Contents

1 9-10-2014

1.1 Measure Theory

1.1 Problem 6.1a.
Consider on $\mathbb{R}$ the family Σ of all Borel sets which are symmetric w.r.t. the origin. Show that Σ is a σ-algebra.

Proof.

1. To show that $\mathbb{R} \in \Sigma$, note that $\mathbb{R}$ is a Borel set that is symmetric w.r.t. to the origin.
2. To show that $A \in \Sigma \Rightarrow A^c \in \Sigma$, it suffices to show that
$$\forall x \in A : -x \in A \Longrightarrow \forall y \in A^c : -y \in A^c,$$
which is equivalent with showing that
$$\forall x \in A : -x \in A \Longrightarrow \forall y \notin A : -y \notin A,$$
which is equivalent with showing that
$$\exists y \notin A : -y \in A \Longrightarrow \exists x \in A : -x \notin A.$$
This last statement hold if we set $x := -y$.
3. To show that Σ is stable under countable unions, assume $A_j = B_j \cup B_j$ for some $B_j \in \mathcal{B}([0, \infty)$. We have
$$\bigcup_{j\in\mathbb{N}} A_j = \bigcup_{j\in\mathbb{N}} B_j \cup \bigcup_{j\in\mathbb{N}} -B_j \in \Sigma$$

□

1.2 Problem 6.3i.

Show that non-void open sets in $\mathbb{R}^n$ have always strictly positive Lebesgue measure.

Proof.
First remember that

1. $\lambda^n[a,b) = \prod_{j=1}^{n}(b_j - a_j)$
2. λ^n is a pre-measure that can be extended to a measure on $\mathcal{B}(\mathbb{R}^n)$.
3. λ^n is invariant under translations
4. $A \subseteq B \Longrightarrow \mu(A) \leq \mu(B)$
5. $Q_\epsilon = [-\epsilon, \epsilon)$

To show that $\lambda^n(U) > 0$ it suffices

$$\lambda^n(U') > 0$$

where $0 \in U'$ and $U' = x + U$ for some $x \in \mathbb{R}^n$. To show that it suffices to show that

$$\lambda^n(B_\epsilon(0)) > 0$$

where $B_\epsilon(0) \subseteq U$. To show that it suffices to show that $Q_{\epsilon'} \subseteq B_\epsilon$ for some $\epsilon' > 0$. This holds if we set $\epsilon' := \frac{\epsilon}{\sqrt{2n}}$. □

1.3 Problem 6.3ii.

Is 6.3i still true for closed sets ?

Proof.
No, take $\{0\}$, then $\lambda\{x\} = 0$. □

1.4 Problem 6.4i.

Show that $\lambda(a,b) = b - a$ for all $a, b \in \mathbb{R}, a \leq b$.

Proof.

$$\begin{aligned}
\lambda(a,b) &= \lambda([b-a) - \{b\}) & \\
&= \lambda[b,a) - \lambda\{b\} & \text{T4.3iii} \\
&= b - a - 0 & \text{Problem 4.11i}
\end{aligned}$$

$\square$

1.5 Problem 6.4ii.
Let $H \subseteq \mathbb{R}^2$ be a hyperplane which is perpendicular to the x_1-direction (that is to say: H is a translate of the x_2 axis). Show that

1. $H \in \mathcal{B}(\mathbb{R}^2)$
2. $\lambda^2(H) = 0$

Proof.

1. To show that $H \in \mathcal{B}(\mathbb{R}^2)$, it suffices to show that H is writable as an intersection of countable half-open sets. Note that:

$$H := \{y\} \times \mathbb{R} = \bigcap_{j \in \mathbb{N}} [y, y + 1/j) \times \mathbb{R}$$

2. We have that for any $\epsilon > 0$:

$$\begin{aligned} \lambda^2(H) &= \lambda^2(\{y\} \times \mathbb{R}) \\ &\leq \lambda^2\Big(\bigcup_{n \in \mathbb{N}} [y, y + \epsilon_n) \times [-n, n) \Big) \\ &\leq 2 \sum_{n \in \mathbb{N}} \epsilon_n n \\ &= \epsilon L \end{aligned}$$

This follows if we choose $\epsilon_n := \frac{\epsilon}{2^n}$. Therefore $\lambda^2(H) = 0$.

