# Finite element method for magnetodynamic applications

2019

# Contents

# Symbols

$\boldsymbol{A}$     $[\mathrm{Wb} \cdot \mathrm{m}^{-1}]$     magnetic vector potential

$\boldsymbol{B}$     $[\mathrm{T}]$     magnetic flux density

$f$     $[\mathrm{Hz}]$     frequency

$\boldsymbol{J}$     $[\mathrm{A} \cdot \mathrm{m}^{-2}]$     current density

$l$     $[\mathrm{m}]$     line

$L$     $[\mathrm{m}]$     length of the model (z coordinate)

$\boldsymbol{n}$     [ - ]     normal vector

$S$     $[\mathrm{m}^2]$     surface

$t$     $[\mathrm{s}]$     time

$T, \mathbb{T}$     $[\mathrm{N} \cdot \mathrm{m}^{-2}]$     Maxwell stress tensor

$\boldsymbol{v}$     $[\mathrm{m} \cdot \mathrm{s}^{-1}]$     velocity

$V$     $[\mathrm{m}^3]$     volume

$\gamma$     $[\mathrm{S} \cdot \mathrm{m}^{-1}]$     conductivity

$\lambda$     [ - ]     basis function

$\mu_0$     $[\mathrm{H} \cdot \mathrm{m}^{-1}]$     permeability of vacuum

$\mu$     $[\mathrm{H} \cdot \mathrm{m}^{-1}]$     permeability

$\boldsymbol{\varphi}$     [ - ]     test function

$\Omega$     [ - ]     closed region

$\partial\Omega$     [ - ]     boundary of closed region

# 1    Theory

## 1.1    Magnetodynamic equation

3D Cartesian coordinate system $(x, y, z)$:

$$\nabla \times \left( \frac{1}{\mu} \left( \nabla \times \boldsymbol{A} \right) \right) + \gamma \frac{\partial \boldsymbol{A}}{\partial t} - \gamma \left( \boldsymbol{v} \times \left( \nabla \times \boldsymbol{A} \right) \right) = \boldsymbol{J} \tag{1.1}$$

Reduction to 2D $(x, y)$:

$$\boldsymbol{A} = (0, 0, A_z) \qquad\qquad \rightarrow \quad A = A_z \tag{1.2}$$
$$\boldsymbol{J} = (0, 0, J_z) \qquad\qquad \rightarrow \quad J = J_z \tag{1.3}$$
$$\boldsymbol{v} = (v_x, v_y, 0) \qquad\qquad \rightarrow \quad \boldsymbol{v} = (v_x, v_y) \tag{1.4}$$
$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = (B_x, B_y, 0) \qquad \rightarrow \quad \boldsymbol{B} = (B_x, B_y) \tag{1.5}$$

For 2D, time-dependent problems are $A, J, \boldsymbol{v}$ and $\boldsymbol{B}$ functions of coordinates $(x, y)$ and time $t$. Both $A$ and $J$ are scalar fields $\mathbb{R}^2 \rightarrow \mathbb{R}^1$, $\boldsymbol{v}$ and $\boldsymbol{B}$ are vector fields $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.

In a linear material, the $\mu$ and the $\gamma$ are dependent only on the coordinate system $(x, y)$, in the non-linear material, the $\mu$ is also dependent on the size of $||\boldsymbol{B}||$.

$$\nabla = (\partial_x, \partial_y) \tag{1.6}$$

Curl of scalar field $(\nabla \times A)$ is vector field $(\partial_y A, -\partial_x A)$, so $\mathbb{R}^1 \rightarrow \mathbb{R}^2$, but curl of vector field $(\nabla \times \boldsymbol{B})$ is scalar field $(\partial_x B_y - \partial_y B_x)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

Divergence of vector field $(\nabla \cdot \boldsymbol{B})$ is also scalar field $(\partial_x B_x + \partial_y B_y)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

$$\nabla \times \left( \frac{1}{\mu} \left( \nabla \times A \right) \right) + \gamma \frac{\partial A}{\partial t} - \gamma \left( \boldsymbol{v} \times \left( \nabla \times A \right) \right) = J \tag{1.7}$$

## 1.2    Boundary conditions

Dirichlet condition:
$$A \Big|_{\partial \Omega_1} = f_1 (x, y) \tag{1.8}$$

Neumann condition:
$$\partial_{\boldsymbol{n}} A \Big|_{\partial \Omega_2} = f_2 (x, y) \tag{1.9}$$

For our model, both $f_1$ and $f_2$ are equal to zero.

