Given

- Let $(X, \mathcal{A}, \mu)$ be a measure space.
- Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^1(\mu)$ be an increasing sequence of integrable functions $u_1 \le u_2 \le \dots$ with limit $u := \sup_{j\in\mathbb{N}} u_j$.

Tools

$(u_j \uparrow) \subseteq \mathcal{M}^+_{\bar{\mathbb{R}}} \Rightarrow \int \sup_{j\in\mathbb{N}} u_j \; d\mu = \sup_{j\in\mathbb{N}} \int u_j \; d\mu$

Assume

$\sup_{j\in\mathbb{N}} u_j \in \mathcal{L}^1(\mu)$

$$\boxed{\sup_{j\in\mathbb{N}} \int u_j \; d\mu < \infty}$$

$$\Uparrow$$

$$\sup_{j\in\mathbb{N}} \int u_j \; d\mu = \int \sup_{j\in\mathbb{N}} u_j \; d\mu$$

$$\Uparrow$$

$$\sup_{j\in\mathbb{N}} \left(\int u_j \; - u_1 \; d\mu \right) + \int u_1 \; d\mu = \int \sup_{j\in\mathbb{N}} u_j \; d\mu$$

$$\Uparrow$$

$$\int \sup_{j\in\mathbb{N}} (u_j \; - u_1) \; d\mu + \int u_1 \; d\mu = \int \sup_{j\in\mathbb{N}} u_j \; d\mu$$

Given

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- Let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^1(\mu)$ be an increasing sequence of integrable functions $u_1 \le u_2 \le \dots$ with limit $u := \sup_{j\in\mathbb{N}} u_j$.

Tools

$(u_j \uparrow) \subseteq \mathcal{M}^+_{\bar{\mathbb{R}}} \Rightarrow \int \sup_{j\in\mathbb{N}} u_j \; d\mu = \sup_{j\in\mathbb{N}} \int u_j \; d\mu$

Assume

$\sup_{j\in\mathbb{N}} \int u_j \; d\mu < \infty$

$$\boxed{\sup_{j\in\mathbb{N}} u_j \in \mathcal{L}^1(\mu)}$$

$$\Uparrow$$

$$\int \sup_{j\in\mathbb{N}} u_j \; d\mu < \infty$$

$$\Uparrow$$

$$\int \sup_{j\in\mathbb{N}} u_j \; d\mu = \sup_{j\in\mathbb{N}} \int u_j \; d\mu$$

$$\Uparrow$$

$$\int \sup_{j\in\mathbb{N}} (u_j \; - u_1) \; d\mu + \int u_1 \; d\mu = \sup_{j\in\mathbb{N}} \int u_j \; d\mu$$

$$\Uparrow$$

$$\sup_{j\in\mathbb{N}} \left(\int u_j \; - u_1 \; d\mu \right) + \int u_1 \; d\mu = \sup_{j\in\mathbb{N}} \int u_j \; d\mu$$

Given

- Let $(X, \mathcal{A}, \mu)$ be a measure space.
- Let $(v_k)_{k\in\mathbb{N}} \subseteq \mathcal{L}^1(\mu)$ be a decreasing sequence of integrable functions $v_1 \geq v_2 \geq \dots$ with limit $v := \inf_{k\in\mathbb{N}} v_k$.

Tools

- $\inf\limits_{j\in\mathbb{N}} f_j(x) = -\sup\limits_{j\in\mathbb{N}}(-f_j(x))$
- $\sup\limits_{j\in\mathbb{N}} u_j \in \mathcal{L}^1(\mu) \iff \sup\limits_{j\in\mathbb{N}} \int u_j d\mu < \infty$
  if $(u_j \uparrow)_{j\in\mathbb{N}} \subseteq \mathcal{L}^1(\mu)$

$$\boxed{\inf_{j\in\mathbb{N}} v_j \in \mathcal{L}^1(\mu) \iff \inf_{j\in\mathbb{N}} \int v_j d\mu < \infty}$$

$$\Uparrow$$

$$-\sup_{j\in\mathbb{N}} -v_j \in \mathcal{L}^1(\mu) \iff -\sup_{j\in\mathbb{N}} \int -v_j d\mu < \infty$$

