Finite element method for magnetodynamic applications

2019

Contents

Symbols

$\boldsymbol{A}$	$[\mathrm{Wb} \cdot \mathrm{m}^{-1}]$	magnetic vector potential
$\boldsymbol{B}$	[T]	magnetic flux density
f	[Hz]	frequency
$\boldsymbol{J}$	$[\mathrm{A} \cdot \mathrm{m}^{-2}]$	current density
l	[m]	line
L	[m]	length of the model (z coordinate)
$\boldsymbol{n}$	[-]	normal vector
S	$[\mathrm{m}^2]$	surface
t	[s]	time
$T, \mathbb{T}$	$[\mathrm{N} \cdot \mathrm{m}^{-2}]$	Maxwell stress tensor
$\boldsymbol{v}$	$[\mathrm{m} \cdot \mathrm{s}^{-1}]$	velocity
V	$[\mathrm{m}^3]$	volume
γ	$[\mathrm{S} \cdot \mathrm{m}^{-1}]$	conductivity
λ	[-]	basis function
μ_0	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability of vacuum
μ	$[\mathrm{H} \cdot \mathrm{m}^{-1}]$	permeability
$\boldsymbol{\varphi}$	[-]	test function
Ω	[-]	closed region
$\partial\Omega$	[-]	boundary of closed region

1 Theory

1.1 Magnetodynamic equation

3D Cartesian coordinate system (x, y, z):

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times \boldsymbol{A}) \right) + \gamma \frac{\partial \boldsymbol{A}}{\partial t} - \gamma (\boldsymbol{v} \times (\nabla \times \boldsymbol{A})) = \boldsymbol{J} \tag{1.1}$$

Reduction to 2D (x, y):

$$\boldsymbol{A} = (0, 0, A_z) \quad \rightarrow \quad A = A_z \tag{1.2}$$

$$\boldsymbol{J} = (0, 0, J_z) \quad \rightarrow \quad J = J_z \tag{1.3}$$

$$\boldsymbol{v} = (v_x, v_y, 0) \quad \rightarrow \quad \boldsymbol{v} = (v_x, v_y) \tag{1.4}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = (B_x, B_y, 0) \quad \rightarrow \quad \boldsymbol{B} = (B_x, B_y) \tag{1.5}$$

For 2D, time-dependent problems are $A, J, \boldsymbol{v}$ and $\boldsymbol{B}$ functions of coordinates (x, y) and time t. Both A and J are scalar fields $\mathbb{R}^2 \rightarrow \mathbb{R}^1$, $\boldsymbol{v}$ and $\boldsymbol{B}$ are vector fields $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.

In a linear material, the μ and the γ are dependent only on the coordinate system (x, y), in the non-linear material, the μ is also dependent on the size of $||\boldsymbol{B}||$.

$$\nabla = (\partial_x, \partial_y) \tag{1.6}$$

Curl of scalar field $(\nabla \times A)$ is vector field $(\partial_y A, -\partial_x A)$, so $\mathbb{R}^1 \rightarrow \mathbb{R}^2$, but curl of vector field $(\nabla \times \boldsymbol{B})$ is scalar field $(\partial_x B_y - \partial_y B_x)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

Divergence of vector field $(\nabla \cdot \boldsymbol{B})$ is also scalar field $(\partial_x B_x + \partial_y B_y)$, so $\mathbb{R}^2 \rightarrow \mathbb{R}^1$.

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) + \gamma \frac{\partial A}{\partial t} - \gamma (\boldsymbol{v} \times (\nabla \times A)) = J \tag{1.7}$$

1.2 Boundary conditions

Dirichlet condition:

$$A \bigg|_{\partial \Omega_1} = f_1 (x, y) \tag{1.8}$$

Neumann condition:

$$\partial_{\boldsymbol{n}} A \bigg|_{\partial \Omega_2} = f_2 (x, y) \tag{1.9}$$

For our model, both f_1 and f_2 are equal to zero.