## 1.3   Initial condition

$$A\bigg|_{t=0} = 0 \tag{1.10}$$

## 1.4   Force

$$T_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}B^2 \delta_{ij}\right) \tag{1.11}$$

$$\mathbb{T} = \frac{1}{\mu_0}\begin{pmatrix} B_x B_x & B_x B_y \\ B_y B_x & B_y B_y \end{pmatrix} - \frac{1}{2\mu_0}\begin{pmatrix} B_x^2 + B_y^2 & 0 \\ 0 & B_x^2 + B_y^2 \end{pmatrix} \tag{1.12}$$

$$\boldsymbol{F}(x,y) = L\oint_{\partial\Omega}\mathbb{T}\boldsymbol{n}\,\mathrm{d}l = L\int_{\Omega}(\nabla\cdot\mathbb{T})\,\mathrm{d}S \tag{1.13}$$

Variable L is the length of the model into the third dimension $z$.

$$
\begin{aligned}
\nabla\cdot\mathbb{T} &= \frac{1}{\mu_0}\begin{pmatrix} \frac{1}{2}\partial_x\left(B_x^2 - B_y^2\right) + \partial_y\left(B_y B_x\right) \\ \frac{1}{2}\partial_y\left(B_y^2 - B_x^2\right) + \partial_x\left(B_x B_y\right) \end{pmatrix} \\
&= \frac{1}{\mu_0}\begin{pmatrix} \partial_x B_x B_x - \partial_x B_y B_y + \partial_y B_y B_x + \partial_y B_x B_y \\ \partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x \end{pmatrix}
\end{aligned} \tag{1.14}
$$

# 2    Weak formulation

To simplify the problem, let us first consider a magnetostacic case ($\gamma = 0$). This is true for models without motion and without time variation of currents.

# 3    Topology optimization

## 3.1    Problem formulation in continuous space

$$
\begin{aligned}
\text{minimize} \quad & F_y^p \quad (\text{p} - \text{plunger}) \\
\text{subject to} \quad & -\frac{1}{\mu}\Delta A = J \\
& B_x = \partial_y A \\
& B_y = -\partial_x A
\end{aligned}
$$

$$
F_y^p = (0,1)\int_{\Omega_p} (\nabla \cdot \mathbb{T})\, \mathrm{d}S = \int_{\Omega_p} \frac{1}{\mu_0}\left(\partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x\right)\mathrm{d}S \quad (3.1)
$$

## 3.2    Problem formulation in discrete space

$$
\begin{aligned}
\text{minimize} \quad & F_y^p \\
\text{subject to} \quad & SA = MJ \\
& MB_x = C_y A \\
& MB_y = -C_x A
\end{aligned}
$$

$$
F_y^p = \frac{1}{\mu_0}\left(B_y^\top C_y^p B_y - B_x^\top C_y^p B_x + B_y^\top C_x^p B_x + B_x^\top C_x^p B_y\right) \qquad (3.2)
$$

Lagrange multipliers are $(\alpha, \beta, \gamma)$.

$$
\begin{aligned}
SA = MJ \quad &\rightarrow \quad \alpha \\
MB_x = C_y A \quad &\rightarrow \quad \beta \\
MB_y = -C_x A \quad &\rightarrow \quad \gamma
\end{aligned}
$$

The $\varphi$ is the topology function.

