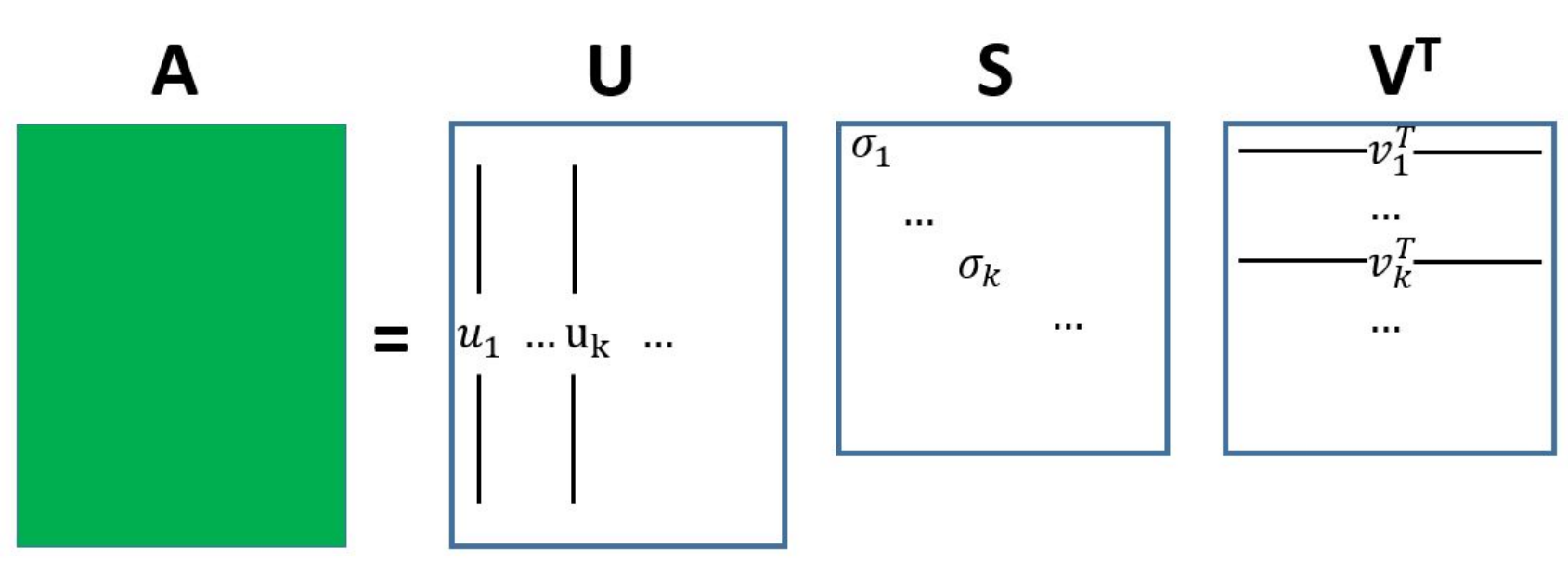
RANDOMISED ALGORITHMS FOR LOW-RANK APPROXIMATION

Kaspar Märtens Sherman Ip

6 November 2015

SVD

Singular value decomposition for matrix $A \in \mathbb{R}^{m,n}$



Truncated SVD

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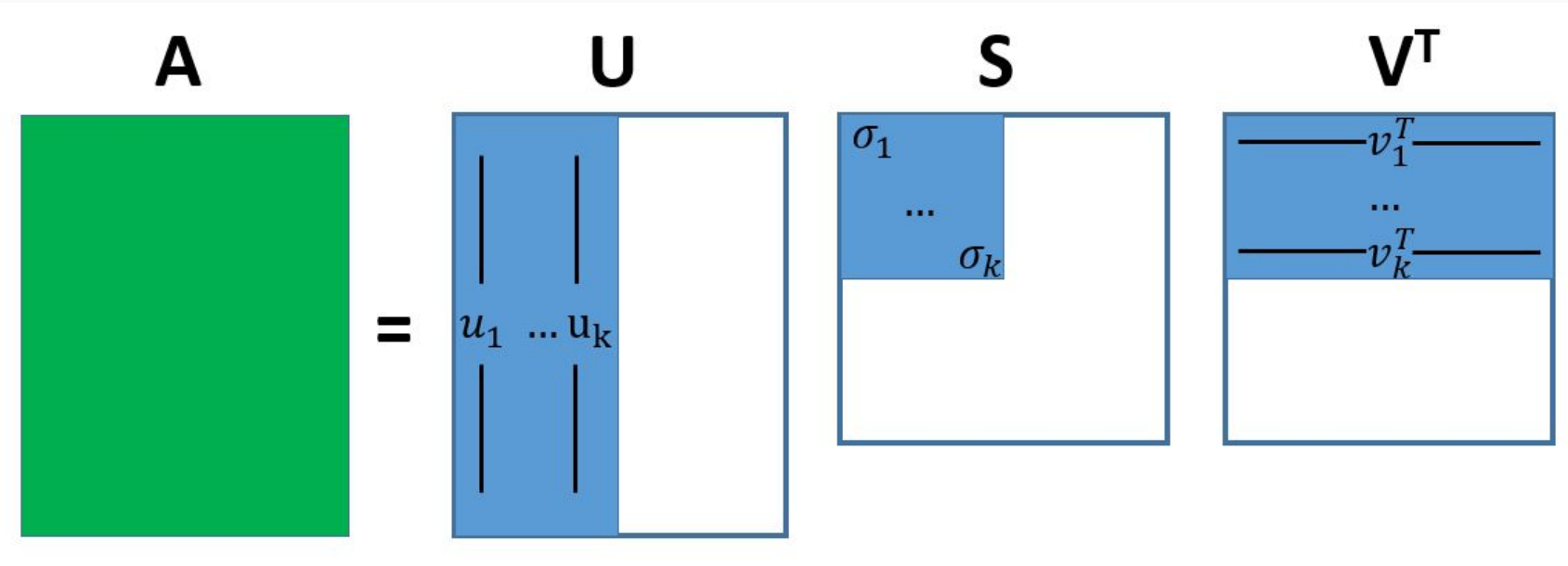

