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# 1 9-10-2014

## 1.1 Measure Theory Chapter 6

### 1.1 Problem 6.1a.

Consider on  $\mathbb{R}$  the family  $\Sigma$  of all Borel sets which are symmetric w.r.t. the origin. Show that  $\Sigma$  is a  $\sigma$ -algebra.

**Proof.**

1. To show that  $\mathbb{R} \in \Sigma$ , note that  $\mathbb{R}$  is a Borel set that is symmetric w.r.t. to the origin.
2. To show that  $A \in \Sigma \Rightarrow A^c \in \Sigma$ , it suffices to show that

$$\forall x \in A : -x \in A \implies \forall y \in A^c : -y \in A^c,$$

which is equivalent with showing that

$$\forall x \in A : -x \in A \implies \forall y \notin A : -y \notin A,$$

which is equivalent with showing that

$$\exists y \notin A : -y \in A \implies \exists x \in A : -x \notin A.$$

This last statement hold if we set  $x := -y$ .

3. To show that  $\Sigma$  is stable under countable unions, assume  $A_j = B_j \cup B_j$  for some  $B_j \in \mathcal{B}([0, \infty))$ . We have

$$\bigcup_{j \in \mathbb{N}} A_j = \bigcup_{j \in \mathbb{N}} B_j \cup \bigcup_{j \in \mathbb{N}} -B_j \in \Sigma$$

□

**1.2 Problem 6.3i.**

Show that non-void open sets in  $\mathbb{R}^n$  have always strictly positive Lebesgue measure.

**Proof.**

First remember that

1.  $\lambda^n[a, b] = \prod_{j=1}^n (b_j - a_j)$
2.  $\lambda^n$  is a pre-measure that can be extended to a measure on  $\mathcal{B}(\mathbb{R}^n)$ .
3.  $\lambda^n$  is invariant under translations
4.  $A \subseteq B \implies \mu(A) \leq \mu(B)$
5.  $Q_\epsilon = [-\epsilon, \epsilon)$

To show that  $\lambda^n(U) > 0$  it suffices

$$\lambda^n(U') > 0$$

where  $0 \in U'$  and  $U' = x + U$  for some  $x \in \mathbb{R}^n$ . To show that it suffices to show that

$$\lambda^n(B_\epsilon(0)) > 0$$

where  $B_\epsilon(0) \subseteq U$ . To show that it suffices to show that  $Q_{\epsilon'} \subseteq B_\epsilon$  for some  $\epsilon' > 0$ . This holds if we set  $\epsilon' := \frac{\epsilon}{\sqrt{2n}}$ .  $\square$

**1.3 Problem 6.3ii.**

Is 6.3i still true for closed sets ?

**Proof.**

No, take  $\{0\}$ , then  $\lambda\{x\} = 0$ .  $\square$

**1.4 Problem 6.4i.**

Show that  $\lambda(a, b) = b - a$  for all  $a, b \in \mathbb{R}, a \leq b$ .

**Proof.**

$$\begin{aligned} \lambda(a, b) &= \lambda([b - a] - \{b\}) \\ &= \lambda[b, a] - \lambda\{b\} && \text{T4.3iii} \\ &= b - a - 0 && \text{Problem 4.11i} \end{aligned}$$

□

**1.5 Problem 6.4ii.**

Let  $H \subseteq \mathbb{R}^2$  be a hyperplane which is perpendicular to the  $x_1$ -direction (that is to say:  $H$  is a translate of the  $x_2$  axis). Show that

1.  $H \in \mathcal{B}(\mathbb{R}^2)$
2.  $\lambda^2(H) = 0$

**Proof.**

1. To show that  $H \in \mathcal{B}(\mathbb{R}^2)$ , it suffices to show that  $H$  is writable as an intersection of countable half-open sets. Note that:

$$H := \{y\} \times \mathbb{R} = \bigcap_{j \in \mathbb{N}} [y, y + 1/j) \times \mathbb{R}$$

2. We have that for any  $\epsilon > 0$ :

$$\begin{aligned} \lambda^2(H) &= \lambda^2(\{y\} \times \mathbb{R}) \\ &\leq \lambda^2\left(\bigcup_{n \in \mathbb{N}} [y, y + \epsilon_n) \times [-n, n)\right) \\ &\leq 2 \sum_{n \in \mathbb{N}} \epsilon_n n \\ &= \epsilon L \end{aligned}$$

This follows if we choose  $\epsilon_n := \frac{\epsilon}{2^n}$ . Therefore  $\lambda^2(H) = 0$ .

□

**1.6 Definition.**

Let  $(X, \mathcal{A}, \mu)$  be a measure space such that all singletons  $\{x\} \in \mathcal{A}$ . A point  $x$  is called an atom, if  $\mu\{x\} > 0$ . A measure is called *non-atomic* or *diffuse*, there are no atoms.

**1.7 Problem 6.5i.**

Show that  $\lambda^1$  is diffuse.

**Proof.**

We've already shown that  $\lambda\{x\} = 0$  for any  $x \in \mathbb{R}$ . □

**1.8 Problem 6.5iii.**

Show that for a diffuse measure  $\mu$  on  $(X, \mathcal{A})$  all countable sets are null sets.

**Proof.**

All countable sets are writable as

$$\bigcup_{j=0}^{\infty} \{x_j\}$$

where  $x_i \neq x_j$ . So we get

$$\lambda\left(\bigcup_{j=0}^{\infty} \{x_j\}\right) = \sum_{j=0}^{\infty} \lambda\{x_j\} = 0.$$

□

**1.9 Definition.**

A set  $A \subseteq \mathbb{R}^n$  is called *bounded* if it can be contained in a ball  $B_r \supseteq A$  of finite radius  $r$ . A set  $A \subseteq \mathbb{R}^n$  is called *connected*, if we can go along a curve from any point  $a \in A$  to any point  $a' \in A$  without ever leaving  $A$ .

**1.10 Problem 6.6a.**

Construct an open and unbounded set in  $\mathbb{R}$  with finite, strictly positive Lebesgue measure.

**Proof.**

Consider the set

$$U := \bigcup_{n=1}^{\infty} \left(n - \frac{1}{2^n}, n + \frac{1}{2^n}\right).$$

This is an open set, as it union of countable open sets. It is unbounded, for any  $B_r(0)$  we have that  $r + 1 \in U$  and not in  $B_r(0)$ . We have to show that it has finite lebesgue measure.

$$\begin{aligned} \lambda(U) &= \bigcup_{n=1}^{\infty} \left(n - \frac{1}{2^n}, n + \frac{1}{2^n}\right) \\ &= \sum_{n=1}^{\infty} \frac{2}{2^n} = 2. \end{aligned}$$

□

**1.11 Problem 6.6ii.**

Construct an open, unbounded and connected set in  $\mathbb{R}$  with finite, strictly positive Lebesgue measure.

**Proof.**

Consider

$$U = \bigcup_{j \in \mathbb{N}} [0, 0 + \epsilon/(2^j)) \times [-j, j)$$

then

$$\begin{aligned} \lambda^2(U) &= \left( \bigcup_{j \in \mathbb{N}} \left(-\frac{1}{2^j}, \frac{1}{2^j}\right) \times (-j, j) \right) \\ &\leq \sum_{j \in \mathbb{N}} \frac{4j}{2^j} \end{aligned}$$

Note that

$$\sum_{j \in \mathbb{N}} \frac{j}{2^j}$$

converges.

□

**1.12 Problem 6.6iii.**

Is there a connected, open and unbounded set in  $\mathbb{R}$  with finite, strictly positive Lebesgue measure ?

**Proof.**

No, this is impossible. Since we are in one dimension, connectedness forces us to go between points in a straight, uninterrupted line. Since the set is unbounded, this means we must have a line of the sort  $(a, \infty)$  or  $(-\infty, b)$  in our set and in both cases Lebesgue measure is infinite. □

**1.13 Definition.**

Let  $A \subset X$ . The closure of  $A$ , denoted by  $\bar{A}$ , is the smallest closed set containing  $A$ , i.e.

$$\bar{A} = \bigcap_{\substack{F \in \mathcal{C} \\ F \supset A}} F$$

**1.14 Definition.**

A set  $A \subseteq X$  is dense in  $X$  if  $\bar{A} = X$

**1.15 Problem 6.7.**

Let  $\lambda := \lambda^1|_{[0,1]}$  be a Lebesgue measure on  $([0, 1], \mathcal{B}[0, 1])$ . Show that for every  $\epsilon > 0$  there is a dense open set  $U \subseteq [0, 1]$  with  $\lambda(U) \leq \epsilon$ .

