Finite element method for magnetodynamic applications

2019

Contents

Symbols

$\boldsymbol{A}$	$[\mathrm{Wb} \cdot \mathrm{m}^{-1}]$	magnetic vector potential
$\boldsymbol{B}$	[T]	magnetic flux density
f	[Hz]	frequency
$\boldsymbol{J}$	$[\mathrm{A} \cdot \mathrm{m}^{-2}]$	current density
l	[m]	line
L	[m]	length of the model (z coordinate)
$\boldsymbol{n}$	[-]	normal vector
S	$[\mathrm{m}^2]$	surface
t	[s]	time
$T, \mathbb{T}$	$[\mathrm{N} \cdot \mathrm{m}^{-2}]$	Maxwell stress tensor
$\boldsymbol{v}$	$[\mathrm{m} \cdot \mathrm{s}^{-1}]$	velocity
V	$[\mathrm{m}^3]$	volume
γ	$[\mathrm{S} \cdot \mathrm{m}^{-1}]$	conductivity
λ	[-]	basis function
μ_0	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability of vacuum
μ	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability
$\boldsymbol{\varphi}$	[-]	test function
Ω	[-]	closed region
$\partial\Omega$	[-]	boundary of closed region

1 Theory

1.1 Magnetodynamic equation

3D Cartesian coordinate system (x, y, z):

$$\nabla \times \left(\frac{1}{\mu}(\nabla \times \boldsymbol{A})\right) + \gamma \frac{\partial \boldsymbol{A}}{\partial t} - \gamma (\boldsymbol{v} \times (\nabla \times \boldsymbol{A})) = \boldsymbol{J} \tag{1.1}$$

Reduction to 2D (x, y):

$$\boldsymbol{A} = (0, 0, A_z) \quad \rightarrow \quad A = A_z \tag{1.2}$$

$$\boldsymbol{J} = (0, 0, J_z) \quad \rightarrow \quad J = J_z \tag{1.3}$$

$$\boldsymbol{v} = (v_x, v_y, 0) \quad \rightarrow \quad \boldsymbol{v} = (v_x, v_y) \tag{1.4}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = (B_x, B_y, 0) \quad \rightarrow \quad \boldsymbol{B} = (B_x, B_y) \tag{1.5}$$

For 2D, time-dependent problems are $A, J, \boldsymbol{v}$ and $\boldsymbol{B}$ functions of coordinates (x, y) and time t. Both A and J are scalar fields $\mathbb{R}^2 \rightarrow \mathbb{R}^1$, $\boldsymbol{v}$ and $\boldsymbol{B}$ are vector fields $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.

In a linear material, the μ and the γ are dependent only on the coordinate system (x, y), in the non-linear material, the μ is also dependent on the size of $||\boldsymbol{B}||$.

$$\nabla = (\partial_x, \partial_y) \tag{1.6}$$

Curl of scalar field $(\nabla \times A)$ is vector field $(\partial_y A, -\partial_x A)$, so $\mathbb{R}^1 \rightarrow \mathbb{R}^2$, but curl of vector field $(\nabla \times \boldsymbol{B})$ is scalar field $(\partial_x B_y - \partial_y B_x)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

Divergence of vector field $(\nabla \cdot \boldsymbol{B})$ is also scalar field $(\partial_x B_x + \partial_y B_y)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

$$\nabla \times \left(\frac{1}{\mu}(\nabla \times A)\right) + \gamma \frac{\partial A}{\partial t} - \gamma (\boldsymbol{v} \times (\nabla \times A)) = J \tag{1.7}$$

1.2 Boundary conditions

Dirichlet condition:

$$A\Big|_{\partial\Omega_1} = f_1(x, y) \tag{1.8}$$

Neumann condition:

$$\partial_{\boldsymbol{n}} A\Big|_{\partial\Omega_2} = f_2(x, y) \tag{1.9}$$

For our model, both f_1 and f_2 are equal to zero.

