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1.1 Measure Theory Chapter 6

1.1 Problem 6.1a.

Consider on \mathbb{R} the family Σ of all Borel sets which are symmetric w.r.t. the origin. Show that Σ is a σ -algebra.

Proof.

1. To show that $\mathbb{R} \in \Sigma$, note that \mathbb{R} is a Borel set that is symmetric w.r.t. to the origin.
2. To show that $A \in \Sigma \Rightarrow A^c \in \Sigma$, it suffices to show that

$$\forall x \in A : -x \in A \implies \forall y \in A^c : -y \in A^c,$$

which is equivalent with showing that

$$\forall x \in A : -x \in A \implies \forall y \notin A : -y \notin A,$$

which is equivalent with showing that

$$\exists y \notin A : -y \in A \implies \exists x \in A : -x \notin A.$$

This last statement hold if we set $x := -y$.

3. Showing that Σ is stable under countable unions is equivalent with showing that

$$(A_j)_{j \in \mathbb{N}} \in \Sigma \implies \bigcup_{j=1}^{\infty} A_j \in \Sigma.$$

It suffices to show that

$$\forall x_j \in A_j : -x_j \in A_j \implies \forall x \in \bigcup_{j=1}^{\infty} A_j : -x \in \bigcup_{j=1}^{\infty} A_j.$$

This holds: Let $x \in \bigcup_{j=1}^{\infty} A_j$. Then for some $j \in \mathbb{N}$ we have $x \in A_j$. By hypothesis, $-x \in A_j$. But then $-x \in \bigcup_{j=1}^{\infty} A_j$.

□

1.2 Problem 6.3i.

Show that non-void open sets in \mathbb{R}^n have always strictly positive Lebesgue measure.

Proof.

First recall that

1. $\lambda^n[a, b) = \prod_{j=1}^n (b_j - a_j)$
2. λ^n is a pre-measure that can be extended to a measure on $\mathcal{B}(\mathbb{R}^n)$.
3. λ^n is invariant under translations
4. $A \subseteq B \implies \mu(A) \leq \mu(B)$

Remember we can contain a ball $B_\epsilon(x)$ in any non-empty open set U . Hence, to show that $\lambda^n(U) > 0$ for any non-empty open set U , it suffices to show that $\lambda^n(B_\epsilon(x)) > 0$.

But as λ^n is invariant under translations, it suffices to show that $\lambda^n(B_\epsilon(0)) > 0$.

We have that $\lambda^n[\dot{\epsilon}, \dot{\epsilon}) > 0$ for any $\dot{\epsilon} > 0$. Hence, it suffices to show that there exists such an $\dot{\epsilon}$ such that $[\dot{\epsilon}, \dot{\epsilon}) \subseteq B_\epsilon(0)$.

This holds: Let $x \in [\dot{\epsilon}, \dot{\epsilon})$. Then $|x_j| < 2\dot{\epsilon}$. Then $x_j^2 < 4\dot{\epsilon}^2$. Then

$$x_1^2 + x_2^2 + \cdots + x_n^2 < 4n\dot{\epsilon}^2.$$

So if we set $\dot{\epsilon} := \frac{\epsilon}{2\sqrt{n}}$. Then

$$x_1^2 + x_2^2 + \cdots + x_n^2 < \epsilon.$$

And $x \in B_\epsilon(0)$.

□

1.3 Problem 6.3ii.

Is 6.3i still true for closed sets ?

Proof.

No, take $\{0\}$, then $\lambda\{x\} = 0$.

□

1.4 Problem 6.4ii.

Let $H \subseteq \mathbb{R}^2$ be a hyperplane which is perpendicular to the x_1 -direction (that is to say: H is a translate of the x_2 axis). Show that

1. $H \in \mathcal{B}(\mathbb{R}^2)$
2. $\lambda^2(H) = 0$

Proof.

