

FINITE ELEMENT METHOD FOR MAGNETODYNAMIC APPLICATIONS

2019

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Symbols

\mathbf{A}	$[\text{Wb} \cdot \text{m}^{-1}]$	magnetic vector potential
\mathbf{B}	$[\text{T}]$	magnetic flux density
f	$[\text{Hz}]$	frequency
\mathbf{J}	$[\text{A} \cdot \text{m}^{-2}]$	current density
l	$[\text{m}]$	line
L	$[\text{m}]$	length of the model (z coordinate)
\mathbf{n}	$[-]$	normal vector
S	$[\text{m}^2]$	surface
t	$[\text{s}]$	time
T, \mathbb{T}	$[\text{N} \cdot \text{m}^{-2}]$	Maxwell stress tensor
\mathbf{v}	$[\text{m} \cdot \text{s}^{-1}]$	velocity
V	$[\text{m}^3]$	volume
γ	$[\text{S} \cdot \text{m}^{-1}]$	conductivity
λ	$[-]$	basis function
μ_0	$[\text{H} \cdot \text{m}^{-1}]$	permeability of vacuum
μ	$[\text{H} \cdot \text{m}^{-1}]$	permeability
φ	$[-]$	test function
Ω	$[-]$	closed region
$\partial\Omega$	$[-]$	boundary of closed region

1 Theory

1.1 Magnetodynamic equation

3D Cartesian coordinate system (x, y, z) :

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times \mathbf{A}) \right) + \gamma \frac{\partial \mathbf{A}}{\partial t} - \gamma (\mathbf{v} \times (\nabla \times \mathbf{A})) = \mathbf{J} \quad (1.1)$$

Reduction to 2D (x, y) :

$$\mathbf{A} = (0, 0, A_z) \quad \rightarrow \quad A = A_z \quad (1.2)$$

$$\mathbf{J} = (0, 0, J_z) \quad \rightarrow \quad J = J_z \quad (1.3)$$

$$\mathbf{v} = (v_x, v_y, 0) \quad \rightarrow \quad \mathbf{v} = (v_x, v_y) \quad (1.4)$$

$$\mathbf{B} = \nabla \times \mathbf{A} = (B_x, B_y, 0) \quad \rightarrow \quad \mathbf{B} = (B_x, B_y) \quad (1.5)$$

For 2D, time-dependent problems are A, J, \mathbf{v} and \mathbf{B} functions of coordinates (x, y) and time t . Both A and J are scalar fields $\mathbb{R}^2 \rightarrow \mathbb{R}^1$, \mathbf{v} and \mathbf{B} are vector fields $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.

In a linear material, the μ and the γ are dependent only on the coordinate system (x, y) , in the non-linear material, the μ is also dependent on the size of $\|\mathbf{B}\|$.

$$\nabla = (\partial_x, \partial_y) \quad (1.6)$$

Curl of scalar field $(\nabla \times A)$ is vector field $(\partial_y A, -\partial_x A)$, so $\mathbb{R}^1 \rightarrow \mathbb{R}^2$, but curl of vector field $(\nabla \times \mathbf{B})$ is scalar field $(\partial_x B_y - \partial_y B_x)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

Divergence of vector field $(\nabla \cdot \mathbf{B})$ is also scalar field $(\partial_x B_x + \partial_y B_y)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) + \gamma \frac{\partial A}{\partial t} - \gamma (\mathbf{v} \times (\nabla \times A)) = J \quad (1.7)$$

1.2 Boundary conditions

Dirichlet condition:

$$A \Big|_{\partial\Omega_1} = f_1(x, y) \quad (1.8)$$

Neumann condition:

$$\partial_{\mathbf{n}} A \Big|_{\partial\Omega_2} = f_2(x, y) \quad (1.9)$$

For our model, both f_1 and f_2 are equal to zero.