$$\Uparrow$$

$$\sup_{j\in\mathbb{N}} -v_j \in \mathcal{L}^1(\mu) \iff \sup_{j\in\mathbb{N}} \int -v_j \; d\mu < \infty$$

$$\Uparrow$$

$$(-v_j \uparrow)_{j\in\mathbb{N}} \subseteq \mathcal{L}^1(\mu)$$

Given

- Let $(X, \mathcal{A}, \mu)$ be a measure space.
- $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{L}^1(\mu) : |u_j| \leq w$ for some $w \in \mathcal{L}^1_+(\mu)$
- $u(x) := \lim_{j\to\infty} u_j(x)$ exists for a.e. $x \in X$

Tools

- $\liminf\limits_{j\to\infty}(-f_j) = -\limsup\limits_{j\to\infty} f_j$
- $\big(u \in \mathcal{M}_{\bar{\mathbb{R}}}, v \in \mathcal{L}^1_+(\mu) : |u| \leq v \; a.e.\big)$
  $\implies u \in \mathcal{L}^1(\mu)$
- $(u_j) \subseteq \mathcal{M}^+_{\bar{\mathbb{R}}} \implies$
  $\int \liminf\limits_{j\to\infty} u_j d\mu \leq \liminf\limits_{j\to\infty} \int u_j d\mu$

$$\boxed{\lim_{j\to\infty} u_j \in \mathcal{L}^1(\mu)}$$

$$\Uparrow$$

$$|\lim_{j\to\infty} u_j| \leq w$$

$$\Uparrow$$

$$|u_j| \leq w$$

$$\boxed{\lim_{j\to\infty} \int |u_j - u| \; d\mu = 0}$$

$$\Uparrow$$

$$0 \leq \liminf_{j\to\infty} \int |u_j - u| d\mu \leq \limsup_{j\to\infty} \int |u_j - u| \; d\mu \leq 0$$

$$\Uparrow$$

$$\limsup_{j\to\infty} \int |u_j - u| \; d\mu \leq 0$$

$$\Uparrow$$

$$0 \leq \liminf_{j\to\infty} \int -|u_j - u| \; d\mu$$

$$\Uparrow$$

$$\int 2w \; d\mu \leq \int 2w \; d\mu + \liminf_{j\to\infty} \int -|u_j - u| \; d\mu$$

$$\Uparrow$$

$$\int \liminf_{j\to\infty}(2w - |u_j - u|) \; d\mu \; \leq \liminf_{j\to\infty} \int 2w - |u_j - u| \; d\mu$$

$$\Uparrow$$

$$2w - |u_j - u| \in \mathcal{M}^+_{\bar{\mathbb{R}}}$$

Given

- $\varnothing \neq (a,b) \subseteq \mathbb{R}$
- $u : (a,b) \times X \to \mathbb{R}$ a function such that:
  - $x \mapsto u(t,x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a,b)$.
  - $t \mapsto u(t,x)$ is continuous for every fixed $x \in X$
  - $|u(t,x)| \leq w(x)$ for all $(t,x) \in (a,b) \times X$ and some $w \in \mathcal{L}^1_+(\mu)$.
- the function $v : (a,b) \to \mathbb{R}$ given by

$$t \mapsto v(t) := \int u(t,x)\ \mu(dx)$$

Random:

- $t \in (a,b),\ (t_j)_{j\in\mathbb{N}} \subseteq (a,b) : \lim_{j\to\infty} t_j = t$
- $x \in X$

$$v \text{ is continuous}$$

$$\Uparrow$$

$$\lim_{j\to\infty} v(t_j) = v(t)$$

$$\Uparrow$$

$$\lim_{j\to\infty} \int u(t_j,x) = \int u(t,x)$$

$$\Uparrow$$

$$\lim_{j\to\infty} \int u(t_j,x) = \int \lim_{j\to\infty} u(t_j,x)$$

$$\Uparrow$$

$$|u(t_j,x)| \leq w(x) \ \text{ and } \ \lim_{j\to\infty} u(t_j,x) \ \text{ exists } a.e.$$

