



Ben-Gurion University of the Negev
The Faculty of Natural Science
The department of Physics

Theoretical model for hysteresis and jumps in disordered 2D superconductors near SIT

Shahar Kasirer
Adviser: Professor Yigal Meir

In this talk I would present my master thesis project. This work was done under the advisement of professor Yigal Meir.

It was originally motivated by experimental results from Dan Shahar's research group, at the Weizmann institute.

This is a work in progress, so I will also indicate work that is still planned for the near future.

1 Introduction and Motivation

I would start by presenting the experimental results that had motivated this project and review relevant existing theoretical models.

2 Proposed model - Single island

The presentation of our theoretical model would be done in 2 parts. First, I would present a simple version of the model, a single conducting island connected to electrodes by tunneling junctions.

3 Results - Single island

Then, I would show how we generalized this model into a disordered array of conducting islands.

4 Proposed model - Random array

I'll conclude with suggesting ways, in which this work could be used for future research.

5 Results - Random array

6 Discussion and conclusion

I'll start by a brief introduction to SIT - Superconducting to insulating transition

1 Introduction and Motivation

2 Proposed model - Single island

3 Results - Single island

4 Proposed model - Random array

5 Results - Random array

6 Discussion and conclusion

SIT

- Some amorphous 2D materials are superconductors in low temperature. ($\sim 3K$).

It is known, for more than a century now, that many metals and semiconductors become superconducting when cooled down to low temperatures.

Usually, when increasing the magnetic field, or the disorder of the sample, superconductivity breaks, and the sample returns to its normal, metallic or semiconducting, phase.

¹Haviland, D. B. et al. Onset of superconductivity in the two-dimensional limit. *Physical Review Letters* **62**, 2180–2183 (May 1989).

SIT

- Some amorphous 2D materials are superconductors in low temperature. ($\sim 3K$).
- When external magnetic field, or disorder, increases, few of those materials (Bi, In:O, TiN) undergo a phase transition, becoming a very good insulators ($R < 10^6\Omega$).

But, some materials behave differently. Instead of returning to their normal phase, they turn into very good insulators. This was observed, for example, in films of Indium oxide, Bismuth, and TiN (which is a composite of titanium and nitrogen-hydrite).

¹Haviland, D. B. et al. Onset of superconductivity in the two-dimensional limit. *Physical Review Letters* **62**, 2180–2183 (May 1989).

SIT

This is the superconductor to insulator transition.

- Some amorphous 2D materials are superconductors in low temperature. ($\sim 3K$).
- When external magnetic field, or disorder, increases, few of those materials (Bi, In:O, TiN) undergo a phase transition, becoming a very good insulators ($R < 10^6\Omega$).
- This is called SIT (Superconductor to Insulator Transition).

¹Haviland, D. B. et al. Onset of superconductivity in the two-dimensional limit. *Physical Review Letters* **62**, 2180–2183 (May 1989).

SIT

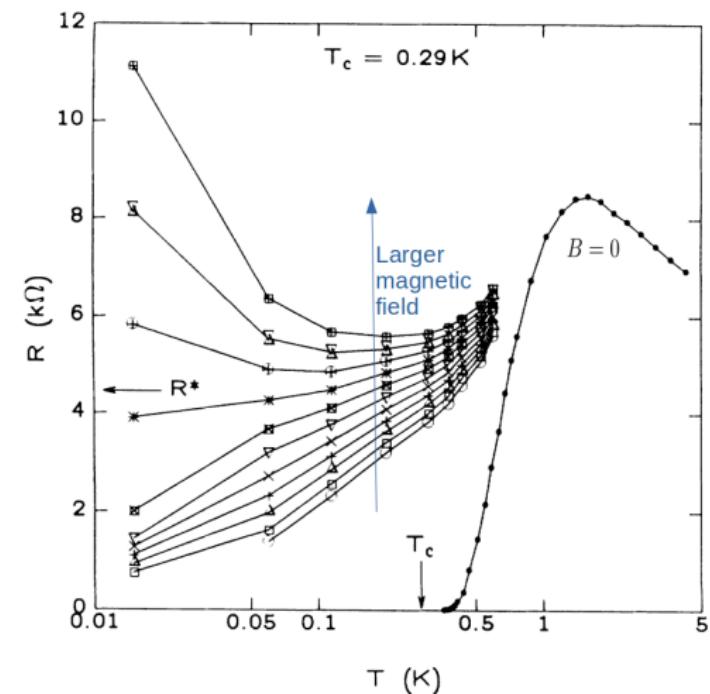
The first measurement of this transition was made 3 decades ago, by David Haviland, while he was a PhD student of Allen Goldman in Minnesota.

The insulating phase can prove to be quite unusual, as we will see next.

- Some amorphous 2D materials are superconductors in low temperature. ($\sim 3K$).
- When external magnetic field, or disorder, increases, few of those materials (Bi, In:O, TiN) undergo a phase transition, becoming a very good insulators ($R < 10^6\Omega$).
- This is called SIT (Superconductor to Insulator Transition).
- It was first measured by Haviland et. al.¹ in 1989.

¹Haviland, D. B. et al. Onset of superconductivity in the two-dimensional limit. *Physical Review Letters* 62, 2180–2183 (May 1989).

SIT - Experimental results



SIT as a function of magnetic field, for In:O films¹.

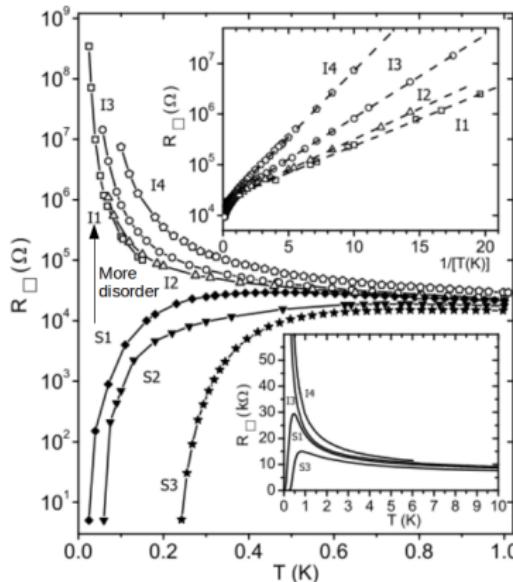
Here we see measurement results of resistance as a function of temperature for different magnetic fields ($0.4 - 0.6\text{ }T$).

Resistance was measured by applying a small voltage to the sample and measuring the current.

For zero external magnetic field, this sample is superconducting under 0.29 degrees Kelvin. When increasing the external magnetic field, low temperature resistance increases.

¹Hebard, A. F. & Paalanen, M. A. Magnetic-field-tuned superconductor-insulator transition in two-dimensional films. *Physical Review Letters* **65**, 927–930 (Aug. 1990).

SIT - Experimental results



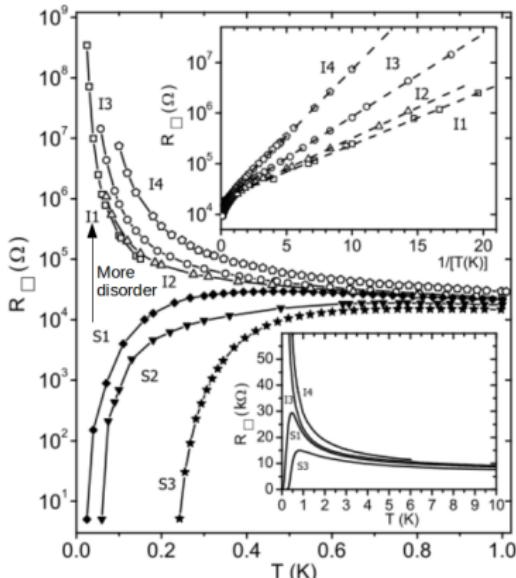
SIT as a function of sample disorder, for TiN films¹.

Here we see measurement results of resistance as a function of temperature for different samples, with different disorder.

In the superconducting phase, resistance goes to zero while temperature decreases. In the insulating phase, it diverges.

¹Baturina, T. I. et al. Quantum-critical region of the disorder-driven superconductor-insulator transition. *Physica C: Superconductivity. Proceedings of the Workshop on Fluctuations and Phase Transitions in Superconductors* 468, 316–321 (Feb. 2008).

SIT - Experimental results



SIT as a function of sample disorder, for TiN films¹.

- For the insulating phase - resistance approximately follows Arrhenius law: $R \propto \exp(T_0/T)$ for low temperatures.

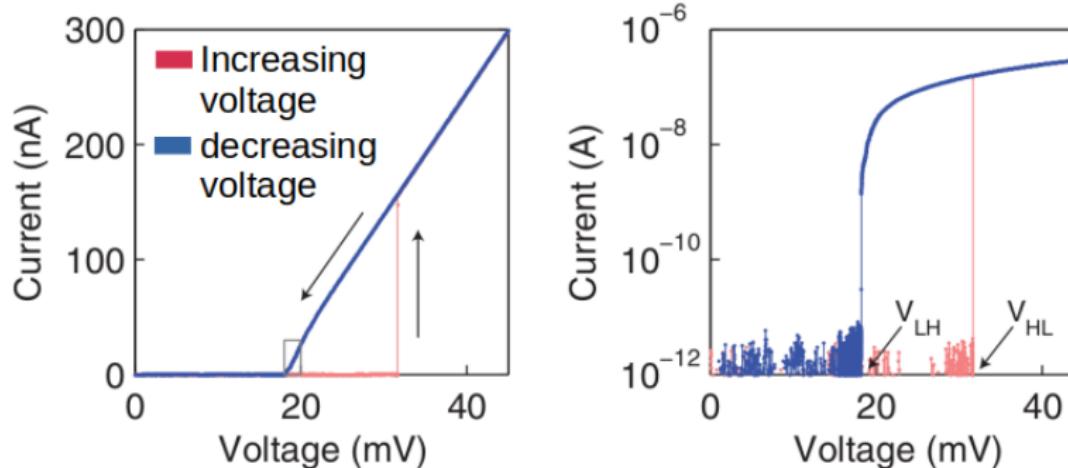
In the inset, we see a semi logarithmic plot of resistance as a function of $1/T$, we see that for low temperatures, it follows Arrhenius law.

A simple measurement one could do, is to apply voltage differences to both sides of a sample and measure the resulting current. For metals, the resulted I-V curves are usually linear, according to Ohm's law.

¹Baturina, T. I. et al. Quantum-critical region of the disorder-driven superconductor-insulator transition. *Physica C: Superconductivity. Proceedings of the Workshop on Fluctuations and Phase Transitions in Superconductors* 468, 316–321 (Feb. 2008).

I-V Curves

- I-V curves of the insulating phase are not trivial.



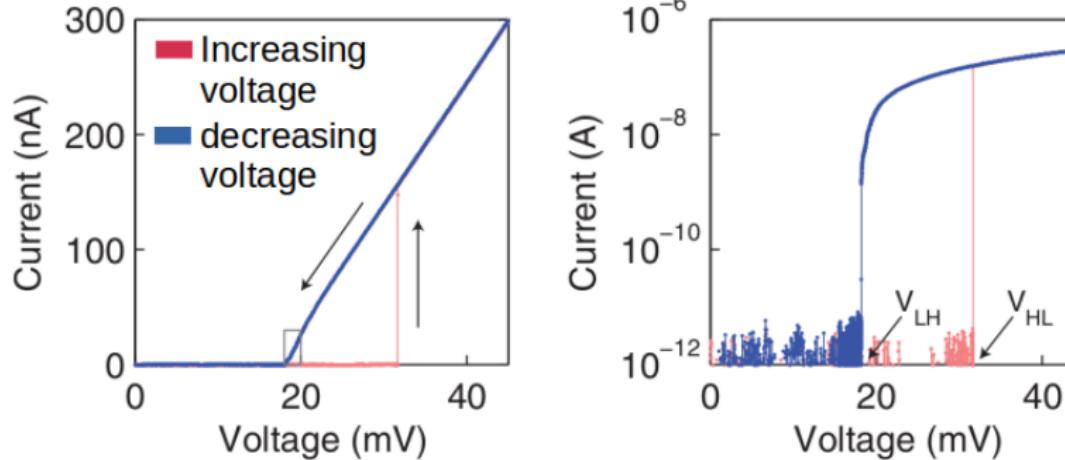
I-V curves of In:O, near SIT¹.

But the observed I-V curves for the insulating phase, near SIT, are far from linear. Those are measurement results, for a sample of Indium oxide, in its insulating phase, published by Cohen et. al. from Dan Shahar's research group.

¹Cohen, O. et al. Electric breakdown effect in the current-voltage characteristics of amorphous indium oxide thin films near the superconductor-insulator transition. *Physical Review B* 84, 100507 (Sept. 2011).

I-V Curves

- I-V curves of the insulating phase are not trivial.
- I-V curves may include **hysteresis** and **current jumps**.



I-V curves of In:O, near SIT¹.

We see a non-linear curve, including a big current jump and hysteresis.

A few theoretical models were suggested to account for the observed behavior.

¹Cohen, O. et al. Electric breakdown effect in the current-voltage characteristics of amorphous indium oxide thin films near the superconductor-insulator transition. *Physical Review B* 84, 100507 (Sept. 2011).

Electrons overheating model

- A phenomenological model, that was suggested by Altshuler¹

One interesting phenomenological model, that was suggested by Boris Altshuler, is the model of overheated electrons.

¹Altshuler, B. L. et al. Jumps in Current-Voltage Characteristics in Disordered Films. *Physical Review Letters* **102**, 176803 (Apr. 2009).

Electrons overheating model

It is based on a few simple assumptions

- A phenomenological model, that was suggested by Altshuler¹

Model assumptions:

¹Altshuler, B. L. et al. Jumps in Current-Voltage Characteristics in Disordered Films. *Physical Review Letters* 102, 176803 (Apr. 2009).

Electrons overheating model

- A phenomenological model, that was suggested by Altshuler¹

Model assumptions:

- ① Strong electron-electron interactions. Electrons have a defined temperature T_{el} .

The electron-electron interaction is strong enough for electrons to be mutually thermalized, and to have a well defined temperature.

¹Altshuler, B. L. et al. Jumps in Current-Voltage Characteristics in Disordered Films. *Physical Review Letters* **102**, 176803 (Apr. 2009).

Electrons overheating model

- A phenomenological model, that was suggested by Altshuler¹

Model assumptions:

- ① Strong electron-electron interactions. Electrons have a defined temperature T_{el} .
- ② Weak electron-phonon coupling, $T_{el} \neq T_{ph}$.

Electron-Phonon thermalization is inefficient at low temperatures. Therefore the temperature of electrons and the temperature of phonons are not necessarily equal.

¹Altshuler, B. L. et al. Jumps in Current-Voltage Characteristics in Disordered Films. *Physical Review Letters* 102, 176803 (Apr. 2009).

Electrons overheating model

The current follows Ohm's law, with temperature dependent resistance.

- A phenomenological model, that was suggested by Altshuler¹

Model assumptions:

- ① Strong electron-electron interactions. Electrons have a defined temperature T_{el} .
- ② Weak electron-phonon coupling, $T_{el} \neq T_{ph}$.
- ③ Current satisfy Ohm's law $IR(T_{el}) = V$, with electrons temperature dependent resistance.

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Electrons overheating model

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Model assumptions:

- ① Strong electron-electron interactions. Electrons have a defined temperature T_{el} .
- ② Weak electron-phonon coupling, $T_{el} \neq T_{ph}$.
- ③ Current satisfy Ohm's law $IR(T_{el}) = V$, with electrons temperature dependent resistance.
- ④ T_{el} is determined by the heat balance equation

$$\underbrace{\frac{V^2}{R(T_{el})}}_{\text{Joule heating}} = \underbrace{\Gamma \Omega (T_{el}^\beta - T_{ph}^\beta)}_{\text{e-ph cooling}}$$

The temperature of electrons is determined by the balance between Joule heating, heating by the current, and the cooling by electron phonon interactions.

Ω is the volume of the sample, Γ is the electron-phonon coupling strength.

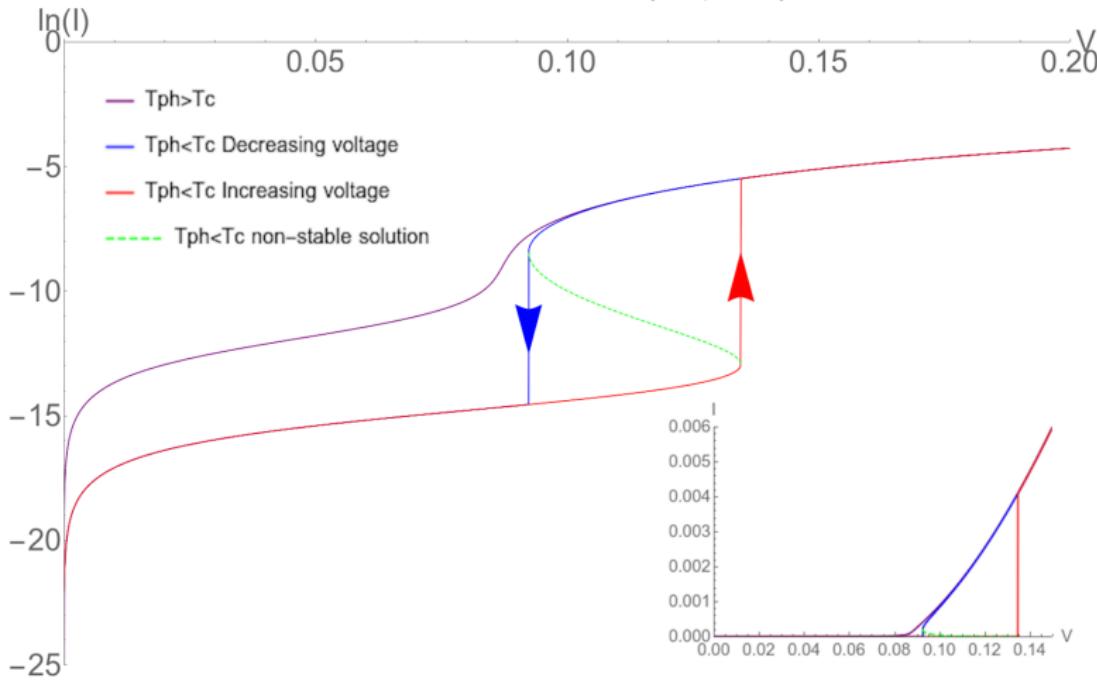
β is a phenomenological parameter, it was calculated to be 6 for the case where thermal phonon wavelength exceeds electrons elastic mean-free-path.

To find the relation between resistance and electrons temperature, resistance should be measured for small voltages, as a function of temperature. For low voltages, electrons and phonons temperatures are almost equal, so the results would give us the right temperature dependency for R .

¹Altshuler, B. L. et al. Jumps in Current-Voltage Characteristics in Disordered Films. *Physical Review Letters* 102, 176803 (Apr. 2009).

Electrons overheating model - simulation results

$$\beta = 6, R = R_0 \exp(T_0/T_{el})$$



In many cases, the relation between resistance and temperature is well fitted by Arrhenius law. Therefore, this is also the relation that was used in Altshuler's paper.

Here we see the resulted I-V curves.

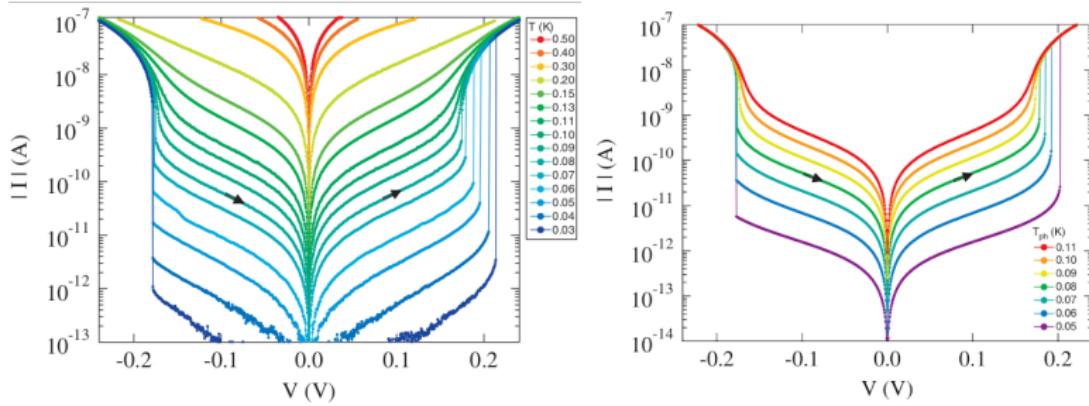
For temperatures above a critical value, there is no hysteresis and no jump, although there is a transition from high to low resistance states.

For temperatures below T_c we see a bi-stable region. It separates a high resistance regime, in which electron and phonon temperatures are similar, and a low resistance regime, in which electrons are overheated, which means that they are much hotter than the phonons.

In the bi-stable region, electron temperature follows phonon temperature when voltage is increased, but once electrons were overheated, it takes a lower voltage value for them to cool back down, and so current follows the low resistance solution when voltage is decreased.

This explains the big current jump, and the hysteresis that were observed in the lab

Electrons overheating model - experimental results



Observed (left) and simulated (right) I-V curves¹

The I-V curves predicted by this model, show a good agreement with some measurements.

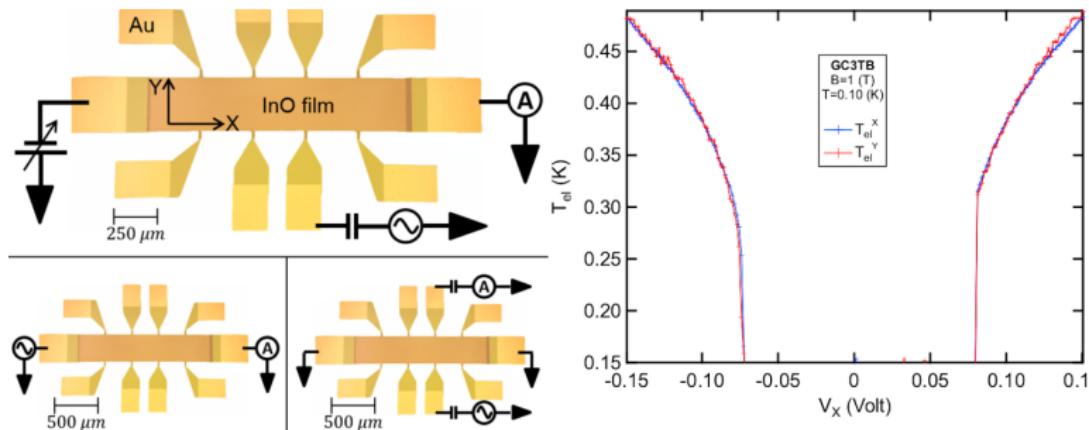
Here, on the left side, we see the observed I-V curves for a sample of indium oxide, from a paper by Ovadia et. al. from Dan Shahar's research group.

On the right side, we see the curves that were simulated using electron overheating model equations. The different parameters in the model, and the resistance-temperature relation, were chosen empirically, according to measurement results from the same sample.

For low temperatures, the curves are not symmetric around zero voltage, which indicates hysteretic behavior.

¹Ovadia, M. et al. Electron-Phonon Decoupling in Disordered Insulators. *Physical Review Letters* 102, 176802 (Apr. 2009).