$\square$

1.6 Definition.
Let $(X, \mathcal{A}, \mu)$ be a measure space such that all singletons $\{x\} \in \mathcal{A}$. A point x is called an atom, if $\mu\{x\} > 0$. A measure is called *non-atomic* or *diffuse*, there are no atoms.

1.7 Problem 6.5i.
Show that λ^1 is diffuse.

Proof.
We've already shown that $\lambda\{x\} = 0$ for any $x \in \mathbb{R}$. $\square$

1.8 Problem 6.5iii.
Show that for a diffuse measure μ on $(X, \mathcal{A})$ all countable sets are null sets.

Proof.
All countable sets are writable as

$$\bigcup_{j=0}^{\infty} \{x_j\}$$

where $x_i \neq x_j$. So we get

$$\lambda\left(\bigcup_{j=0}^{\infty} \{x_j\}\right) = \sum_{j=0}^{n} \lambda\{x_j\} = 0.$$

$\square$

1.9 Definition.
A set $A \subseteq \mathbb{R}^n$ is called *bounded* if it can be contained in a ball $B_r \supseteq A$ of finite radius r. A set $A \subseteq \mathbb{R}^n$ is called *connected*, if we can go along a curve from any point $a \in A$ to any point $a' \in A$ without ever leaving A.

1.10 Problem 6.6a.
Construct an open and unbounded set in $\mathbb{R}$ with finite, strictly positive Lebesgue measure.

Proof.
Consider the set

$$U := \bigcup_{n=1}^{\infty} \left(n - \frac{1}{2^n}, n + \frac{1}{2^n}\right).$$

This is an open set, as it union of countable open sets. It is unbounded, for any $B_r(0)$ we have that $r + 1 \in U$ and not in $B_r(0)$. We have to show that it has finite lebesgue measure.

$$\begin{aligned} \lambda(U) &= \bigcup_{n=1}^{\infty} \left(n - \frac{1}{2^n}, n + \frac{1}{2^n}\right) \\ &= \sum_{n=1}^{\infty} \frac{2}{2^n} = 2. \end{aligned}$$

$\square$

1.11 Problem 6.6ii.
Construct an open, unbounded and connected set in $\mathbb{R}$ with finite, strictly positive Lebesgue measure.

Proof.
Consider
$$U = \bigcup_{j \in \mathbb{N}} [0, 0 + \epsilon/(2^j)) \times [-j, j)$$
then
$$\begin{aligned} \lambda^2(U) &= \left(\bigcup_{j \in \mathbb{N}} (-\frac{1}{2^j}, \frac{1}{2^j}) \times (-j, j) \right) \\ &\leq \sum_{j \in \mathbb{N}} \frac{4j}{2^j} \end{aligned}$$
Note that
$$\sum_{j \in \mathbb{N}} \frac{j}{2^j}$$
converges. $\square$

1.12 Problem 6.6iii.
Is there a connected, open and unbounded set in $\mathbb{R}$ with finite, strictly positive Lebesgue measure ?

Proof.
No, this is impossible. Since we are in one dimension, connectedness forces us to go between points in a straight, uninterrupted line. Since the set is unbounded, this means we must have a line of the sort (a, ∞) or $(-\infty, b)$ in our set and in both cases Lebesgue measure is infinite. $\square$

1.13 Definition.
Let $A \subset X$. The closure of A, denoted by $\bar{A}$, is the smallest closed set containing A, i.e.
$$\bar{A} = \bigcap_{\substack{F \in \mathcal{C} \\ F \supset A}} F$$

1.14 Definition.
A set $A \subseteq X$ is dense in X if $\bar{A} = X$

1.15 Problem 6.7.
Let $\lambda := \lambda^1|_{[0,1]}$ *be a Lebesgue measure on* $([0,1], \mathcal{B}[0,1])$. *Show that for every* $\epsilon > 0$ *there is a dense open set* $U \subseteq [0,1]$ *with* $\lambda(U) \leq \epsilon$.