$$J(\varphi) = J_1(\varphi) + J_2(\varphi) + J_3(\varphi) + J_4(\varphi) \tag{3.3}$$

$$J_1 = \frac{1}{\mu_0} \left( B_y^\top(\varphi) C_y^p B_y(\varphi) - B_x^\top(\varphi) C_y^p B_x(\varphi) + B_y^\top(\varphi) C_x^p B_x(\varphi) + B_x^\top(\varphi) C_x^p B_y(\varphi) \right) \tag{3.4}$$

$$J_2 = \alpha^\top \left( S(\varphi) A(\varphi) - MJ(\varphi) \right) = 0 \tag{3.5}$$

$$J_3 = \beta^\top \left( MB_x(\varphi) - C_y A(\varphi) \right) = 0 \tag{3.6}$$

$$J_4 = \gamma^\top \left( MB_y(\varphi) + C_x A(\varphi) \right) = 0 \tag{3.7}$$

$$\partial_{\varphi_i} J(\varphi) = \partial_{\varphi_i} J_1(\varphi) + \partial_{\varphi_i} J_2(\varphi) + \partial_{\varphi_i} J_3(\varphi) + \partial_{\varphi_i} J_4(\varphi) \tag{3.8}$$

$$\mu_0 \partial_{\varphi_i} J_1 = \partial_{\varphi_i} \left( B_y^\top C_y^p B_y \right) - \partial_{\varphi_i} \left( B_x^\top C_y^p B_x \right) + \partial_{\varphi_i} \left( B_y^\top C_x^p B_x \right) + \partial_{\varphi_i} \left( B_x^\top C_x^p B_y \right) \tag{3.9}$$

$$\partial_{\varphi_i} \left( B_y^\top C_y^p B_y \right) = \partial_{\varphi_i} B_y^\top C_y^p B_y + B_y^\top C_y^p \partial_{\varphi_i} B_y = B_y^\top \left( C_y^p \right)^\top \partial_{\varphi_i} B_y + B_y^\top C_y^p \partial_{\varphi_i} B_y \tag{3.10}$$

$$\partial_{\varphi_i} J_1 = \frac{1}{\mu_0} \left( B_x^\top, B_y^\top \right) \begin{pmatrix} -\left(C_y^p\right)^\top - C_y^p \\ \left(C_x^p\right)^\top + C_x^p \end{pmatrix} \partial_{\varphi_i} B_x$$
$$+ \frac{1}{\mu_0} \left( B_x^\top, B_y^\top \right) \begin{pmatrix} \left(C_x^p\right)^\top + C_x^p \\ \left(C_y^p\right)^\top + C_y^p \end{pmatrix} \partial_{\varphi_i} B_y \tag{3.11}$$

$$\partial_{\varphi_i} J_2 = \alpha^\top \partial_{\varphi_i}(SA) = \alpha^\top \partial_{\varphi_i} SA + \alpha^\top S \partial_{\varphi_i} A \tag{3.12}$$

$$\partial_{\varphi_i} J_3 = \beta^\top M \partial_{\varphi_i} B_x - \beta^\top C_y \partial_{\varphi_i} A \tag{3.13}$$

$$\partial_{\varphi_i} J_4 = \gamma^\top M \partial_{\varphi_i} B_y + \gamma^\top C_x \partial_{\varphi_i} A \tag{3.14}$$

$$\partial_{\varphi_i} B_x : \quad \frac{1}{\mu_0} \left(B_x^\top, B_y^\top\right) \begin{pmatrix} -\left(C_y^p\right)^\top - C_y^p \\ \left(C_x^p\right)^\top + C_x^p \end{pmatrix} \partial_{\varphi_i} B_x + \beta^\top M \partial_{\varphi_i} B_x \tag{3.15}$$

$$\partial_{\varphi_i} B_y : \quad \frac{1}{\mu_0} \left(B_x^\top, B_y^\top\right) \begin{pmatrix} \left(C_x^p\right)^\top + C_x^p \\ \left(C_y^p\right)^\top + C_y^p \end{pmatrix} \partial_{\varphi_i} B_y + \gamma^\top M \partial_{\varphi_i} B_y \tag{3.16}$$

$$\partial_{\varphi_i} A : \quad \alpha^\top S \partial_{\varphi_i} A - \beta^\top C_y \partial_{\varphi_i} A + \gamma^\top C_x \partial_{\varphi_i} A \tag{3.17}$$

$$\partial_{\varphi_i} S : \quad \alpha^\top \partial_{\varphi_i} S A \tag{3.18}$$

