

FINITE ELEMENT METHOD FOR MAGNETODYNAMIC APPLICATIONS

2019

Contents

1	Theory	4
1.1	Magnetodynamic equation	4
1.2	Boundary conditions	4
1.3	Initial condition	5
1.4	Force	5
2	Weak formulation	6
3	Topology optimization	7
3.1	Problem formulation in continuous space	7
3.2	Problem formulation in discrete space	7
	Appendix	10

Symbols

\mathbf{A}	$[\text{Wb} \cdot \text{m}^{-1}]$	magnetic vector potential
\mathbf{B}	$[\text{T}]$	magnetic flux density
f	$[\text{Hz}]$	frequency
\mathbf{J}	$[\text{A} \cdot \text{m}^{-2}]$	current density
l	$[\text{m}]$	line
L	$[\text{m}]$	length of the model (z coordinate)
\mathbf{n}	$[-]$	normal vector
S	$[\text{m}^2]$	surface
t	$[\text{s}]$	time
T, \mathbb{T}	$[\text{N} \cdot \text{m}^{-2}]$	Maxwell stress tensor
\mathbf{v}	$[\text{m} \cdot \text{s}^{-1}]$	velocity
V	$[\text{m}^3]$	volume
γ	$[\text{S} \cdot \text{m}^{-1}]$	conductivity
λ	$[-]$	basis function
μ_0	$[\text{H} \cdot \text{m}^{-1}]$	permeability of vacuum
μ	$[\text{H} \cdot \text{m}^{-1}]$	permeability
φ	$[-]$	test function
Ω	$[-]$	closed region
$\partial\Omega$	$[-]$	boundary of closed region

1 Theory

1.1 Magnetodynamic equation

3D Cartesian coordinate system (x, y, z) :

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times \mathbf{A}) \right) + \gamma \frac{\partial \mathbf{A}}{\partial t} - \gamma (\mathbf{v} \times (\nabla \times \mathbf{A})) = \mathbf{J} \quad (1.1)$$

Reduction to 2D (x, y) :

$$\mathbf{A} = (0, 0, A_z) \quad \rightarrow \quad A = A_z \quad (1.2)$$

$$\mathbf{J} = (0, 0, J_z) \quad \rightarrow \quad J = J_z \quad (1.3)$$

$$\mathbf{v} = (v_x, v_y, 0) \quad \rightarrow \quad \mathbf{v} = (v_x, v_y) \quad (1.4)$$

$$\mathbf{B} = \nabla \times \mathbf{A} = (B_x, B_y, 0) \quad \rightarrow \quad \mathbf{B} = (B_x, B_y) \quad (1.5)$$

For 2D, time-dependent problems are A, J, \mathbf{v} and \mathbf{B} functions of coordinates (x, y) and time t . Both A and J are scalar fields $\mathbb{R}^2 \rightarrow \mathbb{R}^1$, \mathbf{v} and \mathbf{B} are vector fields $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.

In a linear material, the μ and the γ are dependent only on the coordinate system (x, y) , in the non-linear material, the μ is also dependent on the size of $\|\mathbf{B}\|$.

$$\nabla = (\partial_x, \partial_y) \quad (1.6)$$

Curl of scalar field $(\nabla \times A)$ is vector field $(\partial_y A, -\partial_x A)$, so $\mathbb{R}^1 \rightarrow \mathbb{R}^2$, but curl of vector field $(\nabla \times \mathbf{B})$ is scalar field $(\partial_x B_y - \partial_y B_x)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

Divergence of vector field $(\nabla \cdot \mathbf{B})$ is also scalar field $(\partial_x B_x + \partial_y B_y)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) + \gamma \frac{\partial A}{\partial t} - \gamma (\mathbf{v} \times (\nabla \times A)) = J \quad (1.7)$$

1.2 Boundary conditions

Dirichlet condition:

$$A \Big|_{\partial\Omega_1} = f_1(x, y) \quad (1.8)$$

Neumann condition:

$$\partial_{\mathbf{n}} A \Big|_{\partial\Omega_2} = f_2(x, y) \quad (1.9)$$

For our model, both f_1 and f_2 are equal to zero.