$$\tilde{A}_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

TRUNCATED SVD ILLUSTRATION

$k \in \{3, 5, 10, 30, 50, 100\}$

RANDOMISED ALGORITHM

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We want the orthonormal columns to span the range of A.

Bases in the range of A

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It turns out that it is useful to replace k with $k + p$, $p \geq 1$.

Implementation

Putting all of this together, the algorithm only takes two lines in MATLAB

$$Q = \texttt{orth}(A \times \texttt{randn}(n, k + p))$$

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$$A \approx QQ^{\mathrm{T}} A$$

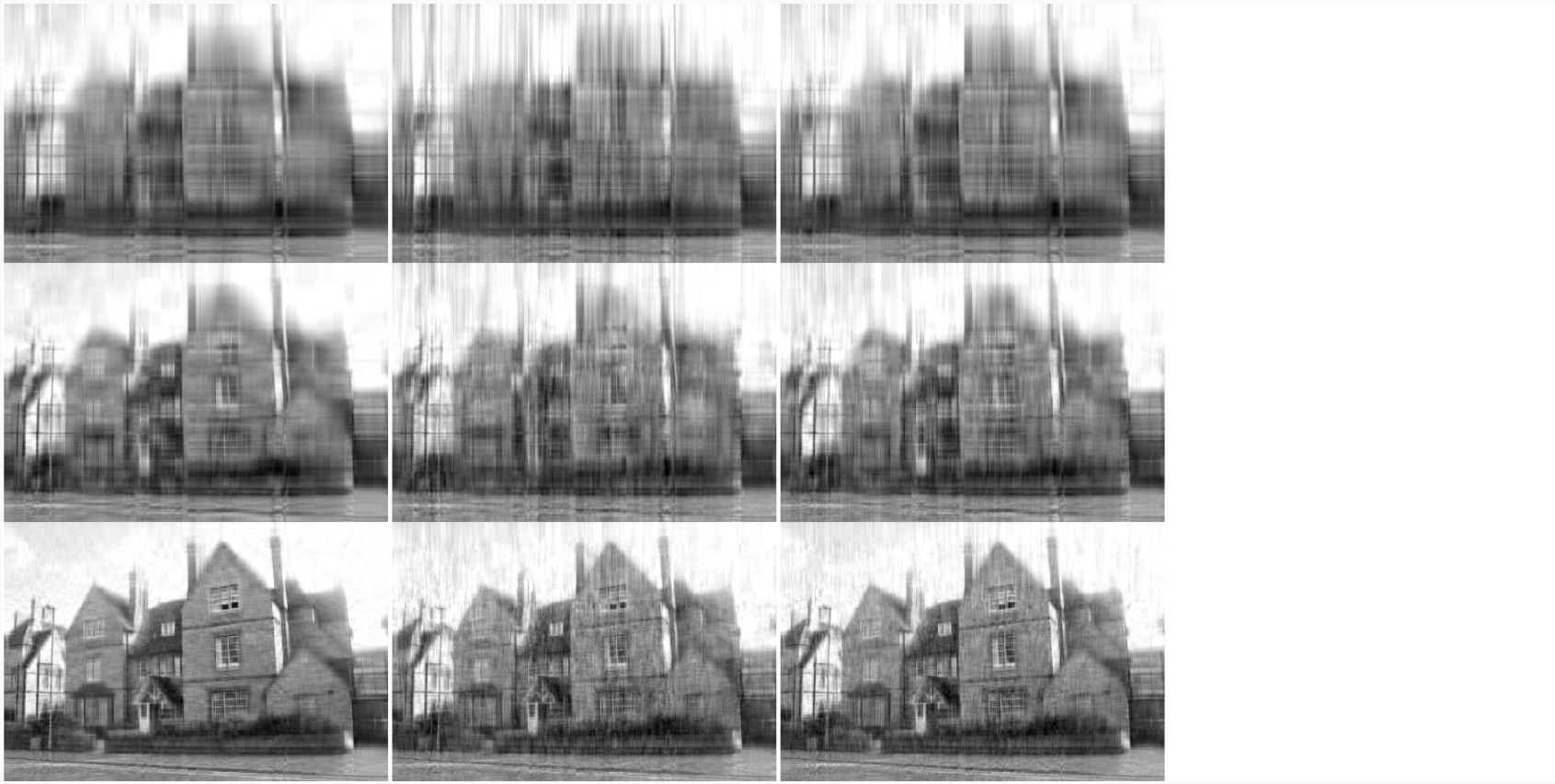
Approximate SVD

This gives us an algorithm for approximate SVD:

Construct Q

$$Q = \texttt{orth}(A \times \texttt{randn}(n, k+p))$$

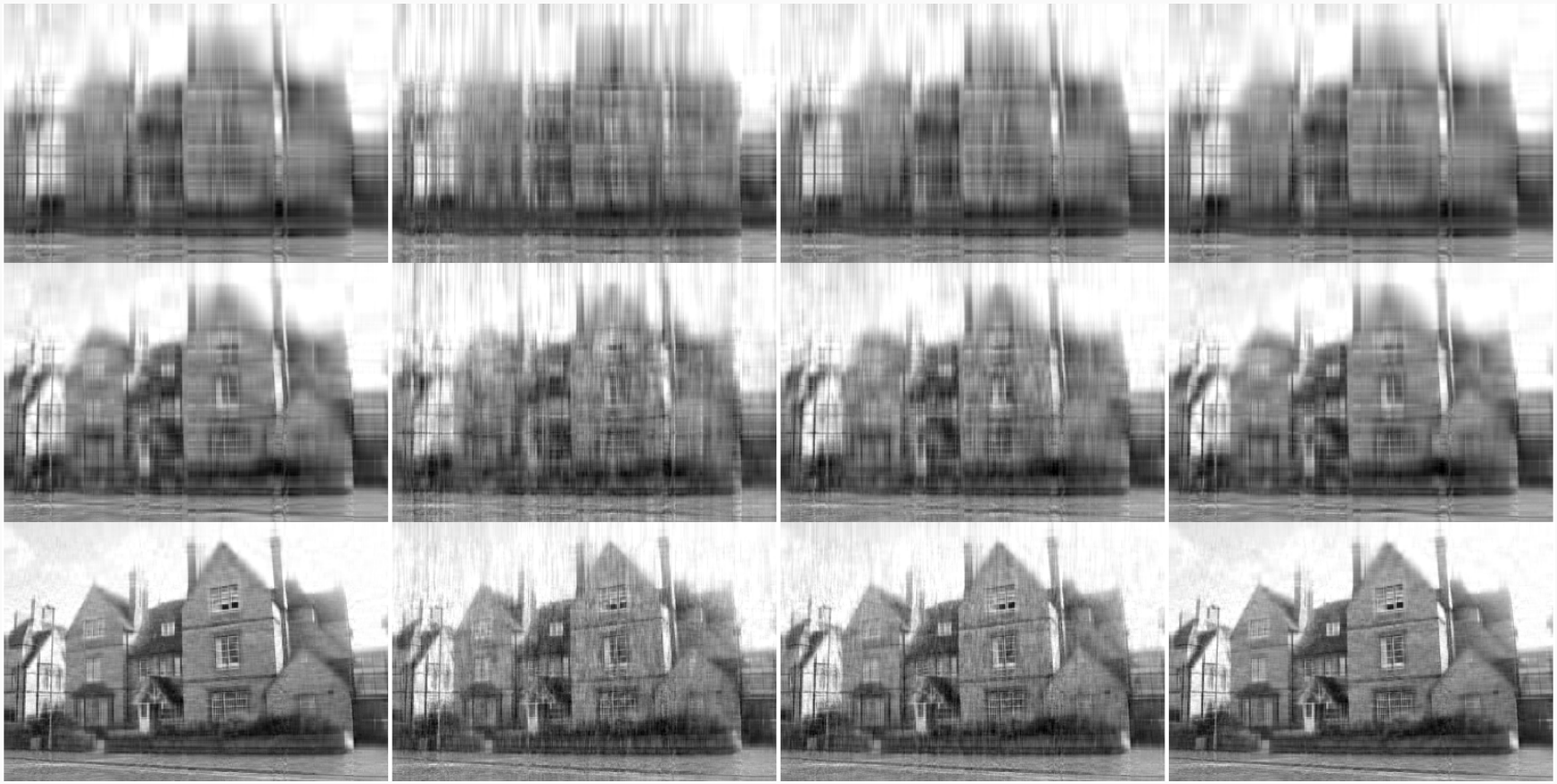
Instead of constructing SVD for $A \approx QQ^{\mathrm{T}}A$, obtain SVD for a small matrix $Q^{\mathrm{T}}A$.