**Proof.**

Note that  $\mathbb{Q}$  is dense. We are going to make an open set contained in  $\mathbb{Q}$ . Consider

$$U := \bigcup_{j=1}^{\infty} (q_j - \epsilon_j, q_j + \epsilon_j)$$

Then

$$\lambda(U) = \lambda\left(\bigcup_{j=1}^{\infty} (q_j - \epsilon_j, q_j + \epsilon_j)\right) \leq \sum 2\epsilon_j.$$

So set  $\epsilon_j := \frac{\epsilon}{2^j - 1}$ . And we are done. □

**1.16 Problem 6.10i.**

Let  $\mu$  be a measure on  $\mathcal{A} = \{\emptyset, [0, 1), [1, 2), [0, 2)\}$  of  $X = [0, 2)$ . Such that

$$\mu[0, 1) = \mu[1, 2) = 1/2 \quad \mu[0, 2) = 1.$$

Define for each  $A \subseteq [0, 2)$  the family of countable  $\mathcal{A}$ -coverings of  $A$

$$\mathcal{C}(A) := \{(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A} : \bigcup_{j \in \mathbb{N}} A_j \supseteq A\}$$

and set

$$\mu^*(A) := \inf \left\{ \sum_{j \in \mathbb{N}} \mu(S_j) : (S_j)_{j \in \mathbb{N}} \in \mathcal{C}(A) \right\}.$$

Define  $\mathcal{A}^* := \{A \subseteq [0, 2) : \mu^*(B) = \mu^*(B \cap A) + \mu^*(B - A) \quad \forall B \subseteq X\}$

Show that

1. Find  $\mu^*(a, b), \mu^*\{a\}$
2.  $(0, 1), \{0\} \notin \mathcal{A}^*$

Note that in T6.1 it is proven that:

- $\mathcal{A} \subseteq \mathcal{A}^*$
- $\mu^*(A) = \mu(A) \quad \forall A \in \mathcal{A}$
- $\mathcal{A}^*$  is a  $\sigma$ -algebra and  $\mu^*$  is a measure on  $([0, 2], \mathcal{A}^*)$

**Proof.**

1. We have

$$\begin{aligned}\mu^*(a, b) &= \mu[0, 1) && \text{if } a, b \in [0, 1) \\ \mu^*(a, b) &= \mu[1, 2) && \text{if } a, b \in [1, 2) \\ \mu^*(a, b) &= \mu[0, 2) && \text{if } a \in [0, 1), b \in [1, 2)\end{aligned}$$

In the case of a singleton  $\{a\}$  the best possible cover is always either  $[0, 1)$  or  $[1, 2)$  so that  $\mu^*\{a\} = 1/2$ .

2. Suppose that  $(0, 1) \in \mathcal{A}^*$  then we would have that

$$\{0\} = [0, 1) - (0, 1) \in \mathcal{A}^*.$$

But this gives

$$\frac{1}{2} = \mu^*[0, 1) = \mu^*(0, 1) + \mu^*\{0\} = 1$$

□



## 2 10-10-2014

### 2.1 Measure Theory Chapter 7

#### 2.1 Definition.

Let  $(X, \mathcal{A}), (X', \mathcal{A}')$  be two measurable spaces. A map  $T : X \rightarrow X'$  is called  $\mathcal{A}/\mathcal{A}'$ -measurable (or *measurable* unless this is too ambiguous) if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A} \quad \forall A' \in \mathcal{A}'.$$

We often denote this by  $T^{-1}(\mathcal{A}') \subseteq \mathcal{A}$ .

#### 2.2 Definition.

A *random variable* is a measurable map from a probability space (i.e.  $\mu(X) = 1$ ) to any measurable space.

#### 2.3 Lemma 7.2.

Let  $(X, \mathcal{A}), (X', \mathcal{A}')$  be measurable spaces and let  $\mathcal{A}' = \sigma(\mathcal{G}')$ . Then  $T : X \rightarrow X'$  is  $\mathcal{A}/\mathcal{A}'$ -measurable if and only if

$$T^{-1}(G') \in \mathcal{A} \quad \forall G' \in \mathcal{G}'.$$

#### 2.4 Problem 7.1.

Show that

$$\tau_x : \mathbb{R}^n \rightarrow \mathbb{R}^n : B \mapsto B - x$$

is a  $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$  measurable map.

#### Proof.

Showing that

$$\tau_x : \mathcal{B}(\mathbb{R}^n) \rightarrow \mathcal{B}(\mathbb{R}^n) : B \mapsto B - x$$

is  $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$  measurable, is equivalent with showing that

$$\tau_x^{-1}(B) \in \mathcal{B}(\mathbb{R}^n) \quad \forall B \in \mathcal{B}(\mathbb{R}^n),$$

which in turn is equivalent with showing that

$$x + B \in \mathcal{B}(\mathbb{R}^n) \quad \forall B \in \mathcal{J}(\mathbb{R}^n),$$

which in turn is equivalent with showing that

$$x + [a, b] \in \mathcal{B}(\mathbb{R}^n) \quad \forall a, b \in \mathbb{R}^n.$$

This follows as  $x + [a, b] = [x + a, x + b] \in \mathcal{J}(\mathbb{R}^n) \subseteq \mathcal{B}(\mathbb{R}^n)$ .  $\square$

## 2.5 Theorem.

*Every continuous map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is  $\mathcal{B}^n/\mathcal{B}^m$  measurable.*

### Proof.

Showing that

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

is  $\mathcal{B}^n/\mathcal{B}^m$  measurable, is equivalent with showing that

$$T^{-1}(\mathcal{O}^m) \subseteq \mathcal{B}^n.$$

As  $\mathcal{O}^n \subseteq \sigma(\mathcal{O}^n) = \mathcal{B}^n$ , it suffices to show that

$$T^{-1}(\mathcal{O}^m) \subseteq \mathcal{O}^n,$$

which follows from the continuity of  $T$ .  $\square$

## 2.6 Definition.

Let  $(T_i)_{i \in I}$  be arbitrarily many mappings  $T_i : X \rightarrow X_i$  from the same space  $X$  into measurable spaces  $(X_i, \mathcal{A}_i)$ . The smallest  $\sigma$ -algebra on  $X$  that makes all  $T_i$  simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)\right).$$

We say that  $\sigma(T_i : i \in I)$  is *generated by the family  $(T_i)_{i \in I}$* .

## 2.7 Theorem.

*Let  $(X_j, \mathcal{A}_j)$ ,  $j = 1, 2, 3$ , be measurable spaces and  $T : X_1 \rightarrow X_2$ ,  $S : X_2 \rightarrow X_3$  be  $\mathcal{A}_1/\mathcal{A}_2$ - resp.  $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps. Then  $S \circ T : X_1 \rightarrow X_3$  is  $\mathcal{A}_1/\mathcal{A}_3$ -measurable.*

### 2.8 Problem 7.4.

Let  $X$  be a set,  $(X_i, \mathcal{A}_i), i \in I$ , be arbitrarily many measurable spaces, and  $T_i : X \rightarrow X_i$  be a family of maps. Show that a map  $f$  from a measurable space  $(F, \mathcal{F})$  to  $(X, \sigma(T_i : i \in I))$  is measurable if, and only if, all maps  $T_i \circ f$  are  $\mathcal{F}/\mathcal{A}_i$ -measurable.

#### Proof of $\implies$ .

To show that all maps  $T_i \circ f$  are  $\mathcal{F}/\mathcal{A}_i$ -measurable, it suffices to show that  $T_i : X \rightarrow X_i$  is  $\sigma(T_i : i \in I)/\mathcal{A}_i$ -measurable and  $f : F \rightarrow X$  is  $\mathcal{F}/\sigma(T_i : i \in I)$ -measurable.

By hypothesis, it suffices to show that  $T_i : X \rightarrow X_i$  is  $\sigma(T_i : i \in I)/\mathcal{A}_i$ -measurable, which is equivalent with showing that

$$T_i^{-1}(A_i) \in \sigma(T_i : i \in I) \quad \forall A_i \in \mathcal{A}_i.$$

It suffices to assume  $A_i \in \mathcal{A}_i$  and show that

$$T_i^{-1}(A_i) \in \bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \quad \checkmark.$$

□

#### Proof of $\impliedby$ .

To show that a map  $f$  from a measurable space  $(F, \mathcal{F})$  to  $(X, \sigma(T_i : i \in I))$  is measurable, it suffices to show that

$$f^{-1}\left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)\right) \subseteq \mathcal{F}$$

To show this it suffices to show that

$$\bigcup_{i \in I} f^{-1}(T_i^{-1}(\mathcal{A}_i)) \subseteq \mathcal{F},$$

to show this it suffices to show that

$$f^{-1}(T_i^{-1}(\mathcal{A}_i)) \subseteq \mathcal{F},$$

to show this it suffices to show that

$$(T_i \circ f)^{-1}(\mathcal{A}_i) \subseteq \mathcal{F}.$$

This follows by hypothesis.