1.3 Initial condition

$$A\Big|_{t=0} = 0 \tag{1.10}$$

1.4 Force

$$T_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}B^2\delta_{ij}\right) \tag{1.11}$$

$$\mathbb{T} = \frac{1}{\mu_0}\begin{pmatrix} B_x B_x & B_x B_y \\ B_y B_x & B_y B_y \end{pmatrix} - \frac{1}{2\mu_0}\begin{pmatrix} B_x^2 + B_y^2 & 0 \\ 0 & B_x^2 + B_y^2 \end{pmatrix} \tag{1.12}$$

$$\boldsymbol{F}(x, y) = L \oint_{\partial\Omega} \mathbb{T}\boldsymbol{n}\,\mathrm{d}l = L\int_{\Omega} (\nabla\cdot\mathbb{T})\,\mathrm{d}S \tag{1.13}$$

Variable L is the length of the model into the third dimension z.

$$\begin{aligned}
\nabla\cdot\mathbb{T} &= \frac{1}{\mu_0}\begin{pmatrix} \frac{1}{2}\left(B_x^2 - B_y^2\right) & B_x B_y \\ B_y B_x & \frac{1}{2}\left(B_y^2 - B_x^2\right) \end{pmatrix}^{\top}\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \\
&= \frac{1}{\mu_0}\begin{pmatrix} \frac{1}{2}\partial_x\left(B_x^2 - B_y^2\right) + \partial_y\left(B_y B_x\right) \\ \frac{1}{2}\partial_y\left(B_y^2 - B_x^2\right) + \partial_x\left(B_x B_y\right) \end{pmatrix} \\
&= \frac{1}{\mu_0}\begin{pmatrix} \partial_x B_x B_x - \partial_x B_y B_y + \partial_y B_y B_x + \partial_y B_x B_y \\ \partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x \end{pmatrix}
\end{aligned} \tag{1.14}$$

2 Weak formulation

To simplify the problem, let us first consider a magnetostacic case ($\gamma = 0$). This is true for models without motion and without time variation of currents.

$$\int_\Omega \frac{1}{\mu} (\nabla \times \boldsymbol{A}) (\nabla \times \boldsymbol{\theta}) \, \mathrm{d}S \approx \sum_{n=1}^{N} \int_{T_n} \frac{1}{\mu} A^n (\nabla \times \lambda^n) \, \theta^m (\nabla \times \lambda^m) \, \mathrm{d}S \tag{2.1}$$

$$\int_\Omega \boldsymbol{J}\boldsymbol{\theta} \, \mathrm{d}S \approx \sum_{n=1}^{N} \int_{T_n} J^n \lambda^n \theta^m \lambda^m \, \mathrm{d}S \tag{2.2}$$

2.1 Magnetostatic case

$$(\nabla \times \lambda^n) (\nabla \times \lambda^m) = (\partial_y \lambda^n, -\partial_x \lambda^n) (\partial_y \lambda^m, -\partial_x \lambda^m) \tag{2.3}$$

$$(\partial_y \lambda^n, -\partial_x \lambda^n) (\partial_y \lambda^m, -\partial_x \lambda^m) = \partial_y \lambda^n \partial_y \lambda^m + \partial_x \lambda^n \partial_x \lambda^m \tag{2.4}$$

$$S_{T_n} = S_{y,T_n} + S_{x,T_n} = \int_{T_n} (\partial_y \lambda^n \partial_y \lambda^m) \, \mathrm{d}S + \int_{T_n} (\partial_x \lambda^n \partial_x \lambda^m) \, \mathrm{d}S \tag{2.5}$$

Basis functions on reference triangle (r, s, t)

$$\beta_1 = 1 - r - s \tag{2.6}$$

$$\beta_2 = r \tag{2.7}$$

$$\beta_3 = s \tag{2.8}$$

$$\partial \beta = \begin{pmatrix} \partial_r \beta_1 & \partial_r \beta_2 & \partial_r \beta_3 \\ \partial_s \beta_1 & \partial_s \beta_2 & \partial_s \beta_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{2.9}$$

Coordinate transformation

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \tag{2.10}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi \begin{pmatrix} r \\ s \end{pmatrix} \tag{2.11}$$

$$\begin{pmatrix} r \\ s \end{pmatrix} = \Phi^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.12}$$

$$\lambda\left(x, y\right) = \left(\beta \circ \Phi^{-1}\right)\left(x, y\right) \tag{2.13}$$

$$\left|\det\left(\nabla\Phi\right)\right| = \left|\det\begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}\right| \tag{2.14}$$

$$\left(\nabla\Phi\right)^{-1} = \frac{1}{\left|\det\left(\nabla\Phi\right)\right|}\begin{pmatrix} y_3 - y_1 & -\left(x_3 - x_1\right) \\ -\left(y_2 - y_1\right) & x_2 - x_1 \end{pmatrix} \tag{2.15}$$

$$S_{y,T_n} = \int_{T_n} \left(\partial_y \lambda^n \partial_y \lambda^m\right) \,\mathrm{d}S = \int_{T_n} \left(\partial_y \left(\beta^n \circ \Phi^{-1}\right) \partial_y \left(\beta^m \circ \Phi^{-1}\right)\right) \,\mathrm{d}S \tag{2.16}$$

