1 27-10-2014

1.1 Measure Theory Chapter 9

1.1 Proposition.
Given $f \in \mathcal{E}^+$. Let $\sum_{j=0}^{M} y_j 1_{A_j}$ and $\sum_{k=0}^{N} z_k 1_{B_k}$ be two standard representations of f. Then

$$\sum_{j=0}^{M} y_j \mu(A_j) = \sum_{k=0}^{N} z_k \mu(B_k).$$

Proof.

$$\sum_{j=0}^{M} y_j \mu(A_j) = \sum_{k=0}^{N} z_k \mu(B_k)$$

$$\Uparrow$$

$$\sum_{j=0}^{M} y_j \sum_{k=0}^{N} \mu(A_j \cap B_k) = \sum_{k=0}^{N} z_k \sum_{j=0}^{M} \mu(A_j \cap B_k)$$

$$\Uparrow$$

$$y_j \mu(A_j \cap B_k) = z_k \mu(A_j \cap B_k) \qquad \forall (j,k)$$

$$\Uparrow$$

$$\sum_{j=0}^{M} y_j 1_{A_j}(x) = \sum_{k=0}^{N} z_k 1_{B_k}(x) \qquad \forall x \in X$$

□

1.2 Definition.
Let $f = \sum_{j=0}^{M} y_j 1_{A_j} \in \mathcal{E}^+$ be a simple function in standard representation. Then the number

$$I_\mu(f) := \sum_{j=0}^{M} y_j \mu(A_j) \in [0, \infty]$$

is called the (μ-)*integral* of f.

1.3 Proposition.

$$I_\mu(1_A) = \mu(A) \qquad \forall A \in \mathcal{A}$$
$$I_\mu(\lambda f) = \lambda I_\mu(f) \qquad \forall \lambda \geq 0$$

Proof.
$I_\mu(f) := \sum_{j=0}^{M} y_j \mu(A_j) \in [0,\infty]$ by definition. $\square$

1.4 Proposition.

$$f, g \in \mathcal{E}^+ \Longrightarrow I_\mu(f+g) = I_\mu(f) + I_\mu(g)$$

Proof.

$$I_\mu(f+g) = I_\mu(f) + I_\mu(g)$$
$$\Uparrow$$
$$I_\mu\Big(\sum_{j=0}^{M} y_j 1_{A_j}(x) + \sum_{k=0}^{N} z_k 1_{B_k}(x)\Big) = I_\mu\Big(\sum_{j=0}^{M} y_j 1_{A_j}(x)\Big) + I_\mu\Big(\sum_{k=0}^{N} z_k 1_{B_k}\Big)$$
$$\Uparrow$$
$$\sum_{j=0}^{M}\sum_{k=0}^{N}(y_j + z_k)\mu(A_j \cap B_k) = \sum_{j=0}^{M} y_j \mu(A_j) + \sum_{k=0}^{N} z_k \mu(B_k)$$
$$\Uparrow$$
$$\sum_{j=0}^{M}\sum_{k=0}^{N}(y_j + z_k)\mu(A_j \cap B_k) = \sum_{j=0}^{M} y_j \sum_{k=0}^{N} \mu(A_j \cap B_k) + \sum_{k=0}^{N} z_k \sum_{j=0}^{M} \mu(A_j \cap B_k)$$

$\square$

1.5 Proposition.

$$f \leq g \Longrightarrow I_\mu(f) \leq I_\mu(g)$$

Proof.

$$\begin{aligned} I_\mu(f) &\leq I_\mu(g) \\ &\Uparrow [g - f \in \mathcal{E}^+] \\ I_\mu(f) &\leq I_\mu(f) + I_\mu(g - f) \end{aligned}$$

$\square$

1.6 Definition.
Let $(X, \mathcal{A}, \mu)$ be a measure space. The μ-integral of a positve numerical function $u \in \mathcal{M}_{\bar{\mathbb{R}}^+}$ is given by

$$\int u \; d\mu := \sup\{I_\mu(g) : g \leq u, g \in \mathcal{E}^+\} \in [0, \infty].$$

If we need to emphasize the *integration variable*, we also write $\int u(x)\mu(dx)$ or $\int u(x)d\mu(x)$.