□

### 2.9 Problem 7.8.

Let  $T : X \rightarrow Y$  be any map. Show that

$$T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G}))$$

holds for arbitrary families of  $\mathcal{G}$  of subsets of  $Y$ .

#### Proof.

To show that

$$T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G}))$$

it suffices to show:

1.  $T^{-1}(\sigma(\mathcal{G})) \subseteq \sigma(T^{-1}(\mathcal{G}))$
2.  $\sigma(T^{-1}(\mathcal{G})) \subseteq T^{-1}(\sigma(\mathcal{G}))$

To show

$$T^{-1}(\sigma(\mathcal{G})) \subseteq \sigma(T^{-1}(\mathcal{G})),$$

it suffices to show that  $T$  is  $\sigma(T^{-1}(\mathcal{G}))/\sigma(\mathcal{G})$  measurable.

To show that it suffices to show that

$$T^{-1}(\mathcal{G}) \subseteq \sigma(T^{-1}(\mathcal{G})) \quad \checkmark.$$

To show

$$\sigma(T^{-1}(\mathcal{G})) \subseteq T^{-1}(\sigma(\mathcal{G})),$$

it suffices to show that

$$T^{-1}(\mathcal{G}) \subseteq T^{-1}(\sigma(\mathcal{G})) \quad \checkmark.$$

□

## 2.2 Measure Theory Chapter 5

### 2.10 Definition.

A family  $\mathcal{D} \subseteq \mathcal{P}(X)$  is a *Dynkin system* if

$$\begin{aligned} X &\in \mathcal{D} \\ D \in \mathcal{D} &\implies D^c \in \mathcal{D} \\ (D_j)_{j \in \mathbb{N}} \subseteq \mathcal{D} \text{ pairwise disjoint} &\implies \bigcup_{j \in \mathbb{N}} D_j \in \mathcal{D} \end{aligned}$$

**2.11 Definition.**

Let  $\mathcal{G} \subseteq \mathcal{P}(X)$ . Then there is a smallest Dynkin system  $\delta(\mathcal{G})$  containing  $\mathcal{G}$ .  $\delta(\mathcal{G})$  is called the *Dynkin system generated by  $\mathcal{G}$* .

**2.12 Proposition.**

Show that

$$\mathcal{G} \subseteq \delta(\mathcal{G}) \subseteq \sigma(\mathcal{G}).$$

**Proof.**

We have that  $\mathcal{G} \subseteq \sigma(\mathcal{G})$ . And therefore  $\delta(\mathcal{G}) \subseteq \delta(\sigma(\mathcal{G})) = \sigma(\mathcal{G})$ .  $\square$

**2.13 Theorem.**

A Dynkin system  $\mathcal{D}$  is a  $\sigma$ -algebra if, and only if, it is stable under finite intersections:  $D, E \in \mathcal{D} \implies D \cap E \in \mathcal{D}$

**Proof.**

It suffices to show that a  $\cap$ -stable Dynkin system is stable under countable unions. To show this, it suffices to show that given  $(D_j)_{j \in \mathbb{N}} \in \mathcal{D}$ , we have

$$D := \bigcup_{j \in \mathbb{N}} D_j \in \mathcal{D}.$$

Set  $E_1 = D_1 \in \mathcal{D}$ . And  $E_2 := D_2 \cap D_1^c$ . And  $E_3 = D_3 \cap D_2^c \cap D_1^c$ . And so on. Then

$$D = \bigcup_{j \in \mathbb{N}} E_j \in \mathcal{D}.$$

$\square$

**2.14 Theorem.**

If  $\mathcal{G} \subseteq \mathcal{P}(X)$  is stable under finite intersections, then  $\delta(\mathcal{G}) = \sigma(\mathcal{G})$ .

**Proof.**

It suffices to show that  $\sigma(\mathcal{G}) \subseteq \delta(\mathcal{G})$ . As  $\mathcal{G} \subseteq \delta(\mathcal{G})$  it suffices to show that  $\delta(\mathcal{G})$  is a  $\sigma$ -algebra. To show that  $\delta(\mathcal{G})$  is a  $\sigma$ -algebra, it suffices to show that  $\delta(\mathcal{G})$  is stable under finite intersections.

Fix  $D \in \delta(\mathcal{G})$ . Consider  $\mathcal{D}_D := \{Q \subseteq X : Q \cap D \in \delta(\mathcal{G})\}$ . It suffices to show that  $\delta(\mathcal{G}) \subseteq \mathcal{D}_D$ . To show that it suffices to show that  $\mathcal{D}_D$  is a Dynkin system and that  $\mathcal{G} \subseteq \mathcal{D}_D$ .

To show that  $\mathcal{G} \subseteq \mathcal{D}_D$ , it suffices to show that

$$G \cap D \in \delta(\mathcal{G}) \quad \forall G \in \mathcal{G},$$

to show that it suffices to show that

$$\delta(\mathcal{G}) \subseteq \mathcal{D}_G \quad \forall G \in \mathcal{G},$$

to show that it suffices to show that (as  $\mathcal{D}_G$  is a dynkin system)

$$\mathcal{G} \subseteq \mathcal{D}_G \quad \forall G \in \mathcal{G}.$$

This follows from  $\mathcal{G} \subseteq \delta(\mathcal{G})$  and  $\mathcal{G}$  is  $\cap$ -stable. □

### 2.15 Proposition.

$$A_j \uparrow A \implies A_j \cap B \uparrow A \cap B$$

#### Proof.

To show that

$$A_j \uparrow A \implies A_j \cap B \uparrow A \cap B,$$

it suffices to show that

$$A = \bigcup_j A_j \implies A \cap B = \bigcup_j A_j \cap B,$$

which is equivalent with showing that

$$\left( \bigcup_j A_j \right) \cap B = \bigcup_j A_j \cap B \quad \checkmark.$$

□

### 2.16 Definition.

An *exhausting sequence*  $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$  is an increasing sequence of sets  $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$  such that  $\bigcup_{j \in \mathbb{N}} A_j = X$ .

**2.17 Theorem.**

Assume that  $(X, \mathcal{A})$  is a measurable space and that  $\mathcal{A} = \sigma(\mathcal{G})$  is generated by a family  $\mathcal{G}$  such that

- $\mathcal{G}$  is stable under finite intersections  $G, H \in \mathcal{G} \implies G \cap H \in \mathcal{G}$
- there exists an exhausting sequence  $(G_j)_{j \in \mathbb{N}} \subseteq \mathcal{G}$  with  $G_j \uparrow X$

Any two measure  $\mu, \nu$  that coincide on  $\mathcal{G}$  and are finite for all members of the exhausting sequence  $\mu(G_j) = \nu(G_j) < \infty$ , are equal on  $\mathcal{A}$ , i.e.

$$\mu(A) = \nu(A) \quad \forall A \in \mathcal{A}.$$

**Proof.**

Remember that for any increasing sequence  $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$  with  $A_j \uparrow A \in \mathcal{A}$  we have

$$\mu(A) = \lim_{j \in \mathbb{N}} \mu(A_j).$$

To show that

$$\mu(A) = \nu(A) \quad \forall A \in \mathcal{A}$$

it suffices to show that (as  $G_j \cap A \uparrow X \cap A$ )

$$\lim_{j \in \mathbb{N}} \mu(G_j \cap A) = \lim_{j \in \mathbb{N}} \nu(G_j \cap A) \quad \forall A \in \mathcal{A}$$

To show that it suffices to show that

$$\mu(G_j \cap A) = \nu(G_j \cap A) \quad \forall j \in \mathbb{N}, \quad \forall A \in \mathcal{A}.$$

Consider  $\mathcal{D}_j := \{A \in \mathcal{A} : \mu(G_j \cap A) = \nu(G_j \cap A)\}$ . It suffices to show that  $\mathcal{A} \subseteq \mathcal{D}_j$ , which is equivalent with showing  $\sigma(\mathcal{G}) \subseteq \mathcal{D}_j$ .

As  $\mathcal{G}$  is stable under finite intersections, it suffices to show that  $\delta(\mathcal{G}) \subseteq \mathcal{D}_j$ .

As  $\mathcal{G}$  is stable under finite intersections and  $\mu(\mathcal{G}) = \nu(\mathcal{G})$ , we have that  $\mathcal{G} \subseteq \mathcal{D}_j$  and therefore it suffices to show that  $\mathcal{D}_j$  is a Dynkin system.

