0, 1]$ with the supremum metric. Let C denote the collection of constants functions in F. Show that $\partial C = C$.

Solution. Let a be the constant function and $f_n = a + \frac{1}{n}$ in $[0, \frac{1}{n}]$ and $f_n = a$ in $(\frac{1}{n}, 1]$, then we see that $f_n' s$ are not constant and they tend to a. Hence $dist(a, C^c) = 0$. Clearly, the other way can be done with $f_n = a + \frac{1}{n}$ which are constants. Hence $C \subset \partial C$.

Now, suppose there is a non-constant function $g \in \partial C$, then we need to have $g \in acc(C)$. Hence, for any $\epsilon = \frac{1}{n} > 0$, we have $a_n \in [0, 1]$ such that $\sup |g(x) - a_n| \leq \frac{1}{n}$. But then, supposing $\lim a_n = a$, we need to have $\sup |g(x) - a| = 0$, which implies that $g = a$ identically. Hence, g is a constant function. Thus, $\partial C = C$.

Exercise 10. Show that every countable subset of $\mathbb{R}$ has empty interior in $\mathbb{R}$ and is therefore included in its own boundary.

Solution. Let S such a set and $s \in S$. Consider the neighborhood $B_n = B_{\frac{1}{n}}(s)$. This contains uncountable number of elements, hence there is an element $s_n \in B_n$ with $s_n \in S^c$. Then, if we consider the sequence $\{s_i\}_{i=1}^{\infty}$, we see that $dist(s, S^c) = 0$, hence $s \in \partial S^c$. But then, $S^o = S/acc(S^c) = \emptyset$.

Exercise 11. Suppose X is a metric space and S is a subset of X. Show that $diam(S^o)$ need not be the same as $diam(S)$.

Solution. We will give a counterexample. Let $A = (0, 1) \cup \{10\}$. We see that $diam(S^o) = diam((0, 1)) = 1$, and $diam(S) = 10$.

Exercise 12. Suppose that X is a metric space and $S \subset X$. Show that $\overline{S} = iso(S) \cup acc(S)$.

Solution. First, we observe that $iso(S) \subset S \subset \overline{S}$, and $acc(S) \subset \overline{S}$, hence we have $iso(S) \cup acc(S) \subset \overline{S}$.

Now, we need to prove the inclusion in the other direction. Let $x \in \overline{S}$. If $x \in iso(S)$, then we are done. Suppose $x \notin iso(S)$. Hence we know that $dist(x, S) = 0$, hence $x \in acc(S)$, and we are done.

Exercise 13. Suppose (X, d) is a metric space and $A, B \subset X$. Prove that

$$dist(\overline{A}, \overline{B}) = dist(A, B).$$

Exercise 14. Suppose X is a metric space and $A \subset X$. Must $(A^o)^c$ equal $Cl((\overline{A})^c)$?

Solution. Consider the set $\mathbb{Q}$ in $\mathbb{R}$ with the Euclidean metric. We observe that $\mathbb{Q}^o = \emptyset$, hence $(\mathbb{Q}^o)^c = \mathbb{R}$.

On the other hand, we have $\overline{Q} = \mathbb{R}$, which yields $\overline{Q}^c = \emptyset$, hence we have $Cl(\emptyset) = \emptyset$. They are not equal always.

Question to think: What can be changed to equalize them? When are they equal as is?(I mean what conditions can we add to the set A?)

3. PS-III

Exercise 15. Show that the points of a discrete metric space are all isolated.

Solution. Since (X, d) is a discrete metric space any subset of X is open. Then for any $a \in X$ there exists an r_a such that $B(a, r_a) \cap \{a\} = \{a\}$. Hence a is an isolated point.

Exercise 16. Suppose (X, d) is a metric space. Show that the metric $\dfrac{d(a, b)}{d(a, b) + 1}$ generates the same topology as $d(a, b)$.

Solution. We will show that an open ball in one can be put into an open ball in the other, hence the basis will be the same. Hence are topologically equivalent. For the first metric, open balls will be denoted by B, and for the second metric by D.

Suppose we have an open ball in the second metric,i.e

$$D_\epsilon(a) := \{x \in X : e(a, x) < \epsilon\}.$$

We then observe that $d(a, x) < \dfrac{d(a, x)}{d(a, x) + 1} = e(a, x) < \epsilon$. Hence, the open ball $B_\epsilon(a) \subset D_\epsilon(a)$.

Suppose we have an open ball in the first metric, i.e.

$$B_\epsilon(a) = \{x \in X : d(a, x) < \epsilon\}.$$

We want to find an open ball in the second metric that is in $B\epsilon_(a)$,i.e.

$$D_\delta(a) := \{x \in X : e(a, x) < \delta\}$$

such that $D_\delta(a) \subset B_r(a)$. We have

$$\frac{d(x, a)}{d(x, a) + 1} < \delta,$$

which implies that $d(a, x) < \dfrac{\delta}{1 - \delta}$. Hence if we pick $\delta = \dfrac{\epsilon}{1 - \epsilon}$ we are done.

