DTFS.

$$x[n] = x[n+N].$$

- periodic discrete time signal

$$\omega_0 = \frac{2\pi}{N}$$

$N \to$ integer.

$\omega_0 \to$ fundamental frequency.

Basis functions.

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n}. \quad k = 0, \pm 1, \pm 2, \pm 3 \ldots$$

Note: $\phi_k[n] \to$ periodic. $\forall k$. (in time)

$$e^{j2\pi.\frac{k}{N}(n+N)} = e^{j2\pi k} \cdot e^{j\frac{2\pi kn}{N}}.$$

$$= e^{j\frac{2\pi kn}{N}}.$$

$$\therefore \phi_k[n+N] = \phi_k[n].$$

Also. $\phi_{k+rN}[n] = \phi_k[n]$. (Periodic in frequency).

$$\phi_{k+rN}[n] = e^{j2\pi(k+rN).\frac{n}{N}}$$

$$= e^{j2\pi\frac{kn}{N}} \cdot e^{j2\pi rn} = e^{j\frac{2\pi kn}{N}} = \phi_k[n]$$

$$\therefore \quad x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] \longrightarrow \text{DTFS.}$$

Contrasts F.S. for continuous signals:

$$k = \langle N \rangle \rightarrow k = 0 \text{ to } N-1$$

$$1 \text{ to } N$$

$$2 \text{ to } N+1.$$

$$\vdots$$

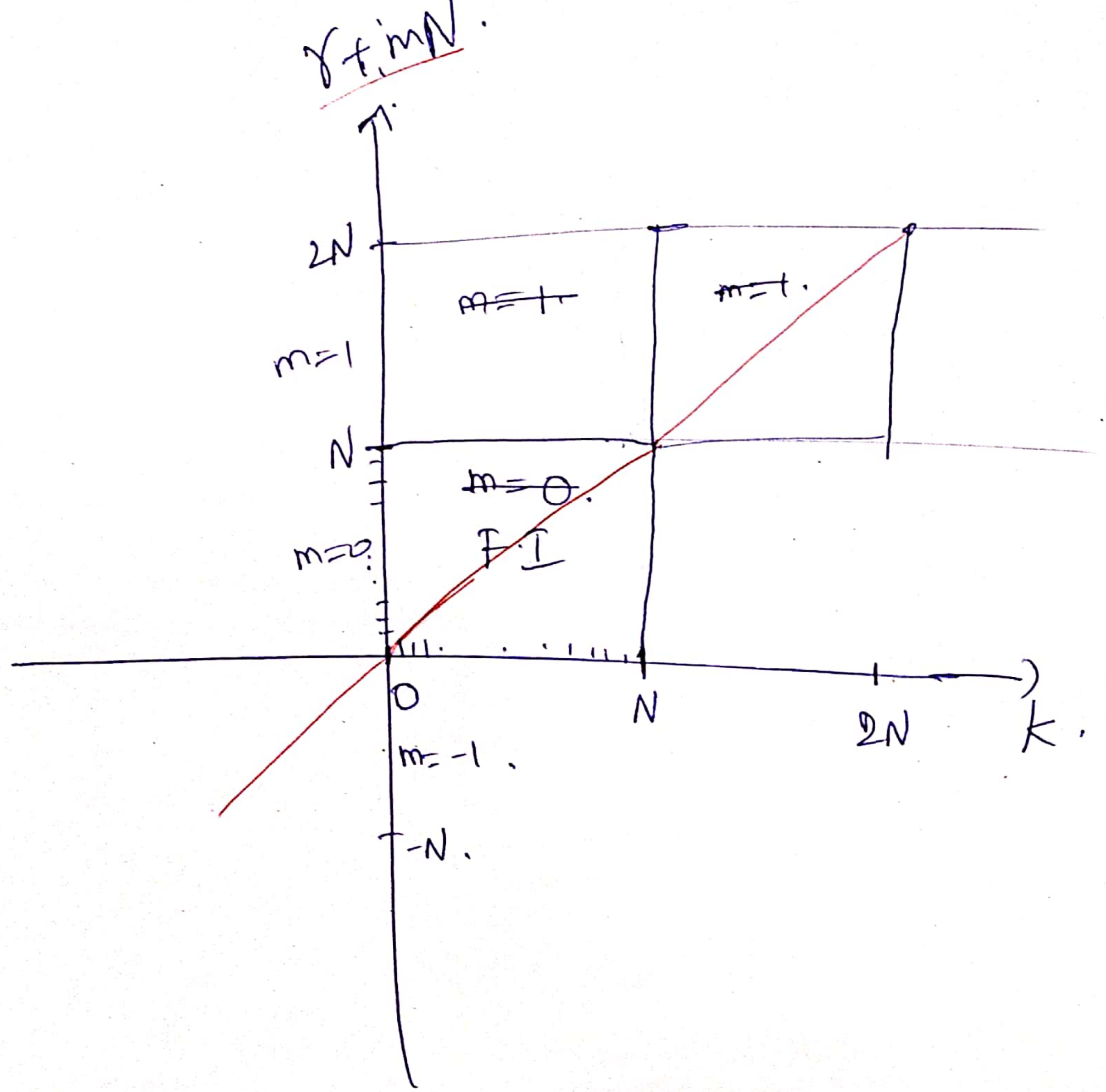
$$-(N-1) \text{ to } 0.$$

Reason. (1)

→ Periodicity in frequency.

Validation.

$$x[n] = \sum_{k=0}^{N-1} a_k \phi_k[n] = \sum_{k=0}^{N-1} a_k e^{j2\pi kn/N}$$



Note: $a_r = a_{r+N} = a_{r+2N} \cdots$

$\therefore a_r = a_{r \pm mN}. \quad m \in \mathbb{Z}.$

Fourier Coefficients.

Goal: To prove that

$$a_r = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jr2\pi n/N}.$$

Def:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk2\pi n/N}.$$

$$\sum_{n=\langle N \rangle} x[n] e^{-jr2\pi n/N} = \frac{1}{N} \sum_{n} \sum_{k=\langle N \rangle} a_k e^{jk2\pi n/N} \cdot e^{-jr2\pi n/N}$$

$$= \frac{1}{N} \sum_{k=\langle N \rangle} a_k \cdot \sum_{n=\langle N \rangle} e^{j(k-r)2\pi n/N}.$$