1.7 Proposition.
For all $f \in \mathcal{E}^+$ we have $\int f d\mu = I_\mu(f)$.

Proof.

$$\int f d\mu = I_\mu(f)$$

$$\Uparrow$$

$$\sup\{I_\mu(g) : g \leq f, g \in \mathcal{E}^+\} = I_\mu(f)$$

$$\Uparrow$$

$$I_\mu(f) \leq \sup\{I_\mu(g) : g \leq f, g \in \mathcal{E}^+\} \leq I_\mu(f)$$

$$\Uparrow$$

$$g \leq f \Longrightarrow I_\mu(g) \leq I_\mu(f)$$

$\square$

1.8 Proposition.
Let $(X, \mathcal{A})$ *be a measurable space. Let* $\mu = \delta_y$ *be the Dirac measure for fixed* $y \in X$. *Show that*

$$\int u \; d\delta_y = u(y) \qquad \forall u \in \mathcal{M}^+_{\overline{\mathbb{R}}}.$$

Proof.
By theorem 8.8, there exists increasing function $(f_j)_{j\in\mathbb{N}} \subseteq \mathcal{E}^+$ with $f_j \leq u$ and $\lim_{j\to\infty} f_j = u$. Therefore:

$$\int u \; d\delta_y = u(y)$$

$$\Uparrow$$

$$\int \lim_{j\to\infty} f_j \; d\delta_y = \lim_{j\to\infty} f_j(y)$$

$$\Uparrow$$

$$\lim_{j\to\infty} \int f_j \; d\delta_y = \lim_{j\to\infty} f_j(y)$$

$$\Uparrow$$

$$\int f_j \; d\delta_y = f_j(y) \qquad \forall j \in \mathbb{N}$$

$$\Uparrow$$

$$\sum_{k=0}^{N} y_{k_j} \; \delta_y(A_{k_j}) = f_j(y) \qquad \forall j \in \mathbb{N}$$

$\square$

1.9 Theorem.
Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}^+_{\overline{\mathbb{R}}}$ *be a sequence of positive measurable functions. Then* $u := \liminf_{j\to\infty} u_j$ *is measurable and*

$$\int \liminf_{j\to\infty} u_j \; d\mu \leq \liminf_{j\to\infty} \int u_j \; d\mu$$

Proof.

Recall that $\liminf_{j\to\infty} u_j = \sup_{k\in\mathbb{N}} \inf_{j\geq k} u_j$. Therefore:

$$\int \liminf_{j\to\infty} u_j \; d\mu \leq \liminf_{j\to\infty} \int u_j \; d\mu$$
$$\Uparrow$$
$$\int \sup_{k\in\mathbb{N}} \inf_{j\geq k} u_j \; d\mu \leq \sup_{k\in\mathbb{N}} \inf_{j\geq k} \int u_j \; d\mu$$
$$\Uparrow \text{ T9.6}$$
$$\sup_{k\in\mathbb{N}} \int \inf_{j\geq k} u_j \; d\mu \leq \sup_{k\in\mathbb{N}} \inf_{j\geq k} \int u_j \; d\mu$$
$$\Uparrow$$
$$\int \inf_{j\geq k} u_j \; d\mu \leq \inf_{l\geq k} \int u_l \; d\mu \qquad \forall k \in \mathbb{N}$$
$$\Uparrow$$
$$\int \inf_{j\geq k} u_j \; d\mu \leq \int u_l \; d\mu \qquad \forall l \geq k$$
$$\Uparrow$$
$$\inf_{j\geq k} u_j \;\; \leq u_l \qquad \forall l \geq k$$

