MATH 338 – Complex Analysis I

MIDTERM - 06.05.2021

1. Are there any analytic functions f such that $f(z) = x^2y^2 + i\,v(x,y)$ on its domain, where $z = x + iy$ and v is a real valued function on two real variables? Explain.

2. (a) Let $(a_n)_{n>0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists. Show that $\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

   (b) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$.

   (c) Find a power series expansion (around 0) for the rational function $f(z) = \frac{z^2}{2+z^2}$.

3. Let f be an entire function such that $|f'(z)| \leq |z|$ for all $z \in \mathbb{C}$. Show that $f(z) = Az^2 + B$ where $A, B \in \mathbb{C}$ with $|A| \leq \frac{1}{2}$.

4. Let $S = \{z = x + iy \in \mathbb{C} : 2x < y < 3 - x^2\}$. Suppose that f is a continuous function on $\bar{S}$ that is analytic on S. Show that $f(x + i2x) \neq |x|$ for some $x \in [-3, 1]$.

5. For $r > 0$, let $I(r) = \int_{C_r} \frac{dz}{z^2+1}$, where C_r is the curve given by $t \mapsto re^{it}$ for $t \in [0, \pi]$. Calculate $\lim_{r\to\infty} I(r)$.