$$\sum_{n=\langle N \rangle} e^{j(k-r)2\pi n/N} = \begin{cases} 0 & k-r \neq mN. \ \cancel{k \neq r} \\ N & \cancel{k=r}. \text{ or } k-r = mN \end{cases}$$

For $k \neq r$. $e^{j(k-r)2\pi n/N} = \left[e^{j(k-r)2\pi/N}\right]^n.$

base → roots of unity.

$\downarrow$

$b.$

$$\sum_{n=0}^{N-1} b^n = \frac{1-b^N}{1-b}$$

S

$$\therefore b^N = e^{j(k-r)\frac{2\pi}{N}\cdot N} = \left(e^{j2\pi}\right)^{k-r} = \underline{\underline{1}}.$$

$$\therefore \sum_{n=0}^{N-1} b^n = 0. \quad \text{for } \cancel{k \neq r}.$$

$$k-r \neq mN.$$

$$\text{sum} \sum_{n=\langle N\rangle} e^{j(k-r)2\pi n/N} = N \;\Big|\; k-r = mN.$$

for $n = \langle N \rangle$.

$$\rightarrow \underline{\underline{\delta(k-r-mN)}}.$$

$$= N \sum_{k=\langle N\rangle} a_k \cdot \cancel{\delta(r-k+mN)} \rightarrow \delta[r-(k-mN)]$$

$$\delta(k-r-mN)$$

for $k = 0, 1, \ldots, N-1.$

$r+mN. \rightarrow: r, \; r+N, \; r+2N, \; r-N$

If $r \in [0, N-1]$ and $r \in \mathbb{Z}$:

then, only $a_r \rightarrow$ valid r ~~exists~~. (matches.)

$\therefore = a_r N$

Summary

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk2\pi/N \cdot n} \rightarrow \text{Synthesis}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk2\pi n/N} \rightarrow \text{Analysis.}$$

$$a_{k+rN} = a_k \rightarrow \text{periodic in } k.$$

$$a_{k+rN} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j(k+rN)\frac{2\pi}{N}n}.$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \cdot e^{-jk2\pi n/N} = a_k.$$

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e.g. $x[n] = \sin \omega_0 n$. given $\omega_0 = \frac{2\pi}{N}$. → condition.