Which you can check. □

**2.18 Theorem.**

The  $n$ -dimensional Lebesgue measure  $\lambda^n$  is invariant under translations, i.e.

$$\lambda^n(x + B) = \lambda^n(B) \quad \forall x \in \mathbb{R}^n, \forall B \in \mathcal{B}(\mathbb{R}^n).$$

**Proof.**

Set  $\nu(B) := \lambda^n(x + B)$  for some fixed  $x \in \mathbb{R}^n$ . It suffices to show that

$$\lambda^n(B) = \nu(B) \quad B \in \mathcal{B}.$$

To show that, it suffices to show that

1.  $\mathcal{J}$  is  $\cap$ -stable    ✓
2.  $\mathcal{J}$  admits an exhausting sequence
  - $[-j, j) \uparrow \mathbb{R}^n$     ✓
3.  $\lambda^n|_{\mathcal{J}} = \nu|_{\mathcal{J}}$

$$\begin{aligned} \nu([a, b)) &= \lambda^n[x + a, x + b) \\ &= \lambda^n[a, b) \end{aligned}$$

4.  $\nu$  is a measure on  $\mathcal{B}^n$

To show that  $\nu$  is a measure on  $\mathcal{B}^n$ , it suffices to show that

$$\nu\left(\bigcup_{j \in \mathbb{N}} B_j\right) = \sum_{j \in \mathbb{N}} \nu(B_j),$$

which is equivalent with

$$\lambda^n\left(x + \bigcup_{j \in \mathbb{N}} B_j\right) = \sum_{j \in \mathbb{N}} \lambda^n(x + B_j).$$

It suffices to show

$$B \in \mathcal{B}^n \implies x + B \in \mathcal{B}^n.$$

Which we have already proven. □

**2.19 Theorem.**

Let  $(X, \mathcal{A}), (X, \mathcal{A}')$  be measurable spaces and  $T : X \rightarrow X'$  be an  $\mathcal{A}/\mathcal{A}'$  measurable map. For every measure  $\mu$  on  $(X, \mathcal{A})$ ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}'.$$

The measure  $\mu'$  is called the image measure of  $\mu$  under  $T$  and is denoted by  $T \circ \mu$  or  $\mu \circ T^{-1}$ .



## 2.3 Measure Theory Chapter 7

### 2.20 Problem 7.7.

Use image measures to give a new proof of Problem 5.8, i.e. to show that

$$\lambda^n(t \cdot B) = t^n \lambda^n(B) \quad \forall B \in \mathcal{B}(\mathbb{R}^n), \forall t > 0$$

**Proof.**

Set  $\nu(B) := t^n \lambda^n(B)$  for some fixed  $x \in \mathbb{R}^n$ . It suffices to show that

$$\lambda^n(tB) = \nu(B) \quad \forall B \in \mathcal{B}.$$

To show that, it suffices to show that

1.  $\mathcal{J}$  is  $\cap$ -stable    ✓
2.  $\mathcal{J}$  admits an exhausting sequence
  - $[-j, j) \uparrow \mathbb{R}^n$     ✓
3.  $\lambda^n|_{\mathcal{J}} = \nu|_{\mathcal{J}}$

$$\begin{aligned} \nu([a, b)) &= \lambda^n[ta, tb) \\ &= t^n \lambda^n[a, b) \end{aligned}$$

4.  $\nu$  is a measure on  $\mathcal{B}^n$  as it is a composition of the inverse of a measurable map and a measure.

□

### 3 11-10-2014

#### 3.1 Measure Theory Chapter 8

##### 3.1 Definition.

Note that:  $u^{-1}[a, \infty) = \{x \in X : u(x) \in [a, \infty)\} = \{x \in X : u(x) \geq a\}$ . We define:

$$\{u(x) \geq a\} = u^{-1}[a, \infty).$$

##### 3.2 Theorem.

Let  $(X, \mathcal{A})$  be a measurable space. The function  $u : X \rightarrow \mathbb{R}$  is  $\mathcal{A}/\mathcal{B}$ -measurable if, and only if, one, hence all, of the following conditions hold

1.  $\{u \geq a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$
2.  $\{u > a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$
3.  $\{u \leq a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$
4.  $\{u < a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$

##### 3.3 Definition.

We define the *extended real line*  $\bar{\mathbb{R}} := [-\infty, \infty]$  with the following rules for all  $x \in \mathbb{R}$ :

$$\begin{aligned} x + \infty &= \infty + x = \infty & x + -\infty &= -\infty + x = -\infty \\ \infty + \infty &= \infty & -\infty - \infty &= -\infty \end{aligned}$$

And for  $x \in (0, \infty]$ :

$$\begin{aligned} \pm x \cdot \infty &= \infty \cdot \pm x = \pm\infty \\ \pm x \cdot -\infty &= -\infty \cdot \pm x = \mp\infty \\ 0 \cdot \pm\infty &= \pm\infty \cdot 0 = 0 \\ \frac{1}{\pm\infty} &= 0 \end{aligned}$$

##### 3.4 Definition.

Functions which take values in  $\bar{\mathbb{R}}$  are called *numerical functions*.

### 3.5 Definition.

The Borel  $\sigma$ -algebra  $\bar{\mathcal{B}} = \mathcal{B}(\bar{\mathbb{R}})$  is defined by:

$$\bar{\mathcal{B}} := \left\{ B \cup S : B \in \mathcal{B} \text{ and } S \in \left\{ \emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\} \right\} \right\}$$

### 3.6 Theorem.

We have  $\mathcal{B}(\mathbb{R}) = \mathbb{R} \cap \mathcal{B}(\bar{\mathbb{R}})$ . Moreover  $\bar{\mathcal{B}}$  is generated by all sets of the form  $[a, \infty]$  or  $(a, \infty]$  or  $[-\infty, a]$  or  $[-\infty, a)$  where  $a \in \mathbb{R}$

### 3.7 Definition.

Let  $(X, \mathcal{A})$  be a measurable space. We write  $\mathcal{M} := \mathcal{M}(\mathcal{A})$  and  $\mathcal{M}_{\bar{\mathbb{R}}} := \mathcal{M}_{\bar{\mathbb{R}}}(\mathcal{A})$  for the families of real valued  $\mathcal{A}/\mathcal{B}$ -measurable and numerical  $\mathcal{A}/\bar{\mathcal{B}}$ -measurable functions on  $X$ .

### 3.8 Definition.

A *simple function*  $g : X \rightarrow \mathbb{R}$  on a measurable space  $(X, \mathcal{A})$  is a function of the form

$$g(x) := \sum_{j=1}^M y_j \mathbf{1}_{A_j}(x)$$

with finitely many sets  $A_1, \dots, A_M \in \mathcal{A}$  and  $y_1, \dots, y_M \in \mathbb{R}$ . The set of simple functions is denoted by  $\mathcal{E}$  or  $\mathcal{E}(\mathcal{A})$ .

If the sets  $A_1, \dots, A_M$  are mutually disjoint we call

$$\sum_{j=0}^M y_j \mathbf{1}_{A_j}(x)$$

with  $y_0 := 0$  and  $A_0 := (A_1 \cup \dots \cup A_M)^c$  a *standard representation* of  $g$ . Caution, this representation is not unique.

### 3.9 Theorem.

Let  $(X, \mathcal{A})$  be a measurable space. Every  $\mathcal{A}/\bar{\mathcal{B}}$ -measurable numerical function  $u : X \rightarrow \bar{\mathbb{R}}$  is the pointwise limit of simple functions:

$$u(x) = \lim_{j \rightarrow \infty} f_j(x)$$

where  $f_j \in \mathcal{E}(\mathcal{A})$  and  $|f_j| \leq |u|$ .

If  $u \geq 0$ , all  $f_j$  can be chosen to be positive and increasing towards  $u$  so that  $u = \sup_{j \in \mathbb{N}} f_j$ .

**3.10 Theorem.**

Let  $(X, \mathcal{A})$  be a measurable space. If  $u_j : X \rightarrow \bar{\mathbb{R}}, j \in \mathbb{N}$  are measurable functions, then so are

$$\sup_{j \in \mathbb{N}} u_j \quad \inf_{j \in \mathbb{N}} u_j \quad \limsup_{j \rightarrow \mathbb{N}} u_j \quad \liminf_{j \rightarrow \mathbb{N}} u_j$$

and whenever it exists

$$\lim_{j \rightarrow \infty} u_j.$$

**3.11 Theorem.**

Let  $u, v$  be  $\mathcal{A}/\bar{\mathcal{B}}$ -measurable functions. Then the functions

$$u \pm v \quad uv \quad u \vee v := \max\{u, v\} \quad u \wedge v := \min\{u, v\}$$

are  $\mathcal{A}/\bar{\mathcal{B}}$ -measurable (whenever they are defined).

