# Contents

# 1 9-10-2014

## 1.1 Measure Theory Chapter 6

**1.1 Problem 6.1a.**
*Consider on $\mathbb{R}$ the family $\Sigma$ of all Borel sets which are symmetric w.r.t. the origin. Show that $\Sigma$ is a $\sigma$-algebra.*

**Proof.**

1. To show that $\mathbb{R} \in \Sigma$, note that $\mathbb{R}$ is a Borel set that is symmetric w.r.t. to the origin.
2. To show that $A \in \Sigma \Rightarrow A^c \in \Sigma$, it suffices to show that
$$\forall x \in A : -x \in A \Longrightarrow \forall y \in A^c : -y \in A^c,$$
which is equivalent with showing that
$$\forall x \in A : -x \in A \Longrightarrow \forall y \notin A : -y \notin A,$$
which is equivalent with showing that
$$\exists y \notin A : -y \in A \Longrightarrow \exists x \in A : -x \notin A.$$
This last statement hold if we set $x := -y$.
3. To show that $\Sigma$ is stable under countable unions, assume $A_j = B_j \cup B_j$ for some $B_j \in \mathcal{B}([0, \infty)$. We have
$$\bigcup_{j \in \mathbb{N}} A_j = \bigcup_{j \in \mathbb{N}} B_j \cup \bigcup_{j \in \mathbb{N}} -B_j \in \Sigma$$

$\square$

**1.2 Problem 6.3i.**
*Show that non-void open sets in $\mathbb{R}^n$ have always strictly positive Lebesgue measure.*

**Proof.**
First remember that

1. $\lambda^n[a,b) = \prod_{j=1}^{n}(b_j - a_j)$
2. $\lambda^n$ is a pre-measure that can be extended to a measure on $\mathcal{B}(\mathbb{R}^n)$.
3. $\lambda^n$ is invariant under translations
4. $A \subseteq B \Longrightarrow \mu(A) \leq \mu(B)$
5. $Q_\epsilon = [-\epsilon, \epsilon)$

To show that $\lambda^n(U) > 0$ it suffices

$$\lambda^n(U') > 0$$

where $0 \in U'$ and $U' = x + U$ for some $x \in \mathbb{R}^n$. To show that it suffices to show that

$$\lambda^n(B_\epsilon(0)) > 0$$

where $B_\epsilon(0) \subseteq U$. To show that it suffices to show that $Q_{\epsilon'} \subseteq B_\epsilon$ for some $\epsilon' > 0$. This holds if we set $\epsilon' := \frac{\epsilon}{\sqrt{2n}}$. $\square$

**1.3 Problem 6.3ii.**
*Is 6.3i still true for closed sets ?*

**Proof.**
No, take $\{0\}$, then $\lambda\{x\} = 0$. $\square$

**1.4 Problem 6.4i.**
*Show that $\lambda(a,b) = b - a$ for all $a, b \in \mathbb{R}, a \leq b$.*

**Proof.**

$$\begin{aligned}
\lambda(a,b) &= \lambda([b-a) - \{b\}) && \\
&= \lambda[b,a) - \lambda\{b\} && \text{T4.3iii} \\
&= b - a - 0 && \text{Problem 4.11i}
\end{aligned}$$

$\square$

**1.5 Problem 6.4ii.**
*Let $H \subseteq \mathbb{R}^2$ be a hyperplane which is perpendicular to the $x_1$-direction (that is to say: $H$ is a translate of the $x_2$ axis). Show that*

1. $H \in \mathcal{B}(\mathbb{R}^2)$
2. $\lambda^2(H) = 0$

**Proof.**

1. To show that $H \in \mathcal{B}(\mathbb{R}^2)$, it suffices to show that $H$ is writable as an intersection of countable half-open sets. Note that:

$$H := \{y\} \times \mathbb{R} = \bigcap_{j \in \mathbb{N}} [y, y + 1/j) \times \mathbb{R}$$

2. We have that for any $\epsilon > 0$:

$$\begin{aligned}\lambda^2(H) &= \lambda^2(\{y\} \times \mathbb{R}) \\ &\leq \lambda^2\Bigg(\bigcup_{n \in \mathbb{N}} [y, y + \epsilon_n) \times [-n, n)\Bigg) \\ &\leq 2 \sum_{n \in \mathbb{N}} \epsilon_n n \\ &= \epsilon L\end{aligned}$$

This follows if we choose $\epsilon_n := \frac{\epsilon}{2^n}$. Therefore $\lambda^2(H) = 0$.

$\square$

**1.6 Definition.**
Let $(X, \mathcal{A}, \mu)$ be a measure space such that all singletons $\{x\} \in \mathcal{A}$. A point $x$ is called an atom, if $\mu\{x\} > 0$. A measure is called *non-atomic* or *diffuse*, there are no atoms.

**1.7 Problem 6.5i.**
*Show that $\lambda^1$ is diffuse.*

**Proof.**
We've already shown that $\lambda\{x\} = 0$ for any $x \in \mathbb{R}$. $\square$

**1.8 Problem 6.5iii.**
*Show that for a diffuse measure $\mu$ on $(X, \mathcal{A})$ all countable sets are null sets.*

**Proof.**
All countable sets are writable as

$$\bigcup_{j=0}^{\infty} \{x_j\}$$

where $x_i \neq x_j$. So we get

$$\lambda\left(\bigcup_{j=0}^{\infty} \{x_j\}\right) = \sum_{j=0}^{n} \lambda\{x_j\} = 0.$$

$\square$

**1.9 Definition.**
A set $A \subseteq \mathbb{R}^n$ is called *bounded* if it can be contained in a ball $B_r \supseteq A$ of finite radius $r$. A set $A \subseteq \mathbb{R}^n$ is called *connected*, if we can go along a curve from any point $a \in A$ to any point $a' \in A$ without ever leaving $A$.

**1.10 Problem 6.6a.**
*Construct an open and unbounded set in $\mathbb{R}$ with finite, strictly positive Lebesgue measure.*

**Proof.**
Consider the set

$$U := \bigcup_{n=1}^{\infty} \left(n - \frac{1}{2^n}, n + \frac{1}{2^n}\right).$$

This is an open set, as it union of countable open sets. It is unbounded, for any $B_r(0)$ we have that $r + 1 \in U$ and not in $B_r(0)$. We have to show that it has finite lebesgue measure.

$$\begin{aligned}\lambda(U) &= \bigcup_{n=1}^{\infty} \left(n - \frac{1}{2^n}, n + \frac{1}{2^n}\right) \\ &= \sum_{n=1}^{\infty} \frac{2}{2^n} = 2.\end{aligned}$$

$\square$

**1.11 Problem 6.6ii.**
*Construct an open, unbounded and connected set in $\mathbb{R}$ with finite, strictly positive Lebesgue measure.*

**Proof.**
Consider

$$U = \bigcup_{j\in\mathbb{N}} [0, 0+\epsilon/(2^j)) \times [-j, j)$$

then

$$\begin{aligned}\lambda^2(U) &= \left(\bigcup_{j\in\mathbb{N}} (-\frac{1}{2^j}, \frac{1}{2^j}) \times (-j, j)\right) \\ &\leq \sum_{j\in\mathbb{N}} \frac{4j}{2^j}\end{aligned}$$

Note that

$$\sum_{j\in\mathbb{N}} \frac{j}{2^j}$$

converges. $\square$

**1.12 Problem 6.6iii.**
*Is there a connected, open and unbounded set in $\mathbb{R}$ with finite, strictly positive Lebesgue measure ?*

**Proof.**
No, this is impossible. Since we are in one dimension, connectedness forces us to go between points in a straight, uninterrupted line. Since the set is unbounded, this means we must have a line of the sort $(a, \infty)$ or $(-\infty, b)$ in our set and in both cases Lebesgue measure is infinite. $\square$

**1.13 Definition.**
Let $A \subset X$. The closure of $A$, denoted by $\bar{A}$, is the smallest closed set containing $A$, i.e.

$$\bar{A} = \bigcap_{\substack{F\in\mathcal{C} \\ F\supset A}} F$$

**1.14 Definition.**
A set $A \subseteq X$ is dense in $X$ if $\bar{A} = X$

**1.15 Problem 6.7.**
*Let $\lambda := \lambda^1|_{[0,1]}$ be a Lebesgue measure on $([0,1], \mathcal{B}[0,1])$. Show that for every $\epsilon > 0$ there is a dense open set $U \subseteq [0,1]$ with $\lambda(U) \leq \epsilon$.*

**Proof.**
Note that $\mathbb{Q}$ is dense. We are going to make an open set contained in $\mathbb{Q}$. Consider

$$U := \bigcup_{j=1}^{\infty} (q_j - \epsilon_j, q_j + \epsilon_j)$$

Then

$$\lambda(U) = \lambda(\bigcup_{j=1}^{\infty} (q_j - \epsilon_j, q_j + \epsilon_j)) \leq \sum 2\epsilon_j.$$

So set $\epsilon_j := \frac{\epsilon}{2^{j-1}}$. And we are done. $\square$

**1.16 Problem 6.10i.**
*Let $\mu$ be a measure on $\mathcal{A} = \{\varnothing, [0,1), [1,2), [0,2)\}$ of $X = [0,2)$. Such that*

$$\mu[0,1) = \mu[1,2) = 1/2 \qquad \mu[0,2) = 1.$$

*Define for each $A \subseteq [0,2)$ the family of countable $\mathcal{A}$-coverings of $A$*

$$\mathcal{C}(A) := \{(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A} : \bigcup_{j\in\mathbb{N}} A_j \supseteq A\}$$

*and set*

$$\mu^*(A) := inf\left\{ \sum_{j\in\mathbb{N}} \mu(S_j) : (S_j)_{j\in\mathbb{N}} \in \mathcal{C}(A) \right\}.$$

*Define $\mathcal{A}^* := \{A \subseteq [0,2) : \mu^*(B) = \mu^*(B \cap A) + \mu^*(B - A) \quad \forall B \subseteq X\}$*

*Show that*

1. *Find $\mu^*(a,b), \mu^*\{a\}$*
2. $(0,1), \{0\} \notin \mathcal{A}^*$

Note that in T6.1 it is proven that:

- $\mathcal{A} \subseteq \mathcal{A}^*$
- $\mu^*(A) = \mu(A) \quad \forall A \in \mathcal{A}$
- $\mathcal{A}^*$ is a $\sigma$-algebra and $\mu^*$ is a measure on $([0,2), \mathcal{A}^*)$

**Proof.**

1. We have

$$\begin{aligned} \mu^*(a,b) &= \mu[0,1) \quad \text{if } a,b \in [0,1) \\ \mu^*(a,b) &= \mu[1,2) \quad \text{if } a,b \in [1,2) \\ \mu^*(a,b) &= \mu[0,2) \quad \text{if } a \in [0,1), b \in [1,2) \end{aligned}$$

In the case of a singleton $\{a\}$ the best posibble cover is always either $[0,1)$ or $[1,2)$ so that $\mu^*\{a\} = 1/2$.

2. Suppose that $(0,1) \in \mathcal{A}^*$ then we would have that

$$\{0\} = [0,1) - (0,1) \in \mathcal{A}^*.$$

But this gives

$$\frac{1}{2} = \mu^*[0,1) = \mu^*(0,1) + \mu^*\{0\} = 1$$

$\square$

# 2 10-10-2014

## 2.1 Measure Theory Chapter 7

**2.1 Definition.**
Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be two measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-*measureable* (or *measurable* unlesss this is too amiguous) if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A} \qquad \forall A' \in \mathcal{A}'.$$

We often denote this by $T^{-1}(\mathcal{A}') \subseteq \mathcal{A}'$.