Proof.
Note that $\mathbb{Q}$ is dense. We are going to make an open set contained in $\mathbb{Q}$. Consider

$$U := \bigcup_{j=1}^{\infty} (q_j - \epsilon_j, q_j + \epsilon_j)$$

Then

$$\lambda(U) = \lambda(\bigcup_{j=1}^{\infty} (q_j - \epsilon_j, q_j + \epsilon_j)) \leq \sum 2\epsilon_j.$$

So set $\epsilon_j := \frac{\epsilon}{2^{j-1}}$. And we are done. □

1.16 Problem 6.10i.
Let μ *be a measure on* $\mathcal{A} = \{\varnothing, [0,1), [1,2), [0,2)\}$ *of* $X = [0,2)$. *Such that*

$$\mu[0,1) = \mu[1,2) = 1/2 \qquad \mu[0,2) = 1.$$

Define for each $A \subseteq [0,2)$ *the family of countable* $\mathcal{A}$*-coverings of* A

$$\mathcal{C}(A) := \{(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A} : \bigcup_{j \in \mathbb{N}} A_j \supseteq A\}$$

and set

$$\mu^*(A) := inf\left\{ \sum_{j \in \mathbb{N}} \mu(S_j) : (S_j)_{j \in \mathbb{N}} \in \mathcal{C}(A) \right\}.$$

Define $\mathcal{A}^* := \{A \subseteq [0,2) : \mu^*(B) = \mu^*(B \cap A) + \mu^*(B - A) \quad \forall B \subseteq X\}$

Show that

1. *Find* $\mu^*(a,b), \mu^*\{a\}$
2. $(0,1), \{0\} \notin \mathcal{A}^*$

Note that in T6.1 it is proven that:

- $\mathcal{A} \subseteq \mathcal{A}^*$
- $\mu^*(A) = \mu(A) \quad \forall A \in \mathcal{A}$
- $\mathcal{A}^*$ is a σ-algebra and μ^* is a measure on $([0,2), \mathcal{A}^*)$

Proof.

1. We have

$$\begin{aligned} \mu^*(a,b) &= \mu[0,1) \quad \text{if } a,b \in [0,1) \\ \mu^*(a,b) &= \mu[1,2) \quad \text{if } a,b \in [1,2) \\ \mu^*(a,b) &= \mu[0,2) \quad \text{if } a \in [0,1), b \in [1,2) \end{aligned}$$

In the case of a singleton $\{a\}$ the best posibble cover is always either $[0,1)$ or $[1,2)$ so that $\mu^*\{a\} = 1/2$.

2. Suppose that $(0,1) \in \mathcal{A}^*$ then we would have that

$$\{0\} = [0,1) - (0,1) \in \mathcal{A}^*.$$

But this gives

$$\frac{1}{2} = \mu^*[0,1) = \mu^*(0,1) + \mu^*\{0\} = 1$$

□

2 10-10-2014

2.1 Definition.
Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be two measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$*-measureable* (or *measurable* unlesss this is too amiguous) if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A} \qquad \forall A' \in \mathcal{A}'.$$

We often denote this by $T^{-1}(\mathcal{A}') \subseteq \mathcal{A}'$.

2.2 Definition.
A *random variable* is a measurable map from a probability space (i.e. $\mu(X) = 1$) to any measurable space.

2.3 Lemma 7.2.
Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable if and only if

$$T^{-1}(G') \in \mathcal{A} \qquad \forall G' \in \mathcal{G}'.$$

2.4 Problem 7.1.
Show that

$$\tau_x : \mathbb{R}^n \to \mathbb{R}^n : B \mapsto B - x$$

is a $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$ measurable map.

Proof.
Showing that

$$\tau_x : \mathcal{B}(\mathbb{R}^n) \to \mathcal{B}(\mathbb{R}^n) : B \mapsto B - x$$

is $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$ measurable, is equivalent with showing that

$$\tau_x^{-1}(B) \in \mathcal{B}(\mathbb{R}^n) \qquad \forall B \in \mathcal{B}(\mathbb{R}^n),$$

which in turn is equivalent with showing that

$$x + B \in \mathcal{B}(\mathbb{R}^n) \qquad \forall B \in \mathcal{J}(\mathbb{R}^n),$$

which in turn is equivalent with showing that

$$x + [a, b) \in \mathcal{B}(\mathbb{R}^n) \qquad \forall a, b \in \mathbb{R}^n.$$

This follows as $x + [a, b) = [x + a, x + b) \in \mathcal{J}(\mathbb{R}^n) \subseteq \mathcal{B}(\mathbb{R}^n)$. $\square$

