M3

(DSAA)

Relation between Fourier Series & transform.

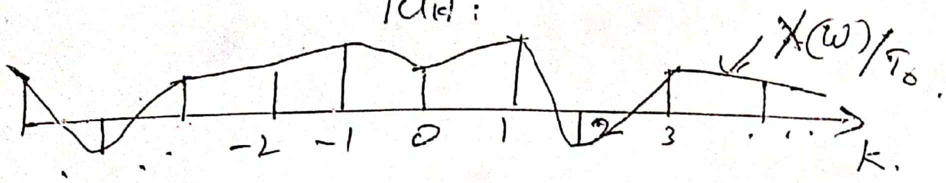
Fourier series.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \rightarrow \text{Synthesis}$$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt. \quad \rightarrow \text{analysis.}$$

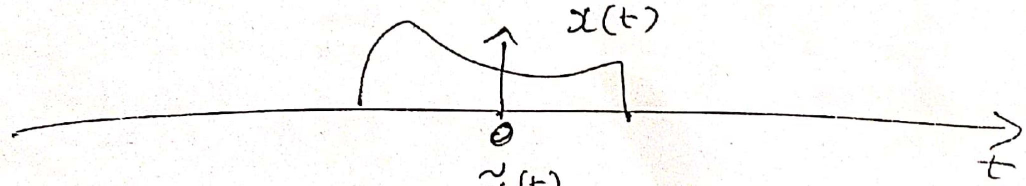
$\rightarrow$ ①

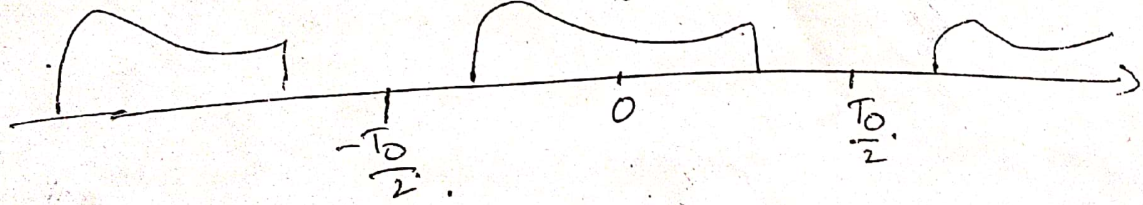
$|a_k|$:



Consider $x(t)$. $\rightarrow$ aperiodic signal.

Let $\tilde{x}(t) \rightarrow$ periodic form of $x(t)$.





Now.- clearly $\tilde{x}(t) = x(t)$. for. $|t| < \frac{T_0}{2}$ — ②

Now $a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt.$

② in ①

$\rightarrow$ Note limits change
$x(t) \rightarrow 0$ outside $[-T_0/2, T_0/2]$

$$a_k = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\cdot e^{-jk\omega_0 t}\, dt.$$

~~Now $T_0 = \frac{2\pi}{\omega_0}$.~~

Title	Topic	Author	Section	Date.
FT ⇌ FS -	Signals & System.	A.V.O	4.4.1.	03/01/15.

TH

let us denote the envelope of $a_k T_0$ as $X(\omega)$,

then. $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt. \quad \text{--- (3)}$$
of envelope.

ie; $$X(k\omega_0) = a_k T_0 = \int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt.$$

ie; ω replaces $k\omega_0$ for envelope.

For synthesis ;. since $a_k = \frac{1}{T_0} X(k\omega_0)$ → a sampled form of $X(\omega)$.