1.3 Initial condition

$$A\Big|_{t=0} = 0 \tag{1.10}$$

1.4 Force

$$T_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}B^2\delta_{ij}\right) \tag{1.11}$$

$$\mathbb{T} = \frac{1}{\mu_0}\begin{pmatrix} B_x B_x & B_x B_y \\ B_y B_x & B_y B_y \end{pmatrix} - \frac{1}{2\mu_0}\begin{pmatrix} B_x^2 + B_y^2 & 0 \\ 0 & B_x^2 + B_y^2 \end{pmatrix} \tag{1.12}$$

$$\boldsymbol{F}\left(x, y\right) = L\oint_{\partial\Omega} \mathbb{T}\boldsymbol{n}\,\mathrm{d}l = L\int_{\Omega}\left(\nabla\cdot\mathbb{T}\right)\,\mathrm{d}S \tag{1.13}$$

Variable L is the length of the model into the third dimension z.

$$\begin{aligned}\nabla\cdot\mathbb{T} &= \frac{1}{\mu_0}\begin{pmatrix} \frac{1}{2}\partial_x\left(B_x^2 - B_y^2\right) + \partial_y\left(B_y B_x\right) \\ \frac{1}{2}\partial_y\left(B_y^2 - B_x^2\right) + \partial_x\left(B_x B_y\right) \end{pmatrix} \\ &= \frac{1}{\mu_0}\begin{pmatrix} \partial_x B_x B_x - \partial_x B_y B_y + \partial_y B_y B_x + \partial_y B_x B_y \\ \partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x \end{pmatrix}\end{aligned} \tag{1.14}$$

1.5 Permanent magnets

$$\nabla\times\left(\frac{1}{\mu}\left(\nabla\times A\right) - \frac{1}{\mu}\boldsymbol{B_r}\right) + \gamma\frac{\partial A}{\partial t} - \gamma\left(\boldsymbol{v}\times\left(\nabla\times A\right)\right) = J \tag{1.15}$$

Permanent magnets are usually considered as a source of magnetic field, so it is better to move them to the right hand side of the equation.

$$\nabla\times\left(\frac{1}{\mu}\left(\nabla\times A\right)\right) + \gamma\frac{\partial A}{\partial t} - \gamma\left(\boldsymbol{v}\times\left(\nabla\times A\right)\right) = J + \frac{1}{\mu}\left(\nabla\times\boldsymbol{B_r}\right) \tag{1.16}$$

2 Weak formulation

To simplify the problem, let us first consider a magnetostacic case ($\gamma = 0$). This is true for models without motion and without time variation of currents.

3 Topology optimization

3.1 Problem formulation in continuous space

$$
\begin{aligned}
\text{minimize} \quad & F_y^p \quad (\mathrm{p} - \text{plunger}) \\
\text{subject to} \quad & -\frac{1}{\mu}\Delta A = J \\
& B_x = \partial_y A \\
& B_y = -\partial_x A
\end{aligned}
$$

$$
F_y^p = (0,1)\int_{\Omega_p} (\nabla \cdot \mathbb{T})\,\mathrm{d}S = \int_{\Omega_p} \frac{1}{\mu_0}\left(\partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x\right)\mathrm{d}S \tag{3.1}
$$

3.2 Problem formulation in discrete space

$$
\begin{aligned}
\text{minimize} \quad & F_y^p \\
\text{subject to} \quad & SA = MJ \\
& MB_x = C_y A \\
& MB_y = -C_x A
\end{aligned}
$$

$$
F_y^p = \frac{1}{\mu_0}\left(B_y^\top C_y^p B_y - B_x^\top C_y^p B_x + B_y^\top C_x^p B_x + B_x^\top C_x^p B_y\right) \tag{3.2}
$$

Lagrange multipliers are (α, β, γ).

$$
\begin{aligned}
SA = MJ \quad & \rightarrow \quad \alpha \\
MB_x = C_y A \quad & \rightarrow \quad \beta \\
MB_y = -C_x A \quad & \rightarrow \quad \gamma
\end{aligned}
$$

The φ is the topology function.

