Properties of CT-FS:

(1) Linearity

Given $x(t)$ and $y(t)$ with period T

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS} b_k.$$

Then $z(t) = A\,x(t) + B\,y(t)$

$$\xleftrightarrow{FS} A a_k + B b_k.$$

Proof:

let $z(t) = \sum_k c_k e^{jk\omega_0 t}.$

then $c_k = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt$

$$= \frac{1}{T}\int_T A\,x(t)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T}\int B\,y(t)\, e^{-jk\omega_0 t}\, dt$$

$$\therefore \boxed{c_k = A a_k + B b_k.}$$

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Time shifting

Given $x(t) \xleftrightarrow{F.S.} a_k$ Period $= T$

then $y(t) = x(t - t_0) \xleftrightarrow{F.S.} e^{-jk\frac{2\pi}{T}t_0} \cdot a_k$.

Proof: Let $y(t) \xleftrightarrow{FS} b_k$.

$$b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t - t_0)\, e^{-jk\omega_0 t}\, dt$$

$t - t_0 = t'$

$$= \frac{1}{T}\int_T x(t')\, e^{-jk\omega_0 (t' + t_0)} \cdot dt'$$

$$= e^{-jk\omega_0 t_0} \cdot \frac{1}{T}\int_T x(t')\, e^{-jk\omega_0 t'}\, dt'$$

$$\boxed{b_k = e^{-jk\frac{2\pi}{T}t_0} \cdot a_k}$$

Time scaling

$x(t) \xleftrightarrow{F.S.} a_k.$

$x(\alpha t) \xleftrightarrow{F.S.} a_k$ but.

$\omega_0 \rightarrow \alpha\omega_0.$

Proof.

let $b_k = \frac{1}{T'} \int_{T'} x(\alpha t)\, e^{-jk\omega_0' t}\, dt$

$T' = \frac{T}{\alpha}; \quad \omega_0' = \alpha\omega_0.$

$\alpha t = t'$

$dt = \frac{dt'}{\alpha}.$

$t \rightarrow t_0$

$t \rightarrow t_0 + T'.$

$\alpha t \rightarrow \alpha t_0$

$\alpha t \rightarrow \alpha t_0 + T.$

$$b_k = \frac{1}{T/\alpha} \int_{\alpha t_0}^{\alpha t_0 + T} x(t') \cdot e^{-jk\omega_0 t'} \frac{dt'}{\alpha}.$$

$$= \frac{1}{T} \int_{\alpha t_0}^{\alpha t_0 + T} x(t')\, e^{-jk\omega_0 t'}\, dt'$$

$$\boxed{b_k = a_k.}$$

$$\therefore x(\alpha t) = \sum_k a_k\, e^{jk(\alpha\omega_0)t}.$$

Time Reversal

If $x(t) \xleftrightarrow{F.S.} a_k$

then $x(-t) \xleftrightarrow{F.S.} a_{-k}$.

Proof: let $x(-t) \longleftrightarrow b_k$.

$$b_k = \frac{1}{T}\int_T x(-t)\, e^{-jk\omega_0 t}\, dt$$

$-t = t'$

$$= \frac{1}{T}\int_T x(t')\, e^{-j(-k)\omega_0 t'}\, dt'$$

$$\boxed{b_k = a_{-k}}$$

Multiplication

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS.} b_k.$$

then $$x(t)\,y(t) \xleftrightarrow{FS.} \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}.$$

$$= a_k * b_k.$$

Proof: Let $x(t)\,y(t) \longleftrightarrow c_k.$

$$c_k = \frac{1}{T}\int_T x(t)\,y(t)\, e^{-jk\omega_0 t}\,dt.$$

note $x(t) = \sum a_k e^{jk\omega_0 t}$, $y(t) = \sum b_k e^{jk\omega_0 t}$

$$\therefore c_k = \frac{1}{T}\int_T \sum_l a_l e^{jl\omega_0 t} \cdot \sum_m b_m e^{jm\omega_0 t}\, e^{-jk\omega_0 t}\,dt.$$

$$= \frac{1}{T}\sum_l\sum_m a_l\, b_m \cdot \int_T e^{-j(k-l-m)\omega_0 t}\,dt.$$

$$= \frac{1}{T}\sum_l\sum_m a_l\, b_m\, T\,\delta(k-l-m).$$

$$\boxed{c_k = \frac{\cancel{T}}{\cancel{T}}\sum_l a_l \cdot b_{k-l}.}$$

$$\sum_m b_m\,\delta(m-(k-l))$$

$$= b_{k-l}.$$

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Conjugation

If $x(t) \xleftrightarrow{F.S.} a_k$.

then $x^*(t) \xleftrightarrow{F.S.} a^*_{-k}$

Proof: Let $x^*(t) \xleftrightarrow{F.S.} b_k$

$$b_k = \frac{1}{T}\int_T x^*(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \left[\frac{1}{T}\int_T x(t)\, e^{-j(-k)\omega_0 t}\, dt\right]^*$$

$$= [a_{-k}]^*$$

$$\boxed{\therefore\ b_k = a^*_{-k}}$$

Parseval's theorem

$$\rightarrow \quad \frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2 .$$

Proof:

$$LHS \rightarrow \frac{1}{T}\int |x(t)|^2\, dt$$

$$= \frac{1}{T}\int \left| \sum a_k e^{jk\omega_0 t} \right|^2 dt$$

$$= \frac{1}{T}\int \sum_k a_k e^{jk\omega_0 t} \sum_l a_l^* e^{-jl\omega_0 t}\, dt$$

$$= \frac{1}{T} \sum_k a_k \sum_l a_l^* \int_T e^{-j(l-k)\omega_0 t}\, dt$$

$$= \frac{\cancel{T}}{\cancel{T}} \sum_k a_k \sum_l a_l^* \cdot \delta(l-k)$$

$$= \sum_k a_k a_k^* . = \sum_k |a_k|^2 .$$