If we choose $\alpha, \beta, \gamma$ such that (3.15), (3.16) and (3.17) are equal to 0, then:
From (3.15), ($M^\top = M$, $S^\top = S$ ):

$$\beta = -\frac{1}{\mu_0} M^{-1} \begin{pmatrix} -\left(C_y^p\right)^\top - C_y^p \\ \left(C_x^p\right)^\top + C_x^p \end{pmatrix}^\top \begin{pmatrix} B_x^\top \\ B_y^\top \end{pmatrix} \tag{3.19}$$

From (3.16):

$$\gamma = -\frac{1}{\mu_0} M^{-1} \begin{pmatrix} \left(C_x^p\right)^\top + C_x^p \\ \left(C_y^p\right)^\top + C_y^p \end{pmatrix}^\top \begin{pmatrix} B_x^\top \\ B_y^\top \end{pmatrix} \tag{3.20}$$

From (3.17):

$$\alpha = S^{-1} \left(C_y^\top \beta - C_x^\top \gamma\right) \tag{3.21}$$

$$\partial_{\varphi_i} J = \alpha^\top \partial_{\varphi_i} S A \tag{3.22}$$

# Appendix

## Weak formulation

For $\gamma = 0$.

$$\nabla \times \left( \frac{1}{\mu} \left( \nabla \times A \right) \right) = J \qquad (3.23)$$

The $\theta$ is test function.

$$\int_\Omega \nabla \times \left( \frac{1}{\mu} \left( \nabla \times A \right) \right) \theta \, \mathrm{d}S = \int_\Omega J\theta \, \mathrm{d}S \qquad (3.24)$$

$$\nabla \times A = \left( \partial_y A, -\partial_x A \right) = \boldsymbol{B}$$
$$\nabla \times \boldsymbol{B} = \partial_x B_y - \partial_y B_x = -\partial_x \partial_x A - \partial_y \partial_y A = -\Delta A \qquad (3.25)$$

$$\int_\Omega -\frac{1}{\mu} \Delta A \theta \, \mathrm{d}S = \int_\Omega J\theta \, \mathrm{d}S \qquad (3.26)$$

Green's first identity:

$$\int_\Omega \Delta F \theta \, \mathrm{d}S + \int_\Omega \nabla F \nabla \theta \, \mathrm{d}S = \int_{\partial\Omega} \left( \nabla F \boldsymbol{n} \right) \theta \mathrm{d}l \qquad (3.27)$$

$$\int_\Omega -\frac{1}{\mu} \Delta A \theta \, \mathrm{d}S = \int_\Omega \frac{1}{\mu} \nabla A \nabla \theta \, \mathrm{d}S - \int_{\partial\Omega_1} \frac{1}{\mu} \left( \nabla A \boldsymbol{n} \right) \theta \mathrm{d}l - \int_{\partial\Omega_2} \frac{1}{\mu} \left( \nabla A \boldsymbol{n} \right) \theta \mathrm{d}l \qquad (3.28)$$

$$\theta \in H^1 : \theta|_{\partial\Omega_1} = 0, \theta|_{\partial\Omega_2} = 0 \rightarrow \int_{\partial\Omega_1} \frac{1}{\mu} \left( \nabla A \boldsymbol{n} \right) \theta \mathrm{d}l = 0, \int_{\partial\Omega_2} \frac{1}{\mu} \left( \nabla A \boldsymbol{n} \right) \theta \mathrm{d}l = 0$$

