Graphs and Spanning Trees

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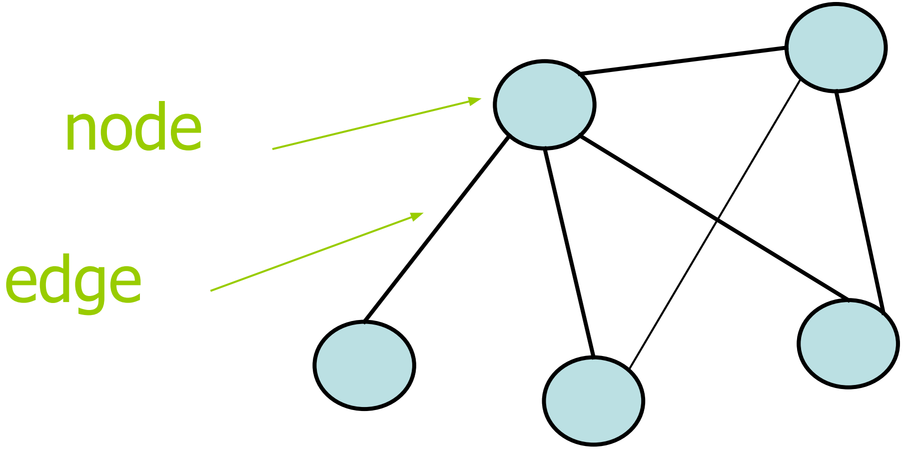
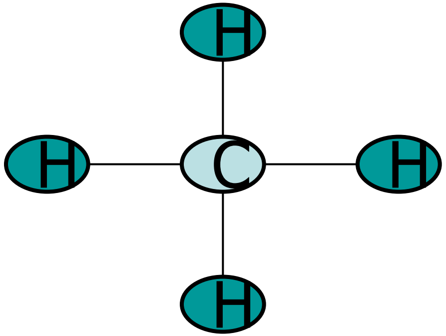
What is a graph?

- Graph represents relationship among the data items
- A graph G consists of
  - a set V of nodes (vertices)
  - a set E of edges (each edge connects two nodes)

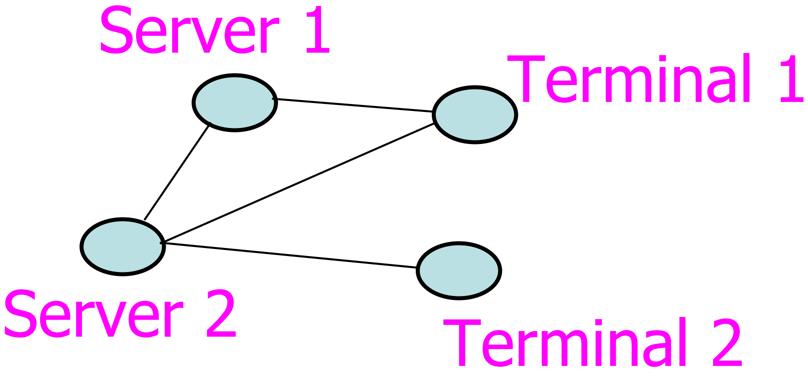


Examples of graphs

Molecular Structure



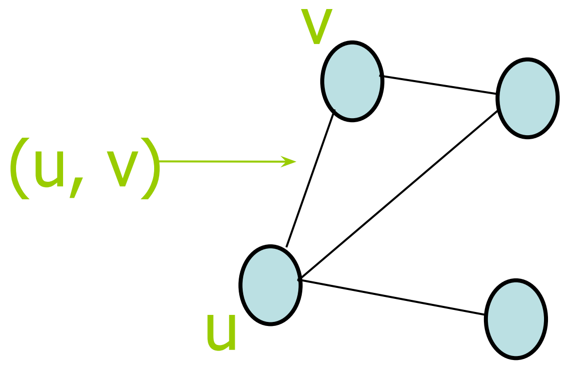
Computer Network



Other examples: electrical and communication networks, airline routes, flow chart, etc.

