OBLIGATORISK INNLEVERING 1/4 - MAT 111

Kasper Melheim

KASPER MELHEIM

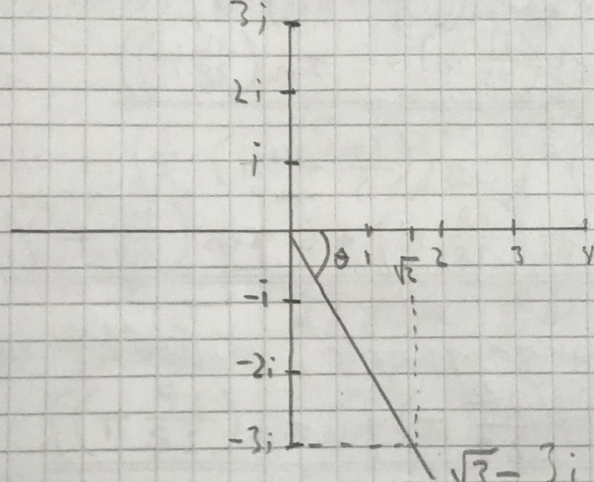
GRUPPE 12

a $w = \sqrt{3} - 3i$

$r = |w| = \sqrt{\sqrt{3}^2 + (-3)^2} = \sqrt{12} = \underline{2\sqrt{3}}$

$w = 2\sqrt{3}\,(\cos\theta + i\sin\theta)$

$\text{arc}(w) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-3}{\sqrt{3}}\right) = -\frac{\pi}{3}$



$$\underline{\underline{w = 2\sqrt{3}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 2\sqrt{3}\,e^{-\frac{\pi}{3}i}}}$$

$$w^7 = (\sqrt{3} - 3i)^7 = (2\sqrt{3})^7\left(\cos\left(-\frac{\pi}{3}\cdot 7\right) + i\sin\left(-\frac{\pi}{3}\cdot 7\right)\right)$$

$$= (2\sqrt{3})^7\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \underline{\underline{-5184 + 2993i}}$$

bi $(3+5i)(\overline{2+3i}) = (3+5i)(2-3i) = (3\cdot 2 - 5(-3)) + (3(-3) + 5\cdot 2)$

$= (6+15) + (-9+10)i = \underline{\underline{21 + i}}$

ii $\frac{25i}{3+4i} - 3i = \frac{25i(3-4i)}{(3+4i)(3-4i)} - 3i = \frac{75i + 100}{3^2 + 4^2} - 3i$

$= \frac{100 + 75i}{25} - 3i = \frac{100 + 75i - 75i}{25} = \frac{100}{25} = \underline{\underline{4}}$

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c Finn alle komplekse løsninger:

$$z^7 - 2 + 2i = 0$$

$$z^7 = 2 - 2i$$

$$|z|^7 = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\arg(z) = \tan^{-1}\left(\frac{2}{-2}\right) + \pi = \frac{3\pi}{4}$$

$$z = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

$$z_0 = (2\sqrt{2})^{\frac{1}{7}}\left(\cos\left(\frac{3\pi}{4}\cdot\frac{1}{7}\right) + i\sin\left(\frac{3\pi}{4}\cdot\frac{1}{7}\right)\right)$$

$$= \underline{\underline{2^{\frac{3}{14}}\left(\cos\left(\frac{3\pi}{28}\right) + i\sin\left(\frac{3\pi}{28}\right)\right)}}$$

$$z_1 = (2\sqrt{2})^{\frac{1}{7}}\left(\cos\left(\left(\frac{3\pi}{4} + 2\pi\right)\cdot\frac{1}{7}\right) + i\sin\left(\left(\frac{3\pi}{4} + 2\pi\right)\cdot\frac{1}{7}\right)\right)$$

$$= \underline{\underline{2^{\frac{3}{14}}\left(\cos\left(\frac{11}{28}\pi\right) + i\sin\left(\frac{11}{28}\pi\right)\right)}}$$

$$z_2 = (2\sqrt{2})^{\frac{1}{7}}\left(\cos\left(\left(\frac{3\pi}{4} + 4\pi\right)\cdot\frac{1}{7}\right) + i\sin\left(\left(\frac{3\pi}{4} + 4\pi\right)\cdot\frac{1}{7}\right)\right)$$

$$= \underline{\underline{2^{\frac{3}{14}}\left(\cos\left(\frac{19}{28}\pi\right) + i\sin\left(\frac{19}{28}\pi\right)\right)}}$$

