# Stoke's theorem.

Statement: If $S$ be an open surface bounded by a closed curve $C$ and $F$ be any continous and differentiable vector point function then

$$\int_C F \cdot dR = \int_S \text{Curl}\, F \cdot N\, ds.$$

where $N = \cos\alpha\, \hat{i} + \cos\beta\, \hat{j} + \cos\gamma\, k$ is a unit normal at any point of $S$.

Proof :- Let $F = f_1 i + f_2 j + f_3 k$.

Consider $\int_C F \cdot dR = \int (f_1 i + f_2 j + f_3 k) \cdot (dx\, i + dy\, j + dz\, k)$

$$= \int f_1\, dx + f_2\, dy + f_3\, dz$$

Consider $\int_S \text{curl}\, F \cdot N\, ds.$

$$\text{curl}\, F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - j\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + k\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + j\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + k\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

now $N = i\cos\alpha + j\cos\beta + k\cos\gamma$

So. $\int_S \text{curl}\, F \cdot N\, ds = \int_S \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma \cdot ds$

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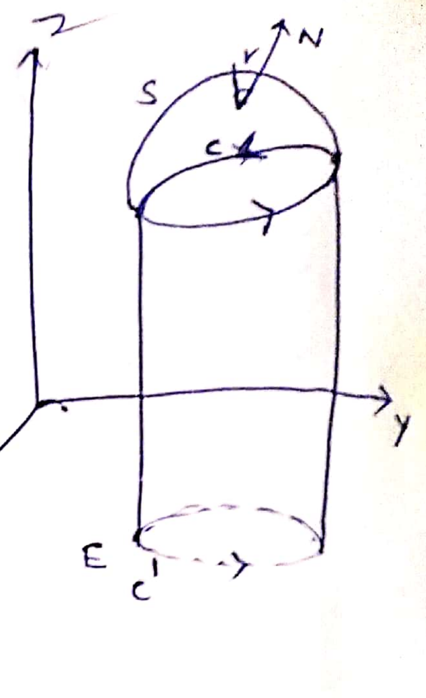
now let us prove that

$$\oint_C f_1\, dx = \int_S \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial y}\cos\gamma\right) ds$$




let $z = g(x,y)$ be the equation of $S$ whose projection is on $xy$-plane.
& projection $C$ is $C'$ on $xy$ plane.

$$\therefore \int_C f_1(x,y,z)\, dx = \int_{C'} f_1\{x,y,g(x,y)\}\, dx$$

$$= -\iint \frac{\partial}{\partial y}\left(f_1(x,y,g)\right) dx\, dy$$

($\because$ by green's theorm)

$$= -\iint \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial g}{\partial y}\right) dx\, dy$$

(differentiation of implicit function)

Now directional cosine of normal to the Surface $z = g(x,y)$ are given by

$$\frac{\cos\alpha}{-\partial g/\partial x} = \frac{\cos\beta}{-\partial g/\partial y} = \frac{\cos\gamma}{1}.$$

& $\frac{dx\,dy}{N\cdot K} = \frac{ds}{\cancel{N\cdot K}} = \frac{dx\,dy}{\cos\gamma}$ i.e., $ds = \frac{dx\,dy}{\cos\gamma}$

therefore r.H.S becomes.

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$$-\iint\left(\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\frac{\partial g}{\partial y}\right)dx\,dy.$$

Since $dx\,dy = ds\cos r$

$$= -\iint\left(\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\cdot\frac{\partial g}{\partial y}\right)ds.\cos r.$$

$$= -\iint\left(\frac{\partial f_1}{\partial y}.\cos r+\frac{\partial f_1}{\partial z}\frac{\partial g}{\partial y}.\cos r\right).ds$$

from the direction.cosines of normal to $S$,

we have $\dfrac{\cos r}{1} = \dfrac{\cos\beta}{-\partial g/\partial y}$

$$\frac{\partial g}{\partial y}\cos r = -\cos\beta.$$

$$= -\iint\left(\frac{\partial f_1}{\partial y}.\cos r-\frac{\partial f_1}{\partial z}.\cos\beta\right)ds$$

$$= \iint\left(\frac{\partial f_1}{\partial z}\cos\beta-\frac{\partial f_1}{\partial y}\cos r\right)ds$$

So we proved $\oint_c f_1\,dx = \int_S\left(\frac{\partial f_1}{\partial z}\cos\beta-\frac{\partial f_1}{\partial y}\cos r\right)ds$

Similarly prove for $f_2$ & $f_3$. and by adding all the results we get (1)

Corollarly Green's theorem in a plane is a special case of stoke's theorem.

