Vector Identities:-

Del Applied Twice to point functions.

We have grad f. & curl F as vectors and div F is a scalar point functions. So we can find div & curl of both grad f & curl F and grad of div F.

So following are the possible cases of:

1) div grad f $= \nabla\cdot(\nabla f) = \nabla^2 f$

$$= \nabla\cdot\left(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)f = \nabla^2 f.$$

2) curl grad f $= \nabla\times(\nabla f) = 0$.

$$= \nabla\times\left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} =$$

$$= i\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial y\partial z}\right) - j\left(\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial x\partial z}\right) + k\left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial x\partial y}\right)$$

$$= i(0) - j(0) + k(0) = 0.$$

Scanned by CamScanner

3). div curl $F = \nabla\cdot(\nabla\times F)$. let $F = fi + \phi j + \psi k$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix} = 0$$

4) curl curl $F = \nabla\times(\nabla\times F)$

directly by using the vector triple product formula i.e.,

$$(a\times(b\times c)) = b(a\cdot c) - (a\cdot b)c$$

So., $(\underset{a}{\nabla}\times(\underset{b}{\nabla}\times \underset{c}{F})) = \nabla(\nabla\cdot F) - (\nabla\cdot\nabla)F$

$$= \nabla(\nabla\cdot F) - \nabla^2 F$$

curl curl F = grad div $F - \nabla^2 F$

5) grad div $F = \nabla(\nabla\cdot F)$ = curl curl $F + \nabla^2 F$

from the identity 4.

Del Applied to product of point functions.

let f, g be the scalar point functions & F, G be a vector point functions. So possible products are

fg, fG, $F \cdot G$, $F \times G$

$fg \downarrow$ Scalar. $fG \downarrow$ vector. $F \cdot G \downarrow$ Scalar. $F \times G \downarrow$ Vector

So (1) grad (fg). (2) div (fG) & (3) Curl (fG) ~~div (F.G)~~ (4) grad$(F \cdot G)$. (5) div$(F \times G)$ (6) Curl $(F \times G)$

So totally there are 6 identities.

(1) grad $(fg) = \nabla(fg) = f$ grad $g + g$ grad f.

proof $\nabla(fg) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial y}\right)\left(fg\right)$

$$= i\frac{\partial fg}{\partial x} + j\frac{\partial fg}{\partial y} + k\frac{\partial fg}{\partial y}$$

$$= i\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right) + j\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) + k\left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)$$

$$= f\left(i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z}\right) + g\left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)$$

$$= f\nabla g + g\nabla f$$

$= f$ grad $g + g$ grad f.

Scanned by CamScanner

2). $\operatorname{div}(fG) = (\operatorname{grad} f)\cdot G + f(\operatorname{div} G)$

Proof: $\nabla\cdot(fG)$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(fG)$$

$$= i\cdot\frac{\partial fG}{\partial x} + j\cdot\frac{\partial fG}{\partial y} + k\cdot\frac{\partial fG}{\partial z}$$

$$= i\cdot\left(f\frac{\partial G}{\partial x} + G\frac{\partial f}{\partial x}\right) + j\cdot\left(f\frac{\partial G}{\partial y} + G\frac{\partial f}{\partial y}\right) + k\cdot\left(f\frac{\partial G}{\partial z} + G\frac{\partial f}{\partial z}\right)$$

$$= f\left(i\cdot\frac{\partial G}{\partial x} + j\cdot\frac{\partial G}{\partial y} + k\cdot\frac{\partial G}{\partial z}\right) + G\cdot\left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)$$

$$= f\operatorname{div} G + G\cdot\operatorname{grad} f.$$

another proof :-

consider $\nabla\cdot(fG)$.

$f_c\ \nabla G + G_c\ \nabla f$. (treating f_c & G_c as constants)

$f(\nabla\cdot G) + G_c\cdot(\nabla f)$.

$G\cdot(\operatorname{grad} f) + f(\operatorname{div} G)$.

3) ~~Curl fG~~ $\operatorname{Curl}(fG) = \nabla\times(fG)$

Similar to above proof

$f(\nabla\times G) + G\times(\nabla f)$

$f\operatorname{curl} G + (G\times\operatorname{grad} f)$

Scanned by CamScanner

4) grad $(F.G) = (F.\nabla)G + (G.\nabla)F + F \times \text{Curl } G + G \times \text{Curl} F$

Proof :- $\nabla(F.G) = F_c(\nabla.G) + G_c(\nabla.F)$

$= F(\nabla.G) + G(\nabla.F)$ (treating F_c & G_c as constants)

from vector triple product.