**2.2 Definition.**
A *random variable* is a measurable map from a probability space (i.e. $\mu(X) = 1$) to any measurable space.

**2.3 Lemma 7.2.**
*Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable if and only if*

$$T^{-1}(G') \in \mathcal{A} \qquad \forall G' \in \mathcal{G}'.$$

**2.4 Problem 7.1.**
*Show that*

$$\tau_x : \mathbb{R}^n \to \mathbb{R}^n : B \mapsto B - x$$

*is a $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$ measurable map.*

**Proof.**
Showing that

$$\tau_x : \mathcal{B}(\mathbb{R}^n) \to \mathcal{B}(\mathbb{R}^n) : B \mapsto B - x$$

is $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$ measurable, is equivalent with showing that

$$\tau_x^{-1}(B) \in \mathcal{B}(\mathbb{R}^n) \qquad \forall B \in \mathcal{B}(\mathbb{R}^n),$$

which in turn is equivalent with showing that

$$x + B \in \mathcal{B}(\mathbb{R}^n) \qquad \forall B \in \mathcal{J}(\mathbb{R}^n),$$

which in turn is equivalent with showing that

$$x + [a, b) \in \mathcal{B}(\mathbb{R}^n) \qquad \forall a, b \in \mathbb{R}^n.$$

This follows as $x + [a, b) = [x + a, x + b) \in \mathcal{J}(\mathbb{R}^n) \subseteq \mathcal{B}(\mathbb{R}^n)$. □

**2.5 Theorem.**
*Every continuous map $T : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{B}^n/\mathcal{B}^m$ measurable.*

**Proof.**
Showing that

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

is $\mathcal{B}^n/\mathcal{B}^m$ measurable, is equivalent with showing that

$$T^{-1}(\mathcal{O}^m) \subseteq \mathcal{B}^n.$$

As $\mathcal{O}^n \subseteq \sigma(\mathcal{O}^n) = \mathcal{B}^n$, it suffices to show that

$$T^{-1}(\mathcal{O}^m) \subseteq \mathcal{O}^n,$$

which follows from the continuity of $T$. □

**2.6 Definition.**
Let $(T_i)_{i\in I}$ be arbitrarily many mappings $T_i : X \to X_i$ from the same space $X$ into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest $\sigma$-algebra on $X$ that makes all $T_i$ simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma\Big(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\Big).$$

We say that $\sigma(T_i : i \in I)$ is *generated by the family* $(T_i)_{i\in I}$.

**2.7 Theorem.**
*Let $(X_j, A_j), j = 1, 2, 3,$ be measurable spaces and $T : X_1 \to X_2, S : X_2 \to X_3$ be $\mathcal{A}_1/\mathcal{A}_2-$ resp. $\mathcal{A}_2/\mathcal{A}_3$-measurable maps. Then $S \circ T : X_1 \to X_3$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable.*

**2.8 Problem 7.4.**

*Let $X$ be a set, $(X_i, \mathcal{A}_i), i \in I$,be arbitrarily many measurable spaces, and $T_i : X \to X_i$ be a family of maps. Show that a map $f$ from a measurable space $(F, \mathcal{F})$ to $(X, \sigma(T_i : i \in I)$ is measurable if, and only if, all maps $T_i \circ f$ are $\mathcal{F}/\mathcal{A}_i$-measurable.*

**Proof of $\Longrightarrow$.**

To show that all maps $T_i \circ f$ are $\mathcal{F}/\mathcal{A}_i$-measurable, it suffices to show that $T_i : X \to X_i$ is $\sigma(T_i : i \in I)/\mathcal{A}_i$-measurable and $f : F \to X$ is $\mathcal{F}/\sigma(T_i : i \in I)$-measurable.

By hypothesis, is suffices to show that $T_i : X \to X_i$ is $\sigma(T_i : i \in I)/\mathcal{A}_i$-measurable, which is equivalent with showing that

$$T_i^{-1}(A_i) \in \sigma(T_i : i \in I) \qquad \forall A_i \in \mathcal{A}_i.$$

It suffices to assume $A_i \in \mathcal{A}_i$ and show that

$$T_i^{-1}(A_i) \in \bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \qquad \checkmark.$$

$\square$

**Proof of $\Longleftarrow$.**

To show that a map $f$ from a measurable space $(F, \mathcal{F})$ to $(X, \sigma(T_i : i \in I)$ is measurable, it suffices to show that

$$f^{-1}(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)) \subseteq \mathcal{F}$$

To show this it suffices to show that

$$\bigcup_{i \in I} f^{-1}(T_i^{-1}(\mathcal{A}_i)) \subseteq \mathcal{F},$$

to show this it suffices to show that

$$f^{-1}(T_i^{-1}(\mathcal{A}_i)) \subseteq \mathcal{F},$$

to show this it suffices to show that

$$(T_i \circ f)^{-1}(\mathcal{A}_i) \subseteq \mathcal{F}.$$

This follows by hypothesis. $\square$

**2.9 Problem 7.8.**
*Let $T : X \to Y$ be any map. Show that*

$$T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G}))$$

*holds for arbitrary families of $\mathcal{G}$ of subsets of $Y$.*

**Proof.**
To show that

$$T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G}))$$

it suffices to show:

1. $T^{-1}(\sigma(\mathcal{G})) \subseteq \sigma(T^{-1}(\mathcal{G}))$
2. $\sigma(T^{-1}(\mathcal{G})) \subseteq T^{-1}(\sigma(\mathcal{G}))$

To show

$$T^{-1}(\sigma(\mathcal{G})) \subseteq \sigma(T^{-1}(\mathcal{G})),$$

it suffices to show that $T$ is $\sigma(T^{-1}(\mathcal{G}))/\sigma(\mathcal{G})$ measurable.

To show that it suffices to show that

$$T^{-1}(\mathcal{G}) \subseteq \sigma(T^{-1}(\mathcal{G})) \qquad \checkmark.$$

To show

$$\sigma(T^{-1}(\mathcal{G})) \subseteq T^{-1}(\sigma(\mathcal{G})),$$

it suffices to show that

$$T^{-1}(\mathcal{G}) \subseteq T^{-1}(\sigma(\mathcal{G})) \qquad \checkmark.$$

$\square$

## 2.2 Measure Theory Chapter 5

**2.10 Definition.**
A family $\mathcal{D} \subseteq \mathcal{P}(X)$ is a *Dynkin system* if

$$X \in \mathcal{D}$$
$$D \in \mathcal{D} \Longrightarrow D^c \in \mathcal{D}$$
$$(D_j)_{j\in\mathbb{N}} \subseteq \mathcal{D} \text{ pairwise disjoint } \Longrightarrow \bigcup_{j\in\mathbb{N}} D_j \in \mathcal{D}$$

**2.11 Definition.**
Let $\mathcal{G} \subseteq \mathcal{P}(X)$. Then there is a smallest Dynkin system $\delta(\mathcal{G})$ containing $\mathcal{G}$. $\delta(\mathcal{G})$ is called the *Dynkin system generated by* $\mathcal{G}$.

**2.12 Proposition.**
*Show that*

$$\mathcal{G} \subseteq \delta(\mathcal{G}) \subseteq \sigma(\mathcal{G}).$$

**Proof.**
We have that $\mathcal{G} \subseteq \sigma(\mathcal{G})$. And therefore $\delta(\mathcal{G}) \subseteq \delta(\sigma(\mathcal{G})) = \sigma(\mathcal{G})$. $\square$

**2.13 Theorem.**
*A Dynkin system $\mathcal{D}$ is a $\sigma$-algebra if, and only if, it is stable under finite intersections: $D, E \in \mathcal{D} \Longrightarrow D \cap E \in \mathcal{D}$*

**Proof.**
It suffices to show that a $\cap$-stable Dynkin system is stable under countable unions. To show this, it suffices to show that given $(D_j)_{j\in\mathbb{N}} \in \mathcal{D}$, we have

$$D := \bigcup_{j\in\mathbb{N}} D_j \in \mathcal{D}.$$

Set $E_1 = D_1 \in \mathcal{D}$. And $E_2 := D_2 \cap D_1^c$. And $E_3 = D_3 \cap D_2^c \cap D_1^c$. And so on. Then

$$D = \bigcup_{j\in\mathbb{N}} E_j \in \mathcal{D}.$$

$\square$

**2.14 Theorem.**
*If $\mathcal{G} \subseteq \mathcal{P}(X)$ is stable under finite intersections, then $\delta(\mathcal{G}) = \sigma(\mathcal{G})$.*

**Proof.**
It suffices to show that $\sigma(\mathcal{G}) \subseteq \delta(\mathcal{G})$. As $\mathcal{G} \subseteq \delta(\mathcal{G})$ it suffices to show that $\delta(\mathcal{G})$ is a $\sigma$-algebra. To show that $\delta(\mathcal{G})$ is a $\sigma$-algebra, it suffices to show that $\delta(\mathcal{G})$ is stable under finite intersections.

Fix $D \in \delta(G)$. Consider $\mathcal{D}_D := \{Q \subseteq X : Q \cap D \in \delta(\mathcal{G})\}$. It suffices to show that $\delta(\mathcal{G}) \subseteq \mathcal{D}_D$. To show that it suffices to show that $\mathcal{D}_D$ is a Dynkin system and that $\mathcal{G} \subseteq \mathcal{D}_D$.

To show that $\mathcal{G} \subseteq \mathcal{D}_D$, it suffices to show that

$$G \cap D \in \delta(\mathcal{G}) \qquad \forall G \in \mathcal{G},$$

to show that it suffices to show that

$$\delta(\mathcal{G}) \subseteq \mathcal{D}_G \qquad \forall G \in \mathcal{G},$$

to show that it suffices to show that (as $\mathcal{D}_G$ is a dynkin system)

$$\mathcal{G} \subseteq \mathcal{D}_G \qquad \forall G \in \mathcal{G}.$$

This follows from $\mathcal{G} \subseteq \delta(\mathcal{G})$ and $\mathcal{G}$ is $\cap$-stable. $\square$

**2.15 Proposition.**

$$A_j \uparrow A \Longrightarrow A_j \cap B \uparrow A \cap B$$

**Proof.**
To show that

$$A_j \uparrow A \Longrightarrow A_j \cap B \uparrow A \cap B,$$

it suffices to show that

$$A = \bigcup_j A_j \Longrightarrow A \cap B = \bigcup_j A_j \cap B,$$

which is equivalent with showing that

$$\left(\bigcup_j A_j\right) \cap B = \bigcup_j A_j \cap B \qquad \checkmark.$$

$\square$

**2.16 Definition.**
An *exhausting sequence* $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ is an increasing sequence of sets $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ such that $\bigcup_{j\in\mathbb{N}} A_j = X$.