F.C.S $x[n] = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$.

$\therefore a_1 = \frac{1}{2j}$, $a_{-1} = \frac{-1}{2j}$.

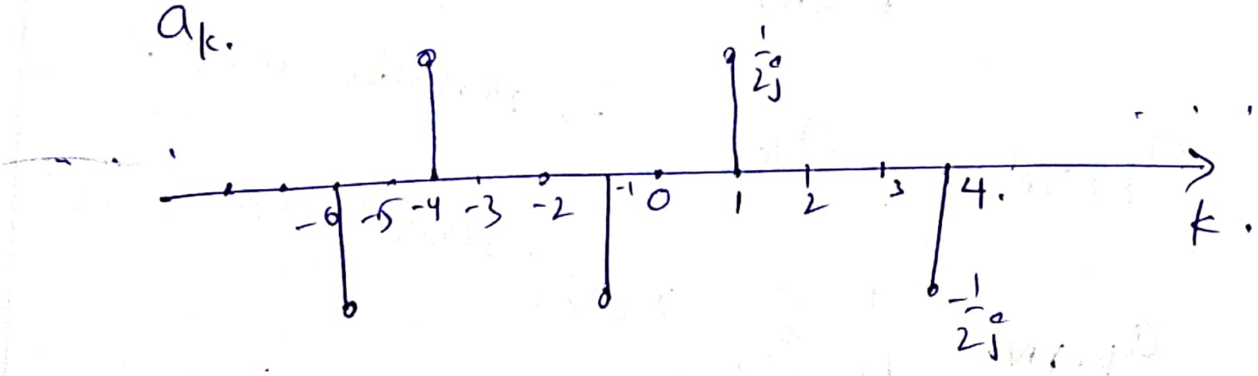
$a_k = 0 \quad k \neq \pm 1$.

$N = 5.$

$a_0 = 0., \; a_1 = \frac{1}{2j}, \ldots a_4 = \frac{-1}{2j}.$

because $a_4 = a_{-1+5} = a_{-1}.$

$a_k.$



Case (2) $\quad \omega_0 = \frac{2\pi M}{N}$ ie; $\frac{\omega_0}{2\pi} = \frac{M}{N}$ a rational

then

$$x[n] = \frac{e^{j\frac{M 2\pi}{N}.n.} - e^{-j\frac{M 2\pi}{N}n.}}{2j}$$

$\therefore \; a_M = \frac{1}{2j} \quad a_{-M} = \frac{-1}{2j}.$

$M = 3, \; N = 5$

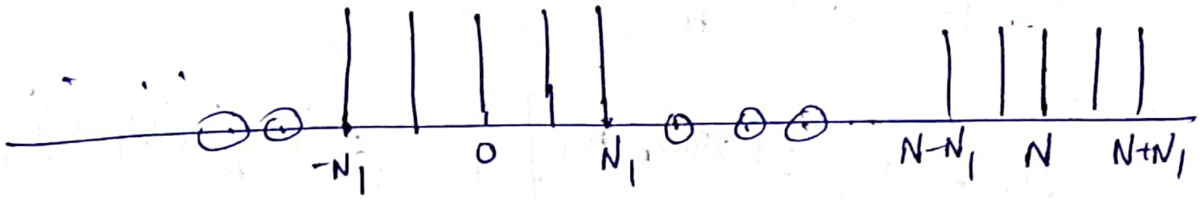
$\boxed{a_2 = a_{-3+5} = a_{-3}}$

$\left[a_0, a_1 \; a_2 \checkmark a_3 \checkmark \quad a_4 = 0 \right]. \; \cancel{a_5 = 0}$

$\downarrow, \downarrow,$
0

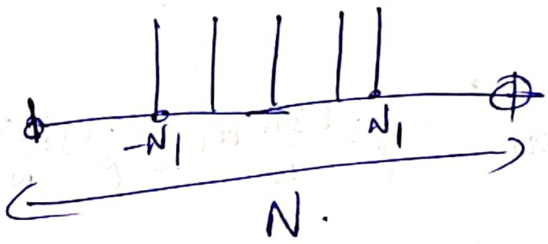
e.g. 3.12

$$x[n] = \begin{cases} 1 & -N_1 \le n \le N_1 \\ 0 & \text{else where.} \end{cases}$$ (in one period N.)