1.3 Initial condition

$$A \Big|_{t=0} = 0 \quad (1.10)$$

1.4 Force

$$T_{ij} = \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} B^2 \delta_{ij} \right) \quad (1.11)$$

$$\mathbb{T} = \frac{1}{\mu_0} \begin{pmatrix} B_x B_x & B_x B_y \\ B_y B_x & B_y B_y \end{pmatrix} - \frac{1}{2\mu_0} \begin{pmatrix} B_x^2 + B_y^2 & 0 \\ 0 & B_x^2 + B_y^2 \end{pmatrix} \quad (1.12)$$

$$\mathbf{F}(x, y) = L \oint_{\partial\Omega} \mathbb{T} \mathbf{n} \, dl = L \int_{\Omega} (\nabla \cdot \mathbb{T}) \, dS \quad (1.13)$$

Variable L is the length of the model into the third dimension z .

$$\begin{aligned} \nabla \cdot \mathbb{T} &= \frac{1}{\mu_0} \begin{pmatrix} \frac{1}{2} \partial_x (B_x^2 - B_y^2) + \partial_y (B_y B_x) \\ \frac{1}{2} \partial_y (B_y^2 - B_x^2) + \partial_x (B_x B_y) \end{pmatrix} \\ &= \frac{1}{\mu_0} \begin{pmatrix} \partial_x B_x B_x - \partial_x B_y B_y + \partial_y B_y B_x + \partial_y B_x B_y \\ \partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x \end{pmatrix} \end{aligned} \quad (1.14)$$

1.5 Permanent magnets

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times A) - \frac{1}{\mu} \mathbf{B}_r \right) + \gamma \frac{\partial A}{\partial t} - \gamma (\mathbf{v} \times (\nabla \times A)) = J \quad (1.15)$$

Permanent magnets are usually considered as a source of magnetic field, so it is better to move them to the right hand side of the equation.

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) + \gamma \frac{\partial A}{\partial t} - \gamma (\mathbf{v} \times (\nabla \times A)) = J + \frac{1}{\mu} (\nabla \times \mathbf{B}_r) \quad (1.16)$$

2 Weak formulation

To simplify the problem, let us first consider a magnetostatic case ($\gamma = 0$). This is true for models without motion and without time variation of currents.

3 Topology optimization

3.1 Problem formulation in continuous space

$$\begin{aligned}
& \text{minimize} && F_y^p \quad (\text{p - plunger}) \\
& \text{subject to} && -\frac{1}{\mu} \Delta A = J \\
& && B_x = \partial_y A \\
& && B_y = -\partial_x A
\end{aligned}$$

$$F_y^p = (0, 1) \int_{\Omega_p} (\nabla \cdot \mathbb{T}) \, dS = \int_{\Omega_p} \frac{1}{\mu_0} (\partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x) \, dS \quad (3.1)$$

3.2 Problem formulation in discrete space

$$\begin{aligned}
& \text{minimize} && F_y^p \\
& \text{subject to} && SA = MJ \\
& && MB_x = C_y A \\
& && MB_y = -C_x A
\end{aligned}$$

$$F_y^p = \frac{1}{\mu_0} (B_y^\top C_y^p B_y - B_x^\top C_y^p B_x + B_y^\top C_x^p B_x + B_x^\top C_x^p B_y) \quad (3.2)$$

Lagrange multipliers are (α, β, γ) .

$$\begin{aligned}
SA = MJ & \rightarrow \alpha \\
MB_x = C_y A & \rightarrow \beta \\
MB_y = -C_x A & \rightarrow \gamma
\end{aligned}$$

The φ is the topology function.

$$J(\varphi) = J_1(\varphi) + J_2(\varphi) + J_3(\varphi) + J_4(\varphi) \quad (3.3)$$

$$J_1 = \frac{1}{\mu_0} (B_y^\top(\varphi) C_y^p B_y(\varphi) - B_x^\top(\varphi) C_y^p B_x(\varphi) + B_y^\top(\varphi) C_x^p B_x(\varphi) + B_x^\top(\varphi) C_x^p B_y(\varphi)) \quad (3.4)$$

$$J_2 = \alpha^\top (S(\varphi) A(\varphi) - M J(\varphi)) = 0 \quad (3.5)$$

$$J_3 = \beta^\top (M B_x(\varphi) - C_y A(\varphi)) = 0 \quad (3.6)$$

$$J_4 = \gamma^\top (M B_y(\varphi) + C_x A(\varphi)) = 0 \quad (3.7)$$

$$\partial_{\varphi_i} J(\varphi) = \partial_{\varphi_i} J_1(\varphi) + \partial_{\varphi_i} J_2(\varphi) + \partial_{\varphi_i} J_3(\varphi) + \partial_{\varphi_i} J_4(\varphi) \quad (3.8)$$

