#5

A1

Aproksymacja metodą najmniejszych kwadratów

x	$f(x) = y$
x_1	$f(x_1) = y_1$
x_2	$f(x_2) = y_2$
$\vdots$	
x_n	$f(x_n) = y_n$

$F(x; \alpha_0, \dots, \alpha_m)$ - funkcja aproksymująca

$$S(\alpha_0, \dots, \alpha_m) = \sum_{i=1}^{n} \left(F(x_i; \alpha_0, \dots \alpha_m) - y_i \right)^2$$

ta wartość jak najmniejsza

$$\frac{\partial S}{\partial \alpha_j} = 0 \quad , j = 0, \dots, m$$

min.. $\frac{\partial^2 S}{\partial \alpha_j^2} > 0$

1. <u>Metoda liniowa</u>: F jest liniowe względem $\alpha_0, \ldots, \alpha_m$ czyli

$$F(X; \alpha_0, \ldots, \alpha_m) = \sum_{k=0}^{m} \alpha_k X^k =$$

$$= \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \ldots + \alpha_m X^m$$

2. <u>Metoda nieliniowa</u>: F jest nieliniowe względem $\alpha_0, \ldots \alpha_m$, np.

$$F(X; \alpha_0, \alpha_1, \alpha_2) = \frac{\alpha_0}{1 + 4(X-\alpha_1)^2/\alpha_2^2}$$

A3

$$S(\alpha_0 \dots \alpha_m) = \sum_{i=1}^{n} \left(\sum_{k=0}^{m} \alpha_k x_i^k - y_i \right)^2$$

$$\frac{\partial S}{\partial \alpha_j} = 0 \quad , \quad j = 0, \dots, m$$

$$\frac{\partial S}{\partial \alpha_j} = \frac{\partial}{\partial \alpha_j} \sum_{i=1}^{n} \left(\sum_{k=0}^{m} \alpha_k x_i^k - y_i \right)^2 =$$

$$= \sum_{i=1}^{n} 2 \left(\sum_{k=0}^{m} \alpha_k x_i^k - y_i \right) x_i^j \, , \quad \text{bo}$$

$$\frac{\partial}{\partial \alpha_j} \sum_{k=0}^{m} \alpha_k x_i^k = \sum_{k=0}^{m} \frac{\partial \alpha_k}{\partial \alpha_j} x_i^k =$$

$$= \sum_{k=0}^{m} \delta_{kj} x_i^k = x_i^j \, .$$

Czyli

$$\sum_{i=1}^{n} \left(\sum_{k=0}^{m} \alpha_k x_i^{k+j} - x_i^j y_i \right) = 0$$

$j = 0, 1, \dots, m$

$j = 0, 1 \ldots m$

$$\sum_{i=1}^{n} \sum_{k=0}^{m} \alpha_k x_i^{k+j} = \sum_{i=1}^{n} x_i^j y_i$$

lub

$$\sum_{k=0}^{m} \alpha_k \sum_{i=1}^{n} x_i^{k+j} = \sum_{i=1}^{n} x_i^j y_i$$

układ $m+1$ równań liniowych
na $m+1$ niewiadomych α_j

#6

A5

Ad. 2

$$S(\alpha_0,\dots,\alpha_m)=\sum_{i=1}^{n}\left(F(x_i;\alpha_0,\dots\alpha_m)-y_i\right)^2$$

Nasze zadanie sprowadza się do znalezienia minimum funkcji S

Np. pierwszy sposób

$\alpha_0^0, \alpha_1^0, \dots, \alpha_m^0$ — zerowe przybliżenie

$\downarrow$

$\alpha_0^1, \alpha_1^0, \dots, \alpha_m^0$

$\alpha_0^1, \alpha_1^1, \dots, \alpha_m^0$

$\vdots$

$\alpha_0^1, \alpha_1^1 \dots \alpha_m^1$ — pierwsze przybliżenie

$\downarrow$

i td

Np. drugi sposób prowadzący do układu r-ń liniowych A6

Ozn. $\alpha_0, \alpha_1, \ldots \alpha_m = \alpha$ i $\alpha_0^0, \alpha_1^0, \ldots \alpha_m^0 = \alpha^0$

$$0 = \frac{\partial S(\alpha)}{\partial \alpha_j} \approx \frac{\partial S(\alpha^0)}{\partial \alpha_j} + (\alpha_0 - \alpha_0^0) \frac{\partial}{\partial \alpha_0} \frac{\partial S(\alpha^0)}{\partial \alpha_j} +$$

$$(\alpha_1 - \alpha_1^0) \frac{\partial}{\partial \alpha_1} \frac{\partial S(\alpha^0)}{\partial \alpha_j} + \ldots + (\alpha_m - \alpha_m^0) \frac{\partial}{\partial \alpha_m} \frac{\partial S(\alpha^0)}{\partial \alpha_j}$$

