Volume Integral.

Let $F(R)$ be a continous vector function and S be the surface enclosing region E. then the volume integral of $F(R) = f\,i + \phi\,j + \psi\,k$ is given by $\int_E F\,dv = i\iiint_E f\,dx\,dy\,dz + j\iiint_E \phi\,dx\,dy\,dz + k\iiint_E \psi\,dx\,dy\,dz$

Evaluate $\int_0^1\int_0^1\int_0^1 (xi + yj + zk)\,dx\,dy\,dz$.

$$= i\int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz + j\int_0^1\int_0^1\int_0^1 y\,dx\,dy\,dz + k\int_0^1\int_0^1\int_0^1 z\,dx\,dy\,dz$$

$$= i.\int_0^1 x\,dx + j\int_0^1 y\,dy + k\int_0^1 z\,dz$$

$$= i\left[\frac{x^2}{2}\right]_0^1 + j\left[\frac{y^2}{2}\right]_0^1 + k\left[\frac{z^2}{2}\right]_0^1$$

$$= \frac{i}{2} + \frac{j}{2} + \frac{k}{2}.$$

$$= \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k.$$ is a vector.

For integrating a vector function over a volume, we integrate each component and the result is a vector.

Gauss Divergence theorem.

* this theorem gives the relation between Surface and Volume integral.

<u>Statement</u> :- If F is a continous and differentiable vector function in the region E enclosed by a <u>closed</u> Surface S, then

$$\int_S F \cdot N \, ds = \int_E \text{div } F \, dv.$$

where N is the unit outward normal

<u>Proof</u> :- let $F(R) = f(x,y,z)\,i + \phi(x,y,z)\,j + \psi(x,y,z)\,k$

we need to prove that, $\int_S F \cdot N \, ds = \int_E \text{div } F \, dv$

Consider LHS $= \int (f\,i + \phi\,j + \psi\,k) \cdot (\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k) \cdot ds$

$$= \int (f\cos\alpha + \phi\cos\beta + \psi\cos\gamma)\, ds.$$

$$= \int f\,dy\,dz + \phi\,dz\,dx + \psi\,dx\,dy \qquad \left(\because \begin{array}{l} \cos\alpha\, ds = dy\,dz \\ \cos\beta\, ds = dx\,dz \\ \cos\gamma\, ds = dx\,dy \end{array}\right)$$

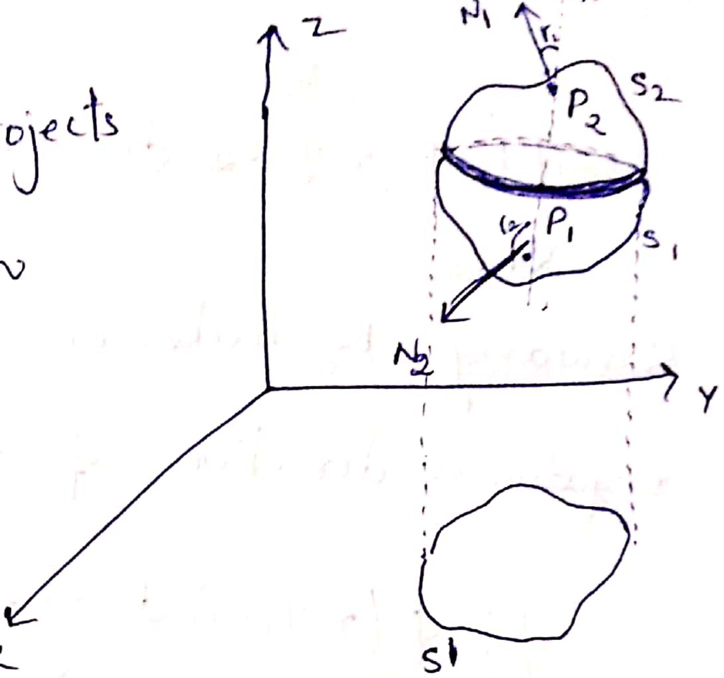
now, R.H.S $\int \text{div } F \, dv = \int \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}\right) dx\,dy\,dz$