$$\mu_0 \partial_{\varphi_i} J_1 = \partial_{\varphi_i} (B_y^\top C_y^p B_y) - \partial_{\varphi_i} (B_x^\top C_y^p B_x) + \partial_{\varphi_i} (B_y^\top C_x^p B_x) + \partial_{\varphi_i} (B_x^\top C_x^p B_y) \quad (3.9)$$

$$\partial_{\varphi_i} (B_y^\top C_y^p B_y) = \partial_{\varphi_i} B_y^\top C_y^p B_y + B_y^\top C_y^p \partial_{\varphi_i} B_y = B_y^\top (C_y^p)^\top \partial_{\varphi_i} B_y + B_y^\top C_y^p \partial_{\varphi_i} B_y \quad (3.10)$$

$$\begin{aligned} \partial_{\varphi_i} J_1 = & \frac{1}{\mu_0} (B_x^\top, B_y^\top) \begin{pmatrix} - (C_y^p)^\top - C_y^p \\ (C_x^p)^\top + C_x^p \end{pmatrix} \partial_{\varphi_i} B_x \\ & + \frac{1}{\mu_0} (B_x^\top, B_y^\top) \begin{pmatrix} (C_x^p)^\top + C_x^p \\ (C_y^p)^\top + C_y^p \end{pmatrix} \partial_{\varphi_i} B_y \end{aligned} \quad (3.11)$$

$$\partial_{\varphi_i} J_2 = \alpha^\top \partial_{\varphi_i} (SA) = \alpha^\top \partial_{\varphi_i} S A + \alpha^\top S \partial_{\varphi_i} A \quad (3.12)$$

$$\partial_{\varphi_i} J_3 = \beta^\top M \partial_{\varphi_i} B_x - \beta^\top C_y \partial_{\varphi_i} A \quad (3.13)$$

$$\partial_{\varphi_i} J_4 = \gamma^\top M \partial_{\varphi_i} B_y + \gamma^\top C_x \partial_{\varphi_i} A \quad (3.14)$$

$$\partial_{\varphi_i} B_x : \frac{1}{\mu_0} (B_x^\top, B_y^\top) \begin{pmatrix} -(C_y^p)^\top - C_y^p \\ (C_x^p)^\top + C_x^p \end{pmatrix} \partial_{\varphi_i} B_x + \beta^\top M \partial_{\varphi_i} B_x \quad (3.15)$$

$$\partial_{\varphi_i} B_y : \frac{1}{\mu_0} (B_x^\top, B_y^\top) \begin{pmatrix} (C_x^p)^\top + C_x^p \\ (C_y^p)^\top + C_y^p \end{pmatrix} \partial_{\varphi_i} B_y + \gamma^\top M \partial_{\varphi_i} B_y \quad (3.16)$$

$$\partial_{\varphi_i} A : \alpha^\top S \partial_{\varphi_i} A - \beta^\top C_y \partial_{\varphi_i} A + \gamma^\top C_x \partial_{\varphi_i} A \quad (3.17)$$

$$\partial_{\varphi_i} S : \alpha^\top \partial_{\varphi_i} S A \quad (3.18)$$

If we choose α, β, γ such that (3.15), (3.16) and (3.17) are equal to 0, then:

From (3.15), ($M^\top = M$, $S^\top = S$):

$$\beta = -\frac{1}{\mu_0} M^{-1} \begin{pmatrix} -(C_y^p)^\top - C_y^p \\ (C_x^p)^\top + C_x^p \end{pmatrix}^\top \begin{pmatrix} B_x \\ B_y \end{pmatrix} \quad (3.19)$$

From (3.16):

$$\gamma = -\frac{1}{\mu_0} M^{-1} \begin{pmatrix} (C_x^p)^\top + C_x^p \\ (C_y^p)^\top + C_y^p \end{pmatrix}^\top \begin{pmatrix} B_x \\ B_y \end{pmatrix} \quad (3.20)$$

