Finite element method for magnetodynamic applications

2019

Contents

Symbols

$\boldsymbol{A}$	$[\mathrm{Wb} \cdot \mathrm{m}^{-1}]$	magnetic vector potential
$\boldsymbol{B}$	[T]	magnetic flux density
f	[Hz]	frequency
$\boldsymbol{J}$	$[\mathrm{A} \cdot \mathrm{m}^{-2}]$	current density
l	[m]	line
L	[m]	length of the model (z coordinate)
$\boldsymbol{n}$	[-]	normal vector
S	$[\mathrm{m}^2]$	surface
t	[s]	time
$T, \mathbb{T}$	$[\mathrm{N} \cdot \mathrm{m}^{-2}]$	Maxwell stress tensor
$\boldsymbol{v}$	$[\mathrm{m} \cdot \mathrm{s}^{-1}]$	velocity
V	$[\mathrm{m}^3]$	volume
γ	$[\mathrm{S} \cdot \mathrm{m}^{-1}]$	conductivity
λ	[-]	basis function
μ_0	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability of vacuum
μ	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability
$\boldsymbol{\varphi}$	[-]	test function
Ω	[-]	closed region
$\partial\Omega$	[-]	boundary of closed region

1 Theory

1.1 Magnetodynamic equation

3D Cartesian coordinate system (x, y, z):

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times \boldsymbol{A}) \right) + \gamma \frac{\partial \boldsymbol{A}}{\partial t} - \gamma (\boldsymbol{v} \times (\nabla \times \boldsymbol{A})) = \boldsymbol{J} \tag{1.1}$$

Reduction to 2D (x, y):

$$\boldsymbol{A} = (0, 0, A_z) \quad \rightarrow \quad A = A_z \tag{1.2}$$

$$\boldsymbol{J} = (0, 0, J_z) \quad \rightarrow \quad J = J_z \tag{1.3}$$

$$\boldsymbol{v} = (v_x, v_y, 0) \quad \rightarrow \quad \boldsymbol{v} = (v_x, v_y) \tag{1.4}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = (B_x, B_y, 0) \quad \rightarrow \quad \boldsymbol{B} = (B_x, B_y) \tag{1.5}$$

For 2D, time-dependent problems are $A, J, \boldsymbol{v}$ and $\boldsymbol{B}$ functions of coordinates (x, y) and time t. Both A and J are scalar fields $\mathbb{R}^2 \rightarrow \mathbb{R}^1$, $\boldsymbol{v}$ and $\boldsymbol{B}$ are vector fields $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.

In a linear material, the μ and the γ are dependent only on the coordinate system (x, y), in the non-linear material, the μ is also dependent on the size of $||\boldsymbol{B}||$.

$$\nabla = (\partial_x, \partial_y) \tag{1.6}$$

Curl of scalar field $(\nabla \times A)$ is vector field $(\partial_y A, -\partial_x A)$, so $\mathbb{R}^1 \rightarrow \mathbb{R}^2$, but curl of vector field $(\nabla \times \boldsymbol{B})$ is scalar field $(\partial_x B_y - \partial_y B_x)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

Divergence of vector field $(\nabla \cdot \boldsymbol{B})$ is also scalar field $(\partial_x B_x + \partial_y B_y)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) + \gamma \frac{\partial A}{\partial t} - \gamma (\boldsymbol{v} \times (\nabla \times A)) = J \tag{1.7}$$

1.2 Boundary conditions

Dirichlet condition:

$$A \bigg|_{\partial \Omega_1} = f_1 (x, y) \tag{1.8}$$

Neumann condition:

$$\partial_{\boldsymbol{n}} A \bigg|_{\partial \Omega_2} = f_2 (x, y) \tag{1.9}$$

For our model, both f_1 and f_2 are equal to zero.

