

4.10 Improper Integrals

4.10.1 Introduction

In Calculus II, students defined the integral $\int_a^b f(x) dx$ over a finite interval $[a, b]$. The function f was assumed to be continuous, or at least bounded, otherwise the integral was not guaranteed to exist. Under these conditions, assuming an antiderivative of f could be found, $\int_a^b f(x) dx$ always existed, and was a number. In this section, we investigate what happens when these conditions are not met.

Definition 337 (Improper Integral) *An integral is an **improper integral** if either the interval of integration is not finite (**improper integral of type 1**) or if the function to integrate is not continuous (not bounded that is becomes infinite) in the interval of integration (**improper integral of type 2**).*

Example 338 $\int_0^\infty e^{-x} dx$ is an improper integral of type 1 since the upper limit of integration is infinite.

Example 339 $\int_0^1 \frac{dx}{x}$ is an improper integral of type 2 because $\frac{1}{x}$ is not continuous at 0.

Example 340 $\int_0^\infty \frac{dx}{x-1}$ is an improper integral of type 1 since the upper limit of integration is infinite. It is also an improper integral of type 2 because $\frac{1}{x-1}$ is not continuous at 1 and 1 is in the interval of integration.

Example 341 $\int_{-2}^2 \frac{dx}{x^2-1}$ is an improper integral of type 2 because $\frac{1}{x^2-1}$ is not continuous at -1 and 1 .

Example 342 $\int_0^\pi \tan x dx$ is an improper integral of type 2 because $\tan x$ is not continuous at $\frac{\pi}{2}$.

We now look how to handle each type of improper integral.

4.10.2 Improper Integrals of Type 1

These are easy to identify. Simply look at the interval of integration. If either the lower limit of integration, the upper limit of integration or both are not finite, it will be an improper integral of type 1.

Definition 343 (improper integral of type 1) *Improper integrals of type 1 are evaluated as follows:*

1. If $\int_a^t f(x) dx$ exists for all $t \geq a$, then we define

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

provided the limit exists as a finite number. In this case, $\int_a^\infty f(x) dx$ is said to be **convergent** (or to **converge**). Otherwise, $\int_a^\infty f(x) dx$ is said to be **divergent** (or to **diverge**).

2. If $\int_t^b f(x) dx$ exists for all $t \leq b$, then we define

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

provided the limit exists as a finite number. In this case, $\int_{-\infty}^b f(x) dx$ is said to be **convergent** (or to **converge**). Otherwise, $\int_{-\infty}^b f(x) dx$ is said to be **divergent** (or to **diverge**).

3. If both $\int_a^\infty f(x) dx$ and $\int_{-\infty}^b f(x) dx$ converge, then we define

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx$$

The integrals on the right are evaluated as shown in 1. and 2.

4.10.3 Improper Integrals of Type 2

These are more difficult to identify. One needs to look at the interval of integration, and determine if the integrand is continuous or not in that interval. Things to look for are fractions for which the denominator becomes 0 in the interval of integration. Keep in mind that some functions do not contain fractions explicitly like $\tan x$, $\sec x$.

Definition 344 (improper integral of type 2) *Improper integrals of type 2 are evaluated as follows:*

1. if f is continuous on $[a, b)$ and not continuous at b then we define

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

provided the limit exists as a finite number. In this case, $\int_a^b f(x) dx$ is said to be **convergent** (or to **converge**). Otherwise, $\int_a^b f(x) dx$ is said to be **divergent** (or to **diverge**).

2. if f is continuous on $(a, b]$ and not continuous at a then we define

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

provided the limit exists as a finite number. In this case, $\int_a^b f(x) dx$ is said to be **convergent** (or to **converge**). Otherwise, $\int_a^b f(x) dx$ is said to be **divergent** (or to **diverge**).

3. If f is not continuous at c where $a < c < b$ and both $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ converge then we define

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

The integrals on the right are evaluated as shown in 1. and 2.

We now look at some examples.

4.10.4 Examples

- Evaluating an improper integral is really two problems. It is an integral problem and a limit problem. It is best to do them separately.
- When breaking down an improper integral to evaluate it, make sure that each integral is improper at only one place, that place should be either the lower limit of integration, or the upper limit of integration.

Example 345 $\int_1^\infty \frac{dx}{x^2}$

This is an improper integral of type 1. We evaluate it by finding $\lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^2}$.

First, $\int_1^t \frac{dx}{x^2} = \left(1 - \frac{1}{t}\right)$ and $\lim_{t \rightarrow \infty} \left(1 - \frac{1}{t}\right) = 1$ hence $\int_1^\infty \frac{dx}{x^2} = 1$.