$$\partial_y \left(\beta^n \circ \Phi^{-1}\right) = \left(\partial_{\Phi^{-1}} \beta^n\right)\left(\partial_y \Phi^{-1}\right) \tag{2.17}$$

$$S_{y,T_n} = \int_{T_n} \left(\partial_{\Phi^{-1}} \beta^n\right)\left(\partial_y \Phi^{-1}\right)\left(\partial_{\Phi^{-1}} \beta^m\right)\left(\partial_y \Phi^{-1}\right) \,\mathrm{d}S \tag{2.18}$$

$$S_{y,T_r} = \left|\det\left(\nabla\Phi\right)\right|\left(0,1\right)\int_{T_r} \partial\beta\left(\nabla\Phi\right)^{-1} \,\mathrm{d}S \tag{2.19}$$

$$S_{x,T_r} = \frac{1}{2}\left(1,0\right)\partial\beta\left(\nabla\Phi\right)^{-1}\left|\det\left(\nabla\Phi\right)\right| \tag{2.20}$$

$$S_{y,T_r} = \frac{1}{2}\left(0,1\right)\partial\beta\left(\nabla\Phi\right)^{-1}\left|\det\left(\nabla\Phi\right)\right| \tag{2.21}$$

LocalMatrices.m

$\texttt{edet} = \left|\det\left(\nabla\Phi\right)\right|$

$\texttt{dFinv} = \left(\nabla\Phi\right)^{-1}$

$\texttt{dphi} = \left(\nabla\Phi\right)^{-1}\partial\beta$

$\texttt{slocxx} = S_{x,T_r} =$ `1/2 * dphi(1,:)' * dphi(1,:) * edet` `slocyy` $= S_{y,T_r} =$ `1/2 * dphi(2,:)' * dphi(2,:) * edet`

3 Topology optimization

3.1 Problem formulation in continuous space

$$\begin{aligned} \text{minimize} \quad & F_y^p \quad (\text{p} - \text{plunger}) \\ \text{subject to} \quad & \nabla \times \left(\frac{1}{\mu} (\nabla \times A) \right) = J \\ & B_x = \partial_y A \\ & B_y = -\partial_x A \end{aligned}$$

$$F_y^p = (0,1) \int_{\Omega p} (\nabla \cdot \mathbb{T}) \, \mathrm{d}S = \int_{\Omega p} \frac{1}{\mu_0} (\partial_y B_y B_y - \partial_y B_x B_x + \partial_x B_x B_y + \partial_x B_y B_x) \, \mathrm{d}S \quad (3.1)$$

3.2 Problem formulation in discrete space

$$\begin{aligned} \text{minimize} \quad & F_y^p \\ \text{subject to} \quad & SA = MJ \\ & MB_x = C_y A \\ & MB_y = -C_x A \end{aligned}$$

$$F_y^p = \frac{1}{\mu_0} \left(B_y^\top C_y^p B_y - B_x^\top C_y^p B_x + B_y^\top C_x^p B_x + B_x^\top C_x^p B_y \right) \quad (3.2)$$

Lagrange multipliers are (α, β, γ).

$$\begin{aligned} SA = MJ \quad & \rightarrow \quad \alpha \\ MB_x = C_y A \quad & \rightarrow \quad \beta \\ MB_y = -C_x A \quad & \rightarrow \quad \gamma \end{aligned}$$

The φ_i is the topology function.

$$J\left(\varphi_i\right) = J_1\left(\varphi_i\right) + J_2\left(\varphi_i\right) + J_3\left(\varphi_i\right) + J_4\left(\varphi_i\right) \tag{3.3}$$

$$J_1 = \frac{1}{\mu_0}\left(B_y^\top C_y^p B_y - B_x^\top C_y^p B_x + B_y^\top C_x^p B_x + B_x^\top C_x^p B_y\right) \tag{3.4}$$

$$J_2 = \alpha^\top\left(SA - MJ\right) = 0 \tag{3.5}$$

$$J_3 = \beta^\top\left(MB_x - C_y A\right) = 0 \tag{3.6}$$

$$J_4 = \gamma^\top\left(MB_y + C_x A\right) = 0 \tag{3.7}$$

$$\partial_{\varphi_i} J\left(\varphi_i\right) = \partial_{\varphi_i} J_1\left(\varphi_i\right) + \partial_{\varphi_i} J_2\left(\varphi_i\right) + \partial_{\varphi_i} J_3\left(\varphi_i\right) + \partial_{\varphi_i} J_4\left(\varphi_i\right) \tag{3.8}$$

