(M8) Continuous FT

Properties of (M3)

F.T.

DSAB

Time & Freq. Scaling.

$x(t) \longleftrightarrow X(\omega)$.

scaling $\quad x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$.

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t}\, dt.$$

$\tau = at$ / $a>0$

Limits unchanged

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} \cdot \frac{d\tau}{a}.$$

$$= \frac{1}{a} \cdot X\left(\frac{\omega}{a}\right)$$

For $a<0$ $\quad \tau = at$. $\quad t = \tau/a$. $(-) -\infty$ to ∞

Limits flip to ∞ to $-\infty$

τ.

but integration $-\infty$ to ∞

fix limit order

$$\therefore \mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} \frac{d\tau}{a} \cdot (-1).$$

$$= \frac{-1}{a} X\left(\frac{\omega}{a}\right).$$

$\therefore \quad x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

Duality of Fourier Transform.

(1) Comes from definitions of FT pair.

$$X(\omega) = \int x(t) e^{-j\omega t} dt.$$

$$x(t) = \frac{1}{2\pi} \int X(\omega) e^{j\omega t} d\omega.$$

Definite symmetry:

Given $g(t) \leftrightarrow f(\omega)$

then. $f(t) \leftrightarrow 2\pi g(-\omega)$.

$$g(t) = \frac{1}{2\pi} \int f(\omega) e^{j\omega t} d\omega. \quad -①$$

$$F\{f(t)\} = \int f(t) e^{-j\omega t} dt. \quad -②$$

from ①, ②

$$= 2\pi \cdot \left[\frac{1}{2\pi} \int f(t) e^{j(-\omega)t} dt\right]$$

similar to ① $[\omega \to t, t \to \omega]$

$$= 2\pi \; g(-\omega).$$

$$\frac{1}{2\pi} \int f(u) e^{j(-v)u} du$$

$$\downarrow$$

$$g(-v).$$

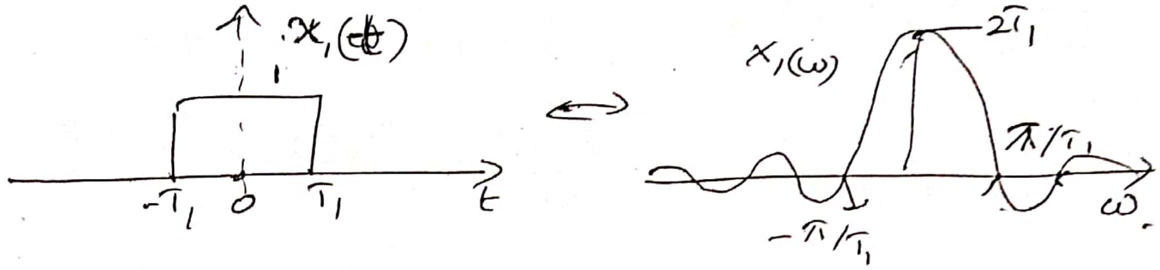
$\boxed{\text{ex: Tutorials}}$

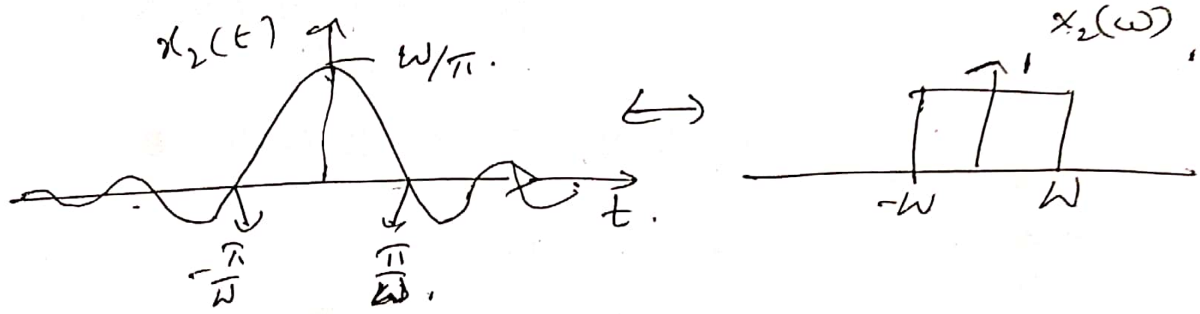
e.g. $x_1(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$

