SMA 2205

Numerical Analysis

Chapter 4

Interpolation

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4.1 Polynomial Interpolation

What is interpolation?

Given ($(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$), find the value of ($y'$) at a value ($x'$) that is not given, where ($y = f(x)$).

Interpolant

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate
- Integrate

4.1.1 Spline Interpolation Method

A spline is a chain of curves joined end to end.

Why Spline?

Splines are preferred over high-order polynomials (e.g., 10th order) because high-order polynomials can oscillate wildly, as shown in the comparison between a true function and a 10th-order interpolating polynomial.

Interpolation

Estimate intermediate values between precise data points using simple functions.

Spline Concept

- Use low order (most often 2nd order), piecewise polynomial.

4.1.1.1 Linear Spline Interpolation

- By similar triangles: $\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$
- Rearrange: $f_1(x) = f(x_0) + m (x - x_0)$, where $m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

Linear Splines

- $f(x) = f(x_0) + m_0 (x - x_0)$ for $x_0 \leq x \leq x_1$
- $f(x) = f(x_1) + m_1 (x - x_1)$ for $x_1 \leq x \leq x_2$
- ...
- $f(x) = f(x_{}) + m_{} (x - x_{})$ for $x_{} \leq x \leq x_n$
- Main drawback: lack of smoothness.

4.1 Example

A trunnion is cooled 80°F to -180°F. Given below is the table of coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at $T = -14°F$ using linear interpolation.

Temperature(F)	Thermal expansion coefficient
80	0
0	-60
-60	-160
-160	-260
-260	-340

Using linear interpolation between -60°F (-160) and 0°F (-60), the value at -14°F can be estimated.

4.1.1.2 Quadratic Spline Interpolation

Quadratic splines have continuous first derivatives at knots.

- For $(n + 1)$ data points there are (n) intervals involved, thus there are $(3n)$ coefficients to evaluate.
- Thus, $(3n)$ conditions are required.

Quadratic Splines: Condition 1

The function values must be equal at the interior points.

- (f(x_) = a_ x_^2 + b_ x_ + c_)
- (f(x_) = a_i x_^2 + b_i x_ + c_i)

Quadratic Splines: Condition 2

The first derivatives must be equal at the interior points.

- (2a_ x_ + b_ = 2a_i x_ + b_i)

Quadratic Splines: Condition 3

The second derivatives must be continuous at the interior points.

- (2a_ = 2a_i)

Quadratic Splines: Condition 4

Boundary conditions (e.g., natural spline where second derivative at endpoints is zero).

- (2a_0 = 0), (2a_n = 0)

4.1.1.3 Cubic Spline Interpolation

Cubic splines use third-order polynomials and ensure continuous first and second derivatives at knots, providing smoother interpolation.

Cubic Splines

- For (n + 1) data points, (n) intervals, and (4n) coefficients, (4n) conditions are required (function values, first and second derivative continuity, and boundary conditions).

4.2 Example

Solve for a cubic spline interpolating points ((0, 0), (1, 1), (2, 0), (3, 1)) with natural boundary conditions.

4.1.2 Newton’s Polynomial Interpolation

Newton’s polynomial interpolation is a successive extension of spline interpolation methods.

4.1.2.1 Linear Interpolation

- Uses a first-order polynomial to connect two points.
- $f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0)$

4.1.2.2 Quadratic Interpolation

- Extends to a second-order polynomial using three points.
- $f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) + \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} (x - x_0)(x - x_1)$

4.1.2.3 General Form of Newton's Interpolating Polynomials

- $P_n(x) = b_0 + b_1 (x - x_0) + b_2 (x - x_0)(x - x_1) + \cdots + b_n (x - x_0)(x - x_1)\cdots(x - x_{})$
- Where b_k are the divided differences.

Divided Difference Table

- $f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{}]}{x_n - x_0}$

4.6 Example

Given points $(0, 0), (1, 1), (2, 8), (3, 27), (4, 64)$, construct the divided difference table and find the interpolating polynomial.

4.1.3 Lagrange's Interpolation Method

Lagrange Interpolating Polynomial

- $P_n(x) = \sum_{}^n y_i l_i(x)$, where $l_i(x) = \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j}$
- Each $l_i(x)$ is a basis polynomial that is 1 at x_i and 0 at other x_j.

Generalized Lagrange Interpolating Polynomial

- Extends to any number of points, ensuring the polynomial passes through all given data points.