Formal Definition of graph

- The set of nodes is denoted as V
- For any nodes u and v, if u and v are connected by an edge, such edge is denoted as (u, v)
- The set of edges is denoted as E
- A graph G is defined as a pair (V, E)

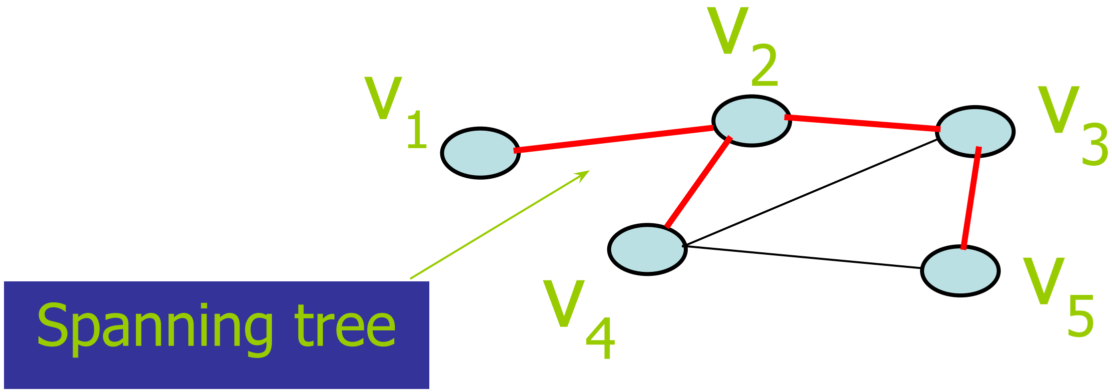


Problems related to Graph

- Spanning Tree
- Minimum Spanning Tree
- Shortest Path

Spanning Tree

- Given a connected undirected graph G, a spanning tree of G is a subgraph of G that contains all of G's nodes and enough of its edges to form a tree.



Spanning tree is not unique!

$E=\{(v_1,v_2),(v_2, v_3), (v_2, v_4), (v_3, v_5)\}$

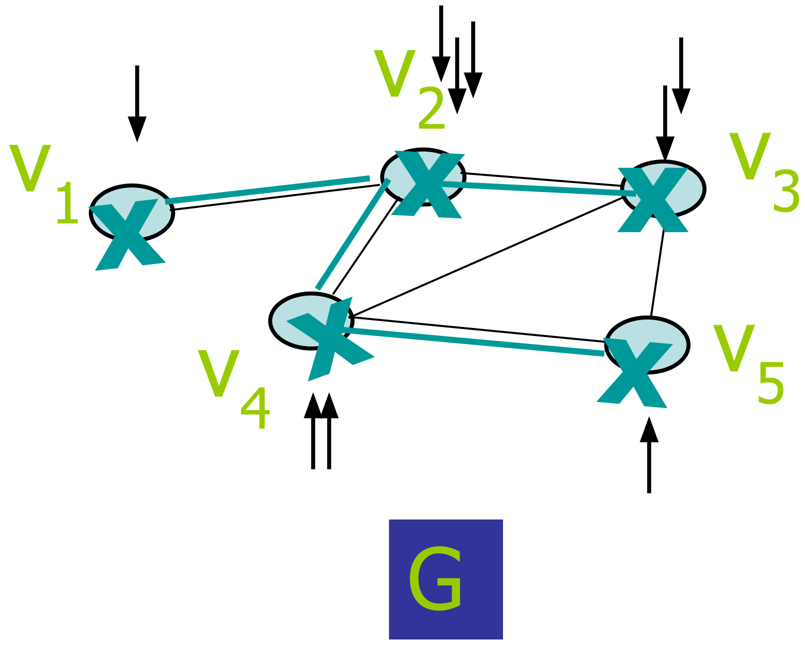
DFS spanning tree

```
Algorithm dfsSpanningTree(v)
mark v as visited;
for (each unvisited node u adjacent to v) {
    mark the edge from u to v;
    dfsSpanningTree(u);
}
```

- Similar to DFS, the spanning tree edges can be generated based on BFS traversal.

Generating SP based on DFS

	stack
v_3	v_3
v_2	v_3, v_2
v_1	v_3, v_2, v_1
backtrack	v_3, v_2
v_4	v_3, v_2, v_4
v_5	v_3, v_2, v_4, v_5
backtrack	v_3, v_2, v_4
backtrack	v_3, v_2
backtrack	v_3
backtrack	empty

