**Abstract**

Starting from a frequency diffusion process for a tagged photon which simulates relaxation to the Planck law, we introduce a resetting where photons lower their frequency at random times. We consider two versions, one where the resetting to low frequency is independent of the existing frequency and a second case where the reduction in frequency scales with the original frequency. The result is a nonlinear Markov process where the stationary distribution modifies the Planck law by abundanace of low frequency occupation. The physical relevance of such photon resetting processes can be found in explorations of nonequilibrium effects, e.g. via random expansions of a confined plasma or photon gas.

# Resetting photons

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## Contents

## 1 Introduction

Resetting has been introduced and added to diffusion processes for a variety of reasons since its original conception, [**?**]. Typically, optimizing search strategies have been the underlying motivation, but one can also imagine physical resettings. By physical resettings we mean the result of a time-dependent potential, for which there are random moments of confinement, or the random appearance in certain locatons of attractors, or the random contraction of an enclosed volume. In the present paper, we investigate a new scenario where resetting is applied in frequency space of photons. In that way we also explore physically motivated nonequilibrium effects on the Planck distribution.

Resetting photons to a lower frequency refers to reducing the wave vector (in the reciprocal lattice), which, in real space, refers to an expansion. The physical mechanism we imagine here is that of a confined plasma where at random moments the confinement is lifted....

## 2 The Kompaneets process

### 2.1 Kompaneets equation

As reference process we consider a fluctuation dynamics which realizes the Kompaneets equation as its nonlinear Fokker-Planck equation. Here the physical realization is relax-

ation towards equilibrium of a photon gas in contact with a nondegenerate, nonrelativistic electron bath in thermal equilibrium at temperature $T$. We concentrate here on Compton scattering as main mechanism In 1957 Kompaneets [11] stared from a semi-classical Boltzmann equation to arrive at an equation for $n(t, \omega)$, the occupation number distribution function at frequency $\omega$ of the photon gas at time $t$:

$$\omega^2 \frac{\partial n}{\partial t}(t, \omega) = \frac{n_e \sigma_T c}{m_e c^2} \frac{\partial}{\partial \omega} \omega^4 \left\{ k_B T \frac{\partial n}{\partial \omega}(t, \omega) + \hbar \left[ 1 + n(t, \omega) \right] n(t, \omega) \right\} \tag{1}$$

The constant $\sigma_T$ is the Thomson total cross section, and $n_e, m_e$ are the density and mass of the electrons, respectively. Note the stimulated emission (induced Compton scattering, [12, 2]) in the nonlinearity (in the second term) of (1). Stationarity is achieved when $n(t, \omega)$ follows the Bose-Einstein distribution.

The Kompaneets equation remains essential for the dynamical understanding of the cosmic microwave backgrond and related Sunyaev-Zeldovich effect [18, 19]. We refer to excellent reviews [17, 6, 20] for more detaile. We skip here the many possible extensions and generalizations; see e.g. [4, 16, 1, 3, 8, 9, 5, 13, 15, 14, 7].

A dimensionless Kompaneets equation employs an average photon occupation number $n(t, x)$ at dimensionless frequency $x = \hbar \omega / k_B T$,

$$x^2 \frac{\partial n}{\partial y}(y, x) = \frac{\partial}{\partial x} x^4 \left\{ \frac{\partial n}{\partial x}(y, x) + \left[ 1 + n(y, x) \right] n(y, x) \right\} \tag{2}$$

with dimensionless Compton optical depth

$$y = \frac{k_B T}{m_e c^2} n_e \sigma_T c\, t := \frac{t}{\tau_C}$$

where $\tau_C$ is the characteristic time in which photons update their frequency as the result of Compton scattering with thermal electrons. Note here that the scattering can be interpreted as a Doppler shift

$$\left\langle \frac{1}{2\tau} \left( \frac{\Delta \omega}{\omega} \right)^2 \right\rangle \approx \frac{k_B T}{m_e c^2} n_e \sigma_T c = \frac{1}{\tau_C} \tag{3}$$

for $\tau = \ell / c$, the average collision rate in terms of the mean free path of photons $\ell = (n_e \sigma_T)^{-1}$.