Now, is this enough? We need to find an open neighborhood around any point, but considering the distance around a' to the boundary of $B_\epsilon(a)$, we can fit another ball using the same techniques. Thus, this finishes the proof.

Exercise 17. Suppose X is a metric space. Let C denote the collection of all dense subsets of X. Show that $\cap C = iso(X)$.

Solution. Let $s \in iso(X)$, and S be an arbitrary dense subset of X. We know that $\overline{S} = X$, hence $s \in \overline{S}$, but since s is an isolated point, we need to have $s \in S$. This proves that $iso(X) \subset \cap C$. For the other direction, consider a point $x \in X/iso(X)$. Then we see that $X/\{x\}$ is a dense subset, hence $x \notin \cap C$. This proves the claim.

Exercise 18. Suppose X is a metric space and S is a subset of X. We say that S is nowhere dense in X if and only if the closure of S in X has empty interior. Show that every nowhere dense subset of X has dense complement and that every closed dense subset of X has nowhere dense complement.

Solution. Let S be a nowhere dense subset of X. We want to prove that $\overline{S^c} = X$. We know that $\overline{S}^o = \emptyset$. Now let $x \in (\overline{S^c})^c$. Then, we know that $x \notin \overline{S^c}$, thus $x \notin S^c \cup \partial(S^c) = S^c \cup \partial S$. The first one means that $x \in S$, and the second part means $x \notin \partial S$. Hence, $x \in S^o$, which implies $x \in \overline{S}^o$. Hence, $(\overline{S^c})^c \subset \overline{S}^o = \emptyset$. This proves the first claim.

I do not get the second part of the question. Since it says closed, we already have $S = \overline{S} = X$, and the question becomes trivial.

Exercise 19. Show that $\mathbb{N}$ is nowhere dense in $\mathbb{R}$.

Solution. $\mathbb{N}$ is a discrete set in $\mathbb{R}$, thus consists of isolated points. Hence, its closure is itself which has empty interior.

Exercise 20. Suppose X is a normed linear space and $S \subset X$. For each subset $A \subset X$, we write $-A := \{-a : a \in A\}$. Show that $\partial(-S) = -\partial S, (-S)^o = -S^o$, and $\overline{-S} = -\overline{S}$.

Solution. Let $a \in \partial(-S)$. Then we know that $dist(a, -S) = dist(a, (-S)^c) = 0$. Then we see that $\inf_{s \in -S}(\|a - s\|) = \inf_{s \in -S}(\|s - a\|) = \inf_{s' \in S}(\|(-s') - a\|) = dist(-a, S) = 0$. Similarly, we have $\inf_{x \in S^c}(\|a - x\|) = \inf_{x \in S^c} \|x - a\| = \inf_{x' \in -S^c} \|-x' - a\| = dist(-a, -S^c) = 0$. Hence $-a \in \partial S$ which means that $a \in -\partial S$. The other inclusion is done the same way.

4. PS-IV

4.1. Part I

We first define Cantor set inductively. Take $I = [0, 1]$. In the first step remove the middle third, i.e. define $I_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. In the second step, remove the middle thirds of the remaining intervals, i.e. $I_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Continue in this manner, then the Cantor set is defined to be the limiting process:

$$C := \cap_{n=1}^{\infty} I_n.$$

<u>*Observation* $1 : |C^c| = 1$.</u> What we mean by this is that the length of the complement of Cantor set (measure) is 1. We observe that at each step we

delete intervals of length 3^{-k}, and the number of intervals we delete is 2^{k-1}. Hence, in the end we have deleted

$$\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{2} \frac{2/3}{1-2/3} = 1.$$

<u>*Observation* 2 :</u> We observe that the elements of Cantor set are exactly the elements in [0,1] whose ternary expansion can be written using only 0 and 2's, i.e. $x = \sum \frac{a_k}{3^k}$ where $a_k \in \{0, 2\}$.

<u>*Observation* 3 : *C is closed.*</u> Observe that at each step, we are removing an open interval from [0,1], and the resulting I_n is itself closed. Hence, the Cantor set which is the intersection of countably many closed sets is itself closed.

<u>*Observation* 4 : *C is nowhere dense.*</u> We use Observation 3 and 1. By 3, we know that $\overline{C} = C$, hence we need to look at C^o. Suppose there is an open set in C^o, but then this would mean that there exists an open interval of length ϵ in C which contradicts Observation 1. Hence C is nowhere dense.

<u>*Observation* 5 : $\partial C = C$</u>. One way to see this is by using Observation 3 and 4. We know that $C^o = \emptyset$ and $\overline{C} = C$. We also know $\overline{C} = C^o \cup \partial C$, hence $C = \partial C$. A more direct and (I think more beautiful) way to see this is by expanding the elements in ternary and constructing elements both from the Cantor set and from the complement that converges to the point.