1. To show that $H \in \mathcal{B}(\mathbb{R}^2)$, it suffices to show that H is writable as an intersection of countable half-open sets. Note that:

$$H := \{y\} \times \mathbb{R} = \bigcap_{j \in \mathbb{N}} [y, y + 1/j) \times \mathbb{R}$$

2. To show that $\lambda^2(H) = 0$, it suffices to show that

$$\forall \epsilon > 0 : \lambda^2(H) < \epsilon.$$

Let $\epsilon > 0$. Note that for any positive sequence (ϵ_n) , we have that:

$$\begin{aligned} \lambda^2(H) &= \lambda^2(\{y\} \times \mathbb{R}) \\ &\leq \lambda^2\left(\bigcup_{n \in \mathbb{N}} [y, y + \epsilon_n) \times [-n, n)\right) \\ &\leq \sum_{n=1}^{\infty} \lambda^2([y, y + \epsilon_n) \times [-n, n)) \\ &= \sum_{n=1}^{\infty} \epsilon_n 2n. \end{aligned}$$

So if we set $\epsilon_n := \frac{\epsilon}{2^n 2n}$. Then $\lambda^2(H) < \epsilon$.

□

1.5 Problem 6.4i.

Show that $\lambda(a, b) = b - a$ for all $a, b \in \mathbb{R}, a \leq b$.

Proof.

$$\begin{aligned}
 \lambda(a, b) &= \lambda([b - a] - \{b\}) \\
 &= \lambda[b, a] - \lambda\{b\} && \text{T4.3iii} \\
 &= b - a - 0 && \text{Problem 4.11i}
 \end{aligned}$$

□

1.6 Definition.

Let (X, \mathcal{A}, μ) be a measure space such that all singletons $\{x\} \in \mathcal{A}$. A point x is called an atom, if $\mu\{x\} > 0$. A measure is called *non-atomic* or *diffuse*, there are no atoms.

1.7 Problem 6.5i.

Show that λ^1 is diffuse.

Proof.

We've already shown that $\lambda\{x\} = 0$ for any $x \in \mathbb{R}$.

□

1.8 Problem 6.5iii.

Show that for a diffuse measure μ on (X, \mathcal{A}) all countable sets are null sets.

Proof.

All countable sets are writable as

$$\bigcup_{j=0}^{\infty} \{x_j\}$$

where $x_i \neq x_j$. So we get

$$\lambda\left(\bigcup_{j=0}^{\infty} \{x_j\}\right) = \sum_{j=0}^{\infty} \lambda\{x_j\} = 0.$$

□

1.9 Definition.

A set $A \subseteq \mathbb{R}^n$ is called *bounded* if it can be contained in a ball $B_r \supseteq A$ of finite radius r . A set $A \subseteq \mathbb{R}^n$ is called *connected*, if we can go along a curve from any point $a \in A$ to any point $a' \in A$ without ever leaving A .

1.10 Problem 6.6a.

Construct an open and unbounded set in \mathbb{R} with finite, strictly positive Lebesgue measure, i.e. $0 < \lambda(U) < \infty$.

Proof.

Consider the set

$$U := \bigcup_{n=1}^{\infty} \left(n - \frac{1}{\epsilon_n}, n + \frac{1}{\epsilon_n} \right).$$

This is an open set, as it union of countable open sets. It is unbounded, for any $B_r(0)$ we have that $\lceil r + 1 \rceil \in U$ and not in $B_r(0)$. We have to show that it has positive finite lebesgue measure.

$$\begin{aligned} \lambda(U) &= \bigcup_{n=1}^{\infty} \left(n - \frac{1}{\epsilon_n}, n + \frac{1}{\epsilon_n} \right) \\ &= \sum_{n=1}^{\infty} \frac{2}{\epsilon_n}. \end{aligned}$$

So set $\epsilon_n := 1/2^n$. And we then have $0 < \lambda(U) = 2 < \infty$.

□

1.11 Problem 6.6ii.

Construct an open, unbounded and connected set in \mathbb{R}^2 with finite, strictly positive Lebesgue measure.

Proof.

Let (ϵ_n) be a positive sequence of real numbers. Consider

$$U = \bigcup_{n=1}^{\infty} (0, \epsilon_n) \times (-n, n).$$

This is an open set, as it is a countable union of open sets. It is clearly connected and unbounded in the y -axis. We now have that then

$$\begin{aligned}\lambda^2(U) &= \left(\bigcup_{n \in \mathbb{N}} (0, \epsilon_n) \times (-n, n) \right) \\ &\leq \sum_{n \in \mathbb{N}} \epsilon_n 2n\end{aligned}$$

Set $\epsilon_n := \frac{1}{2^n 2n}$. Then $\lambda^2(U) = 1$ is a positive finite real number. \square

1.12 Problem 6.6iii.

Is there a connected, open and unbounded set in \mathbb{R} with finite, strictly positive Lebesgue measure?