$\square$

1.10 Theorem.
Let $(X, \mathcal{A}, \mu)$ be a measure space. For an increasing sequence of numerical functions $(f_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}^+_{\bar{\mathbb{R}}}$ where $0 \leq f_j \leq f_{j+1} \leq \dots$, we have

$$\int \sup_{j\in\mathbb{N}} f_j \; d\mu = \sup_{j\in\mathbb{N}} \int f_j \; d\mu$$

and

$$\int \lim_{j\to\infty} f_j \; d\mu = \lim_{j\to\infty} \int f_j \; d\mu.$$

1.11 Theorem.
Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}_{\bar{\mathbb{R}}^+}$. Then $\sum_{j=1}^{\infty} u_j$ is measurable and we have

$$\int \sum_{j=1}^{\infty} u_j d\mu = \sum_{j=1}^{\infty} \int u_j \; d\mu.$$

1.2 Measure Theory Chapter 10

1.12 Definition.
A function $u : X \to \bar{\mathbb{R}}$ on a measure space $(X, \mathcal{A}, \mu)$ is said to (μ-)-integrable, if it is $\mathcal{A}/\bar{\mathcal{B}}$-measurable and if the integrals $\int u^+ d\mu, \int u^- d\mu < \infty$ are finite. In this case we call

$$\int u \, d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty)$$

the μ-integral of u.

1.13 Definition.
We write $\mathcal{L}^1(\mu)$ $[\mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)]$ for the set of all real-valued [numerical] μ-integrable functions.

1.14 Theorem.
Let $u \in \mathcal{M}_{\bar{\mathbb{R}}}$. Then the following conditions are equivalent:

1. $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$
2. $u^+, u^- \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$
3. $|u| \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$
4. $\exists w \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu),\ w \geq 0$ *such that* $|u| \leq w$

Proof.

$$u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu) \iff u^\pm \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu)$$
$$\Updownarrow$$
$$u \in \mathcal{M}^+_{\bar{\mathbb{R}}} \ \wedge \ \int u^\pm d\mu < \infty \iff u^\pm \in \mathcal{M}^+_{\bar{\mathbb{R}}} \ \wedge \ \int u^\pm \, d\mu < \infty$$

$$u^\pm \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu) \implies |u| \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu)$$
$$\Updownarrow$$
$$u^\pm \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu) \implies u^+ + u^- \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu)$$
$$\Updownarrow$$
$$u^\pm \in \mathcal{M}^+_{\bar{\mathbb{R}}} \ \text{ and } \ \int u^\pm \, d\mu < \infty \implies u^+ + u^- \in \mathcal{M}^+_{\bar{\mathbb{R}}} \ \text{ and } \ \int u^+ + u^- \, d\mu < \infty$$

$3 \Rightarrow 4$ is obvious

$$
\begin{array}{c}
\exists w \geq 0 \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu) : |u| \leq w \implies u^{\pm} \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu) \\
\Uparrow \\
\int u^{\pm} d\mu < \infty \\
\Uparrow \\
u^{\pm} \leq w
\end{array}
$$

$\square$

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2.1 Measure Theory Chapter 10

2.1 Proposition.

$$u \in \mathcal{M}_{\bar{\mathbb{R}}} \iff u^{\pm} \in \mathcal{M}^+_{\bar{\mathbb{R}}}$$

2.2 Definition.
A function $u : X \to \bar{\mathbb{R}}$ on a measure space $(X, \mathcal{A}, \mu)$ is said to be μ-integrable, if

1. $u \in \mathcal{M}_{\bar{\mathbb{R}}}$
2. $\int u^{\pm}\, d\mu < \infty$

We write $\mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$ for the set of all numerical μ-integrable functions.

2.3 Definition.
If $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$, then we call

$$\int u\, d\mu := \int u^+ d\mu - \int u^- d\mu$$

the μ-integral of u.

2.4 Proposition.
For $u \geq 0$ we have

$$u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu) \iff u \in \mathcal{M}^+_{\bar{\mathbb{R}}} \text{ and } \int u\, d\mu < \infty$$

2.5 Proposition.
Given $u \in \mathcal{M}_{\bar{\mathbb{R}}}$.

$$\int |u|\, d\mu < \infty \iff u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$$

2.6 Theorem.
Let $(X, \mathcal{A}, \mu)$ be a measure space and $u, v \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$ and $\alpha \in \mathbb{R}$. Then

1. *$\alpha u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$ and $\int \alpha u\, d\mu = \alpha \int u\, d\mu$*
2. *$u + v \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$ and $\int (u+v)\, d\mu = \int u\, d\mu + \int v\, d\mu$*

3. $\min\{u,v\}, \max\{u,v\} \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$