The Kompaneets equation (1) can be rewritten still in terms of the photon density. Here we assume confinement of photons in a box of volume $V$ with periodic boundary conditions. The density of states is

$$g(\mathbf{k}) \mathrm{d}^3 \mathbf{k} = \frac{2V}{(2\pi)^3} \mathrm{d}^3 \mathbf{k} = \frac{2V}{(2\pi)^3} 4\pi k^2 \mathrm{d}k$$

where $\mathbf{k}$ is the wave vector. Under isotropy, which we will always assume, and for $x = \hbar \omega / k_B T$,

$$g(x) \mathrm{d}x = \frac{2V}{(2\pi)^3} \left( \frac{k_B T}{\hbar c} \right)^3 4\pi x^2 \mathrm{d}x$$

It allows to define the spectral number density, i.e., the number of photons with energy between $E$ and $E + \mathrm{d}E$ as

$$u(y,x) = g(x)n(y,x) = V\frac{1}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 x^2 n(y,x) \tag{4}$$

The total number of photons is then

$$N = \int_0^\infty \mathrm{d}x\, u(y,x)$$

and the spectral probability density equals

$$\rho(y,x) = \frac{V}{N}\frac{1}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 x^2 n(y,x) = \frac{x^2 n(y,x)}{2\zeta(3)Z} \tag{5}$$

We put $\zeta(3) = 1/2\int_0^\infty \mathrm{d}x\, x^2/(e^x - 1) \simeq 1.202$ so that $Z$ depends on the temperature and is proportional to the number of photons per volume, and serves to normalize $\rho$, It can be interpreted as the ratio of the photon density to that of the Planck distribution corresponding to the same temperature. When $n(y,x) = n_{BE}(x) = 1/(\exp(x) - 1)$ (Bose-Einstein distribution) the photon number equals

$$N_{BE} = 2\zeta(3)\frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3$$

corresponding to a spectral probability density with $Z = 1, \rho_{BE}(x) = \frac{x^2 n_{BE}(x)}{2\zeta(3)}$.
Then, in terms of the photon spectral density (5), the Kompaneets equation (2) becomes

$$\frac{\partial \rho}{\partial y}(y,x) = -\frac{\partial}{\partial x}\left[\left(4x - x^2\left(1 + 2\zeta(3)Z\frac{\rho(y,x)}{x^2}\right)\right)\rho(y,x)\right] + \frac{\partial^2}{\partial x^2}\left[x^2\rho(y,x)\right] \tag{6}$$

Note that for $Z \to 0$, the Wien distribution $\rho_{\mathrm{Wien}}(x) = x^2 e^{-x}/2$ becomes stationary, which gives less weight to low frequency compared to the Planck law.

## 2.2 Stochastic process

Tthe Kompaneets equation is **positivity** preserving [10]. It allows therefore a probabilistic interpretation as nonlinear Fokker-Planck equation. That was explicitly realized in [?].

We constructed there the tagged photon diffusion process,

$$\dot{x} = \frac{\mathrm{d}B}{\mathrm{d}x}(x) - \beta B(x)U'(x)(1 + n(x,t)) + 2\frac{B(x)}{x} + \sqrt{2B(x)}\,\xi_t \tag{7}$$

where $\xi_t$ is standard white noise. That Itô stochastic process (7) is our Kompaneets process, a (nonlinear)Langevin dynamics associated to the Kompaneets equation (6) in the case where

$$U(x) = x, \qquad B(x) = x^2$$

One can interpret it as a mean-field Markov process for the tagged photon, where the field $n(t,x)$ represents the empirical occupations, followinf the ideas of e.g, the McKean-Vlasov equation [?]. The specific mult-photon diffusion limit was discussed in [?].
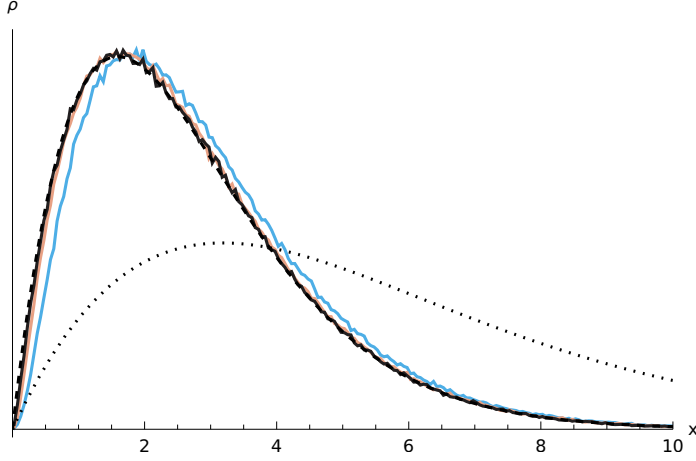
Figure 1: Result of Kompaneets, for three units of time. Needs indication of what in axes. Mention maximum excess of 1/1000. Mention very fastg prethermalization, slow final thermalization — low frequency going like $x$.