Proof.

No, this is impossible. Since we are in one dimension, connectedness forces us to go between points in a straight, uninterrupted line. Since the set is unbounded, this means we must have a line of the sort (a, ∞) or $(-\infty, b)$ in our set and in both cases Lebesgue measure is infinite. \square

1.13 Definition.

Let $A \subset X$. The closure of A , denoted by \bar{A} , is the smallest closed set containing A , i.e.

$$\bar{A} = \bigcap_{\substack{F \in \mathcal{C} \\ F \supset A}} F$$

1.14 Definition.

A set $A \subseteq X$ is dense in X if $\bar{A} = X$

1.15 Problem 6.7.

Let $\lambda := \lambda^1|_{[0,1]}$ be a Lebesgue measure on $([0, 1], \mathcal{B}[0, 1])$. Show that for every $\epsilon > 0$ there is a dense open set $U \subseteq [0, 1]$ with $\lambda(U) \leq \epsilon$.

Proof.

Note that \mathbb{Q} is dense. We are going to make an open set contained in \mathbb{Q} . Consider

$$U := \bigcup_{j=1}^{\infty} (q_j - \epsilon_j, q_j + \epsilon_j)$$

Then

$$\lambda(U) = \lambda\left(\bigcup_{j=1}^{\infty} (q_j - \epsilon_j, q_j + \epsilon_j)\right) \leq \sum 2\epsilon_j.$$

So set $\epsilon_j := \frac{\epsilon}{2^{j-1}}$. And we are done. \square

1.16 Problem 6.10i.

Let μ be a measure on $\mathcal{A} = \{\emptyset, [0, 1), [1, 2), [0, 2)\}$ of $X = [0, 2)$. Such that

$$\mu[0, 1) = \mu[1, 2) = 1/2 \quad \mu[0, 2) = 1.$$

Define for each $A \subseteq [0, 2)$ the family of countable \mathcal{A} -coverings of A

$$\mathcal{C}(A) := \{(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A} : \bigcup_{j \in \mathbb{N}} A_j \supseteq A\}$$

and set

$$\mu^*(A) := \inf \left\{ \sum_{j \in \mathbb{N}} \mu(A_j) : (A_j)_{j \in \mathbb{N}} \in \mathcal{C}(A) \right\}.$$

Define $\mathcal{A}^* := \{A \subseteq [0, 2) : \mu^*(B) = \mu^*(B \cap A) + \mu^*(B - A) \quad \forall B \subseteq X\}$

Show that

1. Find $\mu^*(a, b), \mu^*\{a\}$
2. $(0, 1), \{0\} \notin \mathcal{A}^*$

Note that in T6.1 it is proven that:

- $\mathcal{A} \subseteq \mathcal{A}^*$
- $\mu^*(A) = \mu(A) \quad \forall A \in \mathcal{A}$
- \mathcal{A}^* is a σ -algebra and μ^* is a measure on $([0, 2), \mathcal{A}^*)$

Proof.

1. We have

$$\begin{aligned}\mu^*(a, b) &= \mu[0, 1) && \text{if } a, b \in [0, 1) \\ \mu^*(a, b) &= \mu[1, 2) && \text{if } a, b \in [1, 2) \\ \mu^*(a, b) &= \mu[0, 2) && \text{if } a \in [0, 1), b \in [1, 2)\end{aligned}$$

In the case of a singleton $\{a\}$ the best possible cover is always either $[0, 1)$ or $[1, 2)$ so that $\mu^*\{a\} = 1/2$.

2. Suppose that $(0, 1) \in \mathcal{A}^*$ then we would have that

$$\{0\} = [0, 1) - (0, 1) \in \mathcal{A}^*.$$

But this gives

$$\frac{1}{2} = \mu^*[0, 1) = \mu^*(0, 1) + \mu^*\{0\} = 1$$

□

2 10-10-2014

2.1 Measure Theory Chapter 7

2.1 Definition.

Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be two measurable spaces. A map $T : X \rightarrow X'$ is called \mathcal{A}/\mathcal{A}' -measurable (or *measurable* unless this is too ambiguous) if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A} \quad \forall A' \in \mathcal{A}'.$$

We often denote this by $T^{-1}(\mathcal{A}') \subseteq \mathcal{A}$.

2.2 Definition.

A *random variable* is a measurable map from a probability space (i.e. $\mu(X) = 1$) to any measurable space.