4. $u \le v \implies \int u \; d\mu \le \int v \; d\mu$

5. $|\int u \; d\mu| \le \int |u| d\mu$

Proof.

$$\begin{gathered}
\alpha u \in \mathcal{L}^1_{\mathbb{R}}(\mu) \\
\Uparrow \\
\int |au| \; d\mu < \infty \\
\Uparrow \\
|a| \int |u| \; d\mu < \infty \\
\Uparrow \\
|u| \in \mathcal{L}^1_{\bar{\mathbb{R}}}
\end{gathered}$$

$$\begin{gathered}
u + v \in \mathcal{L}^1_{\bar{\mathbb{R}}} \\
\Uparrow \\
\int |u+v| \; d\mu < \infty \\
\Uparrow \\
\int |u| + |v| \; d\mu < \infty
\end{gathered}$$

$$\begin{gathered}
\min\{u,v\} \in \mathcal{L}^1_{\bar{\mathbb{R}}} \\
\Uparrow \\
\int |\min\{u,v\}| \; d\mu < \infty \\
\Uparrow \\
\int |u| + |v| \; d\mu < \infty
\end{gathered}$$

$\square$

2.7 Proposition.
On $(X, \mathcal{A}, \delta_y)$ *where* $y \in X$ *fixed, we have* $\int u(x)\delta_y(dx) = u(y)$ *and*

$$u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\delta_y) \iff u \in \mathcal{M}_{\bar{\mathbb{R}}} \quad \text{and} \quad |u(y)| < \infty$$

2.8 Proposition.
Let $(\Omega, \mathcal{A}, P)$ *be a probability space. Then every bounded measurable function (random variable)* $\xi \in \mathcal{M}(\mathcal{A})$ *with* $S := \sup_{\omega \in \Omega} |\xi(\omega)| < \infty$ *is* P*-integrable.*

Proof.

$$\begin{gathered} \xi \in \mathcal{L}^1_{\mathbb{R}}(P) \\ \Uparrow \\ \int |\xi| \; dP < \infty \\ \Uparrow \\ \int S \; dP < \infty \\ \Updownarrow \\ S \int 1_\Omega \; dP < \infty \end{gathered}$$

$\square$

2.9 Definition.
Let (X, A, μ) be a measure space and $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$ or $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathcal{A})$. Then

$$\int_A u \; d\mu := \int 1_A u \; d\mu = \int 1_A(x)u(x)\mu(dx) \qquad \forall A \in \mathcal{A}$$

2.10 Proposition.

$$\int_X u \; d\mu = \int u \; d\mu$$

2.11 Theorem.
On the measure space $(X, \mathcal{A}, \mu)$ *let* $u \in \mathcal{M}^+$*. The set-function*

$$\nu : A \mapsto \int_A u \; d\mu = \int 1_A u \; d\mu \qquad A \in \mathcal{A}$$

is a measure on $(X, \mathcal{A})$. It is called the measure with density (function) u with respect to μ and denoted by $\nu = u\mu$.

2.12 Definition.
If ν has a density function w.r.t. μ one writes tradionionally $d\nu/d\mu$ for the density function. This notation is to be understood in a purely symbolical way.

2.13 Definition.
Let $(X, \mathcal{A}, \mu)$ be a measure space. A μ-null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathcal{A}$ satisfying

$$N \in \mathcal{N}_\mu \Longleftrightarrow N \in \mathcal{A} \quad \text{and} \quad \mu(N) = 0.$$

2.14 Definition.
If a property $\Pi(x)$ is true for all $x \in X$ apart from some x contained in a null set $N \in \mathcal{N}_\mu$, we say that $\Pi(x)$ holds a.e. (allmost everyhwere).

