1 27-10-2014

1.1 Measure Theory Chapter 9

1.1 Proposition.
Given $f \in \mathcal{E}^+$. *Let* $\sum_{j=0}^M y_j 1_{A_j}$ *and* $\sum_{k=0}^N z_k 1_{B_k}$ *be two standard representations of* f. *Then*

$$\sum_{j=0}^M y_j \mu(A_j) = \sum_{k=0}^N z_k \mu(B_k).$$

Proof.

$$\sum_{j=0}^M y_j \mu(A_j) = \sum_{k=0}^N z_k \mu(B_k)$$

$$\Uparrow$$

$$\sum_{j=0}^M y_j \sum_{k=0}^N \mu(A_j \cap B_k) = \sum_{k=0}^N z_k \sum_{j=0}^M \mu(A_j \cap B_k)$$

$$\Uparrow$$

$$y_j \mu(A_j \cap B_k) = z_k \mu(A_j \cap B_k) \qquad \forall (j,k)$$

$$\Uparrow$$

$$\sum_{j=0}^M y_j 1_{A_j}(x) = \sum_{k=0}^N z_k 1_{B_k}(x) \qquad \forall x \in X$$

$\square$

1.2 Definition.
Let $f = \sum_{j=0}^M y_j 1_{A_j} \in \mathcal{E}^+$ be a simple function in standard representation. Then the number

$$I_\mu(f) := \sum_{j=0}^M y_j \mu(A_j) \in [0, \infty]$$

is called the (μ-)*integral* of f.

1.3 Proposition.

$$I_\mu(1_A) = \mu(A) \qquad \forall A \in \mathcal{A}$$
$$I_\mu(\lambda f) = \lambda I_\mu(f) \qquad \forall \lambda \geq 0$$

Proof.
$I_\mu(f) := \sum_{j=0}^{M} y_j \mu(A_j) \in [0, \infty]$ by definition. $\square$

1.4 Proposition.

$$f, g \in \mathcal{E}^+ \Longrightarrow I_\mu(f+g) = I_\mu(f) + I_\mu(g)$$

Proof.

$$I_\mu(f+g) = I_\mu(f) + I_\mu(g)$$
$$\Uparrow$$
$$I_\mu\Big(\sum_{j=0}^{M} y_j 1_{A_j}(x) + \sum_{k=0}^{N} z_k 1_{B_k}(x)\Big) = I_\mu\Big(\sum_{j=0}^{M} y_j 1_{A_j}(x)\Big) + I_\mu\Big(\sum_{k=0}^{N} z_k 1_{B_k}\Big)$$
$$\Uparrow$$
$$\sum_{j=0}^{M}\sum_{k=0}^{N}(y_j + z_k)\mu(A_j \cap B_k) = \sum_{j=0}^{M} y_j \mu(A_j) + \sum_{k=0}^{N} z_k \mu(B_k)$$
$$\Uparrow$$
$$\sum_{j=0}^{M}\sum_{k=0}^{N}(y_j + z_k)\mu(A_j \cap B_k) = \sum_{j=0}^{M} y_j \sum_{k=0}^{N} \mu(A_j \cap B_k) + \sum_{k=0}^{N} z_k \sum_{j=0}^{M} \mu(A_j \cap B_k)$$

$\square$

1.5 Proposition.

$$f \leq g \Longrightarrow I_\mu(f) \leq I_\mu(g)$$

Proof.

$$\begin{aligned} I_\mu(f) &\leq I_\mu(g) \\ &\Uparrow [g - f \in \mathcal{E}^+] \\ I_\mu(f) &\leq I_\mu(f) + I_\mu(g - f) \end{aligned}$$

□

1.6 Definition.
Let $(X, \mathcal{A}, \mu)$ be a measure space. The μ-integral of a positve numerical function $u \in \mathcal{M}_{\bar{\mathbb{R}}^+}$ is given by

$$\int u \; d\mu := \sup\{I_\mu(g) : g \leq u, g \in \mathcal{E}^+\} \in [0, \infty].$$

If we need to emphasize the *integration variable*, we also write $\int u(x)\mu(dx)$ or $\int u(x)d\mu(x)$.

1.7 Proposition.
For all $f \in \mathcal{E}^+$ *we have* $\int f d\mu = I_\mu(f)$.

Proof.

$$\begin{gathered} \int f d\mu = I_\mu(f) \\ \Uparrow \\ \sup\{I_\mu(g) : g \leq f, g \in \mathcal{E}^+\} = I_\mu(f) \\ \Uparrow \\ I_\mu(f) \leq \sup\{I_\mu(g) : g \leq f, g \in \mathcal{E}^+\} \leq I_\mu(f) \\ \Uparrow \\ g \leq f \Longrightarrow I_\mu(g) \leq I_\mu(f) \end{gathered}$$

□

1.8 Proposition.
Let $(X, \mathcal{A})$ be a measurable space. Let $\mu = \delta_y$ be the Dirac measure for fixed $y \in X$. Show that

$$\int u \; d\delta_y = u(y) \qquad \forall u \in \mathcal{M}^+_{\mathbb{R}}.$$

Proof.
By theorem 8.8, there exists increasing function $(f_j)_{j\in\mathbb{N}} \subseteq \mathcal{E}^+$ with $f_j \leq u$ and $\lim_{j\to\infty} f_j = u$. Therefore:

$$\int u \; d\delta_y = u(y)$$

$$\Uparrow$$

$$\int \lim_{j\to\infty} f_j \; d\delta_y = \lim_{j\to\infty} f_j(y)$$

$$\Uparrow$$

$$\lim_{j\to\infty} \int f_j \; d\delta_y = \lim_{j\to\infty} f_j(y)$$

$$\Uparrow$$

$$\int f_j \; d\delta_y = f_j(y) \qquad \forall j \in \mathbb{N}$$

$$\Uparrow$$

$$\sum_{k=0}^{N} y_{k_j} \; \delta_y(A_{k_j}) = f_j(y) \qquad \forall j \in \mathbb{N}$$