1.3 Initial condition

$$A \Big|_{t=0} = 0 \quad (1.10)$$

1.4 Force

$$T_{ij} = \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} B^2 \delta_{ij} \right) \quad (1.11)$$

$$\mathbb{T} = \frac{1}{\mu_0} \begin{pmatrix} B_x B_x & B_x B_y \\ B_y B_x & B_y B_y \end{pmatrix} - \frac{1}{2\mu_0} \begin{pmatrix} B_x^2 + B_y^2 & 0 \\ 0 & B_x^2 + B_y^2 \end{pmatrix} \quad (1.12)$$

$$\mathbf{F}(x, y) = L \oint_{\partial\Omega} \mathbb{T} \mathbf{n} \, dl = L \int_{\Omega} (\nabla \cdot \mathbb{T}) \, dS \quad (1.13)$$

Variable L is the length of the model into the third dimension z .

$$\begin{aligned} \nabla \cdot \mathbb{T} &= \frac{1}{\mu_0} \begin{pmatrix} \frac{1}{2} (B_x^2 - B_y^2) & B_x B_y \\ B_y B_x & \frac{1}{2} (B_y^2 - B_x^2) \end{pmatrix}^\top \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \\ &= \frac{1}{\mu_0} \begin{pmatrix} \frac{1}{2} \partial_x (B_x^2 - B_y^2) + \partial_y (B_y B_x) \\ \frac{1}{2} \partial_y (B_y^2 - B_x^2) + \partial_x (B_x B_y) \end{pmatrix} \\ &= \frac{1}{\mu_0} \begin{pmatrix} \partial_x B_x B_x - \partial_x B_y B_y + \partial_y B_y B_x + \partial_y B_x B_y \\ \partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x \end{pmatrix} \end{aligned} \quad (1.14)$$

2 Weak formulation

To simplify the problem, let us first consider a magnetostatic case ($\gamma = 0$). This is true for models without motion and without time variation of currents.

3 Topology optimization

3.1 Problem formulation in continuous space

$$\begin{aligned}
& \text{minimize} && F_y^p \quad (\text{p - plunger}) \\
& \text{subject to} && \nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) = J \\
& && B_x = \partial_y A \\
& && B_y = -\partial_x A
\end{aligned}$$

$$F_y^p = (0, 1) \int_{\Omega_p} (\nabla \cdot \mathbb{T}) \, dS = \int_{\Omega_p} \frac{1}{\mu_0} (\partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x) \, dS \quad (3.1)$$

3.2 Problem formulation in discrete space

$$\begin{aligned}
& \text{minimize} && F_y^p \\
& \text{subject to} && SA = MJ \\
& && MB_x = C_y A \\
& && MB_y = -C_x A
\end{aligned}$$

$$F_y^p = \frac{1}{\mu_0} (B_y^\top C_y^p B_y - B_x^\top C_y^p B_x + B_y^\top C_x^p B_x + B_x^\top C_x^p B_y) \quad (3.2)$$

Lagrange multipliers are (α, β, γ) .

$$\begin{aligned}
SA = MJ & \rightarrow \alpha \\
MB_x = C_y A & \rightarrow \beta \\
MB_y = -C_x A & \rightarrow \gamma
\end{aligned}$$

The φ_i is the topology function.