Samme fremgangsmåte, mindre mellomregning:

$$z_3 = \underline{\underline{2^{\frac{3}{14}}\left(\cos\left(\frac{27}{28}\pi\right) + i\sin\left(\frac{27}{28}\pi\right)\right)}}$$

$$z_4 = \underline{\underline{2^{\frac{3}{14}}\left(\cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right)\right)}}$$

$$z_5 = \underline{\underline{2^{\frac{3}{14}}\left(\cos\left(\frac{43}{28}\pi\right) + i\sin\left(\frac{43}{28}\pi\right)\right)}}$$

$$z_6 = \underline{\underline{2^{\frac{3}{14}}\left(\cos\left(\frac{51}{28}\pi\right) + i\sin\left(\frac{51}{28}\pi\right)\right)}}$$

d Finn faktorisering av $p(z) = z^3 - 3z^2 + z - 3$ i lineære faktorer over $\mathbb{C}$ og i lineære og kvadratiske over $\mathbb{R}$.

$$p(3) = 3^3 - 3\cdot 3^2 + 3 - 3 = \underline{0}$$

$$(z^3 - 3z^2 + z - 3) : (z - 3) = z^2 + 1$$

$$\underline{z^3 - 3z^2}$$

$$0 + z - 3$$

$$\underline{z - 3}$$

$$0$$

$$\underline{\underline{(z-3)(z^2+1)}} \quad \text{OVER } \mathbb{R}$$

$$\underline{\underline{(z-3)(z-i)(z+i)}} \quad \text{OVER } \mathbb{C}$$

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2a $f(x) = \ln x$ Bruk induksjon til å begrunne hvorfor: $f^{(9)}(x) = 1$ og $f^{(9)}(1) = 40320$ og $f^{(15)}(1) = 87178291300$

$f'(x) = \frac{1}{x}$ $\qquad f^{(k+1)}(x) = \frac{d}{dx} f^{(k)}(x)$

$f''(x) = \frac{(1)' \cdot x - 1 \cdot (x)'}{(x)^2} = -\frac{1}{x^2}$

$f'''(x) = \frac{2}{x^3}$ $\qquad \boxed{(-1)^{n-1} (n-1)! \; x^{-n}}$

$f''''(x) = -\frac{6}{x^4}$ $\qquad$ Formelen for den nte deriverte.

Vil vise ved induksjon for $n \geq 1$:

Må vise:

1) $P(1)$

2) For alle $k \geq 1$ $P(k) \Rightarrow P(k+1)$

1) $P(1)$: $f^{(1)}(x) = -1^{1-1} (1-1)! \cdot x^{-1}$
$= 1 \cdot 1 \cdot x^{-1} = \frac{1}{x}$ JA!

2) Anta $P(k) = (-1)^{k-1} (k-1)! \cdot x^{-k}$

Vil vise:

$P(k+1): (-1)^{((k+1)-1)} ((k+1)-1)! \cdot x^{-(k+1)}$

$f^{(k+1)} = \left((-1)^{k-1} (k-1)! \cdot x^{-k}\right)'$

$= (k-1)! \cdot (k-1) \cdot (-1)^{((k-1)-1)} \cdot (-k) \cdot x^{-(k+1)} \cdot (-1)$

$= (-1)^{((k-1)-1)} (k-1)! \cdot x^{-(k+1)} \cdot (k-1) \cdot (-k) \cdot (-1)$

Ser at svaret vi får ikkje blir lik

$P(k+1): (-1)^{((k+1)-1)} ((k+1)-1)! \cdot x^{-(k+1)}$

<u>Stemmer ikke for $n = k+1$</u>

2b $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$ Kasper Melheim

I: La $n=1$:

$\frac{1}{1\cdot(1+1)}=\frac{1}{2}\Rightarrow$ Venstre side

$\frac{1}{1+1}=\frac{1}{2}\Rightarrow$ Høgre side

Induksjonsgrunnlaget stemmer $n=1$.