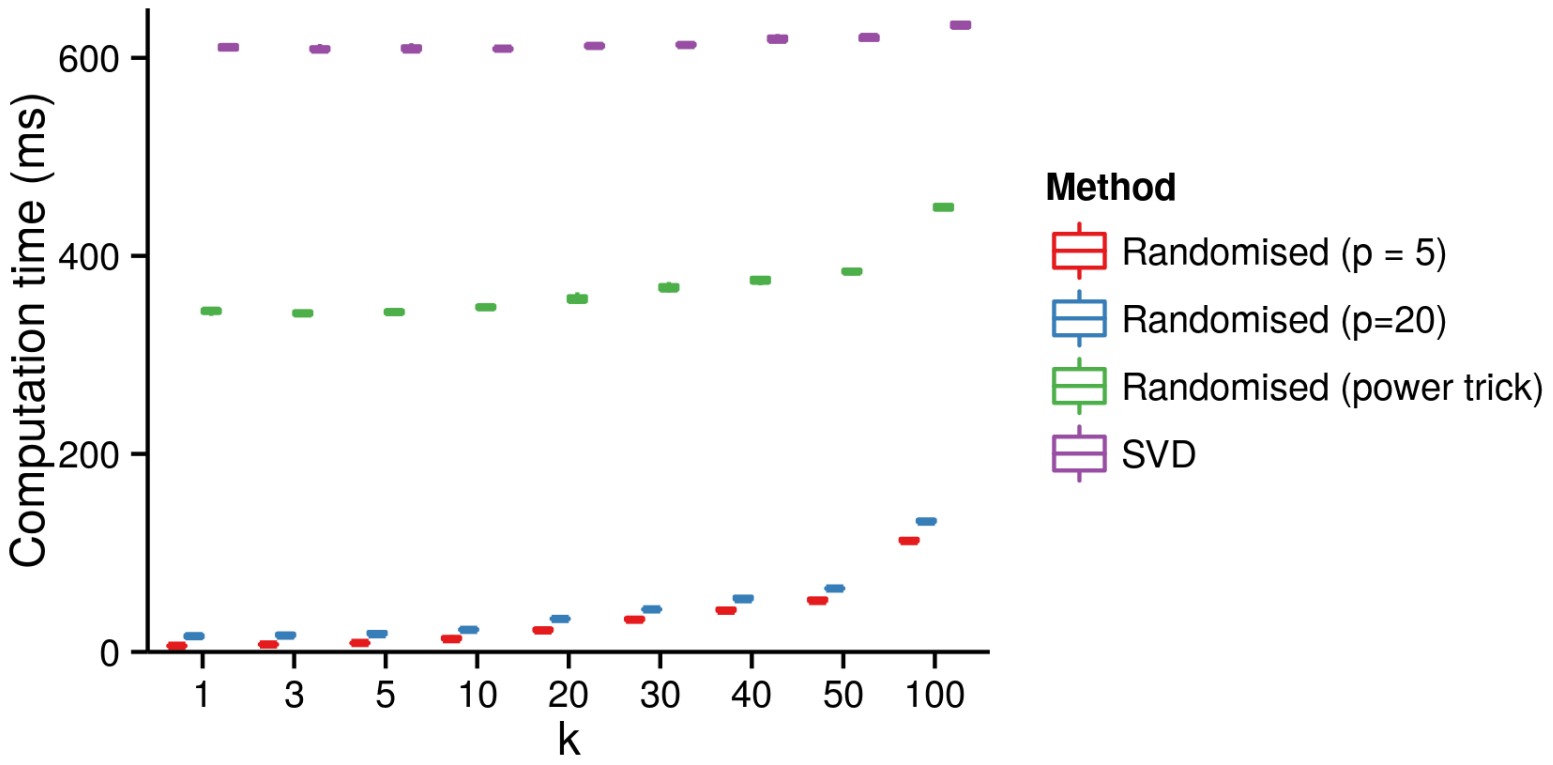
Power trick: improving approximate SVD

Power trick becomes useful when the singular values of A do not decay rapidly enough.

Idea: instead of approximating the range of A, approximate the range of $(AA^T)^q A$ for some small $q \in \{1, 2, 3\}$.



