Finite element method for magnetodynamic applications

2019

Contents

Symbols

$\boldsymbol{A}$	$[\mathrm{Wb} \cdot \mathrm{m}^{-1}]$	magnetic vector potential
$\boldsymbol{B}$	[T]	magnetic flux density
f	[Hz]	frequency
$\boldsymbol{J}$	$[\mathrm{A} \cdot \mathrm{m}^{-2}]$	current density
l	[m]	line
L	[m]	length of the model (z coordinate)
$\boldsymbol{n}$	[-]	normal vector
S	$[\mathrm{m}^2]$	surface
t	[s]	time
$T, \mathbb{T}$	$[\mathrm{N} \cdot \mathrm{m}^{-2}]$	Maxwell stress tensor
$\boldsymbol{v}$	$[\mathrm{m} \cdot \mathrm{s}^{-1}]$	velocity
V	$[\mathrm{m}^3]$	volume
γ	$[\mathrm{S} \cdot \mathrm{m}^{-1}]$	conductivity
λ	[-]	basis function
μ_0	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability of vacuum
μ	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability
$\boldsymbol{\varphi}$	[-]	test function
Ω	[-]	closed region
$\partial\Omega$	[-]	boundary of closed region

1 Theory

1.1 2D Magnetodynamic equations

Cartesian coordinate system (x, y, z)

$$\nabla \times \left(\frac{1}{\mu}(\nabla \times \boldsymbol{A})\right) + \gamma \frac{\partial \boldsymbol{A}}{\partial t} - \gamma\left(\boldsymbol{v} \times (\nabla \times \boldsymbol{A})\right) = \boldsymbol{J} \tag{1.1}$$

$$\boldsymbol{A} = (0, 0, A_z) \tag{1.2}$$

$$\boldsymbol{J} = (0, 0, J_z) \tag{1.3}$$

$$\boldsymbol{v} = (v_x, v_y, 0) \tag{1.4}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = (\partial_y A_x, -\partial_x A_y, 0) = (B_x, B_y, 0) \tag{1.5}$$

$$\boldsymbol{v} \times \boldsymbol{B} = (0, 0, v_x B_y - v_y B_x) \tag{1.6}$$

$$\boldsymbol{A}, \boldsymbol{J}, \boldsymbol{v} = f(x, y, t) \tag{1.7}$$

$$\text{Linear} \quad \mu, \gamma = f(x, y) \qquad \text{Nonlinear} \quad \mu = f(x, y, ||\boldsymbol{B}||) \tag{1.8}$$

1.2 Boundary condition

$$A_z\Big|_{\partial\Omega} = 0 \tag{1.9}$$

1.3 Initial condition

$$A_z\Big|_{t=0} = 0 \tag{1.10}$$

1.4 Force

$$T_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}B^2 \delta_{ij}\right) \tag{1.11}$$

$$\mathbb{T} = \frac{1}{\mu_0}\begin{pmatrix} B_x B_x & B_x B_y \\ B_y B_x & B_y B_y \end{pmatrix} - \frac{1}{2\mu_0}\begin{pmatrix} B_x^2 + B_y^2 & 0 \\ 0 & B_x^2 + B_y^2 \end{pmatrix} \tag{1.12}$$

$$\text{3D} \qquad \boldsymbol{F}\left(x, y, z\right) = \oint_{\partial\Omega} \mathbb{T}\,\mathrm{d}\boldsymbol{S} = \int_{\Omega} \left(\nabla \cdot \mathbb{T}\right)\,\mathrm{d}V \tag{1.13}$$

$$\text{2D} \qquad \boldsymbol{F}\left(x, y\right) = L\oint_{\partial\Omega} \mathbb{T}\,\mathrm{d}\boldsymbol{l} = L\int_{\Omega} \left(\nabla \cdot \mathbb{T}\right)\,\mathrm{d}S \tag{1.14}$$

$$\nabla \cdot \mathbb{T} = \frac{1}{\mu_0}\left(\partial_x, \partial_y\right)\begin{pmatrix} \frac{1}{2}\left(B_x^2 - B_y^2\right) & B_x B_y \\ B_y B_x & \frac{1}{2}\left(B_y^2 - B_x^2\right) \end{pmatrix} \tag{1.15}$$

