# Kwadratury Gaussa. Zamiana zmiennych

1. $$\int_{-1}^{+1} f(x)\,dx = \sum_{i=1}^{m} w_i f_i$$

2. $$\int_{0}^{\infty} e^{-x} f(x)\,dx = \sum_{i=1}^{m} w_i f_i$$

3. $$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx = \sum_{i=1}^{m} w_i f_i$$

Ad. 1.

CN-9

$$\int_a^b f(y)\,dy = \therefore$$

$y \in \langle a, b \rangle$ , $x \in \langle -1, 1 \rangle$

$$x = \frac{y-b}{b-a} + \frac{y-a}{b-a} \quad \text{stąd} \quad y = \frac{(b-a)x + a + b}{2}$$

$$dy = \frac{1}{2}(b-a)\,dx$$

$$\therefore = \frac{b-a}{2}\int_{-1}^{+1} f(y(x))\,dx = \frac{b-a}{2}\int_{-1}^{+1} g(x)\,dx$$

Ad. 2

$$\int_a^\infty e^{-y} f(y)\,dy = \therefore$$

$y = \langle a, \infty )$ , $x \in \langle 0, \infty )$

$x = y - a$ stąd $y = x + a$

$dy = dx$

$$\therefore = \int_0^\infty e^{-x-a} f(y(x))\,dx = e^{-a}\int_0^\infty e^{-x} g(x)\,dx$$

CN-6



this work, *Gaussian Quadrature Formulas* by Stroud and Secrest, actually presents the data to 30 digits.

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**TABLE 4.1 Gauss–Legendre Quadrature: Weights and Abscissas**

$$\int_{-1}^{1} f(x)\,dx = \sum_{m=1}^{N} W_m\, f(x_m)$$

| $x_m$ | $W_m$ |
|---|---|
| $N = 2$ | |
| ±0.57735 02691 89626 | 1.00000 00000 00000 |
| $N = 3$ | |
| ±0.77459 66692 41483 | 0.55555 55555 55556 |
| 0.00000 00000 00000 | 0.88888 88888 88889 |
| $N = 4$ | |
| ±0.86113 63115 94053 | 0.34785 48451 37454 |
| ±0.33998 10435 84856 | 0.65214 51548 62546 |
| $N = 5$ | |
| ±0.90617 98459 38664 | 0.23692 68850 56189 |
| ±0.53846 93101 05683 | 0.47862 86704 99367 |
| 0.00000 00000 00000 | 0.56888 88888 88889 |
| $N = 6$ | |
| ±0.93246 95142 03152 | 0.17132 44923 79170 |
| ±0.66120 93864 66265 | 0.36076 15730 48139 |
| ±0.23861 91860 83197 | 0.46791 39345 72691 |
| $N = 7$ | |
| ±0.94910 79123 42759 | 0.12948 49661 68870 |
| ±0.74153 11855 99394 | 0.27970 53914 89277 |
| ±0.40584 51513 77397 | 0.38183 00505 05119 |
| 0.00000 00000 00000 | 0.41795 91836 73469 |
| $N = 8$ | |
| ±0.96028 98564 97536 | 0.10122 85362 90376 |
| ±0.79666 64774 13627 | [illegible] |

NC-7



In order to develop a Gauss-style integration formula, we need a set of functions that are orthogonal over the region $[0, \infty]$ with the weighting function $w(x) = e^{-x}$. Proceeding as before, we can *construct* a set of polynomials that has precisely this characteristic! Beginning (again) with the set $u_m = x^m$, we first consider the function $\phi_0 = \alpha_{00} u_0$, and the integral

$$\int_0^\infty w(x)\, \phi_0(x)\, \phi_0(x)\, dx = \alpha_{00}^2 \int_0^\infty e^{-x}\, dx = \alpha_{00}^2 = C_0. \tag{4.101}$$

With $C_0$ set to unity, we find $\phi_0(x) = 1$. We then consider the next polynomial,

$$\phi_1(x) = u_1(x) + \alpha_{10}\phi_0(x), \tag{4.102}$$

and require that

$$\int_0^\infty e^{-x} \phi_0(x) \phi_1(x)\, dx = 0, \tag{4.103}$$

and so on. This process constructs the *Laguerre* polynomials; the zeros of these functions can be found, the appropriate weights for the integration determined. These can then be tabulated, as in Table 4.2. We have thus found the sought-after Gauss–Laguerre integration formulas,

$$\int_0^\infty e^{-x} f(x)\, dx = \sum_{m=1}^{N} W_m\, f(x_m). \tag{4.104}$$

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**TABLE 4.2 Gauss–Laguerre Quadrature: Weights and Abscissas**

$$\int_0^\infty e^{-x}\, f(x)\, dx = \sum_{m=1}^{N} W_m\, f(x_m)$$

| $x_m$ | $W_m$ |
|---|---|
| $N = 2$ | |
| 5.85786 43762 69050(-1) | 8.53553 39059 32738(-1) |
| 3.41421 35623 73095 | 1.46446 60940 67262(-1) |
| $N = 4$ | |