$$J\left(\varphi\right)=J_1\left(\varphi\right)+J_2\left(\varphi\right)+J_3\left(\varphi\right)+J_4\left(\varphi\right) \tag{3.3}$$

$$J_1=\frac{1}{\mu_0}\left(B_y^\top\left(\varphi\right)C_y^pB_y\left(\varphi\right)-B_x^\top\left(\varphi\right)C_y^pB_x\left(\varphi\right)+B_y^\top\left(\varphi\right)C_x^pB_x\left(\varphi\right)+B_x^\top\left(\varphi\right)C_x^pB_y\left(\varphi\right)\right) \tag{3.4}$$

$$J_2=\alpha^\top\left(S\left(\varphi\right)A\left(\varphi\right)-MJ\left(\varphi\right)\right)=0 \tag{3.5}$$

$$J_3=\beta^\top\left(MB_x\left(\varphi\right)-C_yA\left(\varphi\right)\right)=0 \tag{3.6}$$

$$J_4=\gamma^\top\left(MB_y\left(\varphi\right)+C_xA\left(\varphi\right)\right)=0 \tag{3.7}$$

$$\partial_{\varphi_i}J\left(\varphi\right)=\partial_{\varphi_i}J_1\left(\varphi\right)+\partial_{\varphi_i}J_2\left(\varphi\right)+\partial_{\varphi_i}J_3\left(\varphi\right)+\partial_{\varphi_i}J_4\left(\varphi\right) \tag{3.8}$$

$$\mu_0\partial_{\varphi_i}J_1=\partial_{\varphi_i}\left(B_y^\top C_y^pB_y\right)-\partial_{\varphi_i}\left(B_x^\top C_y^pB_x\right)+\partial_{\varphi_i}\left(B_y^\top C_x^pB_x\right)+\partial_{\varphi_i}\left(B_x^\top C_x^pB_y\right) \tag{3.9}$$

$$\partial_{\varphi_i}\left(B_y^\top C_y^pB_y\right)=\partial_{\varphi_i}B_y^\top C_y^pB_y+B_y^\top C_y^p\partial_{\varphi_i}B_y=B_y^\top\left(C_y^p\right)^\top\partial_{\varphi_i}B_y+B_y^\top C_y^p\partial_{\varphi_i}B_y \tag{3.10}$$

$$\begin{aligned}\partial_{\varphi_i}J_1=&\frac{1}{\mu_0}\left(B_x^\top,B_y^\top\right)\begin{pmatrix}-\left(C_y^p\right)^\top-C_y^p\\ \left(C_x^p\right)^\top+C_x^p\end{pmatrix}\partial_{\varphi_i}B_x\\&+\frac{1}{\mu_0}\left(B_x^\top,B_y^\top\right)\begin{pmatrix}\left(C_x^p\right)^\top+C_x^p\\ \left(C_y^p\right)^\top+C_y^p\end{pmatrix}\partial_{\varphi_i}B_y\end{aligned} \tag{3.11}$$

$$\partial_{\varphi_i}J_2=\alpha^\top\partial_{\varphi_i}\left(SA\right)=\alpha^\top\partial_{\varphi_i}SA+\alpha^\top S\partial_{\varphi_i}A \tag{3.12}$$

$$\partial_{\varphi_i}J_3=\beta^\top M\partial_{\varphi_i}B_x-\beta^\top C_y\partial_{\varphi_i}A \tag{3.13}$$

$$\partial_{\varphi_i}J_4=\gamma^\top M\partial_{\varphi_i}B_y+\gamma^\top C_x\partial_{\varphi_i}A \tag{3.14}$$

$$\partial_{\varphi_i} B_x: \quad \frac{1}{\mu_0}\left(B_x^\top, B_y^\top\right)\begin{pmatrix} -\left(C_y^p\right)^\top - C_y^p \\ \left(C_x^p\right)^\top + C_x^p \end{pmatrix} \partial_{\varphi_i} B_x + \beta^\top M \partial_{\varphi_i} B_x \tag{3.15}$$

$$\partial_{\varphi_i} B_y: \quad \frac{1}{\mu_0}\left(B_x^\top, B_y^\top\right)\begin{pmatrix} \left(C_x^p\right)^\top + C_x^p \\ \left(C_y^p\right)^\top + C_y^p \end{pmatrix} \partial_{\varphi_i} B_y + \gamma^\top M \partial_{\varphi_i} B_y \tag{3.16}$$

$$\partial_{\varphi_i} A: \quad \alpha^\top S \partial_{\varphi_i} A - \beta^\top C_y \partial_{\varphi_i} A + \gamma^\top C_x \partial_{\varphi_i} A \tag{3.17}$$

$$\partial_{\varphi_i} S: \quad \alpha^\top \partial_{\varphi_i} S A \tag{3.18}$$