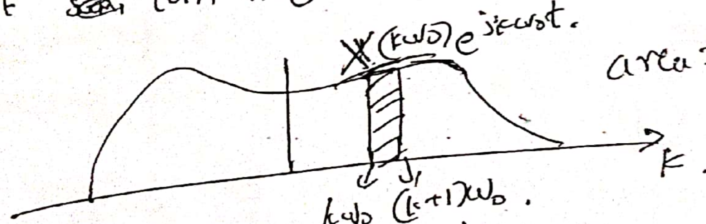
From (1) $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(k\omega_0)\, e^{jk\omega_0 t}$$

Next $T_0 = \frac{2\pi}{\omega_0}$. $\Rightarrow$ $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} X(k\omega_0) e^{jk\omega_0 t} \quad \text{--- (4)}$$

Now as $T_0 \to \infty \Rightarrow \omega_0 \to 0$.

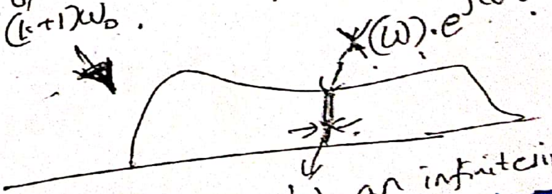
$\tilde{x}(t) \to x(t)$.

Next term in (4) $X(k\omega_0) e^{jk\omega_0 t}$; Can be seen as



area $= \omega_0 . X(k\omega_0)\, e^{jk\omega_0 t}$.

now as $\omega_0 \to 0$



$d\omega$. an infinitesimal.

Now infinitesimal area $= X(\omega)\, e^{j\omega t} d\omega$. and $\sum_{-\infty}^{\infty} \to \int_{-\infty}^{\infty}$

$\therefore$ $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega.$$
Total area ..

Ex. 3.5.

$$x_2(t) = x_1(1-t) + x_1(t-1) \quad \text{— } ①$$

Given $x_1(t) \longleftrightarrow a_k$ with freq. ω_1.

goal ⓐ find ω_2 the freq. of $x_2(t)$.

and ⓑ $x_2(t) \longleftrightarrow b_k$. find b_k in terms of a_k.

ⓐ freq. of $x_2(t)$.

From eq. $x_2(t) = x_1(1-t) + x_1(t-1)$

Since period of $x_1(t) = T_1 = \frac{2\pi}{\omega_1}$.

period of $x_1(1-t)$ is also T_1.

Proof: $x_1(1-t+T_1) = x_1(t'+T_1)$

$= x_1(t') = x_1(1-t)$

let $t' = 1-t$.

period of $x(t-1)$ is also T_1. (delay does not alter period.)

$T_1 \to$ lcm of (T_1, T_1).

$$\therefore \omega_2 = \frac{2\pi}{T_1} = \omega_1.$$

(b). Let $b_k = \frac{1}{T_1}\int_{T_1} x_2(t)\, e^{-jk\omega_0 t}\, dt$.

$$= \underbrace{\frac{1}{T_1}\int_{T_1} x_1(1-t)\cdot e^{-jk\omega_0 t}\, dt}_{T①} + \underbrace{\frac{1}{T_1}\int_{T_1} x_1(t-1)\, e^{-jk\omega_0 t}\, dt}_{T②}$$

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T_1 ①

$$\frac{1}{T_1}\int_{T_1} x_1(1-t)\,e^{-jk\omega_0 t}\,dt$$

let $1-t = t'$ | integration limits are reversed.

$dt = -dt'$. | $1 \to 1-T_1$.

$$= \frac{1}{T_1}\int_{1-T_1}^{1} x_1(t')\,e^{-jk\omega_0(1-t')}\,dt'.$$

$$= \frac{1}{T_1}e^{-jk\omega_0}\int_{T_1} x_1(t')\,e^{-j(-k)\omega_0 t'}\,dt'.$$

$\therefore$ T① $e^{-jk\omega_0}a_{-k}$.

From time-shifting

T② $= e^{-jk\omega_0}a_k$.

$\therefore\ b_k = e^{-jk\omega_0}a_{-k} + e^{-jk\omega_0}a_k$.

ex

Ex. 3.6.

(a). Verify if the signal is real or not.