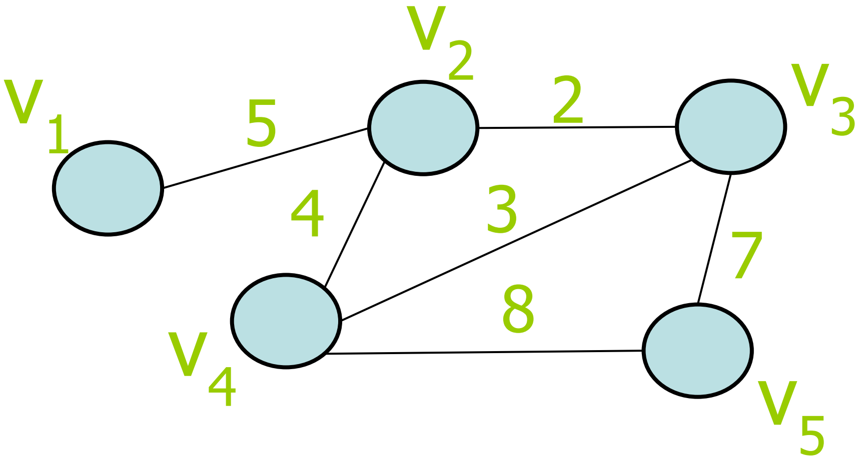


Minimum Spanning Tree (MST)

- Consider a connected undirected graph where
  - Each node x represents a country x
  - Each edge (x, y) has a number which measures the cost of placing telephone line between country x and country y
- Problem: connecting all countries while minimizing the total cost
- Solution: find a spanning tree with minimum total weight, that is, minimum spanning tree

Formal definition of MST

- Given a connected undirected graph G.
- Let T be a spanning tree of G.
- $\text{cost}(T) = \sum_{e \in T} \text{weight}(e)$
- MST is a spanning tree T which minimizes cost(T)



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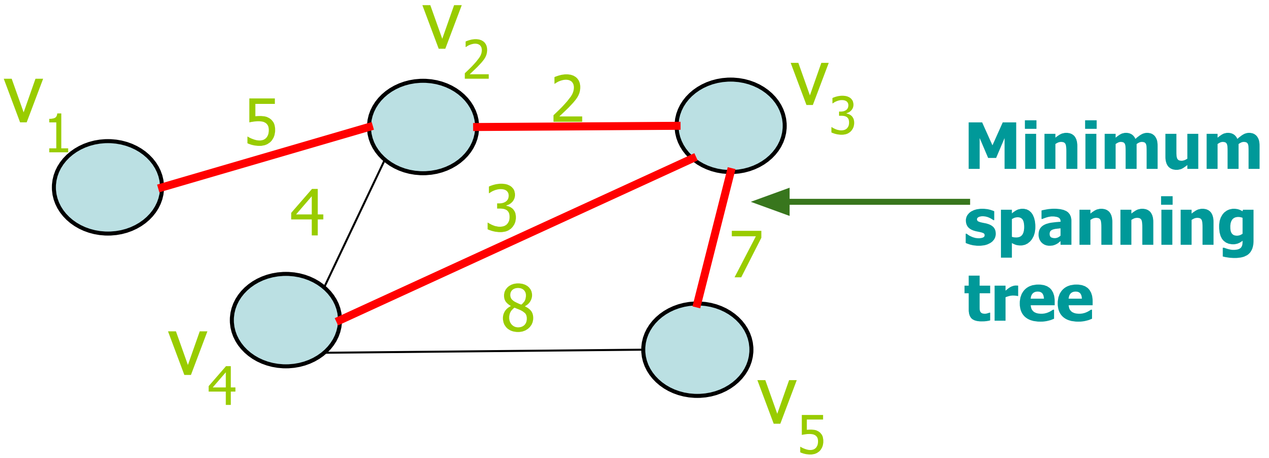


Prim’s algorithm

Algorithm PrimAlgorithm(v)

- Mark node **v** as visited and include it in the minimum spanning tree;
- while (there are unvisited nodes)

  {

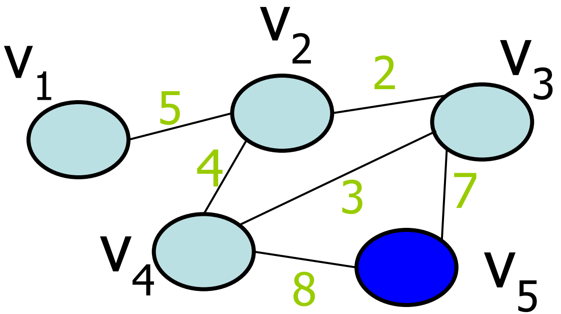
  – find the minimum edge **(v, u)** between a **visited node v** and an **unvisited node u**;

  – mark **u** as visited;