$$\nabla \cdot \mathbb{T} = \frac{1}{\mu_0}\left(\frac{1}{2}\partial_x\left(B_x^2 - B_y^2\right) + \partial_y\left(B_y B_x\right), \frac{1}{2}\partial_y\left(B_y^2 - B_x^2\right) + \partial_x\left(B_x B_y\right)\right) \tag{1.16}$$

$$\nabla \cdot \mathbb{T}\left(x\right) = \frac{1}{\mu_0}\left(\partial_x B_x B_x - \partial_x B_y B_y + \partial_y B_y B_x + \partial_y B_x B_y\right) \tag{1.17}$$

$$\nabla \cdot \mathbb{T}\left(y\right) = \frac{1}{\mu_0}\left(\partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x\right) \tag{1.18}$$

1.5 Weak formulation

$$\begin{aligned} \int_{\Omega} \nabla \times \left(\frac{1}{\mu}\left(\nabla \times \boldsymbol{A}\right)\right)\boldsymbol{\varphi}\,\mathrm{d}S = \int_{\Omega} \frac{1}{\mu}\left(\nabla \times \boldsymbol{A}\right)\left(\nabla \times \boldsymbol{\varphi}\right)\,\mathrm{d}S - \\ - \int_{\partial\Omega} \frac{1}{\mu}\left(\boldsymbol{n} \times \left(\nabla \times \boldsymbol{A}\right)\right)\boldsymbol{\varphi}\,\mathrm{d}l \end{aligned} \tag{1.19}$$

$\boldsymbol{\varphi} \in f\left(\Omega\right) : \boldsymbol{\varphi}|_{\partial\Omega} = 0 \rightarrow -\int_{\partial\Omega} \frac{1}{\mu}\left(\boldsymbol{n} \times \left(\nabla \times \boldsymbol{A}\right)\right)\boldsymbol{\varphi}\,\mathrm{d}l = 0$

$$\boldsymbol{A}\left(x,y,t\right) \approx \sum_{n=1}^{N} A^{n}\left(t\right)\lambda^{n}\left(x,y\right) \tag{1.20}$$

$$\boldsymbol{J}\left(x,y,t\right) \approx \sum_{n=1}^{N} J^{n}\left(t\right)\lambda^{n}\left(x,y\right) \tag{1.21}$$

$$\boldsymbol{\varphi}\left(x,y,t\right) \approx \sum_{n=1}^{N} \varphi^{n}\left(t\right)\lambda^{n}\left(x,y\right) \tag{1.22}$$

$$\int_{\Omega} \frac{1}{\mu}\left(\nabla\times\boldsymbol{A}\right)\left(\nabla\times\boldsymbol{\varphi}\right)\,\mathrm{d}S \approx \sum_{n=1}^{N}\int_{T_n}\frac{1}{\mu}A^{n}\left(\nabla\times\lambda^{n}\right)\varphi^{m}\left(\nabla\times\lambda^{m}\right)\,\mathrm{d}S \tag{1.23}$$

$$\int_{\Omega} \boldsymbol{J}\boldsymbol{\varphi}\,\mathrm{d}S \approx \sum_{n=1}^{N}\int_{T_n} J^{n}\lambda^{n}\varphi^{m}\lambda^{m}\,\mathrm{d}S \tag{1.24}$$

1.6 Magnetostatic case

$$\sum_{n=1}^{N}\int_{T_n}\frac{1}{\mu}A^{n}\left(\nabla\times\lambda^{n}\right)\left(\nabla\times\lambda^{m}\right)\,\mathrm{d}S = \sum_{n=1}^{N}\int_{T_n} J^{n}\lambda^{n}\lambda^{m}\,\mathrm{d}S \tag{1.25}$$

$$\frac{1}{\mu}\left[S\right]\left[A\right] = \left[M\right]\left[J\right] \tag{1.26}$$

$$\left(\nabla\times\lambda^{n}\right)\left(\nabla\times\lambda^{m}\right) = \left(\partial_y\lambda^{n}, -\partial_x\lambda^{n}\right)\left(\partial_y\lambda^{m}, -\partial_x\lambda^{m}\right) \tag{1.27}$$