Example 346 $\int_1^\infty \frac{dx}{x}$

This is an improper integral of type 1. We evaluate it by finding $\lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x}$

First, $\int_1^t \frac{dx}{x} = \ln t$ and $\lim_{t \rightarrow \infty} \ln t = \infty$ hence $\int_1^\infty \frac{dx}{x}$ diverges.

Example 347 $\int_{-\infty}^\infty \frac{dx}{1+x^2}$

This is an improper integral of type 1. Since both limits of integration are infinite, we break it into two integrals.

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$$

Note that since the function $\frac{1}{1+x^2}$ is even, we have $\int_{-\infty}^0 \frac{dx}{1+x^2} = \int_0^\infty \frac{dx}{1+x^2}$; we only need to do $\int_0^\infty \frac{dx}{1+x^2}$.

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{t \rightarrow \infty} \int_0^t \frac{dx}{1+x^2}$$

and

$$\begin{aligned} \int_0^t \frac{dx}{1+x^2} &= \tan^{-1} x \Big|_0^t \\ &= \tan^{-1} t - \tan^{-1} 0 \\ &= \tan^{-1} t \end{aligned}$$

Thus

$$\begin{aligned} \int_0^\infty \frac{dx}{1+x^2} &= \lim_{t \rightarrow \infty} (\tan^{-1} t) \\ &= \frac{\pi}{2} \end{aligned}$$

It follows that

$$\begin{aligned} \int_{-\infty}^\infty \frac{dx}{1+x^2} &= \frac{\pi}{2} + \frac{\pi}{2} \\ &= \pi \end{aligned}$$

Example 348 $\int_0^{\frac{\pi}{2}} \sec x dx$

This is an improper integral of type 2 because $\sec x$ is not continuous at $\frac{\pi}{2}$. We

evaluate it by finding $\lim_{t \rightarrow \frac{\pi}{2}^-} \int_0^t \sec x dx$.

First,

$$\begin{aligned}\int_0^t \sec x \, dx &= \ln |\sec x + \tan x| \Big|_0^t \\ &= \ln |\sec t + \tan t|\end{aligned}$$

and $\lim_{t \rightarrow \frac{\pi}{2}^-} (\ln |\sec t + \tan t|) = \infty$ hence $\int_0^{\frac{\pi}{2}} \sec x \, dx$ diverges.

Example 349 $\int_0^\pi \sec^2 x \, dx$

This is an improper integral of type 2, $\sec^2 x$ is not continuous at $\frac{\pi}{2}$. Thus,

$$\int_0^\pi \sec^2 x \, dx = \int_0^{\frac{\pi}{2}} \sec^2 x \, dx + \int_{\frac{\pi}{2}}^\pi \sec^2 x \, dx$$

First, we evaluate $\int_0^{\frac{\pi}{2}} \sec^2 x \, dx$.

$$\int_0^{\frac{\pi}{2}} \sec^2 x \, dx = \lim_{t \rightarrow \frac{\pi}{2}^-} \int_0^t \sec^2 x \, dx$$

$$\begin{aligned}\int_0^t \sec^2 x \, dx &= \tan t - \tan 0 \\ &= \tan t\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sec^2 x \, dx &= \lim_{t \rightarrow \frac{\pi}{2}^-} (\tan t) \\ &= \infty\end{aligned}$$

It follows that $\int_0^{\frac{\pi}{2}} \sec^2 x \, dx$ diverges, therefore, $\int_0^\pi \sec^2 x \, dx$ also diverges.

Remark 350 If we had failed to see that the above integral is improper, and had evaluated it using the Fundamental Theorem of Calculus, we would have obtained a completely different (and wrong) answer.

$$\begin{aligned}\int_0^\pi \sec^2 x \, dx &= \tan \pi - \tan 0 \\ &= 0 \text{ (this is not correct)}\end{aligned}$$

Example 351 $\int_{-\infty}^\infty \frac{dx}{x^2}$

This integral is improper for several reasons. First, the interval of integration is not finite. The integrand is also not continuous at 0. To evaluate it, we break it so that each integral is improper at only one place, that place being one of the limits of integration, as follows: $\int_{-\infty}^\infty \frac{dx}{x^2} = \int_{-\infty}^{-1} \frac{dx}{x^2} + \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} + \int_1^\infty \frac{dx}{x^2}$. We then evaluate each improper integral. The reader will verify that it diverges.

Several important results about improper integrals are worth remembering, they will be used with infinite series. The proof of these results is left as an exercise.

Theorem 352 $\int_1^\infty \frac{dx}{x^p}$ converges if $p > 1$, it diverges otherwise.