If we choose α, β, γ such that (3.15), (3.16) and (3.17) are equal to 0, then:
From (3.15), ($M^\top = M$, $S^\top = S$):

$$\beta = -\frac{1}{\mu_0} M^{-1} \begin{pmatrix} -\left(C_y^p\right)^\top - C_y^p \\ \left(C_x^p\right)^\top + C_x^p \end{pmatrix}^\top \begin{pmatrix} B_x \\ B_y \end{pmatrix} \tag{3.19}$$

From (3.16):

$$\gamma = -\frac{1}{\mu_0} M^{-1} \begin{pmatrix} \left(C_x^p\right)^\top + C_x^p \\ \left(C_y^p\right)^\top + C_y^p \end{pmatrix}^\top \begin{pmatrix} B_x \\ B_y \end{pmatrix} \tag{3.20}$$

From (3.17):

$$\alpha = S^{-1}\left(C_y^\top \beta - C_x^\top \gamma\right) \tag{3.21}$$

$$\partial_{\varphi_i} J = \alpha^\top \partial_{\varphi_i} S A \tag{3.22}$$

In the linear case the term μ_{Fe} corresponds to $\mu_0\mu_r$, in nonlinear case, μ_{Fe} is function of A and BH curve of the material. Matrix $\hat{S}$ is no longer dependent on φ.

$$S = \frac{1}{\mu}\hat{S} = \left(\left(1-\varphi\right)\mu_0 + \varphi^p \mu_{Fe}\right)^{-1} \hat{S} \tag{3.23}$$

Linear case:

$$\partial_{\varphi_i} S = \partial_{\varphi_i} \frac{1}{\mu}\hat{S} = \frac{\mu_0 - p\varphi_i^{p-1}\mu_{Fe}}{\left(\left(1-\varphi\right)\mu_0 + \varphi^p \mu_{Fe}\right)^2}\hat{S} \tag{3.24}$$

If matrix S is dependent on A, the expression (3.12) must be extended. We modify the formula to differentiate with respect to A, which we will use in the future.

$$\partial_{\varphi_i} J_2 = \alpha^\top \partial_{\varphi_i}\left(SA\right) = \alpha^\top \partial_{A(\varphi_i)}\left(SA\right)\partial_{\varphi_i} A + \alpha^\top \partial_{\varphi_i} S A \tag{3.25}$$

$$\partial_A\left(SA\right) = \partial_A S A + S\partial_A A = \partial_A S A + S \tag{3.26}$$

In this case, the BH curve is the curve that returns μ_{Fe} for the given $||B||$. To avoid square root in $||B|| = \sqrt{B_x^2 + B_y^2}$, the curve is defined directly for $B_x^2 + B_y^2$. The magnetic flux density B_x is equal to $C_y A$ and B_y is equal to $C_x A$, then:

$$\partial_A S = \frac{-\varphi^p \partial_A \mu_{Fe} \left(2C_y A C_y + 2C_x A C_x\right)}{\left(\left(1-\varphi\right)\mu_0 + \varphi^p \mu_{Fe}\right)^2} \hat{S} \tag{3.27}$$

New α:

$$\begin{aligned} \partial_{\varphi_i} A: \quad & \partial_A S A \partial_{\varphi_i} A + S \partial_{\varphi_i} A - \beta^\top C_y \partial_{\varphi_i} A + \gamma^\top C_x \partial_{\varphi_i} A \\ & \alpha = \left(\partial_A S A + S\right)^{-1} \left(C_y^\top \beta - C_x^\top \gamma\right) \end{aligned} \tag{3.28}$$