$(a \times (b \times c)) = b\cdot(a.c) - c(a.b)$

so $b(a.c) = (a \times (b \times c)) + c(a.b)$

by substituting

$= F(\nabla.G) + G(\nabla.F)$ becomes

$= F \times (\nabla \times G) + G(\nabla.F) + G \times (\nabla \times F) + F(\nabla.G)$

$= (F \times \text{Curl } G) + G(\text{div} F) + G \times \text{Curl} F + F \text{ div} F.$

5) div $(F \times G) = \nabla\cdot(F \times G)$

$F.(\nabla \cdot G) + G(\nabla F)$

$F\cdot(\nabla \times G) + G\cdot(\nabla \times F)$ (placing the operators at appropriate class)

Scanned by CamScanner

Identity 6. $\nabla \times (F \times G)$

$= \nabla \times (F_c \times G) + \nabla \times (F \times G_c)$ (∵ by treating F_c & G_c as constant)

by using triple product of vectors

i.e., $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$

So $\nabla \times (F_c \times G) = (\nabla \cdot G) F_c - (F_c \cdot \nabla) G$.

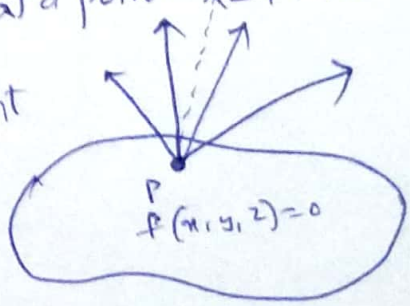
Similarly $\nabla \times (G_c \times F) = (\nabla \cdot F) G_c - (G_c \cdot \nabla) F$

So $\nabla \times (F \times G) = (\nabla \cdot G) F - \underline{(F \cdot \nabla)} G + (\nabla \cdot F) G - \underline{(G \cdot \nabla)} F$

are scalar operators on F & G.

Maximum Derivative of a scaplar point punction

Maximum derivative of a Surface or a Scalar point function occurs in the direction normal vector to the surface at a point P → max.



* we know that for a Scalar point function / Surface ∇f or grad f is the normal to the surface at a perticuler point P.

. Maximum value is magnitude of ∇f. i.e

$\| \nabla f \|$.

Scanned by CamScanner

Integration of vectors

if $F(t) \cdots G(t)$ be such that

$$\frac{d}{dt}(G(t)) = F(t)$$, then

$G(t)$ is called integral of $F(t)$

& $\int F(t) = G(t)$.

Similarly if $\vec{c}$ is any arbitrary constant

& $\frac{d}{dt} G(t) = F(t)$

$F(t) = \frac{d}{dt}(G(t) + \vec{c})$. then

$\int F(t)\,dt = G(t) + c$ is called indefinite integral.

if $\frac{d}{dt} G(t) = F(t)$ for all values in (a, b)

then $\int_a^b F(t)\,dt = \left[G(t)\right]_a^b =$

$G(b) - G(a)$ is called

definite integral

Scanned by CamScanner

Line integral

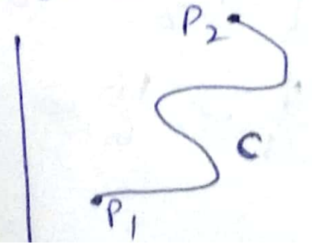
consider a continous function F defined at each point of curve c in space..with joining points P_1 & P_2 with values u_1 & u_2 respectively. Then the tangential integral of vector point function F along the curve c is

$$\int_C F(R)\cdot dR = \int_{P_1}^{P_2} (f\,i + \phi\,j + \psi\,k)\cdot(dx\,i + dy\,j + dz\,k).$$

$$= \int_{P_1}^{P_2} (f\,dx + \phi\,dy + \psi\,dz)$$

* If F represents the force acting on a particle moving along the arc P_1P_2 then the total work done during the displacement from P_1 to P_2 is given by $\int_{P_1}^{P_2} F\cdot dR$.

* If the curve is a closed curve then the integration is represented as $\oint F\cdot dR$



Scanned by CamScanner

① $F = 3xy\,I - y^2\,J$ evaluate $\int F \cdot dR$ where C is the curve in the xy plane $y = 2x^2$ from (0,0) to (1,2)

Sol

Given $F = 3xy\,\hat{i} - y^2\,\hat{j}$.