We remark that the white noise can lead to (unphysical) negative values of $x$. The *ad hoc* remedy is to puch by hand (in the simulation) the frequecny value positive, effectively using a boundary condition at $x = 0$. Physically speaking, reaction processes such as Bremsstrahlung and double Compton scattering control the photon number density but we ignore here the detailed implementation of these processes.

As an illustration of the soundness of the process... we reproduce here....

# 3 Resetting Kompaneets process

## 3.1 Process

Physical motivation plus definiton of two versions of resettings.

We will consider the same dynamics as before, adding Poissonian resetting with a constant rate. We introduce the resetting rate $r$. Three methods are introduced for resetting, the so-called "division" method, where the energy of the photon is divided by a constant factor $d$, the "uniform" method, where the photon's energy is reset to a random value uniformly distributed between 0 and a cutoff $x_0$, and finally the "exponential" method, where the new energy of the photon follows an exponential distribution with scale $x_0$.

**Kasper:** Is there a point in having multiple methods? Looking at figure 5, with the right tuning of parameters all three methods of resetting give identical results. The choices of the scale, interval and divisor are such that the expected value of a photon after resetting is the same, 0.05 in this case.

This implies that the position of the photon is updated by the following stochastic

rules, in an infinitesimal interval $dt$ we have

$$x(t + dt) = x(t) + D(x, t, n_t)dt + \sqrt{2B(x)}\, \xi_t \sqrt{dt} \qquad\qquad \text{with probability } (1 - r)dt$$

$$\tag{8}$$

$$= \begin{cases} x(t)/d & \text{"division" method} \\ x \sim \text{Uniform}(0, x_0) & \text{"uniform" method} \\ x \sim \text{Exp}(x_0) & \text{"exponential" method} \end{cases} \qquad \text{with probability } rdt$$

To simulate these dynamics, we revisit the procedure from (cite vorige paper?). Adding to this, at every timestep, for every particle, we perform the resetting step with a probability $r\Delta t$. The timestep $\Delta t$ is chosen such that this product is much smaller than 1, making sure that the probability of two resets happening in the same timestep remains negligible.

One particular difficulty here lies in the fact that these dynamics are quite prone to the formation of condensates. In particular, the Planck distribution at zero chemical potential is on the knife-edge of this phenomenon, if there is any excess in particles, a non-zero probability flux will come into being at the origin, leading to a Dirac delta mass at $x = 0$. More precisely, letting $f = x^2 n$, the Kompaneets equation takes the form

$$\dot{f} = \partial_x j \qquad\qquad \text{with} \qquad\qquad j = x^2 f' - 2xf + x^2 f + f^2.$$

If we substitute $f = \frac{x^2}{e^x - 1} + \delta$ we find that $j = \delta^2 + O(x)$, hence no probability can be allowed near $x = 0$ to first order. To remedy this, we assume the photon gas is slightly rarefied, having a particle density that is 90% that of a photon gas following the Planck distribution at the same temperature. This is also the distribution with which we compare the results of the simulation

$$\rho(x) = \frac{1}{0.9 \cdot 2\zeta(3)} \frac{1}{e^{x+\mu} - 1} \qquad\qquad \text{with } \mu \approx 0.08 \tag{9}$$

## 3.2   Discussion of results

# 4   Conclusions

# References

[1] D. Barbosa. A note on Compton scattering. *The Astrophysical Journal*, 254:301–308, 1982.

[2] R. D. Blandford and E. T. Scharlemann. On induced Compton scattering by relativistic particles. *Astrophysics and Space Science*, 36(2):303–317, Sept. 1975.

[3] L. S. Brown and D. L. Preston. Leading relativistic corrections to the kompaneets equation. *Astroparticle Physics*, 35(11):742–748, 2012.