4 Nonlinear solver

Linearization:

$$y = \partial_x f\left(x - x_n\right) + f\left(x_n\right) \tag{4.1}$$

$$0 = SA - MJ - CB_r = SA - f \tag{4.2}$$

$$\begin{aligned} 0 &= \partial_A\left(SA_n - f\right)\left(A_{n+1} - A_n\right) + \left(SA_n - f\right) \\ A_{n+1} &= A_n - \partial_A\left(SA_n - f\right)^{-1}\left(SA_n - f\right) \end{aligned}$$

Right hand side f is not dependent on A, $\partial_A f = 0$.

$$A_{n+1} = A_n - \partial_A\left(SA_n\right)^{-1}\left(SA_n - f\right) \tag{4.3}$$

Derivative $\partial_A\left(SA\right)$ was already derived in chapter 3, (3.26) and (3.27).

5 Robustness

Approximation of $B - \mu$ curve by Weibull distribution :

$$\mu = \mu_0 + a_3 \left(B - a_1\right)^{a_2 - 1} e^{-\left(B - a_1\right)^{a_2}} \tag{5.1}$$

$$\partial_B \mu = a_3 e^{-\left(B - a_1\right)^{a_2}} \left(\left(a_2 - 1\right) \left(B - a_1\right)^{a_2 - 2} - a_2 \left(B - a_1\right)^{2a_2 - 2} \right) \tag{5.2}$$

Derivatives of S with respect to parameters of Weibull distribution:

$$\partial_{a_i} S = \frac{-\varphi^p \partial_{a_i} \mu_{Fe}}{\left(\left(1 - \varphi\right) \mu_0 + \varphi^p \mu_{Fe}\right)^2} \hat{S} \tag{5.3}$$

$$\begin{aligned}
\partial_{a_1} \mu_{Fe} &= a_3 e^{-\left(B - a_1\right)^{a_2}} \left(a_2 \left(B - a_1\right)^{2a_2 - 2} - \left(a_2 - 1\right) \left(B - a_1\right)^{a_2 - 2} \right) \\
\partial_{a_2} \mu_{Fe} &= \ln\left(B - a_1\right) a_3 e^{-\left(B - a_1\right)^{a_2}} \left(\left(B - a_1\right)^{a_2 - 1} - \left(x - a_1\right)^{2a_2 - 1} \right) \\
\partial_{a_3} \mu_{Fe} &= e^{-\left(B - a_1\right)^{a_2}} \left(B - a_1\right)^{a_2 - 1}
\end{aligned}$$

Appendix

Weak formulation

For $\gamma = 0$.

$$\nabla \times \left(\frac{1}{\mu}(\nabla \times A)\right) = J \tag{5.4}$$

The θ is test function.

$$\int_\Omega \nabla \times \left(\frac{1}{\mu}(\nabla \times A)\right)\theta \, \mathrm{d}S = \int_\Omega J\theta \, \mathrm{d}S \tag{5.5}$$

$$\begin{aligned} \nabla \times A &= (\partial_y A, -\partial_x A) = \boldsymbol{B} \\ \nabla \times \boldsymbol{B} &= \partial_x B_y - \partial_y B_x = -\partial_x\partial_x A - \partial_y\partial_y A = -\Delta A \end{aligned} \tag{5.6}$$

$$\int_\Omega -\frac{1}{\mu}\Delta A\theta \, \mathrm{d}S = \int_\Omega J\theta \, \mathrm{d}S \tag{5.7}$$