$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk 2\pi/N \cdot n}.$$

choose $n \to$



$$\therefore a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 \cdot e^{-jk \frac{2\pi}{N} n}.$$

let $m = n + N_1$.

$n = -N_1 \to m = 0$

$n = N_1 \quad m = 2N_1$

$$= \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk \frac{2\pi}{N}(m - N_1)}.$$

$$= \frac{1}{N} e^{+jk \frac{2\pi N_1}{N}} \cdot \sum_{m=0}^{2N_1} e^{-jk \frac{2\pi}{N} m}.$$

Term①

$$\sum_{m=0}^{2N_1} e^{-jk\frac{2\pi m}{N}} \rightarrow (2N_1+1) \quad \text{for } k = mN$$

$$= \frac{1 - e^{-jk\frac{2\pi}{N}(2N_1+1)}}{1 - e^{-jk\frac{2\pi}{N}}} \rightarrow k \neq mN$$

$$\therefore a_k = \frac{1}{N} e^{jk2\pi/N \cdot N_1} \left[\frac{1 - e^{-jk\frac{2\pi}{N}(2N_1+1)}}{1 - e^{-jk\frac{2\pi}{N}}} \right]$$

Numerator $\rightarrow e^{jk\frac{2\pi}{N}N_1} - e^{+jk\frac{2\pi}{N}(-N_1-1)}$

$$\frac{k2\pi}{N}(N_1 - 2N_1 - 1) = \frac{(-N_1-1)k2\pi}{N}$$

$$= e^{jk\frac{2\pi}{N}(N_1+\frac{1}{2}-\frac{1}{2})} - e^{jk\frac{2\pi}{N}(-N_1-\frac{1}{2}-\frac{1}{2})}$$

$$= e^{+jk\frac{2\pi}{N}(-\frac{1}{2})} \cdot \left[e^{jk\frac{2\pi}{N}(N_1+\frac{1}{2})} - e^{-jk\frac{2\pi}{N}(N_1+\frac{1}{2})} \right]$$

$$= e^{-jk\frac{2\pi}{N}} \cdot 2j \sin\left[\frac{k2\pi}{N}(N_1+\frac{1}{2})\right].$$

$$1 - e^{-jk\frac{2\pi}{N}} = \cancel{e^{jk\pi\pi}} \; 1 - e^{-j\frac{2\pi k}{N}(\frac{1}{2}+\frac{1}{2})}.$$

$$= e^{\frac{j2\pi k}{N}(\frac{1}{2}-\frac{1}{2})} - e^{-j\frac{2\pi k}{N}(\frac{1}{2}+\frac{1}{2})}$$

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$$= e^{-j\frac{2\pi k}{N}\frac{1}{2}}\left[e^{j\frac{2\pi k}{N}\frac{1}{2}} - e^{-j\frac{2\pi k}{N}\frac{1}{2}}\right]$$

$$= 2j\, e^{-j\frac{2\pi k}{2N}} \cdot \sin\left(\frac{2\pi k}{2N}\right)$$

$$\therefore a_k = \frac{1}{N} \cdot \frac{\sin\left[\frac{2\pi k}{N}\left(N_1 + \frac{1}{2}\right)\right]}{\sin\left[\frac{\pi k}{N}\right]} \qquad k \neq mN.$$

$$a_k = \frac{1}{N} \cdot (2N_1 + 1) \qquad k = mN.$$

6. Properties of Discrete Time Fourier Series.

Property	Periodic Signal $x[n]$, $y[n]$	Fourier Series Coefficients a_k, b_k.
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$.
Time shifting	$x[n-n_0]$	$a_k e^{-j(k2\pi/N)n_0}$.
Frequency shifting	$e^{jM\frac{2\pi}{N}n} x[n]$	a_{k-M}.
Conjugation	$x^*[n]$	a^*_{-k}.
Time-reversal	$x[-n]$	a_{-k}.
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$.

Any 3 properties with proof: (2) each (x 3 = (6).

6.)

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$.
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi \frac{m}{N}$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency ω_0/m.
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$; $\frac{2\pi}{\omega_0}$	Fundamental period $\omega_0 = 0$: undefined. $\omega_0 \neq 0$: $m\frac{2\pi}{\omega_0}$.

(3)

*(3 marks for any 3 points written).

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