**3.12 Theorem.**

A function  $u$  is  $\mathcal{A}/\bar{\mathcal{B}}$  measurable if, and only if,  $u^\pm$  are  $\mathcal{A}/\bar{\mathcal{B}}$  measurable.

**3.13 Theorem.**

Let  $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$  be an  $\mathcal{A}/\mathcal{A}'$ -measurable map and let  $\sigma(T) \subseteq \mathcal{A}$  be the  $\sigma$ -algebra generated by  $T$ . Then  $u = w(T)$  for some  $\mathcal{A}'/\bar{\mathcal{B}}$  measurable function  $w : X' \rightarrow \bar{\mathbb{R}}$  if and only if  $u : X \rightarrow \bar{\mathbb{R}}$  is  $\sigma(T)/\bar{\mathcal{B}}$ -measurable.

**3.14 Proposition.**

Let  $(X, \mathcal{A})$  be a measurable space. We define the indicator function:

$$1_A : X \rightarrow \bar{\mathbb{R}} : x \in A \mapsto 1 \quad x \in X - A \mapsto 0$$

Show that the indicator function is measurable if, and only if,  $A \in \mathcal{A}$ .

**Proof.**

To show that  $1_A$  is measurable, it suffices to show that

$$1_A^{-1}(a, \infty) \in \mathcal{A}.$$

Note that

$$1_A^{-1}(a, \infty) = \{x \in X : 1_A(x) \in (a, \infty)\} = \{1_A > a\}$$

If  $a \geq 1$ , then  $1_A^{-1}(a, \infty) = \emptyset$ .

If  $a \in [0, 1)$ , then  $1_A^{-1}(a, \infty) = A$ .

If  $a < 0$ , then  $1_A^{-1}(a, \infty) = X$ . □

**3.15 Proposition.**

Let  $A_1, \dots, A_M \in \mathcal{A}$  be mutually disjoint sets and  $y_1, \dots, y_M \in \mathbb{R}$ . Then the function

$$g : X \rightarrow \mathbb{R} : x \mapsto \sum_{j=1}^M y_j 1_{A_j}(x)$$

is measurable.

**Proof.**

To show that  $g$  is measurable it suffices to show that

$$\{g > a\} \in \mathcal{A}$$

i.e.

$$\left\{x \in X : \sum_{j=1}^M y_j 1_{A_j}(x) > a\right\} = \bigcup_{j: y_j > a} A_j \in \mathcal{A}.$$

□

**3.16 Problem 8.3i.**

Let  $(X, \mathcal{A})$  be a measurable space. Let  $f, g : X \rightarrow \mathbb{R}$  be measurable functions. Show that for every  $A \in \mathcal{A}$  the functions  $h(x) := f(x)$  if  $x \in A$  and  $h(x) := g(x)$ , if  $x \notin A$ , is measurable.

**Proof.**

Note that

$$h(x) := 1_A(x)f(x) + 1_{A^c}(x)g(x).$$

And remember that sums and products of measurable functions are again measurable. □

**3.17 Problem 8.3ii.**

Let  $(f_j)_{j \in \mathbb{N}}$  be a sequence of measurable functions and let  $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$  such that  $\bigcup_{j \in \mathbb{N}} A_j = X$ . Suppose that  $f_j|_{A_j \cap A_k} = f_k|_{A_j \cap A_k}$  for all  $j, k \in \mathbb{N}$  and set  $f(x) := f_j(x)$  if  $x \in A_j$ . Show that  $f : X \rightarrow \mathbb{R}$  is measurable.

**Proof.**

We have that:

$$f^{-1}(B) = \bigcup_{j \in \mathbb{N}} A_j \cap f^{-1}(B) = \bigcup_{j \in \mathbb{N}} A_j \cap f_j^{-1}(B) \in \mathcal{A}$$

□

**3.18 Problem 8.4.**

Let  $(X, \mathcal{A})$  be a measurable space and let  $\mathcal{B} \subset \mathcal{A}$  be a sub- $\sigma$ -algebra. Show that  $\mathcal{M}(\mathcal{B}) \subset \mathcal{M}(\mathcal{A})$ .

**Proof.**

To show that

$$\mathcal{M}(\mathcal{B}) \subset \mathcal{M}(\mathcal{A})$$

it suffices to show there exists a  $\mathcal{A}$ -measurable function that is not  $\mathcal{B}$ -measurable. By hypothesis, we have an element  $A \in \mathcal{A}$ , that is not in  $\mathcal{B}$ , i.e.  $A \notin \mathcal{B}$ . Since  $1_A$  is  $\mathcal{B}$ -measurable if, and only if,  $B \in \mathcal{B}$ , we have find the  $\mathcal{A}$ -measurable function where we were looking for. □

**3.19 Theorem.**

Let  $u : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable. Explain why  $u$  and  $u' = du/dx$  are measurable.

**Proof.**

If  $u$  is differentiable, it is continuous, hence measurable. Since  $u'$  exists, we can write it in the form

$$u'(x) = \lim_{k \rightarrow \infty} \frac{u(x + 1/k) - u(x)}{1/k}$$

i.e. as limit of measurable functions. Thus,  $u'$  is also measurable. □

**3.20 Problem 8.17.**

Show that the measurability of  $|u|$  does not, in general, imply the measurability of  $u$ .

**Proof.**

Let  $A \subseteq \mathbb{R}$  be such that  $A \notin \mathcal{B}$ . Then it is clear that

$$u(x) := 1_A(x) - 1_{A^c}(x)$$

is not measurable. Take

$$\{u = 1\} = A \notin \mathcal{A}.$$

But  $|u(x)| = 1$ , which is a continuous function and therefore measurable.

□

**3.21 Problem 8.14.**

Consider  $(\mathbb{R}, \mathcal{B})$  and  $u : \mathbb{R} \rightarrow \mathbb{R}$ . Show that  $\{x\} \in \sigma(u)$  for all  $x \in \mathbb{R}$  if, and only if,  $u$  is injective.

**Proof.**

To show that  $u$  is injective, it suffices to assume  $x, y \in \mathbb{R}$  and show that

$$u(x) = u(y) \implies x = y.$$

Showing that is equivalent with showing that

$$|\{u = u(x_0)\}| = 1.$$

We surely have that  $\{x_0\} \subseteq \{u = u(x_0)\}$ . And note that

$$\{x_0\} \in \sigma(u) = \sigma(u^{-1}(\mathcal{B}))$$

just means that  $\{x_0\} = u^{-1}(B)$  for some  $B \in \mathcal{B}$ . □

**Proof.**

Assume that  $u$  is injective,. This means that every point in the range  $u(\mathbb{R})$  comes exactly from unique defined  $x \in \mathbb{R}$ . This can be expressed by saying that  $\{x\} = u^{-1}(\{u(x)\}) = \{u(x)\}$ . But then

$$\{x\} \in \sigma(u) = \sigma(u^{-1}(\mathcal{B})).$$

□

## 4 12-10-2014

### 4.1 Measure Theory Chapter 9

#### 4.1 Definition.

Let  $f = \sum_{j=0}^M y_j 1_{A_j} \in \mathcal{E}^+$  be a simple function in standard representation. Then the number

$$I_\mu(f) := \sum_{j=0}^M y_j \mu(A_j) \in [0, \infty]$$

is called the  $(\mu)$ -integral of  $f$ .

#### 4.2 Theorem.

1.  $I_\mu(1_A) = \mu(A) \quad \forall A \in \mathcal{A}$
2.  $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0$
3.  $I_\mu(f + g) = I_\mu(f) + I_\mu(g)$
4.  $f \leq g \implies I_\mu(f) \leq I_\mu(g)$

#### 4.3 Definition.

Let  $(X, \mathcal{A}, \mu)$  be a measure space. The  $(\mu)$ -integral of a positive numerical function  $u \in \mathcal{M}_{\mathbb{R}}^+$  is given by

$$\int u d\mu := \sup\{I_\mu(g) : g \leq u, g \in \mathcal{E}^+\} \in [0, \infty].$$

If we need to emphasize the integration variable, we also write

$$\int u(x) \mu(dx) \quad \text{or} \quad \int u(x) d\mu(x)$$

#### 4.4 Theorem.

For all  $f \in \mathcal{E}^+$  we have  $\int f du = I_\mu(f)$ .