2.3 Lemma 7.2.

Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \rightarrow X'$ is \mathcal{A}/\mathcal{A}' -measurable if and only if

$$T^{-1}(G') \in \mathcal{A} \quad \forall G' \in \mathcal{G}'.$$

2.4 Problem 7.1.

Show that

$$\tau_x : \mathbb{R}^n \rightarrow \mathbb{R}^n : B \mapsto B - x$$

is a $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$ measurable map.

Proof.

Showing that

$$\tau_x : \mathcal{B}(\mathbb{R}^n) \rightarrow \mathcal{B}(\mathbb{R}^n) : B \mapsto B - x$$

is $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$ measurable, is equivalent with showing that

$$\tau_x^{-1}(B) \in \mathcal{B}(\mathbb{R}^n) \quad \forall B \in \mathcal{B}(\mathbb{R}^n),$$

which in turn is equivalent with showing that

$$x + B \in \mathcal{B}(\mathbb{R}^n) \quad \forall B \in \mathcal{J}(\mathbb{R}^n),$$

which in turn is equivalent with showing that

$$x + [a, b] \in \mathcal{B}(\mathbb{R}^n) \quad \forall a, b \in \mathbb{R}^n.$$

This follows as $x + [a, b] = [x + a, x + b] \in \mathcal{J}(\mathbb{R}^n) \subseteq \mathcal{B}(\mathbb{R}^n)$. \square

2.5 Theorem.

Every continuous map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is $\mathcal{B}^n/\mathcal{B}^m$ measurable.

Proof.

Showing that

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

is $\mathcal{B}^n/\mathcal{B}^m$ measurable, is equivalent with showing that

$$T^{-1}(\mathcal{O}^m) \subseteq \mathcal{B}^n.$$

As $\mathcal{O}^n \subseteq \sigma(\mathcal{O}^n) = \mathcal{B}^n$, it suffices to show that

$$T^{-1}(\mathcal{O}^m) \subseteq \mathcal{O}^n,$$

which follows from the continuity of T . \square

2.6 Definition.

Let $(T_i)_{i \in I}$ be arbitrarily many mappings $T_i : X \rightarrow X_i$ from the same space X into measurable spaces (X_i, \mathcal{A}_i) . The smallest σ -algebra on X that makes all T_i simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)\right).$$

We say that $\sigma(T_i : i \in I)$ is *generated by the family $(T_i)_{i \in I}$* .

2.7 Theorem.

Let (X_j, \mathcal{A}_j) , $j = 1, 2, 3$, be measurable spaces and $T : X_1 \rightarrow X_2$, $S : X_2 \rightarrow X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ - resp. $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps. Then $S \circ T : X_1 \rightarrow X_3$ is $\mathcal{A}_1/\mathcal{A}_3$ -measurable.

2.8 Problem 7.4.

Let X be a set, $(X_i, \mathcal{A}_i), i \in I$, be arbitrarily many measurable spaces, and $T_i : X \rightarrow X_i$ be a family of maps. Show that a map f from a measurable space (F, \mathcal{F}) to $(X, \sigma(T_i : i \in I))$ is measurable if, and only if, all maps $T_i \circ f$ are $\mathcal{F}/\mathcal{A}_i$ -measurable.

Proof of \implies .

To show that all maps $T_i \circ f$ are $\mathcal{F}/\mathcal{A}_i$ -measurable, it suffices to show that $T_i : X \rightarrow X_i$ is $\sigma(T_i : i \in I)/\mathcal{A}_i$ -measurable and $f : F \rightarrow X$ is $\mathcal{F}/\sigma(T_i : i \in I)$ -measurable.

By hypothesis, it suffices to show that $T_i : X \rightarrow X_i$ is $\sigma(T_i : i \in I)/\mathcal{A}_i$ -measurable, which is equivalent with showing that

$$T_i^{-1}(A_i) \in \sigma(T_i : i \in I) \quad \forall A_i \in \mathcal{A}_i.$$

It suffices to assume $A_i \in \mathcal{A}_i$ and show that

$$T_i^{-1}(A_i) \in \bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \quad \checkmark.$$

□

Proof of \impliedby .

