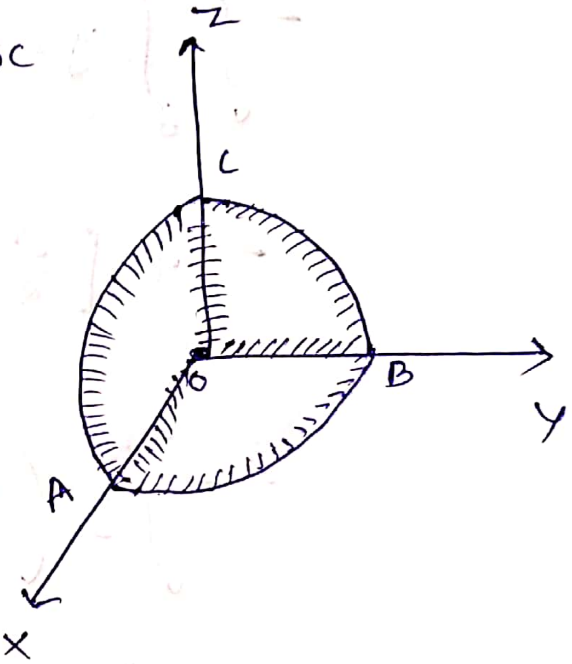
Few more problems on Vector calculus.

1) Evaluate $\int_S yz\,i + zx\,j + xy\,k \cdot ds$ where S is the surface of sphere $x^2+y^2+z^2=a^2$ in 1^{st} octant

Sol: The surface of region V=OABC is piecewise smooth. Let



S_1 - circular quadrant OBC in YZ plane.

S_2 - circular quadrant OAC in zx plane.

S_3 - circular quadrant OAB in xy plane

S - surface ABC of sphere in first octant

also $F = yz\,i + zx\,j + xy\,k$.

by divergence theorem,

$$\int_V \operatorname{div} F = \int_{S_1} F\cdot ds + \int_{S_2} F\cdot ds + \int_{S_3} F\cdot ds + \int_{S} F\cdot ds.$$

$$\operatorname{div} F = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0$$

for now S_1. $\int_{S_1} F\cdot ds = \iint_{S_1} F\cdot(-i)\,\cancel{dx}\,dy\,dz$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} (yz\,i)(-i)\, dy\, dz \qquad (\because x=0)$$

$$= -\int_0^a \int_0^{\sqrt{a^2-y^2}} yz\, dy\, dz.$$

$$= -\int_0^a y \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy.$$

$$= -\int_0^a y\left(\frac{a^2-y^2}{2}\right) dy.$$

$$= -\int_0^a \left(\frac{ya^2}{2} - \frac{y^3}{2}\right) dy.$$

$$= -\frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{-1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{-1}{2}\,\frac{a^4}{\cancel{8}\,4}$$

$$= \frac{-a^4}{8}.$$

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Similarly $\int_{S_2} F\cdot ds = \int_{S_3} F\cdot ds = \frac{-a^4}{8}$.

So $\int_{S_1} F\cdot ds + \int_{S_2} F\cdot ds + \int_{S_3} F\cdot ds + \int_{S_4} F\cdot ds = \int \text{div} F\, ds$

$$-\frac{3a^4}{8} + \int F\cdot ds = 0$$

$$\int F\cdot ds = \frac{3a^4}{8}.$$

2) Use stoke's theorem to evaluate the line integral $\oint y^3 dx - x^3 dy + z^3 dz$, where the curve C is the intersection of cylinder $x^2+y^2=a^2$ and the plane $x+y+z=b$.

Sol:- By Stoke's theorem, we have

$$\int F\cdot dR = \int \text{curl} F\cdot N ds.$$

$N = \text{grad}(\phi)$. where ϕ is the equation of plane i.e., $x+y+z=b$.

$$N = \frac{1\cdot i + 1\cdot j + 1\cdot k}{\sqrt{1^2+1^2+1^2}} = \frac{i+j+k}{\sqrt{3}}$$

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$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & -x^3 & z^3 \end{vmatrix}$$

$$= i(0-0) + j(0) + k(-3x^2-3y^2)$$

$$= -3(x^2+y^2)k.$$

Now by applying Stoke's theorem, we get

$$I = \int \text{curl } F \cdot N \, ds$$

$$= \int -3(x^2+y2) k \cdot \left(\frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k\right) ds$$

$$= -\sqrt{3} \iint (x^2+y^2) \, ds.$$

Taking projection on xy plane

$$= -\sqrt{3} \iint (x^2+y^2) \frac{dx\,dy}{n \cdot k}$$

$$= -\sqrt{3} \iint (x^2+y2) \frac{dx\,dy}{1/\sqrt{3}}$$

$$= -3 \iint (x^2+y^2) \, dx\,dy$$

where the region of integration is a circle with radius a changing to polar form

$$= -3 \iint r^2 \cos^2\theta + r^2 \sin^2\theta \, r \, dr \, d\theta$$

$$= -3 \int_0^a \int_0^{2\pi} r^2 \cdot r \cdot dr \, d\theta .$$

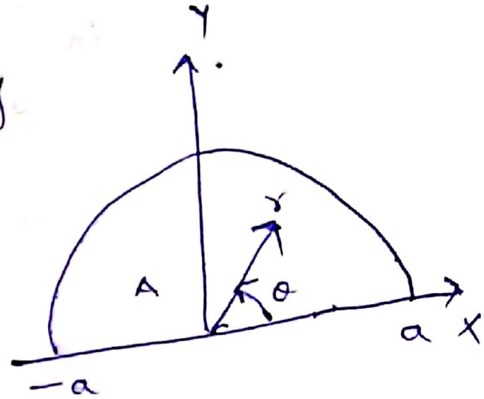
$$= -3 \left(\frac{r^4}{4}\right)_0^a (2\pi - 0)$$

$$= \frac{-3\pi a^4}{2}$$

Apply Green's theorem to evaluate $\int_c (2x^2-y^2)dx + (x^2+y^2)dy$ where c is the boundary of area enclosed by x-axis and upper half of circle $x^2+y^2=a^2$.

By Green's theorem

$$\int P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$



$$= \iint \frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2-y^2)\; dx\,dy$$

$$= \cancel{2\iint (x+y) - (0-y)}$$

$$= \iint 2x + 2y \; dx\,dy$$

$$= 2\iint (x+y)\, dx\,dy.$$

put $x = r\cos\theta$ $\quad y = r\sin\theta$

where r goes from 0 to a

θ goes from 0 to π

$$= 2\int_0^a\int_0^\pi r(\cos\theta+\sin\theta)\,\underbrace{r}_{=|J|}\,d\theta\,dr.$$

$$= 2\int_0^a r^2\,dr \int_0^\pi (\cos\theta+\sin\theta)\,d\theta$$

$$2\int_0^a r^2 dr \int_0^{\pi} (\cos\theta + \sin\theta)\, d\theta$$

$$= 2.\int_0^a r^2.dr \left(\sin\theta - \cos\theta\right)_0^{\pi}$$

$$= 2.\int_0^a r^2 dr \left(1+1\right)$$

$$= 4.\int_0^a r^2.dr$$

$$= 4\left[\frac{r^3}{3}\right]_0^a$$

$$= \frac{4a^3}{3}.$$

Exam pattern total 20M.

4 marks × 2Q = 8.M.

3 marks × 2Q = 6M.

1 Mark × 6Q = 6M

20 M.

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