Green's first identity:

$$\int_\Omega \Delta F\theta \, \mathrm{d}S + \int_\Omega \nabla F \nabla\theta \, \mathrm{d}S = \int_{\partial\Omega} (\nabla F\boldsymbol{n})\,\theta \mathrm{d}l \tag{5.8}$$

$$\int_\Omega -\frac{1}{\mu}\Delta A\theta \, \mathrm{d}S = \int_\Omega \frac{1}{\mu}\nabla A\nabla\theta \, \mathrm{d}S - \int_{\partial\Omega_1} \frac{1}{\mu}(\nabla A\boldsymbol{n})\,\theta \mathrm{d}l - \int_{\partial\Omega_2} \frac{1}{\mu}(\nabla A\boldsymbol{n})\,\theta \mathrm{d}l \tag{5.9}$$

$$\theta \in H^1 : \theta|_{\partial\Omega_1} = 0, \theta|_{\partial\Omega_2} = 0 \to \textstyle\int_{\partial\Omega_1} \frac{1}{\mu}(\nabla A\boldsymbol{n})\,\theta \mathrm{d}l = 0, \int_{\partial\Omega_2} \frac{1}{\mu}(\nabla A\boldsymbol{n})\,\theta \mathrm{d}l = 0$$

Discretization

$$A\left(x,y,t\right)\approx\sum_{n=1}^{N}A^{n}\left(t\right)\lambda^{n}\left(x,y\right) \tag{5.10}$$

$$J\left(x,y,t\right)\approx\sum_{n=1}^{N}J^{n}\left(t\right)\lambda^{n}\left(x,y\right) \tag{5.11}$$

$$\theta\left(x,y,t\right)\approx\sum_{n=1}^{N}\theta^{n}\left(t\right)\lambda^{n}\left(x,y\right) \tag{5.12}$$

$$\int_{T_k}\frac{1}{\mu}\nabla\left(\sum_{n=1}^{N}A^{n}\lambda^{n}\right)\nabla\left(\sum_{n=1}^{N}\theta^{n}\lambda^{n}\right)\,\mathrm{d}S=\int_{T_k}\sum_{n=1}^{N}J^{n}\lambda^{n}\sum_{n=1}^{N}\theta^{n}\lambda^{n}\,\mathrm{d}S \tag{5.13}$$

T_k is one discrete element of geometry, in our case it is a triangle.

$$\begin{aligned}\sum_{n=1}^{N}\int_{T_k}\frac{1}{\mu}\nabla\left(A^{n}\lambda^{n}\right)\nabla\left(\theta^{m}\lambda^{m}\right)\,\mathrm{d}S&=\sum_{n=1}^{N}\int_{T_k}J^{n}\lambda^{n}\theta^{m}\lambda^{m}\,\mathrm{d}S\\ \sum_{n=1}^{N}A^{n}\int_{T_k}\frac{1}{\mu}\nabla\lambda^{n}\nabla\lambda^{m}\,\mathrm{d}S&=\sum_{n=1}^{N}J^{n}\int_{T_k}\lambda^{n}\lambda^{m}\,\mathrm{d}S\end{aligned} \tag{5.14}$$

We can put A^n and J^n out of ∇ and out of $\int_{T_k}$, because they no longer depend on coordinate system.

$$S=\sum_{n=1}^{N}\int_{T_k}\frac{1}{\mu}\nabla\lambda^{n}\nabla\lambda^{m}\,\mathrm{d}S$$
$$M=\sum_{n=1}^{N}\int_{T_k}\lambda^{n}\lambda^{m}\,\mathrm{d}S$$

$$SA=MJ\rightarrow A=SMJ \tag{5.15}$$

If we discretize the problem with triangular elements and select the first-order polynomial (linear function), as an approximation of the scalar field A, we get three basis functions for each triangle. The coordinates (r,s) correspond to the reference triangle with vertices $V_1=(0,0),V_2=(0,1)$ and $V_3=(1,1)$ numbered in counter-clockwise direction.

$$\beta_1 = 1 - r - s \tag{5.16}$$

$$\beta_2 = r \tag{5.17}$$

$$\beta_3 = s \tag{5.18}$$

If we differentiate them in respect to coordinates, we get:

$$\nabla\beta = \begin{pmatrix} \partial_r\beta_1 & \partial_r\beta_2 & \partial_r\beta_3 \\ \partial_s\beta_1 & \partial_s\beta_2 & \partial_s\beta_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{5.19}$$

The matrices, fields and functions of the reference triangle must then be transformed into our coordinate system $(r, s) \to (x, y)$ and put into right place. Vertices of the triangle in (x, y) coordinates are $V_1 = (x_1, y_1), V_2 = (x_2, y_2)$ and $V_3 = (x_3, y_3)$.

