Finite element method for magnetodynamic applications

2019

Contents

Symbols

$\boldsymbol{A}$	$[\mathrm{Wb} \cdot \mathrm{m}^{-1}]$	magnetic vector potential
$\boldsymbol{B}$	[T]	magnetic flux density
f	[Hz]	frequency
$\boldsymbol{J}$	$[\mathrm{A} \cdot \mathrm{m}^{-2}]$	current density
l	[m]	line
L	[m]	length of the model (z coordinate)
$\boldsymbol{n}$	[-]	normal vector
S	$[\mathrm{m}^2]$	surface
t	[s]	time
$T, \mathbb{T}$	$[\mathrm{N} \cdot \mathrm{m}^{-2}]$	Maxwell stress tensor
$\boldsymbol{v}$	$[\mathrm{m} \cdot \mathrm{s}^{-1}]$	velocity
V	$[\mathrm{m}^3]$	volume
γ	$[\mathrm{S} \cdot \mathrm{m}^{-1}]$	conductivity
λ	[-]	basis function
μ_0	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability of vacuum
μ	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability
$\boldsymbol{\varphi}$	[-]	test function
Ω	[-]	closed region
$\partial\Omega$	[-]	boundary of closed region

1 Theory

1.1 2D Magnetodynamic equations

Cartesian coordinate system (x, y, z)

$$\nabla \times \left(\frac{1}{\mu}(\nabla \times \boldsymbol{A})\right) + \gamma \frac{\partial \boldsymbol{A}}{\partial t} - \gamma\left(\boldsymbol{v} \times (\nabla \times \boldsymbol{A})\right) = \boldsymbol{J} \tag{1.1}$$

$$\boldsymbol{A} = (0, 0, A_z) \tag{1.2}$$
$$\boldsymbol{J} = (0, 0, J_z) \tag{1.3}$$
$$\boldsymbol{v} = (v_x, v_y, 0) \tag{1.4}$$
$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = (\partial_y A_x, -\partial_x A_y, 0) = (B_x, B_y, 0) \tag{1.5}$$
$$\boldsymbol{v} \times \boldsymbol{B} = (0, 0, v_x B_y - v_y B_x) \tag{1.6}$$

$$\boldsymbol{A}, \boldsymbol{J}, \boldsymbol{v} = f(x, y, t) \tag{1.7}$$
$$\text{Linear} \quad \mu, \gamma = f(x, y) \qquad \text{Nonlinear} \quad \mu = f(x, y, ||\boldsymbol{B}||) \tag{1.8}$$

1.2 Boundary condition

$$A_z\Big|_{\partial\Omega} = 0 \tag{1.9}$$

1.3 Initial condition

$$A_z\Big|_{t=0} = 0 \tag{1.10}$$

1.4 Force

$$T_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2} B^2 \delta_{ij}\right) \tag{1.11}$$

$$\mathbb{T} = \frac{1}{\mu_0} \begin{pmatrix} B_x B_x & B_x B_y \\ B_y B_x & B_y B_y \end{pmatrix} - \frac{1}{2\mu_0} \begin{pmatrix} B_x^2 + B_y^2 & 0 \\ 0 & B_x^2 + B_y^2 \end{pmatrix} \tag{1.12}$$

$$\text{3D} \qquad \boldsymbol{F}(x, y, z) = \oint_{\partial\Omega} \mathbb{T} \, \mathrm{d}\boldsymbol{S} = \int_{\Omega} (\nabla \cdot \mathbb{T}) \, \mathrm{d}V \tag{1.13}$$

$$\text{2D} \qquad \boldsymbol{F}(x, y) = L \oint_{\partial\Omega} \mathbb{T} \, \mathrm{d}\boldsymbol{l} = L \int_{\Omega} (\nabla \cdot \mathbb{T}) \, \mathrm{d}S \tag{1.14}$$

$$\nabla \cdot \mathbb{T} = \frac{1}{\mu_0} (\partial_x, \partial_y) \begin{pmatrix} \frac{1}{2}\left(B_x^2 - B_y^2\right) & B_x B_y \\ B_y B_x & \frac{1}{2}\left(B_y^2 - B_x^2\right) \end{pmatrix} \tag{1.15}$$

$$\nabla \cdot \mathbb{T} = \frac{1}{\mu_0} \left(\frac{1}{2} \partial_x \left(B_x^2 - B_y^2\right) + \partial_y (B_y B_x), \frac{1}{2} \partial_y \left(B_y^2 - B_x^2\right) + \partial_x (B_x B_y) \right) \tag{1.16}$$

$$\nabla \cdot \mathbb{T}(x) = \frac{1}{\mu_0} (\partial_x B_x B_x - \partial_x B_y B_y + \partial_y B_y B_x + \partial_y B_x B_y) \tag{1.17}$$

$$\nabla \cdot \mathbb{T}(y) = \frac{1}{\mu_0} (\partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x) \tag{1.18}$$