**2.17 Theorem.**
*Assume that $(X, \mathcal{A})$ is a measurable space and that $\mathcal{A} = \sigma(\mathcal{G})$ is generated by a family $\mathcal{G}$ such that*

- *$\mathcal{G}$ is stable under finite intersections $G, H \in \mathcal{G} \Longrightarrow G \cap H \in \mathcal{G}$*
- *there exists an exhausting sequence $(G_j)_{j\in\mathbb{N}} \subseteq \mathcal{G}$ with $G_j \uparrow X$*

*Any two measure $\mu, \nu$ that coincide on $\mathcal{G}$ and are finite for all members of the exhausting sequence $\mu(G_j) = \nu(G_j) < \infty$, are equal on $\mathcal{A}$, i.e.*

$$\mu(A) = \nu(A) \qquad \forall A \in \mathcal{A}.$$

**Proof.**
Remember that for any increasing sequence $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ with $A_j \uparrow A \in \mathcal{A}$ we have

$$\mu(A) = \lim_{j\in\infty} \mu(A_j).$$

To show that

$$\mu(A) = \nu(A) \qquad \forall A \in \mathcal{A}$$

it suffices to show that (as $G_j \cap A \uparrow X \cap A$)

$$\lim_{j\in\infty} \mu(G_j \cap A) = \lim_{j\in\infty} \mu(G_j \cap A) \qquad \forall A \in \mathcal{A}$$

To show that it suffices to show that

$$\mu(G_j \cap A) = \nu(G_j \cap A) \qquad \forall j \in \mathbb{N}, \quad \forall A \in \mathcal{A}.$$

Consider $\mathcal{D}_j := \{A \in \mathcal{A} : \mu(G_j \cap A) = \nu(G_j \cap A)\}$. It suffices to show that $\mathcal{A} \subseteq \mathcal{D}_j$, which is equivalent with showing $\sigma(\mathcal{G}) \subseteq \mathcal{D}_j$.

As $\mathcal{G}$ is stable under finite intersections, it suffices to show that $\delta(\mathcal{G}) \subseteq \mathcal{D}_j$.

As $\mathcal{G}$ is stable under finite intersections and $\mu(\mathcal{G}) = \nu(\mathcal{G})$, we have that $\mathcal{G} \subseteq D_j$ and therefore it suffices to show that $\mathcal{D}_j$ is a Dynkin system.

Which you can check. □

**2.18 Theorem.**
*The $n$-dimensional Lebesgue measure $\lambda^n$ is invariant under translations, i.e.*

$$\lambda^n(x + B) = \lambda^n(B) \qquad \forall x \in \mathbb{R}^n, \forall B \in \mathcal{B}(\mathbb{R}^n).$$

**Proof.**
Set $\nu(B) := \lambda^n(x + B)$ for some fixed $x \in \mathbb{R}^n$. It suffices to show that

$$\lambda^n(B) = \nu(B) \qquad B \in \mathcal{B}.$$

To show that, it suffices to show that

1. $\mathcal{J}$ is $\cap$-stable $\checkmark$
2. $\mathcal{J}$ admits an exhausting sequence
   - $[-j, j) \uparrow \mathbb{R}^n$ $\checkmark$
3. $\lambda^n|_{\mathcal{J}} = \nu|_{\mathcal{J}}$

$$\begin{aligned} v([a,b)) &= \lambda^n[x+a, x+b) \\ &= \lambda^n[a,b) \end{aligned}$$

4. $\nu$ is a measure on $\mathcal{B}^n$

To show that $\nu$ is a measure on $\mathcal{B}^n$, it suffices to show that

$$\nu(\bigcup_{j\in\mathbb{N}} B_j) = \sum_{j\in\mathbb{N}} \nu(B_j),$$

which is equivalent with

$$\lambda^n(x + \bigcup_{j\in\mathbb{N}} B_j) = \sum_{j\in\mathbb{N}} \lambda^n(x + B_j).$$

It suffices to show

$$B \in \mathcal{B}^n \Longrightarrow x + B \in \mathcal{B}^n.$$

Which we have already proven.

$\square$

**2.19 Theorem.**
*Let $(X, \mathcal{A}), (X, \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$ measurable map. For every measure $\mu$ on $(X, \mathcal{A})$,*

$$\mu'(A') := \mu(T^{-1}(A')), \qquad A' \in \mathcal{A}'.$$

*The measure $\mu'$ is called the* image measure *of $\mu$ under $T$ and is denoted by $T \circ \mu$ or $\mu \circ T^{-1}$.*

## 2.3 Measure Theory Chapter 7

**2.20 Problem 7.7.**
*Use image measures to give a new proof of Problem 5.8, i.e. to show that*

$$\lambda^n(t \cdot B) = t^n \lambda^n(B) \qquad \forall B \in \mathcal{B}(\mathbb{R}^n), \forall t > 0$$

**Proof.**
Set $\nu(B) := t^n \lambda^n(B)$ for some fixed $x \in \mathbb{R}^n$. It suffices to show that

$$\lambda^n(tB) = \nu(B) \qquad \forall B \in \mathcal{B}.$$

To show that, it suffices to show that

1. $\mathcal{J}$ is $\cap$-stable ✓
2. $\mathcal{J}$ admits an exhausting sequence
   - $[-j, j) \uparrow \mathbb{R}^n$ ✓
3. $\lambda^n|_{\mathcal{J}} = \nu|_{\mathcal{J}}$

$$\begin{aligned} \nu([a,b)) &= \lambda^n[ta, tb) \\ &= t^n \lambda^n[a, b) \end{aligned}$$

4. $\nu$ is a measure on $\mathcal{B}^n$ as it is a composition of the inverse of a measurable map and a measure.

$\square$

# 3 11-10-2014

## 3.1 Measure Theory Chapter 8

**3.1 Definition.**
Note that: $u^{-1}[a,\infty) = \{x \in X : u(x) \in [a,\infty)\} = \{x \in X : u(x) \geq a\}$. We define:

$$\{u(x) \geq a\} = u^{-1}[a,\infty).$$

**3.2 Theorem.**
*Let $(X,\mathcal{A})$ be a measurable space. The function $u : X \to \mathbb{R}$ is $\mathcal{A}/\mathcal{B}$-measurable if, and only if, one, hence all, of the following conditions hold*

1. $\{u \geq a\} \in \mathcal{A} \qquad \forall a \in \mathbb{R}$
2. $\{u > a\} \in \mathcal{A} \qquad \forall a \in \mathbb{R}$
3. $\{u \leq a\} \in \mathcal{A} \qquad \forall a \in \mathbb{R}$
4. $\{u < a\} \in \mathcal{A} \qquad \forall a \in \mathbb{R}$

**3.3 Definition.**
We define the *extended real line* $\bar{\mathbb{R}} := [-\infty,\infty]$ with the following rules for all $x \in \mathbb{R}$:

$$\begin{aligned} x + \infty = \infty + x &= \infty \qquad & x + -\infty = -\infty + x = -\infty \\ \infty + \infty &= \infty \qquad & -\infty - \infty = -\infty \end{aligned}$$

And for $x \in (0,\infty]$ :

$$\begin{aligned} \pm x \cdot \infty &= \infty \cdot \pm x = \pm\infty \\ \pm x \cdot -\infty &= -\infty \cdot \pm x = \mp\infty \\ 0 \cdot \pm\infty &= \pm\infty \cdot 0 = 0 \\ \frac{1}{\pm\infty} &= 0 \end{aligned}$$

**3.4 Definition.**
Functions which take values in $\bar{\mathbb{R}}$ are called *numerical functions*.

**3.5 Definition.**
The Borel $\sigma$-algebra $\bar{\mathcal{B}} = \mathcal{B}(\bar{\mathbb{R}})$ is defined by:

$$\bar{\mathcal{B}} := \Big\{ B \cup S : B \in \mathcal{B} \text{ and } S \in \Big\{ \varnothing, \{-\infty\}, \{\infty\}, \{-\infty, \infty\} \Big\} \Big\}$$

**3.6 Theorem.**
*We have $\mathcal{B}(\mathbb{R}) = \mathbb{R} \cap \mathcal{B}(\bar{\mathbb{R}})$. Moreover $\bar{\mathcal{B}}$ is generated by all sets of the form $[a, \infty]$ or $(a, \infty]$ or $[-\infty, a)$ or $[-\infty, a]$ where $a \in \mathbb{R}$*

**3.7 Definition.**
Let $(X, \mathcal{A})$ be a measurable space. We write $\mathcal{M} := \mathcal{M}(\mathcal{A})$ and $\mathcal{M}_{\bar{\mathbb{R}}} := \mathcal{M}_{\bar{\mathbb{R}}}(\mathcal{A})$ for the families of real valued $\mathcal{A}/\mathcal{B}$-measurable and numerical $\mathcal{A}/\bar{\mathcal{B}}$-measurable functions on $X$.

**3.8 Definition.**
A *simple function* $g : X \to \mathbb{R}$ on a measurable space $(X, \mathcal{A})$ is a function of the form

$$g(x) := \sum_{j=1}^{M} y_j \mathbf{1}_{A_j}(x)$$

with finitely many sets $A_1, \ldots, A_m \in \mathcal{A}$ and $y_1, \ldots, y_M \in \mathbb{R}$. The set of simple functions is denoted by $\mathcal{E}$ or $\mathcal{E}(\mathcal{A})$.

If the sets $A_1, \ldots, A_M$ are mutally disjoint we call

$$\sum_{j=0}^{M} y_j \mathbf{1}_{A_j}(x)$$

with $y_0 := 0$ and $A_0 := (A_1 \cup \ldots \cup A_M)^c$ a *standard representation* of $g$. Caution, this representation is not unique.

**3.9 Theorem.**
*Let $(X, \mathcal{A})$ be a measurable space. Every $\mathcal{A}/\bar{\mathcal{B}}$-measurable numerical function $u : X \to \bar{\mathbb{R}}$ is the pointwise limit of simple functions:*

$$u(x) = \lim_{j \to \infty} f_j(x)$$

*where $f_j \in \mathcal{E}(\mathcal{A})$ and $|f_j| \leq |u|$.*

*If $u \geq 0$, all $f_j$ can be chosen to be positive and increasing towards $u$ so that $u = \sup_{j \in \mathbb{N}} f_j$.*

**3.10 Theorem.**
*Let $(X, \mathcal{A})$ be a measurable space. If $u_j : X \to \bar{\mathbb{R}}, j \in \mathbb{N}$ are measurable functions, then so are*

$$\sup_{j\in\mathbb{N}} u_j \quad \inf_{j\in\mathbb{N}} u_j \quad \limsup_{j\to\mathbb{N}} u_j \quad \liminf_{j\to\mathbb{N}} u_j$$

*and whenever it exists*

$$\lim_{j\to\infty} u_j.$$

**3.11 Theorem.**
*Let $u, v$ be $\mathcal{A}/\bar{\mathcal{B}}$-measurable functions. Then the functions*

$$u \pm v \quad uv \quad u \vee v := \max\{u, v\} \quad u \wedge v := \min\{u, v\}$$

*are $\mathcal{A}/\bar{\mathcal{B}}$-measurable (whenever they are defined).*

**3.12 Theorem.**
*A function $u$ is $\mathcal{A}/\mathcal{B}$ measurable if, and only if, $u^{\pm}$ are $\mathcal{A}/\bar{\mathcal{B}}$ measurable.*

**3.13 Theorem.**
*Let $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ be an $\mathcal{A}/\mathcal{A}'$-measurable map and let $\sigma(T) \subseteq \mathcal{A}$ be the $\sigma$-algebra generated by $T$. Then $u = w(T)$ for some $\mathcal{A}'/\bar{\mathcal{B}}$ measurable function $w : X' \to \bar{\mathbb{R}}$ if and only if $u : X \to \bar{\mathbb{R}}$ is $\sigma(T)/\bar{\mathcal{B}}$-measurable.*

**3.14 Proposition.**
*Let $(X, \mathcal{A})$ be a measurable space. We define the indicator function:*

$$1_A : X \to \mathbb{R} : x \in A \mapsto 1 \quad x \in X - A \mapsto 0$$

*Show that the indicator function is measurable if, and only if, $A \in \mathcal{A}$.*

**Proof.**
To show that $1_A$ is measurable, it suffices to show that

$$1_A^{-1}(a, \infty) \in \mathcal{A}.$$

Note that

$$1_A^{-1}(a, \infty) = \{x \in X : 1_A(x) \in (a, \infty)\} = \{1_A > a\}$$

If $a \geq 1$, then $1_A^{-1}(a, \infty) = \varnothing$.