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \tag{5.20}$$

The Φ is the transformation function. We can use it to transform basis functions of reference triangle T_r to basis functions of the n-th triangle T_k.

$$\lambda\,(x, y) = \left(\beta \circ \Phi^{-1}\right)(x, y) \tag{5.21}$$

We will also need the inverse Jacobian matrix and the “Jacobian”, ie the determinant of the Jacobian matrix.

$$|\det\left(\nabla\Phi\right)| = \left|\det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}\right| \tag{5.22}$$

$$\nabla\Phi^{-1} = \frac{1}{|\det\left(\nabla\Phi\right)|} \begin{pmatrix} y_3 - y_1 & -\left(x_3 - x_1\right) \\ -\left(y_2 - y_1\right) & x_2 - x_1 \end{pmatrix} \tag{5.23}$$

Mass matrix:

$$\begin{aligned} M_{T_k} &= \int_{T_k} \lambda^n \lambda^m \,\mathrm{d}S = \int_{T_k} \left(\beta^n \circ \Phi^{-1}\right)\left(\beta^m \circ \Phi^{-1}\right)\,\mathrm{d}S \\ &= |\det\left(\nabla\Phi\right)| \int_{T_r} \beta^n \beta^m \,\mathrm{d}S \end{aligned} \tag{5.24}$$

$$M_{T_k} = |\det\left(\nabla\Phi\right)| \begin{pmatrix} \int \beta_1\beta_1 \,\mathrm{d}S & \int \beta_1\beta_2 \,\mathrm{d}S & \int \beta_1\beta_3 \,\mathrm{d}S \\ \int \beta_2\beta_1 \,\mathrm{d}S & \int \beta_2\beta_2 \,\mathrm{d}S & \int \beta_2\beta_3 \,\mathrm{d}S \\ \int \beta_3\beta_1 \,\mathrm{d}S & \int \beta_3\beta_2 \,\mathrm{d}S & \int \beta_3\beta_3 \,\mathrm{d}S \end{pmatrix} \tag{5.25}$$

$$M = \sum_{n=1}^{N} |\det(\nabla\Phi)| \begin{pmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{pmatrix} \tag{5.26}$$

Stiffness matrix:

$$\begin{aligned} S_{T_k} &= \int_{T_k} \frac{1}{\mu} \nabla\lambda^n \nabla\lambda^m \, \mathrm{d}S = \int_{T_k} \frac{1}{\mu} \nabla \left(\beta^n \circ \Phi^{-1}\right) \nabla \left(\beta^m \circ \Phi^{-1}\right) \, \mathrm{d}S \\ &= \int_{T_k} \frac{1}{\mu} \partial_x \left(\beta^n \circ \Phi^{-1}\right) \partial_x \left(\beta^m \circ \Phi^{-1}\right) \, \mathrm{d}S + \int_{T_k} \frac{1}{\mu} \partial_y \left(\beta^n \circ \Phi^{-1}\right) \partial_y \left(\beta^m \circ \Phi^{-1}\right) \, \mathrm{d}S \\ &= S_{T_k,x} + S_{T_k,y} \end{aligned} \tag{5.27}$$

$$\begin{aligned} S_{T_k,x} &= \int_{T_k} \frac{1}{\mu} \left(\partial_x \beta^n \left(\Phi^{-1}\right)\right) \left(\partial_x \Phi^{-1}\right) \left(\partial_x \beta^n \left(\Phi^{-1}\right)\right) \left(\partial_x \Phi^{-1}\right) \, \mathrm{d}S \\ &= \frac{1}{\mu} |\det(\nabla\Phi)| \, (1,0) \int_{T_r} \nabla\Phi^{-1} \nabla\beta^\top \nabla\Phi^{-\top} \nabla\beta \, \mathrm{d}S \end{aligned} \tag{5.28}$$

$$\begin{aligned} S =& \sum_{n=1}^{N} \frac{1}{2\mu} |\det(\nabla\Phi)| \, (1,0) \, \nabla\Phi^{-1} \nabla\beta^\top \nabla\Phi^{-\top} \nabla\beta \\ &+ \sum_{n=1}^{N} \frac{1}{2\mu} |\det(\nabla\Phi)| \, (0,1) \, \nabla\Phi^{-1} \nabla\beta^\top \nabla\Phi^{-\top} \nabla\beta \end{aligned} \tag{5.29}$$