$\cancel{X(\omega)}$

$$X_1(\omega) = \frac{2\sin \omega T_1}{\omega} = 2T_1 \operatorname{sinc}\left[\frac{\omega T_1}{\pi}\right].$$

ie,





Given. $x_1(t) \longleftrightarrow X_1(\omega)$

Find $\mathcal{F}\{x_2(t)\}$. define ~~$x_2(t) = \frac{W}{\pi} \cdot \frac{\sin(Wt/\pi)}{t}$~~ ?

Given. $X_2(\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$

$$x_2(t) = \frac{1}{2\pi}\int X_2(\omega)\, e^{j\omega t}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{1}{2\pi} \left.\frac{e^{j\omega t}}{jt}\right|_{-W}^{W}$$

$$= \frac{1}{2\pi} \cdot \frac{1}{jt} \cdot \left[e^{jWt} - e^{-jWt}\right] = \frac{1}{\pi t}\sin(Wt)$$

$$\therefore\ x_2(t) = \frac{W}{\pi}\,\frac{\sin(Wt)}{Wt}.$$

~~Not enough. — F.T.Q.~~

Parseval's theorem.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

$$LHS \rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

$$= \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt.$$

$$= \int_{-\infty}^{\infty} x(t) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \cdot dt.$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t} \cdot \int_{-\infty}^{\infty} X(\theta)\, e^{+j\theta t} d\theta\, d\omega\, dt$$

$$= \frac{1}{(2\pi)^2} \iint X(\theta)\, X^*(\omega) \cdot \underline{\int_{-\infty}^{\infty} e^{j(\theta-\omega)t} dt}\, d\theta\, d\omega.$$

$$= \frac{1}{(2\pi)^2} \iint X(\theta) X^*(\omega) \cdot 2\pi\, \delta(\theta-\omega)\, d\theta\, d\omega.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

section. date.

(2) $x(t) = \delta(t)$.

$X(\omega) = \int \delta(t)\, e^{-j\omega t}\, dt = 1$.

$\therefore\ \delta(t) \longleftrightarrow 1$.

From duality $1 \longleftrightarrow 2\pi\, \delta(\omega)$.

$x(t) = \sin \omega_0 t \quad \forall\, t > 0$

$$X(\omega) = \int x(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} \sin \omega_0 t\; e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j2}\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2j} \int_0^{\infty} \left[e^{-j(\omega-\omega_0)t} - e^{-j(\omega+\omega_0)t} \right] dt$$

$$= \frac{1}{2j} \left[\frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)} \Big|_0^{\infty} - \frac{e^{-j(\omega+\omega_0)t}}{-j(\omega+\omega_0)} \Big|_0^{\infty} \right]$$

$$= \frac{1}{2j} \left(\frac{-1}{j(\omega-\omega_0)} - \frac{1}{j(\omega+\omega_0)} \right) = \frac{-1}{2}\left(\frac{1}{\omega-\omega_0} - \frac{1}{\omega+\omega_0} \right)$$

$$= \frac{-\omega_0}{\omega^2 - \omega_0^2} = \frac{\omega_0}{\omega_0^2 - \omega^2}$$

Convolution property

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t).$$

(1)

$$y(t) = h(t) * x(t).$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

$$Y(\omega) = F\{y(t)\} = \int y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \cdot e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt \right) \cdot d\tau.$$

$$t-\tau = t'$$

$$= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t')\, e^{-j\omega t'}\, dt' \right) \cdot e^{-j\omega\tau}\, d\tau$$

$$= H(\omega) \cdot \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$\therefore\ Y(\omega) = H(\omega)\, X(\omega).$$

(2) Multiplication property

$$y(t) = x_1(t)\, x_2(t).$$

$$Y(\omega) = F\{y(t)\} = \int x_1(t) x_2(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega_1) e^{j\omega_1 t} d\omega_1 \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega_2) e^{j\omega_2 t} d\omega_2 \, e^{-j\omega t} dt$$