Given

- $\alpha \in \mathbb{R}$
- $f_\alpha(x) := x^\alpha, x > 0$

Random:

- $t \in (a,b),\ (t_j)_{j\in\mathbb{N}} \subseteq (a,b) : \lim_{j\to\infty} t_j = t$
- $x \in X$

Tools

- a continuous function is Borel measurable.
- a positive measurable function is integrable if the integral is finite
- $(u_j \uparrow) \subseteq \mathcal{M}^+_{\bar{\mathbb{R}}} \Longrightarrow \int \lim_{j\to\infty} u_j\, d\mu = \lim_{j\to\infty} \int u_j\, d\mu$
- If u is λ-a.e. continuous, then $u \in \mathcal{L}^1(\lambda)$ and

$$\int_{[a,b]} u\ d\lambda = (R)\int_a^b u(x)\ dx$$

$$f_\alpha \in \mathcal{L}^1(0,1) \iff \alpha > -1$$

$$\Uparrow$$

$$\int 1_{(0,1)}(x)\ x^\alpha d\lambda < \infty \iff \alpha > -1$$

$$\Uparrow$$

$$\int 1_{(0,1)}(x)\ x^\alpha d\lambda = \lim_{j\to\infty} \left(\frac{1}{\alpha+1} - \frac{1}{j^{\alpha+1}(\alpha+1)} \right)$$

$$\Uparrow$$

$$\int \lim_{j\to\infty} 1_{[1/j,1)}(x)\ x^\alpha\ d\lambda = \lim_{j\to\infty} \left[\frac{x^{\alpha+1}}{\alpha+1} \right]_{1/j}^1$$

$$\Uparrow$$

$$\lim_{j\to\infty} \int 1_{[1/j,1)}(x)\ x^\alpha\ d\lambda\ = \lim_{j\to\infty} (R)\int_{1/j}^1 x^\alpha\ dx$$

Given

- $\alpha > -1$ and $\beta \geq 0$
- $f(x) := x^\alpha e^{-\beta x}, x > 0$

Random:

- $x > 0$

Tools

- a continuous function is Borel measurable.
- $u \in \mathcal{M}_{\bar{\mathbb{R}}}$ and $\exists w \in \mathcal{L}^1_{\bar{\mathbb{R}}_+}(\mu) : |u| \leq w$ $\implies u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$
- $e^x \leq 1$ for $x \leq 0$
- $e^x = \sum_{k=0}^\infty x^k/k!$
- $x^\alpha \in \mathcal{L}^1(0,1) \iff \alpha > -1$
- $x^\alpha \in \mathcal{L}^1[1,\infty) \iff \alpha < -1$

$$\boxed{f \in \mathcal{L}^1(0,\infty)}$$

$$\Uparrow$$

$$x^\alpha e^{-\beta x} \leq w(x) \qquad \text{for some } w \in \mathcal{L}^1(0,\infty)$$

$$\boxed{x^\alpha e^{-\beta x} \leq x^\alpha 1_{(0,1)}(x) + \frac{N!}{\beta^n} x^{\alpha-N} 1_{[1,\infty)}(x)}$$

$$\Uparrow$$

$$e^{-\beta x} \leq 1 \quad \text{and} \quad e^{-\beta x} \leq \frac{N!}{\beta^N} x^{-N}$$

$$\Uparrow$$

$$\frac{(\beta x)^n}{N!} \leq e^{\beta x}$$

$$\Uparrow$$

$$\frac{(\beta x)^N}{N!} \leq \sum_{j=0}^\infty \frac{(\beta x)^j}{j!}$$

$$\boxed{x^\alpha 1_{(0,1)}(x) + \frac{N!}{\beta^n} x^{\alpha-N} 1_{[1,\infty)}(x) \in \mathcal{L}^1(0,\infty)}$$

$$\Uparrow$$

$$x^\alpha \in \mathcal{L}^1(0,1) \qquad x^{\alpha-N} \in \mathcal{L}^1[1,\infty)$$

$$\Uparrow$$

$$\alpha > -1 \qquad \alpha - N < -1$$

$$\Uparrow$$

$$\text{Set} \quad N := \lfloor \alpha - 2 \rfloor \in \mathbb{N}$$