Since z=0, we can take $dR = dx\,i + dy\,j$;

$$\int F \cdot dR = \int (3xy\hat{i} - y^2\hat{j}) \cdot (dx\,i + dy\,j)$$

$$= \cancel{\int 3xy - y^2} = \int 3xy\,dx - y^2\,dy$$

now by substituting $y = 2x^2$ & taking x from 0 to 1, we get

$$\int_0^1 3x(2x^2)\,dx - (2x^2)^2 \cdot d(2x^2)$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= 6\left[\frac{x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{18-32}{12} = \frac{-14}{12} = \frac{-7}{6}.$$

Scanned by CamScanner

(2) A vector field is given by $F = \sin y\, I + x(1+\cos y)\, J$ evaluate the line integral over a circular path given by $x^2+y^2=a^2$, $z=0$

Sol:- Since particle moves in xy plane $z=0$.

so now $dR = dx\, i + dy\, j$.

$$\oint_C F \cdot dR = \oint \left(\sin y\, i + x(1+\cos y)\, j\right) \cdot (dx\, i + dy\, j)$$

$$= \oint \sin y\, dx + (1+\cos y)\, x \cdot dy.$$

$$= \oint \underline{\sin y\, dx} + x\, dy + \underline{\cos y \cdot x\, dy}.$$

$$= \oint d(x \sin y) + x\, dy =$$

now let $x = a\cos t$, $y = a \sin t$

$dx = -a\sin t\, dt$, $dy = a\cos t\, dt$

& t varies from 0 to 2π (full sphere)

$$= \int_0^{2\pi} d(a\cos t \cdot \sin(a\sin t)) + a\cos t \cdot a\cos t\, dt$$

$$= \int_0^{2\pi} d(a\cos t \cdot \sin(a\sin t)) + a^2 \cos^2 t \cdot dt$$

$$= \left[a\cos t \cdot \sin(a\sin t)\right]_0^{2\pi} + a^2 \int_0^{2\pi} \frac{(1+\cos 2t)\, dt}{2}$$

$$= 0 + \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = a^2\left[\frac{2\pi}{2}\right] = \pi a^2$$

Scanned by CamScanner

(3) If $\bar{F} = (3x^2+6y)i - 14yzj + 20xz^2k$ evaluate $\int \bar{F}\cdot d\bar{R}$ where

1) C is line joining point (0,0,0) to (1,1,1).

2) C is given by $x=t$, $y=t^2$, $z=t^3$ from (0,0,0) to (1,1,1).

Sol

1) equation of line joining (0,0,0) to (1,1,1)

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \Rightarrow x=t,\ y=t,\ z=t$$

$$d\bar{R} = dx\,i + dy\,j + dz\,k = dt\,i + dt\,j + dt\,k$$

$$= (i+j+k)\,dt$$

$$\int (3x^2+6y)i - 14yzj + 20xz^2k)\cdot(i+j+k)\,dt$$

$$= \int_C (3t^2+6t)i - 14t^2j + 20t^3k)\cdot(i+j+k)\,dt$$

$$= \int_C (3t^2+6t - 14t^2 + 20t^3)\,dt$$

$$= \int_0^1 (3t^2+6t - 14t^2 + 20t^3)\,dt$$

$$= \left(3\frac{t^3}{3} + 6\frac{t^2}{2} - 14\frac{t^3}{3} + 20\frac{t^4}{4}\right]_0^1$$

$$= \left[1 + 3 - \frac{14}{3} + 5\right)$$

$$= \frac{13}{3}$$

Scanned by CamScanner

2) c is the curve given by $x=t,\ y=t^2,\ z=t^3$

from $(0,0,0)$ to $(1,1,1)$.

$$\frac{d\bar{R}}{dt} = \cancel{xi}\cdot \frac{d}{dt}(ti+t^2j+t^3k)$$

$$d\bar{R} = (i+2tj+3t^2k)\cdot dt$$

$$F = (3x^2+6y)i - 14yzj + 20xz^2k$$

$$= \cancel{ai}$$

$$= (3t^2+6t^2)i - 14t^2\cdot t^3 j + 20t\cdot t^{3^2}\cdot k$$

$$= 9t^2 i - 14t^5 j + 20t^7\cdot k$$

$$\int F\cdot dR = \int (9t^2 i - 14t^5 j + 20t^7)\cdot(i+2tj+3t^2k)\,dt$$

$$= \int (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left[\frac{9t^3}{3} - 28\frac{t^7}{7} - 60\frac{t^{10}}{10}\right]_0^1$$

$$= [3-4+6]$$

$$=5$$

Scanned by CamScanner

~~Curl~~

Solenoidal & Irrotational Vectors. point functions.

Solenoidal Vector point function :- A vector point function F, whose divergence is zero is called Solenoidal Vector point function. i-e, $\nabla \cdot F = 0$.

ex:- $F = (x+3y)i + (y-2z)j + (x-2z)k$.

$$\text{div } F = \nabla \cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left((x+3y)i + (y-2z)j + (x-2z)k\right)$$

$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x-2z) = 1+1-2 = 0.$$

Irrotational Vector point function :- A vector point function F, whose curl is zero is called as irrotational vector point function.

i-e, $\nabla \times F = 0$.

ex $F = xi + yj + zk$ then show that curl $F = 0$

$$\nabla \times F = \text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - j\left(\frac{\partial}{\partial z} - \frac{\partial x}{\partial z}\right) + k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0.$$

Scanned by CamScanner