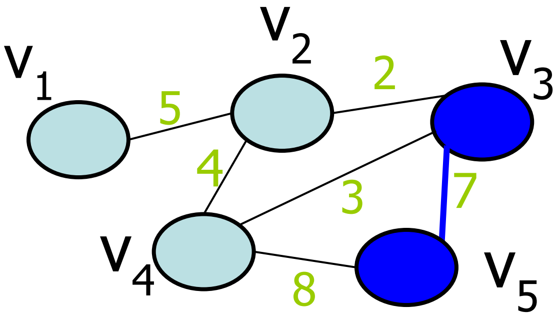
  – add both **v** and **(v, u)** to the minimum spanning tree;

  }

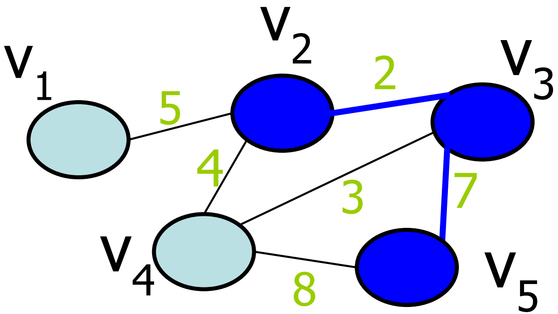
Prim's algorithm



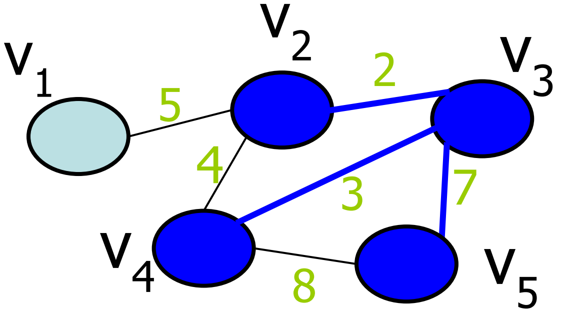
Start from v_5, find the minimum edge attach to v_5



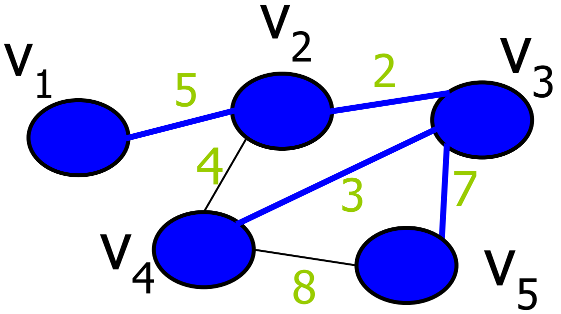
Find the minimum edge attach to v_3 and v_5



Find the minimum edge attach to v_2, v_3 and v_5



Find the minimum edge attach to v_2, v_3 , v_4 and v_5

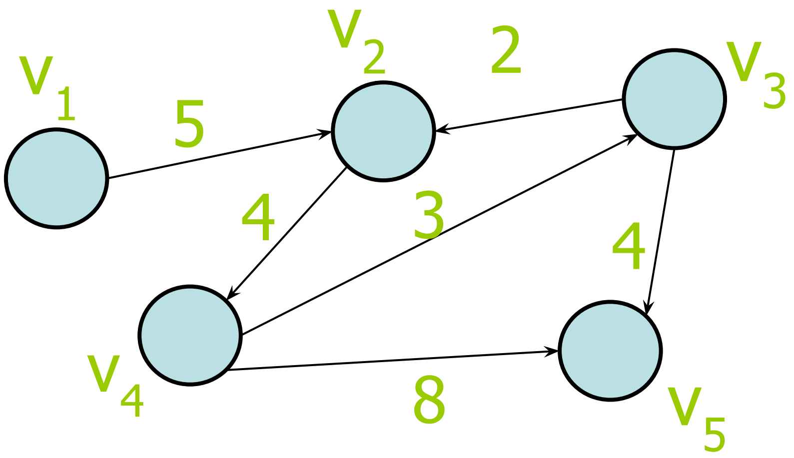


Shortest path

- Consider a weighted directed graph
  - Each node x represents a city x
  - Each edge (x, y) has a number which represent the cost of traveling from city x to city y
- Problem: find the minimum cost to travel from city x to city y
- Solution: find the shortest path from x to y

Formal definition of shortest path

- Given a weighted directed graph G.
- Let P be a path of G from x to y.
- $\mathrm{cost}(P) = \sum_{e \in P} \mathrm{weight}(e)$
- The shortest path is a path P which minimizes cost(P)