If $a \in [0, 1)$, then $1_A^{-1}(a, \infty) = A$.

If $a < 0$, then $1_A^{-1}(a, \infty) = X$. $\square$

**3.15 Proposition.**
*Let $A_1, \ldots, A_M \in \mathcal{A}$ be mutally disjoint sets and $y_1, \ldots, y_M \in \mathbb{R}$. Then the function*

$$g : X \to \mathbb{R} : x \mapsto \sum_{j=1}^{M} y_j 1_{A_j}(x)$$

*is measurable.*

**Proof.**
To show that $g$ is measurable it suffices to show that

$$\{g > a\} \in \mathcal{A}$$

i.e.

$$\left\{x \in X : \sum_{j=1}^{M} y_j 1_{A_j}(x) > a\right\} = \bigcup_{j:y_j>a} A_j \in \mathcal{A}.$$

$\square$

**3.16 Problem 8.3i.**
*Let $(X, \mathcal{A})$ be a measurable space. Let $f, g : X \to \mathbb{R}$ be measurable functions. Show that for every $A \in \mathcal{A}$ the functions $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$, if $x \notin A$, is measurable.*

**Proof.**
Note that

$$h(x) := 1_A(x)f(x) + 1_{A^c}(x)g(x).$$

And remember that sums and products of measurable functions are again measurable. $\square$

**3.17 Problem 8.3ii.**
*Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of measurable functions and let $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ such that $\bigcup_{j\in\mathbb{N}} A_j = X$. Suppose that $f_j|_{A_j\cap A_k} = f_k|_{A_j\cap A_k}$ for all $j, k \in \mathbb{N}$ and set $f(x) := f_j(x)$ if $x \in A_j$. Show that $f : X \to \mathbb{R}$ is measurable.¡br /¿*

**Proof.**
We have that:

$$f^{-1}(B) = \bigcup_{j\in\mathbb{N}} A_j \cap f^{-1}(B) = \bigcup_{j\in\mathbb{N}} A_j \cap f_j^{-1}(B) \in \mathcal{A}$$

$\square$

**3.18 Problem 8.4.**
*Let $(X, \mathcal{A})$ be a measurable space and let $\mathcal{B} \subset \mathcal{A}$ be a sub-$\sigma$-algebra. Show that $\mathcal{M}(\mathcal{B}) \subset \mathcal{M}(\mathcal{A})$.*

**Proof.**
To show that

$$\mathcal{M}(\mathcal{B}) \subset \mathcal{M}(\mathcal{A})$$

it suffices to show there exists a $\mathcal{A}$-measurable function that is not $\mathcal{B}$-measurable. By hypothesis, we have an element $A \in \mathcal{A}$, that is not in $\mathcal{B}$, i.e. $A \notin \mathcal{B}$. Since $1_A$ is $\mathcal{B}$-measurable if, and only if, $B \in \mathcal{B}$, we have find the $\mathcal{A}$-measurable function where we where looking for. $\square$

**3.19 Theorem.**
*Let $u : \mathbb{R} \to \mathbb{R}$ be differentiable. Explain why $u$ and $u' = du/dx$ are measurable.*

**Proof.**
If $u$ is differentiable, it is continuous, hence measurable. Since $u'$ exists, we can write it in the form

$$u'(x) = \lim_{k \to \infty} \frac{u(x + 1/k) - u(x)}{1/k}$$

i.e. as limit of measurable functions. Thus, $u'$ is also measurable. $\square$

**3.20 Problem 8.17.**
*Show that the measurability of $|u|$ does not, in general, imply the measurability of $u$.*

**Proof.**
Let $A \subseteq \mathbb{R}$ be such that $A \notin \mathcal{B}$. Then it is clear that

$$u(x) := 1_A(x) - 1_{A^c}(x)$$

is not measurable. Take

$$\{u = 1\} = A \notin \mathcal{A}.$$

But $|u(x)| = 1$, which is a continuous function and therefore measurable.

$\square$

**3.21 Problem 8.14.**
*Consider* $(\mathbb{R}, \mathcal{B})$ *and* $u : \mathbb{R} \to \mathbb{R}$. *Show that* $\{x\} \in \sigma(u)$ *for all* $x \in \mathbb{R}$ *if, and only if,* $u$ *is injective.*

**Proof.**
To show that $u$ is injective, it suffices to assume $x, y \in \mathbb{R}$ and show that

$$u(x) = u(y) \Longrightarrow x = y.$$

Showing that is equivalent with showing that

$$|\{u = u(x_0)\}| = 1.$$

We surely have that $\{x_0\} \subseteq \{u = u(x_0)\}$. And note that

$$\{x_0\} \in \sigma(u) = \sigma(u^{-1}(\mathcal{B}))$$

just means that $\{x_0\} = u^{-1}(B)$ for some $B \in \mathcal{B}$. □

**Proof.**
Assume that $u$ is injective,. This means that every point in the range $u(\mathbb{R})$ comes exactly from unique defined $x \in \mathbb{R}$. This can be expressed by saying that $\{x\} = u^{-1}(\{u(x)\}) = \{u(x)\}$. But then

$$\{x\} \in \sigma(u) = \sigma(u^{-1}(\mathcal{B})).$$

□

# 4 12-10-2014

## 4.1 Measure Theory Chapter 9

**4.1 Definition.**
Let $f = \sum_{j=0}^{M} y_j 1_{A_j} \in \mathcal{E}^+$ be a simple function in standard representation. Then the number

$$I_\mu(f) := \sum_{j=0}^{M} y_j \mu(A_j) \in [0, \infty]$$

is called the ($\mu$-)integral of $f$.

**4.2 Theorem.**

1. $I_\mu(1_A) = \mu(A) \qquad \forall A \in \mathcal{A}$
2. $I_\mu(\lambda f) = \lambda I_\mu(f) \qquad \forall \lambda \geq 0$
3. $I_\mu(f + g) = I_\mu(f) + I_\mu(g)$
4. $f \leq g \Longrightarrow I_\mu(f) \leq I_\mu(g)$

**4.3 Definition.**
Let $(X, \mathcal{A}, \mu)$ be a measure space. The ($\mu$-)integral of a positive numerical function $u \in \mathcal{M}_{\bar{\mathbb{R}}}^+$ is given by

$$\int u d\mu := \sup\{I_\mu(g) : g \leq u, g \in \mathcal{E}^+\} \in [0, \infty].$$

If we need to emphasize the integration variable, we also write

$$\int u(x)\mu(dx) \quad \text{or} \quad \int u(x) d\mu(x)$$

**4.4 Theorem.**
*For all $f \in \mathcal{E}^+$ we have $\int f du = I_\mu(f)$.*

**4.5 Theorem.**
*Let $(X, A, \mu)$ be a measure space. For an increasing sequence of numerical functions $(u_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}_{\bar{\mathbb{R}}}^+, 0 \leq u_j \leq u_{j+1} \leq \ldots$, we have $u := \sup_{j \in \mathbb{N}} u_j \in \mathcal{M}_{\bar{\mathbb{R}}}^+$ and*

$$\int \sup_{j \in \mathbb{N}} u_j d\mu = \sup_{j \in \mathbb{N}} \int u_j d\mu$$

**4.6 Theorem.**
*Let $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}$. Then*

$$\int u d\mu = \lim_{j\to\infty} \int f_j d\mu$$

*holds for every increasing sequence $(f_j)_{j\in\mathbb{N}} \subseteq \mathcal{E}^+$ with $\lim_{j\to\infty} f_j = u$.*

**4.7 Theorem.**
*Let $u, v \in \mathcal{M}^+_{\bar{\mathbb{R}}}$. Then*

1. $\int 1_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}$
2. $\int \alpha u d\mu = \alpha \int u d\mu$
3. $\int (u+v) d\mu = \int u d\mu + \int v d\mu$
4. $u \leq v \Longrightarrow \int u d\mu \leq \int v d\mu$

**4.8 Theorem.**
*Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}^+_{\bar{\mathbb{R}}}$. Then $\sum_{j=1}^{\infty} u_j$ is measurable and we have*

$$\int \sum_{j=1}^{\infty} u_j d\mu = \sum_{j=1}^{\infty} \int u_j d\mu$$

**4.9 Theorem.**
*Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}^+_{\bar{\mathbb{R}}}$ be a sequence of positive measurable numerical functions. Then $u := \liminf_{j\in\infty} \int u_j d\mu$ is measurable and*

$$\int \liminf_{j\to\infty} u_j d\mu \leq \liminf_{j\to\infty} \int u_j d\mu$$

**4.10 Problem 9.1.**
*Let $f : X \to \mathbb{R}$ be a positive simple function of the form*

$$f(x) = \sum_{j=1}^{m} \xi_j 1_{A_j}(x) \qquad \xi_j \geq 0, A_j \in \mathcal{A}.$$

*Show that*

$$I_\mu(f) = \sum_{j=1}^{m} \xi_j \mu(A_j)$$

**Proof.**

$$I_\mu(f) = I_\mu\left(\sum_{j=1}^m \xi_j 1_{A_j}\right) = \sum_{j=1}^m \xi_j I_\mu(1_{A_j}) = \sum_{j=1}^m \xi_j \mu(A_j)$$

□

**4.11 Problem 9.5.**
*Let $(X, \mathcal{A}, \mu)$ be a measure space and $u \in \mathcal{M}^+(\mathcal{A})$. Show that the set-function*

$$A \mapsto \int 1_A u d\mu \quad A \in \mathcal{A}$$

*is a measure.*

**Proof.**
Set

$$\nu : \mathcal{A} \to [0, \infty] : A \mapsto \int 1_A u d\mu.$$

1. To show that $\nu(\varnothing) = 0$. Notice that $1_\varnothing \equiv 0$.
2. Let $A = \bigcup_{j\in\mathbb{N}} A_j$ a disjoint union of sets $A_j \in \mathcal{A}$. Note that

   $$\sum_{j=1}^\infty 1_{A_j} = 1_A$$

   We have to show that

   $$\begin{aligned} \nu(\bigcup_{j\in\mathbb{N}} A_j) &= \int \left(\sum_{j=1}^\infty 1_{A_j}\right) \cdot u d\mu \\ &= \int \left(\sum_{j=1}^\infty 1_{A_j} u\right) d\mu \\ &= \sum_{j=1}^\infty \int 1_{A_j} u d\mu \\ &= \sum_{j=1}^\infty \nu(A_j). \end{aligned}$$

□

**4.12 Problem 9.8.**
*Let $(X, \mathcal{A}, \mu)$ be a measure space and $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}^+(\mathcal{A})$. If $u_j \leq u$ for all $j \in \mathbb{N}$ and some $u \in \mathcal{M}^+(\mathcal{A})$ with $\int u d\mu < \infty$, then*

$$\limsup_{j\in\infty} \int u_j d\mu \leq \int \limsup_{j\in\infty} u_j d\mu.$$

**Proof.**
Showing that

$$\limsup_{j\in\infty} \int u_j d\mu \leq \int \limsup_{j\in\infty} u_j d\mu$$

is equivalent with showing that

$$-\liminf_{j\in\infty} \int -u_j d\mu \leq -\int \liminf_{j\in\infty} -u_j d\mu$$

which is equivalent with showing that

$$\int \liminf_{j\in\infty} -u_j d\mu \leq \liminf_{j\in\infty} \int -u_j d\mu$$

which is equivalent with showing that

$$\int u d\mu + \int \liminf_{j\in\infty} -u_j d\mu \leq \int u d\mu + \liminf_{j\in\infty} \int -u_j d\mu$$

which is equivalent with showing that

$$\int u + \liminf_{j\in\infty} -u_j d\mu \leq \liminf_{j\in\infty} \left( \int u d\mu + \int -u_j d\mu \right)$$

which is equivalent with showing that

$$\int \liminf_{j\in\infty} \left(u - u_j\right) d\mu \leq \liminf_{j\in\infty} \left( \int u - u_j d\mu \right).$$

By hypothesis, $u_j \leq u$. So we have that $u - u_j$ is a sequence of postive measurable functions and therefore our last statement follows by the theorem of Fatou. □

## 4.2 Measure Theory Chapter 7

**4.13 Proposition.**
*Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable if and only if $T^{-1}(\mathcal{G}') \subseteq \mathcal{A}$.*

It suffices to show assume $T^{-1}(\mathcal{G}') \subseteq \mathcal{A}$ and show that

$$T^{-1}(\mathcal{A}) \subseteq \mathcal{A}.$$

Consider $\Sigma := \{A' \subseteq X' : T^{-1}(A') \in \mathcal{A}\}$. We have that $\mathcal{G}' \subseteq \Sigma$. It suffices to show that

$$\mathcal{A}' \subseteq \Sigma.$$

It suffices to show that $\Sigma$ is a $\sigma$-algebra.