Let $F = f_1 i + f_2 j$ be a vector function which is continously differentiable in a region S of xy plane bounded by curve c

then
$$\int_C F \cdot dR = \int_C (f_1 i + f_2 j) \cdot (dx\, i + dy\, j)$$

$$= \int_C f_1 dx + f_2 dy$$

and
$$\text{Curl } F \cdot N = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ f_1 & f_2 & 0 \end{vmatrix} \cdot k$$

$$= k \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \cdot k$$

$$= \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}.$$

hence Stoke's theorem becomes

$$\int f_1 dx + f_2 dy = \iint_S \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\, dy$$

which is green's theorem in a plane.

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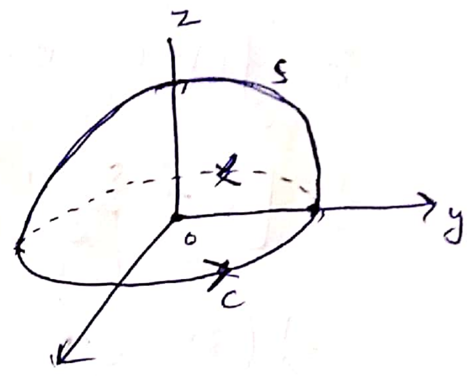
Verify Stoke's theorem for the vector field

Prob. $F = (2x-y)\hat{i} - yz^2\hat{j} - y^2 z k$ over the upper half surface of $x^2+y^2+z^2 = 1$ bounded by it's projection on xy plane.

Sol. projection of upper half of given surface i.e given sphere is on xy plane. $(\because z=0)$




So the projection is $x^2+y^2=1$

now $\oint_C F \cdot dR = \oint_C (2x-y)dx - yz^2 dy - y^2 z\, dz$

Now Since $z=0$ on xy plane

$\oint_C (2x-y)\, dx$.

Taking parametric form $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\, d\theta$

$= \int_0^{2\pi} (2\cos\theta - \sin\theta) -\sin\theta\, d\theta$

$= -\int_0^{2\pi} (2\cos\theta\sin\theta - \sin^2\theta)\, d\theta$

$= \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta)\, d\theta$.

$= \left[\frac{\cos 2\theta}{2}\right]_0^{2\pi} + \int_0^{2\pi} \sin^2\theta \cdot d\theta = 0 + \int_0^{2\pi} \sin^2\theta \cdot d\theta$

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$$= \int_0^{2\pi} \sin^2\theta \cdot d\theta$$

$$= \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi) = \pi .$$

$$\text{Curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= i(-2yz+2yz) + (0-0)j + (0-(-1))k$$

$$= 0+0+k$$

$$= k.$$

$$\int \text{curl} F \cdot N\, ds = \int_S k \cdot N\, ds.$$

$$= \int k \cdot N \cdot \frac{dx\,dy}{|N \cdot k|}.$$

$$= \int_A dx\,dy.$$

area of circle $= \pi$

(2) use Stoke's theorem evaluate $\int_c (x+y)dx + (2x-z)dy + (y+z)dz)$ where $c$ is the boundary of triangle with vertices $(2,0,0)$ $(0,3,0)$ $(0,0,6)$

Sol here $F = (x+y)i + (2x-z)j + (y+z)k$

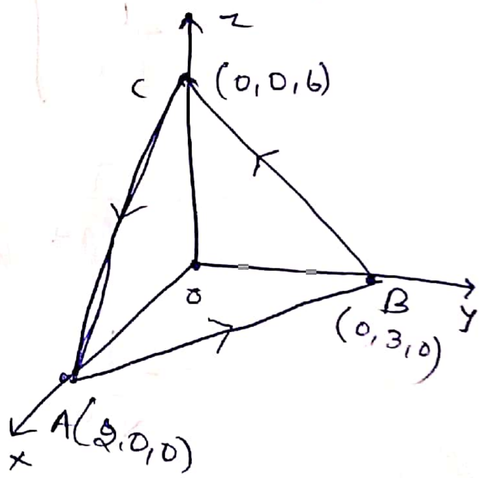
$$\text{Curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z}(2x-z)\right) - j\left(\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial z}(x+y)\right)$$