$j = 0, \ldots m$ Ozn. $\alpha_k - \alpha_k^0 = \delta_k^0$

$$\frac{\partial S(\alpha^0)}{\partial \alpha_j} = S_j, \quad \frac{\partial}{\partial \alpha_k} \frac{\partial S(\alpha^0)}{\partial \alpha_j} = S_{kj}$$

$$S_{0j}\delta_0 + S_{1j}\delta_1 + \ldots + S_{mj}\delta_m = -S_j \qquad j = 0, 1, \ldots, m$$

układ $m+1$ równań
na $\delta_0, \ldots, \delta_m$

$$f(x,y,z) = f(x_0,y_0,z_0) + (x-x_0)\frac{\partial f(x_0,y_0,z_0)}{\partial x}$$

$$+ (y-y_0)\frac{\partial f(x_0,y_0,z_0)}{\partial y} + (z-z_0)\frac{\partial f(x_0,y_0,z_0)}{\partial z}$$

$$+ \frac{(x-x_0)^2}{2!}\frac{\partial^2 f(x_0,y_0,z_0)}{\partial x^2} + \cdots$$

$$f(x,y,z) \equiv \frac{\partial S(x,y,z)}{\partial x}$$

$$f'(x) = \frac{df(x)}{dx} = \frac{f(x+h) - f(x-h)}{2h}$$

$$S_0'(\alpha_0^0, \alpha_1^0, \alpha_2^0) = \frac{\partial S(\alpha_0^0, \alpha_1^0, \alpha_2^0)}{\partial \alpha_0} =$$

$$= \frac{S(\alpha_0^0 + \delta_0, \alpha_1^0, \alpha_2^0) - S(\alpha_0^0 - \delta_0, \alpha_1^0, \alpha_2^0)}{2\delta_0}$$

$$S_1(\alpha_0^0, \alpha_1^0, \alpha_0^0) = \frac{\partial S(\alpha_0^0, \alpha_1^0, \alpha_2^0)}{\partial \alpha_1} =$$

$$= \frac{S(\alpha_0^0, \alpha_1^0 + \delta_1, \alpha_2^0) - S(\alpha_0^0, \alpha_1^0 - \delta_1, \alpha_2^0)}{2\delta_1}$$

$$S_2(\alpha_0^0, \alpha_1^0, \alpha_2^0) = \frac{\partial S(\alpha_0^0, \alpha_1^0, \alpha_2^0)}{\partial \alpha_2} =$$

$$= \frac{S(\alpha_0^0, \alpha_1^0, \alpha_2^0 + \delta_2) - S(\alpha_0^0, \alpha_1^0, \alpha_2^0 - \delta_2)}{2\delta_2}$$

Elementy diagonalne

$$f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$\alpha^0 \equiv \alpha_0^0, \alpha_1^0, \alpha_2^0$$

np. $$S_{00}(\alpha_0^0, \alpha_1^0, \alpha_2^0) = \frac{\partial^2 S(\alpha^0)}{\partial \alpha_0^2} =$$

$$= \frac{S(\alpha_0^0 + \delta_0, \alpha_1^0, \alpha_2^0) - 2S(\alpha_0^0, \alpha_1^0, \alpha_2^0) + S(\alpha_0^0 - \delta_0, \alpha_1, \alpha_2)}{\delta_0^2}$$

$$S_{10}(\alpha^0) = \frac{\partial}{\partial \alpha_1} \frac{\partial S(\alpha^0)}{\partial \alpha_0} = \frac{\partial}{\partial \alpha_1} \frac{S(\alpha_0^0+\delta_0, \alpha_1^0, \alpha_2^0) - S(\alpha_0^0-\delta_0, \alpha_1^0, \alpha_2^0}{2\delta_0}$$

$$= \frac{1}{2\delta_1} \left(\frac{S(\alpha_0^0+\delta_0, \alpha_1^0+\delta_1, \alpha_2^0) - S(\alpha_0^0-\delta_0, \alpha_1^0+\delta_1, \alpha_2^0)}{2\delta_0} + \right.$$

$$\left. - \frac{S(\alpha_0^0+\delta_0, \alpha_1^0-\delta_1, \alpha_2^0) - S(\alpha_0^0-\delta_0, \alpha_1^0-\delta_1, \alpha_2^0)}{2\delta_0} \right)$$

Oznaczenia dla $i, j = 0, 1, 2$

$$\frac{\partial S}{\partial \alpha_i} = S_i$$

$$\frac{\partial}{\partial \alpha_k} \frac{\partial S}{\partial \alpha_j} = \frac{\partial^2 S}{\partial \alpha_k \partial \alpha_j} = \underbrace{S_{kj} = S_{jk}}$$

$$\alpha_i - \alpha_i^0 = h_i$$

(1) $h_0 S_{00} + h_1 S_{01} + h_2 S_{02} = -S_0$

(2) $h_0 S_{10} + h_1 S_{11} + h_2 S_{12} = -S_1$

(3) $h_0 S_{20} + h_1 S_{21} + h_2 S_{22} = -S_2$

Stąd h_0, h_1, h_2 czyli $\alpha_0^{(1)}, \alpha_1^{(1)}, \alpha_2^{(1)}$

Opracować nowe S_{ij} i rozwiązać układ r-ń otrzymując lepsze $\alpha_0^{(2)}, \alpha_1^{(2)}, \alpha_2^{(2)}$ itd, aż do uzyskania określonej dokładności