$$J(\varphi_i) = J_1(\varphi_i) + J_2(\varphi_i) + J_3(\varphi_i) + J_4(\varphi_i) \quad (3.3)$$

$$J_1 = \frac{1}{\mu_0} (B_y^\top C_y^p B_y - B_x^\top C_y^p B_x + B_y^\top C_x^p B_x + B_x^\top C_x^p B_y) \quad (3.4)$$

$$J_2 = \alpha^\top (SA - MJ) = 0 \quad (3.5)$$

$$J_3 = \beta^\top (MB_x - C_y A) = 0 \quad (3.6)$$

$$J_4 = \gamma^\top (MB_y + C_x A) = 0 \quad (3.7)$$

$$\partial_{\varphi_i} J(\varphi_i) = \partial_{\varphi_i} J_1(\varphi_i) + \partial_{\varphi_i} J_2(\varphi_i) + \partial_{\varphi_i} J_3(\varphi_i) + \partial_{\varphi_i} J_4(\varphi_i) \quad (3.8)$$

$$\mu_0 \partial_{\varphi_i} J_1 = \partial_{\varphi_i} (B_y^\top C_y^p B_y) - \partial_{\varphi_i} (B_x^\top C_y^p B_x) + \partial_{\varphi_i} (B_y^\top C_x^p B_x) + \partial_{\varphi_i} (B_x^\top C_x^p B_y) \quad (3.9)$$

$$\partial_{\varphi_i} (B_y^\top C_y^p B_y) = \partial_{\varphi_i} B_y^\top C_y^p B_y + B_y^\top C_y^p \partial_{\varphi_i} B_y = B_y^\top (C_y^p)^\top \partial_{\varphi_i} B_y + B_y^\top C_y^p \partial_{\varphi_i} B_y \quad (3.10)$$

$$\begin{aligned} \partial_{\varphi_i} J_1 = & \frac{1}{\mu_0} (B_x^\top, B_y^\top) \begin{pmatrix} -(C_y^p)^\top & -C_y^p \\ (C_x^p)^\top & C_x^p \end{pmatrix} \begin{pmatrix} \partial_{\varphi_i} B_x \\ \partial_{\varphi_i} B_y \end{pmatrix} \\ & + \frac{1}{\mu_0} (B_x^\top, B_y^\top) \begin{pmatrix} (C_x^p)^\top & C_x^p \\ (C_y^p)^\top & C_y^p \end{pmatrix} \begin{pmatrix} \partial_{\varphi_i} B_x \\ \partial_{\varphi_i} B_y \end{pmatrix} \end{aligned} \quad (3.11)$$

$$\partial_{\varphi_i} J_2 = \alpha^\top \partial_{\varphi_i} (SA) - \alpha^\top \partial_{\varphi_i} (MJ) = \alpha^\top \partial_{\varphi_i} SA + \alpha^\top S \partial_{\varphi_i} A \quad (3.12)$$

$$\partial_{\varphi_i} J_3 = \beta^\top M \partial_{\varphi_i} B_x - \beta^\top C_y \partial_{\varphi_i} A \quad (3.13)$$

$$\partial_{\varphi_i} J_4 = \gamma^\top M \partial_{\varphi_i} B_y + \gamma^\top C_x \partial_{\varphi_i} A \quad (3.14)$$

$$\partial_{\varphi_i} B_x : \quad \frac{1}{\mu_0} (B_x^\top, B_y^\top) \begin{pmatrix} -(C_y^p)^\top & -C_y^p \\ (C_x^p)^\top & C_x^p \end{pmatrix} \begin{pmatrix} \partial_{\varphi_i} B_x \\ \partial_{\varphi_i} B_y \end{pmatrix} + \beta^\top M \partial_{\varphi_i} B_x \quad (3.15)$$

$$\partial_{\varphi_i} B_y : \quad \frac{1}{\mu_0} (B_x^\top, B_y^\top) \begin{pmatrix} (C_x^p)^\top & C_x^p \\ (C_y^p)^\top & C_y^p \end{pmatrix} \begin{pmatrix} \partial_{\varphi_i} B_x \\ \partial_{\varphi_i} B_y \end{pmatrix} + \gamma^\top M \partial_{\varphi_i} B_y \quad (3.16)$$

$$\partial_{\varphi_i} A : \quad \alpha^\top S \partial_{\varphi_i} A - \beta^\top C_y \partial_{\varphi_i} A + \gamma^\top C_x \partial_{\varphi_i} A \quad (3.17)$$

$$\partial_{\varphi_i} S : \quad \alpha^\top \partial_{\varphi_i} S A \quad (3.18)$$