[4] C. Buet, B. Després, and T. Leroy. Anisotropic models and angular moments methods for the Compton scattering. working paper or preprint, Feb. 2018.
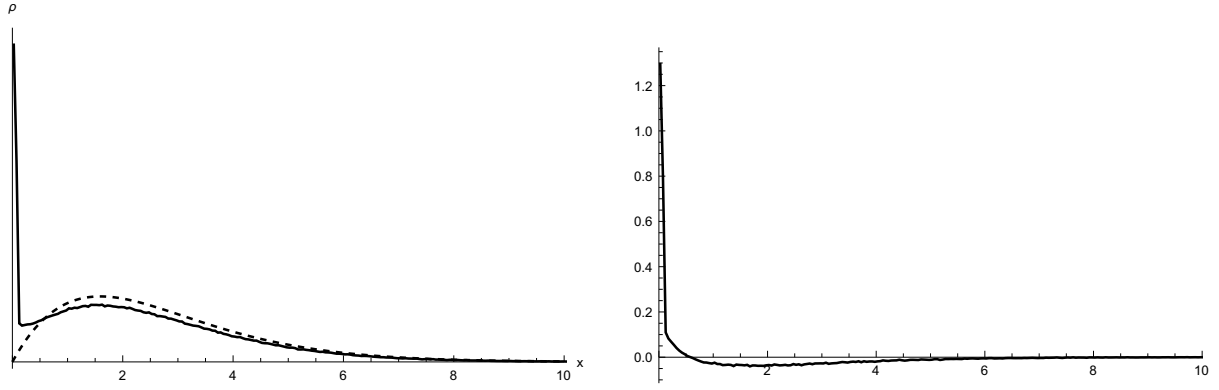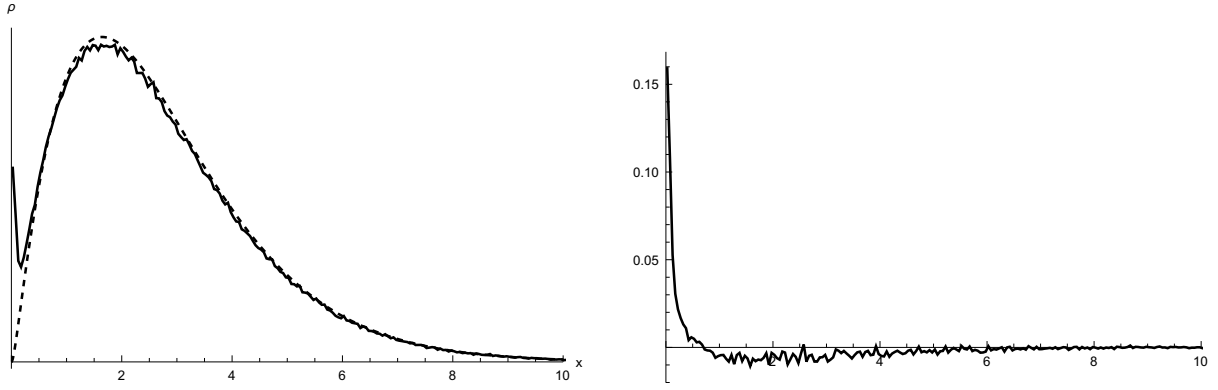
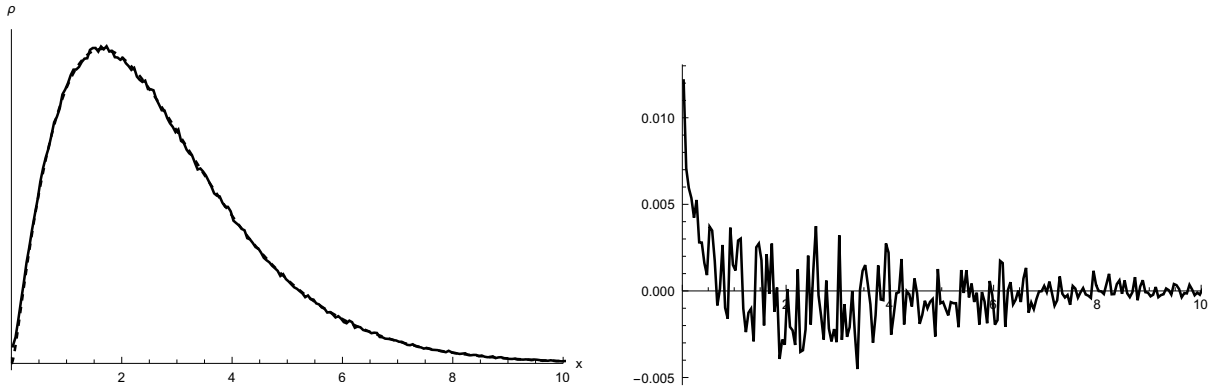(a) $r = 1 \times 10^{-1}$, $d = 10$, simulation stopped
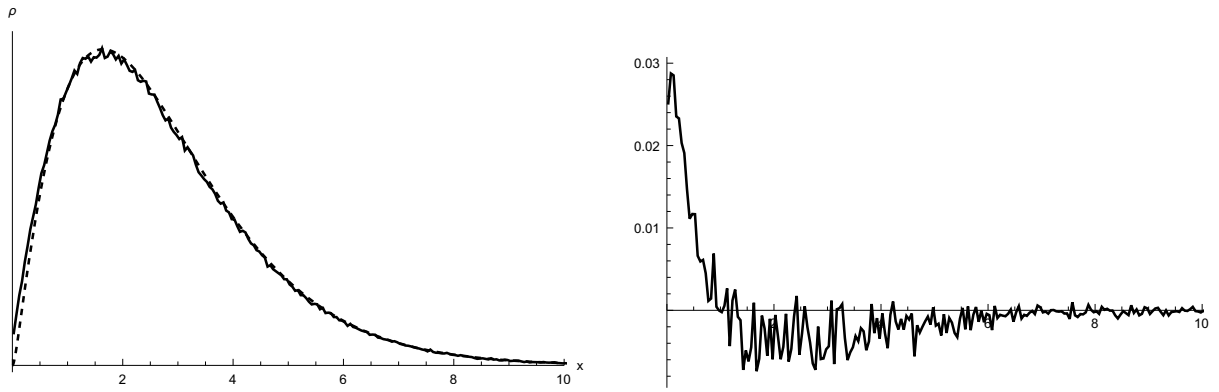