2 Weak formulation

$$\int_\Omega \nabla \times \left(\frac{1}{\mu}\left(\nabla \times \boldsymbol{A}\right)\right)\boldsymbol{\varphi}\,\mathrm{d}S = \int_\Omega \frac{1}{\mu}\left(\nabla \times \boldsymbol{A}\right)\left(\nabla \times \boldsymbol{\varphi}\right)\,\mathrm{d}S - \\ - \int_{\partial\Omega} \frac{1}{\mu}\left(\boldsymbol{n} \times \left(\nabla \times \boldsymbol{A}\right)\right)\boldsymbol{\varphi}\,\mathrm{d}l \tag{2.1}$$

$\boldsymbol{\varphi} \in f\left(\Omega\right) : \boldsymbol{\varphi}|_{\partial\Omega} = 0 \to -\int_{\partial\Omega} \frac{1}{\mu}\left(\boldsymbol{n} \times \left(\nabla \times \boldsymbol{A}\right)\right)\boldsymbol{\varphi}\,\mathrm{d}l = 0$

$$\boldsymbol{A}\left(x, y, t\right) \approx \sum_{n=1}^{N} A^n\left(t\right)\lambda^n\left(x, y\right) \tag{2.2}$$

$$\boldsymbol{J}\left(x, y, t\right) \approx \sum_{n=1}^{N} J^n\left(t\right)\lambda^n\left(x, y\right) \tag{2.3}$$

$$\boldsymbol{\varphi}\left(x, y, t\right) \approx \sum_{n=1}^{N} \varphi^n\left(t\right)\lambda^n\left(x, y\right) \tag{2.4}$$

$$\int_\Omega \frac{1}{\mu}\left(\nabla \times \boldsymbol{A}\right)\left(\nabla \times \boldsymbol{\varphi}\right)\,\mathrm{d}S \approx \sum_{n=1}^{N} \int_{T_n} \frac{1}{\mu} A^n\left(\nabla \times \lambda^n\right)\varphi^m\left(\nabla \times \lambda^m\right)\,\mathrm{d}S \tag{2.5}$$

$$\int_\Omega \boldsymbol{J}\boldsymbol{\varphi}\,\mathrm{d}S \approx \sum_{n=1}^{N} \int_{T_n} J^n\lambda^n\varphi^m\lambda^m\,\mathrm{d}S \tag{2.6}$$

2.1 Magnetostatic case

$$\sum_{n=1}^{N} \int_{T_n} \frac{1}{\mu} A^n\left(\nabla \times \lambda^n\right)\left(\nabla \times \lambda^m\right)\,\mathrm{d}S = \sum_{n=1}^{N} \int_{T_n} J^n\lambda^n\lambda^m\,\mathrm{d}S \tag{2.7}$$

$$\frac{1}{\mu}\left[S\right]\left[A\right] = \left[M\right]\left[J\right] \tag{2.8}$$

$$(\nabla \times \lambda^n)(\nabla \times \lambda^m) = (\partial_y \lambda^n, -\partial_x \lambda^n)(\partial_y \lambda^m, -\partial_x \lambda^m) \tag{2.9}$$

$$(\partial_y \lambda^n, -\partial_x \lambda^n)(\partial_y \lambda^m, -\partial_x \lambda^m) = \partial_y \lambda^n \partial_y \lambda^m + \partial_x \lambda^n \partial_x \lambda^m \tag{2.10}$$

$$[S]_{T_n} = [S_y]_{T_n} + [S_x]_{T_n} = \int_{T_n} (\partial_y \lambda^n \partial_y \lambda^m) \, \mathrm{d}S + \int_{T_n} (\partial_x \lambda^n \partial_x \lambda^m) \, \mathrm{d}S \tag{2.11}$$

