BAYESIAN NON-PARAMETRICS PRIORS WITH DENSITY ESTIMATION

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12th February 2016

Introduction

Why non-parametrics, i.e. infinite-dimensional parameters?

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Bayesian non-parametrics: prior on these objects.

Dirichlet process

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is a random measure

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One solution: Dirichlet process, parameters $\alpha_0 > 0$, G_0, a probability measure.

Dirichlet process

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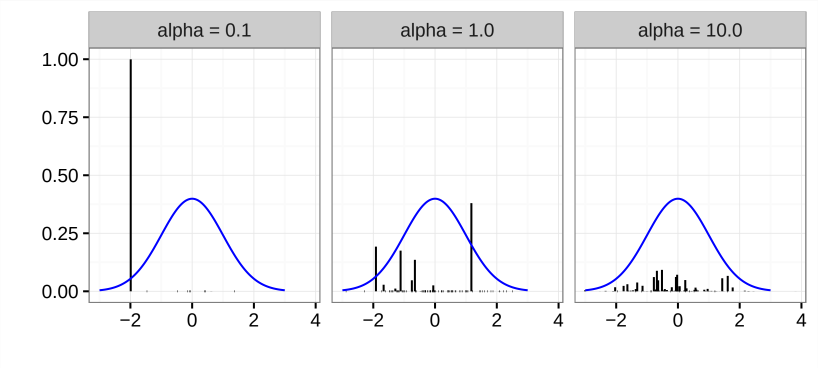
$$G = \sum_{k=1}^{\infty} \pi_k \delta_{\phi_k}$$

Dirichlet process



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$$\hat{\alpha} = \alpha + n$$

$$\hat{G} = \hat{\alpha}^{-1}\big(\alpha G_0 + \sum_{i=1}^{n} \delta_{X_i}\big)$$

Dirichlet process mixture

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$$
\begin{aligned}
G &\sim \mathrm{DP}(\alpha_0, G_0) \\
\phi_i \,|\, G &\sim G \\
x_i \,|\, \phi_i &\sim F(\phi_i)
\end{aligned}
$$

where F is a parametric distribution with density $p(x|\phi)$.

Dirichlet process mixture

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the Dirichlet process mixture specifies

$$p(x) = \sum_{k=1}^{\infty} \pi_k \, p(x|\phi_k)$$

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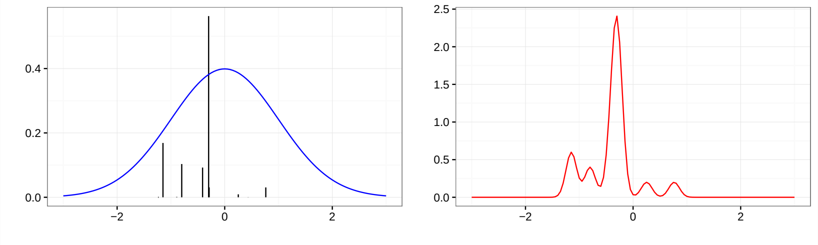
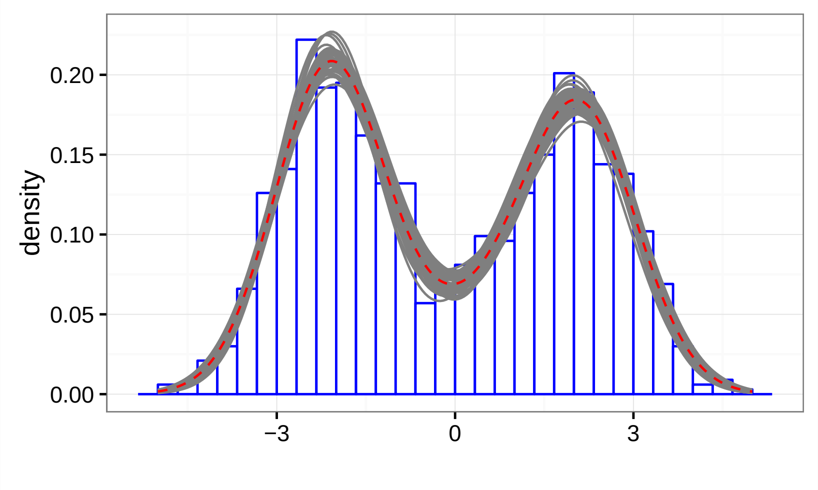


Dirichlet process mixture: fitted to data



Inference for the number of components

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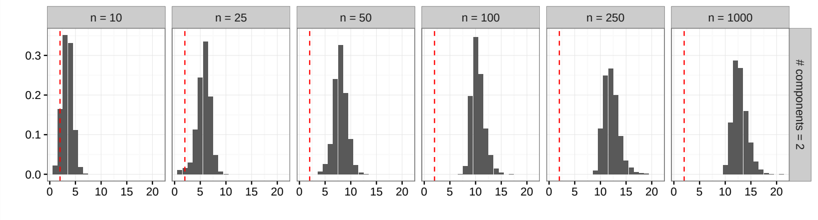