2.5 Theorem.
Every continuous map $T : \mathbb{R}^n \to \mathbb{R}^m$ *is* $\mathcal{B}^n/\mathcal{B}^m$ *measurable.*

Proof.
Showing that

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

is $\mathcal{B}^n/\mathcal{B}^m$ measurable, is equivalent with showing that

$$T^{-1}(\mathcal{O}^m) \subseteq \mathcal{B}^n.$$

As $\mathcal{O}^n \subseteq \sigma(\mathcal{O}^n) = \mathcal{B}^n$, it suffices to show that

$$T^{-1}(\mathcal{O}^m) \subseteq \mathcal{O}^n,$$

which follows from the continuity of T. $\square$

2.6 Definition.
Let $(T_i)_{i\in I}$ be arbitrarily many mappings $T_i : X \to X_i$ from the same space X into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest σ-algebra on X that makes all T_i simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma\Big(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\Big).$$

We say that $\sigma(T_i : i \in I)$ is *generated by the family* $(T_i)_{i\in I}$.

2.7 Theorem.
Let $(X_j, A_j), j = 1, 2, 3,$ *be measurable spaces and* $T : X_1 \to X_2, S : X_2 \to X_3$ *be* $\mathcal{A}_1/\mathcal{A}_2-$ *resp.* $\mathcal{A}_2/\mathcal{A}_3$*-measurable maps. Then* $S \circ T : X_1 \to X_3$ *is* $\mathcal{A}_1/\mathcal{A}_3$*-measurable.*

2.8 Problem 7.4.
Let X be a set, $(X_i, \mathcal{A}_i), i \in I$,be arbitrarily many measurable spaces, and $T_i : X \to X_i$ be a family of maps. Show that a map f from a measurable space $(F, \mathcal{F})$ to $(X, \sigma(T_i : i \in I)$ is measurable if, and only if, all maps $T_i \circ f$ are $\mathcal{F}/\mathcal{A}_i$-measurable.

Proof of $\Longrightarrow$.
To show that all maps $T_i \circ f$ are $\mathcal{F}/\mathcal{A}_i$-measurable, it suffices to show that $T_i : X \to X_i$ is $\sigma(T_i : i \in I)/\mathcal{A}_i$-measurable and $f : F \to X$ is $\mathcal{F}/\sigma(T_i : i \in I)$-measurable.

By hypothesis, is suffices to show that $T_i : X \to X_i$ is $\sigma(T_i : i \in I)/\mathcal{A}_i$-measurable, which is equivalent with showing that

$$T_i^{-1}(A_i) \in \sigma(T_i : i \in I) \qquad \forall A_i \in \mathcal{A}_i.$$

It suffices to assume $A_i \in \mathcal{A}_i$ and show that

$$T_i^{-1}(A_i) \in \bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \qquad \checkmark.$$

$\square$

Proof of $\Longleftarrow$.
To show that a map f from a measurable space $(F, \mathcal{F})$ to $(X, \sigma(T_i : i \in I)$ is measurable, it suffices to show that

$$f^{-1}(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)) \subseteq \mathcal{F}$$

To show this it suffices to show that

$$\bigcup_{i \in I} f^{-1}(T_i^{-1}(\mathcal{A}_i)) \subseteq \mathcal{F},$$

to show this it suffices to show that

$$f^{-1}(T_i^{-1}(\mathcal{A}_i)) \subseteq \mathcal{F},$$

to show this it suffices to show that

$$(T_i \circ f)^{-1}(\mathcal{A}_i) \subseteq \mathcal{F}.$$

This follows by hypothesis. $\square$

2.9 Problem 7.8.
Let $T : X \to Y$ be any map. Show that

$$T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G}))$$

holds for arbitrary families of $\mathcal{G}$ of subsets of Y.

Proof.
To show that

$$T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G}))$$

it suffices to show:

1. $T^{-1}(\sigma(\mathcal{G})) \subseteq \sigma(T^{-1}(\mathcal{G}))$
2. $\sigma(T^{-1}(\mathcal{G})) \subseteq T^{-1}(\sigma(\mathcal{G}))$

To show

$$T^{-1}(\sigma(\mathcal{G})) \subseteq \sigma(T^{-1}(\mathcal{G})),$$

it suffices to show that T is $\sigma(T^{-1}(\mathcal{G}))/\sigma(\mathcal{G})$ measurable.