Proof. This is an improper integral of type 1 since the upper limit of integration is infinite. Thus, we need to evaluate $\lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^p}$. When $p = 1$, we have already seen that the integral diverges. Let us assume that $p \neq 1$. First, we evaluate the integral.

$$\begin{aligned} \int_1^t \frac{dx}{x^p} &= \int_1^t x^{-p} dx \\ &= \left. \frac{x^{1-p}}{1-p} \right|_1^t \\ &= \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \end{aligned}$$

The sign of $1-p$ is important. When $1-p > 0$ that is when $p < 1$, t^{1-p} is in the numerator. Therefore, $\lim_{t \rightarrow \infty} \left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) = \infty$ thus the integral diverges.

When $1-p < 0$ that is when $p > 1$, t^{1-p} is really in the denominator so that $\lim_{t \rightarrow \infty} \left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) = \frac{1}{p-1}$ and therefore $\int_1^\infty \frac{dx}{x^p}$ converges. In conclusion, we have looked at the following cases:

Case 1: $p = 1$. In this case, the integral diverges.

Case 2: $p < 1$. In this case, the integral diverges.

Case 3: $p > 1$. In this case, the integral converges.

■

Theorem 353 $\int_0^1 \frac{dx}{x^p}$ converges if $p < 1$, it diverges otherwise.

Proof. See problems. ■

4.10.5 Comparison Theorems for Improper Integrals

Sometimes an improper integral is too difficult to evaluate. One technique is to compare it with a known integral. The theorem below shows us how to do this.

Theorem 354 (Direct Comparison Test) Suppose that f and g are two continuous functions for $x \geq a$ such that $0 \leq g(x) \leq f(x)$. Then, the following is true:

1. If $\int_a^\infty f(x) dx$ converges then $\int_a^\infty g(x) dx$ also converges.
2. If $\int_a^\infty g(x) dx$ diverges, then $\int_a^\infty f(x) dx$ also diverges.

The theorem is not too difficult to understand if we think about the integral in terms of areas. Since both functions are positive, the integrals simply represent the area of the region below their graph. Let A_f be the area of the region below the graph of f . Use a similar notation for A_g . If $0 \leq g(x) \leq f(x)$, then $A_g \leq A_f$. Part 1 of the theorem is simply saying that if A_f is finite, so is A_g ; this should be obvious from the inequality. Part 2 says that if A_g is infinite, so is A_f .

Remark 355 When using the comparison theorem, the following inequalities are useful:

$$x^2 \geq x \geq \sqrt{x} \geq 1$$

and

$$\ln x \leq x \leq e^x$$

Example 356 Study the convergence of $\int_1^\infty e^{-x^2} dx$

We cannot evaluate the integral directly, e^{-x^2} does not have an antiderivative. We note that

$$\begin{aligned} x \geq 1 &\iff x^2 \geq x \\ &\iff -x^2 \leq -x \\ &\iff e^{-x^2} \leq e^{-x} \end{aligned}$$

Now,

$$\begin{aligned} \int_1^\infty e^{-x} dx &= \lim_{t \rightarrow \infty} \int_1^t e^{-x} dx \\ &= \lim_{t \rightarrow \infty} (e^{-1} - e^{-t}) \\ &= e^{-1} \end{aligned}$$

and therefore converges. It follows that $\int_1^\infty e^{-x^2} dx$ converges by the comparison theorem.

Example 357 Study the convergence of $\int_1^\infty \frac{dx}{1+x^2}$

Note that we did this problem above and found that this integral converges to $\frac{\pi}{2}$. If we only need to know whether it converges or not, we can use a comparison theorem instead. We notice that $0 < \frac{1}{x^2+1} < \frac{1}{x^2}$ when $x \geq 1$ and since $\int_1^\infty \frac{dx}{x^2}$ converges (see above), by the direct comparison test, we conclude that $\int_1^\infty \frac{dx}{1+x^2}$ also converges.

Remark 358 The technique we used in the previous example would not have worked with the integral $\int_1^\infty \frac{dx}{x^2 - \frac{1}{2}}$ because $\frac{1}{x^2 - \frac{1}{2}} > \frac{1}{x^2}$, so the direct comparison test does not allow us to conclude. However, the next comparison test would work in such cases.

Theorem 359 (Limit Comparison Test) Let f and g be two positive and continuous functions on $[a, \infty)$. If $0 < \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} < \infty$ then $\int_a^\infty f(x) dx$ and $\int_a^\infty g(x) dx$ behave the same way, that is they both converge or both diverge.

Remark 360 The theorem does not say anything about what these integrals converge to. In particular, it does not say that if they converge they converge to the same value.