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\frac{2\pi}{50}t}.$$

real signal $a_k^* = a_{-k}$.

$a_1 = \frac{1}{2}$, $a_{-1} = 0$ $\therefore x_1(t)$ is not real.

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi)\, e^{jk\frac{2\pi}{50}t}.$$

$a_k = \cos k\pi$. $\quad a_{-k} = \cos(-k\pi)$

$= \cos(k\pi)$

$= a_k$. ✓

$\therefore x_2(t) \rightarrow$ real.

$$x_3(t) = \sum_{k=-100}^{100} j \sin\frac{k\pi}{2}\, e^{jk\frac{2\pi}{50}t}$$

$a_k = j \sin\frac{k\pi}{2}$. $\quad a_{-k} = -j \sin\frac{k\pi}{2}$.

$\therefore a_k^* = a_{-k}$. $\therefore$

$x_3(t) \rightarrow$ real.

Ex 3·7.

given $x(t) \longleftrightarrow a_k$.

$T = 2$. / $a_0 = 2$.

Find $g(t) \longleftrightarrow b_k$ $g(t) = \frac{d}{dt} x(t)$.

$$x(t) = \sum_k a_k e^{jk\omega_0 t}.$$

$$\frac{d}{dt} x(t) = \sum_k \underline{a_k (jk\omega_0)} \cdot e^{jk\omega_0 t}.$$

$\therefore b_k = jk\omega_0 a_k$. ; $b_0 = 0$.

Ex 3-8 Find a signal that satisfies.

1. $x(t)$ is real and odd
2. $x(t)$ is periodic $T = 2$.

   $x(t) \longleftrightarrow a_k$

3. $a_k = 0$ $|k| > 1$
4. $\frac{1}{2} \int_0^2 |x(t)|^2 dt = 1$.

Sol: $x(t) = 2\sin \pi t$.

$a_1 = -j$ $a_{-1} = j$.

$\frac{1}{2} \int_0^2 4 \cdot |\sin^2 \pi t| dt = 2 \cdot \frac{1}{2} = \underline{\underline{1}}$.

Cor

Continuous Fourier transforms.
(time).

1. $x(t) = e^{j\omega_0 t}$. $\forall t$

since $\int_{-\infty}^{\infty} |x(t)| dt \to \infty$

direct def. cannot be evaluated.

Consider $X(\omega) = 2\pi \delta(\omega - \omega_0)$.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega - \omega_0) e^{j\omega t} d\omega$$

$\Rightarrow x(t) = e^{j\omega_0 t}$.

$\therefore$ $\cancel{X(\omega)}$ $e^{j\omega_0 t} \rightleftarrows 2\pi \delta(\omega - \omega_0)$.

2. $x(t) = \sin \omega_0 t$.

here $x(t) = \text{Im}\{e^{j\omega_0 t}\} \therefore = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$.

$$\therefore F\{x(t)\} = F\left\{\frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})\right\}$$

$$\therefore X(\omega) = \frac{2\pi}{2j}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right].$$

$$= j\pi \left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right]$$

3. $\cos \omega_0 t \rightleftharpoons \frac{2\pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$.

4. $x(t) = e^{-at} u(t) \qquad a > 0$.

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt.$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^{\infty} = \frac{1}{a+j\omega}.$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}. \qquad \angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

~~Amplitude~~ spec. Phase spec.

Magnitude.

5. $x(t) = \cancel{e^{-|at|}}\ e^{-a|t|}$. $\qquad a > 0$ (real x even)

$$X(\omega) = \int_{-\infty}^{0} e^{-j\omega t} e^{at} dt + \int_{0}^{\infty} e^{-at-j\omega t} dt.$$

$$= \left.\frac{e^{(a-j\omega)t}}{a-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}.$$

Exercises.