2.15 Theorem.
Let $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$ be a numerical integrable function on a measure sapce $(X, \mathcal{A}, \mu)$. Then

1. $\int |u| \; d\mu = 0 \iff |u| = 0 \;\; a.e. \iff \mu(\{u \neq 0\}) = 0$
2. $\int_N u \; d\mu = 0 \quad \forall N \in \mathcal{N}_\mu$

2.16 Theorem.
Let $u, v \in \mathcal{M}_{\bar{\mathbb{R}}}$ such that $u = v$ μ-a.e. Then

1. $u, v \geq 0 \implies \int u \; d\mu = \int v \; d\mu$
2. $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu) \implies v \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu) \quad and \quad \int u \; d\mu = \int v d\mu$

Proof.

$$\int u \; d\mu = \int v \; d\mu$$

$$\Uparrow$$

$$\int_{\{u=v\}} u \; d\mu + \int_{\{u\neq v\}} u \; d\mu = \int_{\{u=v\}} v \; d\mu + \int_{\{u\neq v\}} v \; d\mu$$

$$\Uparrow$$

$$\int_{\{u=v\}} u \; d\mu = \int_{\{u=v\}} v \; d\mu$$

$$\int u \; d\mu = \int v \; d\mu$$

$$\Uparrow$$

$$\int v^{\pm} \; d\mu = \int u^{\pm} \; d\mu < \infty$$

$$\Uparrow \text{[apply 1]}$$

$$u^{\pm} = v^{\pm} \; a.e.$$

$$\Uparrow$$

$$u = v \; a.e.$$

$\square$

2.17 Theorem.
If $u \in \mathcal{M}_{\bar{\mathbb{R}}}$ and $v \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu), v \geq 0$ then

$$|u| \leq v \quad a.e. \implies u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu).$$

2.18 Proposition.
For all $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu), A \in \mathcal{A}$ and $c > 0$

$$\mu(\{|u| \geq c\} \cap A) \leq 1/c \int_A |u| d\mu.$$

2.19 Theorem.
If $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$, then u is almost everywhere $\mathbb{R}$-valued. In particular, we can find a version $\tilde{u} \in \mathcal{L}^1(\mu)$ such that $\tilde{u} = u$ a.e. and $\int \tilde{u} d\mu = \int u d\mu$.

2.20 Theorem.
Let $\mathcal{G} \subseteq \mathcal{A}$ be a sub-σ-algebra.

1. *If $u, v \in \mathcal{L}^1(\mathcal{G})$ and if $\int_G u d\mu = \int_G w d\mu$ for all $G \in \mathcal{G}$, then $u = w$ μ -a.e.*
2. *If $u, w \in \mathcal{M}^+(\mathcal{G})$ and if $\int_G u d\mu = \int_G w d\mu$ for all $G \in \mathcal{G}$, then $u = w$ μ-a.e. under the additional assumption that $\mu|_\mathcal{G}$ is σ-finite.*

2.21 Proposition.

$$\int_N u \; d\mu = 0 \quad \forall N \in \mathcal{N}_\mu$$

Proof.

$$\begin{gathered} \int_N u \; d\mu = 0 \\ \Uparrow \\ |\int_N u \; d\mu| \leq 0 \\ \Uparrow \\ \int 1_N |u| \; d\mu \leq 0 \\ \Uparrow \\ \sup_{j \in \mathbb{N}} \int j \; 1_N \; d\mu = 0 \end{gathered}$$

□

2.22 Proposition.

$$|u| = 0 \; a.e. \iff \mu\{u \neq 0\} = 0$$

Proof.

$$
\begin{gathered}
|u| = 0 \ a.e. \\
\Updownarrow \\
\{x : |u(x)| \neq 0\} \in \mathcal{N}_\mu \\
\Updownarrow \\
\mu\{u \neq 0\} = 0
\end{gathered}
$$

$\square$

2.23 Proposition.

$$
|u| = 0 \ a.e. \implies \int |u| \ d\mu = 0
$$

Proof.

$$
\begin{gathered}
\int |u| \ d\mu = 0 \\
\Uparrow \\
\nu(X) = 0 \\
\Uparrow \\
\nu(\{u \neq 0\} \cup \{u = 0\}) = 0 \\
\Uparrow \\
\nu(\{u \neq 0\}) + \nu(\{u = 0\}) = 0 \\
\Uparrow \\
\int_{\{u \neq 0\}} |u| d\mu + \int_{\{u = 0\}} |u| d\mu = 0 \\
\Uparrow \\
\{u \neq 0\} \in \mathcal{N}_\mu \\
\Uparrow \\
|u| = 0 \ a.e.
\end{gathered}
$$

$\square$

2.24 Proposition.

$$\int |u| \; d\mu = 0 \implies \mu\{u \neq 0\} = 0$$

Proof.