$$\mu_0 \partial_{\varphi_i} J_1 = \partial_{\varphi_i}\left(B_y^\top C_y^p B_y\right) - \partial_{\varphi_i}\left(B_x^\top C_y^p B_x\right) + \partial_{\varphi_i}\left(B_y^\top C_x^p B_x\right) + \partial_{\varphi_i}\left(B_x^\top C_x^p B_y\right) \tag{3.9}$$

$$\partial_{\varphi_i}\left(B_y^\top C_y^p B_y\right) = \partial_{\varphi_i} B_y^\top C_y^p B_y + B_y^\top C_y^p \partial_{\varphi_i} B_y = B_y^\top \left(C_y^p\right)^\top \partial_{\varphi_i} B_y + B_y^\top C_y^p \partial_{\varphi_i} B_y \tag{3.10}$$

$$\begin{aligned} \partial_{\varphi_i} J_1 =& \frac{1}{\mu_0}\left(B_x^\top, B_y^\top\right)\begin{pmatrix} -\left(C_y^p\right)^\top & -C_y^p \\ \left(C_x^p\right)^\top & C_x^p \end{pmatrix}\begin{pmatrix} \partial_{\varphi_i} B_x \\ \partial_{\varphi_i} B_x \end{pmatrix} \\ &+ \frac{1}{\mu_0}\left(B_x^\top, B_y^\top\right)\begin{pmatrix} \left(C_x^p\right)^\top & C_x^p \\ \left(C_y^p\right)^\top & C_y^p \end{pmatrix}\begin{pmatrix} \partial_{\varphi_i} B_y \\ \partial_{\varphi_i} B_y \end{pmatrix} \end{aligned} \tag{3.11}$$

$$\partial_{\varphi_i} J_2 = \alpha^\top \partial_{\varphi_i}\left(SA\right) - \alpha^\top \partial_{\varphi_i}\left(MJ\right) = \alpha^\top \partial_{\varphi_i} SA + \alpha^\top S \partial_{\varphi_i} A \tag{3.12}$$

$$\partial_{\varphi_i} J_3 = \beta^\top M \partial_{\varphi_i} B_x - \beta^\top C_y \partial_{\varphi_i} A \tag{3.13}$$

$$\partial_{\varphi_i} J_4 = \gamma^\top M \partial_{\varphi_i} B_y + \gamma^\top C_x \partial_{\varphi_i} A \tag{3.14}$$

$$\partial_{\varphi_i} B_x : \quad \frac{1}{\mu_0} \left(B_x^\top, B_y^\top\right) \begin{pmatrix} -\left(C_y^p\right)^\top & -C_y^p \\ \left(C_x^p\right)^\top & C_x^p \end{pmatrix} \begin{pmatrix} \partial_{\varphi_i} B_x \\ \partial_{\varphi_i} B_x \end{pmatrix} + \beta^\top M \partial_{\varphi_i} B_x \tag{3.15}$$

$$\partial_{\varphi_i} B_y : \quad \frac{1}{\mu_0} \left(B_x^\top, B_y^\top\right) \begin{pmatrix} \left(C_x^p\right)^\top & C_x^p \\ \left(C_y^p\right)^\top & C_y^p \end{pmatrix} \begin{pmatrix} \partial_{\varphi_i} B_y \\ \partial_{\varphi_i} B_y \end{pmatrix} + \gamma^\top M \partial_{\varphi_i} B_y \tag{3.16}$$

$$\partial_{\varphi_i} A : \quad \alpha^\top S \partial_{\varphi_i} A - \beta^\top C_y \partial_{\varphi_i} A + \gamma^\top C_x \partial_{\varphi_i} A \tag{3.17}$$

$$\partial_{\varphi_i} S : \quad \alpha^\top \partial_{\varphi_i} S A \tag{3.18}$$

From (3.15):

$$\beta = -\frac{1}{\mu_0} \left(M^\top\right)^{-1} \left(B_x^\top, B_y^\top\right) \begin{pmatrix} -\left(C_y^p\right)^\top & -C_y^p \\ \left(C_x^p\right)^\top & C_x^p \end{pmatrix} \tag{3.19}$$

From (3.16):

$$\gamma = -\frac{1}{\mu_0} \left(M^\top\right)^{-1} \left(B_x^\top, B_y^\top\right) \begin{pmatrix} \left(C_x^p\right)^\top & C_x^p \\ \left(C_y^p\right)^\top & C_y^p \end{pmatrix} \tag{3.20}$$

From (3.17):

$$\alpha = \left(S^\top\right)^{-1} C_y^\top \beta - \left(S^\top\right)^{-1} C_x^\top \gamma \tag{3.21}$$

