Finite element method for magnetodynamic applications

2019

Contents

Symbols

$\boldsymbol{A}$	$[\mathrm{Wb} \cdot \mathrm{m}^{-1}]$	magnetic vector potential
$\boldsymbol{B}$	[T]	magnetic flux density
f	[Hz]	frequency
$\boldsymbol{J}$	$[\mathrm{A} \cdot \mathrm{m}^{-2}]$	current density
l	[m]	line
L	[m]	length of the model (z coordinate)
$\boldsymbol{n}$	[-]	normal vector
S	$[\mathrm{m}^2]$	surface
t	[s]	time
$T, \mathbb{T}$	$[\mathrm{N} \cdot \mathrm{m}^{-2}]$	Maxwell stress tensor
$\boldsymbol{v}$	$[\mathrm{m} \cdot \mathrm{s}^{-1}]$	velocity
V	$[\mathrm{m}^3]$	volume
γ	$[\mathrm{S} \cdot \mathrm{m}^{-1}]$	conductivity
λ	[-]	basis function
μ_0	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability of vacuum
μ	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability
$\boldsymbol{\varphi}$	[-]	test function
Ω	[-]	closed region
$\partial\Omega$	[-]	boundary of closed region

1 Theory

1.1 2D Magnetodynamic equations

Cartesian coordinate system (x, y, z)

$$\nabla \times \left(\frac{1}{\mu}\left(\nabla \times \boldsymbol{A}\right)\right) + \gamma \frac{\partial \boldsymbol{A}}{\partial t} - \gamma\left(\boldsymbol{v} \times \left(\nabla \times \boldsymbol{A}\right)\right) = \boldsymbol{J} \tag{1.1}$$

$$\boldsymbol{A} = (0, 0, A_z) \tag{1.2}$$

$$\boldsymbol{J} = (0, 0, J_z) \tag{1.3}$$

$$\boldsymbol{v} = (v_x, v_y, 0) \tag{1.4}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = (\partial_y A_x, -\partial_x A_y, 0) = (B_x, B_y, 0) \tag{1.5}$$

$$\boldsymbol{v} \times \boldsymbol{B} = (0, 0, v_x B_y - v_y B_x) \tag{1.6}$$

$$\boldsymbol{A}, \boldsymbol{J}, \boldsymbol{v} = f\left(x, y, t\right) \tag{1.7}$$

$$\text{Linear} \quad \mu, \gamma = f\left(x, y\right) \qquad \text{Nonlinear} \quad \mu = f\left(x, y, ||\boldsymbol{B}||\right) \tag{1.8}$$

1.2 Boundary condition

$$A_z \Big|_{\partial\Omega} = 0 \tag{1.9}$$

1.3 Initial condition

$$A_z \Big|_{t=0} = 0 \tag{1.10}$$

1.4 Force

$$T_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2} B^2 \delta_{ij}\right) \tag{1.11}$$

$$\mathbb{T} = \frac{1}{\mu_0}\begin{pmatrix} B_xB_x & B_xB_y \\ B_yB_x & B_yB_y \end{pmatrix} - \frac{1}{2\mu_0}\begin{pmatrix} B_x^2 + B_y^2 & 0 \\ 0 & B_x^2 + B_y^2 \end{pmatrix} \tag{1.12}$$

$$\text{3D} \qquad \boldsymbol{F}(x,y,z) = \oint_{\partial\Omega} \mathbb{T}\,\mathrm{d}\boldsymbol{S} = \int_{\Omega} (\nabla \cdot \mathbb{T})\,\mathrm{d}V \tag{1.13}$$

$$\text{2D} \qquad \boldsymbol{F}(x,y) = L\oint_{\partial\Omega} \mathbb{T}\,\mathrm{d}\boldsymbol{l} = L\int_{\Omega} (\nabla \cdot \mathbb{T})\,\mathrm{d}S \tag{1.14}$$

$$\nabla \cdot \mathbb{T} = \frac{1}{\mu_0}(\partial_x, \partial_y)\begin{pmatrix} \frac{1}{2}\left(B_x^2 - B_y^2\right) & B_xB_y \\ B_yB_x & \frac{1}{2}\left(B_y^2 - B_x^2\right) \end{pmatrix} \tag{1.15}$$

$$\nabla \cdot \mathbb{T} = \frac{1}{\mu_0}\left(\frac{1}{2}\partial_x\left(B_x^2 - B_y^2\right) + \partial_y(B_yB_x), \frac{1}{2}\partial_y\left(B_y^2 - B_x^2\right) + \partial_x(B_xB_y)\right) \tag{1.16}$$

$$\nabla \cdot \mathbb{T}(x) = \frac{1}{\mu_0}(\partial_xB_xB_x - \partial_xB_yB_y + \partial_yB_yB_x + \partial_yB_xB_y) \tag{1.17}$$

$$\nabla \cdot \mathbb{T}(y) = \frac{1}{\mu_0}(\partial_yB_yB_y - \partial_yB_xB_x + \partial_xB_xB_y + \partial_xB_yB_x) \tag{1.18}$$

1.5 Weak formulation

$$\begin{aligned} \int_{\Omega} \nabla \times \left(\frac{1}{\mu}(\nabla \times \boldsymbol{A})\right)\boldsymbol{\varphi}\,\mathrm{d}S = \int_{\Omega} \frac{1}{\mu}(\nabla \times \boldsymbol{A})(\nabla \times \boldsymbol{\varphi})\,\mathrm{d}S - \\ - \int_{\partial\Omega} \frac{1}{\mu}(\boldsymbol{n} \times (\nabla \times \boldsymbol{A}))\,\boldsymbol{\varphi}\,\mathrm{d}l \end{aligned} \tag{1.19}$$

$\boldsymbol{\varphi} \in f(\Omega) : \boldsymbol{\varphi}|_{\partial\Omega} = 0 \to -\int_{\partial\Omega} \frac{1}{\mu}(\boldsymbol{n} \times (\nabla \times \boldsymbol{A}))\,\boldsymbol{\varphi}\,\mathrm{d}l = 0$

$$\boldsymbol{A}(x,y,t) \approx \sum_{n=1}^{N} A^n(t)\,\lambda^n(x,y) \tag{1.20}$$

$$\boldsymbol{J}(x,y,t) \approx \sum_{n=1}^{N} J^n(t)\,\lambda^n(x,y) \tag{1.21}$$

$$\boldsymbol{\varphi}(x,y,t) \approx \sum_{n=1}^{N} \varphi^n(t)\,\lambda^n(x,y) \tag{1.22}$$

$$\int_\Omega \frac{1}{\mu}(\nabla\times\boldsymbol{A})(\nabla\times\boldsymbol{\varphi})\,\mathrm{d}S \approx \sum_{n=1}^{N}\int_\Omega \frac{1}{\mu}A^n(\nabla\times\lambda^n)\,\varphi^m(\nabla\times\lambda^m)\,\mathrm{d}S \tag{1.23}$$

$$\int_\Omega \boldsymbol{J}\boldsymbol{\varphi}\,\mathrm{d}S \approx \sum_{n=1}^{N}\int_\Omega J^n\lambda^n\varphi^m\lambda^m\,\mathrm{d}S \tag{1.24}$$

$$\int_\Omega \gamma\frac{\partial\boldsymbol{A}}{\partial t}\boldsymbol{\varphi}\,\mathrm{d}S \tag{1.25}$$

$$\int_\Omega \gamma(\boldsymbol{v}\times(\nabla\times\boldsymbol{A}))\,\boldsymbol{\varphi}\,\mathrm{d}S \tag{1.26}$$