From (3.15):

$$\beta = -\frac{1}{\mu_0} (M^\top)^{-1} (B_x^\top, B_y^\top) \begin{pmatrix} -(C_y^p)^\top & -C_y^p \\ (C_x^p)^\top & C_x^p \end{pmatrix} \quad (3.19)$$

From (3.16):

$$\gamma = -\frac{1}{\mu_0} (M^\top)^{-1} (B_x^\top, B_y^\top) \begin{pmatrix} (C_x^p)^\top & C_x^p \\ (C_y^p)^\top & C_y^p \end{pmatrix} \quad (3.20)$$

From (3.17):

$$\alpha = (S^\top)^{-1} C_y^\top \beta - (S^\top)^{-1} C_x^\top \gamma \quad (3.21)$$

If we choose α, β, γ such that (3.15), (3.16) and (3.17) are equal to 0, then:

$$\partial_{\varphi_i} J = \alpha^\top \partial_{\varphi_i} S A \quad (3.22)$$

Appendix

Weak formulation

For $\gamma = 0$.

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) = J \quad (3.23)$$

The θ is test function.

$$\int_{\Omega} \nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) \theta \, dS = \int_{\Omega} J \theta \, dS \quad (3.24)$$

$$\begin{aligned} \nabla \times A &= (\partial_y A, -\partial_x A) = \mathbf{B} \\ \nabla \times \mathbf{B} &= \partial_x B_y - \partial_y B_x = -\partial_x \partial_x A - \partial_y \partial_y A = -\Delta A \end{aligned} \quad (3.25)$$

$$\int_{\Omega} -\frac{1}{\mu} \Delta A \theta \, dS = \int_{\Omega} J \theta \, dS \quad (3.26)$$

Green's first identity:

$$\int_{\Omega} \Delta F \theta \, dS + \int_{\Omega} \nabla F \nabla \theta \, dS = \int_{\partial\Omega} (\nabla F \mathbf{n}) \theta \, dl \quad (3.27)$$

$$\int_{\Omega} -\frac{1}{\mu} \Delta A \theta \, dS = \int_{\Omega} \frac{1}{\mu} \nabla A \nabla \theta \, dS - \int_{\partial\Omega_1} \frac{1}{\mu} (\nabla A \mathbf{n}) \theta \, dl - \int_{\partial\Omega_2} \frac{1}{\mu} (\nabla A \mathbf{n}) \theta \, dl \quad (3.28)$$

$$\theta \in f(\Omega) : \theta|_{\partial\Omega_1} = 0, \theta|_{\partial\Omega_2} = 0 \rightarrow \int_{\partial\Omega_1} \frac{1}{\mu} (\nabla A \mathbf{n}) \theta \, dl = 0, \int_{\partial\Omega_2} \frac{1}{\mu} (\nabla A \mathbf{n}) \theta \, dl = 0$$

Discretization

$$A(x, y, t) \approx \sum_{n=1}^N A^n(t) \lambda^n(x, y) \quad (3.29)$$

$$J(x, y, t) \approx \sum_{n=1}^N J^n(t) \lambda^n(x, y) \quad (3.30)$$

$$\theta(x, y, t) \approx \sum_{n=1}^N \theta^n(t) \lambda^n(x, y) \quad (3.31)$$

$$\int_{T_n} \frac{1}{\mu} \nabla \left(\sum_{n=1}^N A^n \lambda^n \right) \nabla \left(\sum_{n=1}^N \theta^n \lambda^n \right) dS = \int_{T_n} \sum_{n=1}^N J^n \lambda^n \sum_{n=1}^N \theta^n \lambda^n dS \quad (3.32)$$

T_n is one discrete element of geometry, in our case it is a triangle.