Electrons overheating model - perpendicular temperature measurement



Perpendicular T_{el} - experimental setup (left) and results (right)¹

Another experimental support for the electrons overheating model was given in a paper by Levinson et. al., also from Dan Shahar's group

In the first stage, they have measured the current for small voltages in 2 perpendicular directions of their sample, while changing the temperature. Using those measurements, they have calculated the resistance temperature dependency for each direction.

In the second stage, they applied a small, alternating current, to the vertical direction, and a direct voltage to the horizontal direction. They measured the current in both directions while increasing the voltage in the horizontal direction.

Then, they used those measurements to calculate the resistance in both directions as a function of the applied horizontal voltage. Finally, using the pre-calculated resistance-temperature dependencies, They calculated the temperature of electrons in both directions.

The resulted temperatures agree with each other quite well, as expected by the model, in which there is only one temperature for electrons.

¹Levinson, T. et al. Direct determination of the temperature of overheated electrons in an insulator. *Physical Review B* 94. Publisher: American Physical Society, 174204 (Nov. 2016).

Electrons overheating model - problem

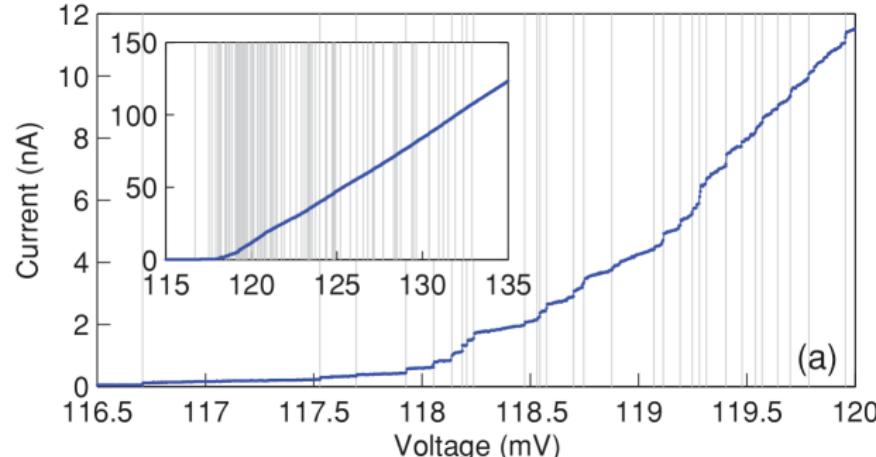
- In some measurements we see multiple jumps.

As we saw, electrons overheating model is quite successful in explaining the observed curves in some cases. But it fails on other cases.

¹Cohen, O. et al. Electric breakdown effect in the current-voltage characteristics of amorphous indium oxide thin films near the superconductor-insulator transition. *Physical Review B* 84, 100507 (Sept. 2011).

Electrons overheating model - problem

- In some measurements we see multiple jumps.
- Here, **92** jumps were detected!



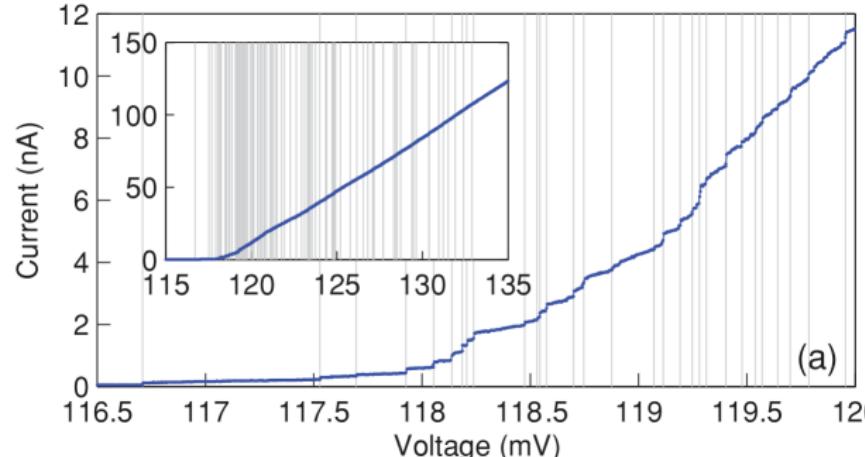
I-V curves of In:O, insulating phase near SIT¹

For example, in the shown measurement results, from a paper by Cohen et al. In this measurement, 92 current jumps were observed.

¹Cohen, O. et al. Electric breakdown effect in the current-voltage characteristics of amorphous indium oxide thin films near the superconductor-insulator transition. *Physical Review B* 84, 100507 (Sept. 2011).

Electrons overheating model - problem

- In some measurements we see multiple jumps.
- Here, **92** jumps were detected!



I-V curves of In:O, insulating phase near SIT¹

- To account for this result, 92 different local T_{el} are needed.

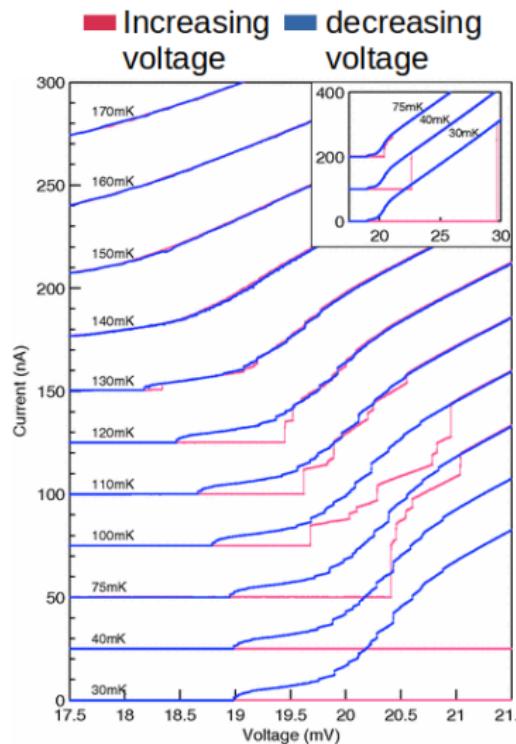
In electrons overheating model, a jump occurs when electron temperature decouples from phonon temperature. For 92 jumps to be accounted by this model, we need to have 92 different local temperatures for electrons!

To explain this result, a different theoretical model is needed.

¹Cohen, O. et al. Electric breakdown effect in the current-voltage characteristics of amorphous indium oxide thin films near the superconductor-insulator transition. *Physical Review B* 84, 100507 (Sept. 2011).

I-V curves - characteristics

- Threshold voltage.



I-V curves of In:O, insulating phase near SIT¹.

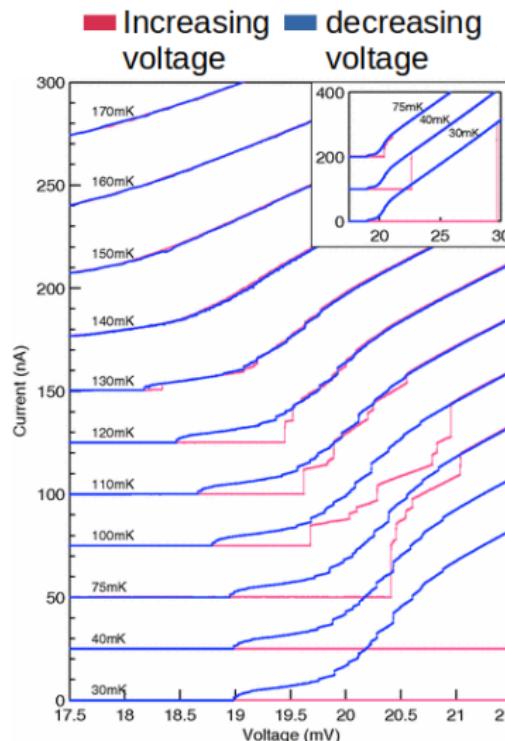
Here is another result, from the same paper, measured for a different sample. We see multiple current jumps and hysteresis loops. Lets follow one of the curves.

For small voltages, current is zero up to a threshold voltage.

¹Cohen, O. et al. Electric breakdown effect in the current-voltage characteristics of amorphous indium

I-V curves - characteristics

- Threshold voltage.



- Current Jumps.

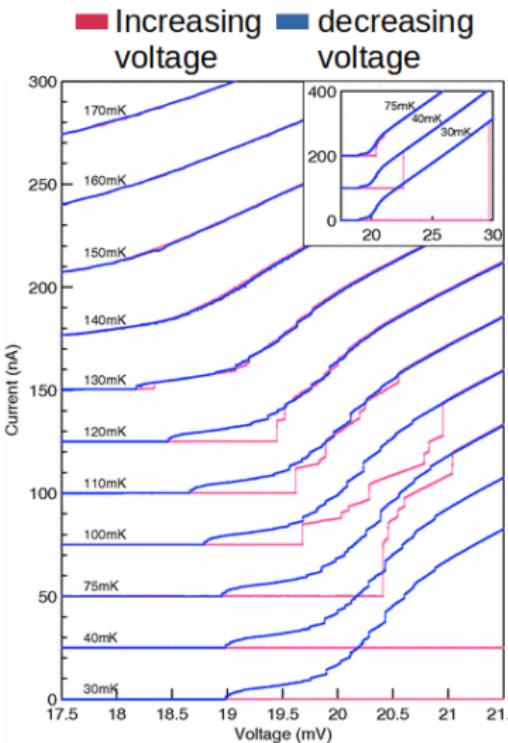
Then, current starts flowing with a big jump, followed by smaller ones. For high voltages we see a linear, ohmic region.

I-V curves of In:O, insulating phase near SIT¹.

¹Cohen, O. et al. Electric breakdown effect in the current-voltage characteristics of amorphous indium

I-V curves - characteristics

- Threshold voltage.



- Current Jumps.

- Hysteresis.

When decreasing voltage back towards zero, current keeps flowing down to a lower threshold value, creating a large hysteresis loop.

When increasing the temperature, hysteresis loop gets smaller and then separates into a few smaller loops.

In addition, both threshold voltages, for increasing and decreasing external voltage, become smaller.

I-V curves of In:O, insulating phase near SIT¹.

¹Cohen, O. et al. Electric breakdown effect in the current-voltage characteristics of amorphous indium

Research Goals

The main goal of this research is to suggest a theoretical model that will reproduce the measured I-V curve features, with emphasis on current jumps and hysteresis.

- Suggest an alternative model that will reproduce the observed I-V curves, with emphasis on hysteresis and multiple current jumps.

Research Goals

In addition, we would like to get predictions that will answer the following questions:

- Suggest an alternative model that will reproduce the observed I-V curves, with emphasis on hysteresis and multiple current jumps.
- Answer the following questions:

Research Goals

When do we expect to measure hysteresis?

- Suggest an alternative model that will reproduce the observed I-V curves, with emphasis on hysteresis and multiple current jumps.
- Answer the following questions:
 - When do we predict hysteresis?

Research Goals

When do we expect to measure current jumps?

- Suggest an alternative model that will reproduce the observed I-V curves, with emphasis on hysteresis and multiple current jumps.
- Answer the following questions:
 - When do we predict hysteresis?
 - When do we predict current jumps?

Research Goals

How does changing different parameters affect the observed I-V curve?

- Suggest an alternative model that will reproduce the observed I-V curves, with emphasis on hysteresis and multiple current jumps.
- Answer the following questions:
 - When do we predict hysteresis?
 - When do we predict current jumps?
 - How does changing different parameters would change the observed I-V curve?

Why an array model?

- SIT can be explained using a model of superconducting islands, connected by tunneling junctions(JJA)¹

For our theoretical model, we chose to use an array of conducting islands. Similar models were frequently used, in relation to SIT.

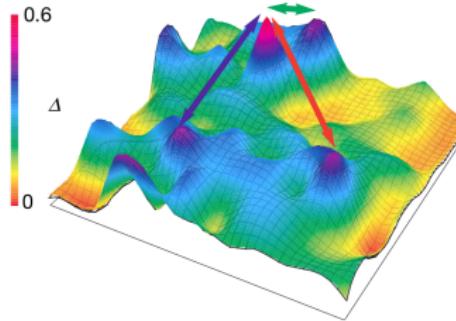
One of the first models that predicted SIT, proposed by Efetov, was a Josephson juctions array, an array of conducting islands connected by tunneling junctions.

¹Efetov, K. B. Phase transition in granulated superconductors. en. *Journal of Experimental and Theoretical Physics*, 8 (1980).

²Dubi, Y. et al. Nature of the superconductor–insulator transition in disordered superconductors. en. *Nature* 449, 876–880 (Oct. 2007).

Why an array model?

- SIT can be explained using a model of superconducting islands, connected by tunneling junctions(JJA)¹
- Disorder + high magnetic fields → SC islands.



Formation of SC islands for high disorder and magnetic field². Δ is given relative to its value for the clean system.

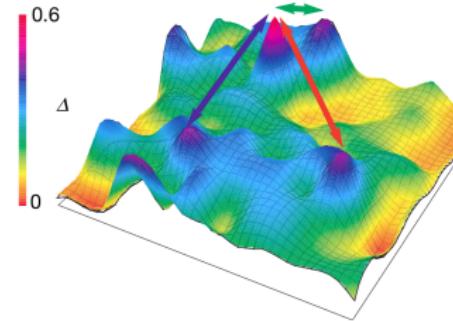
In a nature paper published by Yoni Dubi, while he was a Phd student of Yigal, he showed that for disordered superconductors, the result of applying an external magnetic field is the formation of superconducting islands, regions with higher order parameters, that are not coherently coupled with each other.

¹Efetov, K. B. Phase transition in granulated superconductors. en. *Journal of Experimental and Theoretical Physics*, 8 (1980).

²Dubi, Y. et al. Nature of the superconductor–insulator transition in disordered superconductors. en. *Nature* 449, 876–880 (Oct. 2007).

Why an array model?

- SIT can be explained using a model of superconducting islands, connected by tunneling junctions(JJA)¹
- Disorder + high magnetic fields → SC islands.
- Similar I-V curves were observed for arrays.



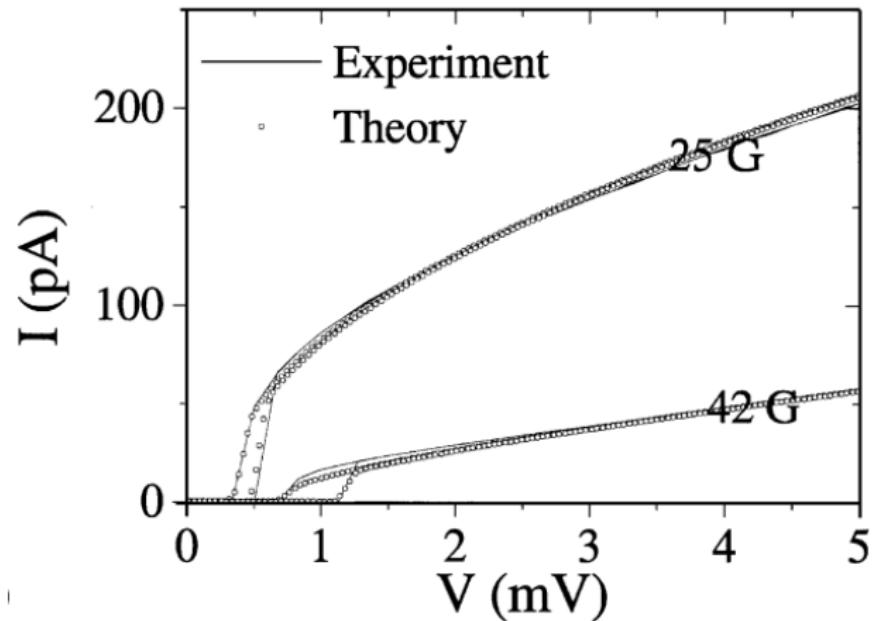
Formation of SC islands for high disorder and magnetic field². Δ is given relative to its value for the clean system.

In addition, I-V curves similar to the ones we saw, were observed for different systems of arrays.

¹Efetov, K. B. Phase transition in granulated superconductors. en. *Journal of Experimental and Theoretical Physics*, 8 (1980).

²Dubi, Y. et al. Nature of the superconductor–insulator transition in disordered superconductors. en. *Nature* 449, 876–880 (Oct. 2007).

Similar I-V curves - Josephson Junction Arrays

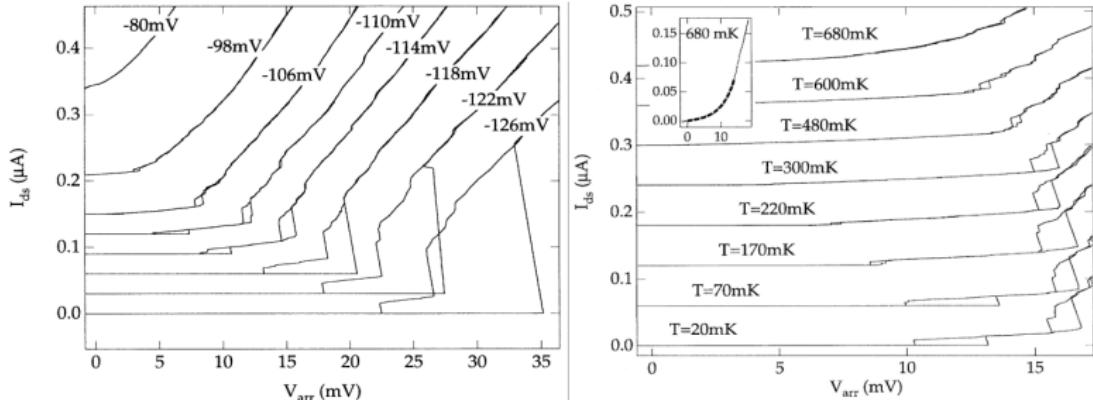


I-V curves of 1D JJA, for different magnetic fields¹

Here we see experimental results published by Agren et. al., from David Haviland's research group. Measurements were done on a one dimensional chain of Josephson junctions, for different magnetic fields. We see a single jump and hysteresis. In this paper, a theoretical model, where hysteresis was accounted for by the addition of induction to each junction, was used to fit the data. But, the induction values that were needed to get a good fit, were way too big to be physically reasonable.

¹Agren, P. et al. Hysteretic current-voltage characteristics and Coulomb blockade in 1D-arrays of Josephson junctions. en. *Physica B: Condensed Matter* 280, 414–415 (May 2000).

Similar I-V curves - Quantum Dot Arrays



I-V curves of 2D QDA, for different gate voltages(left) and temperatures (right)¹

A similar behavior was observed also in a non-superconducting system, 2D Quantum dot array, which is an array of metallic or semiconducting islands, connected by tunneling junctions

For example, we see here the I-V curves measured for a semiconducting quantum dot array, by Duruoz et. al., from Charlie Marcus research group. Measurements were done for different temperatures and different gate voltages.

¹Duruöz, C. I. et al. Conduction Threshold, Switching, and Hysteresis in Quantum Dot Arrays. *Physical Review Letters* **74**, 3237–3240 (Apr. 1995).

Existing array models - no jumps and hysteresis

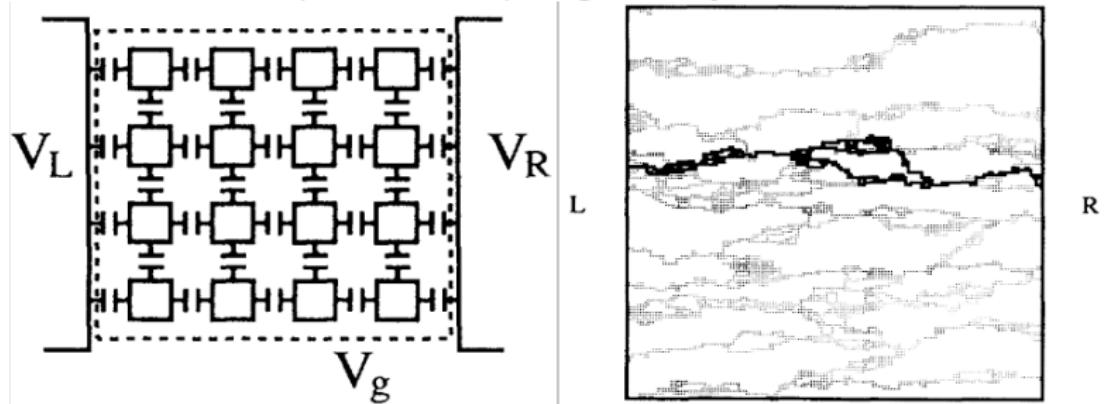
- I-V curves for different array models were calculated before.

This is, ofcourse not the first time an array model is used to calculate I-V curves.

¹Middleton, A. A. & Wingreen, N. S. Collective transport in arrays of small metallic dots. *Physical Review Letters* **71**, 3198–3201 (Nov. 1993).

Existing array models - no jumps and hysteresis

- I-V curves for different array models were calculated before.
- For disordered 2D array current flows in different paths that open gradually above threshold.



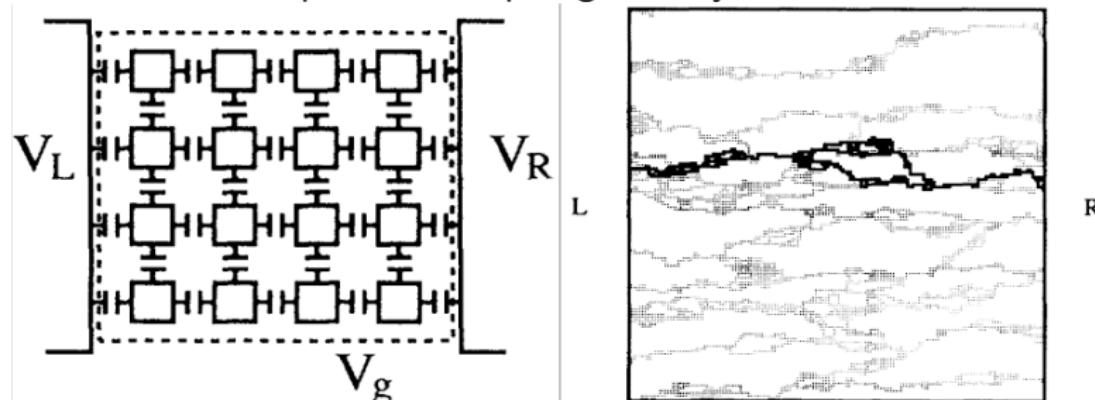
QDA model, by Middleton and Wingreen¹, Schema(left) and current paths slightly above threshold (right).

In a pioneer work by Middleton and Wingreen, a disordered two dimensional array of quantum dots was used. Disorder was realized, in this case, by the addition of random offset charges to each island. Using numerical simulations, they showed that current flows in different paths through the array. Conduction starts when the first path opens (here in black), and later different paths open gradually (in grey).

¹Middleton, A. A. & Wingreen, N. S. Collective transport in arrays of small metallic dots. *Physical Review Letters* **71**, 3198–3201 (Nov. 1993).

Existing array models - no jumps and hysteresis

- I-V curves for different array models were calculated before.
- For disordered 2D array current flows in different paths that open gradually above threshold.



QDA model, by Middleton and Wingreen¹, Schema(left) and current paths slightly above threshold (right).

- A threshold voltage exists in most models, but no jumps nor hysteresis.

Most models predict the existence of a threshold voltage. But no array model, as far as we know, predicted hysteresis and current jumps.

¹Middleton, A. A. & Wingreen, N. S. Collective transport in arrays of small metallic dots. *Physical Review Letters* 71, 3198–3201 (Nov. 1993).

But, there is one model of a single conducting island, that does that.