If we choose α, β, γ such that (3.15), (3.16) and (3.17) are equal to 0, then:

$$\partial_{\varphi_i} J = \alpha^\top \partial_{\varphi_i} S A \tag{3.22}$$

Appendix

Weak formulation

For $\gamma = 0$.

$$\nabla \times \left(\frac{1}{\mu} \left(\nabla \times A \right) \right) = J \tag{3.23}$$

The θ is test function.

$$\int_{\Omega} \nabla \times \left(\frac{1}{\mu} \left(\nabla \times A \right) \right) \theta \mathrm{d}S = \int_{\Omega} J \theta \mathrm{d}S \tag{3.24}$$

$$\begin{aligned} \nabla \times A &= \left(\partial_y A, -\partial_x A \right) = \boldsymbol{B} \\ \nabla \times \boldsymbol{B} &= \partial_x B_y - \partial_y B_x = -\partial_x \partial_x A - \partial_y \partial_y A = -\Delta A \end{aligned} \tag{3.25}$$

$$\int_{\Omega} -\frac{1}{\mu} \Delta A \theta \mathrm{d}S = \int_{\Omega} J \theta \mathrm{d}S \tag{3.26}$$

Green's first identity:

$$\int_{\Omega} \Delta F \theta \mathrm{d}S + \int_{\Omega} \nabla F \nabla \theta \mathrm{d}S = \int_{\partial\Omega} \left(\nabla F \boldsymbol{n} \right) \theta \mathrm{d}l \tag{3.27}$$

$$\int_{\Omega} -\frac{1}{\mu} \Delta A \theta \mathrm{d}S = \int_{\Omega} \frac{1}{\mu} \nabla A \nabla \theta \mathrm{d}S - \int_{\partial\Omega 1} \frac{1}{\mu} \left(\nabla A \boldsymbol{n} \right) \theta \mathrm{d}l - \int_{\partial\Omega 2} \frac{1}{\mu} \left(\nabla A \boldsymbol{n} \right) \theta \mathrm{d}l \tag{3.28}$$

$\theta \in f\left(\Omega\right) : \theta|_{\partial\Omega 1} = 0, \theta|_{\partial\Omega 2} = 0 \rightarrow \int_{\partial\Omega 1} \frac{1}{\mu} \left(\nabla A \boldsymbol{n} \right) \theta \mathrm{d}l = 0, \int_{\partial\Omega 2} \frac{1}{\mu} \left(\nabla A \boldsymbol{n} \right) \theta \mathrm{d}l = 0$

Discretization

$$A\left(x, y, t\right) \approx \sum_{n=1}^{N} A^n\left(t\right) \lambda^n\left(x, y\right) \tag{3.29}$$

$$J\left(x, y, t\right) \approx \sum_{n=1}^{N} J^n\left(t\right) \lambda^n\left(x, y\right) \tag{3.30}$$

$$\theta\left(x, y, t\right) \approx \sum_{n=1}^{N} \theta^n\left(t\right) \lambda^n\left(x, y\right) \tag{3.31}$$

$$\int_{T_n} \frac{1}{\mu} \nabla \left(\sum_{n=1}^{N} A^n \lambda^n \right) \nabla \left(\sum_{n=1}^{N} \theta^n \lambda^n \right) \mathrm{d}S = \int_{T_n} \sum_{n=1}^{N} J^n \lambda^n \sum_{n=1}^{N} \theta^n \lambda^n \mathrm{d}S \tag{3.32}$$

T_n is one descrete element of geometry, in our case it is a triangle.

$$\begin{aligned} \sum_{n=1}^{N} \int_{T_n} \frac{1}{\mu} \nabla \left(A^n \lambda^n \right) \nabla \left(\theta^m \lambda^m \right) \mathrm{d}S &= \sum_{n=1}^{N} \int_{T_n} J^n \lambda^n \theta^m \lambda^m \mathrm{d}S \\ \sum_{n=1}^{N} A^n \int_{T_n} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m \mathrm{d}S &= \sum_{n=1}^{N} J^n \int_{T_n} \lambda^n \lambda^m \mathrm{d}S \end{aligned} \tag{3.33}$$

We can put A^n and J^n out of ∇ and out of $\int_{T_n}$, because they no longer depend on coordinate system.

$$S = \sum_{n=1}^{N} \int_{T_n} \frac{1}{\mu} \nabla \lambda^n \nabla \lambda^m \mathrm{d}S$$
$$M = \sum_{n=1}^{N} \int_{T_n} \lambda^n \lambda^m \mathrm{d}S$$

$$SA = MJ \rightarrow A = S^{\top} MJ \tag{3.34}$$