1.3 Initial condition

$$A\Big|_{t=0} = 0 \tag{1.10}$$

1.4 Force

$$T_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}B^2\delta_{ij}\right) \tag{1.11}$$

$$\mathbb{T} = \frac{1}{\mu_0}\begin{pmatrix} B_x B_x & B_x B_y \\ B_y B_x & B_y B_y \end{pmatrix} - \frac{1}{2\mu_0}\begin{pmatrix} B_x^2 + B_y^2 & 0 \\ 0 & B_x^2 + B_y^2 \end{pmatrix} \tag{1.12}$$

$$\boldsymbol{F}\left(x, y\right) = L\oint_{\partial\Omega} \mathbb{T}\boldsymbol{n}\,\mathrm{d}l = L\int_{\Omega}\left(\nabla\cdot\mathbb{T}\right)\,\mathrm{d}S \tag{1.13}$$

Variable L is the length of the model into the third dimension z.

$$\begin{aligned}
\nabla\cdot\mathbb{T} &= \frac{1}{\mu_0}\begin{pmatrix} \frac{1}{2}\left(B_x^2 - B_y^2\right) & B_x B_y \\ B_y B_x & \frac{1}{2}\left(B_y^2 - B_x^2\right) \end{pmatrix}^{\top}\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \\
&= \frac{1}{\mu_0}\begin{pmatrix} \frac{1}{2}\partial_x\left(B_x^2 - B_y^2\right) + \partial_y\left(B_y B_x\right) \\ \frac{1}{2}\partial_y\left(B_y^2 - B_x^2\right) + \partial_x\left(B_x B_y\right) \end{pmatrix} \\
&= \frac{1}{\mu_0}\begin{pmatrix} \partial_x B_x B_x - \partial_x B_y B_y + \partial_y B_y B_x + \partial_y B_x B_y \\ \partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x \end{pmatrix}
\end{aligned} \tag{1.14}$$

2 Weak formulation

To simplify the problem, let us first consider a magnetostacic case ($\gamma = 0$). This is true for models without motion and without time variation of currents.

3 Topology optimization

3.1 Problem formulation in continuous space

$$\begin{aligned} \text{minimize} \quad & F_y^p \quad (\text{p} - \text{plunger}) \\ \text{subject to} \quad & \nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) = J \\ & B_x = \partial_y A \\ & B_y = -\partial_x A \end{aligned}$$

$$F_y^p = (0,1) \int_{\Omega p} (\nabla \cdot \mathbb{T}) \, \mathrm{d}S = \int_{\Omega p} \frac{1}{\mu_0} \left(\partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x \right) \mathrm{d}S \tag{3.1}$$

3.2 Problem formulation in discrete space

$$\begin{aligned} \text{minimize} \quad & F_y^p \\ \text{subject to} \quad & SA = MJ \\ & MB_x = C_y A \\ & MB_y = -C_x A \end{aligned}$$

$$F_y^p = \frac{1}{\mu_0} \left(B_y^\top C_y^p B_y - B_x^\top C_y^p B_x + B_y^\top C_x^p B_x + B_x^\top C_x^p B_y \right) \tag{3.2}$$

Lagrange multipliers are (α, β, γ).

$$\begin{aligned} SA = MJ \quad & \rightarrow \quad \alpha \\ MB_x = C_y A \quad & \rightarrow \quad \beta \\ MB_y = -C_x A \quad & \rightarrow \quad \gamma \end{aligned}$$

The φ_i is the topology function.

$$J(\varphi_i) = J_1(\varphi_i) + J_2(\varphi_i) + J_3(\varphi_i) + J_4(\varphi_i) \tag{3.3}$$

$$J_1 = \frac{1}{\mu_0}\left(B_y^\top C_y^p B_y - B_x^\top C_y^p B_x + B_y^\top C_x^p B_x + B_x^\top C_x^p B_y\right) \tag{3.4}$$

$$J_2 = \alpha^\top (SA - MJ) = 0 \tag{3.5}$$

$$J_3 = \beta^\top (MB_x - C_y A) = 0 \tag{3.6}$$

$$J_4 = \gamma^\top (MB_y + C_x A) = 0 \tag{3.7}$$

$$\partial_{\varphi_i} J(\varphi_i) = \partial_{\varphi_i} J_1(\varphi_i) + \partial_{\varphi_i} J_2(\varphi_i) + \partial_{\varphi_i} J_3(\varphi_i) + \partial_{\varphi_i} J_4(\varphi_i) \tag{3.8}$$