1 Introduction and Motivation

2 Proposed model - Single island

3 Results - Single island

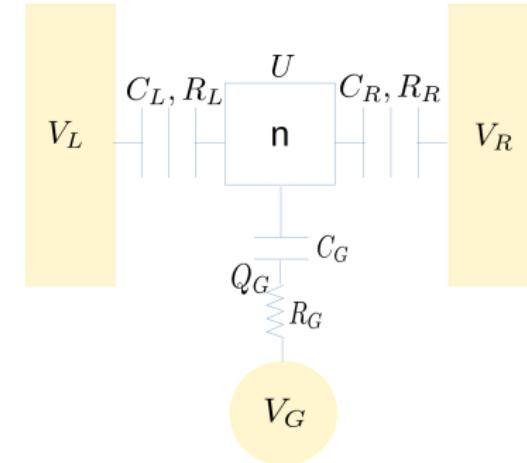
4 Proposed model - Random array

5 Results - Random array

6 Discussion and conclusion

Single island model - introduction

- Our model is base on a model by Korotkov¹, an RC-coupled Single electron transistor (RC-SET).



Schema of the single island model. n is the number of excess electrons on the island. Q_G is the charge on the gate capacitor, connecting the island to the gate voltage V_G , through a resistor with resistance R_G . U is the electric potential on the island.

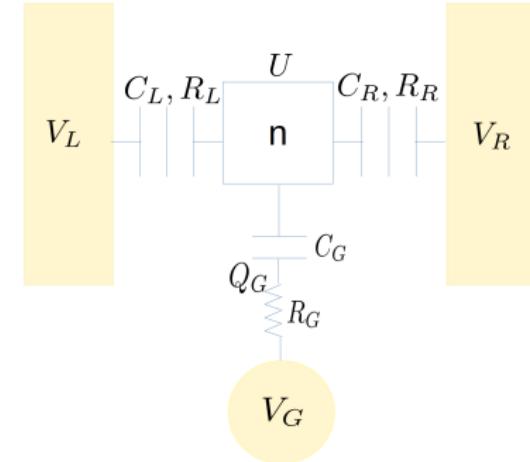
This model was suggested by Korotkov, as a single electron transistor, that can be used also as a storage device.

Here we see a shcema of the model. It is composed of a conducting island that is coupled to a gate voltage trough an RC circuit.

¹Korotkov, A. N. Single-electron transistor controlled with a RC circuit. *Physical Review B* 49, 16518–16522 (June 1994).

Single island model - introduction

- Our model is base on a model by Korotkov¹, an RC-coupled Single electron transistor (RC-SET).
- In this model, electrons can tunnel between an electrode and the island.



Schema of the single island model. n is the number of excess electrons on the island. Q_G is the charge on the gate capacitor, connecting the island to the gate voltage V_G , through a resistor with resistance R_G . U is the electric potential on the island.

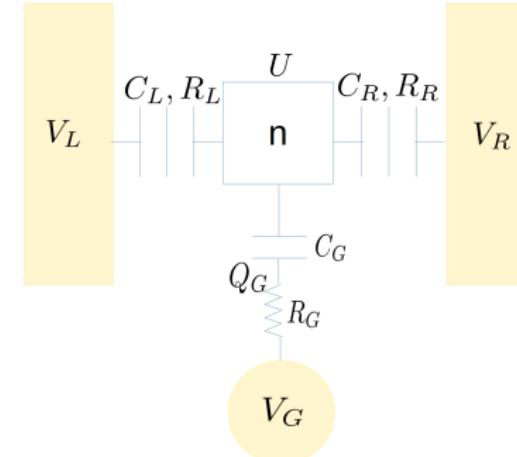
The left and right junctions are tunneling junctions, with tunneling resistance of R_L and R_R , allowing electrons to tunnel from each electrode to the island and vice versa.

No tunneling is allowed to the gate.

¹Korotkov, A. N. Single-electron transistor controlled with a RC circuit. *Physical Review B* 49, 16518–16522 (June 1994).

Single island model - introduction

- Our model is base on a model by Korotkov¹, an RC-coupled Single electron transistor (RC-SET).
- In this model, electrons can tunnel between an electrode and the island.
- Between tunnelings, charge distribution relaxes to equilibrium. The coupled RC-circuit accounts for finite relaxation time.



Schema of the single island model. n is the number of excess electrons on the island. Q_G is the charge on the gate capacitor, connecting the island to the gate voltage V_G , through a resistor with resistance R_G . U is the electric potential on the island.

Between tunnelings, charge relaxes to electrostatic equilibrium, as normal RC circuits do.

There are 2 dynamical variables in this model:
 n - the number of excess electrons on the island.

Q_G - the charge on the gate capacitor.

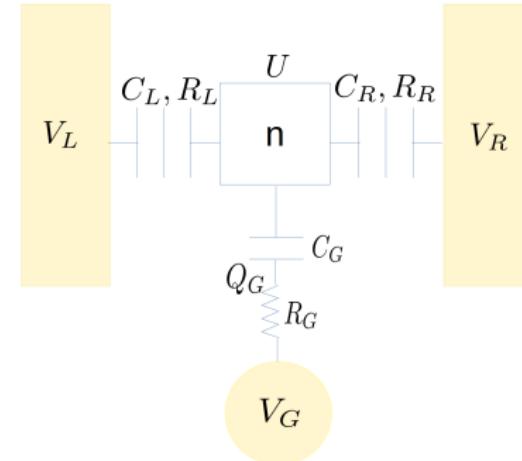
Using Kirchoff's laws, we can calculate all the charges in the system using only n and Q_G .

It would be convenient to understand some features of this model by calculating U , the electric potential on the island.

¹Korotkov, A. N. Single-electron transistor controlled with a RC circuit. *Physical Review B* 49, 16518–16522 (June 1994).

Single island model - introduction

- Our model is base on a model by Korotkov¹, an RC-coupled Single electron transistor (RC-SET).
- In this model, electrons can tunnel between an electrode and the island.
- Between tunnelings, charge distribution relaxes to equilibrium. The coupled RC-circuit accounts for finite relaxation time.
- For convenience - charge carriers with charge $+e$ would be considered.



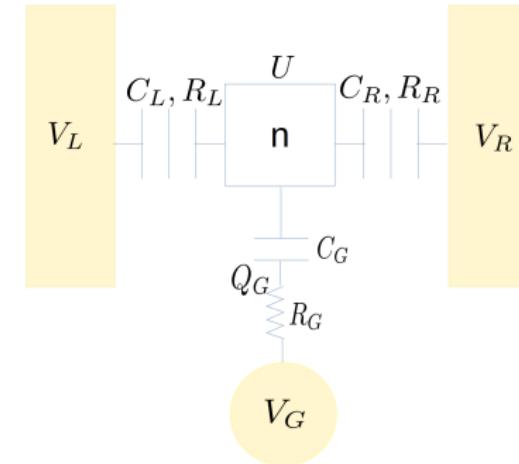
Schema of the single island model. n is the number of excess electrons on the island. Q_G is the charge on the gate capacitor, connecting the island to the gate voltage V_G , through a resistor with resistance R_G . U is the electric potential on the island.

An important note: Throught this talk I will use positive charge carriers with charge of plus e , or "positive electrons", for convenience. This doesn't change any of the results I would show.

¹Korotkov, A. N. Single-electron transistor controlled with a RC circuit. *Physical Review B* 49, 16518–16522 (June 1994).

Single island model - time scales

- There are 3 relevant time scales in this model:
 - 1 Tunneling time
 - 2 Time between tunnelings
 - 3 Relaxation time



Lets speak a little bit about time scales in this model.

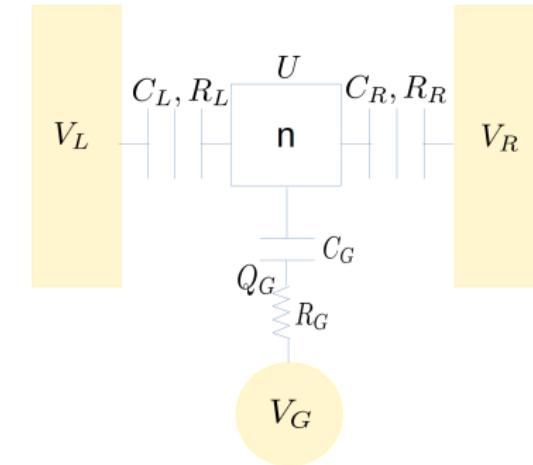
The first time scale is the time it takes for the tunneling itself. This would be considered as much shorter than all other time scales, as is usually done.

The second, is the time passing from one tunneling to the next. It is determined by tunneling rates.

The third, is the relaxation time of charge distribution. In most existing models, charge is assumed to always be in electrostatic equilibrium, which is like assuming that relaxation time is much shorter than the time between tunnelings.

Single island model - time scales

- There are 3 relevant time scales in this model:
 - ① Tunneling time
 - ② Time between tunnelings
 - ③ Relaxation time
- In this work we focus on the slow relaxation limit where **relaxation time** \gg **time between tunnelings**



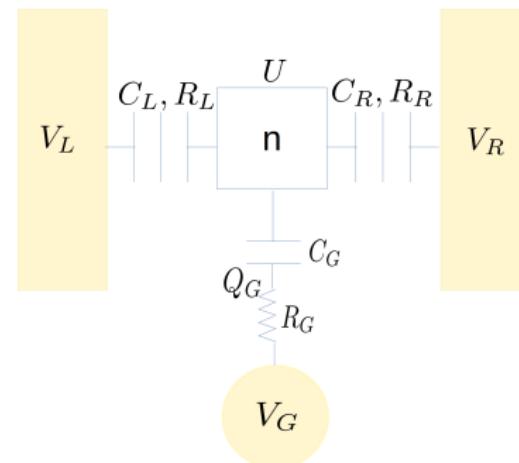
There is no physical reason for why relaxation time should be much smaller than the time between tunnelings. Therefore, in this work, we would like to focus on the opposite limit, where relaxation time is much longer than the time between tunnelings.

I will refer this as the "slow relaxation limit".

Single island - Hamiltonian

$$\mathcal{H}(n, Q_G) = \mathcal{H}_C(n, Q_G) + \mathcal{H}_T$$

This is the general structure of the Hamiltonian. It is composed of two terms.

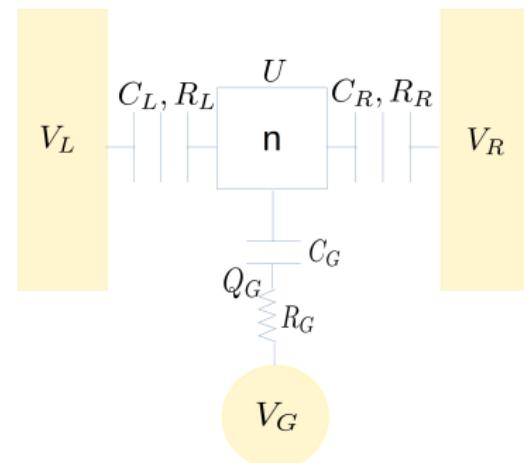


► More details

Single island - Hamiltonian

$$\mathcal{H}(n, Q_G) = \mathcal{H}_C(n, Q_G) + \mathcal{H}_T$$

- \mathcal{H}_C is the electrostatic energy



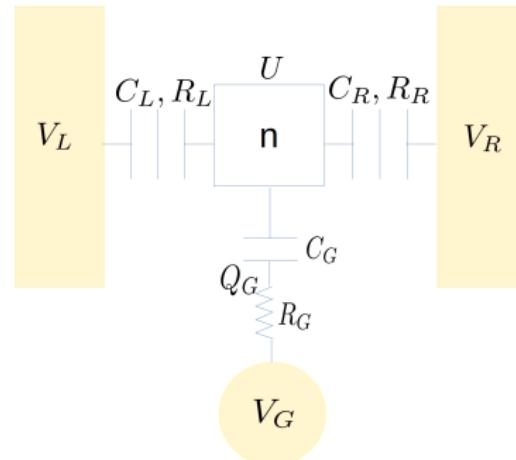
The first term, is the electrostatic energy, which is just the energy stored on capacitors and electrodes. It is affected by n and Q_G

► More details

Single island - Hamiltonian

$$\mathcal{H}(n, Q_G) = \mathcal{H}_C(n, Q_G) + \mathcal{H}_T$$

- \mathcal{H}_C is the electrostatic energy
- \mathcal{H}_T is the tunneling term.
It is considered to be a small perturbation.
Therefore we solve for the basis of \mathcal{H}_C (states with a well defined occupation).



The second, is the tunneling term, allowing electrons to tunnel.

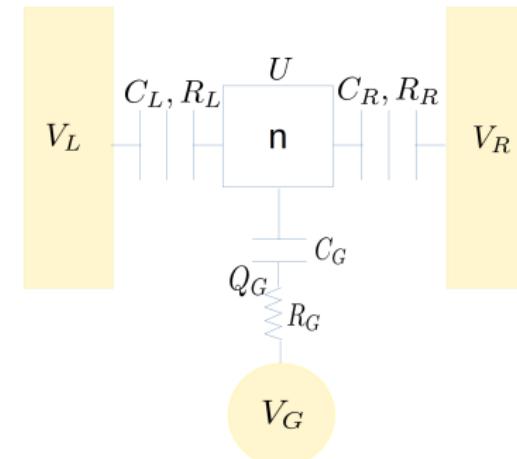
It is considered to be a constant term, independent of n and Q_G .

To find the tunneling rates, we treat the tunneling term as a small perturbation and use "Fermi golden rule".

Single island - Hamiltonian

$$\mathcal{H}(n, Q_G) = \mathcal{H}_C(n, Q_G) + \mathcal{H}_T$$

- \mathcal{H}_C is the electrostatic energy
- \mathcal{H}_T is the tunneling term.
It is considered to be a small perturbation.
Therefore we solve for the basis of \mathcal{H}_C (states with a well defined occupation).
- For superconducting arrays we separate tunneling of quasi-particles and Cooper-pairs.

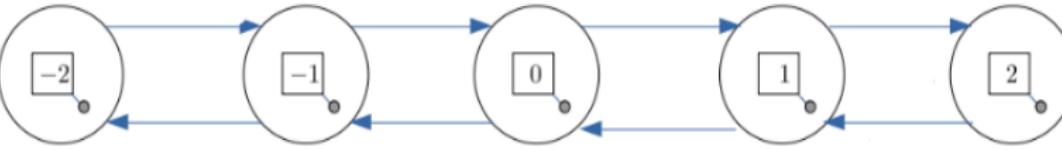


► More details

For superconductin islands, we need to take into account tunneling of Cooper-pairs, for which the Josephson term in the hamiltonian is the tunneling term, and quasi-particles, that behave in a similar way to electrons.

Single island - master equation

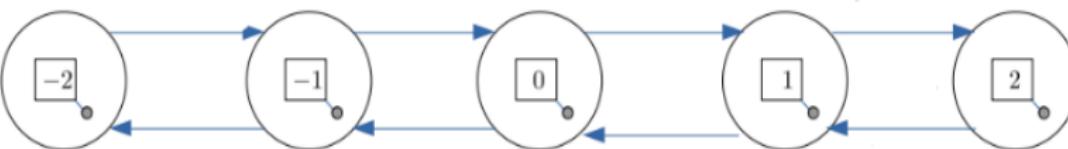
- For a constant Q_G , the probability function for the island distribution satisfy a Markovian master equation. For $T = 0$ the number of possible states is finite.



Since we are working in the slow relaxation limit, Q_G is changing much slower than the electron occupation of the island. Therefore, we would start by treating it as a constant.

Single island - master equation

- For a constant Q_G , the probability function for the island distribution satisfy a Markovian master equation. For $T = 0$ the number of possible states is finite.



- For a steady state solution, the in-flow (probability \times rate) and the out-flow of each node are equal.

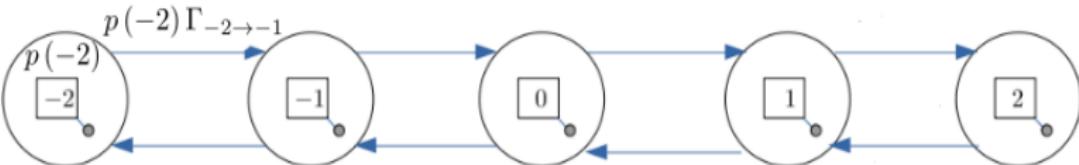
In that case, the probability to find the island with n excess electrons on it, satisfies a Markovian master equation. This is its graph representation.

Each node represents an occupation state.

Each edge represents a possible tunneling.

Single island - master equation

- For a constant Q_G , the probability function for the island distribution satisfy a Markovian master equation. For $T = 0$ the number of possible states is finite.



- For a steady state solution, the in-flow (probability \times rate) and the out-flow of each node are equal.

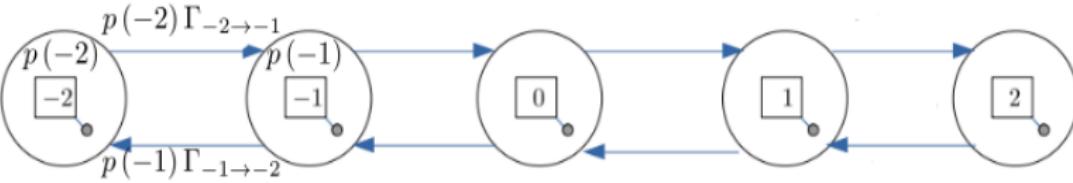
We define the flow on each edge as the matching tunneling rate times the probability of its origin.

To find a steady state probability, we want to assign a probability to each node such that the outgoing-flow and the incoming-flow of each node would be equal. To do that, we start by assigning a probability to the leftmost node.

Single island - master equation

We then choose the probability of the next node such that the in-flow and out-flow of the leftmost node are equal.

- For a constant Q_G , the probability function for the island distribution satisfy a Markovian master equation. For $T = 0$ the number of possible states is finite.

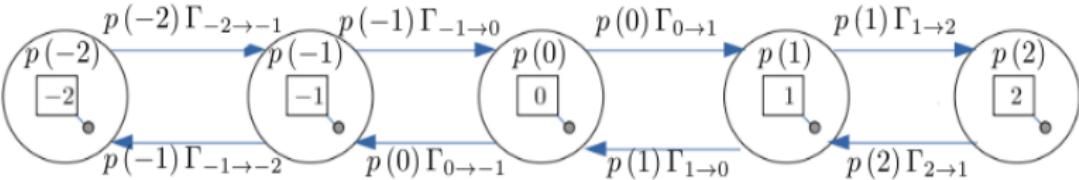


- For a steady state solution, the in-flow (probability \times rate) and the out-flow of each node are equal.

Single island - master equation

We continue to assign probabilities in the same way, and in the end we normalize the probabilities such that they would sum up to 1.

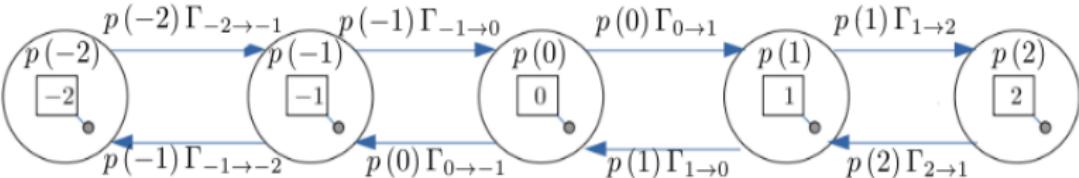
- For a constant Q_G , the probability function for the island distribution satisfy a Markovian master equation. For $T = 0$ the number of possible states is finite.



- For a steady state solution, the in-flow (probability \times rate) and the out-flow of each node are equal.

Single island - master equation

- For a constant Q_G , the probability function for the island distribution satisfy a Markovian master equation. For $T = 0$ the number of possible states is finite.



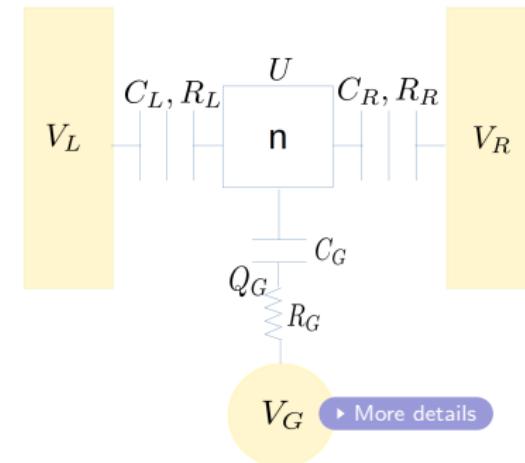
- For a steady state solution, the in-flow (probability \times rate) and the out-flow of each node are equal.
- $I = e \times \langle \Gamma_L \rangle_{Q_G} = e \times \langle \Gamma_R \rangle_{Q_G}$.
 $\langle \cdot \rangle$ - averaging with respect to the Q_G dependent steady-state probability.

Using this probability, we can calculate the steady state current as e times the average tunneling rate through the left or right junction. For a steady state current, both should be equal.

Single island - charge relaxation

- Between tunnelings, Q_G relaxes towards electrostatic equilibrium.

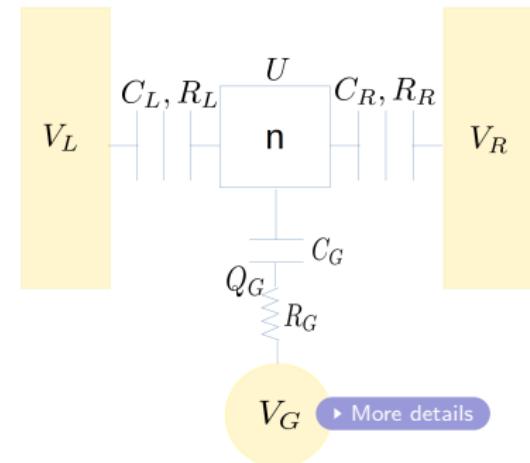
Between tunnelings, the different capacitors charge or discharge towards equilibrium. Like a normal RC circuit.



Single island - charge relaxation

For each n we will get a different steady-state value.

- Between tunnelings, Q_G relaxes towards electrostatic equilibrium.
- Its equilibrium value, Q_n , is n dependent.

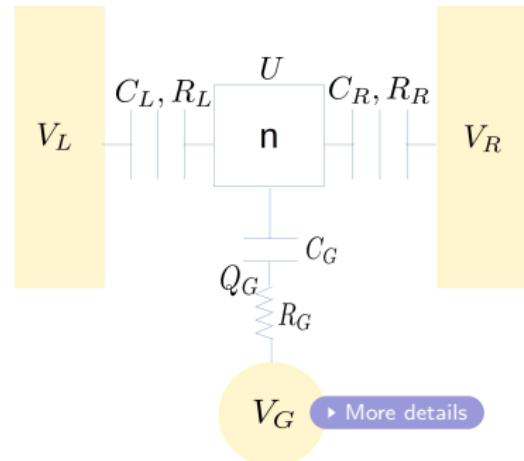


Single island - charge relaxation

- Between tunnelings, Q_G relaxes towards electrostatic equilibrium.
- Its equilibrium value, Q_n , is n dependent.
- In the slow relaxation limit, Q_G relaxes to the average equilibrium value. The steady state condition is given by

$$Q_G = \langle Q_n \rangle_{Q_G}$$

Since n changes much faster than Q_G , Q_G will relax to the average steady-state value, according to the matching steady state probability.