4.1. (a) $x(t) = e^{-2(t-1)} u(t-1)$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-2(t-1)} \cdot u(t-1)\, e^{-j\omega t}\, dt$$

$$= \int_{1}^{\infty} e^{-2(t-1)}\, e^{-j\omega t}\, dt$$

$$= e^{2} \cdot \frac{e^{-(2+j\omega)t}}{-(2+j\omega)} \Bigg|_{1}^{\infty}$$

$$= \frac{e^{2-(2+j\omega)}}{2+j\omega} = \frac{e^{-j\omega}}{2+j\omega}$$

$$|X(\omega)| = \frac{e^{2-2}}{\sqrt{4+\omega^2}} = \frac{1}{\sqrt{4+\omega^2}}$$

Mag-spec

$$\angle X(\omega) = \cancel{\tan} \; -\omega - \tan^{-1}\frac{\omega}{2}$$

roperties
of Fourier transform.

4.6.1 (1) Linearity.

Given $x_1(t) \leftrightarrow X_1(\omega)$.

$x_2(t) \leftrightarrow X_2(\omega)$.

then $a\,x_1(t) + b\,x_2(t) \leftrightarrow a\,X_1(\omega) + b\,X_2(\omega)$

$$F\{a\,x_1(t) + b\,x_2(t)\}$$

$$= \int_{-\infty}^{\infty} (a\,x_1(t) + b\,x_2(t))\,e^{-j\omega t}\,dt$$

$$= a\left[\int x_1(t)\,e^{-j\omega t}\,dt\right] + b\left[\int x_2(t)\,e^{-j\omega t}\,dt\right]$$

$$= a\,X_1(\omega) + b\,X_2(\omega)$$

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(2) Symmetry Property.

$$X(-\omega) = X^*(\omega).$$ when $x(t) \to$ real

$$X(\omega) = \int x(t)\,e^{-j\omega t}\,dt$$

$$X(-\omega) = \int x(t)\,e^{-j(-\omega)t}\,dt$$

$$= \left[\int x(t)\,e^{-j\omega t}\,dt\right]^* = (X(\omega))^* = X^*(\omega).$$

(3) Consequence

$$X(-\omega) = X^*(\omega).$$

$$\Rightarrow \mathrm{Re}\{X(-\omega)\} + j\,\mathrm{Im}\{X(-\omega)\}$$

$$= \mathrm{Re}\,X^*(\omega) + j\,\mathrm{Im}\,X^*(\omega).$$

$$\Rightarrow \mathrm{Re}\{X(-\omega)\} = \mathrm{Re}\{X^*(\omega)\} = \mathrm{Re}\{X(\omega)\}.$$

$$\mathrm{Im}\{X(-\omega)\} = \mathrm{Im}\{X^*(\omega)\} = -\mathrm{Im}\{X(\omega)\}.$$

if $x(t)$ is real & even $X(\omega) \to$ real & even.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \longrightarrow \text{even fn.}$$

$$= \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t}\, dt$$

$$\tau = -t$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{j\omega\tau}\, d\tau = X(-\omega).$$

since $X(\omega) = X(-\omega)$ but $X(-\omega) = X^*(\omega)$

$X(\omega) = X^*(\omega) \Rightarrow \mathrm{Im}[X(\omega)] = 0.$

hence $X(\omega) \to$ real & even.

More Properties of FT.

(i) Time Shifting.

$x(t) \leftrightarrow X(\omega)$ then

$x(t-t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$.

$F\{x(t-t_0)\}$.

$= \int x(t-t_0) e^{-j\omega t} dt$

Let $\tau = t - t_0$.
Limits won't change

$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau+t_0)} d\tau$.

$= e^{-j\omega t_0} . X(\omega)$ → phase shift in Freq. domain.

(ii) Prove $\frac{d x(t)}{dt} \leftrightarrow j\omega X(\omega)$.

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$.