Computation times



QUANTIFYING APPROXIMATION ACCURACY

Residual spectral norm

Given a matrix A and low rank approximation $QQ^{\mathrm{T}}A$, the residual spectral norm can be taken

$$\|A - QQ^{\mathrm{T}}A\| \tag{5}$$

to quantify the approximation accuracy. This is the largest singular value of $\|A - QQ^{\mathrm{T}}A\|$

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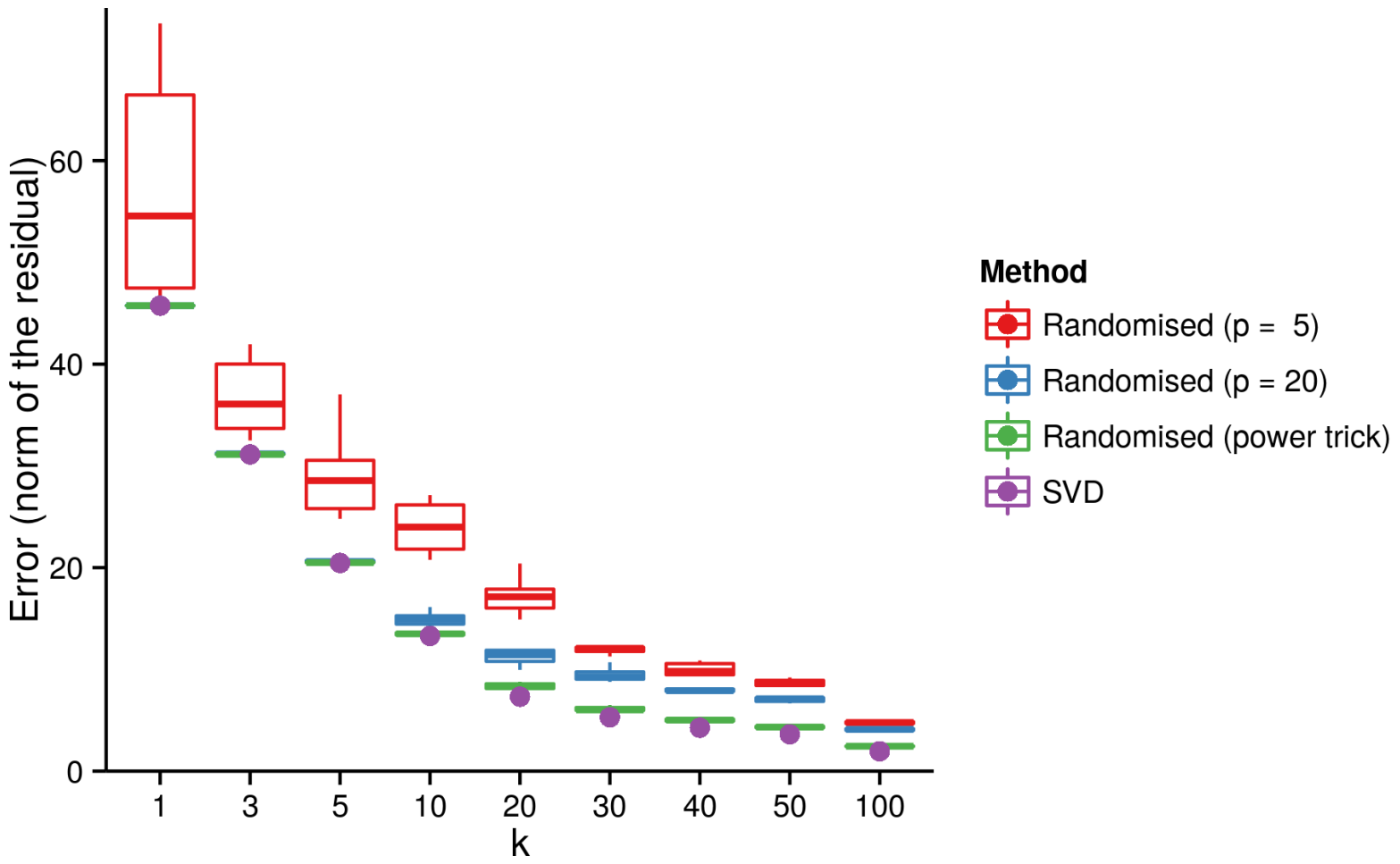
For exact SVD truncation, $\|A - QQ^{\mathrm{T}}A\| = \sigma_{k+1}$.

For the randomised approach, it is a random variable.

Observed $||A - QQ^{\mathsf{T}}A||$ for image example



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Our goal is to bound $\mathbb{E}||A - QQ^{\mathrm{T}}A||$.

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This is not possible. However, this constant exists for a rank $k+p$ approximation ($p \geq 1$).