#### 4.5 Theorem.

Let  $(X, \mathcal{A}, \mu)$  be a measure space. For an increasing sequence of numerical functions  $(u_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}_{\mathbb{R}}^+, 0 \leq u_j \leq u_{j+1} \leq \dots$ , we have  $u := \sup_{j \in \mathbb{N}} u_j \in \mathcal{M}_{\mathbb{R}}^+$  and

$$\int \sup_{j \in \mathbb{N}} u_j d\mu = \sup_{j \in \mathbb{N}} \int u_j d\mu$$



**4.6 Theorem.**

Let  $u \in \mathcal{M}_{\mathbb{R}}^+$ . Then

$$\int u d\mu = \lim_{j \rightarrow \infty} \int f_j d\mu$$

holds for every increasing sequence  $(f_j)_{j \in \mathbb{N}} \subseteq \mathcal{E}^+$  with  $\lim_{j \rightarrow \infty} f_j = u$ .

**4.7 Theorem.**

Let  $u, v \in \mathcal{M}_{\mathbb{R}}^+$ . Then

1.  $\int 1_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}$
2.  $\int \alpha u d\mu = \alpha \int u d\mu$
3.  $\int (u + v) d\mu = \int u d\mu + \int v d\mu$
4.  $u \leq v \implies \int u d\mu \leq \int v d\mu$

**4.8 Theorem.**

Let  $(u_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}_{\mathbb{R}}^+$ . Then  $\sum_{j=1}^{\infty} u_j$  is measurable and we have

$$\int \sum_{j=1}^{\infty} u_j d\mu = \sum_{j=1}^{\infty} \int u_j d\mu$$

**4.9 Theorem.**

Let  $(u_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}_{\mathbb{R}}^+$  be a sequence of positive measurable numerical functions.

Then  $u := \liminf_{j \in \mathbb{N}} \int u_j d\mu$  is measurable and

$$\int \liminf_{j \rightarrow \infty} u_j d\mu \leq \liminf_{j \rightarrow \infty} \int u_j d\mu$$

**4.10 Problem 9.1.**

Let  $f : X \rightarrow \mathbb{R}$  be a positive simple function of the form

$$f(x) = \sum_{j=1}^m \xi_j 1_{A_j}(x) \quad \xi_j \geq 0, A_j \in \mathcal{A}.$$

Show that

$$I_{\mu}(f) = \sum_{j=1}^m \xi_j \mu(A_j)$$

**Proof.**

$$I_\mu(f) = I_\mu\left(\sum_{j=1}^m \xi_j 1_{A_j}\right) = \sum_{j=1}^m \xi_j I_\mu(1_{A_j}) = \sum_{j=1}^m \xi_j \mu(A_j)$$

□

#### 4.11 Problem 9.5.

Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $u \in \mathcal{M}^+(\mathcal{A})$ . Show that the set-function

$$A \mapsto \int 1_A u d\mu \quad A \in \mathcal{A}$$

is a measure.

**Proof.**

Set

$$\nu : \mathcal{A} \rightarrow [0, \infty] : A \mapsto \int 1_A u d\mu.$$

1. To show that  $\nu(\emptyset) = 0$ . Notice that  $1_\emptyset \equiv 0$ .
2. Let  $A = \bigcup_{j \in \mathbb{N}} A_j$  a disjoint union of sets  $A_j \in \mathcal{A}$ . Note that

$$\sum_{j=1}^{\infty} 1_{A_j} = 1_A$$

We have to show that

$$\begin{aligned} \nu\left(\bigcup_{j \in \mathbb{N}} A_j\right) &= \int \left(\sum_{j=1}^{\infty} 1_{A_j}\right) \cdot u d\mu \\ &= \int \left(\sum_{j=1}^{\infty} 1_{A_j} u\right) d\mu \\ &= \sum_{j=1}^{\infty} \int 1_{A_j} u d\mu \\ &= \sum_{j=1}^{\infty} \nu(A_j). \end{aligned}$$

□

#### 4.12 Problem 9.8.

Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $(u_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}^+(\mathcal{A})$ . If  $u_j \leq u$  for all  $j \in \mathbb{N}$  and some  $u \in \mathcal{M}^+(\mathcal{A})$  with  $\int u d\mu < \infty$ , then

$$\limsup_{j \in \mathbb{N}} \int u_j d\mu \leq \int \limsup_{j \in \mathbb{N}} u_j d\mu.$$

#### Proof.

Showing that

$$\limsup_{j \in \mathbb{N}} \int u_j d\mu \leq \int \limsup_{j \in \mathbb{N}} u_j d\mu$$

is equivalent with showing that

$$-\liminf_{j \in \mathbb{N}} \int -u_j d\mu \leq -\int \liminf_{j \in \mathbb{N}} -u_j d\mu$$

which is equivalent with showing that

$$\int \liminf_{j \in \mathbb{N}} -u_j d\mu \leq \liminf_{j \in \mathbb{N}} \int -u_j d\mu$$

which is equivalent with showing that

$$\int u d\mu + \int \liminf_{j \in \mathbb{N}} -u_j d\mu \leq \int u d\mu + \liminf_{j \in \mathbb{N}} \int -u_j d\mu$$

which is equivalent with showing that

$$\int u + \liminf_{j \in \mathbb{N}} -u_j d\mu \leq \liminf_{j \in \mathbb{N}} \left( \int u d\mu + \int -u_j d\mu \right)$$

which is equivalent with showing that

$$\int \liminf_{j \in \mathbb{N}} (u - u_j) d\mu \leq \liminf_{j \in \mathbb{N}} \left( \int u - u_j d\mu \right).$$

By hypothesis,  $u_j \leq u$ . So we have that  $u - u_j$  is a sequence of positive measurable functions and therefore our last statement follows by the theorem of Fatou. □

## 4.2 Measure Theory Chapter 7

### 4.13 Proposition.

Let  $(X, \mathcal{A}), (X', \mathcal{A}')$  be measurable spaces and let  $\mathcal{A}' = \sigma(\mathcal{G}')$ . Then  $T : X \rightarrow X'$  is  $\mathcal{A}/\mathcal{A}'$ -measurable if and only if  $T^{-1}(\mathcal{G}') \subseteq \mathcal{A}$ .

It suffices to show assume  $T^{-1}(\mathcal{G}') \subseteq \mathcal{A}$  and show that

$$T^{-1}(\mathcal{A}) \subseteq \mathcal{A}.$$

Consider  $\Sigma := \{A' \subseteq X' : T^{-1}(A') \in \mathcal{A}\}$ . We have that  $\mathcal{G}' \subseteq \Sigma$ . It suffices to show that

$$\mathcal{A}' \subseteq \Sigma.$$

It suffices to show that  $\Sigma$  is a  $\sigma$ -algebra.

1. To show that  $X' \in \Sigma$ , it suffices to show that  $T^{-1}(X') \in \mathcal{A}$ .
2. Showing that

$$A' \in \Sigma \implies A'^c \in \Sigma$$

is equivalent with showing that

$$T^{-1}(A') \in \mathcal{A} \implies T^{-1}(A'^c) \in \mathcal{A} \quad \checkmark$$

3. Showing that

$$(A'_j)_{j \in \mathbb{N}} \subseteq \Sigma \implies \bigcup_{j \in \mathbb{N}} A'_j \in \Sigma$$

is equivalent with showing that

$$T^{-1}(A'_j) \in \mathcal{A} \implies T^{-1}\left(\bigcup_{j \in \mathbb{N}} A'_j\right) \in \mathcal{A} \quad \checkmark$$

### 4.14 Proposition.

Let  $(X, \mathcal{A}), (X', \mathcal{A}')$  be measurable spaces and  $T : X \rightarrow X'$  be an  $\mathcal{A}/\mathcal{A}'$ -measurable map. For every measure  $\mu$  on  $(X, \mathcal{A})$ ,

$$\mu'(A') := T(\mu)(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}'$$

defines a measure on  $(X', \mathcal{A}')$ .