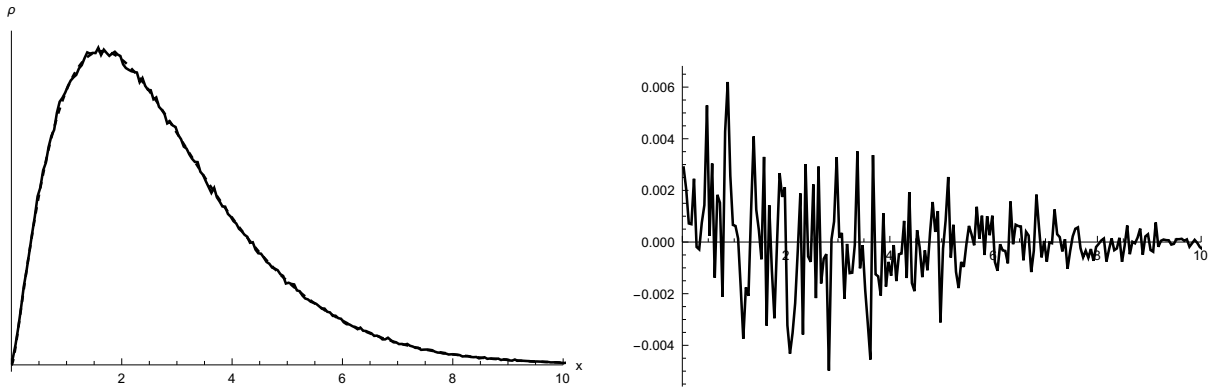(b) $r = 1 \times 10^{-2}$, $d = 10$
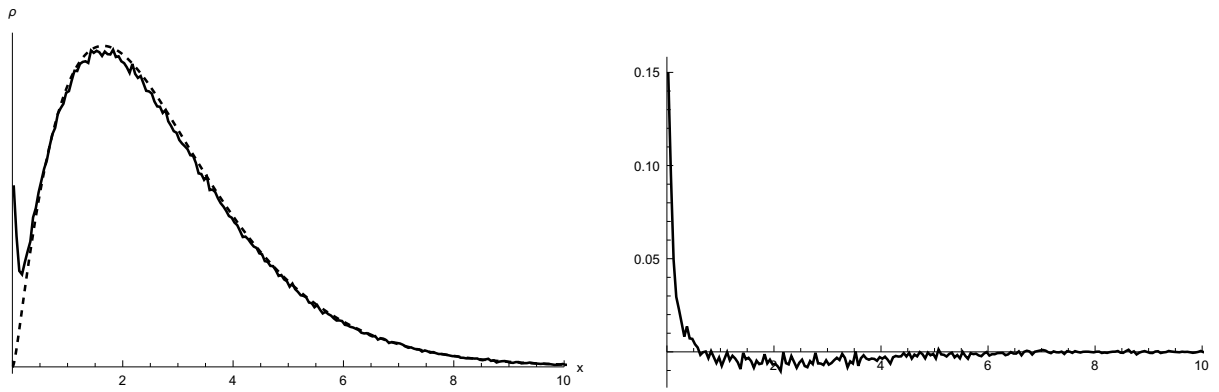