1. To show that $X' \in \Sigma$, it suffices to show that $T^{-1}(X') \in \mathcal{A}$.
2. Showing that

   $$A' \in \Sigma \Longrightarrow A'^c \in \Sigma$$

   is equivalent with showing that

   $$T^{-1}(A') \in \mathcal{A} \Longrightarrow T^{-1}(A'^c) \in \mathcal{A} \qquad \checkmark$$

3. Showing that

   $$(A'_j)_{j\in\mathbb{N}} \subseteq \Sigma \Longrightarrow \bigcup_{j\in\mathbb{N}} A_j \in \Sigma$$

   is equivalent with showing that

   $$T^{-1}(A'_j) \in \mathcal{A} \Longrightarrow T^{-1}\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) \in \mathcal{A} \qquad \checkmark$$

**4.14 Proposition.**
*Let $(X, \mathcal{A}), (X, \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$-measurable map. For every measure $\mu$ on $(X, \mathcal{A})$,*

$$\mu'(A') := T(\mu)(A') := \mu(T^{-1}(A')), \qquad A' \in \mathcal{A}'$$

*defines a measure on $(X', \mathcal{A}')$.*

**Proof.**

1. To show that
$$\mu(T^{-1}(\varnothing)) = 0 \qquad \checkmark$$

2. Assume $(A'_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}'$ mutually disjoint sets and show that
$$\mu'(\bigcup_{j\in\mathbb{N}} A'_j) = \sum_{j\in\mathbb{N}} \mu'(A'_j),$$
which is equivalent with showing that
$$\mu(T^{-1}\Big(\bigcup_{j\in\mathbb{N}} A'_j\Big) = \sum_{j\in\mathbb{N}} \mu(T^{-1}(A'_j)),$$
which is equivalent with showing that
$$\mu(T^{-1}\Big(\bigcup_{j\in\mathbb{N}} A'_j\Big) = \mu\Big(\bigcup_{j\in\mathbb{N}} T^{-1}(A'_j)\Big) \qquad \checkmark.$$

□

**4.15 Problem 7.9i.**
*Let $\mu$ be a measure on $(\mathbb{R}, \mathcal{B})$. Show that*

$$F_\mu(x) := \begin{cases} \mu[0,x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu[x,0) & \text{if } x < 0 \end{cases}$$

1. *is monotonically increasing*
2. *left-continuous funciton*

**Proof.**

1. Showing that $F_\mu$ is monotonically increasing is equivalent with showing that
$$x \le y \Longrightarrow F_\mu(x) \le F_\mu(y).$$

(a) $x \leq 0 \leq y$ :. Then $F_\mu(x) = -\mu[x,0) \leq 0$ and $F_\mu(y) = \mu[0,y) \geq 0$.

(b) $0 < x \leq y$ : Then $[0,x) \subseteq [0,y)$. And $\mu[0,x) \leq \mu[0,y)$.

(c) $x \leq y < 0$ : Then $[y,0) \subseteq [x,0)$. And $\mu[y,0) \leq \mu[x,0)$.

2. Showing that $F_\mu$ is left continuous is equivalent with assuming $(x_k)$ a sequence such that $x_k < x$ and $x_k \uparrow x$ and showing that

$$\lim_{k\to\infty} F_\mu(x_k) = F_\mu(x).$$

If if $x > 0$, it suffices to show that

$$\lim_{k\to\infty} \mu[0,x_k) = \mu[0,x).$$

If $x < 0$ it suffices to show that

$$\lim_{k\to\infty} -\mu[x_k,0) = -\mu[x,0).$$

If $x = 0$ it suffices to show that

$$\lim_{k\to\infty} -\mu[x_k,0) = 0.$$

Remember that:

1. For any increasing sequence $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ with $A_j \uparrow A \in \mathcal{A}$ we have

$$\mu(A) = \mu(\cup A_j) = \lim_{j\in\infty} \mu(A_j)$$

2. For any decreasing sequence $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ with $A_j \downarrow A \in \mathcal{A}$ we have

$$\mu(A) = \mu(\cap A_j) = \lim_{j\in\mathbb{N}} \mu(A_j)$$

$\square$

**4.16 Problem 7.9ii.**

*Let $F : \mathbb{R} \to \mathbb{R}$ be a Stieltjes function. Show that*

$$\nu_F[a,b) = F(b) - F(a) \qquad \forall a,b \in \mathbb{R}, a < b$$

*has a unique extension to a measure on $\mathcal{B}$.*

**Proof.**
By theorem 6.1 it suffices to show that $\nu_F$ is a pre-measure. To show this it suffices to show that

1. $\nu_F(\varnothing) = v_F[a,a) = 0$
2. $\nu_F([a,b) \cup [b,c)) = \nu_F([a,b)) + \nu_F([b,c))$

   •
$$\begin{aligned} \nu_F([a,b)) + \nu_F([b,c)) &= F(b) - F(a) + F(c) - F(b) \\ &= F(c) - F(a) \\ &= \nu_F[a,c) \\ &= \nu_F([a,b) \cup [b,c)) \end{aligned}$$

3. For any decreasing sequence $[a_j, b_j)_{j\in\mathbb{N}} \subseteq \mathcal{J}$ with $[a_j, b) \downarrow [a,b) \in \mathcal{J}$ we have
$$\nu_F([a,b)) = \lim_{j\in\infty} \nu_F[a_j, b).$$
   This last statement is equivalent with
$$F(b) - F(a) = \lim_{j\in\infty}(F(b) - F(a_j)).$$
   Note that since $[a_j, b_j) \downarrow [a,b) \in \mathcal{J}$ we have that $a_j \uparrow a, a_j \leq a$ and therefore
$$\lim_{j\in\infty}(F(b) - F(a_j)) = F(b) - F(a),$$
   as $F$ is left-continous.
4. $\mathcal{J}$ contains an exhausting sequence $[a_j, b_j)$ such that $[a_j, b_j) \uparrow \mathbb{R}$ and $v_F[a_j, b_j) < \infty$

$\square$

# 5 13-10-2014

## 5.1 Markov Chains 1.2

**5.1 Definition.**
We say that $i$ *leads to* $j$ and write $i \to j$ if

$$\mathbb{P}_i(X_n = j \text{ for some } n \geq 0) > 0.$$

**5.2 Definition.**
We say $i$ *communicates with* $j$ and write $i \leftrightarrow j$ if both $i \to j$ and $j \to i$.

**5.3 Theorem.**
*For distinct states $i$ and $j$ the following are equivalent:*

1. $i \to j \iff \mathbb{P}_i(X_n = j \textit{ for some } n \geq 0) > 0$
2. $p_{i_1 i_2} p_{i_2 i_3} \dots p_{i_{n-1} i_n} > 0$ *for some states* $i_1, \dots, i_n$ *with* $i_1 = i$ *and* $i_n = j$
3. $p_{ij}^n > 0$ *for some* $n \geq 0$

**Proof.**
Remember that

$$p_{ij}^n = \mathbb{P}_i(X_n = j) = \sum_{i_2, \dots, i_{n-1}} p_{i i_2} p_{i_2 i_3} \dots p_{i_{n-1} j}.$$

From this everything follows. $\square$

**5.4 Proposition.**
*Show that $i \to i$.*

**Proof.**

$$i \to i \iff \mathbb{P}_i(X_n = i \text{ for some } n \geq 0) > 0$$

This follows, as $\mathbb{P}_i(X_0 = i) = 1$. $\square$

**5.5 Definition.**
The equivalence classes of the equivalence relations $\leftrightarrow$ are called *communicating classes*. We say that a class $C$ is closed if

$$i \in C, i \to j \implies j \in C$$

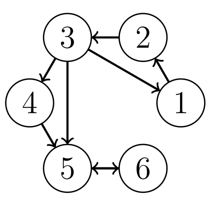
**5.6 Definition.**
A state $i$ is *absorbing* if $\{i\}$ is a closed class.

**5.7 Definition.**
A chain or transition matrix $P$ where $I$ is a single class is called *irreducible*.

**5.8 Example 1.2.2.**
*Find the communicating classes associated to the stochastic matrix*

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

**Proof.**
The solution is obvious from the diagram. The classes being $\{1, 2, 3\}, \{4\}$ and $\{5, 6\}$. With only $\{5, 6\}$ closed.

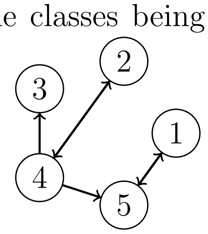



$\square$

**5.9 Exercise 1.2.1.**
*Identify the communicating classes of the following transition matrix:*

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

The solution is obvious from the diagram. The classes being $\{1, 5\}, \{2, 4\}$ and $\{3\}$. With $\{1, 5\}$ closed and $\{3\}$ absorbing.