**Proof.**

1. To show that

$$\mu(T^{-1}(\emptyset)) = 0 \quad \checkmark$$

2. Assume  $(A'_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}'$  mutually disjoint sets and show that

$$\mu'(\bigcup_{j \in \mathbb{N}} A'_j) = \sum_{j \in \mathbb{N}} \mu'(A'_j),$$

which is equivalent with showing that

$$\mu(T^{-1}(\bigcup_{j \in \mathbb{N}} A'_j)) = \sum_{j \in \mathbb{N}} \mu(T^{-1}(A'_j)),$$

which is equivalent with showing that

$$\mu(T^{-1}(\bigcup_{j \in \mathbb{N}} A'_j)) = \mu(\bigcup_{j \in \mathbb{N}} T^{-1}(A'_j)) \quad \checkmark.$$

□

#### 4.15 Problem 7.9i.

Let  $\mu$  be a measure on  $(\mathbb{R}, \mathcal{B})$ . Show that

$$F_\mu(x) := \begin{cases} \mu[0, x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu[x, 0) & \text{if } x < 0 \end{cases}$$

1. is monotonically increasing
2. left-continuous function

**Proof.**

1. Showing that  $F_\mu$  is monotonically increasing is equivalent with showing that

$$x \leq y \implies F_\mu(x) \leq F_\mu(y).$$

- (a)  $x \leq 0 \leq y$  : Then  $F_\mu(x) = -\mu[x, 0] \leq 0$  and  $F_\mu(y) = \mu[0, y] \geq 0$ .
  - (b)  $0 < x \leq y$  : Then  $[0, x] \subseteq [0, y]$ . And  $\mu[0, x] \leq \mu[0, y]$ .
  - (c)  $x \leq y < 0$  : Then  $[y, 0] \subseteq [x, 0]$ . And  $\mu[y, 0] \leq \mu[x, 0]$ .
2. Showing that  $F_\mu$  is left continuous is equivalent with assuming  $(x_k)$  a sequence such that  $x_k < x$  and  $x_k \uparrow x$  and showing that

$$\lim_{k \rightarrow \infty} F_\mu(x_k) = F_\mu(x).$$

If  $x > 0$ , it suffices to show that

$$\lim_{k \rightarrow \infty} \mu[0, x_k] = \mu[0, x].$$

If  $x < 0$  it suffices to show that

$$\lim_{k \rightarrow \infty} -\mu[x_k, 0] = -\mu[x, 0].$$

If  $x = 0$  it suffices to show that

$$\lim_{k \rightarrow \infty} -\mu[x_k, 0] = 0.$$

Remember that:

1. For any increasing sequence  $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$  with  $A_j \uparrow A \in \mathcal{A}$  we have

$$\mu(A) = \mu(\cup A_j) = \lim_{j \in \mathbb{N}} \mu(A_j)$$

2. For any decreasing sequence  $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$  with  $A_j \downarrow A \in \mathcal{A}$  we have

$$\mu(A) = \mu(\cap A_j) = \lim_{j \in \mathbb{N}} \mu(A_j)$$

□

#### 4.16 Problem 7.9ii.

Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be a Stieltjes function. Show that

$$\nu_F[a, b) = F(b) - F(a) \quad \forall a, b \in \mathbb{R}, a < b$$

has a unique extension to a measure on  $\mathcal{B}$ .

**Proof.**

By theorem 6.1 it suffices to show that  $\nu_F$  is a pre-measure. To show this it suffices to show that

1.  $\nu_F(\emptyset) = \nu_F[a, a) = 0$
2.  $\nu_F([a, b) \cup [b, c)) = \nu_F([a, b)) + \nu_F([b, c))$

•

$$\begin{aligned}
 \nu_F([a, b)) + \nu_F([b, c)) &= F(b) - F(a) + F(c) - F(b) \\
 &= F(c) - F(a) \\
 &= \nu_F[a, c) \\
 &= \nu_F([a, b) \cup [b, c))
 \end{aligned}$$

3. For any decreasing sequence  $[a_j, b_j)_{j \in \mathbb{N}} \subseteq \mathcal{J}$  with  $[a_j, b) \downarrow [a, b) \in \mathcal{J}$  we have

$$\nu_F([a, b)) = \lim_{j \in \infty} \nu_F[a_j, b).$$

This last statement is equivalent with

$$F(b) - F(a) = \lim_{j \in \infty} (F(b) - F(a_j)).$$

Note that since  $[a_j, b_j) \downarrow [a, b) \in \mathcal{J}$  we have that  $a_j \uparrow a, a_j \leq a$  and therefore

$$\lim_{j \in \infty} (F(b) - F(a_j)) = F(b) - F(a),$$

as  $F$  is left-continuous.

4.  $\mathcal{J}$  contains an exhausting sequence  $[a_j, b_j)$  such that  $[a_j, b_j) \uparrow \mathbb{R}$  and  $\nu_F[a_j, b_j) < \infty$

□

## 5 13-10-2014

### 5.1 Markov Chains 1.2

#### 5.1 Definition.

We say that  $i$  leads to  $j$  and write  $i \rightarrow j$  if

$$\mathbb{P}_i(X_n = j \text{ for some } n \geq 0) > 0.$$

#### 5.2 Definition.

We say  $i$  communicates with  $j$  and write  $i \leftrightarrow j$  if both  $i \rightarrow j$  and  $j \rightarrow i$ .

#### 5.3 Theorem.

For distinct states  $i$  and  $j$  the following are equivalent:

1.  $i \rightarrow j \iff \mathbb{P}_i(X_n = j \text{ for some } n \geq 0) > 0$
2.  $p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n} > 0$  for some states  $i_1, \dots, i_n$  with  $i_1 = i$  and  $i_n = j$
3.  $p_{ij}^n > 0$  for some  $n \geq 0$

#### Proof.

Remember that

$$p_{ij}^n = \mathbb{P}_i(X_n = j) = \sum_{i_2, \dots, i_{n-1}} p_{ii_2} p_{i_2 i_3} \cdots p_{i_{n-1} j}.$$

From this everything follows. □

#### 5.4 Proposition.

Show that  $i \rightarrow i$ .

#### Proof.

$$i \rightarrow i \iff \mathbb{P}_i(X_n = i \text{ for some } n \geq 0) > 0$$

This follows, as  $\mathbb{P}_i(X_0 = i) = 1$ . □

#### 5.5 Definition.

The equivalence classes of the equivalence relations  $\leftrightarrow$  are called *communicating classes*. We say that a class  $C$  is closed if

$$i \in C, i \rightarrow j \implies j \in C$$



### 5.6 Definition.

A state  $i$  is *absorbing* if  $\{i\}$  is a closed class.

### 5.7 Definition.

A chain or transition matrix  $P$  where  $I$  is a single class is called *irreducible*.

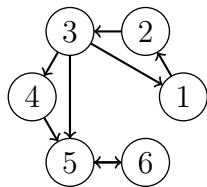
### 5.8 Example 1.2.2.

Find the communicating classes associated to the stochastic matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

**Proof.**

The solution is obvious from the diagram. The classes being  $\{1, 2, 3\}$ ,  $\{4\}$  and  $\{5, 6\}$ . With only  $\{5, 6\}$  closed.

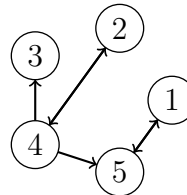
☐

### 5.9 Exercise 1.2.1.

Identify the communicating classes of the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

The solution is obvious from the diagram. The classes being  $\{1, 5\}, \{2, 4\}$



and  $\{3\}$ . With  $\{1, 5\}$  closed and  $\{3\}$  absorbing.

## 5.2 Markov Chains 1.3

### 5.10 Definition.

Let  $X_n$  be a Markov chain with transition matrix  $P$ . The *hitting time* of a subset  $A$  of  $I$  is the random variable

$$H^A : \Omega \rightarrow \{0, 1, 2, \dots, \infty\}$$

given by

$$H^A(\omega) = \inf\{n \geq 0 : X_n(\omega) \in A\}$$

where we agree that the infimum of the empty set  $\emptyset$  is  $\infty$ . The probability starting from  $i$  that  $(X_n)_{n \geq 0}$  ever hits  $A$  is then

$$h_i^A = \mathbb{P}_i(H^A < \infty).$$

When  $A$  is a closed class,  $h_i^A$  is called the *absorption probability*. The mean time taken for  $X_n$  to reach  $A$  is given by

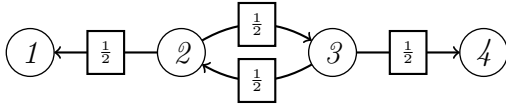
$$k_i^A = E_i(H^A) = \sum_{n < \infty} n \mathbb{P}(H^A = n) + \infty \mathbb{P}(H^A = \infty).$$

We shall often write less formally

$$h_i^A = \mathbb{P}_i(\text{hit } A) \quad k_i^A = E_i(\text{time to hit } A).$$

### 5.11 Example 1.3.1.

Consider the chain with following diagram:



1. Starting from 2, what is the probability of absorption in 4?
2. How long does it take until the chain is absorbed in 1 or 4 ?