$$\begin{gathered}
\mu\{u \neq 0\} = 0 \\
\Uparrow \\
\mu\{|u| > 0\} = 0 \\
\Uparrow \\
\mu\Big(\bigcup_{j \in \mathbb{N}} \{|u| \geq \tfrac{1}{j}\}\Big) = 0 \\
\Uparrow \\
\mu\Big(\{|u| \geq \tfrac{1}{j}\}\Big) = 0 \qquad \forall j \in \mathbb{N} \\
\Uparrow \text{[Markov Inequality]} \\
j \int_X |u| \; d\mu = 0
\end{gathered}$$

□

2.25 Proposition.

$$\mu(\{u \geq c\} \cap A) \leq \frac{1}{c} \int_A |u(x)| \mu(dx)$$

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3.1 Measure Theory Chapter 11

3.1 Theorem.
Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. *Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^1(\mu)$ be an increasing sequence of integrable functions $u_1 \leq u_2 \leq \dots$ with limit $u := \sup_{j\in\mathbb{N}} u_j$. Then $u \in \mathcal{L}^1(\mu)$ if, and only if, $\sup_{j\in\mathbb{N}} \int u_j d\mu < \infty$, in which case*

$$\sup_{j\in\mathbb{N}} \int u_j d\mu = \int \sup_{j\in\mathbb{N}} u_j d\mu$$

2. *Let $(v_k)_{k\in\mathbb{N}} \subseteq \mathcal{L}^1(\mu)$ be a decreasing sequence of integrable functions $v_1 \geq v_2 \geq \dots$ with limit $v := \inf_{k\in\mathbb{N}} d\mu$. Then $v \in \mathcal{L}^1(\mu)$ if, and only if, $\inf_{k\in\mathbb{N}} \int v_k \ d\mu > -\infty$, in which case*

$$\inf_{k\in\mathbb{N}} \int v_k d\mu = \int \inf_{k\in\mathbb{N}} v_k \ d\mu$$

3.2 Theorem.
Let $(X, \mathcal{A}, \mu)$ be measure space and $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^1(\mu)$ be a sequence of functions such that $|u_j| \leq w$ for all $j \in \mathbb{N}$ and some $w \in \mathcal{L}^1_+(\mu)$. If $u(x) = \lim_{j\to\infty} u_j(x)$ exists for almost every $x \in X$ then $u \in \mathcal{L}^1(\mu)$ and we have

1. $\lim_{j\to\infty} \int |u_j - u| \ d\mu = 0$
2. $\lim_{j\to\infty} \int u_j \ d\mu = \int \lim_{j\to\infty} u_j d\mu = \int u d\mu$

3.3 Theorem.
Let $\varnothing \neq (a, b) \subseteq \mathbb{R}$ be a non-degenerate open interval and $u : (a, b) \times X \to \mathbb{R}$ be a function satisfying

1. *$x \mapsto u(t, x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a, b)$.*
2. *$t \mapsto u(t, x)$ is continuous for every fixed $x \in X$*
3. *$|u(t, x)| \leq w(x)$ for all $(t, x) \in (a, b) \times X$ and some $w \in \mathcal{L}^1_+(\mu)$.*

Then the function $v : (a,b) \to \mathbb{R}$ given by

$$t \mapsto v(t) := \int u(t,x)\ \mu(dx)$$

is continuous.

3.4 Theorem.
Let $\varnothing \neq (a,b) \subseteq \mathbb{R}$ be a non-degenerate open interval and $u : (a,b) \times X \to \mathbb{R}$ be a function satsifying:

1. *$x \mapsto u(t,x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a,b)$.*
2. *$t \mapsto u(t,x)$ is differentiable for every fixed $x \in X$*
3. *$|\partial_t u(t,x)| \leq w(x)$ for all $(t,x) \in (a,b) \times X$ and some $w \in \mathcal{L}^1_+(\mu)$*