$$\left(\partial_y\lambda^{n}, -\partial_x\lambda^{n}\right)\left(\partial_y\lambda^{m}, -\partial_x\lambda^{m}\right) = \partial_y\lambda^{n}\partial_y\lambda^{m} + \partial_x\lambda^{n}\partial_x\lambda^{m} \tag{1.28}$$

$$\left[S\right]_{T_n} = \left[S_y\right]_{T_n} + \left[S_x\right]_{T_n} = \int_{T_n}\left(\partial_y\lambda^{n}\partial_y\lambda^{m}\right)\,\mathrm{d}S + \int_{T_n}\left(\partial_x\lambda^{n}\partial_x\lambda^{m}\right)\,\mathrm{d}S \tag{1.29}$$

Basis functions on reference triangle (r, s, t)

$$\beta_1 = 1 - r - s \tag{1.30}$$

$$\beta_2 = r \tag{1.31}$$

$$\beta_3 = s \tag{1.32}$$

$$[\partial\beta] = \begin{pmatrix} \partial_r\beta_1 & \partial_r\beta_2 & \partial_r\beta_3 \\ \partial_s\beta_1 & \partial_s\beta_2 & \partial_s\beta_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.33}$$

Coordinate transformation

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \tag{1.34}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi \begin{pmatrix} r \\ s \end{pmatrix} \tag{1.35}$$

$$\begin{pmatrix} r \\ s \end{pmatrix} = \Phi^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.36}$$

$$\lambda\left(x, y\right) = \left(\beta \circ \Phi^{-1}\right)\left(x, y\right) \tag{1.37}$$

$$\left|\det\left(\nabla\Phi\right)\right| = \left|\det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}\right| \tag{1.38}$$

$$\left(\nabla\Phi\right)^{-1} = \frac{1}{\left|\det\left(\nabla\Phi\right)\right|} \begin{pmatrix} y_3 - y_1 & -\left(x_3 - x_1\right) \\ -\left(y_2 - y_1\right) & x_2 - x_1 \end{pmatrix} \tag{1.39}$$

$$[S_y]_{T_n} = \int_{T_n} \left(\partial_y\lambda^n\partial_y\lambda^m\right)\,\mathrm{d}S = \int_{T_n} \left(\partial_y\left(\beta^n \circ \Phi^{-1}\right)\partial_y\left(\beta^m \circ \Phi^{-1}\right)\right)\,\mathrm{d}S \tag{1.40}$$

$$\partial_y \left(\beta^n \circ \Phi^{-1}\right) = \left(\partial_{\Phi^{-1}} \beta^n\right) \left(\partial_y \Phi^{-1}\right) \tag{1.41}$$

$$[S_y]_{T_n} = \int_{T_n} \left(\partial_{\Phi^{-1}} \beta^n\right) \left(\partial_y \Phi^{-1}\right) \left(\partial_{\Phi^{-1}} \beta^m\right) \left(\partial_y \Phi^{-1}\right) \, \mathrm{d}S \tag{1.42}$$

$$[S_y]_{T_r} = \left|\det\left(\nabla\Phi\right)\right| (0,1) \int_{T_r} [\partial\beta] \left(\nabla\Phi\right)^{-1} \, \mathrm{d}S \tag{1.43}$$

$$[S_x]_{T_r} = \frac{1}{2} (1,0) [\partial\beta] \left(\nabla\Phi\right)^{-1} \left|\det\left(\nabla\Phi\right)\right| \tag{1.44}$$

$$[S_y]_{T_r} = \frac{1}{2} (0,1) [\partial\beta] \left(\nabla\Phi\right)^{-1} \left|\det\left(\nabla\Phi\right)\right| \tag{1.45}$$

LocalMatrices.m

- `edet` $= \left|\det\left(\nabla\Phi\right)\right|$
- `dFinv` $= \left(\nabla\Phi\right)^{-1}$
- `dphi` $= \left(\nabla\Phi\right)^{-1} [\partial\beta]$
- `slocxx` $= [S_x]_{T_r} =$ `1/2 * dphi(1,:)' * dphi(1,:) * edet`
- `slocyy` $= [S_y]_{T_r} =$ `1/2 * dphi(2,:)' * dphi(2,:) * edet`