So we need to prove that

$$\iint_S f\,dy\,dz + \phi\,dz\,dx + \psi\,dx\,dy = \iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}\right) dx\,dy\,dz.$$

Consider surface S, such that a line parallel to z axis cuts it in 2 points say $P_1(x,y,z_1)$ and $P_2(x,y,z_2)$, $(z_1 \le z_2)$.

let the surface S projects on xy plane i.e., S', then



$$\iiint_E \frac{\partial \phi}{\partial z}\, dx\,dy\,dz$$

$$= \iint_{S'} dx\,dy \int_{z_1}^{z_2} \frac{\partial \phi}{\partial z}\, dz$$

$$= \iint_{S'} dx\,dy \left[\phi(x,y,z_2) - \phi(x,y,z_1) \right]$$

$$= \iint_{S'} \phi(x,y,z_2)\, dx\,dy - \iint_{S'} \phi(x,y,z_1)\, dx\,dy$$

Let S_1, S_2 be the lower and upper parts of the surface S corresponding to points P_1 & P_2 respectively and N is unit normal at any point of S.

The outward normal at any point of S_2 makes an acute angle with positive direction of z-axis

So $\iint_{S'} \psi(x,y,z_2)\,dx\,dy = \int_{S_2} \psi\, N.K\, ds$ $\left(\because dx\,dy = \cos r_1\, ds = N.k\,ds\right)$

Similarly the outward normal at any point of S_1 makes an obtuse angle with positive direction of z axis, $\therefore$ so

$$\iint_{S'} \psi(x,y,z_1)\,dx\,dy = -\int_{S_1} \psi\, N.K\, ds$$

So, now we have $\iiint_E \frac{\partial \psi}{\partial z}\,dx\,dy\,dz = \iint \psi(x,y,z_2)\,dx\,dy - \iint \psi(x,y,z_1)\,dx\,dy$

$$= \int_{S_2} \psi N.K\,ds + \int_{S_1} \psi N.K\,ds = \int_S \psi N.K\,ds$$

Similarly we can prove $\iiint_E \frac{\partial f}{\partial x}\,dx\,dy\,dz = \int_S f\, N.i\,ds$

and $\iiint_E \frac{\partial \phi}{\partial y}\,dx\,dy\,dz = \int_S \phi\, N.j\,ds$

by adding three terms we get the required result

$$\iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}\right) dx\,dy\,dz = \int_S (f i + \phi j + \psi k) \cdot N\, ds$$

$$\iiint_E \operatorname{div} F\, dv = \int_S F \cdot N\, ds.$$

1) Verify Divergence theorem for

$$F = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$$

taken over the rectangular parallelepiped

$0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$

Sol

$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$2(x) + 2y + 2z = 2(x+y+z)$$

$$=$$

$$\int_R \operatorname{div} F\, dv = 2\int_0^c \int_0^b \int_0^a (x+y+z)\, dx\,dy\,dz$$

$$= 2\int_0^c \int_0^b \left[\frac{x^2}{2} + yx + zx\right]_0^a dy\,dz$$

$$= 2\int_0^c \int_0^b \frac{a^2}{2} + a(y) + a(z) \cdot dy\,dz$$

$$= 2\int_0^c \left(\frac{a^2 y}{2} + a\frac{y^2}{2} + azy\right) dz$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right) dz \qquad = \qquad abc\,(a+b+c)$$

$$= 2\left(\frac{a^2 b z}{2} + \frac{ab^2 z}{2} + \frac{ab z^2}{2}\right) = 2\left(\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2}\right)$$