From (3.17):

$$\alpha = S^{-1} (C_y^\top \beta - C_x^\top \gamma) \quad (3.21)$$

$$\partial_{\varphi_i} J = \alpha^\top \partial_{\varphi_i} S A \quad (3.22)$$

In the linear case the term μ_{Fe} corresponds to $\mu_0 \mu_r$, in nonlinear case, μ_{Fe} is function of A and BH curve of the material. Matrix \hat{S} is no longer dependent on φ .

$$S = \frac{1}{\mu} \hat{S} = ((1 - \varphi) \mu_0 + \varphi^p \mu_{Fe})^{-1} \hat{S} \quad (3.23)$$

Linear case:

$$\partial_{\varphi_i} S = \partial_{\varphi_i} \frac{1}{\mu} \hat{S} = \frac{\mu_0 - p \varphi_i^{p-1} \mu_{Fe}}{((1 - \varphi) \mu_0 + \varphi^p \mu_{Fe})^2} \hat{S} \quad (3.24)$$

If matrix S is dependent on A , the expression (3.12) must be extended. We modify the formula to differentiate with respect to A , which we will use in the future.

$$\partial_{\varphi_i} J_2 = \alpha^\top \partial_{\varphi_i} (SA) = \alpha^\top \partial_{A(\varphi_i)} (SA) \partial_{\varphi_i} A + \alpha^\top \partial_{\varphi_i} S A \quad (3.25)$$

$$\partial_A (SA) = \partial_A S A + S \partial_A A = \partial_A S A + S \quad (3.26)$$

In this case, the BH curve is the curve that returns μ_{Fe} for the given $\|B\|$. To avoid

square root in $\|B\| = \sqrt{B_x^2 + B_y^2}$, the curve is defined directly for $B_x^2 + B_y^2$. The magnetic flux density B_x is equal to $C_y A$ and B_y is equal to $C_x A$, then:

$$\partial_A S = \frac{-\varphi^p \partial_A \mu_{Fe} (2C_y A C_y + 2C_x A C_x)}{((1 - \varphi) \mu_0 + \varphi^p \mu_{Fe})^2} \hat{S} \quad (3.27)$$

New α :

$$\begin{aligned} \partial_{\varphi_i} A : \quad & \partial_A S A \partial_{\varphi_i} A + S \partial_{\varphi_i} A - \beta^\top C_y \partial_{\varphi_i} A + \gamma^\top C_x \partial_{\varphi_i} A \\ & \alpha = (\partial_A S A + S)^{-1} (C_y^\top \beta - C_x^\top \gamma) \end{aligned} \quad (3.28)$$

Motor:

3.3 Problem formulation in continuous space

$$\begin{aligned} & \text{minimize} \quad T^r \quad (\text{r} - \text{rotor}) \\ & \text{subject to} \quad -\frac{1}{\mu} \Delta A = J \\ & \quad \quad \quad B_x = \partial_y A \\ & \quad \quad \quad B_y = -\partial_x A \end{aligned}$$

$$T^r = \frac{L}{\mu_0 (r_{out} - r_{in})} \int_{\Omega_d} r \left(B_x \frac{x}{r} + B_y \frac{y}{r} \right) \left(-B_x \frac{y}{r} + B_y \frac{x}{r} \right) dS \quad (3.29)$$

3.4 Problem formulation in discrete space

$$\begin{aligned} & \text{minimize} \quad T^r \\ & \text{subject to} \quad SA = MJ \\ & \quad \quad \quad MB_x = C_y A \\ & \quad \quad \quad MB_y = -C_x A \end{aligned}$$

$$T^r = \frac{L}{\mu_0 (r_{out} - r_{in})} r \left(\left(B_x \frac{x}{r} + B_y \frac{y}{r} \right)^\top M^r \left(-B_x \frac{y}{r} + B_y \frac{x}{r} \right) \right) \quad (3.30)$$