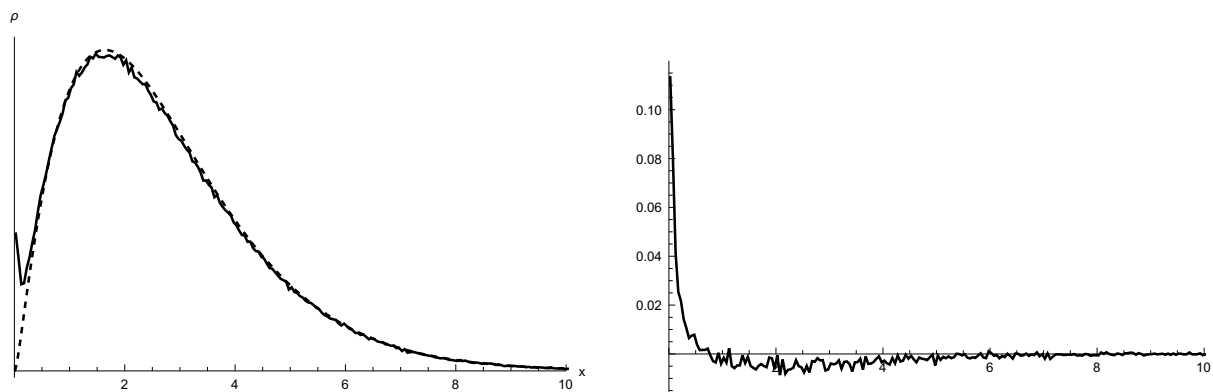(c) $r = 1 \times 10^{-2}$, $d = 100$, simulation stopped

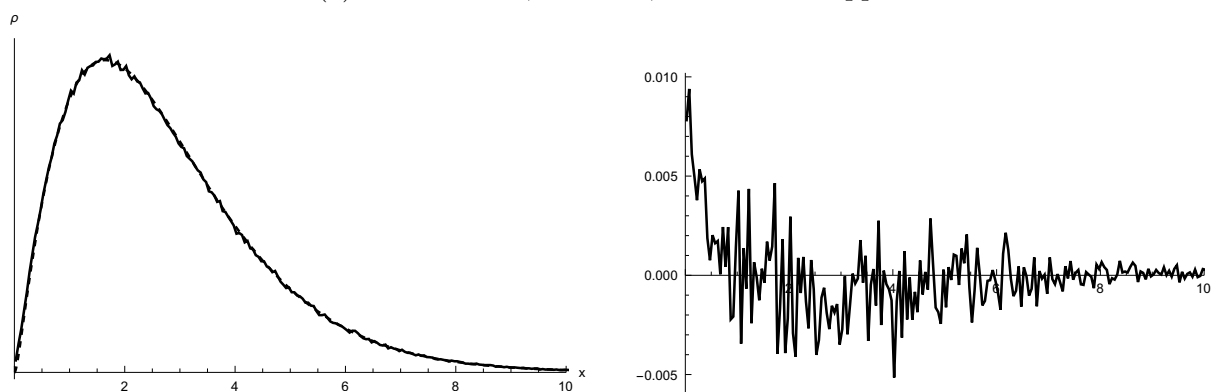(a) $r = 1 \times 10^{-2}$, $d = 1000$, simulation stopped