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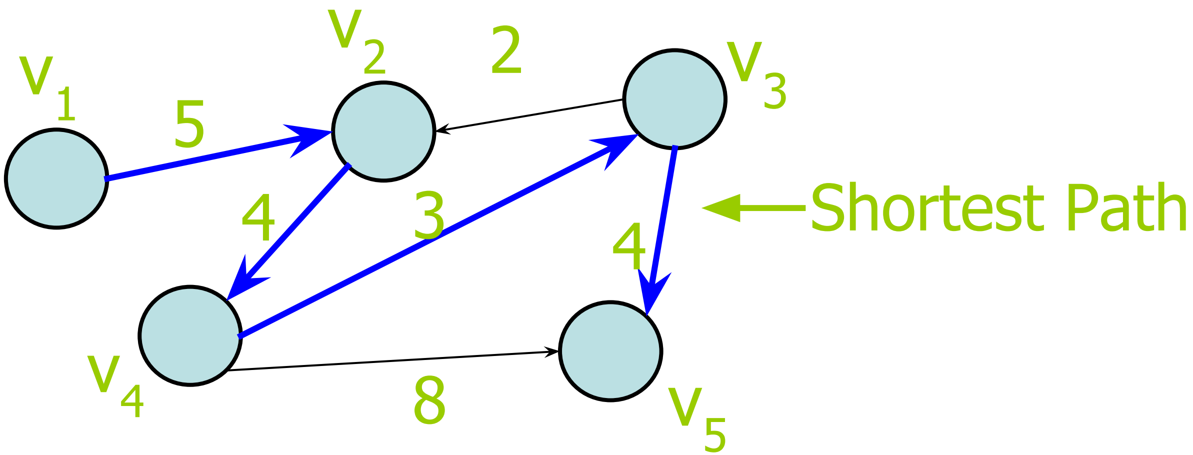


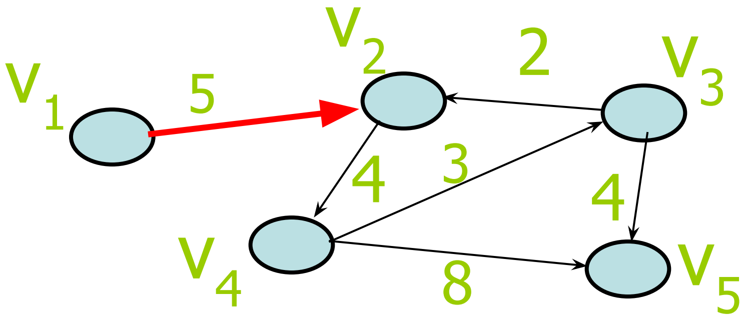
Dijkstra's algorithm

- Let **G** be a graph
- Each edge **(u, v)** has a weight **w(u, v) > 0**
- Find the shortest path starting from $\mathbf{v_1}$ to any node $\mathbf{v_i}$
- Let **VS** be a subset of nodes in **G**
- Let **cost[**$\mathbf{v_i}$**]** be the weight of the shortest path from $\mathbf{v_1}$ **to** $\mathbf{v_i}$ that passes through nodes in **VS** only

Dijkstra's algorithm

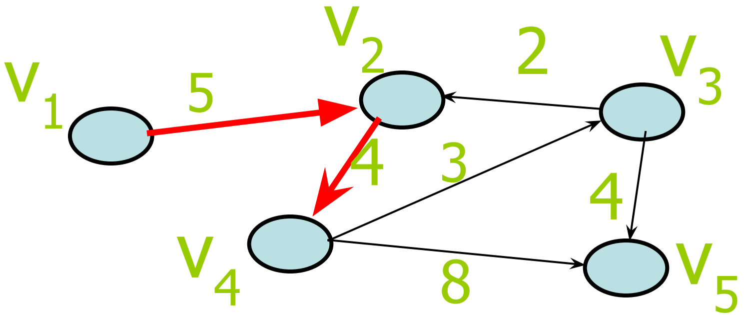
```
Algorithm shortestPath()
n = number of nodes in the graph;
for i = 1 to n
    cost[v_i] = w(v_1, v_i);
VS = { v_1 };
for step = 2 to n {
    find the smallest cost[v_i] s.t. v_i is not in VS;
    include v_i to VS;
    for (all nodes v_j not in VS) {
        if (cost[v_j] > cost[v_i] + w(v_i, v_j))
            cost[v_j] = cost[v_i] + w(v_i, v_j);
    }
}
```