## 5.2 Markov Chains 1.3

**5.10 Definition.**
Let $X_n$ be a Markov chain with transition matrix $P$. The *hitting time* of a subset $A$ of $I$ is the random variable

$$H^A : \Omega \to \{0, 1, 2, \dots, \infty\}$$

given by

$$H^A(\omega) = \inf\{n \geq 0 : X_n(\omega) \in A\}$$

where we agree that the infimum of the empty set $\varnothing$ is $\infty$. The probability starting from $i$ that $(X_n)_{n\geq 0}$ ever hits $A$ is then

$$h_i^A = \mathbb{P}_i(H^A < \infty).$$

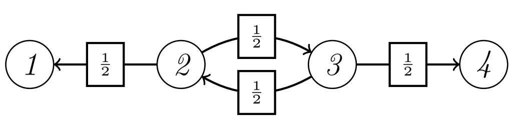
When $A$ is a closed class, $h_i^A$ is called the *absorption probability.* The mean time taken for $X_n$ to reach $A$ is given by

$$k_i^A = E_i(H^A) = \sum_{n<\infty} n\mathbb{P}(H^A = n) + \infty\mathbb{P}(H^A = \infty).$$

We shall often write less formally

$$h_i^A = \mathbb{P}_i(\text{hit } A) \qquad k_i^A = E_i(\text{time to hit } A).$$

**5.11 Example 1.3.1.**
*Consider the chain with following diagram:*




1. *Starting from 2, what is the probability of absorption in* 4*?*
2. *How long does it take until the chain is absorbed in* 1 *or* 4 *?*

**Proof.**

1. Note that $A = \{4\}$ is a closed class. The aborption probability is defined as
$$h_i := h_i^{\{4\}} = \mathbb{P}_i(\text{hit } 4).$$

We have

$$\begin{aligned}
h_1 &= 0 \\
h_4 &= 1 \\
h_2 &= \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3 = \tfrac{1}{2}h_3 \\
h_3 &= \tfrac{1}{2}h_2 + \tfrac{1}{2}h_4 = \tfrac{1}{2}h_2 + \tfrac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
& h_2 = \tfrac{1}{4}h_2 + \tfrac{1}{4} \\
\implies & \tfrac{3}{4}h_2 = \tfrac{1}{4} \\
\implies & h_2 = \tfrac{1}{3}
\end{aligned}$$

2. We need to compute

$$k_2 = k_2^{\{1,4\}} = E_2(\text{time to hit } \{1,4\}).$$

We have

$$\begin{aligned}
k_1 &= 0 \\
k_4 &= 0 \\
k_2 &= 1 + \tfrac{1}{2}k_3 \\
k_3 &= 1 + \tfrac{1}{2}k_2
\end{aligned}$$

Hence

$$\begin{aligned}
& k_2 = \tfrac{3}{2} + \tfrac{1}{4}k_2 \\
\implies & \tfrac{3}{4}k_2 = \tfrac{3}{2} \\
\implies & k_2 = 2
\end{aligned}$$

$\square$

**5.12 Theorem 1.3.2.**
*The vector of hitting probabilities $h^A = (h_i^A : i \in I)$ is the minimal non-negative solution to the system*

$$\begin{cases} h_i^A = 1 & \text{for} i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & \text{for } i \notin A. \end{cases}$$

*Minimality means that if $x = (x_i : i \in I)$ is another solution with $x_i \geq 0$ for all $i$, then $x_i \geq h_i$ for all $i$.*

**5.13 Example 1.3.1(continued).**
*Use theorem 1.3.2 to compute $h_2$ again.*

**Proof.**
The vector of hitting probabilities $h^{\{4\}} = \left(h_1^{\{4\}}, h_2^{\{4\}}, h_3^{\{4\}}, h_4^{\{4\}}\right)$ is the minimal non-negative solution to the system

$$\begin{aligned}
h_1^{\{4\}} &= \sum_{j\in I} p_{ij} h_j^{\{4\}} = h_1^{\{4\}} \\
h_2^{\{4\}} &= \sum_{j\in I} p_{ij} h_j^{\{4\}} = \tfrac{1}{2} h_1^{\{4\}} + \tfrac{1}{2} h_3^{\{4\}} \\
h_3^{\{4\}} &= \sum_{j\in I} p_{ij} h_j^{\{4\}} = \tfrac{1}{2} h_2^{\{4\}} + \tfrac{1}{2} h_4^{\{4\}} \\
h_4^{\{4\}} &= 1
\end{aligned}$$

The minimality condition gives, that $h_1^{\{4\}} = 0$. So that ¡br /¿

$$\begin{aligned}
h_2^{\{4\}} &= \tfrac{1}{2} h_3^{\{4\}} \\
h_3^{\{4\}} &= \tfrac{1}{2} h_2^{\{4\}} + \tfrac{1}{2}
\end{aligned}$$

which gives:

$$\begin{aligned}
h_2^{\{4\}} &= \tfrac{1}{4} h_2^{\{4\}} + \tfrac{1}{4} \Longrightarrow h_2^{\{4\}} = \tfrac{1}{3} \\
h_3^{\{4\}} &= \tfrac{1}{4} h_3^{\{4\}} + \tfrac{1}{2} \Longrightarrow h_3^{\{4\}} = \tfrac{2}{3}
\end{aligned}$$

□

**5.14 Theorem.**
*Consider a recurrence relation of the form*

$$a x_{n+1} + b x_n + c x_{n-1} = 0 \qquad a, c \neq 0.$$

*Let $\alpha, \beta$ be the roots of the quadratic equation*

$$a x^2 + b x + c.$$

*Then the general soltuions is given by*

$$x_n = \begin{cases} A\alpha^n + B\beta^n & \text{if } \alpha \neq \beta \\ (A + nB)\alpha^n & \text{if } \alpha = \beta \end{cases}$$

**5.15 Proposition.**
*Give a general solution for the recurrence relation*

$$\begin{aligned} h_0 &= 1 \\ h_i &= ph_{i+1} + qh_{i-1} \end{aligned}$$

**Proof.**
Note that we have $-ph_{i+1} + h_i - qh_{i-1} = 0$. Consider

$$-px^2 + x - 1 + p = 0$$

We have the roots $\alpha = 1, \beta = \frac{q}{p}$. If $q \neq p$, this gives

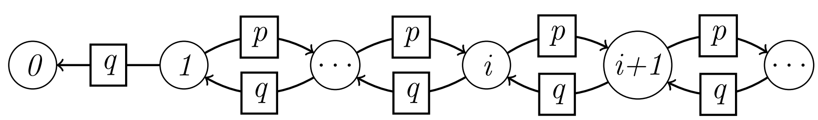
$$h_i = A\alpha^i + B\beta^i = A + B\left(\frac{q}{p}\right)^i.$$

And if $p = q$, then $\alpha = \beta = 1$, and we have

$$h_i = A\alpha^i + B\beta^i = A + iB$$

$\square$

**5.16 Example 1.3.3.**
*Consider the Markov chain with diagram*




*where* $0 < p = 1 - q < 1$. *What is* $h_i = \mathbb{P}_i(\textit{hit } 0)$*?*

**Proof.**
We know that $h$ is the minimal non-negative solution to

$$\begin{aligned} h_0 &= 1 \\ h_i &= ph_{i+1} + qh_{i-1} \end{aligned}$$

We consider some cases:

- Suppose $p = q$, then we have

$$h_i = A\alpha^i + B\beta^i = A + iB$$

and as $0 \leq h_i \leq 1$ is a probability, we must have $B = 0$. We then have

$$h_i = A,$$

and as $h_0 = 1$, we must have $h_i = 1$.

- Suppose $p \neq q$, we then have

$$h_i = A\alpha^i + B\beta^i = A + B\Big(\frac{q}{p}\Big)^i.$$

If $\frac{q}{p} > 1$, then we must set $B = 0$ again.

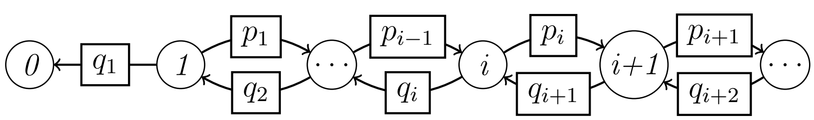
- Suppose $\frac{q}{p} < 1$. We have that $h_0 = 1$, and therefore $A + B = 1$. Hence:

$$h_i = (\tfrac{q}{p})^i + A\Big(1 - (\tfrac{q}{p})^i\Big)$$

So the minimal non-negative solutions is $h_i = (q/p)^i$.

□

**5.17 Example 1.3.4.**
*Consider the Markov chain with diagram*




*where for $i = 1, 2, \ldots,$ we have $0 < p_i = 1 - q_i < 1$. As in the preceding example, 0 is the absorbing state, and we wish to calculate the absorption probability starting form $i$.*

**Proof.**
Consider the system of equations

$$\begin{aligned} &h_0 = 1 \\ &h_i = p_i h_{i+1} + q_i h_{i-1} \qquad \text{for } i = 1, 2, \ldots \end{aligned}$$

Consider

$$u_i : = h_{i-1} - h_i,$$

then

$$\begin{aligned} p_i u_{i+1} &= p_i h_i - p_i h_{i+1} \\ q_i u_i &= q_i h_{i-1} - q_i h_i \end{aligned}$$

$$\implies q_i u_i - p_i u_{i+1} = h_i - q_i h_i - p_i h_i = 0$$

Therefore $p_i u_{i+1} = q_i u_i$ and we have

$$u_{i+1} = \left(\frac{q_i}{p_i}\right) u_i = \prod_{j=1}^{i} \frac{q_i}{p_i} u_1 = \gamma_1 u_1$$

We also have

$$u_1 + \ldots + u_i = h_0 - h_i$$

so

$$h_i = h_0 - (u_1 + \ldots + u_i) = 1 - u_1(\gamma_0 + \ldots + y_{n-1}).$$

At this point $A$ remains to be determined. In the case that $\sum_{i=0}^{\infty} \gamma_i = \infty$, we must have $A = 0$. In the other case, we can't take $A < 0$, but we can take $A > 0$ so long as

$$h_i = 1 - A \sum_{i=0}^{i-1} \gamma_i \geq 0.$$

Which means that

$$A \leq (\sum_{i=0}^{i-1} \gamma_i)^{-1}$$

but still as big as possible. So we get

$$A = (\sum_{i=0}^{\infty} \gamma_i)^{-1}.$$

And therefore

$$h_i = \frac{\sum_{j=i}^{\infty} \gamma_j}{\sum_{j=0}^{\infty} \gamma_j}$$

$\square$

**5.18 Theorem 1.3.5.**

*The vector of mean hitting times* $k^A = (k^A : i \in I)$ *is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} k_i^A = 0 & \textit{for } i \in A \\ k_i^A = 1 + \sum_{j \notin A} p_{ij} k_j^A & \textit{for } \notin A \end{cases}$$

# 6 14-10-2014

## 6.1 Markov Chains 1.4

**6.1 Definition.**
A random variable $T : \Omega \to \{0, 1, 2, \dots, \infty\}$ is called a *stopping time* if the event $\{T = n\}$ depends only on $X_0, X_1, \dots X_n$ for $n = 0, 1, 2, \dots$. Intuitively, by watching the process, you know at the time when $T$ occurs. If asked to stop at $T$, you know when to stop.

**6.2 Proposition.**
*The first passage time*

$$T_j = \inf\{n \geq 1 : X_n = j\}$$

*is a stopping time.*

**Proof.**
To show that $T_j$ is a stopping time, we have to show that

$$\{T_j = n\}$$

depends only on $X_0, \dots, X_n$.

This follows from

$$\{T_j = n\} = \{X_1 \neq j, \dots X_{n-1} \neq j, X_n = j\}$$

$\square$

**6.3 Proposition.**
*The first hitting time*

$$H^A = \inf\{n \geq 0 : X_n \in A\}$$

*is a stopping time.*

**Proof.**
To show that $H^A$ is a stopping time, we have to show that

$$\{H^A = n\}$$

depends only on $X_0, \dots, X_n$. This follows from

$$\{H^A = n\} = \{X_0 \notin A, \dots, X_{n-1} \notin A, X_n \in A\}$$

$\square$

**6.4 Proposition.**
*The last exit time*

$$L^A = \sup\{n \geq 0 : X_n \in A\}$$

*is not in general a stopping time.*

**Proof.**
The event $\{L^A = n\}$ depends on whether $(X_{n+m})_{m\geq 0}$ visits $A$ or not. So we don't have a stopping time. $\square$

**6.5 Theorem 1.4.2 (Strong Markov property).**
*Let $(X_n)_{n\geq 0}$ be Markov$(\lambda, P)$ and let $T$ be a stopping time of $(X_n)_{n\geq 0}$. Then, conditional on $T < \infty$ and $X_T = i, (X_{T+n})_{n\geq 0}$ is Markov$(\delta_i, P)$ and independent of $X_0, \dots, X_T$.*

We now consider an application of the strong Markov property to a Makrov chain $(X_n)_{n\geq 0}$ observed only at certain times. In the first instance suppose that $J$ is some subset of the state space $I$ and that we observe the chain only when it takes values in $J$.