$$J_1 = \frac{L}{\mu_0 (r_{out} - r_{in})} r \left(\left(B_x(\varphi) \frac{x}{r} + B_y(\varphi) \frac{y}{r} \right)^\top M^r \left(-B_x(\varphi) \frac{y}{r} + B_y(\varphi) \frac{x}{r} \right) \right) \quad (3.31)$$

$$\begin{aligned} \frac{\mu_0 (r_{out} - r_{in})}{L r} J_1 &= \left(B_x(\varphi) \frac{x}{r} + B_y(\varphi) \frac{y}{r} \right)^\top M^r \left(-B_x(\varphi) \frac{y}{r} + B_y(\varphi) \frac{x}{r} \right) = \\ &- \left(B_x \frac{x}{r} \right)^\top M^r \left(B_x \frac{y}{r} \right) + \left(B_x \frac{x}{r} \right)^\top M^r \left(B_y \frac{x}{r} \right) - \\ &\left(B_y \frac{y}{r} \right)^\top M^r \left(B_x \frac{y}{r} \right) + \left(B_y \frac{y}{r} \right)^\top M^r \left(B_y \frac{x}{r} \right) = \\ &- \left(\frac{x}{r} \right)^\top B_x^\top M^r \left(\frac{y}{r} \right) B_x + \left(\frac{x}{r} \right)^\top B_x^\top M^r \left(\frac{x}{r} \right) B_y - \\ &\left(\frac{y}{r} \right)^\top B_y^\top M^r \left(\frac{y}{r} \right) B_x + \left(\frac{y}{r} \right)^\top B_y^\top M^r \left(\frac{x}{r} \right) B_y \end{aligned}$$

$$\begin{aligned} \frac{\mu_0 (r_{out} - r_{in})}{L r} \partial_{\varphi_i} J_1 &= - \left(\frac{x}{r} \right)^\top \partial_{\varphi_i} B_x^\top M^r \left(\frac{y}{r} \right) B_x - \left(\frac{x}{r} \right)^\top B_x^\top M^r \left(\frac{y}{r} \right) \partial_{\varphi_i} B_x + \\ &\left(\frac{x}{r} \right)^\top \partial_{\varphi_i} B_x^\top M^r \left(\frac{x}{r} \right) B_y + \left(\frac{x}{r} \right)^\top B_x^\top M^r \left(\frac{x}{r} \right) \partial_{\varphi_i} B_y - \\ &\left(\frac{y}{r} \right)^\top \partial_{\varphi_i} B_y^\top M^r \left(\frac{y}{r} \right) B_x - \left(\frac{y}{r} \right)^\top B_y^\top M^r \left(\frac{y}{r} \right) \partial_{\varphi_i} B_x + \\ &\left(\frac{y}{r} \right)^\top \partial_{\varphi_i} B_y^\top M^r \left(\frac{x}{r} \right) B_y + \left(\frac{y}{r} \right)^\top B_y^\top M^r \left(\frac{x}{r} \right) \partial_{\varphi_i} B_y \end{aligned}$$

4 Nonlinear solver

Linearization:

$$y = \partial_x f(x - x_n) + f(x_n) \quad (4.1)$$

$$0 = SA - MJ - CB_r = SA - f \quad (4.2)$$

$$\begin{aligned} 0 &= \partial_A (SA_n - f) (A_{n+1} - A_n) + (SA_n - f) \\ A_{n+1} &= A_n - \partial_A (SA_n - f)^{-1} (SA_n - f) \end{aligned}$$

Right hand side f is not dependent on A , $\partial_A f = 0$.

$$A_{n+1} = A_n - \partial_A (SA_n)^{-1} (SA_n - f) \quad (4.3)$$

Derivative $\partial_A (SA)$ was already derived in chapter 3, (3.26) and (3.27).