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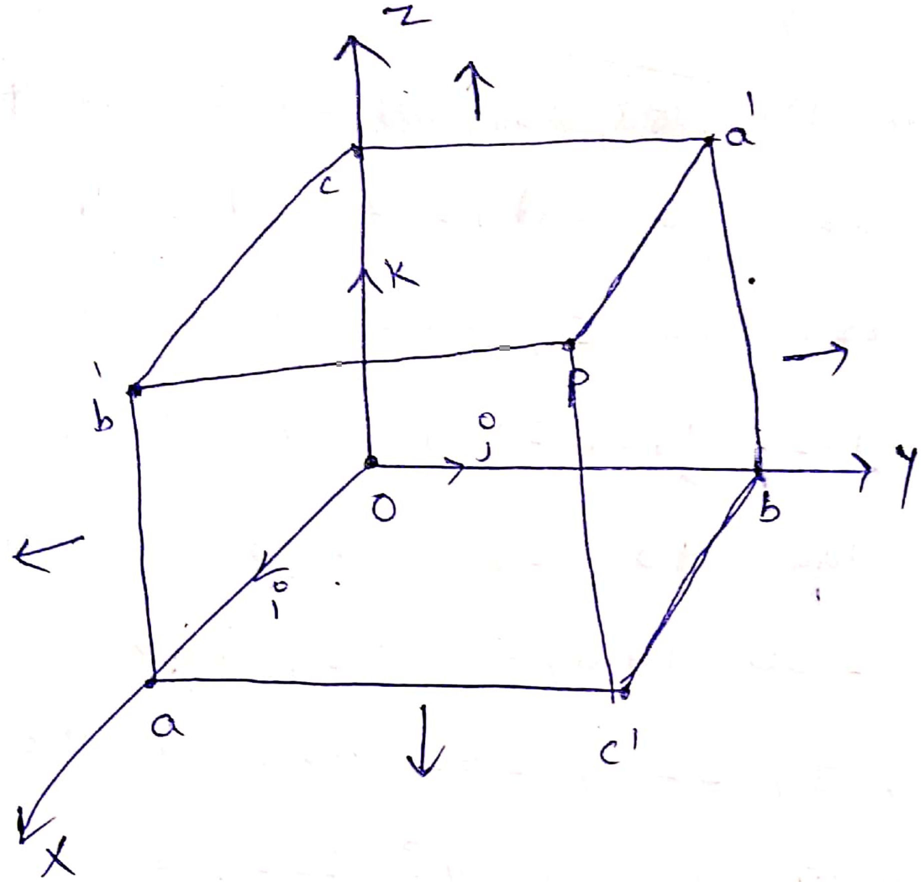

S_1 – $oabc'$

S_2 – $cb'pa'$

S_3 – $oba'c$

S_4 – $ac'b'p$

S_5 – $ocb'a$

S_6 – $ba'pc'$

$$\int_S F.N\,ds = \int_{S_1} F.N\,ds + \int_{S_2} F.N\,ds + \int_{S_3} F.N\,ds + \int_{S_4} F.N\,ds + \int_{S_5} F.N\,ds + \int_{S_6} F.N\,ds$$

S_1 $oabc'$ $\quad \int_{S_1} F.N\,ds = \int_{S_1} F.(-K)\,ds$ (outward normal is $-k$)

$$= \int_{S_1} (z^2 - xy)k.(-k)\,dx\,dy$$

$$= \int_{S_1} xy\,dx\,dy$$

($\because z = 0$ on xy plane)

$$= \int_0^a \int_0^b xy\,dx\,dy = \int_0^a x\left(\frac{y^2}{2}\right)_0^b dx$$

$$= \frac{b^2}{2}\int_0^a x\,dx = \frac{b^2a^2}{4}$$

let S_2 be the Surface $cb'pa'$ which is parallel to $oa'cb$

$= \int_{S_2} F \cdot K\, ds$ $\quad$ ($\because$ normal is in +ve direction of z axis)

$= \int_0^a \int_0^b (c^2 - xy)\, dx\, dy$ $\quad$ ($\because z = c$)

$= \int_0^a \left[c^2 y - x\frac{y^2}{2} \right]_0^b dx$

$= \int_0^a \left(c^2 b - \frac{xb^2}{2} \right) dx$

$= \left[c^2 b\, x - \frac{x^2 b^2}{4} \right]_0^a$

$= abc^2 - \frac{b^2 a^2}{4}.$

Now S_3 be the surface $oba'c$

here $x = 0$, $\int_{S_3} F \cdot N\, ds = \int F\cdot(-i)\, ds$

$= -\int_0^b \int_0^c (x^2 - yz)\, dy\, dz.$

$= +\int_0^b \int_0^c yz\, dy\, dz$ $\quad$ ($x=0$)

$= \frac{b^2 c^2}{4}.$

S_4 be the surface $ac'pb'$ exactly parallel to $oca'b$.