To show that it suffices to show that

$$T^{-1}(\mathcal{G}) \subseteq \sigma(T^{-1}(\mathcal{G})) \qquad \checkmark.$$

To show

$$\sigma(T^{-1}(\mathcal{G})) \subseteq T^{-1}(\sigma(\mathcal{G})),$$

it suffices to show that

$$T^{-1}(\mathcal{G}) \subseteq T^{-1}(\sigma(\mathcal{G})) \qquad \checkmark.$$

□

2.10 Definition.
A family $\mathcal{D} \subseteq \mathcal{P}(X)$ is a *Dynkin system* if

$$X \in \mathcal{D}$$
$$D \in \mathcal{D} \Longrightarrow D^c \in \mathcal{D}$$
$$(D_j)_{j \in \mathbb{N}} \subseteq \mathcal{D} \text{ pairwise disjoint } \Longrightarrow \bigcup_{j \in \mathbb{N}} D_j \in \mathcal{D}$$

2.11 Definition.
Let $\mathcal{G} \subseteq \mathcal{P}(X)$. Then there is a smallest Dynkin system $\delta(\mathcal{G})$ containing $\mathcal{G}$. $\delta(\mathcal{G})$ is called the *Dynkin system generated by* $\mathcal{G}$.

2.12 Proposition.
Show that

$$\mathcal{G} \subseteq \delta(\mathcal{G}) \subseteq \sigma(\mathcal{G}).$$

Proof.
We have that $\mathcal{G} \subseteq \sigma(\mathcal{G})$. And therefore $\delta(\mathcal{G}) \subseteq \delta(\sigma(\mathcal{G})) = \sigma(\mathcal{G})$. $\square$

2.13 Theorem.
A Dynkin system $\mathcal{D}$ is a σ-algebra if, and only if, it is stable under finite intersections: $D, E \in \mathcal{D} \Longrightarrow D \cap E \in \mathcal{D}$

Proof.
It suffices to show that a $\cap$-stable Dynkin system is stable under countable unions. To show this, it suffices to show that given $(D_j)_{j\in\mathbb{N}} \in \mathcal{D}$, we have

$$D := \bigcup_{j\in\mathbb{N}} D_j \in \mathcal{D}.$$

Set $E_1 = D_1 \in \mathcal{D}$. And $E_2 := D_2 \cap D_1^c$. And $E_3 = D_3 \cap D_2^c \cap D_1^c$. And so on. Then

$$D = \bigcup_{j\in\mathbb{N}} E_j \in \mathcal{D}.$$

$\square$

2.14 Theorem.
If $\mathcal{G} \subseteq \mathcal{P}(X)$ is stable under finite intersections, then $\delta(\mathcal{G}) = \sigma(\mathcal{G})$.

Proof.
It suffices to show that $\sigma(\mathcal{G}) \subseteq \delta(\mathcal{G})$. As $\mathcal{G} \subseteq \delta(\mathcal{G})$ it suffices to show that $\delta(\mathcal{G})$ is a σ-algebra. To show that $\delta(\mathcal{G})$ is a σ-algebra, it suffices to show that $\delta(\mathcal{G})$ is stable under finite intersections.

Fix $D \in \delta(G)$. Consider $\mathcal{D}_D := \{Q \subseteq X : Q \cap D \in \delta(\mathcal{G})\}$. It suffices to show that $\delta(\mathcal{G}) \subseteq \mathcal{D}_D$. To show that it suffices to show that $\mathcal{D}_D$ is a Dynkin system and that $\mathcal{G} \subseteq \mathcal{D}_D$.

