

Linear Algebra

Eigen Values and Eigen Vectors

Source: Introduction to linear algrebra by Gilbert Strang

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A . *Certain exceptional vectors x are in the same direction as Ax . Those are the “eigenvectors”.* Multiply an eigenvector by A , and the vector Ax is a number λ times the original x .

The basic equation is $Ax = \lambda x$. The number λ is the “*eigenvalue*”. It tells whether the special vector x is stretched or shrunk or reversed or left unchanged—when it is multiplied by A . We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1 . The eigenvalue λ could be zero! Then $Ax = 0x$ means that this eigenvector x is in the nullspace.

If A is the identity matrix, every vector has $Ax = x$. All vectors are eigenvectors. The eigenvalue (the number lambda) is $\lambda = 1$. This is unusual to say the least.

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix} \quad \dots$$

A

A^2

A^3

$$\begin{bmatrix} .6000 & .6000 \\ .4000 & .4000 \end{bmatrix}$$

A^{100}

$$\mathbf{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad A\mathbf{x}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \mathbf{x}_1 \quad (A\mathbf{x} = \mathbf{x} \text{ means that } \lambda_1 = 1)$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad A\mathbf{x}_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \frac{1}{2}\mathbf{x}_2 \text{ so } \lambda_2 = \frac{1}{2})$$

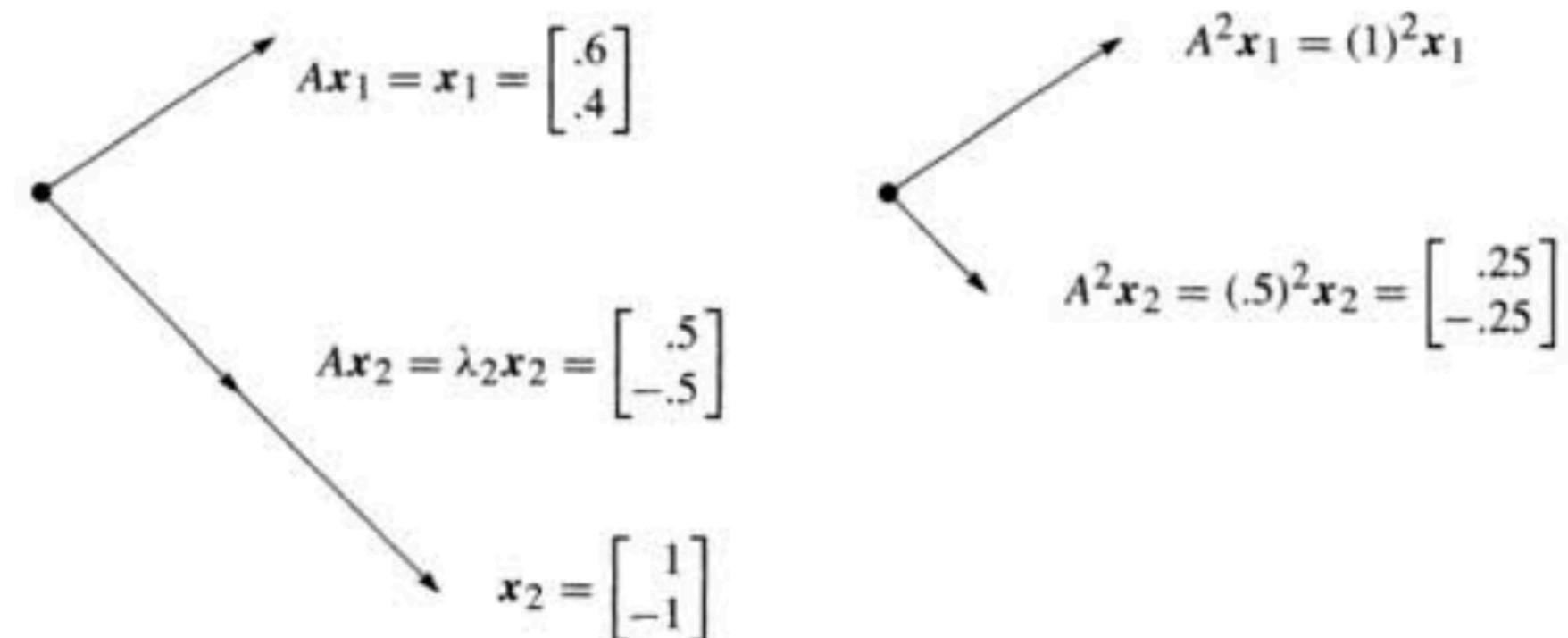


Figure 6.1 The eigenvectors keep their directions. A^2 has eigenvalues 1^2 and $(.5)^2$.