$\int_{S_4} F \cdot N\, ds = \int_{S_4} F \cdot i\, ds$

$$= \int_0^b \int_0^c (x^2 - yz)\, dy\, dz \qquad \text{put } x = a$$

$$= a^2bc - \frac{b^2c^2}{4}$$

Similarly S_5 $ocb'a$, here $y = 0$

$$\int_{S_5} F \cdot N\, ds = \iint F \cdot (-j)\, dx\, dz$$

$$= -\int_0^a \int_0^c (y^2 - xz)\, dx\, dz \qquad (-j \text{ is the normal})$$

$$= \int_0^a \int_0^c xz\, dx\, dz \qquad (\because y = 0)$$

$$= \frac{a^2c^2}{4}.$$

now, S_6 $ba'pc'$ which is parallel to $ocb'a$.

$$\int_{S_6} F \cdot N\, ds = \int_{S_6} F \cdot j\, dx\, dz$$

$$= \int_0^a \int_0^c (y^2 - zx) \cdot dx\, dz$$

$$= ab^2c - \frac{c^2a^2}{4} \qquad \text{put } y = b.$$

now $$\int_S F \cdot N\, ds = \int_{S_1} F \cdot N\, ds + \int_{S_2} F \cdot N\, ds + \int_{S_3} \cdots + \int_{S_5} F \cdot N\, ds + \int_{S_6} F \cdot N\, ds$$

$$= \cancel{\frac{a^2b^2}{4}} + abc^2 - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{b^2c^2}{4}} + a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{c^2a^2}{4}} + ab^2c - \cancel{\frac{c^2a^2}{4}}$$

$$= abc^2 + a^2bc + ab^2c$$

$$= abc(a + b + c) = \int \operatorname{div} F\, dv$$

Apply divergence theorem to evaluate $\int lx^2 + my^2 + nz^2 ds$ taken over sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$, l, m & n being the direction cosines of external normal to the sphere

1 By Gauss divergence theorem, $\int_S F.N ds = \int_V div F dv$

So $\int lx^2 + my^2 + nz^2 \cdot ds = \int (x^2 i + y^2 j + z^2 k) \cdot (li + mj + nk) ds$

$$= \int (x^2 i + y^2 j + z^2 k) . N ds$$

$$= \int div(x^2 + y^2 + z^2) dv.$$

$$= 2 \int x + y + z \, dv. \quad = 2 \int (x+y+z) dx dy dz$$

parametric equations of sphere are

$$x = a + r \sin\theta \cos\phi.$$

$$y = b + r \sin\theta \sin\phi$$

$$z = c + r \cos\theta.$$

where r goes from 0 to ρ

θ goes from 0 to π

ϕ goes from 0 to 2π.

$$|J| \, dx dy dz = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta .$$

$$dx dy dz = \cancel{dx dy dz} \; r^2 \sin\theta \, dr d\theta d\phi$$

now $\quad 2\int (x+y+z)\, dx\, dy\, dz$

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_0^{\rho} (a + r\sin\theta\cos\phi + b + r\sin\theta\sin\phi + c + r\cos\theta)\cdot r^2\sin\theta \cdot dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_0^{\rho} \big((a+b+c) + r\sin\theta\cos\phi + r\sin\theta\sin\phi + r\cos\theta\big)\, r^2\sin\theta \cdot dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_0^{\rho} (a+b+c)\, r^2 \sin\theta \, dr\, d\theta\, d\phi \ +$$

$$2\int_0^{2\pi}\int_0^{\pi}\int_0^{\rho} r^3 \cdot (\sin^2\theta\cos\phi + \sin^2\theta\cdot\sin\phi + \cos\theta\sin\theta)\, dr\, d\theta\, d\phi$$