To show that $\mathcal{G} \subseteq \mathcal{D}_D$, it suffices to show that

$$G \cap D \in \delta(\mathcal{G}) \qquad \forall G \in \mathcal{G},$$

to show that it suffices to show that

$$\delta(\mathcal{G}) \subseteq \mathcal{D}_G \qquad \forall G \in \mathcal{G},$$

to show that it suffices to show that (as $\mathcal{D}_G$ is a dynkin system)

$$\mathcal{G} \subseteq \mathcal{D}_G \qquad \forall G \in \mathcal{G}.$$

This follows from $\mathcal{G} \subseteq \delta(\mathcal{G})$ and $\mathcal{G}$ is $\cap$-stable. □

2.15 Proposition.

$$A_j \uparrow A \Longrightarrow A_j \cap B \uparrow A \cap B$$

Proof.
To show that

$$A_j \uparrow A \Longrightarrow A_j \cap B \uparrow A \cap B,$$

it suffices to show that

$$A = \bigcup_j A_j \Longrightarrow A \cap B = \bigcup_j A_j \cap B,$$

which is equivalent with showing that

$$\left(\bigcup_j A_j\right) \cap B = \bigcup_j A_j \cap B \qquad \checkmark.$$

□

2.16 Definition.
An *exhausting sequence* $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ is an increasing sequence of sets $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ such that $\bigcup_{j\in\mathbb{N}} A_j = X$.

2.17 Theorem.
Assume that $(X, \mathcal{A})$ *is a measurable space and that* $\mathcal{A} = \sigma(\mathcal{G})$ *is generated by a family* $\mathcal{G}$ *such that*

- $\mathcal{G}$ *is stable under finite intersections* $G, H \in \mathcal{G} \Longrightarrow G \cap H \in \mathcal{G}$
- *there exists an exhausting sequence* $(G_j)_{j\in\mathbb{N}} \subseteq \mathcal{G}$ *with* $G_j \uparrow X$

Any two measure μ, ν *that coincide on* $\mathcal{G}$ *and are finite for all members of the exhausting sequence* $\mu(G_j) = \nu(G_j) < \infty$, *are equal on* $\mathcal{A}$, *i.e.*

$$\mu(A) = \nu(A) \qquad \forall A \in \mathcal{A}.$$

Proof.
Remember that for any increasing sequence $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ with $A_j \uparrow A \in \mathcal{A}$ we have

$$\mu(A) = \lim_{j\in\infty} \mu(A_j).$$

To show that

$$\mu(A) = \nu(A) \qquad \forall A \in \mathcal{A}$$

it suffices to show that (as $G_j \cap A \uparrow X \cap A$)

$$\lim_{j\in\infty} \mu(G_j \cap A) = \lim_{j\in\infty} \mu(G_j \cap A) \qquad \forall A \in \mathcal{A}$$

To show that it suffices to show that

$$\mu(G_j \cap A) = \nu(G_j \cap A) \qquad \forall j \in \mathbb{N}, \quad \forall A \in \mathcal{A}.$$

Consider $\mathcal{D}_j := \{A \in \mathcal{A} : \mu(G_j \cap A) = \nu(G_j \cap A)\}$. It suffices to show that $\mathcal{A} \subseteq \mathcal{D}_j$, which is equivalent with showing $\sigma(\mathcal{G}) \subseteq \mathcal{D}_j$.

As $\mathcal{G}$ is stable under finite intersections, it suffices to show that $\delta(\mathcal{G}) \subseteq \mathcal{D}_j$.

As $\mathcal{G}$ is stable under finite intersections and $\mu(\mathcal{G}) = \nu(\mathcal{G})$, we have that $\mathcal{G} \subseteq D_j$ and therefore it suffices to show that $\mathcal{D}_j$ is a Dynkin system.

Which you can check. $\square$

2.18 Theorem.
The n*-dimensional Lebesgue measure* λ^n *is invariant under translations, i.e.*

$$\lambda^n(x + B) = \lambda^n(B) \qquad \forall x \in \mathbb{R}^n, \forall B \in \mathcal{B}(\mathbb{R}^n).$$

Proof.
Set $\nu(B) := \lambda^n(x+B)$ for some fixed $x \in \mathbb{R}^n$. It suffices to show that

$$\lambda^n(B) = \nu(B) \qquad B \in \mathcal{B}.$$

To show that, it suffices to show that

1. $\mathcal{J}$ is $\cap$-stable $\quad\checkmark$
2. $\mathcal{J}$ admits an exhausting sequence
   - $[-j, j) \uparrow \mathbb{R}^n \quad \checkmark$
3. $\lambda^n|_{\mathcal{J}} = \nu|_{\mathcal{J}}$

$$\begin{aligned} v([a,b)) &= \lambda^n[x+a, x+b) \\ &= \lambda^n[a,b) \end{aligned}$$