Single island - charge relaxation

- Between tunnelings, Q_G relaxes towards electrostatic equilibrium.
- Its equilibrium value, Q_n , is n dependent.
- In the slow relaxation limit, Q_G relaxes to the average equilibrium value. The steady state condition is given by

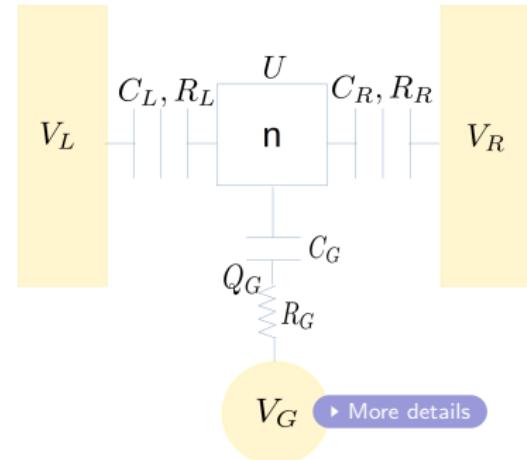
$$Q_G = \langle Q_n \rangle_{Q_G}$$

- We can rewrite this condition in terms of the electric potential on the island, as

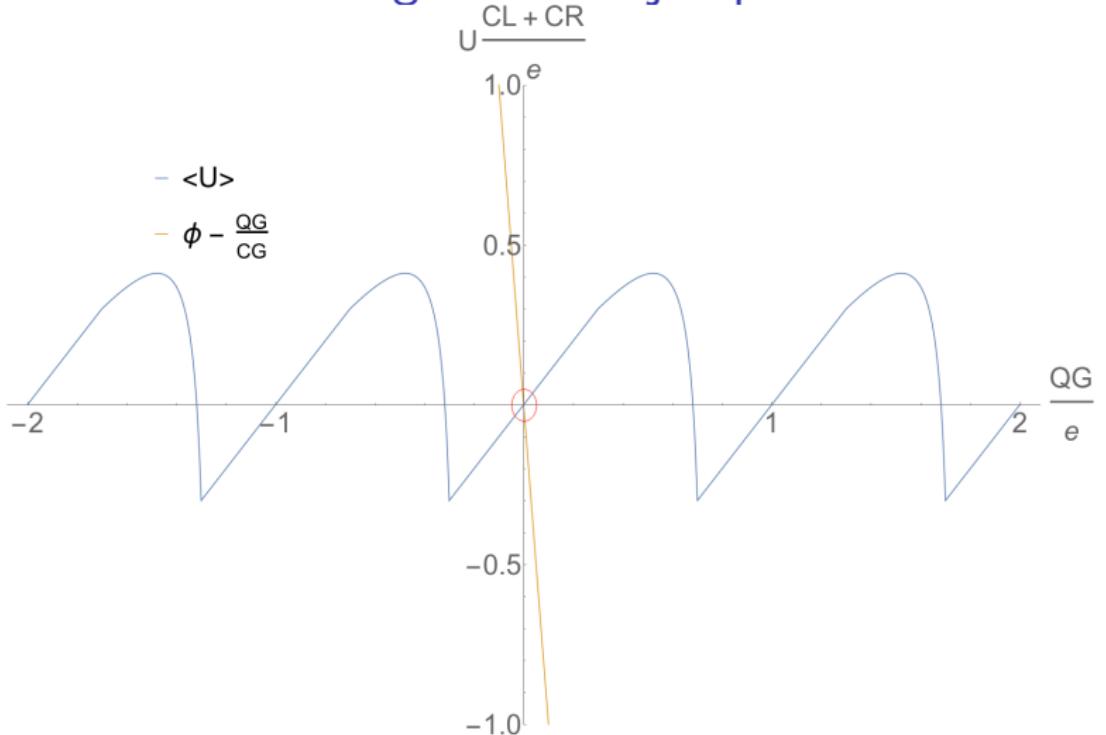
$$V_G - \frac{Q_G}{C_G} = \langle U \rangle_{Q_G}$$

It will be convenient to re-write this steady-state condition for Q_G , in terms of U , the electric potential on the island. This result could be also derived by demanding that, on average, there would be no current through the gate resistor.

To find a solution to this equation, we plot both of its sides, as a function of Q_G .

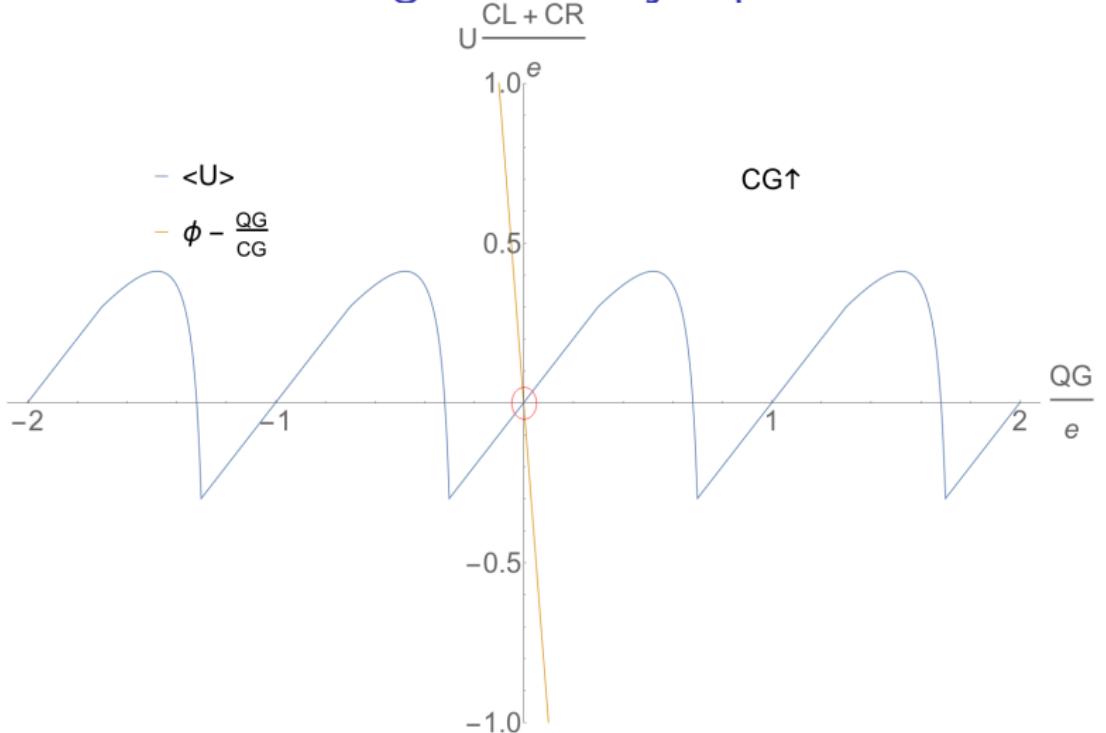


Single island - jumps



The average of U is a periodic function in Q_G , since increasing Q_G by e is the same as decreasing occupation by 1. Any intersection between the two functions, is a steady state solution for Q_G . A red circle is marking the current steady state solution for Q_G

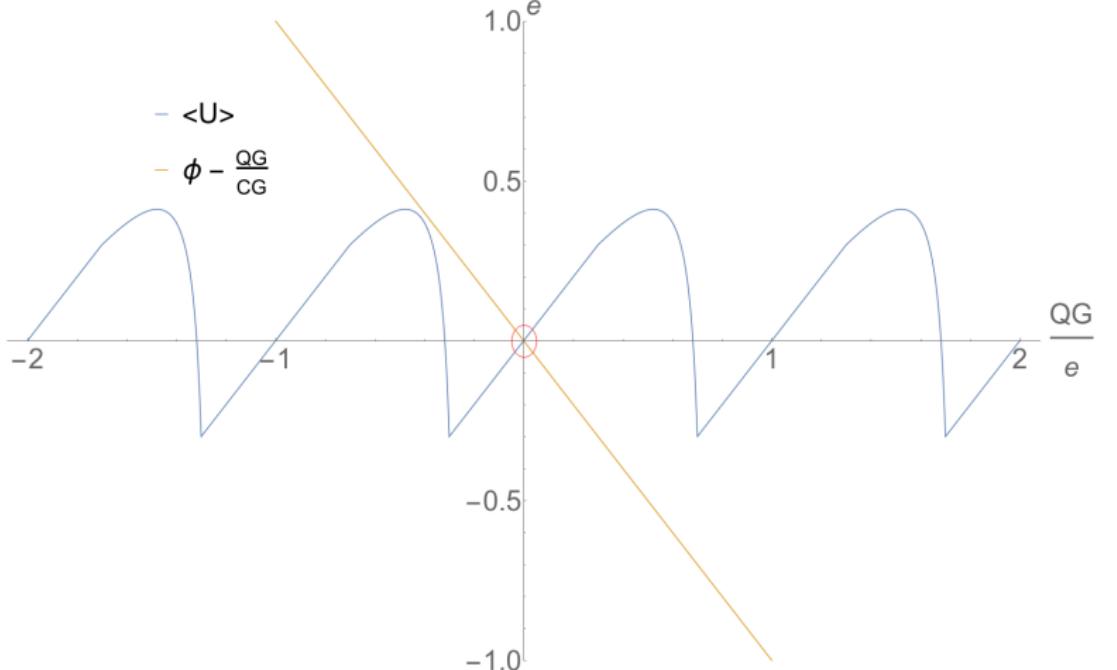
Single island - jumps



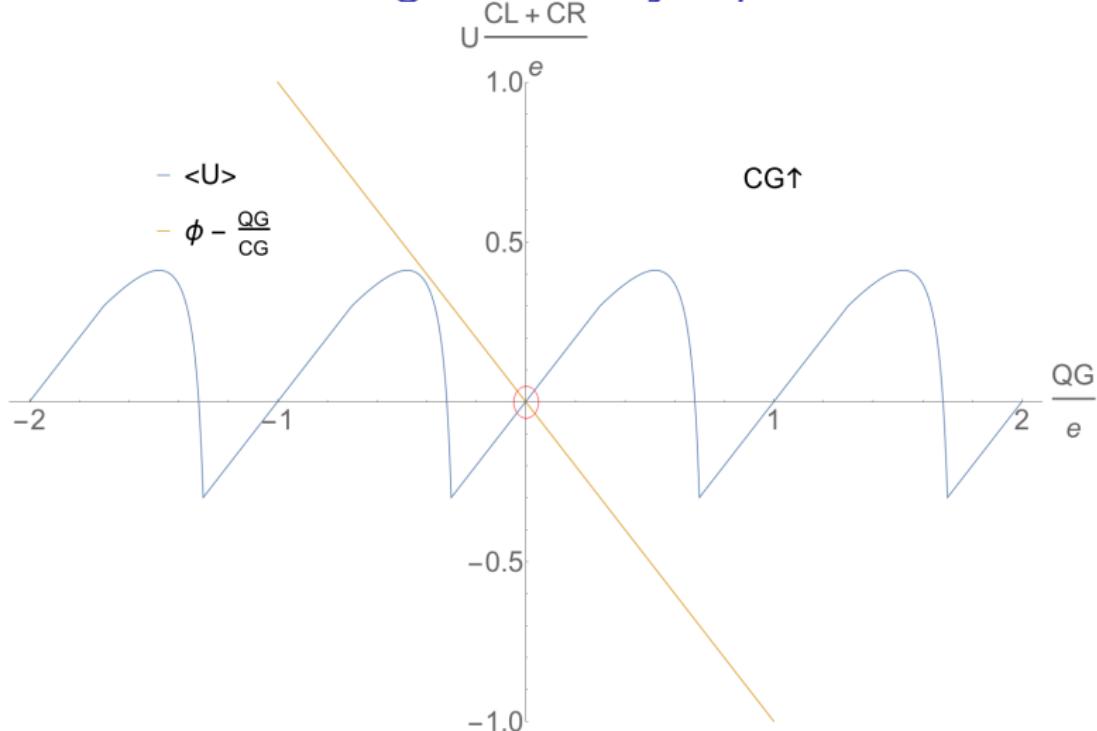
Changing C_G , would change the slope of the linear function.

Single island - jumps

$\frac{CL + CR}{U}$

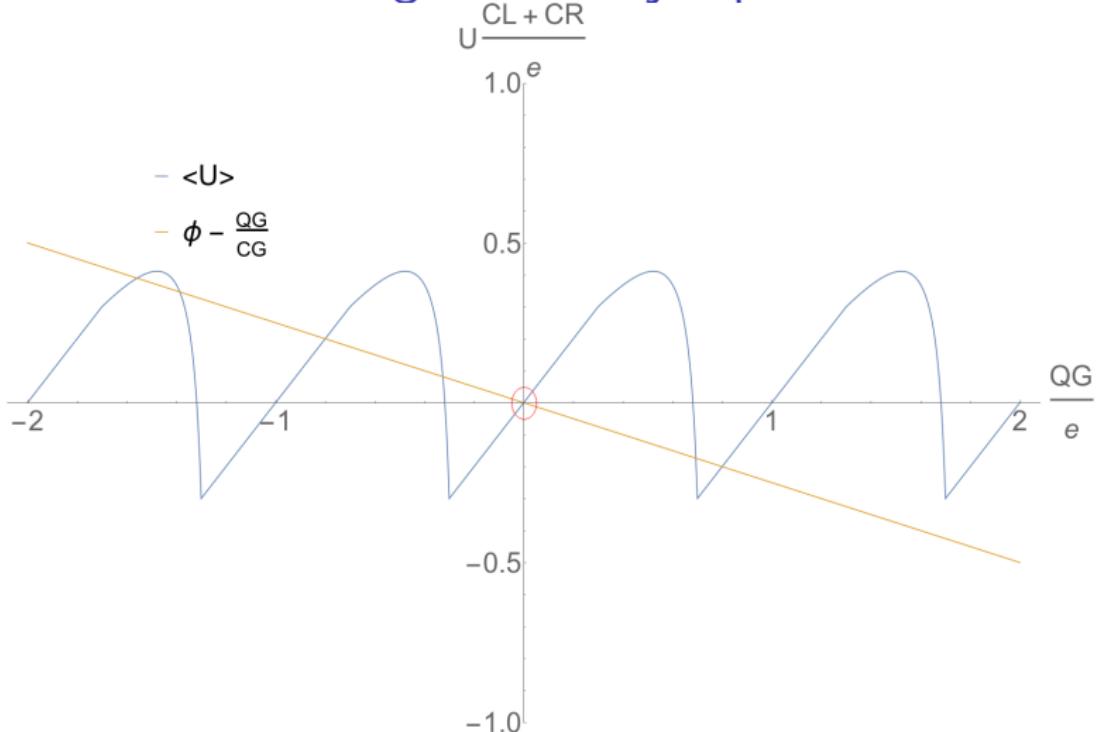


Single island - jumps



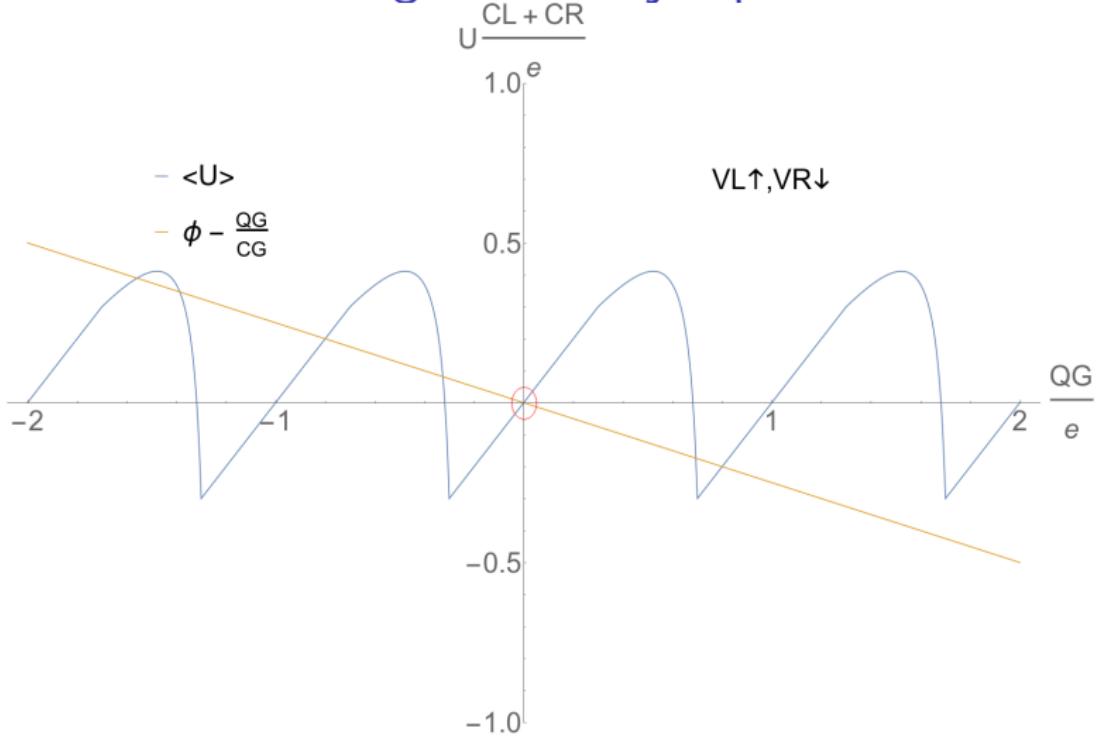
For big enough C_G , the linear function will cross the periodic function more than once.

Single island - jumps



For this case, there is more than one steady state solution for Q_G .

Single island - jumps



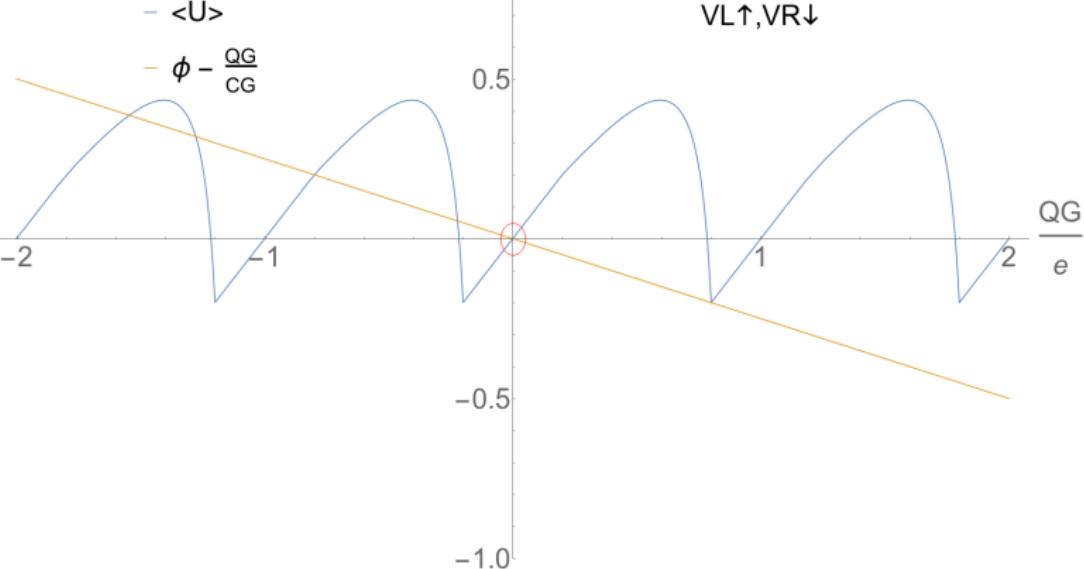
When voltage increases, the average of U moves right and up

Single island - jumps

$\frac{CL + CR}{U}$

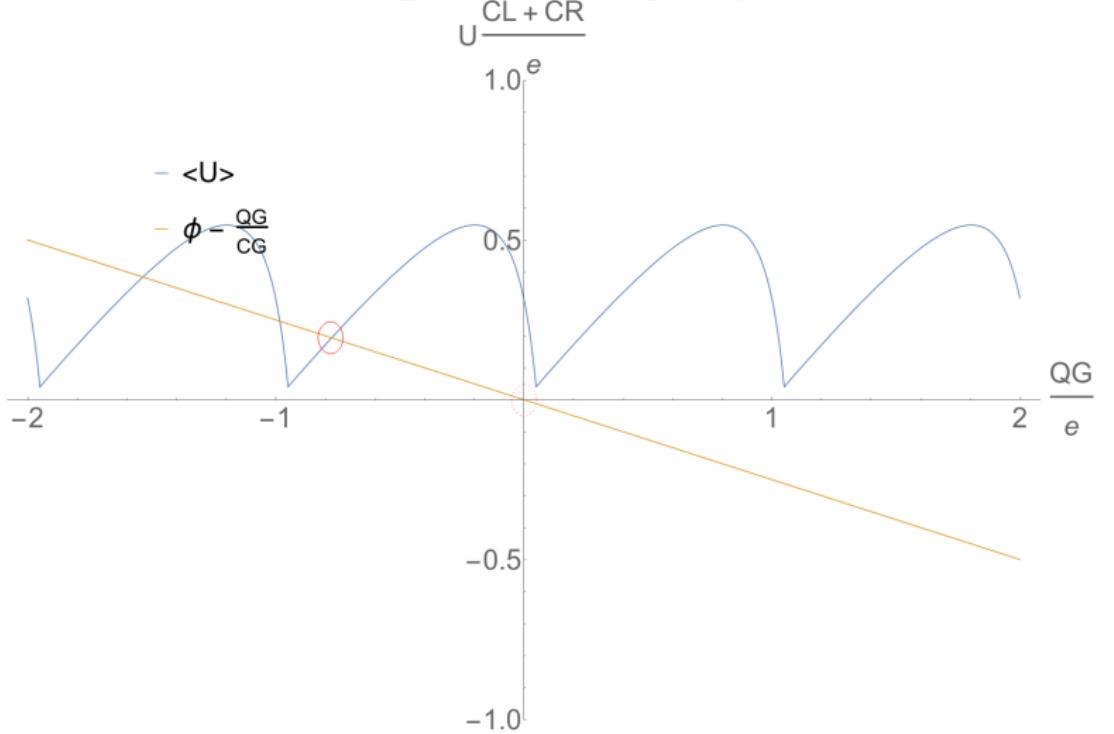
e

$VL \uparrow, VR \downarrow$



until the current solution disappears.