**Proof.**

1. Note that  $A = \{4\}$  is a closed class. The absorption probability is defined as

$$h_i := h_i^{\{4\}} = \mathbb{P}_i(\text{hit } 4).$$

We have

$$\begin{aligned} h_1 &= 0 \\ h_4 &= 1 \\ h_2 &= \frac{1}{2}h_1 + \frac{1}{2}h_3 = \frac{1}{2}h_3 \\ h_3 &= \frac{1}{2}h_2 + \frac{1}{2}h_4 = \frac{1}{2}h_2 + \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} h_2 &= \frac{1}{4}h_2 + \frac{1}{4} \\ \implies \frac{3}{4}h_2 &= \frac{1}{4} \\ \implies h_2 &= \frac{1}{3} \end{aligned}$$

2. We need to compute

$$k_2 = k_2^{\{1,4\}} = E_2(\text{time to hit } \{1,4\}).$$

We have

$$\begin{aligned} k_1 &= 0 \\ k_4 &= 0 \\ k_2 &= 1 + \frac{1}{2}k_3 \\ k_3 &= 1 + \frac{1}{2}k_2 \end{aligned}$$

Hence

$$\begin{aligned} k_2 &= \frac{3}{2} + \frac{1}{4}k_2 \\ \implies \frac{3}{4}k_2 &= \frac{3}{2} \\ \implies k_2 &= 2 \end{aligned}$$

□

### 5.12 Theorem 1.3.2.

The vector of hitting probabilities  $h^A = (h_i^A : i \in I)$  is the minimal non-negative solution to the system

$$\begin{cases} h_i^A = 1 & \text{for } i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & \text{for } i \notin A. \end{cases}$$

Minimality means that if  $x = (x_i : i \in I)$  is another solution with  $x_i \geq 0$  for all  $i$ , then  $x_i \geq h_i$  for all  $i$ .

### 5.13 Example 1.3.1(continued).

Use theorem 1.3.2 to compute  $h_2$  again.

#### Proof.

The vector of hitting probabilities  $h^{\{4\}} = (h_1^{\{4\}}, h_2^{\{4\}}, h_3^{\{4\}}, h_4^{\{4\}})$  is the minimal non-negative solution to the system

$$\begin{aligned}h_1^{\{4\}} &= \sum_{j \in I} p_{ij} h_j^{\{4\}} = h_1^{\{4\}} \\h_2^{\{4\}} &= \sum_{j \in I} p_{ij} h_j^{\{4\}} = \frac{1}{2} h_1^{\{4\}} + \frac{1}{2} h_3^{\{4\}} \\h_3^{\{4\}} &= \sum_{j \in I} p_{ij} h_j^{\{4\}} = \frac{1}{2} h_2^{\{4\}} + \frac{1}{2} h_4^{\{4\}} \\h_4^{\{4\}} &= 1\end{aligned}$$

The minimality condition gives, that  $h_1^{\{4\}} = 0$ . So that

$$\begin{aligned}h_2^{\{4\}} &= \frac{1}{2} h_3^{\{4\}} \\h_3^{\{4\}} &= \frac{1}{2} h_2^{\{4\}} + \frac{1}{2}\end{aligned}$$

which gives:

$$\begin{aligned}h_2^{\{4\}} &= \frac{1}{4} h_2^{\{4\}} + \frac{1}{4} \implies h_2^{\{4\}} = \frac{1}{3} \\h_3^{\{4\}} &= \frac{1}{4} h_3^{\{4\}} + \frac{1}{2} \implies h_3^{\{4\}} = \frac{2}{3}\end{aligned}$$

□

### 5.14 Theorem.

Consider a recurrence relation of the form

$$ax_{n+1} + bx_n + cx_{n-1} = 0 \quad a, c \neq 0.$$

Let  $\alpha, \beta$  be the roots of the quadratic equation

$$ax^2 + bx + c.$$

Then the general solution is given by

$$x_n = \begin{cases} A\alpha^n + B\beta^n & \text{if } \alpha \neq \beta \\ (A + nB)\alpha^n & \text{if } \alpha = \beta \end{cases}$$

### 5.15 Proposition.

Give a general solution for the recurrence relation

$$\begin{aligned} h_0 &= 1 \\ h_i &= ph_{i+1} + qh_{i-1} \end{aligned}$$

#### Proof.

Note that we have  $-ph_{i+1} + h_i - qh_{i-1} = 0$ . Consider

$$-px^2 + x - 1 + p = 0$$

We have the roots  $\alpha = 1, \beta = \frac{q}{p}$ . If  $q \neq p$ , this gives

$$h_i = A\alpha^i + B\beta^i = A + B\left(\frac{q}{p}\right)^i.$$

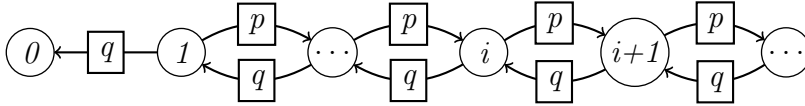
And if  $p = q$ , then  $\alpha = \beta = 1$ , and we have

$$h_i = A\alpha^i + B\beta^i = A + iB$$

□

### 5.16 Example 1.3.3.

Consider the Markov chain with diagram



where  $0 < p = 1 - q < 1$ . What is  $h_i = \mathbb{P}_i(\text{hit } 0)$ ?

#### Proof.

We know that  $h$  is the minimal non-negative solution to

$$\begin{aligned} h_0 &= 1 \\ h_i &= ph_{i+1} + qh_{i-1} \end{aligned}$$

We consider some cases:

- Suppose  $p = q$ , then we have

$$h_i = A\alpha^i + B\beta^i = A + iB$$

and as  $0 \leq h_i \leq 1$  is a probability, we must have  $B = 0$ . We then have

$$h_i = A,$$

and as  $h_0 = 1$ , we must have  $h_i = 1$ .

- Suppose  $p \neq q$ , we then have

$$h_i = A\alpha^i + B\beta^i = A + B\left(\frac{q}{p}\right)^i.$$

If  $\frac{q}{p} > 1$ , then we must set  $B = 0$  again.

- Suppose  $\frac{q}{p} < 1$ . We have that  $h_0 = 1$ , and therefore  $A + B = 1$ . Hence:

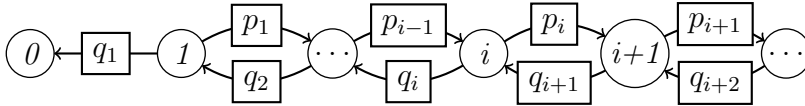
$$h_i = \left(\frac{q}{p}\right)^i + A\left(1 - \left(\frac{q}{p}\right)^i\right)$$

So the minimal non-negative solutions is  $h_i = (q/p)^i$ .

□

### 5.17 Example 1.3.4.

Consider the Markov chain with diagram



where for  $i = 1, 2, \dots$ , we have  $0 < p_i = 1 - q_i < 1$ . As in the preceding example, 0 is the absorbing state, and we wish to calculate the absorption probability starting from  $i$ .

**Proof.**

Consider the system of equations

$$\begin{aligned} h_0 &= 1 \\ h_i &= p_i h_{i+1} + q_i h_{i-1} \quad \text{for } i = 1, 2, \dots \end{aligned}$$

Consider

$$u_i := h_{i-1} - h_i,$$

then

$$\begin{aligned} p_i u_{i+1} &= p_i h_i - p_i h_{i+1} \\ q_i u_i &= q_i h_{i-1} - q_i h_i \end{aligned}$$

$$\implies q_i u_i - p_i u_{i+1} = h_i - q_i h_i - p_i h_i = 0$$

Therefore  $p_i u_{i+1} = q_i u_i$  and we have

$$u_{i+1} = \left(\frac{q_i}{p_i}\right) u_i = \prod_{j=1}^i \frac{q_j}{p_j} u_1 = \gamma_1 u_1$$

We also have

$$u_1 + \dots + u_i = h_0 - h_i$$

so

$$h_i = h_0 - (u_1 + \dots + u_i) = 1 - u_1(\gamma_0 + \dots + \gamma_{n-1}).$$

At this point  $A$  remains to be determined. In the case that  $\sum_{i=0}^{\infty} \gamma_i = \infty$ , we must have  $A = 0$ . In the other case, we can't take  $A < 0$ , but we can take  $A > 0$  so long as

$$h_i = 1 - A \sum_{i=0}^{i-1} \gamma_i \geq 0.$$

Which means that

$$A \leq \left(\sum_{i=0}^{i-1} \gamma_i\right)^{-1}$$

but still as big as possible. So we get

$$A = \left(\sum_{i=0}^{\infty} \gamma_i\right)^{-1}.$$

And therefore

$$h_i = \frac{\sum_{j=i}^{\infty} \gamma_j}{\sum_{j=0}^{\infty} \gamma_j}$$

□

**5.18 Theorem 1.3.5.**

*The vector of mean hitting times  $k^A = (k^A : i \in I)$  is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} k_i^A = 0 & \text{for } i \in A \\ k_i^A = 1 + \sum_{j \notin A} p_{ij} k_j^A & \text{for } i \notin A \end{cases}$$