Basis functions on reference triangle (r, s, t)

$$\beta_1 = 1 - r - s \tag{2.12}$$

$$\beta_2 = r \tag{2.13}$$

$$\beta_3 = s \tag{2.14}$$

$$[\partial \beta] = \begin{pmatrix} \partial_r \beta_1 & \partial_r \beta_2 & \partial_r \beta_3 \\ \partial_s \beta_1 & \partial_s \beta_2 & \partial_s \beta_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{2.15}$$

Coordinate transformation

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \tag{2.16}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi \begin{pmatrix} r \\ s \end{pmatrix} \tag{2.17}$$

$$\begin{pmatrix} r \\ s \end{pmatrix} = \Phi^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.18}$$

$$\lambda(x, y) = \left(\beta \circ \Phi^{-1}\right)(x, y) \tag{2.19}$$

$$|\det(\nabla \Phi)| = \left| \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \right| \tag{2.20}$$

$$(\nabla \Phi)^{-1} = \frac{1}{|\det(\nabla \Phi)|} \begin{pmatrix} y_3 - y_1 & -(x_3 - x_1) \\ -(y_2 - y_1) & x_2 - x_1 \end{pmatrix} \tag{2.21}$$

$$[S_y]_{T_n} = \int_{T_n} \left(\partial_y \lambda^n \partial_y \lambda^m\right) \, \mathrm{d}S = \int_{T_n} \left(\partial_y \left(\beta^n \circ \Phi^{-1}\right) \partial_y \left(\beta^m \circ \Phi^{-1}\right)\right) \, \mathrm{d}S \tag{2.22}$$

$$\partial_y \left(\beta^n \circ \Phi^{-1}\right) = \left(\partial_{\Phi^{-1}} \beta^n\right) \left(\partial_y \Phi^{-1}\right) \tag{2.23}$$

$$[S_y]_{T_n} = \int_{T_n} \left(\partial_{\Phi^{-1}} \beta^n\right) \left(\partial_y \Phi^{-1}\right) \left(\partial_{\Phi^{-1}} \beta^m\right) \left(\partial_y \Phi^{-1}\right) \, \mathrm{d}S \tag{2.24}$$

$$[S_y]_{T_r} = \left|\det\left(\nabla\Phi\right)\right| (0,1) \int_{T_r} [\partial\beta] \left(\nabla\Phi\right)^{-1} \, \mathrm{d}S \tag{2.25}$$

$$[S_x]_{T_r} = \frac{1}{2} (1,0) [\partial\beta] \left(\nabla\Phi\right)^{-1} \left|\det\left(\nabla\Phi\right)\right| \tag{2.26}$$

$$[S_y]_{T_r} = \frac{1}{2} (0,1) [\partial\beta] \left(\nabla\Phi\right)^{-1} \left|\det\left(\nabla\Phi\right)\right| \tag{2.27}$$

LocalMatrices.m

- `edet` $= \left|\det\left(\nabla\Phi\right)\right|$
- `dFinv` $= \left(\nabla\Phi\right)^{-1}$
- `dphi` $= \left(\nabla\Phi\right)^{-1} [\partial\beta]$
- `slocxx` $= [S_x]_{T_r} =$ `1/2 * dphi(1,:)' * dphi(1,:) * edet`
- `slocyy` $= [S_y]_{T_r} =$ `1/2 * dphi(2,:)' * dphi(2,:) * edet`

3 Topology optimization

3.1 Problem formulation in continuous space

$$\begin{aligned} \text{minimize} \quad & F_y^p \quad (\mathrm{p}\,-\,\text{plunger}) \\ \text{subject to} \quad & \nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) = J \\ & B_x = \partial_y A \\ & B_y = -\partial_x A \end{aligned}$$

$$F_y^p = (0,1) \int_{\Omega p} (\nabla \cdot \mathbb{T}) \, \mathrm{d}S = \int_{\Omega p} \frac{1}{\mu_0} (\partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x) \, \mathrm{d}S \tag{3.1}$$

3.2 Problem formulation in discrete space

$$\begin{aligned} \text{minimize} \quad & F_y^p \\ \text{subject to} \quad & SA = MJ \\ & MB_x = C_y A \\ & MB_y = -C_x A \end{aligned}$$

$$F_y^p = \frac{1}{\mu_0} \left(B_y^T C_y^p B_y - B_x^T C_y^p B_x + B_y^T C_x^p B_x + B_x^T C_x^p B_y \right) \tag{3.2}$$