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$$\mathrm{E}\left[\|A - QQ^{\mathrm{T}}A\|\right] \leqslant \left[1 + \frac{4\sqrt{(k+p)\min(m,n)}}{p-1}\right]\sigma_{k+1} \qquad (6)$$

Worst case matrix

R. Witten and E. Candès (2015) improved the bounds by introducing a random variable W such that

$$||A - QQ^{\mathrm{T}} A|| \overset{d}{\leq} \sigma_{k+1} W \tag{7}$$

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with expectation

$$\sup_{A} \mathbb{E}||A - QQ^{\mathrm{T}}A|| = \sigma_{k+1}\, \mathbb{E}\, W\,. \tag{8}$$

Worst case matrix

The expectation can be bounded. Defining $d = \min(m, n)$, the upper bound is given as

$$\sup_A \mathbb{E}||A - QQ^{\mathrm{T}}A|| \leq \sigma_{k+1}\left[1 + (\sqrt{d-k} + \sqrt{k})\, e\frac{\sqrt{k+p}}{p}\right] \tag{9}$$

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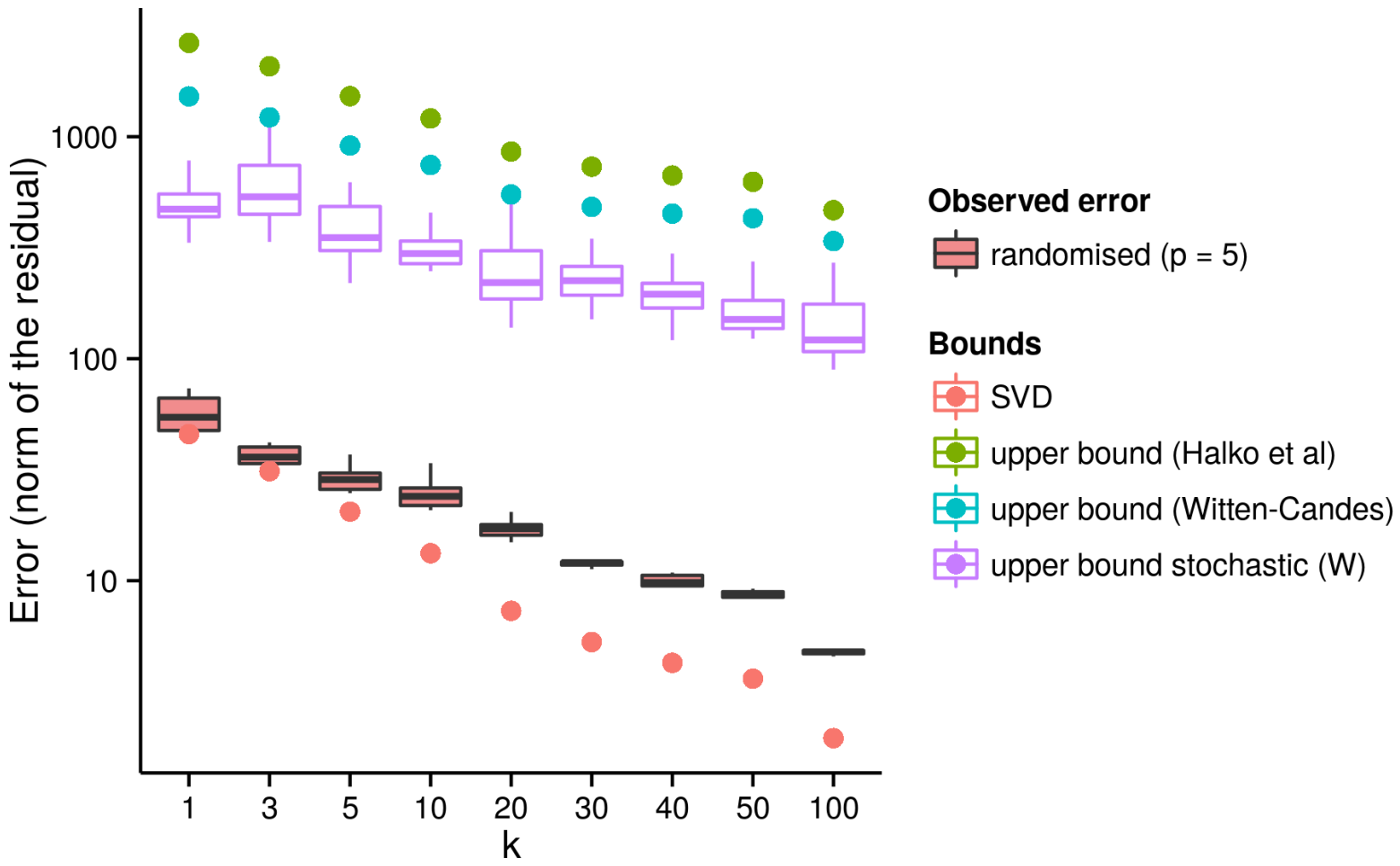
and the lower bound as

$$\sup_A \mathbb{E}||A - QQ^{\mathrm{T}}A|| \geq \sigma_{k+1}\sqrt{\frac{d-(k+p+2)}{p+1}} \tag{10}$$