**6.6 Proposition.**
*Let $(X_n)_{n\geq 0}$ be a markov chain. Consider*

$$T_0 = \inf\{n \geq 0 : X_n \in J\}$$

*and, for $m = 0, 1, 2, \dots$*

$$T_{m+1} = \inf\{n > T_m : X_n \in J\}.$$

*Assume $P(T_m < \infty) = 1$ for all $m$. Show that $Y_m = X_{T_m}$ is a markov chain and compute its transition matrix in terms of the transition matrix $P$ of $X_n$.*

**Proof.**
Showing that $(Y_m)$ is a markov chain is equivalent with showing that

$$\begin{aligned}&\mathbb{P}(Y_{m+1} = i_{m+1} | Y_0 = i_1, \dots, Y_m = i_m)\\ &\quad = \mathbb{P}(Y_{m+1} = i_{m+1} | Y_m = i_m)\end{aligned}$$

which in turn is equivalent with showing that

$$\begin{aligned}&\mathbb{P}(X_{T_{m+1}} = i_{m+1} | X_{T_0} = i_1, \dots, X_{T_m} = i_m)\\ &\quad = \mathbb{P}(X_{T_{m+1}} = i_{m+1} | X_{T_m} = i_m).\end{aligned}$$

The Markov property gives that $(X_{T_m+n})_{n\geq 0}$ is a markov chain and independent of $X_0, \dots, X_{T_m}$, and so surely independent of $X_{T_0} = i_1, \dots, X_{T_{m-1}}$. Now $X_{T_{m+1}} = X_{T_m+n}$ for some $n$. So the equality follows.

Now the question is, strating from $i \in J$ what is the chance that we hit $j \in J$ the first time we hit $J$? Call this chance $h_i^j$ Well this is chance is surely greater than $p_{ij}$ as there is also a chance that we first get outside of $J$ and then next time hit $J$, and so on. With a similar reasoning as in Theorem 1.3.2 we can show that for $j \in J$ the vector $(h_i^j : i \in I)$ is the minimal non-negative solution to

$$h_i^j = p_{ij} + \sum_{k \notin J} p_{ik} h_k^j.$$

$\square$

**6.7 Proposition.**
*Let* $(X_n)_{n\geq 0}$ *be a markov chain. Consider*

$$T_0 = \inf\{n \geq 0 : X_n \neq X_0\}$$

*and, for* $m = 0, 1, 2, \dots$

$$T_{m+1} = \inf\{n \geq T_m : X_n \neq X_{T_m}\}.$$

*Assume* $\mathbb{P}(T_m < \infty) = 1$ *for all* $m$. *Show that* $Y_m = X_{T_m}$ *is a markov chain and compute its transition matrix in terms of the transition matrix* $P$ *of* $X_n$.

**Proof.**
Showing that $(Y_m)$ is a markov chain is equivalent with showing that

$$\mathbb{P}(Y_{m+1} = i_{m+1} | Y_0 = i_1, \ldots, Y_m = i_m)$$
$$= \mathbb{P}(Y_{m+1} = i_{m+1} | Y_m = i_m)$$

which in turn is equivalent with showing that

$$\mathbb{P}(X_{T_{m+1}} = i_{m+1} | X_{T_0} = i_1, \ldots, X_{T_m} = i_m)$$
$$= \mathbb{P}(X_{T_{m+1}} = i_{m+1} | X_{T_m} = i_m).$$

The Markov property gives that $(X_{T_m+n})_{n\geq 0}$ is a markov chain and independent of $X_0, \ldots, X_{T_m}$, and so surely independent of $X_{T_0} = i_1, \ldots, X_{T_{m-1}}$. Now $X_{T_{m+1}} = X_{T_m+n}$ for some $n$. So the equality follows.

Now, the question is, starting from $i$ what is the chance to go to $j$ now, if we set the chance $p_{ii} = 0$. Call this chance $\tilde{p}_{ij}$. We have

$$\tilde{p}_{ij} = \frac{p_{ij}}{\sum\limits_{k\neq i} p_{ik}}$$

□

## 6.2 Markov Chains 1.5

**6.8 Definition.**
Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix $P$. We say that a state $i$ is *recurrent* if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 1.$$

A recurrent state is a state $i$ where you keep coming back.

**6.9 Definition.**
We say that a state $i$ is *transient* if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 0.$$

A transient state is a state $i$ which you eventually leave for ever.

**6.10 Theorem.**
*A state $i$ is either recurrent or transient.*

**6.11 Definition.**
Recall that the first passage time to a state $i$ is the random variable $T_i$ defined by

$$T_i(\omega) = \inf\{n \geq 1 : X_n(\omega) = i\}$$

where $\inf \varnothing = \infty$. We now define inductively the *rth passage time* $T_i^{(r)}$ to state $i$ by

$$T_i^{(0)}(\omega) = 0 \quad T_i^{(1)}(\omega) = T_i(\omega)$$

and for $r = 0, 1, 2, \ldots$,

$$T_i^{(r+1)}(\omega) = \inf\{n \geq T_i^{(r)}(\omega) + 1 : X_n(\omega) = i\}.$$

The length of the rth excursion to $i$ is then

$$S_i^{(r)} = \begin{cases} T_i^{(r)} - T_i^{(r-1)} & \text{if } T_i^{(r-1)} < \infty \\ 0 & \text{otherwise} \end{cases}.$$

**6.12 Lemma 1.5.1.**
*For $r = 2, 3, \ldots$, conditional on $T_i^{(r-1)} < \infty$, $S_i^{(r)}$ is independent of $\{X_m : m \leq T_i^{(r-1)}\}$ and*

$$\mathbb{P}(S_i^{(r)} = n | T_i^{(r-1)} < \infty) = \mathbb{P}_i(T_i = n)$$

**6.13 Definition.**
The *number of visits* to $i$ is denoted by

$$V_i = \sum_{n=0}^{\infty} 1_{\{X_n = i\}}.$$

**6.14 Theorem.**

$$E_i(V_i) = \sum_{n=0}^{\infty} p_{ii}^{(n)}$$

**Proof.**
We have

$$\begin{aligned}E_i(V_i) &= \sum_{n=0}^{\infty} E_i(1_{\{X_n=i\}}) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) \\ &= \sum_{n=0}^{\infty} p_{ii}^{(n)}\end{aligned}$$

□

**6.15 Definition.**
The *return probability* of $i$ is denoted by

$$f_i = \mathbb{P}_i(T_i < \infty).$$

**6.16 Lemma 1.5.2.**
*For $r = 0, 1, 2, \ldots,$ we have $\mathbb{P}_i(V_i > r) = f_i^r$.*

**Proof.**
Showing that

$$\mathbb{P}_i(V_i > r) = f_i^r$$

is equivalent with showing that

$$\mathbb{P}_i(V_i > r) = \mathbb{P}_i(T_i < \infty)^r$$

which in turn is equivalent with

$$\mathbb{P}_i(T_i^{(r)} < \infty) = \mathbb{P}_i(T_i < \infty)^r.$$

This last statement can be proven by induction. □

**6.17 Theorem 1.5.3.**
*The following dichotomy holds:*

1. *if $\mathbb{P}_i(T_i < \infty) = 1$, then $i$ is recurrent and $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$*
2. *if $\mathbb{P}_i(T_i < \infty) < 1$, then $i$ is transient and $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$*

**6.18 Theorem 1.5.4.**
*Let $C$ be a communicating class. Then either all states in $C$ are transient or all are recurrent.*

**6.19 Theorem 1.5.5.**
*Every recurrent class is closed. And the contrapositive:*
*Every class that is not closed, is transient.*

**6.20 Theorem 1.5.6.**
*Every finite closed class is recurrent. And the contrapositive:*
*Every transient class is either infinite or not closed..*

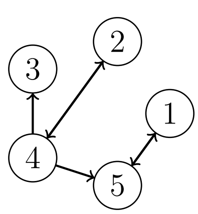
**6.21 Theorem 1.5.7.**
*Suppose $P$ is irreducible and recurrent. Then for all $j \in I$ we have*

$$\mathbb{P}(T_j < \infty) = 1.$$

**6.22 Exercise 1.5.1.**
*Identify the recurrent and transient states of the Markov chain with the following transition matrix:*

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

The solution is obvious from the diagram:




The classes being $\{1,5\}, \{2,4\}$ and $\{3\}$. With $\{1,5\}$ closed and finite, and therefore recurrent. The class $\{3\}$ is absorbing, so closed and finite, and therefore recurrent. The other class $\{2,4\}$ is not closed, and therefore, not recurrent. So we have that $\{2,4\}$ is transient.

# 7 15-10-2014

## 7.1 Markov Chains 1.6

**7.1 Theorem.**
*A state $i$ is recurrent if and only if* $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$.

**7.2 Example 1.6.1 The simple random walk on $\mathbb{Z}$.**
*Compute* $\sum_{n=0}^{\infty} p_{00}^{(n)}$.

**Proof.**
First note that $p_{00}^{(2n+1)} = 0$.

Any given sequence of steps of length $2n$ from 0 to 0 occurs with probability $p^n q^n$, there being $n$ steps up and $n$ steps down. And the number of such sequences is the number of ways of choosing the $n$ steps up from $2n$. Thus

$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{n!^2}(pq)^n.$$

Remember that

$$n! \simeq \sqrt{2\pi n}(n/e)^n \qquad n \to \infty.$$

So

$$\begin{aligned} (2n)! &\simeq \sqrt{4\pi n}(2n/e)^{2n} && n \to \infty \\ n!^2 &\simeq 2\pi n(n/e)^{2n} && n \to \infty. \end{aligned}$$

And therefore

$$\frac{(2n)!}{n!^2}(pq)^n \simeq \frac{(4pq)^n}{\sqrt{\pi n}} \qquad n \to \infty$$

- $p = q$ : Then $p = q = 1/2$, so $4pq = 1$. And we have

  $$p_{00}^{(2n)} \simeq \frac{1}{\sqrt{\pi n}} \qquad n \to \infty$$

  which is equivalent with

  $$\forall \epsilon > 0 \ \exists N : n \geq N \Longrightarrow \frac{1}{\sqrt{\pi}\sqrt{n}} - \epsilon < p_{00}^{(2n)} < \frac{1}{\sqrt{\pi n}} + \epsilon.$$

So there exists a $N$ such that for all $n \geq N$ we have

$$p_{00}^{(2n)} > \frac{1}{2\sqrt{n}}.$$

Therefore

$$\sum_{n=0}^{\infty} p_{00}^{(n)} > \sum_{n=N}^{\infty} p_{00}^{(2n)} > \sum_{n=N}^{\infty} \frac{1}{2\sqrt{n}} = \infty$$

- $p \neq q$ : Then $4pq = r < 1$. And we have

$$p_{00}^{(2n)} \simeq \frac{r^n}{\sqrt{\pi n}} \qquad n \to \infty$$

which is equivalent with

$$\forall \epsilon > 0 \ \exists N : n \geq N \Longrightarrow \frac{r^n}{\sqrt{\pi}\sqrt{n}} - \epsilon < p_{00}^{(2n)} < \frac{r^n}{\sqrt{\pi n}} + \epsilon.$$

So there exists a $N$ such that for all $n \geq N$ we have

$$p_{00}^{(2n)} < r^n$$

Therefore

$$\sum_{n=N}^{\infty} p_{00}^{(n)} = \sum_{n=N}^{\infty} p_{00}^{(2n)} < \sum_{n=N}^{\infty} r^n < \infty.$$