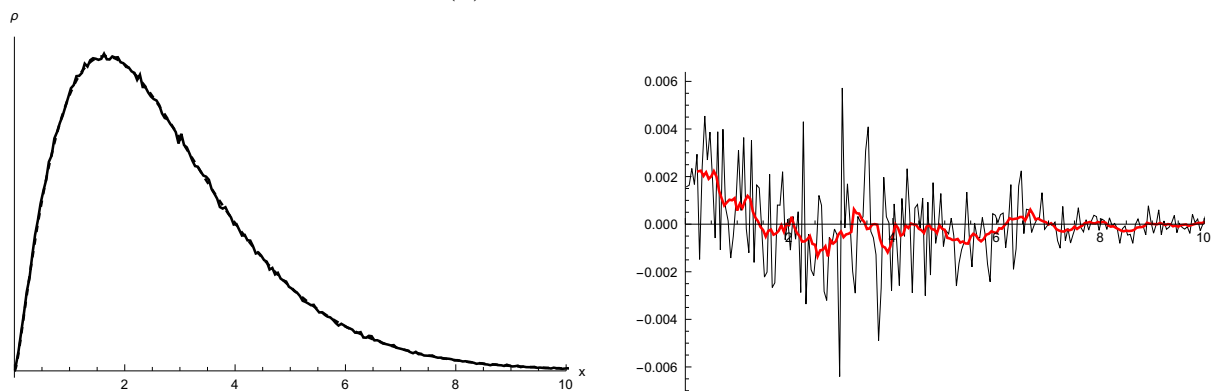(b) $r = 1 \times 10^{-3}$, $d = 1000$



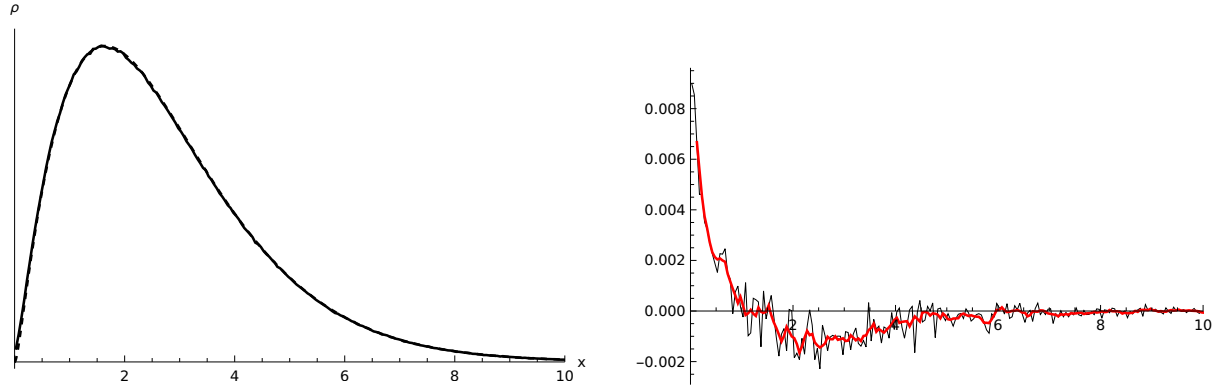(c) $r = 1 \times 10^{-4}$, $d = 1000$

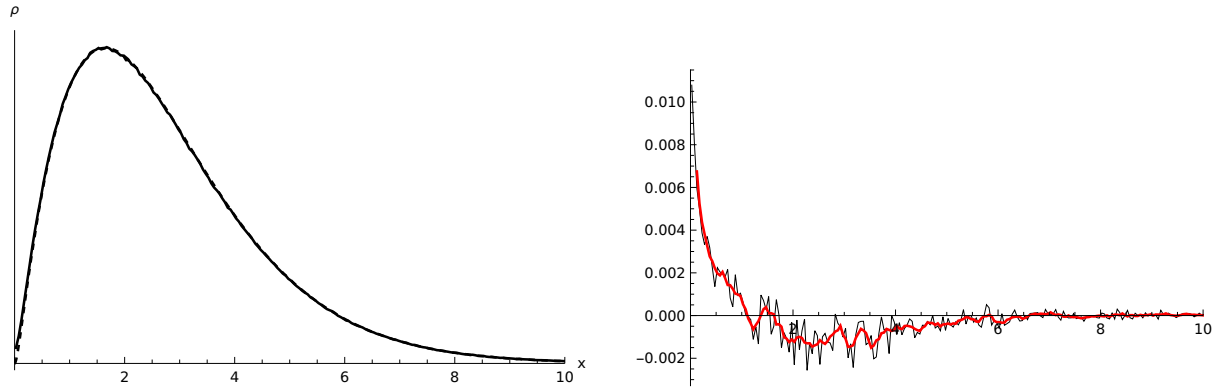(a) $r = 1 \times 10^{-2}$, $x_0 = 0.1$, simulation stopped