$$\mu_0 \partial_{\varphi_i} J_1 = \partial_{\varphi_i}\left(B_y^\top C_y^p B_y\right) - \partial_{\varphi_i}\left(B_x^\top C_y^p B_x\right) + \partial_{\varphi_i}\left(B_y^\top C_x^p B_x\right) + \partial_{\varphi_i}\left(B_x^\top C_x^p B_y\right) \tag{3.9}$$

$$\partial_{\varphi_i}\left(B_y^\top C_y^p B_y\right) = \partial_{\varphi_i} B_y^\top C_y^p B_y + B_y^\top C_y^p \partial_{\varphi_i} B_y = B_y^\top \left(C_y^p\right)^\top \partial_{\varphi_i} B_y + B_y^\top C_y^p \partial_{\varphi_i} B_y \tag{3.10}$$

$$\begin{aligned}\partial_{\varphi_i} J_1 =& \frac{1}{\mu_0}\left(B_x^\top, B_y^\top\right)\begin{pmatrix} -\left(C_y^p\right)^\top & -C_y^p \\ \left(C_x^p\right)^\top & C_x^p \end{pmatrix}\begin{pmatrix} \partial_{\varphi_i} B_x \\ \partial_{\varphi_i} B_x \end{pmatrix} \\ &+ \frac{1}{\mu_0}\left(B_x^\top, B_y^\top\right)\begin{pmatrix} \left(C_x^p\right)^\top & C_x^p \\ \left(C_y^p\right)^\top & C_y^p \end{pmatrix}\begin{pmatrix} \partial_{\varphi_i} B_y \\ \partial_{\varphi_i} B_y \end{pmatrix}\end{aligned} \tag{3.11}$$

$$\partial_{\varphi_i} J_2 = \alpha^\top \partial_{\varphi_i}(SA) - \alpha^\top \partial_{\varphi_i}(MJ) = \alpha^\top \partial_{\varphi_i} S A + \alpha^\top S \partial_{\varphi_i} A \tag{3.12}$$

$$\partial_{\varphi_i} J_3 = \beta^\top M \partial_{\varphi_i} B_x - \beta^\top C_y \partial_{\varphi_i} A \tag{3.13}$$

$$\partial_{\varphi_i} J_4 = \gamma^\top M \partial_{\varphi_i} B_y + \gamma^\top C_x \partial_{\varphi_i} A \tag{3.14}$$

$$\begin{aligned}
\partial_{\varphi_i} B_x : &\quad \frac{1}{\mu_0}\left(B_x^\top, B_y^\top\right)\begin{pmatrix} -\left(C_y^p\right)^\top & -C_y^p \\ \left(C_x^p\right)^\top & C_x^p \end{pmatrix}\begin{pmatrix} \partial_{\varphi_i} B_x \\ \partial_{\varphi_i} B_x \end{pmatrix} + \beta^\top M \partial_{\varphi_i} B_x && (3.15)\\
\partial_{\varphi_i} B_y : &\quad \frac{1}{\mu_0}\left(B_x^\top, B_y^\top\right)\begin{pmatrix} \left(C_x^p\right)^\top & C_x^p \\ \left(C_y^p\right)^\top & C_y^p \end{pmatrix}\begin{pmatrix} \partial_{\varphi_i} B_y \\ \partial_{\varphi_i} B_y \end{pmatrix} + \gamma^\top M \partial_{\varphi_i} B_y && (3.16)\\
\partial_{\varphi_i} A : &\quad \alpha^\top S \partial_{\varphi_i} A - \beta^\top C_y \partial_{\varphi_i} A + \gamma^\top C_x \partial_{\varphi_i} A && (3.17)\\
\partial_{\varphi_i} S : &\quad \alpha^\top \partial_{\varphi_i} S A && (3.18)
\end{aligned}$$