Pòlya tree

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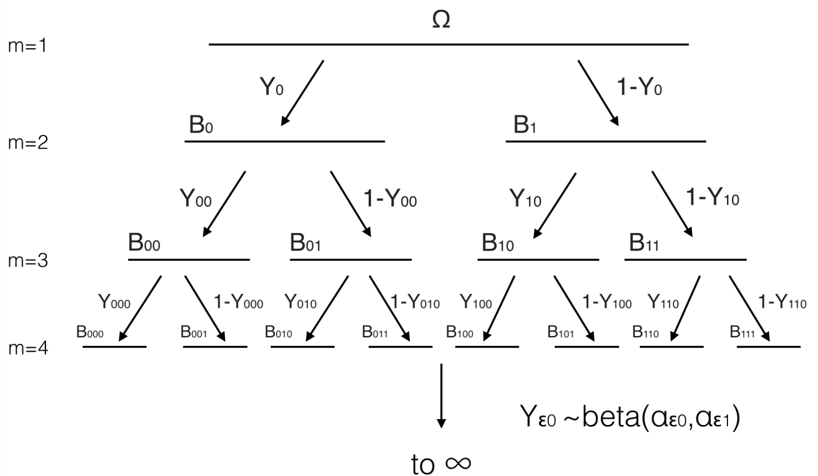
Popular in survival analysis, where censored data is common.

Pólya tree diagram



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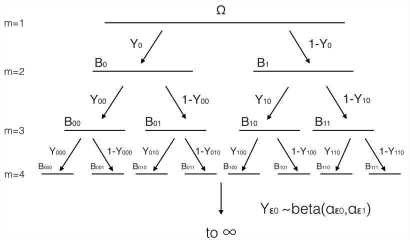

P on Ω have a Pólya tree distribution, with parameters $(\Pi = \{\Pi_m; m = 0, 1, \dots\}, \mathcal{A})$ is written $P \sim \mathrm{PT}(\Pi, \mathcal{A})$

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When $\rho(m) = 2^{-m}$ we obtain the Dirichlet process.

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With this choice, we have that for each B, $\mathbb{E}[P(B)] = G_0(B)$, where $P \sim \mathrm{PT}(\Pi, \mathcal{A})$.

Posterior Update

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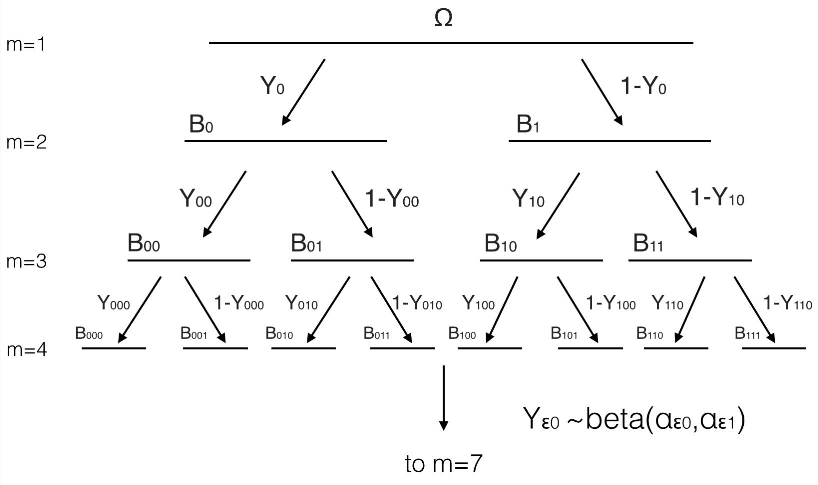
Then the posterior is given by a Pólya tree, $P|x \sim \mathrm{PT}(\Pi, \mathcal{A}^*)$ with:

$$\alpha^*_\epsilon = \begin{cases} \alpha_\epsilon + 1, & \text{if } x \in B_\epsilon \\ \alpha_\epsilon, & \text{otherwise} \end{cases} \tag{2}$$

Finite Pólya tree diagram



Density Estimation with Pólya Tree

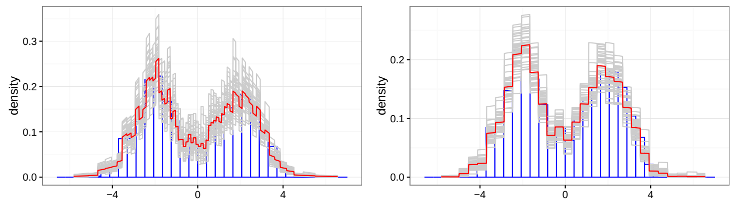


Figure: Pólya trees with two base measures, $G_0 = \mathcal{N}(0, 3^2)$ (left) and $G_0 = \mathcal{N}(0, 10^2)$ (right) with $\alpha_\epsilon = \alpha m^2$, where $\alpha \sim \Gamma(1, 0.01)$.

Summary

We have explored Bayesian non-parametric approaches for density estimation:

1. Dirichlet process mixtures
2. Pólya trees