And therefore

$$\sum_{n=0}^{\infty} p_{00}^{(n)} < \infty.$$

□

**7.3 Example 1.6.2 The simple random walk on $\mathbb{Z}^2$.**
*Compute* $\sum_{n=0}^{\infty} p_{00}^{(n)}$.

**Proof.**
By rotating $\mathbb{Z}^2$ we get that each step is like moving in one step in each of the independent, one dimensional, simple random walks. Hence

$$p_{00}^{(2n)} \simeq \left(\frac{1}{\sqrt{\pi n}}\right)^2 = \frac{1}{\pi n}$$

which is equivalent with

$$\forall \epsilon > 0 \ \exists N : n \geq N \Longrightarrow \frac{1}{\pi n} - \epsilon < p_{00}^{(2n)} < \frac{1}{\pi n} + \epsilon.$$

So there exists a $N$ such that for all $n \geq N$ we have

$$p_{00}^{(2n)} > \frac{1}{4n}.$$

Therefore

$$\sum_{n=0}^{\infty} p_{00}^{(n)} > \sum_{n=N}^{\infty} p_{00}^{(2n)} > \frac{1}{4} \sum_{n=N}^{\infty} \frac{1}{n} = \infty$$

□

## 7.2 Markov Chains 1.7

**7.4 Definition.**
We say that a measure $\lambda = (\lambda_i : i \in I)$ where $\lambda_i \geq 0$ is *invariant* if

$$\lambda P = \lambda.$$

**7.5 Theorem 1.7.1.**
*Let $(X_n)_{n \geq 0}$ be a Markov$(\lambda, P)$ and suppose that $\lambda$ is invariant for $P$. Then $(X_{n+m})_{n \geq 0}$ is also Markov $(\lambda, P)$.*

**7.6 Theorem 1.7.2.**
*Let $I$ be finite. Suppose for some $i \in I$ that for all $j \in J$*

$$p_{ij}^{(n)} \to \pi_j \qquad n \to \infty.$$

*Then $\pi = (\pi_j : j \in I)$ is an invariant distribution.*

**7.7 Proposition.**
*Give a example of infinite state space $I$ where Theorem 1.7.2 doesn't hold.*

**Proof.**
For the random walk in $\mathbb{Z}$ we have for all $i, j \in I$

$$p_{ij}^{n} \to 0 \qquad n \to \infty.$$

Note however that $(0, 0, \ldots)$ is not a distrubution. As the total mass

$$\sum_{i \in I} \lambda_i = 0 \neq 1.$$

$\square$

**7.8 Theorem.**
*Consider a recurrence relation of the form*

$$x_{n+1} = a x_n + b.$$

*The general solution is*

$$x_n = \begin{cases} A a^n + b/(1-a) & a \neq 1 \\ x_n = x_0 + nb & a = 1 \end{cases}$$

**7.9 Example 1.1.4.**
*The most general two-state chain has transition matrix of the form*

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

*Draw the diagram, and find* $p_{11}^{(n)}$.

**Proof.**

$$\begin{tikzcd} 1 \arrow[r, bend left, "\boxed{\alpha}"] & 2 \arrow[l, bend left, "\boxed{\beta}"] \end{tikzcd}$$

We have that

$$P^{n+1} = P^n \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

And therefore

$$p_{11}^{(n+1)} = p_{11}^{(n)}(1-\alpha) + p_{12}^{(n)}\beta.$$

Note that we have

$$p_{12}^{(n)} = 1 - p_{11}^{(n)}.$$

So we have

$$p_{11}^{(n+1)} = p_{11}^{(n)}(1-\alpha-\beta) + \beta.$$

The general solution of this recurrence relation is

$$p_{11}^{(n)} = \begin{cases} A(1-\alpha+\beta)^n + \alpha/(\alpha+\beta) & (1-\alpha+\beta) \neq 1 \\ 1+n\beta & 1-\alpha+\beta = 1 \end{cases}$$

This reduces to:

$$p_{11}^{(n)} = \begin{cases} \frac{\beta}{\alpha+\beta}(1-\alpha+\beta)^n + \alpha/(\alpha+\beta) & \alpha+\beta > 0 \\ 1 & \alpha=\beta=0 \end{cases}$$

□

**7.10 Proposition.**
*Consider the two-state Markov chain with transition matrix*

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

*Ignore the trivial cases* $\alpha = \beta = 0$ *and* $\alpha = \beta = 1$.

**Proof.**
From example 1.1.4 we have

$$p_{11}^{(n)} = \begin{cases} \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n + \beta/(\alpha+\beta) & \alpha+\beta > 0 \\ 1 & \alpha=\beta=0 \end{cases}$$

In a similar we could have shown that

$$p_{12}^{(n)} = \begin{cases} \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n + \alpha/(\alpha+\beta) & \alpha+\beta > 0 \\ 1 & \alpha=\beta=0 \end{cases}$$

Therefore $p_{11}^{(n)} \to \frac{\beta}{\alpha+\beta}$ and $p_{12}^{(n)} \to \frac{\alpha}{\alpha+\beta}$. And by theorem 1.7.2 we get that

$$\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$$

is an invariant distribution. □

**7.11 Definition.**
For a fixed state $k$, consider for each $i$ the expected time spent in $i$ between visits to $k$:

$$\gamma_i^k = E_k \sum_{n=0}^{T_k-1} 1_{\{X_n=i\}}$$

**7.12 Theorem 1.7.5.**
*Let $P$ be irreducible and recurrent. Then*

1. $\gamma_k^k = 1$
2. $\gamma^k = (\gamma_i^k : i \in I)$ *satisfies* $\gamma^k P = \gamma^k$
3. $0 < \gamma_i^k < \infty$ *for all* $i \in I$

**7.13 Theorem 1.7.6.**
*Let $P$ be irreducible and let $\lambda$ be an invariant measure for $P$ with $\lambda_k = 1$. Then $\lambda \geq \gamma^k$. If in addition $P$ is recurrent, then $\lambda = \gamma^k$.*

Recall that a state $i$ is recurrent if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 1$$

and we showed in Theorem 1.5.3 that this is equivalent to

$$\mathbb{P}_i(T_i < \infty) = 1.$$

**7.14 Definition.**
We call a recurrent state $i$ positive recurrent if

$$m_i = E_i(T_i) < \infty.$$

If a recurrent state fails to have this stronger property we call it *null recurrent.*

**7.15 Theorem 1.7.7.**
*Let $P$ be irreducible. Then the following are equivalent:*

1. *every state is positive recurrent*
2. *some state $i$ is positive recurrent*
3. *$P$ has an invariant distribution $\pi$*

**7.16 Theorem.**
*If $P$ has an invariant distribution $\pi$, we have that $m_i = E_i(T_i) = \frac{1}{\pi_i}$ for all $i$.*

**7.17 Example 1.7.8.**
*Show that the simple symmetric random walk on $\mathbb{Z}$ is null recurrent.*

**Proof.**
The simple symmetric random walk on $\mathbb{Z}$ is irreducible and, by example 1.6.1, it also recurrent. Remember that

$$(\pi P)_j = \sum_{i \in I} \pi_i p_{ij}$$

and this equals to

$$\sum_{i \in I} \pi_i p_{ij} = 1/2\pi_{j-1} + 1/2\pi_{j+1}.$$

So

$$\pi_i = \tfrac{1}{2}\pi_{i-1} + \tfrac{1}{2}\pi_{i+1}.$$

We get the equation

$$x^2 - 2x + 1 = 0.$$

And so we have

$$\alpha = \beta = 1.$$

And so the general solutions is

$$\pi_i = A + iB.$$

and the total mass is then

$$\sum_i \pi_i = A \sum_i 1 + B \sum_i i = B\infty.$$

So whatever $A, B$ we choose, we never get the total mass to be zero. So there doesn't exists a invariant distribution, and by Theorem 1.7.7 we have that all states must be null recurrent. $\square$

**7.18 Example 1.7.9.**

*Show that the existence of an invariant measure does not guarantee recurrence.*

**Proof.**

The simple symmetric random walk on $\mathbb{Z}^3$ has an invariant measure. Consider:

$$(\pi P)_j = \sum_{i \in I} \pi_i p_{ij} = 1/4(\pi_a + \pi_b + \pi_c + \pi_d)$$

So if we set $\pi = (1, 1, 1, \ldots)$. Then $\pi$ is invariant. But $\mathbb{Z}^3$ is also transient. $\square$

**7.19 Example 1.7.10.**

*Consider the asymmetric random walk on $\mathbb{Z}$ with transition probabilities $p_{i,i-1} = q < p = p_{i,i+1}$. Show that the walk is null recurrent.*

**Proof.**

We have

$$\begin{aligned}\pi_j = (\pi P)_j = \sum_{i \in I} \pi_i p_{ij} &= \pi_{j-1} p_{j-1,j} + \pi_{j+1} p_{j+1,j} \\ &= \pi_{j-1} p + \pi_{j+1} q\end{aligned}$$

This is a recurrence relation and we have the equation

$$qx^2 - x + p = 0$$
$$D = \sqrt{1 - 4(1-p)p} = \sqrt{1 - 4p + 4p^2} = 1 - 2p$$
$$\alpha = \frac{-1 + 1 - 2p}{-2p} = 1 \qquad \beta = \frac{-2 + 2p}{-2p} = \frac{1-p}{p} = \frac{q}{p}$$

So we have the general solution:

$$\pi_j = A\alpha^j + B\beta^j = A + B\left(\frac{p}{q}\right)^i.$$

And the total mass is therefore

$$\sum_{j=0}^{\infty} \pi_j = \sum_{j=0}^{\infty} A + B\left(\frac{p}{q}\right)^i = B\infty.$$

So whatever $A, B$ we choose, we never get the total mass to be zero. So there doesn't exists a invariant distribution, and by Theorem 1.7.7 we have that all states must be null recurrent.

$\square$

**7.20 Example 1.7.11.**
*Consider a success-run chain on $\mathbb{Z}^+$, whose transition probabilities given by*

$$p_{i,i+1} = p_i \qquad p_{i0} = q_i = 1 - p_i.$$

*And where*

$$p = \prod_{i=0}^{\infty} p_i > 0.$$

*Show that every state is transient.*

**Proof.**
We have

$$\begin{aligned} \pi_j = (\pi P)_j &= \sum_{i \in I} \pi_i p_{ij} \\ &= \pi_{j-1} p_{j-1} \\ &= \pi_0 \prod_{i=0}^{j-1} p_i \end{aligned}$$

and

$$\begin{aligned} \pi_0 &= \sum_{i=0}^{\infty} \pi_i p_{i0} \\ &= \sum_{i=0}^{\infty} (1 - p_i) \pi_i \\ &= \pi_0 \sum_{i=0}^{\infty} (1 - p_i) \prod_{j=0}^{i-1} p_j \\ &= \pi_0 \sum_{i=0}^{\infty} \prod_{j=0}^{i-1} p_j - \prod_{j=0}^{i} p_j \\ &= \pi_0 (1 - \prod_{j=0}^{\infty} p_j). \end{aligned}$$

The last equality need some thinking, but notice that it's a telescoping sum. Define

$$a_i := \prod_{j=0}^{i-1} p_j,$$

the sum is

$$\sum_{i=0}^{\infty}(a_i - a_{i+1}) = a_0 - \lim_{k\to\infty} a_k = 1 - \prod_{j=0}^{\infty} p_j$$

The equation

$$\pi_0 = \pi_0(1 - \prod_{j=0}^{\infty} p_j)$$

forces $\pi_0$ to be 0. And therefore all $\pi_i$ are zero. So there is no invariant distribution and we have that $P$ is transient by theorem 1.7.7. $\square$