CvP - Werkcollege 3

Exercise 1 Consider the grammar

$$\begin{aligned}
\langle prog\rangle &\to \langle prog\rangle\langle stmt\rangle \mid \langle stmt\rangle\\
\langle stmt\rangle &\to \langle var\rangle = \langle expr\rangle\\
&\quad \mid \textbf{if } \langle(guard)\rangle \textbf{ then } \langle prog\rangle \textbf{ else } \langle prog\rangle \textbf{ fi}\\
&\quad \mid \textbf{while } \langle(guard)\rangle \textbf{ do } \langle prog\rangle \textbf{ od}\\
\langle guard\rangle &\to \langle expr\rangle(< \mid = \mid \neq \mid >)\langle expr\rangle\\
\langle expr\rangle &\to \langle expr\rangle(+ \mid -)\langle term\rangle \mid \langle term\rangle\\
\langle term\rangle &\to \langle nat\rangle \mid \langle var\rangle\\
\langle nat\rangle &\to \langle nat\rangle\langle dig\rangle \mid \langle dig\rangle\\
\langle dig\rangle &\to 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9\\
\langle var\rangle &\to x \mid y
\end{aligned}$$

Let $T(\langle x\rangle)$ denote the set of parse trees with root $\langle x\rangle$.

Define the meaning $M_v : T(\langle var\rangle) \to \{x, y\}$ of variables as $M_v(x) = x$ and $M_v(y) = y$.

Define the meaning $M_d : T(\langle dig\rangle) \to \mathbb{N}$ of digits as $M_n(a) = a$, for all $a \in \{0, \ldots, 9\}$.

Define the meaning $M_n : T(\langle nat\rangle) \to \mathbb{N}$ of natural numbers as $M_n(\langle dig\rangle) = M_d(\langle dig\rangle)$ and $M_n(\langle nat\rangle\langle dig\rangle) = 10M_n(\langle nat\rangle) + M_d(\langle dig\rangle)$.

Denote the state space as $S = \{s : X \to \mathbb{N} \mid X \subseteq \{x, y\}\} \cup \{\text{error}\}$. Define the meaning $M_t : T(\langle term\rangle) \to (S \to (\mathbb{N} \cup \{\text{error}\}))$ of terms as $M_t(\langle nat\rangle)(s) = M_n(\langle nat\rangle)$ and

$$M_t(\langle var\rangle)(s) = \begin{cases} s(M_v(\langle var\rangle)) & \text{if } s \neq \text{error and } M_v(\langle var\rangle) \in \text{dom}(s)\\ \text{error} & \text{otherwise}\end{cases}$$

Define the meaning $M_e : T(\langle expr\rangle) \to (S \to (\mathbb{N} \cup \{\text{error}\}))$ of expressions as $M_e(\langle term\rangle)(s) = M_t(\langle term\rangle)(s)$, and

$$M_e(\langle expr\rangle \oplus \langle term\rangle)(s) = \begin{cases} a \oplus b & \text{if } a, b \neq \text{error}\\ \text{error} & \text{otherwise.}\end{cases}$$

where $a = M_e(\langle expr\rangle)(s)$, $b = M_t(\langle term\rangle)(s)$, and $\oplus \in \{+, -\}$.

Define the meaning $M_g : T(\langle guard\rangle) \to (S \to \{\top, \bot, \text{error}\})$ of guards as

$$M_g(\langle expr\rangle_1 \sim \langle expr\rangle_2)(s) = \begin{cases} \top & \text{if } t_1 \sim t_2 \text{ and } t_1, t_2 \neq \text{error} \\ \bot & \text{if not } t_1 \sim t_2 \text{ and } t_1, t_2 \neq \text{error} \\ \text{error} & \text{otherwise.} \end{cases},$$

where $t_i = M_e(\langle expr\rangle_i)(s)$, for $i \in \{1, 2\}$, and $\sim \in \{<, =, \neq, >\}$.

Define the meaning $M_s : T(\langle stmt\rangle) \to (S \to S)$ of statements as

$$M_s(\langle var\rangle = \langle expr\rangle)(s) = \begin{cases} s[a \mapsto n] & \text{if } s, n \neq \text{error} \\ \text{error} & \text{otherwise.} \end{cases}$$

with $a = M_v(\langle var\rangle)(s)$ and $n = M_e(\langle expr\rangle)(s)$; and of the while loop as $M_s(\textbf{while } \langle guard\rangle \textbf{ do } \langle prog\rangle \textbf{ od}) = f$, with f is recursively defined as

$$f(s) = \begin{cases} f(M_p(\langle prog\rangle)(s)) & \text{if } M_g(\langle guard\rangle)(s) = \top \\ s & \text{if } M_g(\langle guard\rangle)(s) = \bot \\ \text{error} & \text{otherwise.} \end{cases}$$

Define the meaning $M_p : T(\langle prog\rangle) \to (S \to S)$ of programs as

$$M_p(\langle prog\rangle\langle stmt\rangle)(s) = M_p(\langle prog\rangle)(M_s(\langle stmt\rangle)(s))$$

and $M_p(\langle stmt\rangle)(s) = M_s(\langle stmt\rangle)(s)$.

Compute the denotational semantics of the program (Euclidean algorithm):

$x := 12;$
$y := 8;$
while $(x \neq y)$ **do**
  if $(x > y)$ **then**
    $x := x - y;$
  else
    $y := y - x;$
  fi
od

Exercise 2 Compute the weakest precondition for the sequence

$$y := 3(x - 4y); x := y + 3;\ \{x > 5\}.$$

Exercise 3 Compute the weakest precondition for the following programs and assertions:

(a) **if** $(x > 0)$ **then** $y := 3;$ **else** $y := 6;$ **fi** $\{y > 1\}$.

(b) **if** $(x > 0)$ **then** $y := 1;$ **else** $y := 2;$ **fi** $\{y = 1\}$.

(c) **if** $(x > 0)$ **then** $y := y - 1;$ **else** $y := y + 1;$ **fi** $\{y > 0\}$.

Exercise 4 Prove the partial correctness of the following assertions:

(a) $\{x \leq 5\}$ **while** $(x < 5)$ **do** $x = x + 1;$ **od** $\{x = 5\}$. (Hint: use the loop invariant $I \equiv x \leq 5$)

(b) $\{x = 5\}$ $y := 0;$ **while** $(y < 10)$ **do** $y := y + 1;$ **od** $\{x = 5\}$.

(c) $\{$**true**$\}$ $x := 0;$ **while** $(x \neq 8)$ **do** $x := x + 1;$ **od** $\{x = 8\}$.

(d) $y := 5;$ **while** $(y < 8)$ **do** $x := x + 1; y := y + 1;$ **od** $\{x < 4\}$. (Hint: use the loop invariant $I \equiv x < y - 4 \wedge y \leq 8$).

Exercise 5 For each of the following programs and assertions, find the weakest precondition that implies total correctness:

(a) **while** $(y < 10)$ **do** $y := y * 2;$ **od** $\{x = 5\}$

(b) $y := 5;$ **while** $(y < 8)$ **do** $x := x + 1; y := y + 1;$ **od** $\{x < 4\}$. (Hint: use the loop invariant $I \equiv x < y - 4 \wedge y \leq 8$).