Upper bounds for the image example



SOME MORE EXPERIMENTS

Simulating matrices

Recall a matrix A can be decomposed using SVD

$$A = USV^{\mathrm{T}} \ . \tag{11}$$

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By setting $m = n = 100$, the matrices U and V were construced randomly

$$U = \texttt{orth}(\texttt{randn}(100)) \tag{12}$$

$$V = \texttt{orth}(\texttt{randn}(100)) \ . \tag{13}$$

Simulating matrices

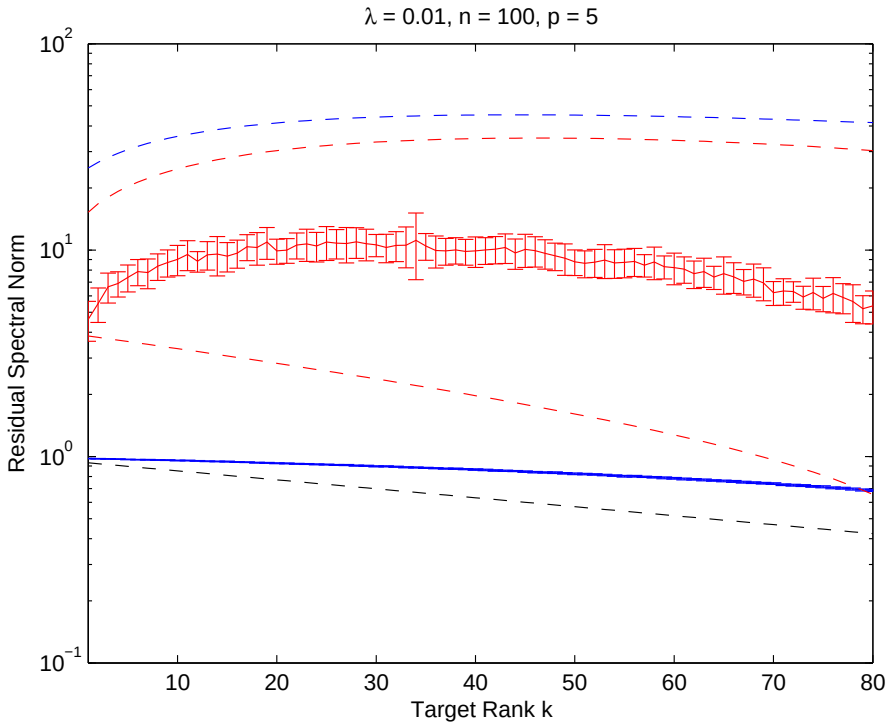
The singular values were predetermined to either to be exponentially decaying

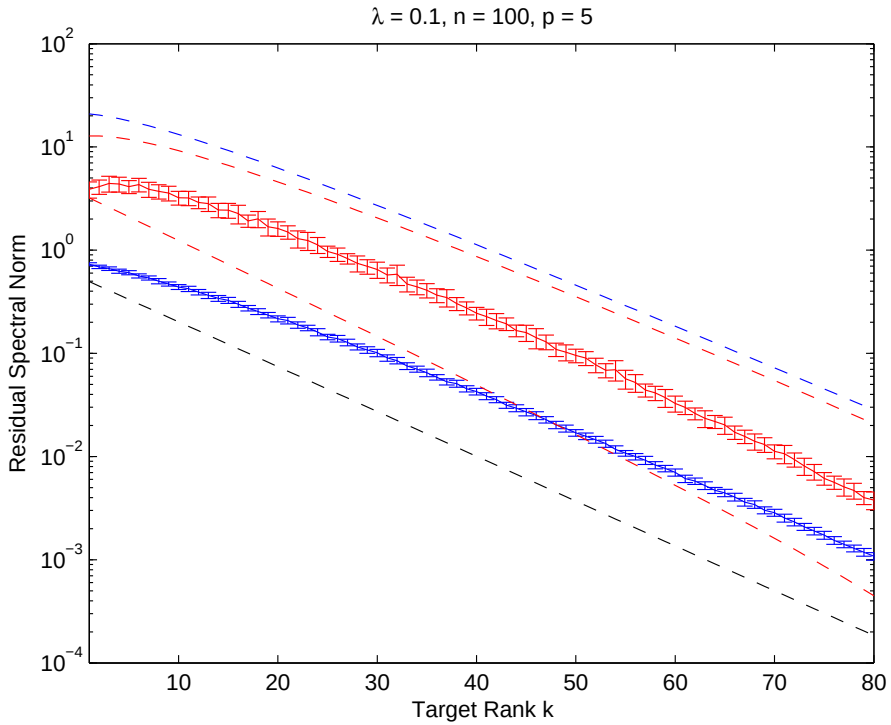
$$S = \begin{pmatrix} \mathrm{e}^{-\lambda \cdot 1} & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{-\lambda \cdot 2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{e}^{-\lambda \cdot 100} \end{pmatrix} \tag{14}$$