Całki niewłaściwe. Zamiana zmiennych

np. $\int_0^\infty f(x)dx = \int_0^\infty x^2 e^{-x} dx$

$$I = \int_0^\infty f(x)dx = \int_0^a f(x)dx + \int_a^\infty f(x)dx$$

$$\int_a^\infty f(x)dx = \because$$

$y = \frac{1}{x}$ stąd

$$x = \frac{1}{y}, \quad dx = -\frac{1}{y^2}dy$$

$$\because = -\int_{1/a}^{0} f\left(\tfrac{1}{y}\right)\frac{dy}{y^2} = \int_0^{1/a} \frac{g(y)}{y^2}dy$$

gdzie $g(y) = f(y^{-1})$

PRZYKŁAD



$$I = \int_0^\infty \frac{dx}{1+x^2} = \int_0^1 \frac{dx}{1+x^2} + \underbrace{\int_1^\infty \frac{dx}{1+x^2}}_{=I_2}$$

$$x = \frac{1}{y}, \quad dx = -\frac{dy}{y^2}$$

$$I_2 = -\int_1^0 \frac{1}{1+\frac{1}{y^2}} \frac{dy}{y^2} = \int_0^1 \frac{dy}{1+y^2}$$

Czasem wygodnie jest wprowadzić inne podstawienie np. $y = e^{-t}$

Inne, $x = \dfrac{1+y}{1-y}$, które przeprowadza odcinek $[0,\infty]$ w $[-1,1]$

lub $x = \dfrac{y}{1-y}$ $\left([0,\infty] \to [0,1]\right)$.

Ważną metodą jest rozwijanie funkcji podcałkowej w szeregi potęgowe. Można też pozbyć się osobliwości poprzez jej odjęcie lub dzielenie (mnożenie).

PRZYKŁAD

$$I = \int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = I_1 + I_2$$

$$I_1 = \int_0^a \frac{dx}{(1+x)\sqrt{x}}$$

$$I_2 = \int_a^\infty \frac{dx}{(1+x)\sqrt{x}}$$

$$I_1 = \int_0^a \left( \frac{1}{(1+x)\sqrt{x}} + \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}} \right) dx = \int_0^a \frac{1}{\sqrt{x}}\, dx +$$

$$+ \int_0^a \frac{-x}{(1+x)\sqrt{x}}\, dx = 2\sqrt{x}\Big|_0^a - \int_0^a \frac{\sqrt{x}}{(1+x)}\, dx$$

Pozbyliśmy się osobliwości poprzez odejmowanie.

$$I_2=\int_a^\infty \frac{dx}{(1+x)\sqrt{x}}$$

dla $a \geqslant 1$

$$\frac{1}{1+x}=\frac{1}{x(1+\frac{1}{x})}=\frac{1}{x}\left(1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}+\dots\right)$$

$$I_2=\int_a^\infty \left(x^{-\frac{3}{2}}-x^{-\frac{5}{2}}+x^{-\frac{7}{2}}-x^{-\frac{9}{2}}+\dots\right)dx$$

$$=-2x^{-\frac{1}{2}}\Big|_a^\infty+\frac{2}{3}x^{-\frac{3}{2}}\Big|_a^\infty-\frac{2}{5}x^{-\frac{5}{2}}\Big|_a^\infty+\frac{2}{7}x^{-\frac{7}{2}}\Big|_a^\infty\dots=$$

$$=2a^{-1/2}-\frac{2}{3}a^{-3/2}+\frac{2}{5}a^{-\frac{5}{2}}-\frac{2}{7}a^{-\frac{7}{2}}\dots$$

Szereg jest zbieżny.

Możemy jeszcze funkcję podcałkową podzielić i pomnożyć przez odpowiednią dobraną funkcję $g(x)$

$$\int f(x)\,dx = \int \frac{f(x)}{g(x)}\, g(x)\,dx$$

i dokonać zamiany zmiennych $x \to y$ tak aby $g(x)\,dx = dy$

Przykład

$$I_1 = \int_0^a \frac{dx}{(1+x)\sqrt{x}} = \int_0^a \frac{\sqrt{x}}{(1+x)\sqrt{x}}\,\frac{dx}{\sqrt{x}}$$

$$dy = \frac{dx}{\sqrt{x}} \to y = 2\sqrt{x}$$

$$I_1 = \int_0^{2\sqrt{a}} \frac{1}{1+\frac{y^2}{4}}\,dy$$

Uogólnienie wzoru Newtona na dowolne potęgi o wykładniku ujemnym lub ułamkowym n

$|b| < a$

$|b| < a$

$$(a+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 +$$

$$+ \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \dots$$

$$\dots + \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} a^{n-k} b^k + \dots$$