# Discretization

$$A(x, y, t) \approx \sum_{n=1}^{N} A^n(t) \lambda^n(x, y) \tag{3.29}$$

$$J(x, y, t) \approx \sum_{n=1}^{N} J^n(t) \lambda^n(x, y) \tag{3.30}$$

$$\theta(x, y, t) \approx \sum_{n=1}^{N} \theta^n(t) \lambda^n(x, y) \tag{3.31}$$

$$\int_{T_k} \frac{1}{\mu} \nabla \left( \sum_{n=1}^{N} A^n \lambda^n \right) \nabla \left( \sum_{n=1}^{N} \theta^n \lambda^n \right) \, \mathrm{d}S = \int_{T_k} \sum_{n=1}^{N} J^n \lambda^n \sum_{n=1}^{N} \theta^n \lambda^n \, \mathrm{d}S \tag{3.32}$$

$T_k$ is one discrete element of geometry, in our case it is a triangle.

$$\sum_{n=1}^{N} \int_{T_k} \frac{1}{\mu} \nabla (A^n \lambda^n) \nabla (\theta^m \lambda^m) \, \mathrm{d}S = \sum_{n=1}^{N} \int_{T_k} J^n \lambda^n \theta^m \lambda^m \, \mathrm{d}S$$

$$\sum_{n=1}^{N} A^n \int_{T_k} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m \, \mathrm{d}S = \sum_{n=1}^{N} J^n \int_{T_k} \lambda^n \lambda^m \, \mathrm{d}S \tag{3.33}$$

We can put $A^n$ and $J^n$ out of $\nabla$ and out of $\int_{T_k}$, because they no longer depend on coordinate system.

$$S = \sum_{n=1}^{N} \int_{T_k} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m \, \mathrm{d}S$$

$$M = \sum_{n=1}^{N} \int_{T_k} \lambda^n \lambda^m \, \mathrm{d}S$$

$$SA = MJ \rightarrow A = SMJ \tag{3.34}$$

If we discretize the problem with triangular elements and select the first-order polynomial (linear function), as an approximation of the scalar field A, we get three basis functions for each triangle. The coordinates $(r, s)$ correspond to the reference triangle with vertices $V_1 = (0, 0), V_2 = (0, 1)$ and $V_3 = (1, 1)$ numbered in counter-clockwise direction.

$$\beta_1 = 1 - r - s \tag{3.35}$$

$$\beta_2 = r \tag{3.36}$$

$$\beta_3 = s \tag{3.37}$$

If we differentiate them in respect to coordinates, we get:

$$\nabla \beta = \begin{pmatrix} \partial_r \beta_1 & \partial_r \beta_2 & \partial_r \beta_3 \\ \partial_s \beta_1 & \partial_s \beta_2 & \partial_s \beta_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{3.38}$$

The matrices, fields and functions of the reference triangle must then be transformed into our coordinate system $(r, s) \to (x, y)$ and put into right place. Vertices of the triangle in $(x, y)$ coordinates are $V_1 = (x_1, y_1), V_2 = (x_2, y_2)$ and $V_3 = (x_3, y_3)$.

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \tag{3.39}$$

The $\Phi$ is the transformation function. We can use it to transform basis functions of reference triangle $T_r$ to basis functions of the n-th triangle $T_k$.

$$\lambda (x, y) = \left( \beta \circ \Phi^{-1} \right) (x, y) \tag{3.40}$$

We will also need the inverse Jacobian matrix and the "Jacobian", ie the determinant of the Jacobian matrix.

$$|\det (\nabla \Phi)| = \left| \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \right| \tag{3.41}$$

$$\nabla \Phi^{-1} = \frac{1}{|\det (\nabla \Phi)|} \begin{pmatrix} y_3 - y_1 & -(x_3 - x_1) \\ -(y_2 - y_1) & x_2 - x_1 \end{pmatrix} \tag{3.42}$$