From (3.15):

$$\beta = -\frac{1}{\mu_0} M^{-\top}\left(B_x^\top, B_y^\top\right)\begin{pmatrix} -\left(C_y^p\right)^\top & -C_y^p \\ \left(C_x^p\right)^\top & C_x^p \end{pmatrix} \tag{3.19}$$

From (3.16):

$$\gamma = -\frac{1}{\mu_0} M^{-\top}\left(B_x^\top, B_y^\top\right)\begin{pmatrix} \left(C_x^p\right)^\top & C_x^p \\ \left(C_y^p\right)^\top & C_y^p \end{pmatrix} \tag{3.20}$$

From (3.17):

$$\alpha = S^{-\top}\left(C_y^\top \beta - C_x^\top \gamma\right) \tag{3.21}$$

If we choose α, β, γ such that (3.15), (3.16) and (3.17) are equal to 0, then:

$$\partial_{\varphi_i} J = \alpha^\top \partial_{\varphi_i} S A \tag{3.22}$$

Appendix

Weak formulation

For $\gamma = 0$.

$$\nabla \times \left(\frac{1}{\mu} \left(\nabla \times A \right) \right) = J \tag{3.23}$$

The θ is test function.

$$\int_{\Omega} \nabla \times \left(\frac{1}{\mu} \left(\nabla \times A \right) \right) \theta \, \mathrm{d}S = \int_{\Omega} J \theta \, \mathrm{d}S \tag{3.24}$$

$$\begin{aligned} \nabla \times A &= \left(\partial_y A, -\partial_x A \right) = \boldsymbol{B} \\ \nabla \times \boldsymbol{B} &= \partial_x B_y - \partial_y B_x = -\partial_x \partial_x A - \partial_y \partial_y A = -\Delta A \end{aligned} \tag{3.25}$$

$$\int_{\Omega} -\frac{1}{\mu} \Delta A \theta \, \mathrm{d}S = \int_{\Omega} J \theta \, \mathrm{d}S \tag{3.26}$$

Green's first identity:

$$\int_{\Omega} \Delta F \theta \, \mathrm{d}S + \int_{\Omega} \nabla F \nabla \theta \, \mathrm{d}S = \int_{\partial \Omega} \left(\nabla F \boldsymbol{n} \right) \theta \mathrm{d}l \tag{3.27}$$

$$\int_{\Omega} -\frac{1}{\mu} \Delta A \theta \, \mathrm{d}S = \int_{\Omega} \frac{1}{\mu} \nabla A \nabla \theta \, \mathrm{d}S - \int_{\partial \Omega 1} \frac{1}{\mu} \left(\nabla A \boldsymbol{n} \right) \theta \mathrm{d}l - \int_{\partial \Omega 2} \frac{1}{\mu} \left(\nabla A \boldsymbol{n} \right) \theta \mathrm{d}l \tag{3.28}$$

$\theta \in f\left(\Omega \right) : \theta|_{\partial \Omega 1} = 0, \theta|_{\partial \Omega 2} = 0 \to \int_{\partial \Omega 1} \frac{1}{\mu} \left(\nabla A \boldsymbol{n} \right) \theta \mathrm{d}l = 0, \int_{\partial \Omega 2} \frac{1}{\mu} \left(\nabla A \boldsymbol{n} \right) \theta \mathrm{d}l = 0$
Discretization

$$A\left(x, y, t \right) \approx \sum_{n=1}^{N} A^n \left(t \right) \lambda^n \left(x, y \right) \tag{3.29}$$

$$J\left(x, y, t \right) \approx \sum_{n=1}^{N} J^n \left(t \right) \lambda^n \left(x, y \right) \tag{3.30}$$

$$\theta\left(x, y, t \right) \approx \sum_{n=1}^{N} \theta^n \left(t \right) \lambda^n \left(x, y \right) \tag{3.31}$$

$$\int_{T_n} \frac{1}{\mu} \nabla \left(\sum_{n=1}^{N} A^n \lambda^n \right) \nabla \left(\sum_{n=1}^{N} \theta^n \lambda^n \right) \,\mathrm{d}S = \int_{T_n} \sum_{n=1}^{N} J^n \lambda^n \sum_{n=1}^{N} \theta^n \lambda^n \,\mathrm{d}S \tag{3.32}$$