Remark 361 Of course, one difficulty of this theorem is to find an integral to compare to.

Example 362 Study the convergence of $\int_1^\infty \frac{dx}{x^2 - \frac{1}{2}}$.

We need to find an integral to compare it to. We see that $\frac{1}{x^2 - \frac{1}{2}}$ is similar to $\frac{1}{x^2}$ for large x . We compute

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2 - \frac{1}{2}}} &= \lim_{x \rightarrow \infty} \frac{x^2 - \frac{1}{2}}{x^2} \\ &= 1 \end{aligned}$$

Hence, the two integrals behave the same way. Since $\int_1^\infty \frac{dx}{x^2}$ converges by theorem 352 it follows that $\int_1^\infty \frac{dx}{x^2 - \frac{1}{2}}$ also converges.

4.10.6 Things to know

- Be able to tell if an integral is improper or not and what type it is.
- Be able to tell if an improper integral converges or diverges. If it converges, be able to find what it converges to.
- Be able to write an improper integral as a limit of definite integral(s).

4.10.7 Problems

1. Determine why each integral below is improper. Then, determine if they are convergent or divergent. When they converge, evaluate them.

(a) $\int_1^{\infty} \frac{dx}{(3x+1)^2}$

(b) $\int_{-\infty}^{-1} \frac{dx}{\sqrt{2-x}}$

(c) $\int_{2\pi}^{\infty} \sin t \, dt$

(d) $\int_{-\infty}^{\infty} x e^{-x^2} \, dx$

(e) $\int_0^{\infty} x e^{-5x} \, dx$

(f) $\int_1^{\infty} \frac{\ln x}{x} \, dx$

(g) $\int_1^{\infty} \frac{\ln x}{x^2} \, dx$

(h) $\int_{-\infty}^{\infty} \frac{x^2}{9+x^6} \, dx$

(i) $\int_0^1 \frac{3}{x^5} \, dx$

(j) $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

(k) $\int_{-1}^1 \frac{e^x}{e^x-1} \, dx$

2. Use one of the comparison theorems to determine if $\int_1^{\infty} \frac{\cos^2 x}{1+x^2} \, dx$ converges or diverges.
3. Use one of the comparison theorems to determine if $\int_1^{\infty} \frac{dx}{x^3+1}$ converges or diverges.
4. Use one of the comparison theorems to determine if $\int_1^{\infty} \frac{dx}{\sqrt{x}+1}$ converges or diverges.

5. Find the values of a for which $\int_0^\infty e^{ax} dx$ converges and diverges.
6. Prove theorem 353.
7. Find the values of p for which $\int_e^\infty \frac{dx}{x(\ln x)^p}$ converges and diverges.

4.10.8 Answers

1. Determine why each integral below is improper. Then, determine if they are convergent or divergent. When they converge, evaluate them.

- (a) $\int_1^\infty \frac{dx}{(3x+1)^2} = \frac{1}{12}$
- (b) $\int_{-\infty}^{-1} \frac{dx}{\sqrt{2-x}}$ diverges
- (c) $\int_{2\pi}^\infty \sin t dt$ diverges
- (d) $\int_{-\infty}^\infty x e^{-x^2} dx = 0$
- (e) $\int_0^\infty x e^{-5x} dx = \frac{1}{25}$
- (f) $\int_1^\infty \frac{\ln x}{x} dx$ diverges
- (g) $\int_1^\infty \frac{\ln x}{x^2} dx = 1$
- (h) $\int_{-\infty}^\infty \frac{x^2}{9+x^6} dx = \frac{1}{9}\pi$
- (i) $\int_0^1 \frac{3}{x^5} dx$ diverges
- (j) $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} = \frac{32}{3}$
- (k) $\int_{-1}^1 \frac{e^x}{e^x-1} dx$ diverges

2. Use the comparison theorem to determine if $\int_1^\infty \frac{\cos^2 x}{1+x^2}$ converges or diverges.
It converges.
3. Use one of the comparison theorems to determine if $\int_1^\infty \frac{dx}{x^3+1}$ converges or diverges.
It converges.
4. Use one of the comparison theorems to determine if $\int_1^\infty \frac{dx}{\sqrt{x}+1}$ converges or diverges.
It diverges.

5. Find the values of a for which $\int_0^\infty e^{ax} dx$ converges and diverges.
It diverges if $a \geq 0$ and converges otherwise.
6. Prove theorem 353.
7. Find the values of p for which $\int_e^\infty \frac{dx}{x (\ln x)^p}$ converges and diverges.
It diverges if $p \leq 1$ and converges otherwise.