4. ν is a measure on $\mathcal{B}^n$

To show that ν is a measure on $\mathcal{B}^n$, it suffices to show that

$$\nu(\bigcup_{j\in\mathbb{N}} B_j) = \sum_{j\in\mathbb{N}} \nu(B_j),$$

which is equivalent with

$$\lambda^n(x + \bigcup_{j\in\mathbb{N}} B_j) = \sum_{j\in\mathbb{N}} \lambda^n(x + B_j).$$

It suffices to show

$$B \in \mathcal{B}^n \Longrightarrow x + B \in \mathcal{B}^n.$$

Which we have already proven.

$\square$

2.19 Theorem.
Let $(X, \mathcal{A}), (X, \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$ measurable map. For every measure μ on $(X, \mathcal{A})$,

$$\mu'(A') := \mu(T^{-1}(A')), \qquad A' \in \mathcal{A}'.$$

The measure μ' is called the image measure *of μ under T and is denoted by $T \circ \mu$ or $\mu \circ T^{-1}$.*

2.20 Problem 7.7.

Use image measures to give a new proof of Problem 5.8, i.e. to show that

$$\lambda^n(t \cdot B) = t^n \lambda^n(B) \qquad \forall B \in \mathcal{B}(\mathbb{R}^n), \forall t > 0$$

Proof.

Set $\nu(B) := t^n \lambda^n(B)$ for some fixed $x \in \mathbb{R}^n$. It suffices to show that

$$\lambda^n(tB) = \nu(B) \qquad \forall B \in \mathcal{B}.$$

To show that, it suffices to show that

1. $\mathcal{J}$ is $\cap$-stable $\quad \checkmark$
2. $\mathcal{J}$ admits an exhausting sequence
   - $[-j, j) \uparrow \mathbb{R}^n \quad \checkmark$
3. $\lambda^n|_{\mathcal{J}} = \nu|_{\mathcal{J}}$

$$\begin{aligned} \nu([a,b)) &= \lambda^n[ta, tb) \\ &= t^n \lambda^n[a, b) \end{aligned}$$

4. ν is a measure on $\mathcal{B}^n$ as it is a composition of the inverse of a measurable map and a measure.

$\square$

3 11-10-2014

3.1 Definition.
Note that: $u^{-1}[a,\infty) = \{x \in X : u(x) \in [a,\infty)\} = \{x \in X : u(x) \geq a\}$. We define:

$$\{u(x) \geq a\} = u^{-1}[a,\infty).$$

3.2 Theorem.
Let $(X, \mathcal{A})$ be a measurable space. The function $u : X \to \mathbb{R}$ is $\mathcal{A}/\mathcal{B}$-measurable if, and only if, one, hence all, of the following conditions hold

1. $\{u \geq a\} \in \mathcal{A} \qquad \forall a \in \mathbb{R}$
2. $\{u > a\} \in \mathcal{A} \qquad \forall a \in \mathbb{R}$
3. $\{u \leq a\} \in \mathcal{A} \qquad \forall a \in \mathbb{R}$
4. $\{u < a\} \in \mathcal{A} \qquad \forall a \in \mathbb{R}$

3.3 Definition.
We define the *extended real line* $\bar{\mathbb{R}} := [-\infty, \infty]$ with the following rules for all $x \in \mathbb{R}$:

$$\begin{aligned} x + \infty = \infty + x = \infty \qquad & x + -\infty = -\infty + x = -\infty \\ \infty + \infty = \infty \qquad & -\infty - \infty = -\infty \end{aligned}$$

And for $x \in (0, \infty]$:

$$\begin{aligned} \pm x \cdot \infty = \infty \cdot \pm x = \pm\infty \\ \pm x \cdot -\infty = -\infty \cdot \pm x = \mp\infty \\ 0 \cdot \pm\infty = \pm\infty \cdot 0 = 0 \\ \frac{1}{\pm\infty} = 0 \end{aligned}$$