T_n is one descrete element of geometry, in our case it is a triangle.

$$\begin{aligned} \sum_{n=1}^{N} \int_{T_n} \frac{1}{\mu} \nabla \left(A^n \lambda^n \right) \nabla \left(\theta^m \lambda^m \right) \,\mathrm{d}S &= \sum_{n=1}^{N} \int_{T_n} J^n \lambda^n \theta^m \lambda^m \,\mathrm{d}S \\ \sum_{n=1}^{N} A^n \int_{T_n} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m \,\mathrm{d}S &= \sum_{n=1}^{N} J^n \int_{T_n} \lambda^n \lambda^m \,\mathrm{d}S \end{aligned} \tag{3.33}$$

We can put A^n and J^n out of ∇ and out of $\int_{T_n}$, because they no longer depend on coordinate system.

$$\begin{aligned} S &= \sum_{n=1}^{N} \int_{T_n} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m \,\mathrm{d}S \\ M &= \sum_{n=1}^{N} \int_{T_n} \lambda^n \lambda^m \,\mathrm{d}S \end{aligned}$$

$$SA = MJ \rightarrow A = S^{\top} MJ \tag{3.34}$$

If we discretize the problem with triangular elements and select the first-order polynomial (linear function), as an approximation of the scalar field A, we get three basis functions for each triangle. The coordinates (r, s) correspond to the reference triangle with vertices $V_1 = (0, 0), V_2 = (0, 1)$ and $V_3 = (1, 1)$ numbered in counter-clockwise direction.

$$\beta_1 = 1 - r - s \tag{3.35}$$

$$\beta_2 = r \tag{3.36}$$

$$\beta_3 = s \tag{3.37}$$

If we differentiate them in respect to coordinates, we get:

$$\partial \beta = \begin{pmatrix} \partial_r \beta_1 & \partial_r \beta_2 & \partial_r \beta_3 \\ \partial_s \beta_1 & \partial_s \beta_2 & \partial_s \beta_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{3.38}$$

The matrices, fields and functions of the reference triangle must then be transformed into our coordinate system $(r, s) \rightarrow (x, y)$ and put into right place. Vertices of the triangle in (x, y) coordinates are $V_1 = (x_1, y_1), V_2 = (x_2, y_2)$ and $V_3 = (x_3, y_3)$.

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \tag{3.39}$$

The Φ is the transformation function. We can use it to transform basis functions of reference triangle T_r to basis functions of the n-th triangle T_n.

$$\lambda\left(x, y\right) = \left(\beta \circ \Phi^{-1}\right)\left(x, y\right) \tag{3.40}$$

We will also need the inverse Jacobian matrix and the "Jacobian", ie the determinant of the Jacobian matrix.

$$\left|\det\left(\nabla\Phi\right)\right| = \left|\det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}\right| \tag{3.41}$$

$$\left(\nabla\Phi\right)^{-1} = \frac{1}{\left|\det\left(\nabla\Phi\right)\right|} \begin{pmatrix} y_3 - y_1 & -\left(x_3 - x_1\right) \\ -\left(y_2 - y_1\right) & x_2 - x_1 \end{pmatrix} \tag{3.42}$$

Mass matrix:

$$\begin{aligned} M_{T_n} &= \int_{T_n} \lambda^n \lambda^m \, \mathrm{d}S = \int_{T_n} \left(\beta \circ \Phi^{-1}\right)^n \left(\beta \circ \Phi^{-1}\right)^m \, \mathrm{d}S \\ &= \left|\det\left(\nabla\Phi\right)\right| \int_{T_r} \beta^n \beta^m \, \mathrm{d}S \end{aligned} \tag{3.43}$$