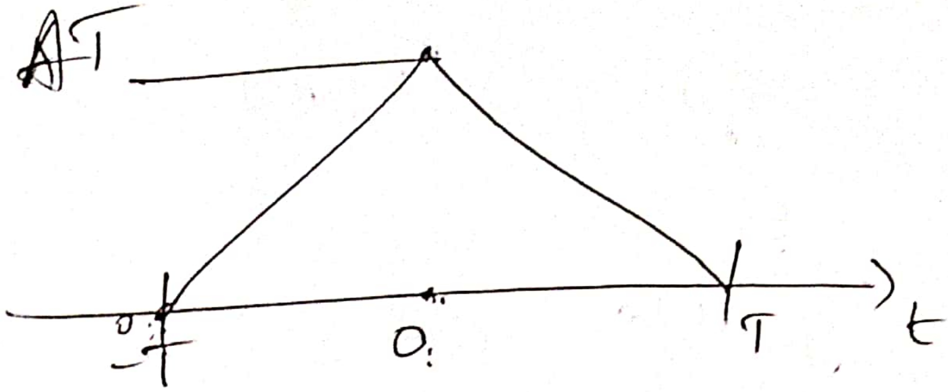
$$= \frac{1}{(2\pi)^2} \iint X_1(\omega_1) X_2(\omega_2) \cdot \int_{-\infty}^{\infty} e^{-j(\omega-\omega_1-\omega_2)t} dt \, d\omega_1 d\omega_2$$

$$= \frac{1}{(2\pi)^{\cancel{2}}} \iint X_1(\omega_1) X_2(\omega_2) \, \delta(\omega-\omega_1-\omega_2) \, d\omega_1 d\omega_2 .$$

$$= \frac{1}{2\pi} \int X_1(\omega_1) \left[\int X_2(\omega_2) \, \delta(\underline{\omega-\omega_1} - \omega_2) \, d\omega_2 \right] d\omega_1$$

$$= \frac{1}{2\pi} \int X_1(\omega_1) \, X_2(\omega-\omega_1) \, d\omega_1$$

$$\therefore Y(\omega) = \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$



$$A(t+T) \qquad -T<t<0$$

$$\cancel{A(t-T)} \quad A(T-t) \qquad 0<t<T$$

$$\int_{-T}^{0} A(t+T)\, e^{-j\omega t}\, dt + \int_{0}^{T} A(T-t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{T} AT\, e^{-j\omega t}\, dt + A\left[\frac{t\, e^{-j\omega t}}{-j\omega}\Bigg|_{-T}^{0} - \int_{-T}^{0} \frac{e^{-j\omega t}}{-j\omega}\, dt \right]$$

$$- A\left[\frac{t\, e^{-j\omega t}}{-j\omega}\Bigg|_{0}^{T} - \int_{0}^{T} \frac{e^{-j\omega t}}{-j\omega}\, dt \right]$$

$$= AT \cdot \frac{e^{-j\omega T} - e^{+j\omega T}}{-j\omega} + A\left[\frac{T}{-j\omega} e^{j\omega T} + \frac{e^{-j\omega t}}{\omega^2}\Bigg|_{-T}^{0} \right]$$

$$- A\left[\frac{T e^{-j\omega T}}{-j\omega} + \frac{e^{-j\omega t}}{\omega^2}\Bigg|_{0}^{T} \right]$$

$$= 2AT\,\frac{\sin\omega T}{\omega} + \frac{AT}{-j\omega}\left[e^{j\omega T} - e^{-j\omega T}\right]$$

$$+ A\,\frac{1 - e^{j\omega T}}{\omega^2} - A\cdot\frac{e^{-j\omega T} - 1}{\omega^2}$$

$$= 2AT\,\frac{\sin\omega T}{\omega} - 2AT\,\frac{\sin\omega T}{\omega} + \frac{A+A}{\omega^2}$$

$$- \frac{A}{\omega^2}\left[e^{j\omega T} + e^{-j\omega T}\right]$$

$$= \frac{2A}{\omega^2} - \frac{2A}{\omega^2}\cos\omega T$$