Mass matrix:

$$M_{T_k} = \int_{T_k} \lambda^n \lambda^m \, \mathrm{d}S = \int_{T_k} \left( \beta^n \circ \Phi^{-1} \right) \left( \beta^m \circ \Phi^{-1} \right) \, \mathrm{d}S$$
$$= |\det (\nabla \Phi)| \int_{T_r} \beta^n \beta^m \, \mathrm{d}S \tag{3.43}$$

$$M_{T_k} = |\det (\nabla \Phi)| \begin{pmatrix} \int \beta_1 \beta_1 \, \mathrm{d}S & \int \beta_1 \beta_2 \, \mathrm{d}S & \int \beta_1 \beta_3 \, \mathrm{d}S \\ \int \beta_2 \beta_1 \, \mathrm{d}S & \int \beta_2 \beta_2 \, \mathrm{d}S & \int \beta_2 \beta_3 \, \mathrm{d}S \\ \int \beta_3 \beta_1 \, \mathrm{d}S & \int \beta_3 \beta_2 \, \mathrm{d}S & \int \beta_3 \beta_3 \, \mathrm{d}S \end{pmatrix} \tag{3.44}$$

$$M = \sum_{n=1}^{N} |\det(\nabla\Phi)| \begin{pmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{pmatrix} \tag{3.45}$$

Stiffness matrix:

$$
\begin{aligned}
S_{T_k} &= \int_{T_k} \frac{1}{\mu} \nabla\lambda^n \nabla\lambda^m \, \mathrm{d}S = \int_{T_k} \frac{1}{\mu} \nabla\left(\beta^n \circ \Phi^{-1}\right) \nabla\left(\beta^m \circ \Phi^{-1}\right) \mathrm{d}S \\
&= \int_{T_k} \frac{1}{\mu} \partial_x \left(\beta^n \circ \Phi^{-1}\right) \partial_x \left(\beta^m \circ \Phi^{-1}\right) \mathrm{d}S + \int_{T_k} \frac{1}{\mu} \partial_y \left(\beta^n \circ \Phi^{-1}\right) \partial_y \left(\beta^m \circ \Phi^{-1}\right) \mathrm{d}S \\
&= S_{T_k,x} + S_{T_k,y}
\end{aligned}
\tag{3.46}
$$

$$
\begin{aligned}
S_{T_k,x} &= \int_{T_k} \frac{1}{\mu} \left(\partial_x \beta^n \left(\Phi^{-1}\right)\right) \left(\partial_x \Phi^{-1}\right) \left(\partial_x \beta^n \left(\Phi^{-1}\right)\right) \left(\partial_x \Phi^{-1}\right) \mathrm{d}S \\
&= \frac{1}{\mu} |\det(\nabla\Phi)| \, (1,0) \int_{T_r} \nabla\Phi^{-1} \nabla\beta^\top \nabla\Phi^{-\top} \nabla\beta \, \mathrm{d}S
\end{aligned}
\tag{3.47}
$$

$$
\begin{aligned}
S &= \sum_{n=1}^{N} \frac{1}{2\mu} |\det(\nabla\Phi)| \, (1,0) \, \nabla\Phi^{-1} \nabla\beta^\top \nabla\Phi^{-\top} \nabla\beta \\
&+ \sum_{n=1}^{N} \frac{1}{2\mu} |\det(\nabla\Phi)| \, (0,1) \, \nabla\Phi^{-1} \nabla\beta^\top \nabla\Phi^{-\top} \nabla\beta
\end{aligned}
\tag{3.48}
$$

If we want to get the vector field $\boldsymbol{B}$ from the scalar field $A$, we need a "curl matrix".

$$\boldsymbol{B} = \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} \partial_y A \\ -\partial_x A \end{pmatrix} \tag{3.49}$$

$$\int_\Omega B_x \theta \, \mathrm{d}S = \int_\Omega \partial_y A \theta \, \mathrm{d}S \tag{3.50}$$