If we again multiply \mathbf{x}_1 by A , we still get \mathbf{x}_1 . Every power of A will give $A^n \mathbf{x}_1 = \mathbf{x}_1$. Multiplying \mathbf{x}_2 by A gave $\frac{1}{2}\mathbf{x}_2$, and if we multiply again we get $(\frac{1}{2})^2\mathbf{x}_2$. **When A is squared, the eigenvectors \mathbf{x}_1 and \mathbf{x}_2 stay the same.** The λ 's are now 1^2 and $(\frac{1}{2})^2$. **The eigenvalues are squared!** This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of A^{100} are the same \mathbf{x}_1 and \mathbf{x}_2 . The eigenvalues of A^{100} are $1^{100} = 1$ and $(\frac{1}{2})^{100} =$ very small number.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1 + (.2)\mathbf{x}_2$:

$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} \text{ is } \mathbf{x}_1 + (.2)\mathbf{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}. \quad (1)$$

Multiplying by A gives the first column of A^2 . Do it separately for \mathbf{x}_1 and $(.2)\mathbf{x}_2$. Of course $A\mathbf{x}_1 = \mathbf{x}_1$. And A multiplies \mathbf{x}_2 by its eigenvalue $\frac{1}{2}$:

$$A \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix} \text{ is } \mathbf{x}_1 + \frac{1}{2}(.2)\mathbf{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .1 \\ -.1 \end{bmatrix}.$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A . We didn't need these eigenvectors to find A^2 . But it is the good way to do 99 multiplications.

At every step \mathbf{x}_1 is unchanged and \mathbf{x}_2 is multiplied by $(\frac{1}{2})$, so we have $(\frac{1}{2})^{99}$:

$$A^{99} \begin{bmatrix} .8 \\ .2 \end{bmatrix} \text{ is really } \mathbf{x}_1 + (.2)(\frac{1}{2})^{99}\mathbf{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}.$$

This is the first column of A^{100} . The number we originally wrote as .6000 was not exact. We left out $(.2)(\frac{1}{2})^{99}$ which wouldn't show up for 30 decimal places.

Example 1 The projection matrix $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ has eigenvalues 1 and 0.

1. Each column of P adds to 1, so $\lambda = 1$ is an eigenvalue.
2. P is singular, so $\lambda = 0$ is an eigenvalue.
3. P is symmetric, so its eigenvectors $(1, 1)$ and $(1, -1)$ are perpendicular.

The only possible eigenvalues of a projection matrix are 0 and 1. The eigenvectors for $\lambda = 0$ (which means $P\mathbf{x} = 0\mathbf{x}$) fill up the nullspace. The eigenvectors for $\lambda = 1$ (which means $P\mathbf{x} = \mathbf{x}$) fill up the column space. The nullspace is projected to zero. The column space projects onto itself.

First move λx to the left side. Write the equation $Ax = \lambda x$ as $(A - \lambda I)x = \mathbf{0}$. The matrix $A - \lambda I$ times the eigenvector x is the zero vector. ***The eigenvectors make up the nullspace of $A - \lambda I$!*** When we know an eigenvalue λ , we find an eigenvector by solving $(A - \lambda I)x = \mathbf{0}$.

Eigenvalues first. If $(A - \lambda I)x = \mathbf{0}$ has a nonzero solution, $A - \lambda I$ is not invertible. ***The determinant of $A - \lambda I$ must be zero.*** This is how to recognize an eigenvalue λ :

6A The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular:

$$\det(A - \lambda I) = 0. \tag{3}$$

This “characteristic equation” involves only λ , not x . When A is n by n , $\det(A - \lambda I) = 0$ is an equation of degree n . Then A has n eigenvalues and each λ leads to x :

For each λ solve $(A - \lambda I)x = \mathbf{0}$ or $Ax = \lambda x$ to find an eigenvector x . (4)

Example 3 $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is already singular (zero determinant). Find its λ 's and x 's.

When A is singular, $\lambda = 0$ is one of the eigenvalues. The equation $Ax = 0x$ has solutions. They are the eigenvectors for $\lambda = 0$. But here is the way to find *all* λ 's and x 's! Always subtract λI from A :

Subtract λ from the diagonal to find $A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix}$.

Take the determinant “ $ad - bc$ ” of this 2 by 2 matrix. From $1 - \lambda$ times $4 - \lambda$, the “ ad ” part is $\lambda^2 - 5\lambda + 4$. The “ bc ” part, not containing λ , is 2 times 2.

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - (2)(2) = \lambda^2 - 5\lambda. \quad (5)$$

Set this determinant $\lambda^2 - 5\lambda$ to zero. One solution is $\lambda = 0$ (as expected, since A is singular). Factoring into λ times $\lambda - 5$, the other root is $\lambda = 5$:

$$\det(A - \lambda I) = \lambda^2 - 5\lambda = 0 \quad \text{yields the eigenvalues} \quad \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 5.$$

Now find the eigenvectors. Solve $(A - \lambda I)x = \mathbf{0}$ separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$$(A - 0I)x = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{yields an eigenvector} \quad \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{for } \lambda_1 = 0$$