$$= 2(a+b+c)\int_{\theta=0}^{\pi}\int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^{\rho} \cdot \sin\theta\, dr\, d\theta\, d\phi \quad + I_2$$

$$= \frac{2}{3}(a+b+c)\rho^3 \cdot \int_0^{\pi} \sin\theta\, [\phi]_0^{2\pi}\, d\theta \quad + I_2$$

$$= \frac{2}{3}(2\pi)(a+b+c)\rho^3(-\cos\theta)_0^{\pi} \quad + I_2$$

$$= \frac{8}{3}\pi\,(a+b+c)\rho^3 \quad + I_2$$

$$= \frac{8}{3}\pi\,(a+b+c)\rho^3 \quad + 0$$

$$I_2 = 2\int_0^{2\pi}\int_0^{\pi}\int_0^{\rho}\left(\sin^2\theta\cos\phi + \sin^2\theta\cdot\sin\phi + \cos\theta\cdot\sin\theta\right)$$

$$r^3\cdot dr\,d\phi\,ds.$$

$$= \frac{2\rho^4}{4}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sin^2\theta(\cos\phi + \sin\phi) + \cos\theta\sin\theta\cdot d\theta\cdot d\phi$$

$$= \frac{\rho^4}{2}\int_0^{\pi}\left[\sin^2\theta\left(\sin\phi - \cos\phi\right)_0^{2\pi}\right] + \cos\theta\cdot\sin\theta\left(\phi\right)_0^{2\pi}\Big]d\theta$$

$$= \frac{\rho^4}{2}\int_0^{\pi}\left(\sin^2\theta\,(0) + \frac{\sin 2\theta}{2}\cdot 2\pi\right) d\theta.$$

$$= \frac{\rho^4}{4}\pi\int_0^{\pi} \sin 2\theta\cdot d\theta$$

$$= \frac{\rho^4\pi}{4}\left[\frac{\cos 2\theta}{2}\right]_0^{\pi}$$

$$= 0.$$

(3) Evaluate $\int_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2} \, ds$ where S the Surface of the ellipsoid $ax^2+by^2+cz^2=1$

Sol Let $\phi = ax^2+by^2+cz^2-1=0$

$$\nabla\phi = 2axi + 2byj + 2czk$$

unit vector normal to the given ellipsoid is

$$N = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(axi+byj+czk)}{\sqrt{(2ax)^2+(2by)^2+(2cz)^2}}$$

$$= \frac{axi+byj+czk}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

$F.N = (a^2x^2+b^2y^2+c^2z^2)^{-1/2}$ Comparing integrand with F.N

So $F.\dfrac{(axi+byj+czk)}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} = (a^2x^2+b^2y^2+c^2z^2)^{-1/2}$

$F.(axi+byj+czk)=1.$

$\Rightarrow$ obviously $F = xi+yj+zk$ $(\because ax^2+by^2+cz^2=1)$

So

So by divergence theorem

$$\int F.ds = \int \text{div} F \, dv = \int \left(\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right) dv$$

$$= 3\int dv = 3.V = 3.\frac{4\pi}{3}\frac{1}{\sqrt{abc}}$$ ($\because$ volume of ellipsoid)

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Volume of the ellipsoid $ax^2+by^2+cz^2=1$

by converting to sphere,

takes $\sqrt{a}\,x=u$ so that $ax^2=u^2$.

$\Rightarrow dx=\frac{du}{\sqrt{a}}$, $dy=\frac{dv}{\sqrt{b}}$, $dz=\frac{dw}{\sqrt{c}}$.

so now the equation changes to $u^2+v^2+w^2=1$

whose radius is 1 and $dx\,dy\,dz=\frac{1}{\sqrt{abc}}du\,dv\,dw$

so volume of ellipsoid $=\frac{1}{\sqrt{abc}}\iiint_{\text{Sphere}} du\,dv\,dw$.

$=\frac{1}{\sqrt{abc}}\frac{4}{3}\pi$..

(Volume of sphere $=\frac{4}{3}\pi$)
of radius 1