$$M_{T_n} = \left|\det\left(\nabla\Phi\right)\right| \begin{pmatrix} \int \beta_1\beta_1 \, \mathrm{d}S & \int \beta_1\beta_2 \, \mathrm{d}S & \int \beta_1\beta_3 \, \mathrm{d}S \\ \int \beta_2\beta_1 \, \mathrm{d}S & \int \beta_2\beta_2 \, \mathrm{d}S & \int \beta_2\beta_3 \, \mathrm{d}S \\ \int \beta_3\beta_1 \, \mathrm{d}S & \int \beta_3\beta_2 \, \mathrm{d}S & \int \beta_3\beta_3 \, \mathrm{d}S \end{pmatrix} \tag{3.44}$$

$$M = \sum_{n=1}^{N} \left|\det\left(\nabla\Phi\right)\right| \begin{pmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{pmatrix} \tag{3.45}$$

Stiffness matrix:

$$\begin{aligned} S_{T_n} &= \int_{T_n} \frac{1}{\mu} \nabla\lambda^n \nabla\lambda^m \, \mathrm{d}S = \int_{T_n} \frac{1}{\mu} \nabla\left(\beta \circ \Phi^{-1}\right)^n \nabla\left(\beta \circ \Phi^{-1}\right)^m \, \mathrm{d}S \\ &= \int_{T_n} \frac{1}{\mu} \partial_x \left(\beta \circ \Phi^{-1}\right)^n \partial_x \left(\beta \circ \Phi^{-1}\right)^m \, \mathrm{d}S + \int_{T_n} \frac{1}{\mu} \partial_y \left(\beta \circ \Phi^{-1}\right)^n \partial_y \left(\beta \circ \Phi^{-1}\right)^m \, \mathrm{d}S \\ &= S_{T_n,x} + S_{T_n,y} \end{aligned} \tag{3.46}$$

$$
\begin{aligned}
S_{T_n,x} &= \int_{T_n} \frac{1}{\mu} \left(\partial_{\Phi^{-1}} \beta^n\right) \left(\partial_x \Phi^{-1}\right) \left(\partial_{\Phi^{-1}} \beta^m\right) \left(\partial_x \Phi^{-1}\right) \, \mathrm{d}S \\
&= \frac{1}{\mu} |\det \left(\nabla \Phi\right)| \left(1,0\right) \int_{T_r} \left(\nabla \Phi\right)^{-1} \partial \beta^\top \left(\nabla \Phi\right)^{-\top} \partial \beta \, \mathrm{d}S
\end{aligned} \tag{3.47}
$$

$$
\begin{aligned}
S =& \sum_{n=1}^{N} \frac{1}{2\mu} |\det \left(\nabla \Phi\right)| \left(1,0\right) \left(\nabla \Phi\right)^{-1} \partial \beta^\top \left(\nabla \Phi\right)^{-\top} \partial \beta \\
&+ \sum_{n=1}^{N} \frac{1}{2\mu} |\det \left(\nabla \Phi\right)| \left(0,1\right) \left(\nabla \Phi\right)^{-1} \partial \beta^\top \left(\nabla \Phi\right)^{-\top} \partial \beta
\end{aligned} \tag{3.48}
$$

If we want to get the vector field $\boldsymbol{B}$ from the scalar field A, we need a “curl matrix”.

$$
\boldsymbol{B} = \nabla \times A \tag{3.49}
$$

$\boldsymbol{\tau} = \left(\partial_y, -\partial_x\right)$

$$
\int_\Omega \boldsymbol{B} \theta \, \mathrm{d}S = \int_\Omega \left(\nabla \times A\right) \theta \, \mathrm{d}S = \int_\Omega \boldsymbol{\tau} A \theta \, \mathrm{d}S \tag{3.50}
$$

Integration by parts:

$$
\int_\Omega \partial F \theta \, \mathrm{d}S = \int_{\partial\Omega} F \theta \mathrm{d}l - \int_\Omega F \partial \theta \, \mathrm{d}S \tag{3.51}
$$

$$
\int_\Omega \boldsymbol{B} \theta \, \mathrm{d}S = \int_\Omega \boldsymbol{\tau} A \theta \, \mathrm{d}S = \int_{\partial\Omega} A \theta \, \mathrm{d}l - \int_\Omega A \boldsymbol{\tau} \theta \, \mathrm{d}S \tag{3.52}
$$