Przykład dla $F(X_i; \alpha_0, \alpha_1, \alpha_2)$

$$S(\alpha_0, \alpha_1, \alpha_2) = \sum_{i=1}^{n} \left(F(X_i; \alpha_0, \alpha_1, \alpha_2) - y_i \right)^2$$

Z warunku koniecznego istnienia minimum

$$\frac{\partial S(\alpha_0, \alpha_1, \alpha_2)}{\partial \alpha_0} = 0, \quad \frac{\partial S(\alpha_0, \alpha_1, \alpha_2)}{\partial \alpha_1} = 0, \quad \frac{\partial S(\alpha_0, \alpha_1, \alpha_2)}{\partial \alpha_2} = 0$$

Z rozw. w szereg Taylora wokół $\alpha_0^0, \alpha_1^0, \alpha_2^0$

$$(1) \quad \frac{\partial S(\alpha_0, \alpha_1, \alpha_2)}{\partial \alpha_0} \approx \frac{\partial S(\alpha_0^0, \alpha_1^0, \alpha_2^0)}{\partial \alpha_0} + (\alpha_0 - \alpha_0^0) \frac{\partial}{\partial \alpha_0} \frac{\partial S(\alpha_0^0, \alpha_1^0, \alpha_2^0)}{\partial \alpha_0} +$$

$$+ (\alpha_1 - \alpha_1^0) \frac{\partial}{\partial \alpha_1} \frac{\partial S(\alpha_0^0, \alpha_1^0, \alpha_2^0)}{\partial \alpha_0} + (\alpha_2 - \alpha_2^0) \frac{\partial}{\partial \alpha_2} \frac{\partial S(\alpha_0^0, \alpha_1^0, \alpha_2^0)}{\partial \alpha_0}$$

PRZYKŁAD (tzw. regresja kwadratowa)

$$F(x;\alpha_0,\alpha_1,\alpha_2)=\alpha_0+\alpha_1 x+\alpha_2 x^2$$

$$S(\alpha_0,\alpha_1,\alpha_2)=\sum_{i=1}^{n}\left(\alpha_0+\alpha_1 x_i+\alpha_2 x_i^2-y_i\right)^2, \quad \frac{\partial S}{\partial\alpha_0}=\frac{\partial S}{\partial\alpha_1}=\frac{\partial S}{\partial\alpha_2}=0 \quad (*)$$

$$\frac{\partial S}{\partial\alpha_0}=\sum_{i=1}^{n}2\left(\alpha_0+\alpha_1 x_i+\alpha_2 x_i^2-y_i\right)\underbrace{\frac{\partial\alpha_0}{\partial\alpha_0}}_{=1}=2n\alpha_0+2\Big(\sum_i x_i\Big)\alpha_1+2\Big(\sum_i x_i^2\Big)\alpha_2-2\sum_i y_i$$

$$\frac{\partial S}{\partial\alpha_1}=\sum_i 2\left(\alpha_0+\alpha_1 x_i+\alpha_2 x_i^2-y_i\right)\frac{\partial\alpha_1}{\partial\alpha_1}x_i=2\Big(\sum_i x_i\Big)\alpha_0+2\Big(\sum_i x_i^2\Big)\alpha_1+2\Big(\sum_i x_i^3\Big)\alpha_2-2\sum_i x_i y_i$$

$$\frac{\partial S}{\partial\alpha_2}=\sum_i 2\left(\alpha_0+\alpha_1 x_i+\alpha_2 x_i^2-y_i\right)\frac{\partial\alpha_2}{\partial\alpha_2}x_i^2=2\Big(\sum_i x_i^2\Big)\alpha_0+2\Big(\sum_i x_i^3\Big)\alpha_1+2\Big(\sum_i x_i^4\Big)\alpha_2-2\sum_i x_i^2 y_i$$

z (*)

$$\begin{cases} n\alpha_0+\Big(\sum_i x_i\Big)\alpha_1+\Big(\sum_i x_i^2\Big)\alpha_2=\sum_i y_i \\ \Big(\sum_i x_i\Big)\alpha_0+\Big(\sum_i x_i^2\Big)\alpha_1+\Big(\sum_i x_i^3\Big)\alpha_2=\sum_i x_i y_i \\ \Big(\sum_i x_i^2\Big)\alpha_0+\Big(\sum_i x_i^3\Big)\alpha_1+\Big(\sum_i x_i^4\Big)\alpha_2=\sum_i x_i^2 y_i \end{cases}$$

Stąd $\alpha_0, \alpha_1, \alpha_2$