Lagrange multipliers are (α, β, γ).

$$\begin{aligned} SA = MJ \quad & \rightarrow \quad \alpha \\ MB_x = C_y A \quad & \rightarrow \quad \beta \\ MB_y = -C_x A \quad & \rightarrow \quad \gamma \end{aligned}$$

The θ is the topology function.

$$J\left(\theta\right)=J_{1}\left(\theta\right)+J_{2}\left(\theta\right)+J_{3}\left(\theta\right)+J_{4}\left(\theta\right) \tag{3.3}$$

$$J_{1}=\frac{1}{\mu_{0}}\left(B_{y}^{T}C_{y}^{p}B_{y}-B_{x}^{T}C_{y}^{p}B_{x}+B_{y}^{T}C_{x}^{p}B_{x}+B_{x}^{T}C_{x}^{p}B_{y}\right) \tag{3.4}$$

$$J_{2}=\alpha^{T}\left(SA-MJ\right)=0 \tag{3.5}$$

$$J_{3}=\beta^{T}\left(MB_{x}-C_{y}A\right)=0 \tag{3.6}$$

$$J_{4}=\gamma^{T}\left(MB_{y}+C_{x}A\right)=0 \tag{3.7}$$

$$\partial_{\theta}J\left(\theta\right)=\partial_{\theta}J_{1}\left(\theta\right)+\partial_{\theta}J_{2}\left(\theta\right)+\partial_{\theta}J_{3}\left(\theta\right)+\partial_{\theta}J_{4}\left(\theta\right) \tag{3.8}$$

$$\begin{aligned}\mu_{0}\partial_{\theta}J_{1}=&\partial_{\theta}\left(\left(B_{y}\circ A\right)^{T}C_{y}^{p}\left(B_{y}\circ A\right)\right)-\partial_{\theta}\left(\left(B_{x}\circ A\right)^{T}C_{y}^{p}\left(B_{x}\circ A\right)\right)\\&+\partial_{\theta}\left(\left(B_{y}\circ A\right)^{T}C_{x}^{p}\left(B_{x}\circ A\right)\right)+\partial_{\theta}\left(\left(B_{x}\circ A\right)^{T}C_{x}^{p}\left(B_{y}\circ A\right)\right)\end{aligned}$$

$$\partial_{\theta}\left(\left(B_{y}\circ A\right)^{T}C_{y}^{p}\left(B_{y}\circ A\right)\right)=\partial_{\theta}\left(B_{y}\circ A\right)^{T}C_{y}^{p}\left(B_{y}\circ A\right)+\left(B_{y}\circ A\right)^{T}C_{y}^{p}\partial_{\theta}\left(B_{y}\circ A\right)$$

$$\partial_{\theta}J_{2}=\alpha^{T}\partial_{\theta}\left(SA\right)-\alpha^{T}\partial_{\theta}\left(MJ\right)=\alpha^{T}\partial_{\theta}SA+\alpha^{T}S\partial_{\theta}A \tag{3.9}$$

$$\partial_{\theta}J_{3}=\beta^{T}M\partial_{\theta}\left(B_{x}\circ A\right)-\beta^{T}C_{y}\partial_{\theta}A \tag{3.10}$$

$$\partial_{\theta}J_{4}=\gamma^{T}M\partial_{\theta}\left(B_{y}\circ A\right)+\gamma^{T}C_{x}\partial_{\theta}A \tag{3.11}$$

$$\begin{aligned}\partial_{\theta}\left(B_{x}\circ A\right):&\quad(...)+\beta^{T}M\partial_{\theta}\left(B_{x}\circ A\right)\\\partial_{\theta}\left(B_{y}\circ A\right):&\quad(...)+\gamma^{T}M\partial_{\theta}\left(B_{y}\circ A\right)\\\partial_{\theta}A:&\quad\alpha^{T}S\partial_{\theta}A-\beta^{T}C_{y}\partial_{\theta}A+\gamma^{T}C_{x}\partial_{\theta}A\\\partial_{\theta}S:&\quad\alpha^{T}\partial_{\theta}SA\end{aligned}$$