<u>*Observation* 6 : *C is compact.*</u> We will prove this after we learn compactness.

4.2. Part II

Let us recall the definition of lim sup and lim inf, and try to understand what they are. Let T_n be the sequence of tails of a sequence $\{x_k\}$, i.e. $T_n = \{x_k : k \geq n\}$. Suppose our sequence is bounded, then we can define $s_n = \sup T_n$ and $i_n = \inf T_n$. We also see that $s_n \geq s_{n+1}$ and $i_{n+1} \geq i_n$. We define

$$\limsup x_n := \lim_{n\to\infty} [\sup\{x_k : k \geq n\}],$$

and

$$\liminf x_n := \lim_{n\to\infty} [\inf\{x_k : k \geq n\}],$$

i.e. $\limsup x_n = \lim s_n$, and $\liminf x_n = \lim i_n$.

<u>*Some Properties* :</u>

(1) $\liminf x_n \le \limsup x_n$, and if they are equal the limit exists, that is $\liminf x_n = \lim x_n = \limsup x_n$.

(2)If $\{a_n\}$ and $\{b_n\}$ are two bounded sequences, we have

$$\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n,$$

$$\liminf(a_n + b_n) \ge \liminf a_n + \liminf b_n.$$

(3) If one of them has a limit, say $b_n \to b$, we have

$$\limsup(a_n + b_n) = \limsup a_n + b, \quad \liminf(a_n + b_n) = \liminf a_n + b,$$

$$\limsup(a_n b_n) = b \limsup a_n, \quad \liminf(a_n b_n) = b \liminf a_n.$$

Moral of the Story: The lim inf and lim sup are respectively the smallest and the greatest of the cluster points.

Of course, one can give a more precise and rigourous definition using ϵ construction.

4.3. Part III

Exercise 21. Define a real sequence recursively by the following equations: $x_1 = 0$, $x_{2n} = \frac{x_{2n-1}}{2}$, and $x_{2n+1} = x_{2n} + \frac{1}{2}$. Find lim sup and lim inf.

Solution. First, let us see how the first couple of terms behave. We have

$$x_1 = 0, x_2 = 0, x_3 = \frac{1}{2}, x_4 = \frac{1}{4}, x_5 = \frac{3}{4}, x_6 = \frac{3}{8}, ...$$

Hence we observe that on the even indices we have

$$x_{2n} = \frac{x_{2n-2}}{2} + \frac{1}{4}.$$

Taking limits we see that

$$e = \frac{e}{2} + \frac{1}{4},$$

where $e = \lim x_{2n}$ is the limit on the even indices. Hence, the sequence converges to $\frac{1}{2}$, on even indices, and on the odd indices, it converges to 1. Thus, $\liminf = \frac{1}{2}$ and $\limsup = 1$.

Exercise 22. Suppose x_n is a sequence of positive real numbers. Prove the following inequalities:

$$\liminf \frac{x_{n+1}}{x_n} \leq \liminf x_n^{1/n} \leq \limsup x_n^{1/n} \leq \limsup \frac{x_{n+1}}{x_n}.$$

Solution. First observe that the middle inequality is trivial, and the first inequality can be proved by considering $-x_n$ sequence. Hence, we prove the last inequality. Suppose the sequence is bounded, and we have $\limsup \frac{x_{n+1}}{x_n} = a$, this means that there is a N such that for all $n > N$, we have $\sup_{k\geq n} \frac{x_{k+1}}{x_k} \leq a + \epsilon$. Since the supremum is less than, we have $\frac{x_{k+1}}{x_k} \leq a + \epsilon$ for all $k \geq n$. Now, given $n > N$, we write

$$\frac{a_n}{a_N} = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \dots \frac{a_{N+1}}{a_N} \leq (a+\epsilon)^{n-N}.$$

Hence we have

$$a_n^{1/n} \leq (a+\epsilon)(a+\epsilon)^{-N/n} a_N^{1/n}.$$

But this means that $\limsup a_n^{1/n} \leq a + \epsilon$ for any ϵ, hence we proved the last inequality. (The first one may be proved in a similar spirit.) If they are not bounded, we have nothing to prove.

Exercise 23. Find a metric space (X, d) and a sequence x_n in X such that x_n has no convergent subsequence but the infimum of the set $\{d(x_n, x_m) : n \neq m\}$ is zero.

Solution. In the near future, we will see that this has something to do with the completeness of the metric space X. Consider for an example $(\mathbb{Q}, d)$ with the Euclidean metric, and the sequence $\{x_n\}$ be the sequence of decimal places upto n^{th} place of $\sqrt{2}$. We observe that any subsequence tries to converge to $\sqrt{2}$, but since $\sqrt{2} \notin \mathbb{Q}$, there is no convergent subsequence. However, we have $d(x_n - x_m) \leq \frac{9}{10^{\min(n,m)}}$, hence has an infimum of 0.