Integration by parts:

$$\int_\Omega \partial F \theta \, \mathrm{d}S = \int_{\partial\Omega} F \theta \boldsymbol{n} \, \mathrm{d}l - \int_\Omega F \partial\theta \, \mathrm{d}S \tag{3.51}$$

$$\int_\Omega B_x \theta \, \mathrm{d}S = \int_\Omega \partial_y A \theta \, \mathrm{d}S = \int_{\partial\Omega} A \theta \boldsymbol{n} \, \mathrm{d}l - \int_\Omega A \partial_y \theta \, \mathrm{d}S \tag{3.52}$$

Constraints for $\theta$ ...

$$\int_\Omega \sum_{n=1}^{N} B_x^n \lambda^n \sum_{n=1}^{N} \theta^n \lambda^n \, \mathrm{d}S = -\int_\Omega \sum_{n=1}^{N} A^n \lambda^n \partial_y \left(\sum_{n=1}^{N} \theta^n \lambda^n\right) \mathrm{d}S \tag{3.53}$$

$$\sum_{n=1}^{N} B_x^n \int_\Omega \lambda^n \lambda^m \, \mathrm{d}S = -\sum_{n=1}^{N} A^n \int_\Omega \lambda^n \partial_y \lambda^m \, \mathrm{d}S \tag{3.54}$$

$$MB_x = -C_y A \rightarrow B_x = -MC_y A \tag{3.55}$$

"Curl matrix":

$$\begin{aligned}
C_{T_k,y} &= \int_{T_k} \lambda^n \partial_y \lambda^m \, \mathrm{d}S = \int_{T_k} \left( \beta^n \circ \Phi^{-1} \right) \partial_y \left( \beta^m \circ \Phi^{-1} \right) \, \mathrm{d}S \\
&= \int_{T_k} \left( \beta^n \circ \Phi^{-1} \right) \left( \partial_y \beta^n \left( \Phi^{-1} \right) \right) \left( \partial_x \Phi^{-1} \right) \, \mathrm{d}S \\
&= \left| \det \left( \nabla \Phi \right) \right| (1,0) \int_{T_r} \beta \nabla \Phi^{-\top} \nabla \beta \, \mathrm{d}S
\end{aligned} \tag{3.56}$$

We got $\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$, because $\int \beta \, \mathrm{d}S = \frac{1}{3}$.

$$\begin{aligned}
C_x &= \sum_{n=1}^{N} \frac{1}{6} \left| \det \left( \nabla \Phi \right) \right| (0,1) \nabla \Phi^{-\top} \nabla \beta \\
C_y &= -\sum_{n=1}^{N} \frac{1}{6} \left| \det \left( \nabla \Phi \right) \right| (1,0) \nabla \Phi^{-\top} \nabla \beta
\end{aligned} \tag{3.57}$$

LocalMatrices.m

$\texttt{edet} = \left| \det \left( \nabla \Phi \right) \right|$

$\texttt{dFinv} = \nabla \Phi^{-1}$

$\texttt{dphi} = \nabla \Phi^{-\top} \nabla \beta$

$\texttt{slocx} = 3 C_{T_k,x} = \texttt{1/2 *  dphi(1,:) * edet}$

$\texttt{slocy} = 3 C_{T_k,y} = \texttt{1/2 *  dphi(2,:) * edet}$

$\texttt{slocxx} = \mu S_{T_k,x} = \texttt{1/2 * dphi(1,:)' * dphi(1,:) * edet}$

$\texttt{slocyy} = \mu S_{T_k,y} = \texttt{1/2 * dphi(2,:)' * dphi(2,:) * edet}$

$\texttt{mloc} = M_{T_k} = \texttt{[1 1/2 1/2;1/2 1 1/2;1/2 1/2 1]/12 * edet}$

$\texttt{clocx} = C_{T_k,x} = \texttt{[slocx;slocx;slocx]/3}$

$\texttt{clocy} = C_{T_k,y} = \texttt{[slocy;slocy;slocy]/3}$