$$\begin{aligned} \sum_{n=1}^N \int_{T_n} \frac{1}{\mu} \nabla (A^n \lambda^n) \nabla (\theta^m \lambda^m) dS &= \sum_{n=1}^N \int_{T_n} J^n \lambda^n \theta^m \lambda^m dS \\ \sum_{n=1}^N A^n \int_{T_n} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m dS &= \sum_{n=1}^N J^n \int_{T_n} \lambda^n \lambda^m dS \end{aligned} \quad (3.33)$$

We can put A^n and J^n out of ∇ and out of \int_{T_n} , because they no longer depend on coordinate system.

$$\begin{aligned} S &= \sum_{n=1}^N \int_{T_n} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m dS \\ M &= \sum_{n=1}^N \int_{T_n} \lambda^n \lambda^m dS \end{aligned}$$

$$SA = MJ \rightarrow A = S^\top MJ \quad (3.34)$$

If we discretize the problem with triangular elements and select the first-order polynomial (linear function), as an approximation of the scalar field A , we get three basis functions for each triangle. The coordinates (r, s) correspond to the reference triangle with vertices $V_1 = (0, 0)$, $V_2 = (0, 1)$ and $V_3 = (1, 1)$ numbered in counter-clockwise direction.

$$\beta_1 = 1 - r - s \quad (3.35)$$

$$\beta_2 = r \quad (3.36)$$

$$\beta_3 = s \quad (3.37)$$

If we differentiate them in respect to coordinates, we get:

$$\partial \beta = \begin{pmatrix} \partial_r \beta_1 & \partial_r \beta_2 & \partial_r \beta_3 \\ \partial_s \beta_1 & \partial_s \beta_2 & \partial_s \beta_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (3.38)$$

The matrices, fields and functions of the reference triangle must then be transformed into our coordinate system $(r, s) \rightarrow (x, y)$ and put into right place. Vertices of the triangle in (x, y) coordinates are $V_1 = (x_1, y_1)$, $V_2 = (x_2, y_2)$ and $V_3 = (x_3, y_3)$.

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \quad (3.39)$$

The Φ is the transformation function. We can use it to transform basis functions of reference triangle T_r to basis functions of the n -th triangle T_n .

$$\lambda(x, y) = (\beta \circ \Phi^{-1})(x, y) \quad (3.40)$$

We will also need the inverse Jacobian matrix and the “Jacobian”, ie the determinant of the Jacobian matrix.

$$|\det(\nabla\Phi)| = \left| \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \right| \quad (3.41)$$

$$(\nabla\Phi)^{-1} = \frac{1}{|\det(\nabla\Phi)|} \begin{pmatrix} y_3 - y_1 & -(x_3 - x_1) \\ -(y_2 - y_1) & x_2 - x_1 \end{pmatrix} \quad (3.42)$$

Mass matrix:

$$\begin{aligned} M_{T_n} &= \int_{T_n} \lambda^n \lambda^m \, dS = \int_{T_n} (\beta \circ \Phi^{-1})^n (\beta \circ \Phi^{-1})^m \, dS \\ &= |\det(\nabla\Phi)| \int_{T_r} \beta^n \beta^m \, dS \end{aligned} \quad (3.43)$$

$$M_{T_n} = |\det(\nabla\Phi)| \begin{pmatrix} \int \beta_1 \beta_1 \, dS & \int \beta_1 \beta_2 \, dS & \int \beta_1 \beta_3 \, dS \\ \int \beta_2 \beta_1 \, dS & \int \beta_2 \beta_2 \, dS & \int \beta_2 \beta_3 \, dS \\ \int \beta_3 \beta_1 \, dS & \int \beta_3 \beta_2 \, dS & \int \beta_3 \beta_3 \, dS \end{pmatrix} \quad (3.44)$$

$$M = \sum_{n=1}^N |\det(\nabla\Phi)| \begin{pmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{pmatrix} \quad (3.45)$$