$\frac{d x(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \frac{d}{dt} e^{j\omega t} d\omega$

$= j\omega . \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

$= j\omega X(\omega)$

Title	Topic	Author	section	date
FT properties contd.	Signals	A.V.O.	4.8.4.	03/03/15

Fourier series Coefficients as samples of Fourier Transform of one period.

Given

In one period $\leftarrow$ $x(t) = \begin{cases} \tilde{x}(t), & -\frac{T_0}{2} < t < \frac{T_0}{2} \\ 0 & -\frac{T_0}{2} < t, \; t > \frac{T_0}{2}. \end{cases}$

Fourier Coeff:

$$a_k = \frac{1}{T_0} \int_{T_0} \tilde{x}(t) \, e^{-jk\omega_0 t} \, dt.$$

$$= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{T_0/2} x(t) \cdot e^{-jk\omega_0 t} \, dt. \quad \text{— } (0)$$

but. $X(\omega) = \int x(t) \, e^{-j\omega t} \, dt.$ For given $x(t)$

$$= \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-j\omega t} \, dt.$$

From (0)

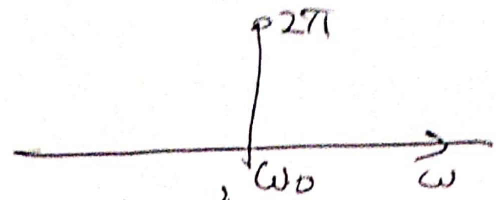
$$\therefore \; a_k = \frac{1}{T_0} X(k\omega_0). \quad \text{— } (1)$$

Also note that (1) is true for. $x(t)$ can be defined as.

$$x(t) = \begin{cases} \tilde{x}(t) & s \le t \le s + T_0 \\ 0 & t < s, \; t > s + T_0 \end{cases}$$

Topic	Author	Section	date
Signals	A.V.O	4.5	03/03/15.

Fourier Transform for periodic signals.



For a single scaled sinusoid at ω_0.

$$X(\omega) = 2\pi\,\delta(\omega-\omega_0) \qquad \text{--- (1)}$$

$$\Rightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega$$

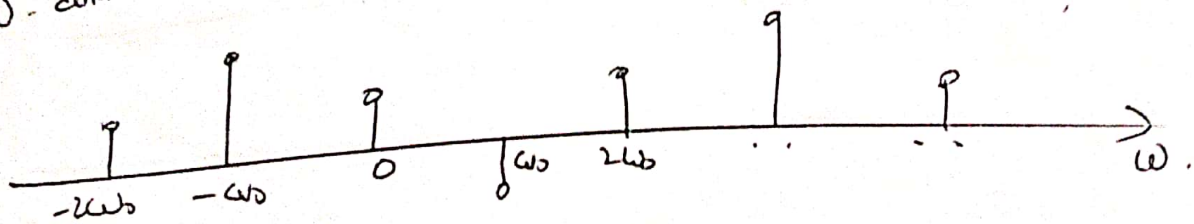
$$= \frac{1}{2\pi}\int 2\pi\,\delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega$$

$$\therefore \quad x(t) = e^{j\omega_0 t}$$

Fourier series from F.T.

For a train of weighted Impulses in ω-domain

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\,a_k\,\delta(\omega-k\omega_0). \qquad \text{--- (2)}$$



then

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \cancel{2\pi}\,a_k\,\delta(\omega-k\omega_0)\,e^{j\omega t}\,d\omega.$$

$$\therefore x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot \int \delta(\omega - k\omega_0)\, e^{j\omega t}\, d\omega$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}.$$

$\hookrightarrow$ synthesis eq. For Fourier Series.

Thus from (2) $X(\omega) = \sum 2\pi a_k \delta(\omega - k\omega_0)$ $\rightarrow$ FT of a periodic signal. $\left(T_0 = \frac{2\pi}{\omega_0}\right)$

$\downarrow$

Train of weighted impulses

$\downarrow$

k^{th} area $\rightarrow 2\pi \cdot a_k$.