Constraints for θ ... , $\boldsymbol{\theta} = \boldsymbol{\tau} \theta$.

$$
\int_\Omega \sum_{n=1}^{N} \boldsymbol{B}^n \lambda^n \sum_{n=1}^{N} \boldsymbol{\theta}^n \lambda^n \, \mathrm{d}S = - \int_\Omega \sum_{n=1}^{N} A^n \lambda^n \boldsymbol{\tau} \left(\sum_{n=1}^{N} \theta^n \lambda^n \right) \, \mathrm{d}S \tag{3.53}
$$

$$
\sum_{n=1}^{N} \boldsymbol{B}^n \int_\Omega \lambda^n \lambda^m \, \mathrm{d}S = - \sum_{n=1}^{N} A^n \int_\Omega \lambda^n \boldsymbol{\tau} \lambda^m \, \mathrm{d}S \tag{3.54}
$$

$$
MB = -CA \rightarrow B = -M^\top CA \tag{3.55}
$$

“Curl matrix”:

$$
\begin{aligned}
C_{T_n} &= \int_{T_n} \lambda^n \boldsymbol{\tau} \lambda^m \,\mathrm{d}S = \int_{T_n} \lambda^n \left(\partial_y \lambda^m, -\partial_x \lambda^m\right) \,\mathrm{d}S \\
&= \int_{T_n} \lambda^n \partial_y \lambda^m \,\mathrm{d}S - \int_{T_n} \lambda^n \partial_x \lambda^m \,\mathrm{d}S = \\
&= \int_{T_n} \left(\beta \circ \Phi^{-1}\right)^n \partial_y \left(\beta \circ \Phi^{-1}\right)^m \,\mathrm{d}S - \int_{T_n} \left(\beta \circ \Phi^{-1}\right)^n \partial_x \left(\beta \circ \Phi^{-1}\right)^m \,\mathrm{d}S \\
&= C_{T_n,x} - C_{T_n,y}
\end{aligned} \tag{3.56}
$$

$$
\begin{aligned}
C_{T_n,x} &= \int_{T_n} \left(\beta \circ \Phi^{-1}\right)^n \left(\partial_{\Phi^{-1}} \beta^n\right) \left(\partial_x \Phi^{-1}\right) \,\mathrm{d}S \\
&= \left|\det\left(\nabla \Phi\right)\right| (1,0) \int_{T_r} \left(\nabla \Phi\right)^{-\top} \partial \beta \,\mathrm{d}S
\end{aligned} \tag{3.57}
$$

$$
C = \sum_{n=1}^{N} \frac{1}{2} \left|\det\left(\nabla \Phi\right)\right| (1,0) \left(\nabla \Phi\right)^{-\top} \partial \beta - \sum_{n=1}^{N} \frac{1}{2} \left|\det\left(\nabla \Phi\right)\right| (0,1) \left(\nabla \Phi\right)^{-\top} \partial \beta \tag{3.58}
$$

LocalMatrices.m

`edet` $= \left|\det\left(\nabla \Phi\right)\right|$

`dFinv` $= \left(\nabla \Phi\right)^{-1}$

`dphi` $= \left(\nabla \Phi\right)^{-\top} \partial \beta$

`slocx` = ... = `1/2 *  dphi(1,:) * edet`

`slocy` = ... = `1/2 *  dphi(2,:) * edet`

`slocxx` $= \mu S_{T_n,x} =$ `1/2 * dphi(1,:)' * dphi(1,:) * edet`

`slocyy` $= \mu S_{T_n,y} =$ `1/2 * dphi(2,:)' * dphi(2,:) * edet`

`mloc` $= M_{T_n} =$ `[1 1/2 1/2;1/2 1 1/2;1/2 1/2 1]/12 * edet`

`clocx` = ... = `[slocx;slocx;slocx]/3`

`clocy` = ... = `[slocy;slocy;slocy]/3`