3.4 Definition.
Functions which take values in $\bar{\mathbb{R}}$ are called *numerical functions.*

3.5 Definition.
The Borel σ-algebra $\bar{\mathcal{B}} = \mathcal{B}(\bar{\mathbb{R}})$ is defined by:

$$\bar{\mathcal{B}} := \Big\{ B \cup S : B \in \mathcal{B} \text{ and } S \in \Big\{ \varnothing, \{-\infty\}, \{\infty\}, \{-\infty, \infty\} \Big\} \Big\}$$

3.6 Theorem.
We have $\mathcal{B}(\mathbb{R}) = \mathbb{R} \cap \mathcal{B}(\bar{\mathbb{R}})$. *Moreover* $\bar{\mathcal{B}}$ *is generated by all sets of the form* $[a, \infty]$ *or* $(a, \infty]$ *or* $[-\infty, a)$ *or* $[-\infty, a]$ *where* $a \in \mathbb{R}$

3.7 Definition.
Let $(X, \mathcal{A})$ be a measurable space. We write $\mathcal{M} := \mathcal{M}(\mathcal{A})$ and $\mathcal{M}_{\bar{\mathbb{R}}} := \mathcal{M}_{\bar{\mathbb{R}}}(\mathcal{A})$ for the families of real valued $\mathcal{A}/\mathcal{B}$-measurable and numerical $\mathcal{A}/\bar{\mathcal{B}}$-measurable functions on X.

3.8 Definition.
A *simple function* $g : X \to \mathbb{R}$ on a measurable space $(X, \mathcal{A})$ is a function of the form

$$g(x) := \sum_{j=1}^{M} y_j \mathbf{1}_{A_j}(x)$$

with finitely many sets $A_1, \ldots, A_m \in \mathcal{A}$ and $y_1, \ldots, y_M \in \mathbb{R}$. The set of simple functions is denoted by $\mathcal{E}$ or $\mathcal{E}(\mathcal{A})$.

If the sets $A_1, \ldots, A_M$ are mutally disjoint we call

$$\sum_{j=0}^{M} y_j \mathbf{1}_{A_j}(x)$$

with $y_0 := 0$ and $A_0 := (A_1 \cup \ldots \cup A_M)^c$ a *standard representation* of g. Caution, this representation is not unique.

3.9 Theorem.
Let $(X, \mathcal{A})$ *be a measurable space. Every* $\mathcal{A}/\bar{\mathcal{B}}$*-measurable numerical function* $u : X \to \bar{\mathbb{R}}$ *is the pointwise limit of simple functions:*

$$u(x) = \lim_{j \to \infty} f_j(x)$$

where $f_j \in \mathcal{E}(\mathcal{A})$ *and* $|f_j| \leq |u|$.

If $u \geq 0$, *all* f_j *can be chosen to be positive and increasing towards* u *so that* $u = \sup_{j \in \mathbb{N}} f_j$.

3.10 Theorem.
Let $(X, \mathcal{A})$ *be a measurable space. If* $u_j : X \to \bar{\mathbb{R}}, j \in \mathbb{N}$ *are measurable functions, then so are*

$$\sup_{j \in \mathbb{N}} u_j \quad \inf_{j \in \mathbb{N}} u_j \quad \limsup_{j \to \mathbb{N}} u_j \quad \liminf_{j \to \mathbb{N}} u_j$$

and whenever it exists

$$\lim_{j\to\infty} u_j.$$

3.11 Theorem.
Let u, v be $\mathcal{A}/\bar{\mathcal{B}}$-measurable functions. Then the functions

$$u \pm v \quad uv \quad u \vee v := \max\{u, v\} \quad u \wedge v := \min\{u, v\}$$

are $\mathcal{A}/\bar{\mathcal{B}}$-measurable (whenever they are defined).

3.12 Theorem.
A function u is $\mathcal{A}/\mathcal{B}$ measurable if, and only if, $u^{\pm}$ are $\mathcal{A}/\bar{\mathcal{B}}$ measurable.

3.13 Theorem.
Let $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ be an $\mathcal{A}/\mathcal{A}'$-measurable map and let $\sigma(T) \subseteq \mathcal{A}$ be the σ-algebra generated by T. Then $u = w(T)$ for some $\mathcal{A}'/\bar{\mathcal{B}}$ measurable function $w : X' \to \bar{\mathbb{R}}$ if and only if $u : X \to \bar{\mathbb{R}}$ is $\sigma(T)/\bar{\mathcal{B}}$-measurable.