5 Robustness

Approximation of $B - \mu$ curve by Weibull distribution :

$$\mu = \mu_0 + a_3 (B - a_1)^{a_2-1} e^{-(B-a_1)^{a_2}} \quad (5.1)$$

$$\partial_B \mu = a_3 e^{-(B-a_1)^{a_2}} \left((a_2 - 1) (B - a_1)^{a_2-2} - a_2 (B - a_1)^{2a_2-2} \right) \quad (5.2)$$

Derivatives of S with respect to parameters of Weibull distribution:

$$\partial_{a_i} S = \frac{-\varphi^p \partial_{a_i} \mu_{Fe}}{((1 - \varphi) \mu_0 + \varphi^p \mu_{Fe})^2} \hat{S} \quad (5.3)$$

$$\begin{aligned} \partial_{a_1} \mu_{Fe} &= a_3 e^{-(B-a_1)^{a_2}} \left(a_2 (B - a_1)^{2a_2-2} - (a_2 - 1) (B - a_1)^{a_2-2} \right) \\ \partial_{a_2} \mu_{Fe} &= \ln(B - a_1) a_3 e^{-(B-a_1)^{a_2}} \left((B - a_1)^{a_2-1} - (x - a_1)^{2a_2-1} \right) \\ \partial_{a_3} \mu_{Fe} &= e^{-(B-a_1)^{a_2}} (B - a_1)^{a_2-1} \end{aligned}$$

Appendix

Weak formulation

For $\gamma = 0$.

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) = J \quad (5.4)$$

The θ is test function.

$$\int_{\Omega} \nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) \theta \, dS = \int_{\Omega} J \theta \, dS \quad (5.5)$$

$$\begin{aligned} \nabla \times A &= (\partial_y A, -\partial_x A) = \mathbf{B} \\ \nabla \times \mathbf{B} &= \partial_x B_y - \partial_y B_x = -\partial_x \partial_x A - \partial_y \partial_y A = -\Delta A \end{aligned} \quad (5.6)$$

$$\int_{\Omega} -\frac{1}{\mu} \Delta A \theta \, dS = \int_{\Omega} J \theta \, dS \quad (5.7)$$

Green's first identity:

$$\int_{\Omega} \Delta F \theta \, dS + \int_{\Omega} \nabla F \nabla \theta \, dS = \int_{\partial\Omega} (\nabla F \mathbf{n}) \theta \, dl \quad (5.8)$$

$$\int_{\Omega} -\frac{1}{\mu} \Delta A \theta \, dS = \int_{\Omega} \frac{1}{\mu} \nabla A \nabla \theta \, dS - \int_{\partial\Omega_1} \frac{1}{\mu} (\nabla A \mathbf{n}) \theta \, dl - \int_{\partial\Omega_2} \frac{1}{\mu} (\nabla A \mathbf{n}) \theta \, dl \quad (5.9)$$

$$\theta \in H^1 : \theta|_{\partial\Omega_1} = 0, \theta|_{\partial\Omega_2} = 0 \rightarrow \int_{\partial\Omega_1} \frac{1}{\mu} (\nabla A \mathbf{n}) \theta \, dl = 0, \int_{\partial\Omega_2} \frac{1}{\mu} (\nabla A \mathbf{n}) \theta \, dl = 0$$

Discretization

$$A(x, y, t) \approx \sum_{n=1}^N A^n(t) \lambda^n(x, y) \quad (5.10)$$

$$J(x, y, t) \approx \sum_{n=1}^N J^n(t) \lambda^n(x, y) \quad (5.11)$$

$$\theta(x, y, t) \approx \sum_{n=1}^N \theta^n(t) \lambda^n(x, y) \quad (5.12)$$

$$\int_{T_k} \frac{1}{\mu} \nabla \left(\sum_{n=1}^N A^n \lambda^n \right) \nabla \left(\sum_{n=1}^N \theta^n \lambda^n \right) dS = \int_{T_k} \sum_{n=1}^N J^n \lambda^n \sum_{n=1}^N \theta^n \lambda^n dS \quad (5.13)$$

T_k is one discrete element of geometry, in our case it is a triangle.

$$\begin{aligned} \sum_{n=1}^N \int_{T_k} \frac{1}{\mu} \nabla (A^n \lambda^n) \nabla (\theta^m \lambda^m) dS &= \sum_{n=1}^N \int_{T_k} J^n \lambda^n \theta^m \lambda^m dS \\ \sum_{n=1}^N A^n \int_{T_k} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m dS &= \sum_{n=1}^N J^n \int_{T_k} \lambda^n \lambda^m dS \end{aligned} \quad (5.14)$$