or has a sudden drop in value

$$S = \begin{pmatrix} I_{50} & 0 \\ 0 & \beta I_{50} \end{pmatrix} \tag{15}$$

Exponentially decaying singular values



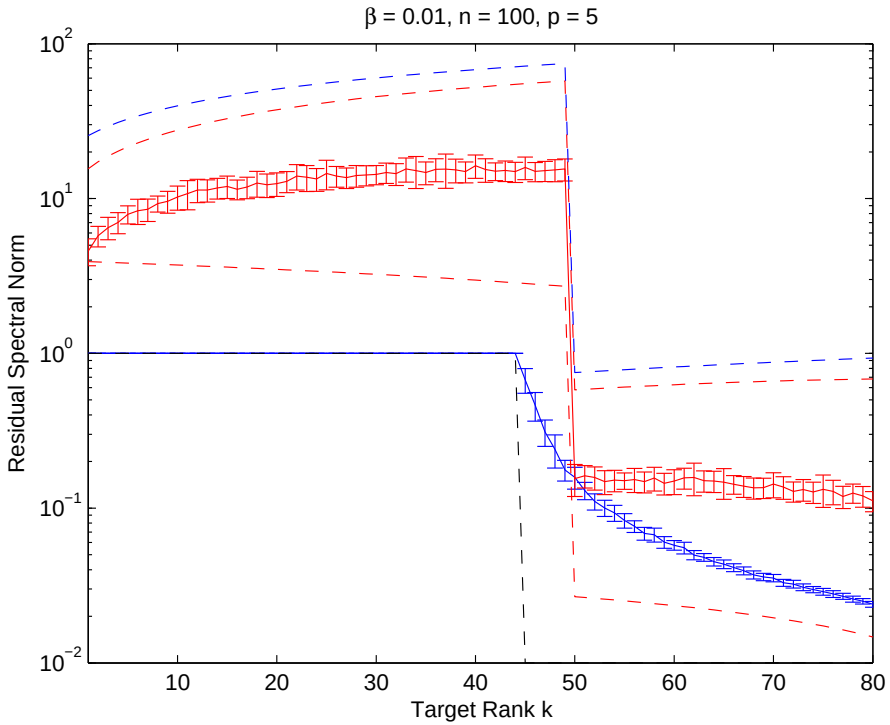


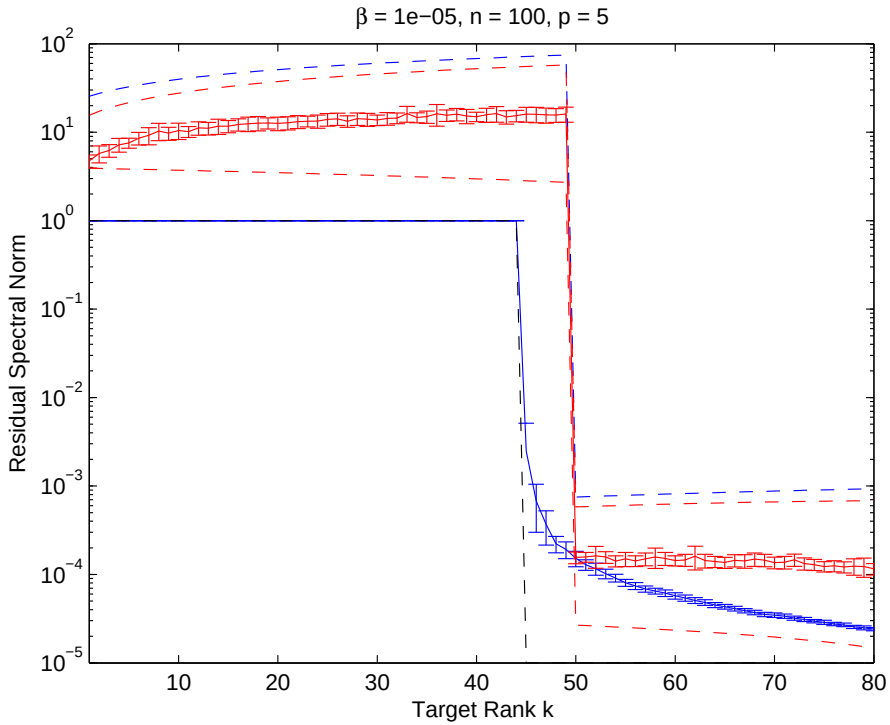
Blue line: Randomised algorithm on a matrix

Red line: Randomised algorithm on the worst case matrix

Black line: Truncated SVD

Sudden drop singular values





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