Dijkstra's algorithm



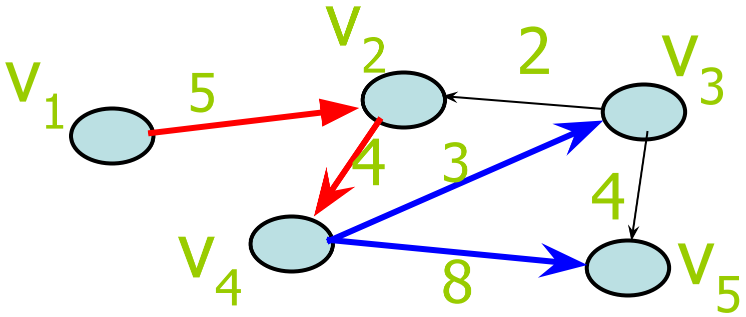
	v	VS	cost[v_1]	cost[v_2]	cost[v_3]	cost[v_4]	cost[v_5]
1		[v_1]	0	5	∞	∞	∞

Dijkstra's algorithm



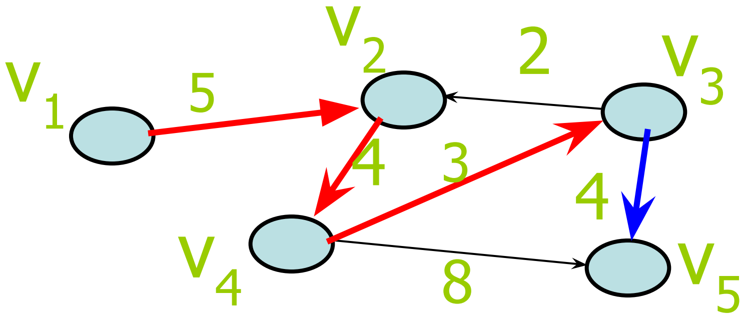
	v	VS	cost[v_1]	cost[v_2]	cost[v_3]	cost[v_4]	cost[v_5]
1		[v_1]	0	5	∞	∞	∞
2	v_2	[v_1, v_2]	0	5	∞	9	∞

Dijkstra's algorithm



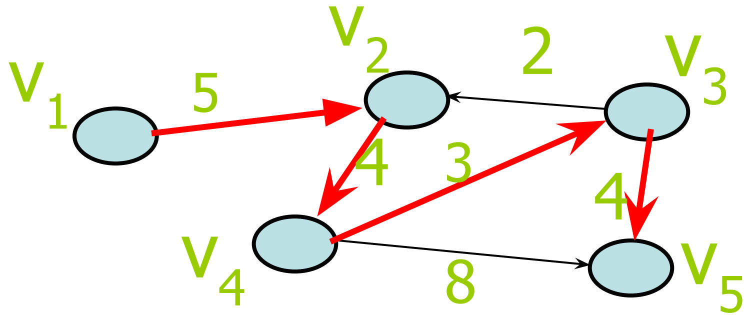
	v	VS	cost[v_1]	cost[v_2]	cost[v_3]	cost[v_4]	cost[v_5]
1		[v_1]	0	5	∞	∞	∞
2	v_2	[v_1, v_2]	0	5	∞	9	∞
3	v_4	[v_1, v_2, v_4]	0	5	12	9	17

Dijkstra's algorithm



	v	VS	cost[v_1]	cost[v_2]	cost[v_3]	cost[v_4]	cost[v_5]
1		[v_1]	0	5	∞	∞	∞
2	v_2	[v_1, v_2]	0	5	∞	9	∞
3	v_4	[v_1, v_2, v_4]	0	5	12	9	17
4	v_3	[v_1, v_2, v_4, v_3]	0	5	12	9	16

Dijkstra's algorithm



	v	VS	cost[v_1]	cost[v_2]	cost[v_3]	cost[v_4]	cost[v_5]
1		[v_1]	0	5	∞	∞	∞
2	v_2	[v_1, v_2]	0	5	∞	9	∞
3	v_4	[v_1, v_2, v_4]	0	5	12	9	17
4	v_3	[v_1, v_2, v_4, v_3]	0	5	12	9	16
5	v_5	[v_1, v_2, v_4, v_3, v_5]	0	5	12	9	16

Summary

- We have studied some basic concepts and algorithms
  - **Spanning Tree**
  - **Minimum Spanning Tree**
  - **Shortest Path**