We can put A^n and J^n out of ∇ and out of \int_{T_k} , because they no longer depend on coordinate system.

$$\begin{aligned} S &= \sum_{n=1}^N \int_{T_k} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m dS \\ M &= \sum_{n=1}^N \int_{T_k} \lambda^n \lambda^m dS \\ SA = MJ &\rightarrow A = SMJ \end{aligned} \quad (5.15)$$

If we discretize the problem with triangular elements and select the first-order polynomial (linear function), as an approximation of the scalar field A , we get three basis functions for each triangle. The coordinates (r, s) correspond to the reference triangle with vertices $V_1 = (0, 0)$, $V_2 = (0, 1)$ and $V_3 = (1, 1)$ numbered in counter-clockwise direction.

$$\beta_1 = 1 - r - s \quad (5.16)$$

$$\beta_2 = r \quad (5.17)$$

$$\beta_3 = s \quad (5.18)$$

If we differentiate them in respect to coordinates, we get:

$$\nabla\beta = \begin{pmatrix} \partial_r\beta_1 & \partial_r\beta_2 & \partial_r\beta_3 \\ \partial_s\beta_1 & \partial_s\beta_2 & \partial_s\beta_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (5.19)$$

The matrices, fields and functions of the reference triangle must then be transformed into our coordinate system $(r, s) \rightarrow (x, y)$ and put into right place. Vertices of the triangle in (x, y) coordinates are $V_1 = (x_1, y_1)$, $V_2 = (x_2, y_2)$ and $V_3 = (x_3, y_3)$.

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \quad (5.20)$$

The Φ is the transformation function. We can use it to transform basis functions of reference triangle T_r to basis functions of the n-th triangle T_k .

$$\lambda(x, y) = (\beta \circ \Phi^{-1})(x, y) \quad (5.21)$$

We will also need the inverse Jacobian matrix and the “Jacobian”, ie the determinant of the Jacobian matrix.

$$|\det(\nabla\Phi)| = \left| \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \right| \quad (5.22)$$

$$\nabla\Phi^{-1} = \frac{1}{|\det(\nabla\Phi)|} \begin{pmatrix} y_3 - y_1 & -(x_3 - x_1) \\ -(y_2 - y_1) & x_2 - x_1 \end{pmatrix} \quad (5.23)$$

Mass matrix:

$$\begin{aligned} M_{T_k} &= \int_{T_k} \lambda^n \lambda^m dS = \int_{T_k} (\beta^n \circ \Phi^{-1})(\beta^m \circ \Phi^{-1}) dS \\ &= |\det(\nabla\Phi)| \int_{T_r} \beta^n \beta^m dS \end{aligned} \quad (5.24)$$

$$M_{T_k} = |\det(\nabla\Phi)| \begin{pmatrix} \int \beta_1\beta_1 dS & \int \beta_1\beta_2 dS & \int \beta_1\beta_3 dS \\ \int \beta_2\beta_1 dS & \int \beta_2\beta_2 dS & \int \beta_2\beta_3 dS \\ \int \beta_3\beta_1 dS & \int \beta_3\beta_2 dS & \int \beta_3\beta_3 dS \end{pmatrix} \quad (5.25)$$

$$M = \sum_{n=1}^N |\det(\nabla\Phi)| \begin{pmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{pmatrix} \quad (5.26)$$

Stiffness matrix:

$$\begin{aligned} S_{T_k} &= \int_{T_k} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m \, dS = \int_{T_k} \frac{1}{\mu} \nabla (\beta^n \circ \Phi^{-1}) \nabla (\beta^m \circ \Phi^{-1}) \, dS \\ &= \int_{T_k} \frac{1}{\mu} \partial_x (\beta^n \circ \Phi^{-1}) \partial_x (\beta^m \circ \Phi^{-1}) \, dS + \int_{T_k} \frac{1}{\mu} \partial_y (\beta^n \circ \Phi^{-1}) \partial_y (\beta^m \circ \Phi^{-1}) \, dS \\ &= S_{T_k,x} + S_{T_k,y} \end{aligned} \quad (5.27)$$