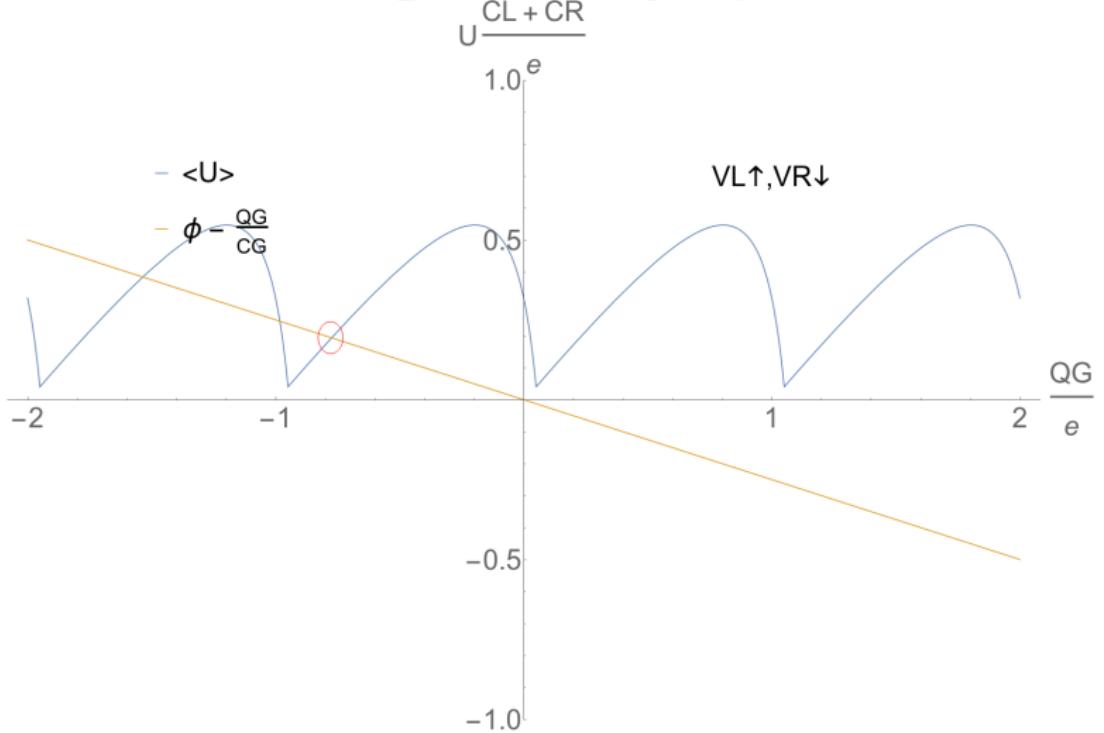
Single island - jumps



In that case, Q_G relaxes to the next stable solution. This will cause a jump in the I-V curve.

When decreasing the voltage back towards zero, Q_G would remain in this solution, and U would be higher than it was when we started. This will cause hysteresis in the I-V curve, as we will see next.

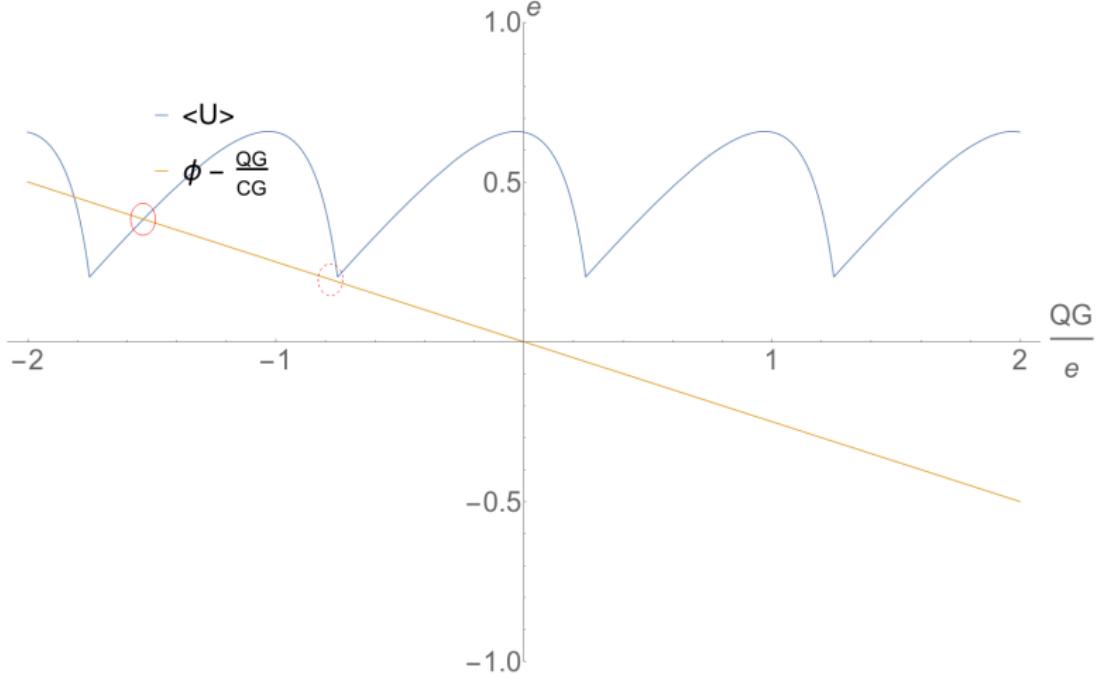
Single island - jumps



For high voltages

Single island - jumps

$U^{\text{CL} + \text{CR}}$



Theoretical model for hysteresis and jumps in disordered 2D superconductors near SIT

Shahar Kasirer

Introduction and Motivation

Proposed model - Single island

Results - Single island

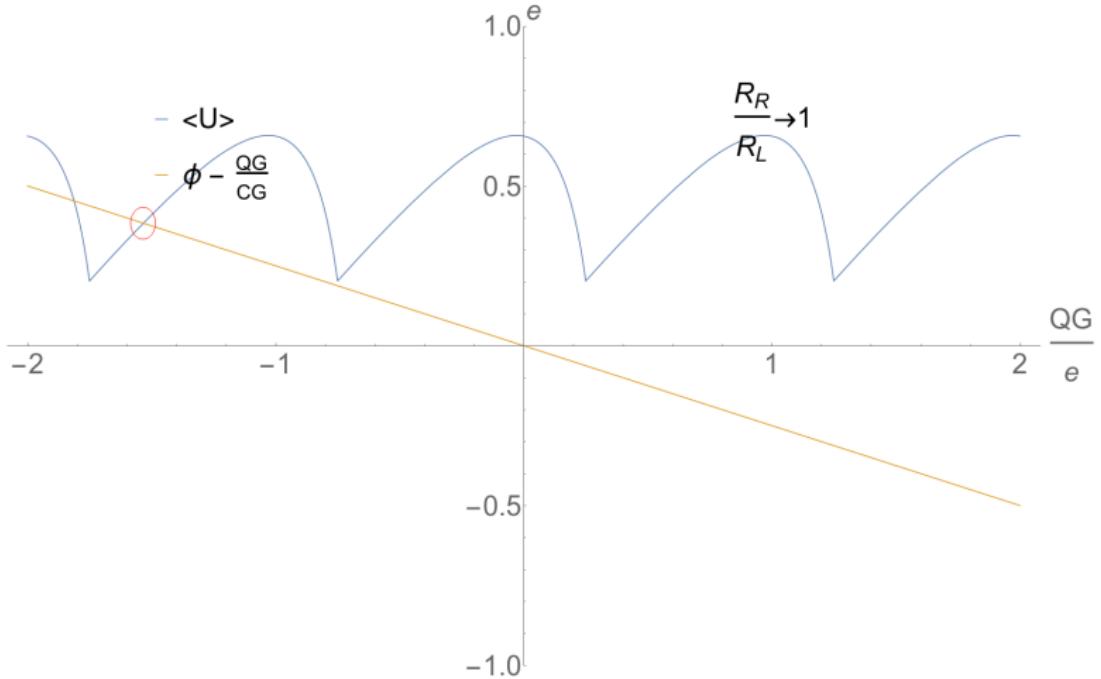
Proposed model - Random array

Results - Random array

Discussion and conclusion

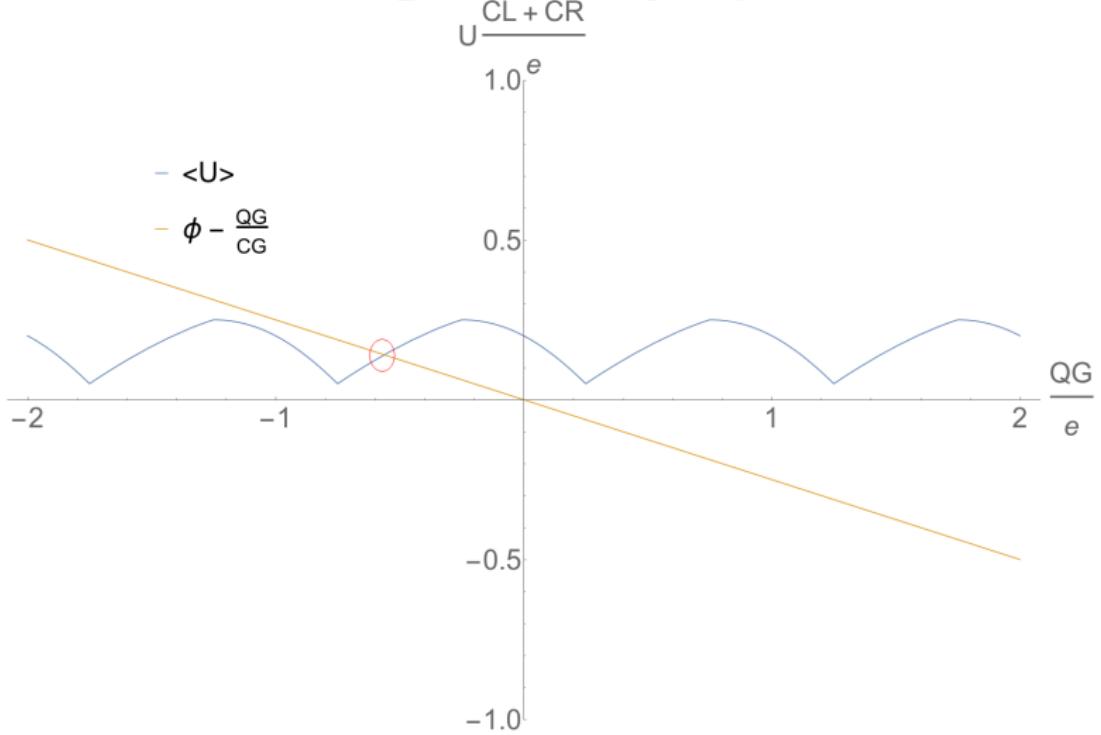
Single island - jumps

$$U^{\text{CL} + \text{CR}}$$



And for a more symmetric tunneling resistances

Single island - jumps

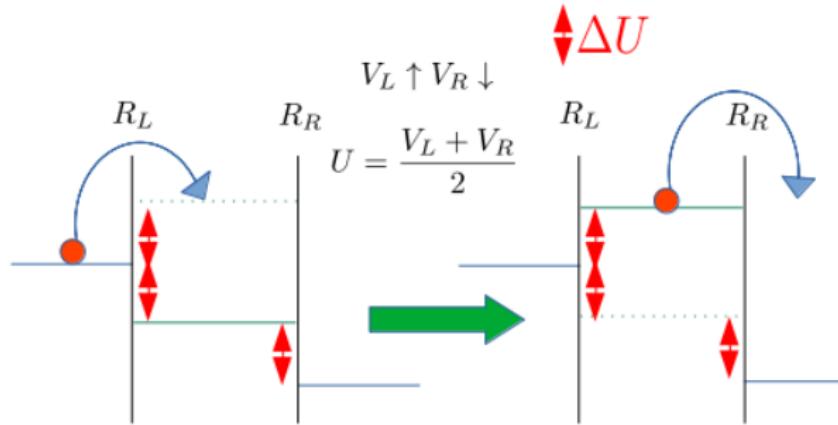


we are left with only one solution.

Therefore, we will get hysteresis and jumps only for small voltages and large tunneling resistance assymetry.

Single island - hysteresis

- For tunneling in zero temperature, a voltage difference of $\Delta U = \frac{e^2/2}{C_L+C_R}$ between electrode and island is needed.

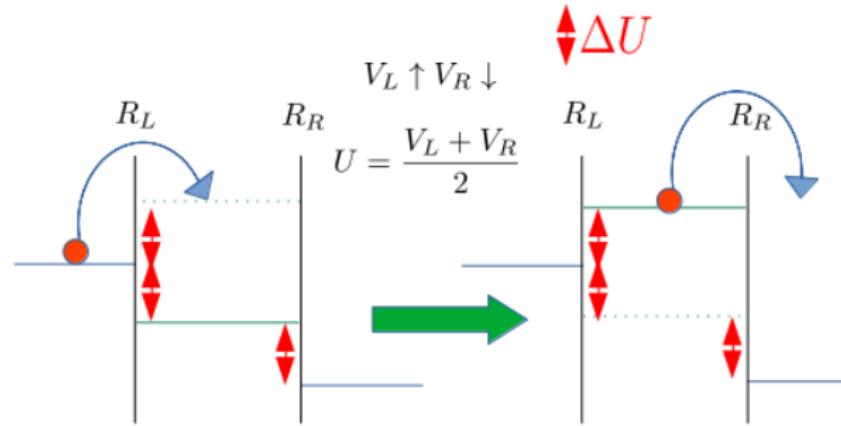


Now I would like to explain intuitively, why we would get hysteresis in the I-V curve.

Starting from $V = 0$, there are no excess electrons on the island, and the steady state potential on the island is right between the potentials on electrodes.

Single island - hysteresis

- For tunneling in zero temperature, a voltage difference of $\Delta U = \frac{e^2/2}{C_L+C_R}$ between electrode and island is needed.



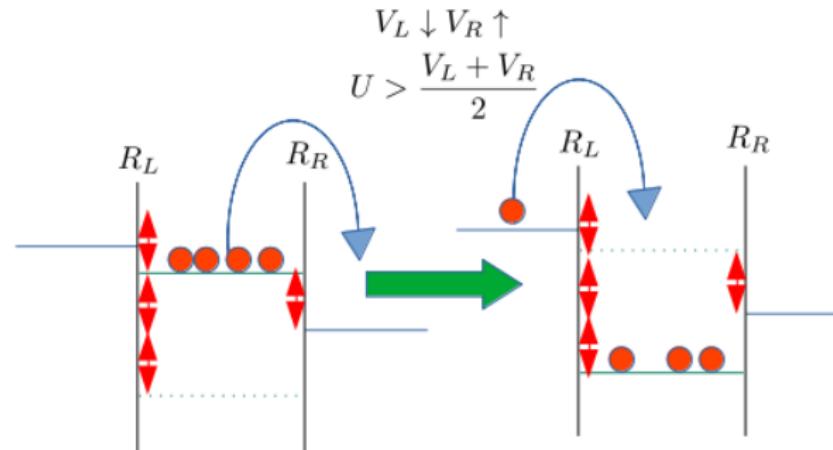
- For increasing voltage, U is between V_L and V_R . For tunneling we need $V = V_L - V_R > 2\Delta U$.

To get conduction, we would have to raise the voltage up to twice ΔU . Reaching this threshold, one electron will tunnel to the island, increasing U by twice ΔU and then tunnel from the island to the right electrode

Single island - hysteresis

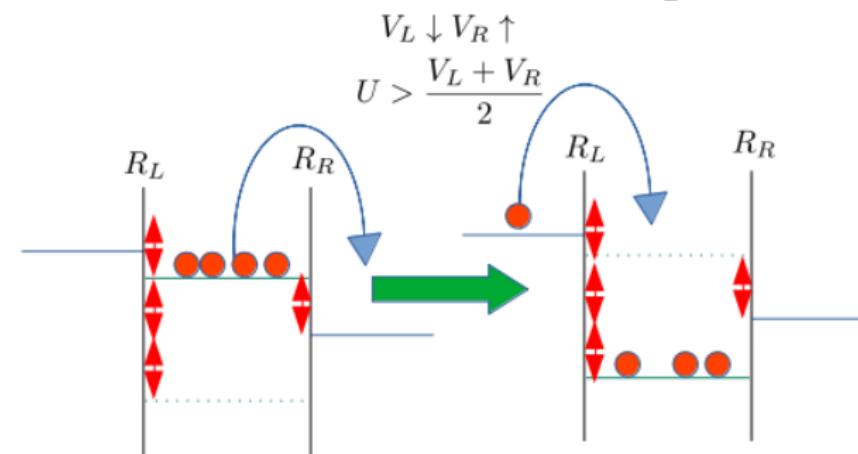
When decreasing the voltage back towards zero, the average number of electrons on island is bigger than zero, and U is bigger than it was for $V = 0$, as we saw earlier.

- When decreasing the voltage, $U > \frac{V_L + V_R}{2}$.



Single island - hysteresis

- When decreasing the voltage, $U > \frac{V_L + V_R}{2}$.



- For tunneling we need only $U - V_R > \Delta U$

In this case, current can still flow, by the indicated process, Tunneling from island to right electrode, and then from right electrode to island. This process would be possible until the voltage difference between the island and **both** electrodes would become smaller than ΔU .

This means that threshold voltage would be smaller when the voltage is decreased, and thus we would get hysteresis.

Now I would briefly describe the kinetic monte carlo simulation we had used to find a steady state current. The same method was also used to simulate the current in arrays, that we will see later.

Simulation method

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹

Our simulation is based on an algorithm that was proposed by Gillespie, for tracking the number of molecules in a system with chemical reactions. In our case the analog of a chemical reaction is tunneling, it changes the number of molecules in the system, which is analogous to the number of electrons on the island.

► More details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977).

Simulation method

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹
- At each step, one tunneling is simulated. Between tunnelings, Q_G is updated according to its RC relaxation.

At each simulation step, we simulate one tunneling. Then we generate a waiting time until the next tunneling and update Q_G according to its RC relaxation, with respect to the current island occupation.

► More details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977).

Simulation method

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹
- At each step, one tunneling is simulated. Between tunnelings, Q_G is updated according to its RC relaxation.
- We run the simulation for each external voltage, until reaching a steady state.

We run the simulation for each external voltage value, with the initial state being the state where simulation had finished for the last voltage value. Simulation runs until reaching a steady state. A major set back of this model, is that it takes many steps to reach a steady state, because of the slow relaxation. This is not much of a problem for the single island, but it would be a problem for arrays, where each simulation step is computationally expensive.

► More details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977).

Simulation method

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹
- At each step, one tunneling is simulated. Between tunnelings, Q_G is updated according to its RC relaxation.
- We run the simulation for each external voltage, until reaching a steady state.
- Once a steady state was reached, current is calculated, by averaging the results from different time steps.

When a steady state had been reached, we keep running the simulation and calculate the current, as a weighted average, weighted by the time of each step.

► More details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977).

Simulation method

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹
- At each step, one tunneling is simulated. Between tunnelings, Q_G is updated according to its RC relaxation.
- We run the simulation for each external voltage, until reaching a steady state.
- Once a steady state was reached, current is calculated, by averaging the results from different time steps.
- For $T = 0$ we can also solve the master equation to find the steady state current.

For zero temperature, we can find the current either by simulations, or by the method we had described before, solving the Q_G dependent the master equation and use a numerical solution to find a steady state Q_G .

► More details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977).

We used both methods to calculate the current for different gate capaci-
tance and tunneling resistance.

1 Introduction and Motivation

2 Proposed model - Single island

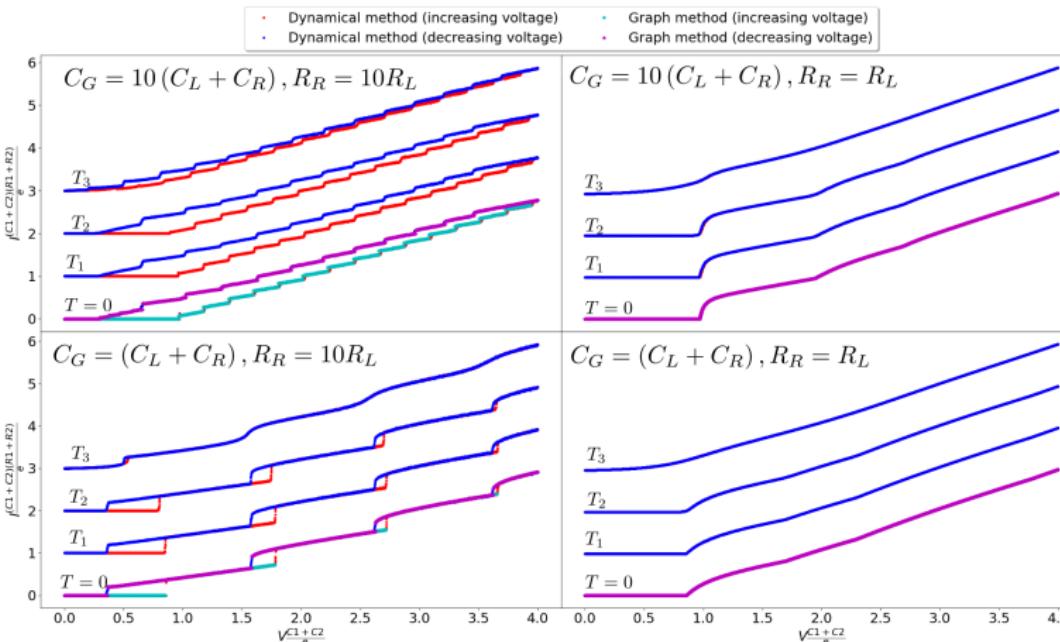
3 Results - Single island

4 Proposed model - Random array

5 Results - Random array

6 Discussion and conclusion

Single island - simulation results



I-V simulation results for a single island. Each plot shows results for the same system at different temperatures. $k_B T_1 = 0.001 \frac{e^2}{C_L + C_R}$, $k_B T_2 = 0.01 \frac{e^2}{C_L + C_R}$, $k_B T_3 = 0.1 \frac{e^2}{C_L + C_R}$.

At each square, the calculated current for the same gate capacitance and tunneling resistance, at different temperatures, is plotted.

For zero temperature, the results from both methods are plotted on top of each other.

Starting from the lower right plot, where we see no hysteresis and jumps for small C_G and equal resistances.

In the upper right plot, we see no hysteresis and jumps also for large C_G , although conductance starts with a steep slope.

For right tunneling resistance, smaller than the left one, and small C_G , we see multiple hysteresis loops, each of them is accompanied by a jump.

For a big C_G , the small loops merge into one, that includes many small jumps.

For zero temperature, both methods agree. As temperature raises, threshold and hysteresis become smaller.

Single island - results summary

- Jumps and hysteresis are only measured for small voltages and large resistance asymmetries.

To summarize the single island results:

We only expect hysteresis and jumps to be observed for small voltages and large tunneling resistance asymmetry. This would be translated to large resistance disorder for arrays.

Single island - results summary

When increasing the gate capacitance, hysteresis loops grow, and finally merge into one big loop. In addition, we see more, but smaller jumps. For larger gate capacitance, hysteresis is expected to be observed up to higher voltages.

- Jumps and hysteresis are only measured for small voltages and large resistance asymmetries.
- Increasing $C_G \rightarrow$ Larger hysteresis loops, more but smaller jumps.

Single island - results summary

Increasing temperature resulted in smoothing up of the resulting curves.
Hysteresis loops and jumps became smaller.

- Jumps and hysteresis are only measured for small voltages and large resistance asymmetries.
- Increasing $C_G \rightarrow$ Larger hysteresis loops, more but smaller jumps.
- Increasing temperature \rightarrow Smaller hysteresis loops and jumps.

Now lets see how to generalize this model into an array.

1 Introduction and Motivation

2 Proposed model - Single island

3 Results - Single island

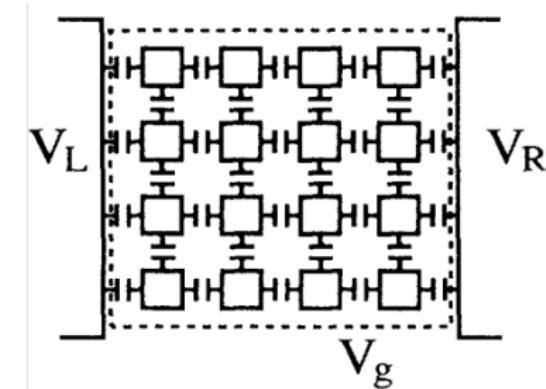
4 Proposed model - Random array

5 Results - Random array

6 Discussion and conclusion

Proposed model - Random array

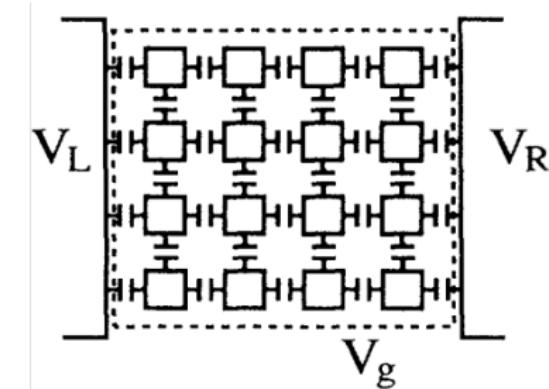
- Square array of conducting islands with electrodes on left and right.