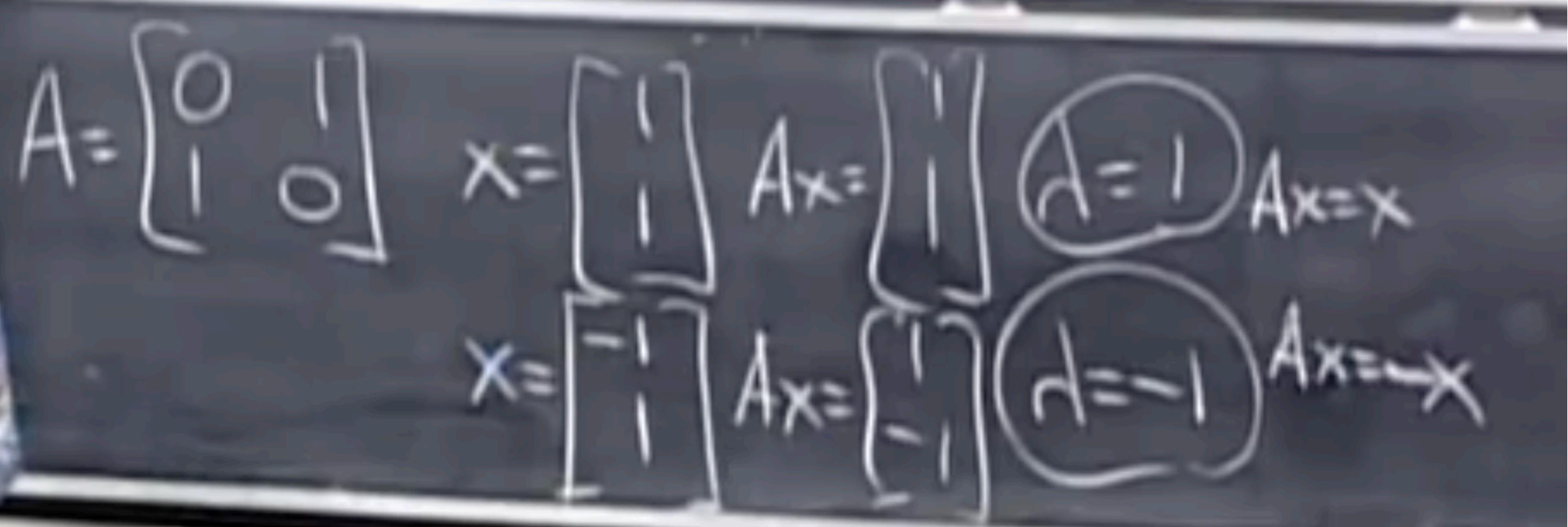
$$(A - 5I)x = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{yields an eigenvector} \quad \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{for } \lambda_2 = 5.$$

The matrices $A - 0I$ and $A - 5I$ are singular (because 0 and 5 are eigenvalues). The eigenvectors $(2, -1)$ and $(1, 2)$ are in the nullspaces: $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is $A\mathbf{x} = \lambda\mathbf{x}$.

We need to emphasize: *There is nothing exceptional about $\lambda = 0$.* Like every other number, zero might be an eigenvalue and it might not. If A is singular, it is. The eigenvectors fill the nullspace: $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$. If A is invertible, zero is not an eigenvalue. We shift A by a multiple of I to *make it singular*. In the example, the shifted matrix $A - 5I$ was singular and 5 was the other eigenvalue.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. *Compute the determinant of $A - \lambda I$.* With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n .
2. *Find the roots of this polynomial,* by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A . They make $A - \lambda I$ singular.
3. For each eigenvalue λ , *solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to find an eigenvector \mathbf{x} .*



$$A = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$$
$$A - 2I = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

st.

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = 4, x_2 = 2x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\# A x = \lambda x$$
$$\therefore (A + 3I)x = \lambda x + 3x = (\lambda + 3)x.$$

Bad news first: If you add a row of A to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the λ 's.* The triangular U has its eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of A ! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 2.$$

Good news second: The *product λ_1 times λ_2 and the sum $\lambda_1 + \lambda_2$ can be found quickly from the matrix*. For this A , the product is 0 times 2. That agrees with the determinant (which is 0). The sum of eigenvalues is 0 + 2. That agrees with the sum down the main diagonal (which is 1 + 1). These quick checks always work:

6B *The product of the n eigenvalues equals the determinant of A .*

6C *The sum of the n eigenvalues equals the sum of the n diagonal entries of A .*
This sum along the main diagonal is called the *trace* of A :

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace} = a_{11} + a_{22} + \cdots + a_{nn}. \quad (6)$$

Example 4 The 90° rotation $Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has no real eigenvectors or eigenvalues. No vector Qx stays in the same direction as x (except the zero vector which is useless). There cannot be an eigenvector, unless we go to *imaginary numbers*. Which we do.

If $Ax = \lambda x$, B has eigenvalues α_1 ,
 $Bx = \alpha x$
 $(A+B)x = (\lambda + \alpha)x \rightarrow$ No
as diff x
True for some x