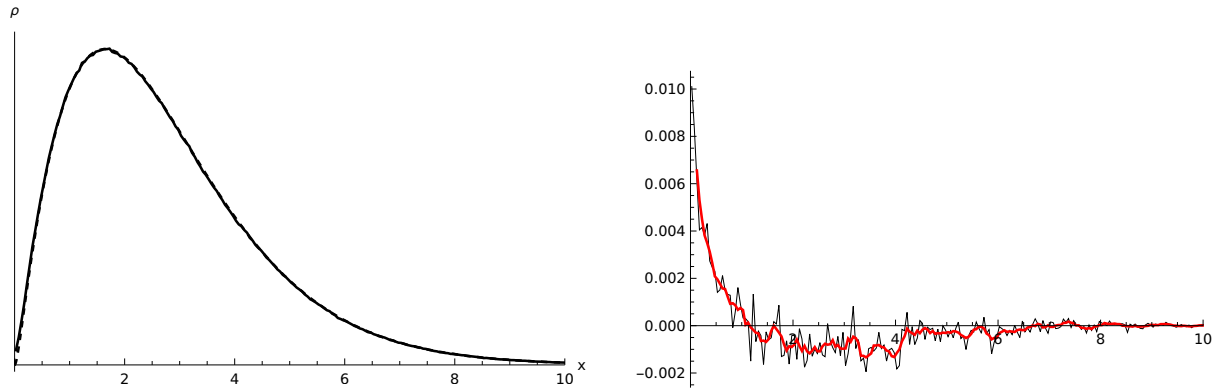(b) $r = 1 \times 10^{-3}$, $x_0 = 0.1$



(c) $r = 1 \times 10^{-4}$, $x_0 = 0.1$

(a) $r = 1 \times 10^{-3}$, "interval", $x_0 = 0.1$



(b) $r = 1 \times 10^{-3}$, "exponential", $x_0 = 0.05$



(c) $r = 1 \times 10^{-4}$, $d = 60$

Figure 5: Comparison between methods

[5] G. Cooper. Compton Fokker-Planck equation for hot plasmas. *Physical Review D*, 3(10):2312, 1971.

[6] G. E. Freire Oliveira. Statistical mechanics of the Kompaneets equation. Master's thesis, Leuven, 2021.

[7] G. E. Freire Oliveira, C. Maes, and K. Meerts. On the derivation of the Kompaneets equation. *Astroparticle Physics*, 133:102644, 2021.

[8] N. Itoh, Y. Kohyama, and S. Nozawa. Relativistic corrections to the Sunyaev-Zeldovich effect for clusters of galaxies. *The Astrophysical Journal*, 502(1):7, 1998.

[9] N. Itoh and S. Nozawa. Relativistic corrections to the Sunyaev-Zeldovich effect for extremely hot clusters of galaxies. *A&A*, 417(3):827–832, 2004.

[10] O. Kavian. Remarks on the Kompaneets equation, a simplified model of the Fokker-Planck equation. In *Nonlinear Partial Differential Equations and their Applications - Collège de France Seminar Volume XIV*, pages 469–487. Elsevier, 2002.

[11] A. S. Kompaneets. The Establishment of Thermal Equilibrium between Quanta and Electrons. *Soviet Journal of Experimental and Theoretical Physics*, 4(5):730–737, May 1957.

[12] D. A. Liedahl. *The X-Ray Spectral Properties of Photoionized Plasma and Transient Plasmas*, volume 520, page 189. 1999.

[13] S. Nozawa and Y. Kohyama. Analytical study on the Sunyaev-Zeldovich effect for clusters of galaxies. *Physical Review D*, 79(8):083005, 2009.

[14] S. Nozawa and Y. Kohyama. Relativistic corrections to the kompaneets equation. *Astroparticle Physics*, 62:30–32, 2015.

[15] S. Nozawa, Y. Kohyama, and N. Itoh. Analytical study on the Sunyaev-Zeldovich effect for clusters of galaxies. ii. comparison of covariant formalisms. *Physical Review D*, 82(10):103009, 2010.

[16] C. Pitrou. Radiative transport of relativistic species in cosmology. *Astroparticle Physics*, page 102494, 2020.

[17] D. G. Shirk. A practical review of the kompaneets equation and its application to Compton scattering. 2006.

[18] R. A. Sunyaev and Y. B. Zeldovich. Distortions of the background radiation spectrum. *Nature*, 223(5207):721–722, Aug. 1969.

[19] R. A. Sunyaev and Y. B. Zeldovich. The Observations of Relic Radiation as a Test of the Nature of X-Ray Radiation from the Clusters of Galaxies. *Comments on Astrophysics and Space Physics*, 4:173, Nov. 1972.

[20] Y. B. Zeldovich. Interaction of free electrons with electromagnetic radiation. *Soviet Physics Uspekhi*, 18(2):79–98, Feb. 1975.