$$+ k\left(\frac{\partial}{\partial x}(2x-z) - \frac{\partial}{\partial y}(x+y)\right)$$

$$= (1+1)i - j(0-0) + k(2-1)$$

$$= 2i + k$$




We find eqn of plane through given points $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

$$3x + 2y + z = 6$$

So normal to this plane is $\nabla(3x + 2y + z - 6)$

$$= 3i + 2j + k.$$

$$N = \frac{3i + 2j + k}{\sqrt{9+4+1}} = \frac{1}{\sqrt{14}}(3i + 2j + k)$$

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Now

$$\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$$

$$= \int F \cdot dR.$$

$$= \int_S \text{curl } F \cdot N\, ds$$

$$= \int_S (2i+k)\left(\frac{3i+2j+k}{\sqrt{14}}\right) ds$$

$$\frac{1}{\sqrt{14}} \int (6+1)\, ds$$

$$= \frac{7}{\sqrt{14}} \int_S ds.$$

$$= \frac{7}{\sqrt{14}} \text{ Area of triangle } ABC = \frac{7}{\sqrt{14}} \cdot \frac{3\sqrt{14}}{2} = 21$$

Area

$A(2,0,0)$ $B(0,3,0)$ $C(0,0,6)$

$$\text{Area of } \Delta ABC = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right|$$

$$\vec{AB} = (0-2)i + (3-0)j + (0-0)k = -2i+3j$$

$$\vec{AC} = (0-2)i + (0-0)j + (6-0)k = -2i+6k$$

$$\text{Area} = \frac{1}{2} \text{ curl } AB \,\&\, AC = \frac{1}{2} \begin{vmatrix} i & j & k \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix}$$

$$= i(18) - j(-12) + k(6)$$

$$\frac{1}{2} \|\vec{AB} \times \vec{AC}\| = \frac{1}{2} \sqrt{18^2+12^2+6^2} = \frac{1}{2} 6\sqrt{3^2+2^2+1} = 3\sqrt{14}$$

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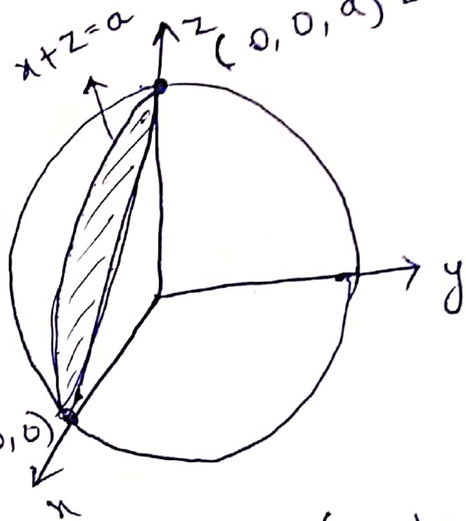
Q :- Apply Stoke's theorem to evaluate $\int y\,dx + z\,dy + x\,dz$ where C is the curve of intersection of $x^2+y^2+z^2 = a^2$ and $x+z=a$

Sol :- Curve is a circle on plane ~~x+y~~ $x+z=a$ with endpoints of diameter as $A(a,0,0)$, $B(0,0,a)$




$$\therefore \int_C y\,dx + z\,dy + x\,dz$$

$$= \int_C (y\,i + z\,j + x\,k) \cdot dR$$

$$= \int_S \text{curl}(y\,i + z\,j + x\,k) \cdot N\,ds$$

(grad of plane eqn)

$$N = \frac{\partial x}{\partial x} i + \frac{\partial 0}{\partial y} j + \frac{\partial z}{\partial z} k$$

where S is circle on AB.

$$\text{unit normal } N = \frac{i+k}{\sqrt{2}} \qquad (\because y = 0)$$

$$\text{curl}(y\,i + z\,j + x\,k) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= i(0-1) - j(1-0) + k(0-1)$$

$$= -(i+j+k)$$

$$= \int_S -(i+j+k) \cdot \left(\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} k\right) ds$$

$$= \int \left(\frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\right) \cdot ds$$

$$= \frac{-2}{\sqrt{2}} \int_S ds = \frac{-2}{\sqrt{2}} \pi \left(\frac{a}{\sqrt{2}}\right)^2$$

$$= \frac{-\pi a^2}{\sqrt{2}}$$

r of circle $= \frac{1}{2}\sqrt{a^2+0^2+a^2} = \frac{1}{2}\sqrt{2a^2} = \frac{a}{\sqrt{2}}$

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