$$\begin{aligned} S_{T_k,x} &= \int_{T_k} \frac{1}{\mu} (\partial_x \beta^n (\Phi^{-1})) (\partial_x \Phi^{-1}) (\partial_x \beta^n (\Phi^{-1})) (\partial_x \Phi^{-1}) \, dS \\ &= \frac{1}{\mu} |\det(\nabla\Phi)| (1,0) \int_{T_r} \nabla \Phi^{-1} \nabla \beta^\top \nabla \Phi^{-\top} \nabla \beta \, dS \end{aligned} \quad (5.28)$$

$$\begin{aligned} S &= \sum_{n=1}^N \frac{1}{2\mu} |\det(\nabla\Phi)| (1,0) \nabla \Phi^{-1} \nabla \beta^\top \nabla \Phi^{-\top} \nabla \beta \\ &\quad + \sum_{n=1}^N \frac{1}{2\mu} |\det(\nabla\Phi)| (0,1) \nabla \Phi^{-1} \nabla \beta^\top \nabla \Phi^{-\top} \nabla \beta \end{aligned} \quad (5.29)$$

If we want to get the vector field \mathbf{B} from the scalar field A , we need a “curl matrix”.

$$\mathbf{B} = \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} \partial_y A \\ -\partial_x A \end{pmatrix} \quad (5.30)$$

$$\int_{\Omega} B_x \theta \, dS = \int_{\Omega} \partial_y A \theta \, dS \quad (5.31)$$

Integration by parts:

$$\int_{\Omega} \partial F \theta \, dS = \int_{\partial\Omega} F \theta \mathbf{n} \, dl - \int_{\Omega} F \partial \theta \, dS \quad (5.32)$$

$$\int_{\Omega} B_x \theta \, dS = \int_{\Omega} \partial_y A \theta \, dS = \int_{\partial\Omega} A \theta \mathbf{n} \, dl - \int_{\Omega} A \partial_y \theta \, dS \quad (5.33)$$

Constraints for θ ...

$$\int_{\Omega} \sum_{n=1}^N B_x \lambda^n \sum_{n=1}^N \theta^n \lambda^n \, dS = - \int_{\Omega} \sum_{n=1}^N A^n \lambda^n \partial_y \left(\sum_{n=1}^N \theta^n \lambda^n \right) \, dS \quad (5.34)$$

$$\sum_{n=1}^N B_x^n \int_{\Omega} \lambda^n \lambda^m \, dS = - \sum_{n=1}^N A^n \int_{\Omega} \lambda^n \partial_y \lambda^m \, dS \quad (5.35)$$

$$MB_x = -C_y A \rightarrow B_x = -MC_y A \quad (5.36)$$

“Curl matrix”:

$$\begin{aligned} C_{T_k, y} &= \int_{T_k} \lambda^n \partial_y \lambda^m \, dS = \int_{T_k} (\beta^n \circ \Phi^{-1}) \partial_y (\beta^m \circ \Phi^{-1}) \, dS \\ &= \int_{T_k} (\beta^n \circ \Phi^{-1}) (\partial_y \beta^n (\Phi^{-1})) (\partial_x \Phi^{-1}) \, dS \\ &= |\det(\nabla \Phi)| (1, 0) \int_{T_r} \beta \nabla \Phi^{-\top} \nabla \beta \, dS \end{aligned} \quad (5.37)$$

We got $\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$, because $\int \beta \, dS = \frac{1}{3}$.

$$\begin{aligned} C_x &= \sum_{n=1}^N \frac{1}{6} |\det(\nabla \Phi)| (0, 1) \nabla \Phi^{-\top} \nabla \beta \\ C_y &= - \sum_{n=1}^N \frac{1}{6} |\det(\nabla \Phi)| (1, 0) \nabla \Phi^{-\top} \nabla \beta \end{aligned} \quad (5.38)$$

This matrix is also used when permanent magnets are considered as a another magnetic field source. Magnets are represented by their remanent magnetic field density \mathbf{B}_r and in the magnetic field equation they appear as $\nabla \times \mathbf{B}_r$. The right hand side of the equation with both coils and permanent magnets is then: $MJ + (C_y B_{r_x} + C_x B_{r_y})$.