Then the function $v : (a,b) \to \mathbb{R}$ given by

$$t \mapsto v(t) := \int u(t,x)\ \mu(dx)$$

is differentiable and its derivative is

$$\partial_t v(t) = \int \partial_t u(t,x) \mu(dx).$$

3.5 Definition.
Consider on the finite interval $[a,b] \subseteq \mathbb{R}$ the partitions

$$\pi := \{a = t_0 < t_1 < \ldots < t_{k(\pi)} = b,$$

define for a given function $u : [a,b] \to \mathbb{R}$

$$m_j := \inf_{x \in [t_{j-1}, t_j]} u(x) \qquad M_j := \sup_{x \in [t_{j-1}, t_j]} u(x) \qquad j = 1, 2, \ldots, k(\pi)$$

and introduce the lower resp. upper Darboux sums

$$S_\pi[u] := \sum_{j=1}^{k(\pi)} m_j (t_j - t_{j-1}) \qquad S^\pi[u] := \sum_{j=1}^{k(\pi)} M_j (t_j - t_{j-1})$$

3.6 Definition.
A bounded function $u : [a,b] \to \mathbb{R}$ is said to be Riemann integrable, if the values

$$\int_* u := \sup_\pi S_\pi[u] = \inf_\pi S^\pi[u] := \int^* u$$

(sup,inf range over all partitions π of $[a,b]$) conincide and are finite. Their common value is called the Riemann integral of u and denoted by $(R)\int_a^b u(x)\,dx$ or $\int_a^b u(x)\ dx$.

3.7 Proposition.
$S_\pi[u]$ and $S^\pi[u]$ correspond to simple functions $\sigma_\pi[u]$ and $\Sigma^\pi[u]$ given by

$$\sigma_\pi[u](x) = \sum_{j=1}^{k(\pi)} m_j 1_{[t_{j-1},t_j)}(x) \qquad \Sigma^\pi[u](x) = \sum_{j=1}^{k(\pi)} M_j 1_{[t_{j-1},t_j)}(x)$$

which satisfy $\sigma_\pi[u](x) \le u(x) \le \Sigma^\pi[u](x)$ and which increase resp. decrease as π refines.

3.8 Theorem.
Let $u : [a,b] \to \mathbb{R}$ be a measurable function.

1. *If u is Riemann integrable, then u is in $\mathcal{L}^1(\lambda)$ and the Lebesgue and Riemann integrals coincide:*

$$\int_{[a,b]} u\ d\lambda = (R)\int_a^b u(x)\ dx.$$

2. *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if, and only if, the points (a,b) where f is discontinuous are a Lebesgue null set.*

3.9 Definition.
An improper Riemann integral is defined as

$$(R)\int_0^\infty u(x)\ dx := \lim_{a\to\infty} (R)\int_0^a u(x)\ dx$$

provided that the limit exists.

3.10 Theorem.
Let $u : [0,\infty) \to \mathbb{R}$ be a measurable function which is Riemann integrable for every interval $[0,N], n \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0,\infty)$ if, and only if,

$$\lim_{N\to\infty} (R) \int_0^N |u(x)|dx < \infty.$$

In this case, $(R)\int_0^\infty u(x)dx = \int_{[0,\infty)} u \; d\lambda$.

3.11 Proposition.
The function $s : (0,\infty) \to \mathbb{R} : x \mapsto \frac{\sin x}{x}$ is improperly Riemann integrable but not Lebesgue integrable.

3.12 Proposition.
Let $f_\alpha(x) := x^\alpha, x > 0$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} f_\alpha \in \mathcal{L}^1(0,1) &\iff \alpha > -1 \\ f_\alpha \in \mathcal{L}^1[1,\infty) &\iff \alpha < -1 \end{aligned}$$

3.13 Proposition.
The function $f(x) := x^\alpha e^{-\beta x}, x > 0$ is Lebesgue integrable over $(0,\infty)$ for all $\alpha > -1$ and $\beta \geq 0$.

3.14 Proposition.
The parameter-dependent integral

$$\Gamma(t) := \int_{(0,\infty)} x^{t-1}e^{-x}\lambda(dx) \qquad t > 0$$

is called the Gamma function. It has the following properties:

1. *Γ is continuous*
2. *Γ is arbitrarily often differentiable*
3. *$t\Gamma(t) = \Gamma(t+1)$ in particular $\Gamma(n+1) = n!$*
4. *$\ln \Gamma(t)$ is convex*