Here we see a schema of the array model, it is similar to the model used by Middleton and Wingreen.

Proposed model - Random array

- Square array of conducting islands with electrodes on left and right.
- Each island is connected to V_G through an RC circuit.



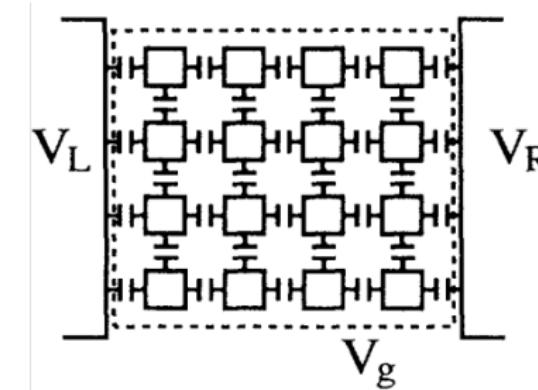
Each island has 4 nearest neighbors (except for bottom and top rows) and each is connected to a gate voltage through an RC -circuit.

Tunneling rates and charge relaxations are calculated in a similar way to the single island case.

Proposed model - Random array

Disorder is realized by randomly choosing tunneling resistance and capacitance for each junction.

- Square array of conducting islands with electrodes on left and right.
- Each island is connected to V_G through an RC circuit.
- Disorder is realized by randomly choosing tunneling resistance and capacitance.



► More details

Electric potentials on different islands

- For zero external voltage, the electric potential on each island are given by

$$\mathbf{U} = \mathcal{C}^{-1} (\mathbf{en} + \mathbf{Q}_G)$$

Where \mathbf{U} , \mathbf{n} , \mathbf{Q}_G are vectors of potentials, occupations and gate charges of each island, respectively.

A key difference between a single island and an array is the addition of interactions between electrons that are located on different islands.

To calculate those interactions, we calculate the electric potential on each island. Here, instead of a single number for each variable, we use vectors that hold the different values for each island. The electron potential is given in terms of the capacitance matrix.

Electric potentials on different islands

The diagonal elements of this matrix are the self-capacitance of each island.

- For zero external voltage, the electric potential on each island are given by

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Where \mathbf{U} , \mathbf{n} , \mathbf{Q}_G are vectors of potentials, occupations and gate charges of each island, respectively.

- \mathcal{C} is the capacitance matrix. It's diagonal elements are the self capacitance of each island.

Electric potentials on different islands

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Where \mathbf{U} , \mathbf{n} , \mathbf{Q}_G are vectors of potentials, occupations and gate charges of each island, respectively.

- \mathcal{C} is the capacitance matrix. It's diagonal elements are the self capacitance of each island.
- It's off diagonal elements are the mutual capacitance between island (non-zero only for nearest neighbors)

off-diagonal terms hold the mutual capacitance, which is the capacitance between islands. The mutual capacitance is not zero only for nearest neighbors.

We see that the interactions between a pair of islands is given by their mutual element in the inverse capacitance matrix.

Interactions

- In the case where R_G and C_G are the same for each island, the electric-potentials in steady-state are given by

$$\mathbf{U}^{eq} = (\mathcal{C} + C_G \mathbb{I})^{-1} (C_G V_G + e \langle \mathbf{n} \rangle) + e \mathcal{C}^{-1} (\mathbf{n} - \langle \mathbf{n} \rangle)$$

Lets examine the simple case, where gate capacitance and resistance are uniform.

In a steady state, gate charges relax to their average equilibrium values, and the electric potential on each island is given by the following relation.

The off-diagonal terms in both inverse matrices determine the interaction strength between pairs of islands.

Luckily, those matrices are invertible for uniform capacitances.

► More details

¹Bakhvalov, N. S. et al. Statics and dynamics of single-electron solitons in two-dimensional arrays of ultrasmall tunnel junctions. en. *Physica B: Condensed Matter* **173**, 319–328 (Sept. 1991).

Interactions

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$$\begin{aligned}\mathbf{U}^{eq} = & (\mathcal{C} + C_G \mathbb{I})^{-1} (C_G V_G + e \langle \mathbf{n} \rangle) \\ & + e \mathcal{C}^{-1} (\mathbf{n} - \langle \mathbf{n} \rangle)\end{aligned}$$

- For uniform capacitance, the matrix $\mathcal{C} + \mathbb{I} C_G$ can be inverted analytically¹

$$\begin{aligned}[(\mathcal{C} + C_G \mathbb{I})^{-1}]_{i,j} \propto & \begin{cases} \ln(\frac{1}{r}) & 1 \ll r \ll \lambda \\ \frac{\exp(-r/\lambda)}{\sqrt{r}} & \lambda \ll r \end{cases} \\ \lambda \equiv & -1/\ln \left(1 + \frac{C_G}{2C} - \sqrt{\left(\frac{C_G}{2C}\right)^2 + \frac{C_G}{C}} \right)\end{aligned}$$

r is the euclidean distance between islands i and j (in units of nearest neighbors distance).

► More details

The off diagonal terms of the result depend on an interaction length, λ , that is a function of the ratio between gate and tunneling capacitance.

For distances smaller than λ we get logarithmic decay. This is just a Coulomb interaction in 2D.

For distances larger than λ , we get exponential decay, a screened coulomb interaction.

λ is small for $C < C_G$ and big for $C > C_G$.

Examining the steady state voltages again, we see that the gate charge partially screens the electrons on each island.

For a big C_G , average occupation is screened, and has a short range effect, while deviations from it are unscreened.

We can conclude that long range interactions happen only between conducting islands. Since the occupation in non-conducting islands doesn't change, and so average n , and instantenous n are the same for those islands.

¹Bakhvalov, N. S. et al. Statics and dynamics of single-electron solitons in two-dimensional arrays of ultrasmall tunnel junctions. en. *Physica B: Condensed Matter* 173, 319–328 (Sept. 1991).

1 Introduction and Motivation

2 Proposed model - Single island

3 Results - Single island

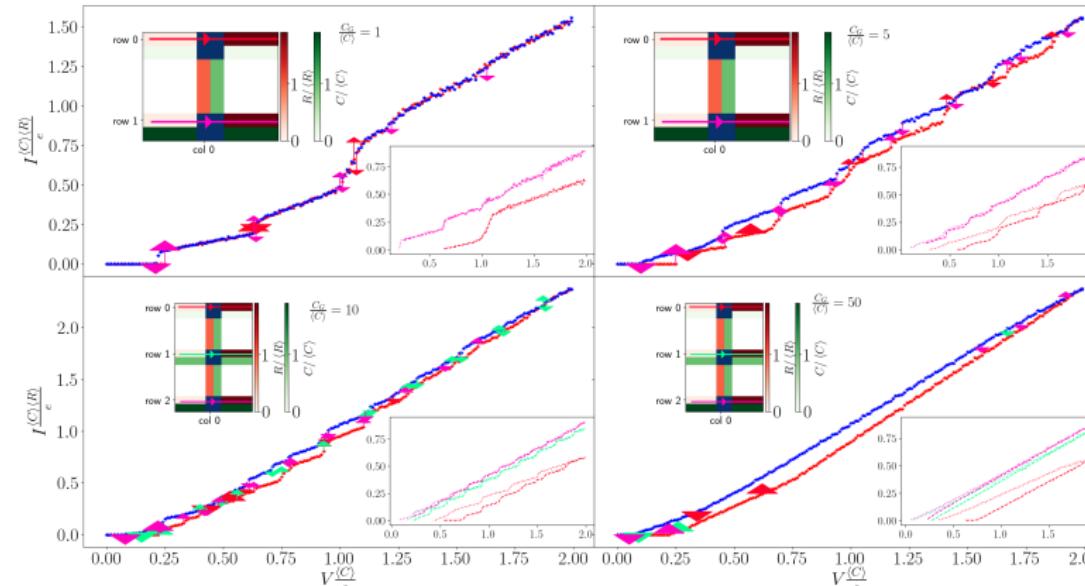
4 Proposed model - Random array

5 Results - Random array

6 Discussion and conclusion

Now we are finally ready to examine some simulation results for random arrays. As mentioned before, the long running time that was needed to reach a steady state had limited our ability to simulate large arrays, and so the larger array we used is only 10 by 10 islands.

Results - single island paths



Simulation results for 2×1 and 3×1 arrays. $T = 0$

Main plots - I-V curves, jumps in the current through each path are indicated by arrows.

Upper insets - Array parameters and paths indicated by arrows with different colors.

Lower insets - Current through each path, colored by the same path color as in the upper inset.

First, we ran our simulations for vertical arrays, where electrons only need to cross one island to get from left to right.

The total current is plotted in the main plots. The lower inner plots shows the current through each island.

The tunneling capacitance and resistance of each junction are plotted in the upper left. A color for each current path is also indicated here.

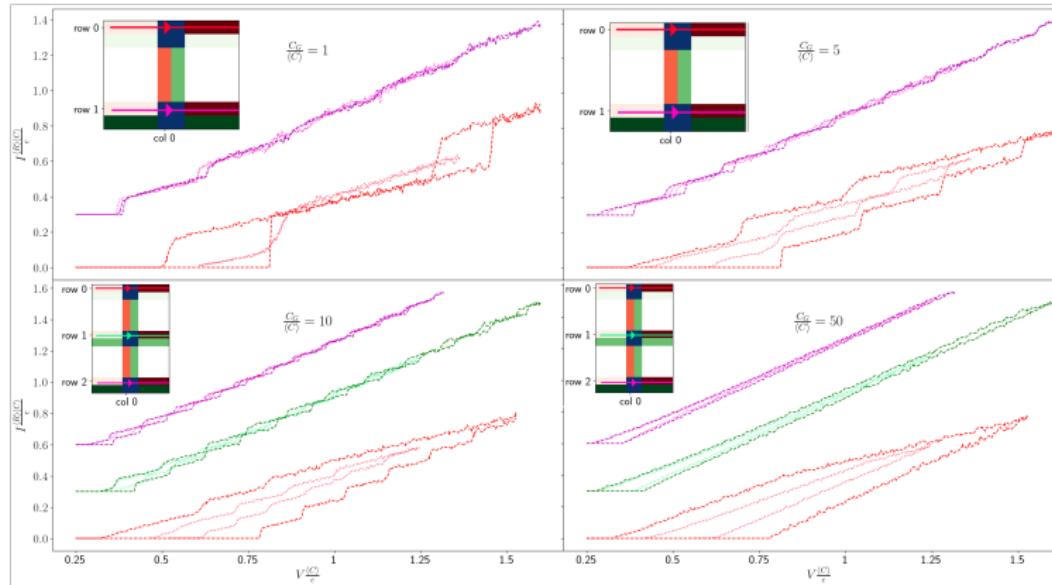
We can see that the current trough each island in our array behaves in a similar way to the current through a single island.

For smaller C_G , relative to the tunneling capacitances, we get smaller hysteresis.

For bigger C_G we get more hysteresis loops and jumps,

untill, for a big C_G we get one big hysteresis loop and jumps are too small to seperate.

Results - interactions



Comparison between single island results (dashed lines) and current through paths in a 2×1 and 3×1 arrays. $T = 0$.
Upper insets - Array parameters and paths indicated by arrows with different colors.

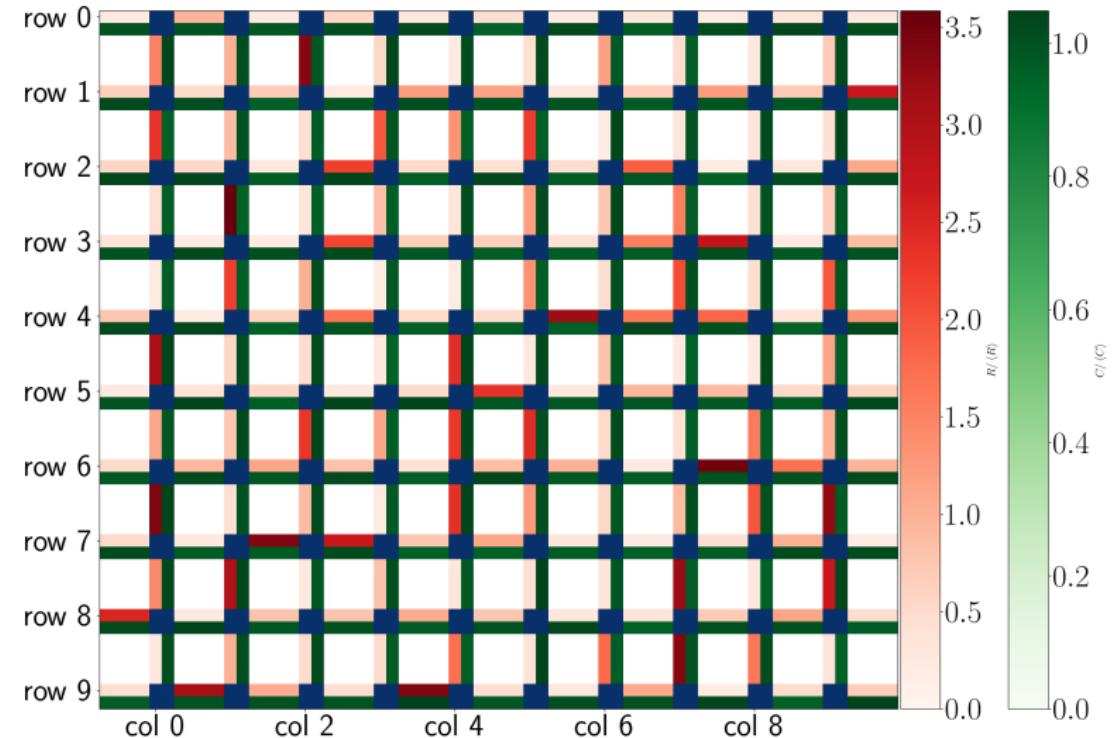
But, currents through different paths are not really independent, do to interactions.

Here we compare the current through each path in the vertical arrays we just saw, shown as a dotted line.

to the current through a single island with the same parameters, shown as a dashed line.

We see that interactions are causing a kind of averaging, and hysteresis becomes smaller. This is more apparent for smaller C_G , as interaction length is bigger. It is also more apparent for paths that open later (red vs. purple path for example.) this is because interactions are stronger between conducting paths. For the first path to open, there are no other conducting paths, and thus less interactions.

Results - local charge and currents



Simulation results for 10×10 array. $k_B T = 0.002 \frac{e^2}{\langle C \rangle}$.

Using our simulation, we can follow the local charges and currents in the array.

In this video we see the charges and currents in a 10×10 array while we increase the voltage on its left side, and decrease the voltage on its right.

Here we see the average occupations and currents on different islands and junctions, respectively. Positive current flows from left to right and from top to bottom.

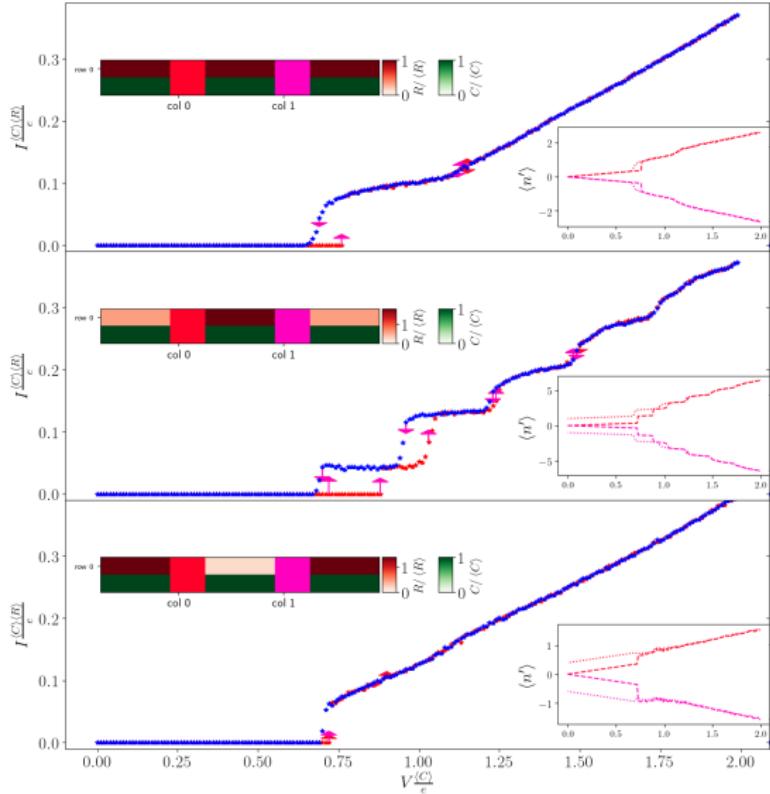
As voltage increases, electrons tunnel into the array from its left side, and out from its right side. The right side charges negatively, while the left charges positively.

When the positive and negative regions meet, conduction is starting. In this example, capacitance disorder is small, and since capacitance determines the electric potential on each island, conduction starts at once in many paths.

As voltage decreases, conduction stops gradually.

When voltage is back to zero, the system doesn't return to its initial state, it stays locally charged in some regions.

Results - the effect of individual tunneling resistance - 1D



Simulation results for 1×2 arrays.
 $T = 0$

Main plots - I-V curves. jumps in the occupation of each island are indicated by arrows.

Upper insets - Array parameters.

Lower insets - Average occupation on each island, using the same colors as in the upper inset.

We saw that resistance asymmetry was important for getting hysteresis in the single island case. To examine what role it plays for an array, we ran simulations for horizontal arrays.

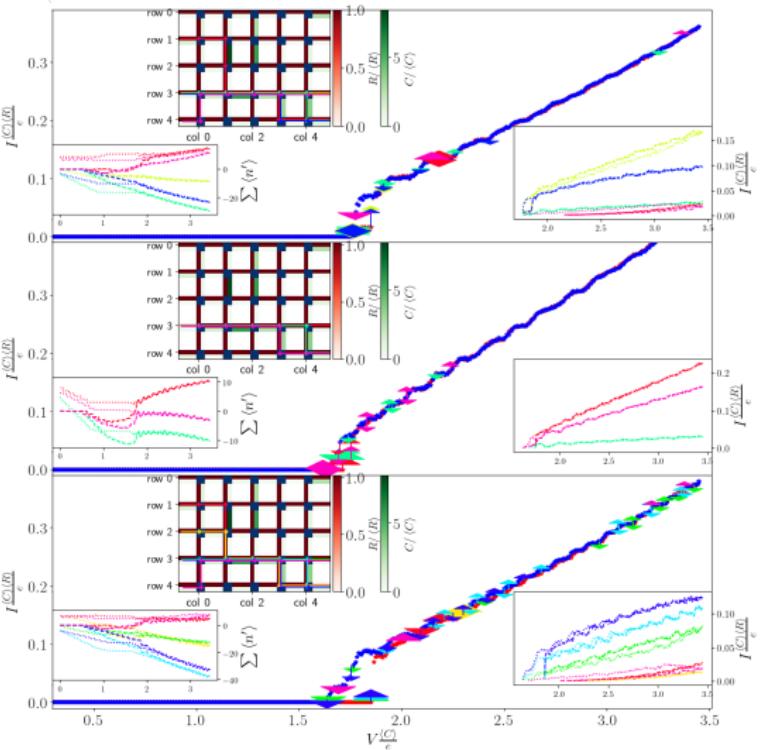
Remember that what was important for getting hysteresis in the single island is that the island won't be symmetric and either charge positively or negatively.

For uniform resistance, we get one hysteresis loop.

For small resistance in the beginning and the end of a path it would be easy to tunnel between the system and the electrodes. In this case the left side would charge positively, and the right side would charge negatively. Therefore, we get many hysteresis loops

For big resistance at the ends of a path, it is hard for electrons to change the electric potentials on the islands. In this case we get almost no hysteresis.

Results - the effect of individual tunneling resistance - 2D



Simulation results for 5×5 arrays. $T = 0$

Main plots - I-V curves, jumps in the current through each path are indicated by arrows. **Upper insets** - Array parameters. **Lower left insets**

- Average total occupation on the island in each path, using the same colors as in the upper inset.

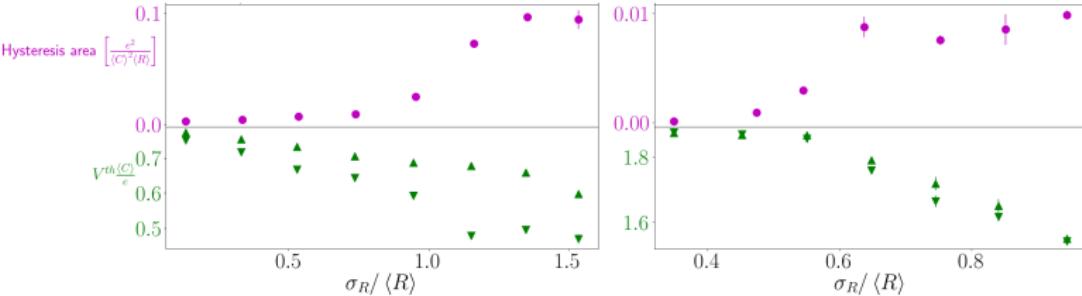
Lower right insets - Current on each path, using the same colors as in the upper inset.

The same thing is true for bigger arrays. In those examples we started with uniform resistance, and changed the resistance of a single junction each time.

In this case, decreasing both the resistance at the beginning of a current path, and on its end, led to bigger hysteresis loop.

In this example, because of tunneling capacitance realization, it was easier for electrons to exit from the right of the array, then it was for them to enter from the left. Therefore decreasing the resistance at the end of a path had a more pronounced effect.

Results - the effect of resistance disorder



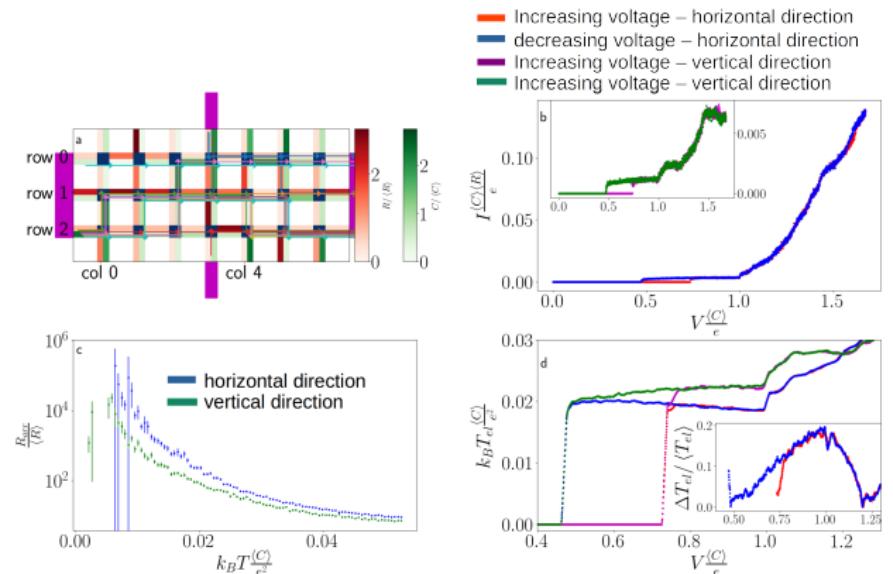
Average results for hysteresis area and threshold voltage (V^{th}) as a function of tunneling resistance disorder. For 3×3 arrays at $T = 0$ (left) and 10×10 arrays at $k_B T = 0.002 \frac{e^2}{\langle C \rangle}$. Each point is the average of $N > 10$ simulations. Up/down pointing triangles mark threshold for increasing/decreasing voltage.