II: Setter $n=k$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{k(k+1)}=\underline{\frac{k}{k+1}}$$

og set $n=k+1$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{(k+1)((k+1)+1)}+\frac{k}{k+1}=\frac{k+1}{((k+1)+1)}$$

$$=\underline{\underline{\frac{k+1}{k+2}}}$$

$$\frac{1}{(k+1)((k+1)+1)}+\frac{k}{k+1}$$

$$=\frac{1}{(k+1)(k+2)}+\frac{k}{k+1}=\frac{1}{(k+1)(k+2)}+\frac{k(k+2)}{(k+1)(k+2)}$$

$$=\frac{1+k^2+2k}{(k+1)(k+2)}=\frac{\cancel{(k+1)}(k+1)}{\cancel{(k+1)}(k+2)}=\underline{\underline{\frac{k+1}{k+2}}}\quad \text{Q.E.D}$$

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3 a $\lim_{x\to 2} \frac{x^2-x-2}{\sqrt{x^2-3x+2}} = \lim_{x\to 2} \frac{(x^2-x-2)\cdot\sqrt{x^2-3x+2}}{\sqrt{x^2-3x+2}\cdot\sqrt{x^2-3x+2}}$

$= \lim_{x\to 2} \frac{(x+1)\cancel{(x-2)}\sqrt{x^2-3x+2}}{(x-1)\cancel{(x-2)}} = \lim_{x\to 2} \frac{(x+1)\sqrt{x^2-3x+2}}{(x-1)}$

$= \frac{(2+1)\sqrt{2^2-3\cdot 2+2}}{(2-1)} = \frac{3\cdot\sqrt{0}}{1} = \underline{\underline{0}}$

b $\lim_{x\to -1^-} \frac{x^2-x-2}{x^2-3x+2} = \lim_{x\to -1^-} \frac{(x+1)\cancel{(x-2)}}{(x-1)\cancel{(x-2)}}$

$= \lim_{x\to -1^-} \frac{x+1}{x-1} = \frac{0}{-2} = 0$

~~Får negative ledd, derfor $= \infty$~~

Ser at $x+1 \to 0$ og $x-1 \to -2$ frå negativ side.

c $\lim_{x\to 2} \cos\left(\frac{x^2-x-2}{\sqrt{x^2-3x+2}}\right) = \lim_{x\to 2} \cos\left(\frac{(x+1)\sqrt{x^2-3x+2}}{(x-1)}\right) \to$ Fra a)

$= \cos(0) = \underline{\underline{1}}$

d $\lim_{x\to\infty} \frac{2x+3}{\sqrt{x^2+3}} = \lim_{x\to\infty} \frac{2x+3}{\sqrt{x^2(1+\frac{3}{x^2})}}$

$= \lim_{x\to\infty} \frac{2x+3}{|x|\sqrt{1+\frac{3}{x^2}}} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{\sqrt{1+\frac{3}{x^2}}} = \frac{2+0}{\sqrt{1+0}} = \underline{\underline{2}}$

e $\lim_{x\to 1} (x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right)$. Bruke skviseteoremet:

$\left|\left(1+\sin\left(\frac{1}{x^2-1}\right)\right)\right| \leq 1$ for alle $x \neq 0$ og dermed er

$\left|(x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right)\right| \leq |x|$ for alle $x \neq 0$ s.a.

$-|x| \leq (x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right) \leq |x|$ for alle $x \neq 0$

Siden $\lim_{x\to 1} |(x-1)| = 0$ gir det at

$\lim_{x\to 1} (x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right) = \underline{0}$

4 a Vis ved $\varepsilon - \delta$ at Kasper Methe[illegible]

$$\lim_{x \to 1} (x^2 + x + 1) = 3$$

Gitt $\varepsilon > 0$, må vise at det finnes $\delta > 0$ S. A.

$$0 < |x-1| < \delta \Rightarrow |(x^2 + x + 1) - 3| < \varepsilon$$

$$|(x^2 + x + 1) - 3| = |x^2 + x - 2| = |(x-1)(x+2)|$$
$$= |x-1| \cdot |x+2|$$

Sett $\delta \leq 1$ og $|x-1| < \delta$, da har vi

$$|x-1| < 1 \Rightarrow 0 < x < 2 \Rightarrow 2 < x + 2 < 4$$
$$\Rightarrow |x+2| < 4$$

Har da:

$$|f(x) - 3| < 4|x-1| \quad \text{om } |x-1| < \delta \leq 1$$

$$4|x-1| < \varepsilon \quad \text{om } |x-1| < \frac{\varepsilon}{4}$$

Sett $\delta = \min\{1, \frac{\varepsilon}{4}\}$, da får vi

$$|f(x) - 3| < 4|x-1| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon \quad \text{om } |x-1| < \delta$$