Stiffness matrix:

$$\begin{aligned} S_{T_n} &= \int_{T_n} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m \, dS = \int_{T_n} \frac{1}{\mu} \nabla (\beta \circ \Phi^{-1})^n \nabla (\beta \circ \Phi^{-1})^m \, dS \\ &= \int_{T_n} \frac{1}{\mu} \partial_x (\beta \circ \Phi^{-1})^n \partial_x (\beta \circ \Phi^{-1})^m \, dS + \int_{T_n} \frac{1}{\mu} \partial_y (\beta \circ \Phi^{-1})^n \partial_y (\beta \circ \Phi^{-1})^m \, dS \\ &= S_{T_n, x} + S_{T_n, y} \end{aligned} \quad (3.46)$$

$$\begin{aligned}
S_{T_n, x} &= \int_{T_n} \frac{1}{\mu} (\partial_{\Phi^{-1}} \beta^n) (\partial_x \Phi^{-1}) (\partial_{\Phi^{-1}} \beta^m) (\partial_x \Phi^{-1}) \, dS \\
&= \frac{1}{\mu} |\det(\nabla \Phi)| (1, 0) (\nabla \Phi)^{-\top} \partial \beta \, dS
\end{aligned} \tag{3.47}$$

$$S = \sum_{n=1}^N \frac{1}{2\mu} |\det(\nabla \Phi)| (1, 0) (\nabla \Phi)^{-\top} \partial \beta + \sum_{n=1}^N \frac{1}{2\mu} |\det(\nabla \Phi)| (0, 1) (\nabla \Phi)^{-\top} \partial \beta \tag{3.48}$$

If we want to get the vector field \mathbf{B} from the scalar field A , we need a “curl matrix”.

$$\mathbf{B} = \nabla \times A \tag{3.49}$$

$$\boldsymbol{\tau} = (\partial_y, -\partial_x)$$

$$\int_{\Omega} \mathbf{B} \theta \, dS = \int_{\Omega} (\nabla \times A) \theta \, dS = \int_{\Omega} \boldsymbol{\tau} A \theta \, dS \tag{3.50}$$

Integration by parts:

$$\int_{\Omega} \partial F \theta \, dS = \int_{\partial \Omega} F \theta \, dl - \int_{\Omega} F \partial \theta \, dS \tag{3.51}$$

$$\int_{\Omega} \mathbf{B} \theta \, dS = \int_{\Omega} \boldsymbol{\tau} A \theta \, dS = \int_{\partial \Omega} A \theta \, dl - \int_{\Omega} A \boldsymbol{\tau} \theta \, dS \tag{3.52}$$

Constraints for $\theta \dots$, $\boldsymbol{\theta} = \boldsymbol{\tau} \theta$.

$$\int_{\Omega} \sum_{n=1}^N \mathbf{B}^n \lambda^n \sum_{n=1}^N \boldsymbol{\theta}^n \lambda^n \, dS = - \int_{\Omega} \sum_{n=1}^N A^n \lambda^n \boldsymbol{\tau} \left(\sum_{n=1}^N \theta^n \lambda^n \right) \, dS \tag{3.53}$$

$$\sum_{n=1}^N \mathbf{B}^n \int_{\Omega} \lambda^n \lambda^m \, dS = - \sum_{n=1}^N A^n \int_{\Omega} \lambda^n \boldsymbol{\tau} \lambda^m \, dS \tag{3.54}$$

$$MB = -CA \rightarrow B = -M^{\top} CA \tag{3.55}$$

LocalMatrices.m

```

edet = |det(∇Φ)|
dFinv = (∇Φ)-1
dphi = (∇Φ)-⊤ ∂β
slocxx = μSTn,x = 1/2 * dphi(1,:)′ * dphi(1,:) * edet
slocyy = μSTn,y = 1/2 * dphi(2,:)′ * dphi(2,:) * edet
mloc = MTn = edet*[1 1/2 1/2; 1/2 1 1/2; 1/2 1/2 1]/12;

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