Running our simulations for different random realizations of tunneling resistance and capacitance we gathered statistical information. Here, average hysteresis loop area and threshold voltages are plotted, as a function of resistance disorder, for small 3×3 arrays, and large 10×10 arrays.

We see that hysteresis loop area becomes larger as disorder grows.

Threshold voltages become lower and drift apart from each other as disorder grows.

Perpendicular electron temperature - our results



Perpendicular temperature simulation results, for 3×7 array.

- (a) system parameters, purple squares indicate locations of electrodes.
- (b) Current in the horizontal (main plot) and vertical (inner plot) directions as a function of the voltage in the horizontal direction.
- (c) Resistance as a function of temperature.
- (d) Calculated temperature of electrons. Inner plot - difference between results in different directions, normalized by their average.

To check the prediction of our model, for the same experimental setup, we used a similar procedure to the one used in the paper by Levinson et. al. we simulate the same system as was used in the experiment.

First, we calculated the resistance in both directions as a function of temperature.

Then, we calculated the current in each direction as a function of the voltage in the horizontal direction, and used this measurement to calculate the resistance in each direction as a function of the horizontal voltage.

Finally, we used those results to calculate the temperature of electrons.

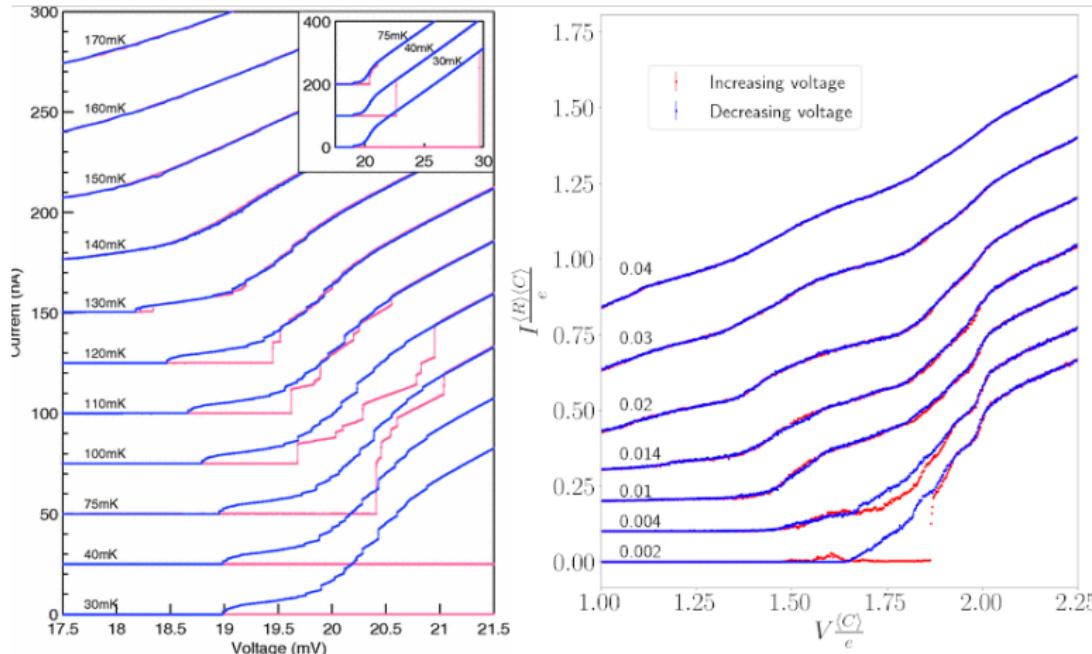
For low voltages, our results agree with the experimental ones, temperature is similar for both directions. For higher voltages, that were not tested in the experiment, temperatures diverge from each other.

The reason for the agreement in low voltages is different between our model and the electrons overheating model.

In our model, the similarity is originated from the main current path, which for low temperatures is the path from the upper and lower electrodes to the right one. Therefore the current is the same in both directions, and so does the calculated temperature.

For higher voltages, more paths open, including paths directly from left to right, and so the calculated temperatures diverge.

Results - I-V curves for different temperatures



Experimental results by Cohen et. al.¹ (left) and simulation results for a 10×10 array at different temperatures (right). Temperature, in units of $\frac{e^2}{\langle C \rangle}$ is indicated above each curve.

Finally, we ran our simulation for the same 10×10 array at different temperatures.

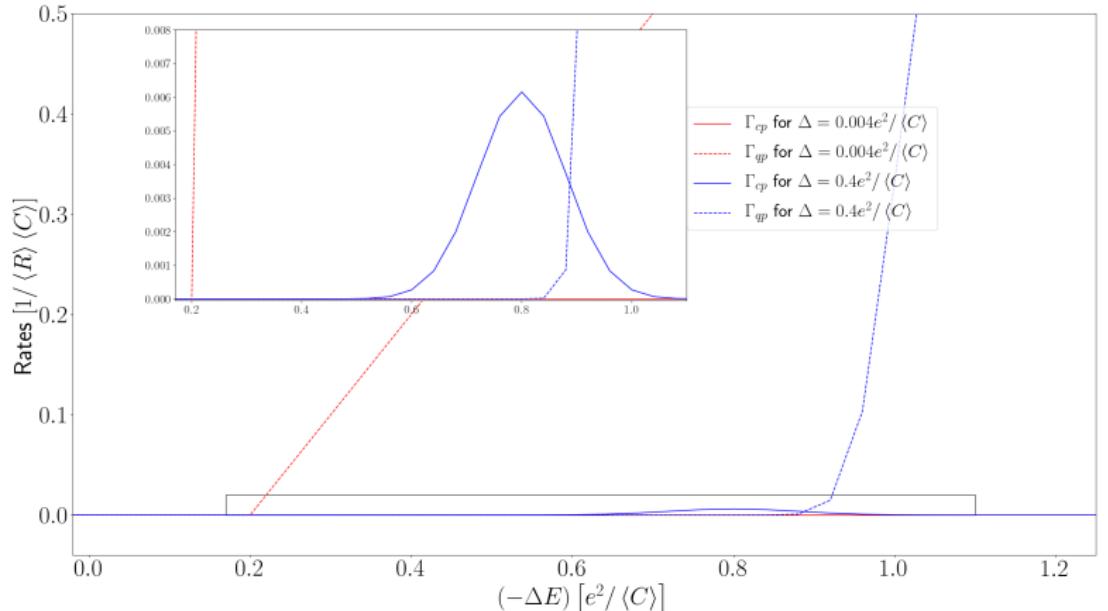
We see that, as for the experimental results, hysteresis and jumps become smaller as temperature increases, and threshold voltages become smaller.

Our results are **qualitatively** similar to the experimental results, although the first jump for low temperatures is much bigger in the experimental results.

We believe that this is due to the small array size we used, and that we would get a bigger jump for larger arrays.

¹Cohen, O. et al. Electric breakdown effect in the current-voltage characteristics of amorphous indium oxide thin films near the superconductor-insulator transition. *Physical Review B* 84, 100507 (Sept. 2011).

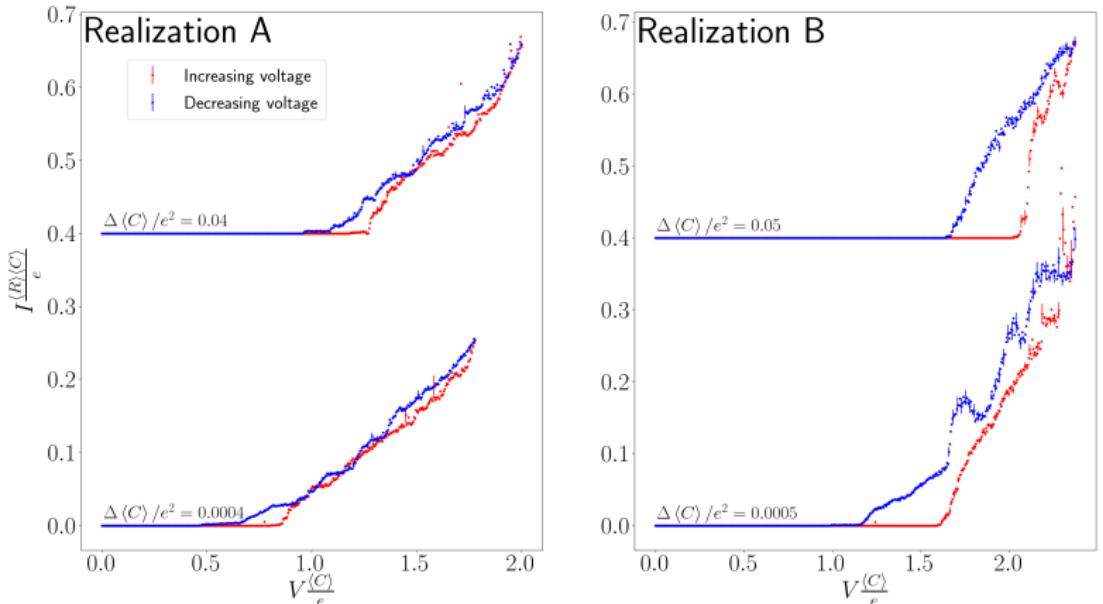
Superconducting array tunneling rates



Tunneling rates for Cooper-pairs and quasi-particles.

$$k_B T = 0.004 \frac{e^2}{\langle C \rangle}.$$

Results - Superconducting arrays



I-V curves for 2 different 5×5 arrays.

(left) $k_B T = 0.004 \frac{e^2}{\langle C \rangle}$, $C_G = 2.5 \langle C \rangle$.

(right): $k_B T = 0.005 \frac{e^2}{\langle C \rangle}$, $C_G = 2 \langle C \rangle$.

Results summary

- To measure hysteresis and jumps, temperature and voltage must be lower than the energy scale for changes in gate charges.

To summarize our results:

We predict hysteresis and jumps only for small temperatures and voltages.
Both need to be smaller than the energy scale for changing Q_G .

Results summary

- To measure hysteresis and jumps, temperature and voltage must be lower than the energy scale for changes in gate charges.
- Small capacitance disorder → paths open together → bigger jumps and hysteresis.

For small capacitance disorder, many paths open together and we get bigger jumps and hysteresis.

Results summary

Threshold voltages decrease, and hysteresis area increases, when resistance disorder increases.

- To measure hysteresis and jumps, temperature and voltage must be lower than the energy scale for changes in gate charges.
- Small capacitance disorder → paths open together → bigger jumps and hysteresis.
- Threshold voltages decrease and hysteresis loop area increases, for larger resistance disorders.

Results summary

- To measure hysteresis and jumps, temperature and voltage must be lower than the energy scale for changes in gate charges.
- Small capacitance disorder → paths open together → bigger jumps and hysteresis.
- Threshold voltages decrease and hysteresis loop area increases, for larger resistance disorders.
- Tunneling-resistance of edge junctions have the most significant effect on hysteresis.

Tunneling resistance of junctions that connect the system to the electrodes, have the most significant affect on hysteresis. Small resistance in those, will increase hysteresis significantly.

Results summary

- To measure hysteresis and jumps, temperature and voltage must be lower than the energy scale for changes in gate charges.
- Small capacitance disorder → paths open together → bigger jumps and hysteresis.
- Threshold voltages decrease and hysteresis loop area increases, for larger resistance disorders.
- Tunneling-resistance of edge junctions have the most significant effect on hysteresis.
- Bigger C_G → bigger threshold voltage and hysteresis loop area, and more jumps.

Bigger gate capacitance would increase the voltage threshold and the hysteresis loop area. We would also get more, but smaller, jumps.

Lets take a step back and conclude this talk.

1 Introduction and Motivation

2 Proposed model - Single island

3 Results - Single island

4 Proposed model - Random array

5 Results - Random array

6 Discussion and conclusion

Overview

Using the slow relaxation limit, we effectively added an extra degree of freedom to our system. This slow degree of freedom acts as a memory, creating hysteresis.

- By assuming slow relaxation we effectively added an extra DOF, which functions as a memory.

Overview

Jumps occur when this degree of freedom has more than one steady-state solutions. When one solution disappears, the relaxation to the next solution creates a jump.

- By assuming slow relaxation we effectively added an extra DOF, which functions as a memory.
- This slow DOF can have more than one stable solution. A jump occurs when it moves from one stable solution to another.

Overview

We believe that our results would not have changed much if we would have used a different relaxation process, instead of the linear RC coupling. The only condition for getting hysteresis is that relaxation would be much longer than the time between tunnelings.

- By assuming slow relaxation we effectively added an extra DOF, which functions as a memory.
- This slow DOF can have more than one stable solution. A jump occurs when it moves from one stable solution to another.
- Hysteresis and jumps should appear for any relaxation process, as long as it is much longer than the time between tunnelings.

Research limitations

This work had 2 main limitations.

Limitations

¹Gillespie, D. T. Approximate accelerated stochastic simulation of chemically reacting systems. *The Journal of Chemical Physics* **115**. Publisher: American Institute of Physics, 1716–1733 (July 2001).

Research limitations

Limitations

- Long running time → limited for small arrays (up to 10×10)

The first is a long running time, that prevented us from running our simulations for arrays bigger than 10 by 10.

The long simulation time comes from slow relaxation, since many tunneling events should be simulated to reach a steady state.

¹Gillespie, D. T. Approximate accelerated stochastic simulation of chemically reacting systems. *The Journal of Chemical Physics* **115**. Publisher: American Institute of Physics, 1716–1733 (July 2001).

Research limitations

Another limitations is, that it would be hard to test some of our predictions in the lab, since it is hard to measure local values as tunneling resistance, capacitance or local charge in the system.

Limitations

- Long running time → limited for small arrays (up to 10×10)
- Hard to test predictions.

¹Gillespie, D. T. Approximate accelerated stochastic simulation of chemically reacting systems. *The Journal of Chemical Physics* **115**. Publisher: American Institute of Physics, 1716–1733 (July 2001).

Research limitations

I would like to propose solutions for these problems, that could help in a future research.

Limitations

- Long running time → limited for small arrays (up to 10×10)
- Hard to test predictions.

Proposed solutions

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Research limitations

Limitations

- Long running time → limited for small arrays (up to 10×10)
- Hard to test predictions.

Proposed solutions

- Use " τ -leaping"¹ approximation to speed up simulation.

We can speed up simulation time by assuming different tunnelings to be independent, and simulating many tunneling events at once, instead of one by one. This is the essence of the τ leaping method, that was suggested by Gillespie for his own algorithm. The problem with this approximation is that it might not be a good approximation do to interactions, Further research is needed to determine when it would be "safe to use".

¹Gillespie, D. T. Approximate accelerated stochastic simulation of chemically reacting systems. *The Journal of Chemical Physics* **115**. Publisher: American Institute of Physics, 1716–1733 (July 2001).

Research limitations

Limitations

- Long running time → limited for small arrays (up to 10×10)
- Hard to test predictions.

Proposed solutions

- Use " τ -leaping"¹ approximation to speed up simulation.
- Include "easier to control" parameters in the model, e.g. external magnetic field, temperature gradient.

To get easier to test predictions we can run our simulations for easy to control parameters, like changing external magnetic field or temperature gradient. Again, further research is needed for incorporating those into our model in the correct way.

¹Gillespie, D. T. Approximate accelerated stochastic simulation of chemically reacting systems. *The Journal of Chemical Physics* **115**. Publisher: American Institute of Physics, 1716–1733 (July 2001).

Future research suggestions

Future research suggestions

Our model and simulation can be used as a basis for future research of disordered material's transport properties.

Our model and simulation could be used as a basis for future research.
Here are few suggestions:

¹Altshuler, B. L. et al. Jumps in Current-Voltage Characteristics in Disordered Films. *Physical Review Letters* **102**, 176803 (Apr. 2009).

Future research suggestions

Future research suggestions

Our model and simulation can be used as a basis for future research of disordered material's transport properties.

- Using different tunneling rates.

The same simulation could be used with different tunneling rates, accounting for different environment impedances. For example, we are currently running simulations for superconducting arrays using different tunneling rates for Cooper-pairs and Quasi-particles.

¹Altshuler, B. L. et al. Jumps in Current-Voltage Characteristics in Disordered Films. *Physical Review Letters* **102**, 176803 (Apr. 2009).

Future research suggestions

Future research suggestions

Our model and simulation can be used as a basis for future research of disordered material's transport properties.

- Using different tunneling rates.
- Using different relaxation processes. e.g. use long thermal relaxation time (as in "electrons overheating model"¹).

A different relaxation process could be used. For example, a slow thermal relaxation of electrons temperature, like in the electrons overheating model

¹Altshuler, B. L. et al. Jumps in Current-Voltage Characteristics in Disordered Films. *Physical Review Letters* 102, 176803 (Apr. 2009).

Future research suggestions

Future research suggestions

Our model and simulation can be used as a basis for future research of disordered material's transport properties.

- Using different tunneling rates.
- Using different relaxation processes. e.g. use long thermal relaxation time (as in "electrons overheating model"¹).
- Predict thermoelectric properties, e.g. Seebeck and Thompson effects.

Another interesting direction is to use our simulations to predict the thermoelectric properties of disordered materials, and their dependency on different parameters.

¹Altshuler, B. L. et al. Jumps in Current-Voltage Characteristics in Disordered Films. *Physical Review Letters* 102, 176803 (Apr. 2009).

Conclusion

Conclusion

A random array model, with slow charge distribution relaxation, can account for jumps and hysteresis in the I-V curve.

The main "take home message" of this talk is that a random array model can reproduce jumps and hysteresis in its I-V curve, if it has a slow-relaxing degree of freedom.

Conclusion

Our code is written in Python and is available to use, for anyone that is interested, at this git repository.

Conclusion

A random array model, with slow charge distribution relaxation, can account for jumps and hysteresis in the I-V curve.

Code

Our simulation code is written in Python and available at

https://github.com/kasirershaharbgu/random_2D_tunneling_arrays

Acknowledgments

- My adviser: **Professor Yigal Meir.**
- Professor Dan Shahar.
- Professor Ned Wingreen, Professor Herman Verlinde and all of the Molecular biology and Physics departments at Princeton University.
- Dr. Amir Erez, Dr. Lailai Zhu, Benjamin Weiner
- All of my classmates here at BGU.

I would like to express my gratitude towards anyone that helped me with this project.

First and foremost, my adviser, Yigal. Thank you for offering me this project and for helping me on the way.

To professor Dan Shahar for sharing his results and some valuable insights.

Part of this project was done in Princeton University. I would like to thank Professor Ned Wingreen, Professor Herman Verlinde, and their departments for their warm hospitality.

A special thanks is given to Dr. Amir Erez, Dr. Lailai Zhu and Benjamin Weiner, for helping me with the simulations.

I would also like to thank all of my classmates at Ben Gurion University, for helping me with this project.

And finally, I would like to thank you for listening!

Thank you!

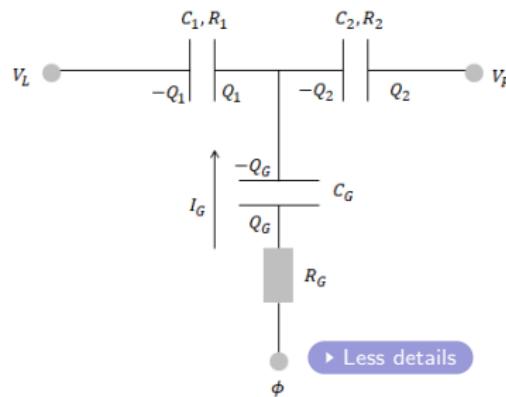
If there are any questions, I would be happy to take them.

Questions?

Single island - model detail

$$\mathcal{H}(n, Q_G) = \mathcal{H}_C(n, Q_G) + \mathcal{H}_T$$

Now for a little bit of math. This is the general structure of the Hamiltonian. It is composed of two terms.



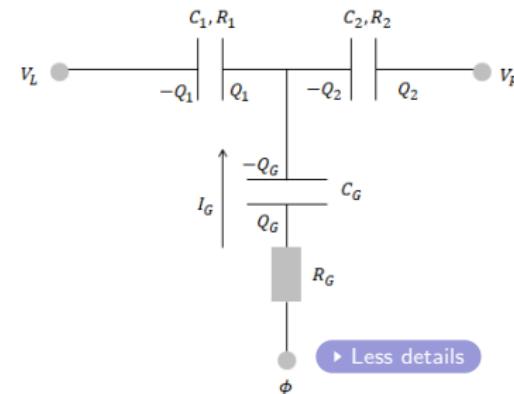
► Less details
 ϕ

Single island - model detail

$$\mathcal{H}(n, Q_G) = \mathcal{H}_C(n, Q_G) + \mathcal{H}_T$$

- \mathcal{H}_C is the electrostatic energy term, in our case:

$$\mathcal{H}_C = \underbrace{\frac{1}{2} \left(\frac{Q_1^2}{C_1} + \frac{Q_2^2}{C_2} + \frac{Q_G^2}{C_G} \right)}_{\text{Electrostatic energy}} + \underbrace{Q_L V_L + Q_R V_R}_{\text{Boundary terms}}$$



The first term, is the electrostatic energy, which is just the energy stored on capacitors plus the energy stored on electrodes.

► Less details

Single island - model detail

$$\mathcal{H}(n, Q_G) = \mathcal{H}_C(n, Q_G) + \mathcal{H}_T$$

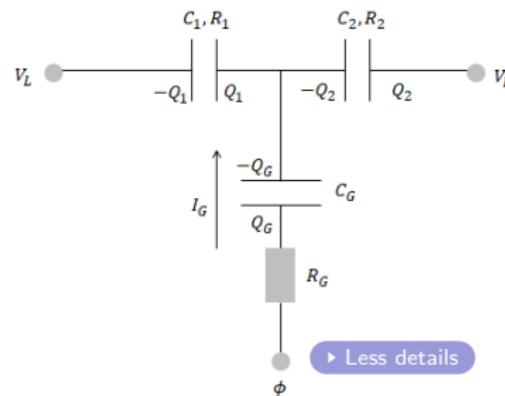
- \mathcal{H}_C is the electrostatic energy term, in our case:

$$\mathcal{H}_C = \underbrace{\frac{1}{2} \left(\frac{Q_1^2}{C_1} + \frac{Q_2^2}{C_2} + \frac{Q_G^2}{C_G} \right)}_{\text{Electrostatic energy}} + \underbrace{Q_L V_L + Q_R V_R}_{\text{Boundary terms}}$$

- \mathcal{H}_T is the tunneling term.

It is considered to be a small perturbation.

Therefore we solve for the basis of \mathcal{H}_C (states with a well defined occupation).



The second, is the tunneling term, allowing electrons to tunnel.