□

1.9 Theorem.
Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}^+_{\mathbb{R}}$ be a sequence of positive measurable functions. Then $u := \liminf_{j\to\infty} u_j$ is measurable and

$$\int \liminf_{j\to\infty} u_j \; d\mu \leq \liminf_{j\to\infty} \int u_j \; d\mu$$

Proof.

Recall that $\liminf_{j\to\infty} u_j = \sup_{k\in\mathbb{N}} \inf_{j\geq k} u_j$. Therefore:

$$\int \liminf_{j\to\infty} u_j \ d\mu \leq \liminf_{j\to\infty} \int u_j \ d\mu$$

$$\Uparrow$$

$$\int \sup_{k\in\mathbb{N}} \inf_{j\geq k} u_j \ d\mu \leq \sup_{k\in\mathbb{N}} \inf_{j\geq k} \int u_j \ d\mu$$

$$\Uparrow \text{T9.6}$$

$$\sup_{k\in\mathbb{N}} \int \inf_{j\geq k} u_j \ d\mu \leq \sup_{k\in\mathbb{N}} \inf_{j\geq k} \int u_j \ d\mu$$

$$\Uparrow$$

$$\int \inf_{j\geq k} u_j \ d\mu \leq \inf_{l\geq k} \int u_l \ d\mu \qquad \forall k\in\mathbb{N}$$

$$\Uparrow$$

$$\int \inf_{j\geq k} u_j \ d\mu \leq \int u_l \ d\mu \qquad \forall l\geq k$$

$$\Uparrow$$

$$\inf_{j\geq k} u_j \ \leq u_l \qquad \forall l\geq k$$

$\square$

1.10 Theorem.
Let $(X, \mathcal{A}, \mu)$ be a measure space. For an increasing sequence of numerical functions $(f_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}^+_{\bar{\mathbb{R}}}$ where $0 \leq f_j \leq f_{j+1} \leq \dots$, we have

$$\int \sup_{j\in\mathbb{N}} f_j \ d\mu = \sup_{j\in\mathbb{N}} \int f_j \ d\mu$$

and

$$\int \lim_{j\to\infty} f_j \ d\mu = \lim_{j\to\infty} \int f_j \ d\mu.$$

1.11 Theorem.
Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}_{\bar{\mathbb{R}}^+}$. Then $\sum_{j=1}^{\infty} u_j$ is measurable and we have

$$\int \sum_{j=1}^{\infty} u_j d\mu = \sum_{j=1}^{\infty} \int u_j \ d\mu.$$

1.2 Measure Theory Chapter 10

1.12 Definition.
A function $u : X \to \bar{\mathbb{R}}$ on a measure space $(X, \mathcal{A}, \mu)$ is said to (μ-)-integrable, if it is $\mathcal{A}/\bar{\mathcal{B}}$-measurable and if the integrals $\int u^+ d\mu, \int u^- d\mu < \infty$ are finite. In this case we call

$$\int u \; d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty)$$

the μ-integral of u.

1.13 Definition.
We write $\mathcal{L}^1(\mu)$ $[\mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)]$ for the set of all real-valued [numerical] μ-integrable functions.

1.14 Theorem.
Let $u \in \mathcal{M}_{\bar{\mathbb{R}}}$. Then the following conditions are equivalent:

1. $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$
2. $u^+, u^- \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$
3. $|u| \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$
4. $\exists w \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu),\; w \geq 0$ *such that* $|u| \leq w$

Proof.

$$\begin{array}{c} u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu) \iff u^\pm \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu) \\ \Updownarrow \\ u \in \mathcal{M}^+_{\bar{\mathbb{R}}} \;\wedge\; \displaystyle\int u^\pm d\mu < \infty \iff u^\pm \in \mathcal{M}^+_{\bar{\mathbb{R}}} \;\wedge\; \displaystyle\int u^\pm \; d\mu < \infty \end{array}$$

$$\begin{array}{c} u^\pm \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu) \implies |u| \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu) \\ \Updownarrow \\ u^\pm \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu) \implies u^+ + u^- \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu) \\ \Updownarrow \\ u^\pm \in \mathcal{M}^+_{\bar{\mathbb{R}}} \text{ and } \displaystyle\int u^\pm \; d\mu < \infty \implies u^+ + u^- \in \mathcal{M}^+_{\bar{\mathbb{R}}} \text{ and } \displaystyle\int u^+ + u^- \; d\mu < \infty \end{array}$$

$3 \Rightarrow 4$ is obvious

$$\begin{array}{c} \exists w \geq 0 \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu) : |u| \leq w \implies u^{\pm} \in \mathcal{L}_{\bar{\mathbb{R}}}(\mu) \\ \Uparrow \\ \displaystyle\int u^{\pm} d\mu < \infty \\ \Uparrow \\ u^{\pm} \leq w \end{array}$$

$\square$