If we want to get the vector field $\boldsymbol{B}$ from the scalar field A, we need a “curl matrix”.

$$\boldsymbol{B} = \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} \partial_y A \\ -\partial_x A \end{pmatrix} \tag{5.30}$$

$$\int_\Omega B_x \theta \, \mathrm{d}S = \int_\Omega \partial_y A \theta \, \mathrm{d}S \tag{5.31}$$

Integration by parts:

$$\int_\Omega \partial F \theta \, \mathrm{d}S = \int_{\partial\Omega} F \theta \boldsymbol{n} \mathrm{d}l - \int_\Omega F \partial\theta \, \mathrm{d}S \tag{5.32}$$

$$\int_\Omega B_x \theta \, \mathrm{d}S = \int_\Omega \partial_y A \theta \, \mathrm{d}S = \int_{\partial\Omega} A \theta \boldsymbol{n} \, \mathrm{d}l - \int_\Omega A \partial_y \theta \, \mathrm{d}S \tag{5.33}$$

Constraints for θ ...

$$\int_\Omega \sum_{n=1}^{N} B_x^n \lambda^n \sum_{n=1}^{N} \theta^n \lambda^n \, \mathrm{d}S = -\int_\Omega \sum_{n=1}^{N} A^n \lambda^n \partial_y \left(\sum_{n=1}^{N} \theta^n \lambda^n \right) \mathrm{d}S \tag{5.34}$$

$$\sum_{n=1}^{N} B_x^n \int_\Omega \lambda^n \lambda^m \,\mathrm{d}S = -\sum_{n=1}^{N} A^n \int_\Omega \lambda^n \partial_y \lambda^m \,\mathrm{d}S \tag{5.35}$$

$$MB_x = -C_y A \to B_x = -MC_y A \tag{5.36}$$

"Curl matrix":

$$\begin{aligned} C_{T_k,y} &= \int_{T_k} \lambda^n \partial_y \lambda^m \,\mathrm{d}S = \int_{T_k} \left(\beta^n \circ \Phi^{-1}\right) \partial_y \left(\beta^m \circ \Phi^{-1}\right) \,\mathrm{d}S \\ &= \int_{T_k} \left(\beta^n \circ \Phi^{-1}\right) \left(\partial_y \beta^n \left(\Phi^{-1}\right)\right) \left(\partial_x \Phi^{-1}\right) \,\mathrm{d}S \\ &= \left|\det\left(\nabla\Phi\right)\right| (1,0) \int_{T_r} \beta \nabla\Phi^{-\top} \nabla\beta \,\mathrm{d}S \end{aligned} \tag{5.37}$$

We got $\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$, because $\int \beta \,\mathrm{d}S = \frac{1}{3}$.

$$\begin{aligned} C_x &= \sum_{n=1}^{N} \frac{1}{6} \left|\det\left(\nabla\Phi\right)\right| (0,1)\, \nabla\Phi^{-\top} \nabla\beta \\ C_y &= -\sum_{n=1}^{N} \frac{1}{6} \left|\det\left(\nabla\Phi\right)\right| (1,0)\, \nabla\Phi^{-\top} \nabla\beta \end{aligned} \tag{5.38}$$

This matrix is also used when permanent magnets are considered as a another magnetic field source. Magnets are represented by their remanent magnetic field density $\boldsymbol{B_r}$ and in the magnetic field equation they appear as $\nabla \times \boldsymbol{B_r}$. The right hand side of the equation with both coils and permanent magnets is then: $MJ + \left(C_y B_{r_x} + C_x B_{r_y}\right)$.