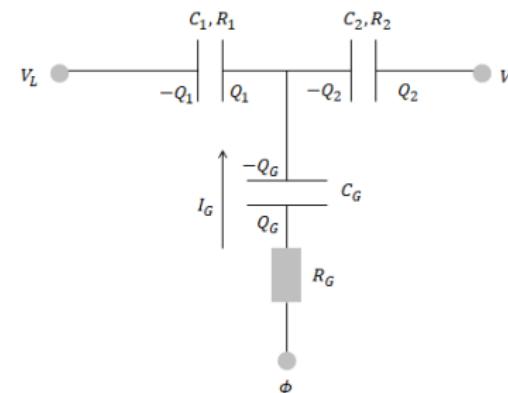
It is considered to be a constant term, independent of n , which the number of excess electrons on the island. It is also independent of Q_G , which is the charge on the gate capacitor.

To find the tunneling rates, we treat the tunneling term as a small perturbation and use "Fermi golden rule".

Single island - tunneling rates

- Tunneling rates are calculated using 1st order perturbation theory (Fermi golden-rule)

$$\Gamma(\Delta E) = \frac{1}{e^2 R_T} \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' f(E) [1 - f(E')] P(-\Delta E - (E' - E))$$



For an electron to tunnel from a state with energy E to one with energy E' , while changing the total energy by ΔE we need:
The first state to be occupied.
The second to be empty.

Assuming thermal equilibrium, those are given by Fermi-Dirac distribution.

P is the probability that the extra energy would be absorbed by the environment. It is a function of temperature and the impedance of the environment.

Single island - tunneling rates

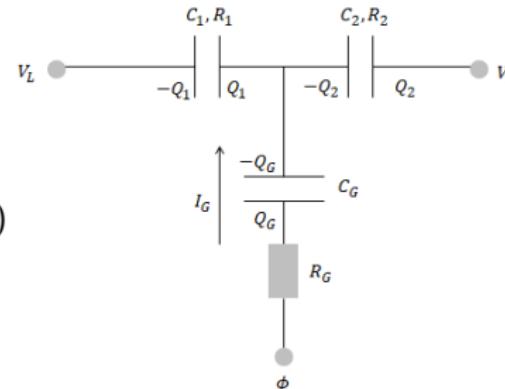
- Tunneling rates are calculated using 1st order perturbation theory (Fermi golden-rule)

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- For a low impedance environment

$$P(E) = \delta(E)$$

$$\Gamma(\Delta E) = \frac{1}{e^2 R_T} \begin{cases} \frac{-\Delta E}{1 - \exp(\Delta E/k_B T)} \\ -\Delta E \Theta(-\Delta E) \end{cases}$$



For low impedance, there is no absorption of energy, and we get the following result.

For zero temperature, tunneling is only allowed if it reduces the total energy. In that case, tunneling rate is proportional to the energy difference.

Single island - tunneling rates

- Tunneling rates are calculated using 1st order perturbation theory (Fermi golden-rule)

$$\Gamma(\Delta E) = \frac{1}{e^2 R_T} \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' f(E) [1 - f(E')] P(-\Delta E - (E' - E))$$

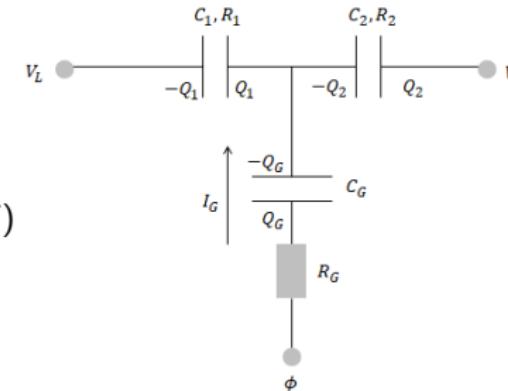
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- The typical time between tunnelings is proportional to R_T .

The typical time between tunnelings is one over the average tunneling rate. It is proportional to the tunnling resistance.



► Less details

Superconducting island tunneling rates

- For a superconducting islands we need to take both Quasi-particles and Cooper-pairs into account.

Superconducting island tunneling rates

- For a superconducting islands we need to take both Quasi-particles and Cooper-pairs into account.
- Quasi-particles act just like electrons, using the proper density of states

$$\frac{N_s(E)}{N(0)} = \begin{cases} \frac{|E|}{\sqrt{E^2 - \Delta^2}} & |E| > \Delta \\ 0 & |E| < \Delta \end{cases}$$

where 2Δ is the superconducting energy gap. we get

$$\Gamma(\Delta E) = \frac{1}{e^2 R_T} \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \frac{N_s(E) N_s(E' - \Delta E)}{N(0)^2} f(E) \times [1 - f(E' - \Delta E)] P(E - E')$$

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- For Cooper pairs, the relevant tunneling term is the Josephson term $\mathcal{H}_T = -E_J \cos(\phi)$, which is a tunneling term in first order perturbation theory.
- The resulting tunneling rates for Cooper-pairs are

$$\Gamma_{cp}(\Delta E) = \frac{\pi}{2\hbar} E_J^2 P(-\Delta E)$$

► Less details

Single island - charge relaxation details

Between tunnelings, charge distribution relaxes to electrostatic equilibrium

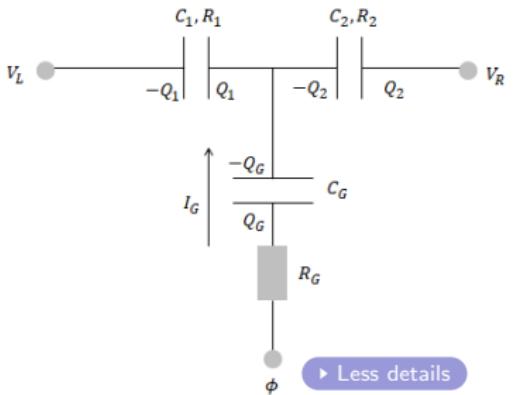
$$Q_G(n, t) = (Q_0 - Q_n) \exp\left(-\frac{t}{\tau}\right) + Q_n$$

$$Q_n = \frac{C_G}{C_1 + C_2 + C_G} [(C_1 + C_2) \phi - C_1 V_L - C_2 V_R - ne]$$

$$\tau = \frac{R_G C_G (C_1 + C_2)}{C_1 + C_2 + C_G}$$

Between tunnelings, the different capacitors charge or discharge towards electrostatic equilibrium.

Using Kirchoff's laws, we get a linear differential equation, with the following solution. The steady state value of gate charge is Q_n . It is n -dependent, for each occupation we get a different steady state solution.



Single island - charge relaxation details

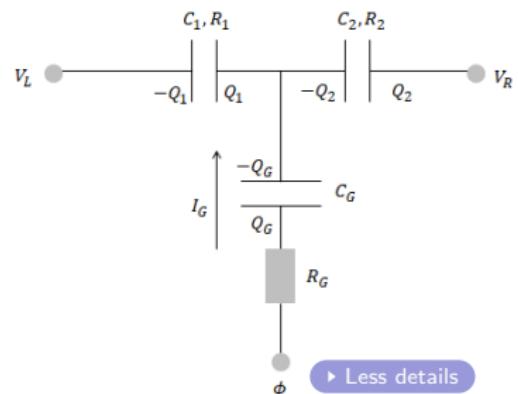
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- Relaxation time is proportional to R_G .



τ is the relaxation time, which is proportional to R_G .

► Less details

Single island - charge relaxation details

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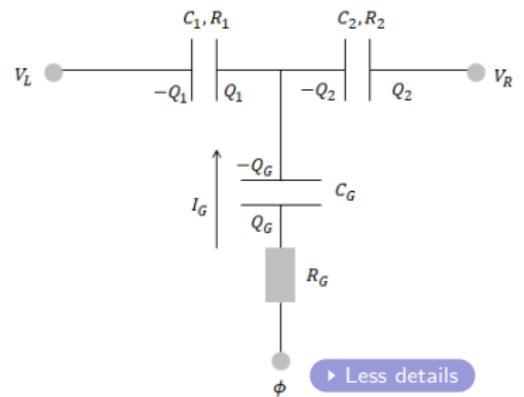
$$\tau = \frac{R_G C_G (C_1 + C_2)}{C_1 + C_2 + C_G}$$

- Relaxation time is proportional to R_G .
- For slow relaxation we require $R_G \gg R_i$.

If R_G is much bigger than all tunneling resistances, we get the slow relaxation limit.

This equation for Q_G , and the tunneling rates we saw earlier, are enough for calculating the current using numerical Kinetic Monte Carlo simulations, as we will see next.

But first I would like to show a more analytical method of solution, that will help us to gain some intuition.



Master equation analytic solution

- **Goal:** finding a steady state current.

Our goal is to find a steady state current. For this we need to find a steady state solution for Q_G and for the occupation probability function, the probability to find the system with a given occupation.

Shahar Kasher

Introduction
and
Motivation

Proposed
model - Single
island

Results -
Single island

Proposed
model -
Random array

Results -
Random array

Discussion and
conclusion

Master equation analytic solution

- **Goal:** finding a steady state current.
- For a fixed Q_G , the probabilities of each occupation satisfy the master equation

$$\begin{aligned}\frac{dp_{Q_G}(n)}{dt} &= \Gamma^+(n-1, Q_G, V) p_{Q_G}(n-1) + \Gamma^-(n+1, Q_G, V) p_{Q_G}(n+1) \\ &\quad - [\Gamma^+(n, Q_G, V) + \Gamma^-(n, Q_G, V)] p_{Q_G}(n) \\ \Gamma^\pm &= \Gamma_1^\pm + \Gamma_2^\pm\end{aligned}$$

Since Q_G changes much slower than occupation, we start by considering it to be constant.

Then, occupation probability satisfy the following master equation.

This equation is just the statement that the probability of occupation n increase when tunneling change the occupation from $n \pm 1$ to n , and decrease when tunneling change it from n to $n \pm 1$.

Master equation analytic solution

- **Goal:** finding a steady state current.
- For a fixed Q_G , the probabilities of each occupation satisfy the master equation

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- For slow relaxation, $p(n)$ relaxes much faster than Q_G . Therefore, Q_G relaxes to the **average** probability.

$$Q_G = \langle Q_n \rangle_{Q_G} \equiv \sum_n p_{Q_G}(n) Q_n$$

A steady state solution for Q_G would be obtained when Q_G equals to its average equilibrium value, according to the probability function that matches the same Q_G .

Having a steady state solution for the master equation, as a function of Q_G , we can solve this equation numerically.

Simulation method details

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹

Our simulation is based on an algorithm that was proposed by Gillespie, for simulating chemical reactions. In our case the analog of a chemical reaction is tunneling and the number of molecules is the number of electrons on the island.

► Less details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977)

Simulation method details

For each simulation step we need to know 2 things

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹
- To simulate our process we need to answer 2 questions:

► Less details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977)

Simulation method details

How much time is passing until next event?

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹
- To simulate our process we need to answer 2 questions:
 - What is the time until next tunneling?

► Less details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977)

Simulation method details

And what is the next event?

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹
- To simulate our process we need to answer 2 questions:
 - What is the time until
 - What is the next
 - next tunneling?
 - tunneling?

► Less details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977)

Simulation method details

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹
 - To simulate our process we need to answer 2 questions:
 - What is the time until next tunneling?
 - What is the next tunneling?
- $$p[\text{waiting time is } t] = \exp(-\sum_i \Gamma_i t)$$

Exactly like for radiative decay, the probability of each event occurring at time t , is the exponent of minus t times its rate. The probability of any event is the product of the probabilities for all possible events.

► Less details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977)

Simulation method details

The probability of each event is proportional to its rate, and we normalize it to get all probabilities to sum to 1.

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- To simulate our process we need to answer 2 questions:

- What is the time until next tunneling?
- What is the next tunneling?

$$p[\text{waiting time is } t] = \exp(-\sum_i \Gamma_i t)$$

$$p[\text{next tunneling is } i] = \frac{\Gamma_i}{\sum_j \Gamma_j}$$

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Simulation method details

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$$p[\text{next tunneling is } i] = \frac{\Gamma_i}{\sum_j \Gamma_j}$$

- At each step, simulation generates a random waiting time and choosing the next tunneling accordingly.

From here, the simulation method is clear. We generate a random waiting time and next event for each step, using these probabilities, and update the system state accordingly.

► Less details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* 81, 2340–2361 (Dec. 1977)

Simulation method details

Between tunnelings, we update Q_G according to the relaxation that we saw earlier.

- Based on a kinetic Monte-Carlo simulation, the specific algorithm was proposed by Gillespie¹
- To simulate our process we need to answer 2 questions:

- What is the time until next tunneling?
- What is the next tunneling?

$$p[\text{waiting time is } t] = \exp(-\sum_i \Gamma_i t)$$

$$p[\text{next tunneling is } i] = \frac{\Gamma_i}{\sum_j \Gamma_j}$$

- At each step, simulation generates a random waiting time and choosing the next tunneling accordingly.
- Between tunnelings, Q_G , and tunneling rates, are updated.

► Less details

¹Gillespie, D. T. Exact stochastic simulation of coupled chemical reactions. en. *The Journal of Physical Chemistry* **81**, 2340–2361 (Dec. 1977)

Simulation method - details

We repeat this for any external voltage value, until reaching a steady state.

- This is done for each external voltage, until reaching a steady state.

Simulation method - details

When we reach a steady state, we keep running the simulation and calculate the current, as a weighted average, weighted by the time of each step.

- This is done for each external voltage, until reaching a steady state.
- Once a steady state was reached, current is calculated, by

$$\langle I \rangle = e \frac{\sum_I [\Gamma_1^+(t_I) - \Gamma_1^-(t_I)] (t_{I+1} - t_I)}{\sum_I (t_{I+1} - t_I)}$$

Simulation method - details

We can also calculate other averages in the same way.

- This is done for each external voltage, until reaching a steady state.
- Once a steady state was reached, current is calculated, by

$$\langle I \rangle = e \frac{\sum_I [\Gamma_1^+(t_I) - \Gamma_1^-(t_I)] (t_{I+1} - t_I)}{\sum_I (t_{I+1} - t_I)}$$

- Other averages, e.g. $\langle n \rangle$, $\langle Q_G \rangle$, can be calculated in the same way.

Simulation method - Alternative

An alternative simulation method, for a single island in zero temperature, is to solve the master equation as described.

- Solving for the steady state occupation probabilities, $p_{Q_G}(n)$, as a function of Q_G .

Simulation method - Alternative

Then we use this solution to solve for a steady state Q_G numerically.

- Solving for the steady state occupation probabilities, $p_{Q_G}(n)$, as a function of Q_G .
- Using this solution, solving numerically for a steady state Q_G .

Simulation method - Alternative

Finally we calculate the current according to the steady state solution we found.

- Solving for the steady state occupation probabilities, $p_{Q_G}(n)$, as a function of Q_G .
- Using this solution, solving numerically for a steady state Q_G .
- Calculating the current using
$$I = e \sum_n p_{Q_G}(n) [\Gamma_1^+(n, Q_G) - \Gamma_1^-(n, Q_G)]$$

Simulation method - Alternative

- Solving for the steady state occupation probabilities, $p_{Q_G}(n)$, as a function of Q_G .
- Using this solution, solving numerically for a steady state Q_G .
- Calculating the current using
$$I = e \sum_n p_{Q_G}(n) [\Gamma_1^+(n, Q_G) - \Gamma_1^-(n, Q_G)]$$
- We compared results from both methods to verify our simulation.

This will work also for a big array in principle, but won't not be feasible in terms of running time and memory, since the size of the graph grows exponentially with the number of islands. Any way, it is good to have another method, to validate the simulations by comparison.

Now lets see how to generalize this model into an array.

1 Introduction and Motivation

2 Proposed model - Single island

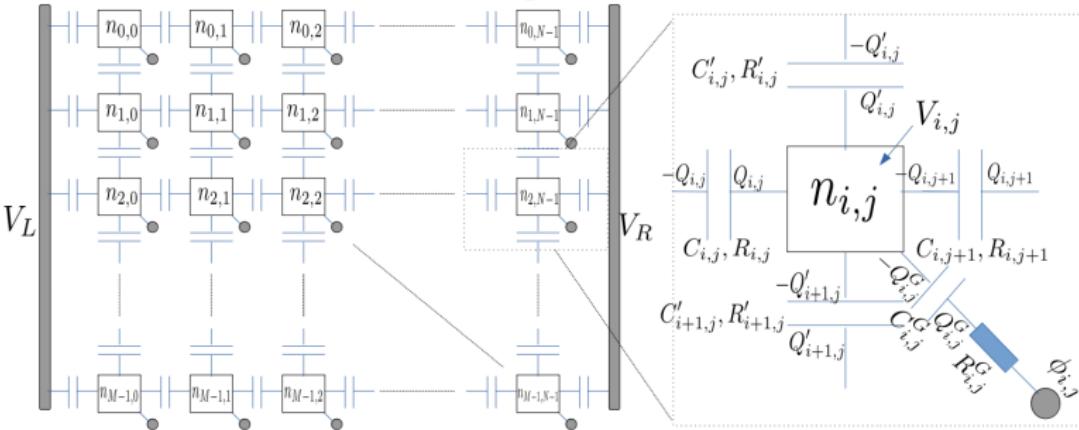
3 Results - Single island

4 Proposed model - Random array

5 Results - Random array

6 Discussion and conclusion

Random array model details



The voltage on island (i, j) is given by

$$V_{i,j} = \sum_{k,l} C_{i,j;k,l}^{-1} (en'_{k,l} + Q^G_{k,l}) , \quad en'_{i,j} = en_{i,j} + \delta_{j,0} C_{i,0} V_L + \delta_{j,N-1} C_{i,N} V_R$$

Where C is the capacitance matrix, defined by

$$C_{i,j;k,l} = \begin{cases} C_{i,j} + C_{i,j+1} + C'_{i,j} + C'_{i+1,j} & k = i, l = j \\ -C_{i,j} & k = i, l = j - 1 \\ -C_{i,j+1} & k = i, l = j + 1 \\ -C'_{i,j} & k = i - 1, l = j \\ -C'_{i+1,j} & k = i + 1, l = j \\ 0 & \text{else} \end{cases}$$

▶ Less details

Here we see a schema of the array model.

Each island has 4 nearest neighbors (except on bottom and top rows) and each is connected to a gate voltage through an RC-circuit.

To simulate the current through an array, we need to calculate 2 things. The first is the energy difference for tunneling, that will be used for calculating tunneling rates.

The second is the gate charge relaxation.

To calculate the energy differences, we will use the capacitance matrix. It relates the charge distribution to the electric potential on each island. I'm using double indexing here, as a size independent indexing method for a matrix.

The diagonal elements of the capacitance matrix are the self capacitance of each island, while off diagonal terms are mutual capacitances, which are the capacitance between islands. The mutual capacitance is not zero only for nearest neighbors.

Random array model - energy differences

- The energy difference for tunneling from island (a, b) to (a', b') is

$$\begin{aligned}\Delta E_{a,b \rightarrow a',b'} = & e(V_{a',b'} - V_{a,b}) \\ & + \frac{e^2}{2} \left(C_{a',b';a',b'}^{-1} + C_{a,b;a,b}^{-1} - 2C_{a,b;a',b'}^{-1} \right)\end{aligned}$$

As for a single island, we can write energy differences in terms of the electric potential on each island. We see that here we also get an energy barrier, that is the sum of self capacitances minus the mutual capacitance.

Note: this is the total energy difference in the array. For those who are familiar with the term, we are using the "global rule", and not the "local" one.

Random array model - energy differences

Tunneling rates are calculated the same way as for a single island.

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- Using this, we can calculate tunneling rates in the same way used for a single island.

Random array model - charge relaxation details

- Charge relaxation, between tunnelings, is given by

$$\frac{dQ_{i,j}^G}{dt} = - \sum_{k,l} \tau_{i,j;k,l}^{-1} (Q_{k,l}^G - Q_{k,l}^n)$$

$$\tau_{i,j;k,l}^{-1} \equiv \frac{1}{R_{i,j}^G} \left(C_{i,j;k,l}^{-1} + \frac{1}{C_{i,j}^G} \delta_{i,j;k,l} \right)$$

$$Q_{i,j}^n \equiv \frac{1}{R_{i,j}^G} \sum_{k,l} \tau_{i,j;k,l} \left[\phi_{k,l} + \frac{e}{C_{k,l}^G} n'_{k,l} \right] - e n'_{i,j}$$

As for charge relaxation, we use Kirchoff's laws to get a set of linear differential equations.

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- This is a set of linear ODEs, solved by diagonalizing the matrix τ^{-1} .

We solve these equations by diagonalizing the inverse τ matrix. The inverse of its eigenvalues are the relaxation times of charge in the system.

As for the single island, we will get slow relaxation if all gate resistances are much bigger than all tunneling resistances.

Random array model - charge relaxation details

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$$\frac{dQ_{i,j}^G}{dt} = - \sum_{k,l} \tau_{i,j;k,l}^{-1} (Q_{k,l}^G - Q_{k,l}^n)$$

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- This is a set of linear ODEs, solved by diagonalizing the matrix τ^{-1} .
- With addition of the calculated energy differences, this is enough for running the KMC simulation.

Using the tunneling rates and this solution, we can run the same simulation as for the single island.

A big difference from the single island case, is that now we would have interactions between electrons on different islands in our system.

Random array model - interactions details

- In the case where R_G and C_G are the same for each island, the voltage in steady-state is

$$V_{i,j}^n = \sum_{k,l} (\mathcal{C}_{i,j;k,l} + C_G \delta_{i,j;k,l})^{-1} (C_G \phi_{k,l} + e \langle n'_{k,l} \rangle) \\ + e \sum_{k,l} \mathcal{C}_{i,j;k,l}^{-1} (n_{k,l} - \langle n_{k,l} \rangle)$$

Lets examine the simple case, where gate capacitance and resistance are uniform.

In a steady state, gate charges relax to their average equilibrium values, and the electric potential on each island is given by the following relation.

The off-diagonal terms in both inverse matrices determine the interaction strength between pairs of islands.

Luckily, those matrices are invertible for uniform capacitances.

► Less details

¹Bakhvalov, N. S. et al. Statics and dynamics of single-electron solitons in two-dimensional arrays of ultrasmall tunnel junctions. en. *Physica B: Condensed Matter* **173**, 319–328 (Sept. 1991).

Random array model - interactions details

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- For uniform capacitance, the matrix $\mathcal{C} + \mathbb{I}C_G$ can be inverted analytically¹

$$(\mathcal{C}_{i,j;k,l} + \delta_{i,j;k,l} C_G)^{-1} \propto \begin{cases} \frac{\ln(\frac{1}{r})}{\sqrt{r}} & 1 \ll r \ll \lambda \\ \frac{\exp(-r/\lambda)}{\sqrt{r}} & \lambda \ll r \end{cases}$$
$$\lambda \equiv -1/\ln \left(1 + \frac{C_G}{2C} - \sqrt{\left(\frac{C_G}{2C} \right)^2 + \frac{C_G}{C}} \right)$$

$r = \sqrt{(i-k)^2 + (j-l)^2}$ is the distance between islands

► Less details

They depend on an interaction length, λ , that is a function of the ratio between gate and tunneling capacitance.

For distances smaller than λ we get logarithmic decay. This is just Coulomb interaction in 2D.

For distances larger than λ , we get exponential decay, a screened coulomb interaction.

λ is small for $C < C_G$ and big for $C > C_G$.

Examining the steady state voltages again, we see that the gate charge partially screens the electrons on each island.

For big C_G , average occupation is screened, and has a short range effect, while deviations from it are unscreened.

We can conclude that long range interactions happen only between conducting islands. Since the occupation in non-conducting islands doesn't change, and so average n and n' are the same for those islands.

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