Dette beviser at $\lim_{x \to 1} f(x) = 3$

$$\underline{\underline{\delta = \frac{\varepsilon}{4}}}$$

Kasper Nelhein

4b $f(a) = g(a) = 0, \quad g'(a) \neq 0$

Begrunn at $\dfrac{f'(a)}{g'(a)} = \lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$

$$\frac{f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}}{g'(a) = \lim\limits_{x \to 0} \dfrac{g(a+h) - f(a)}{h}}$$

$$= \frac{f'(a) = \lim\limits_{h \to 0} f(a+h) - 0}{g'(a) = \lim\limits_{h \to 0} g(a+h) - 0}$$ ← siden $f(a) = g(a) = 0$

$$= \frac{f'(a)}{g'(a)} = \lim\limits_{h \to 0} \frac{f(a+h)}{g(a+h)} = \lim\limits_{x \to a} \frac{f(x)}{g(x)}$$

c Bruk konklusjonen i b.

$$\lim\limits_{x \to 0} \frac{x}{e^x - 1} = \frac{f'(x)}{g'(x)} = \frac{(x)'}{(e^x-1)'} = \frac{1}{e^x}$$

Sett inn $x = 0$ og vi får $\dfrac{1}{e^0} = \underline{\underline{1}}$

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5a $f(x) = 2x^3 + x^2 - 4x - 2 + \cos x$

Finn ligningen for normalen til grafen til f. i punktet $(0,-1) = P$

$\boxed{y - y_0 = a(x - x_0)}$

$f'(x) = 6x^2 + 2x - 4 - \sin x$

$f'(0) = 6 \cdot 0^2 + 2 \cdot 0 - 4 - \sin 0 = \underline{-4}$

Stigningstallet til tangenten til $f(x)$ i $x = 0$ er -4.

$y + 1 = -4(x - 0)$

$y = -4x - 1 \Rightarrow$ Tangenten til f

Slope of the normal $= \frac{-1}{-4} = \underline{\frac{1}{4}}$

Normalen er da $=$ $\underline{\underline{y = \frac{1}{4}x - 1}}$

b Som vist i a er f deriverbar.
f er kontinuerlig overalt siden den er et polynom.

5c Begrunn at $f(x) = 0$ har minst tre løsninger innenfor intervalla $(-2,-1)$, $(-1,0)$ og $(1,2)$

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Løsningene ligger i $[-2,2]$

Må derfor sjekke ulike verdier innenfor det intervallet.

$f(-2) = 2 \cdot (-2)^3 + (-2)^2 - 4 \cdot (-2) - 2 + \cos(-2) = \underline{-6,4}$

$f(-1) = 1,5$

$f(0) = -1$

$f(1) = -2,5$

$f(2) = 9,6$

Ut i fra svara ser vi at 1) $f(-2)$ er negativ og $f(-1)$ er positiv $\Rightarrow$ $\underline{\underline{\text{Nullpunkt i } [-2,-1]}}$

2) $f(-1)$ er positiv og $f(0)$ er negativ $\Rightarrow$ $\underline{\underline{\text{Nullpunkt i } [-1,0]}}$

3) $f(1)$ er negativ og $f(2)$ er positiv $\Rightarrow$ $\underline{\underline{\text{Nullpunkt i } [1,2]}}$

Brukte her IVT teoremet.

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6 $f(x) = \begin{cases} (x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right) & \text{når } x \neq 1 \\ 0 & \text{når } x = 1 \end{cases}$

$f: \mathbb{R} \to \mathbb{R}$

a Definisjonen sier

f er kontinuerlig i 1 $\iff \lim_{x \to 1} f(x) = f(1)$

Vi har $f(1) = 0$ per def. og

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \left((x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right)\right)$$
$$= \underline{0}$$

Ved grensesetningene og grensa vi fikk fra 3e.

$\underline{\underline{f \text{ er da kontinuerlig i } 1}}$

b Avgjør om f er deriverbar i $x=1$. Finn $f'(0)$

f er deriverbar i 0 om $\iff \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$

eksisterer og $f'(0)$ er lik denne grensen.

$$\frac{f(h)-f(0)}{h} = \frac{(h-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right) + 0,16}{h}$$

$$= \frac{h + h\sin\left(\frac{1}{h^2-1}\right) - 1 - \sin\left(\frac{1}{h^2-1}\right) - 0,16}{h}$$