To show that a map f from a measurable space (F, \mathcal{F}) to $(X, \sigma(T_i : i \in I))$ is measurable, it suffices to show that

$$f^{-1}\left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)\right) \subseteq \mathcal{F}$$

To show this it suffices to show that

$$\bigcup_{i \in I} f^{-1}(T_i^{-1}(\mathcal{A}_i)) \subseteq \mathcal{F},$$

to show this it suffices to show that

$$f^{-1}(T_i^{-1}(\mathcal{A}_i)) \subseteq \mathcal{F},$$

to show this it suffices to show that

$$(T_i \circ f)^{-1}(\mathcal{A}_i) \subseteq \mathcal{F}.$$

This follows by hypothesis.

□

2.9 Problem 7.8.

Let $T : X \rightarrow Y$ be any map. Show that

$$T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G}))$$

holds for arbitrary families of \mathcal{G} of subsets of Y .

Proof.

To show that

$$T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G}))$$

it suffices to show:

1. $T^{-1}(\sigma(\mathcal{G})) \subseteq \sigma(T^{-1}(\mathcal{G}))$
2. $\sigma(T^{-1}(\mathcal{G})) \subseteq T^{-1}(\sigma(\mathcal{G}))$

To show

$$T^{-1}(\sigma(\mathcal{G})) \subseteq \sigma(T^{-1}(\mathcal{G})),$$

it suffices to show that T is $\sigma(T^{-1}(\mathcal{G}))/\sigma(\mathcal{G})$ measurable.

To show that it suffices to show that

$$T^{-1}(\mathcal{G}) \subseteq \sigma(T^{-1}(\mathcal{G})) \quad \checkmark.$$

To show

$$\sigma(T^{-1}(\mathcal{G})) \subseteq T^{-1}(\sigma(\mathcal{G})),$$

it suffices to show that

$$T^{-1}(\mathcal{G}) \subseteq T^{-1}(\sigma(\mathcal{G})) \quad \checkmark.$$

□

2.2 Measure Theory Chapter 5

2.10 Definition.

A family $\mathcal{D} \subseteq \mathcal{P}(X)$ is a *Dynkin system* if

$$\begin{aligned} X &\in \mathcal{D} \\ D \in \mathcal{D} &\implies D^c \in \mathcal{D} \\ (D_j)_{j \in \mathbb{N}} \subseteq \mathcal{D} \text{ pairwise disjoint} &\implies \bigcup_{j \in \mathbb{N}} D_j \in \mathcal{D} \end{aligned}$$

2.11 Definition.

Let $\mathcal{G} \subseteq \mathcal{P}(X)$. Then there is a smallest Dynkin system $\delta(\mathcal{G})$ containing \mathcal{G} . $\delta(\mathcal{G})$ is called the *Dynkin system generated by \mathcal{G}* .

2.12 Proposition.

Show that

$$\mathcal{G} \subseteq \delta(\mathcal{G}) \subseteq \sigma(\mathcal{G}).$$

Proof.

We have that $\mathcal{G} \subseteq \sigma(\mathcal{G})$. And therefore $\delta(\mathcal{G}) \subseteq \delta(\sigma(\mathcal{G})) = \sigma(\mathcal{G})$. \square

2.13 Theorem.

A Dynkin system \mathcal{D} is a σ -algebra if, and only if, it is stable under finite intersections: $D, E \in \mathcal{D} \implies D \cap E \in \mathcal{D}$

Proof.

It suffices to show that a \cap -stable Dynkin system is stable under countable unions. To show this, it suffices to show that given $(D_j)_{j \in \mathbb{N}} \in \mathcal{D}$, we have

$$D := \bigcup_{j \in \mathbb{N}} D_j \in \mathcal{D}.$$

Set $E_1 = D_1 \in \mathcal{D}$. And $E_2 := D_2 \cap D_1^c$. And $E_3 = D_3 \cap D_2^c \cap D_1^c$. And so on. Then

$$D = \bigcup_{j \in \mathbb{N}} E_j \in \mathcal{D}.$$

\square

2.14 Theorem.

If $\mathcal{G} \subseteq \mathcal{P}(X)$ is stable under finite intersections, then $\delta(\mathcal{G}) = \sigma(\mathcal{G})$.

Proof.

It suffices to show that $\sigma(\mathcal{G}) \subseteq \delta(\mathcal{G})$. As $\mathcal{G} \subseteq \delta(\mathcal{G})$ it suffices to show that $\delta(\mathcal{G})$ is a σ -algebra. To show that $\delta(\mathcal{G})$ is a σ -algebra, it suffices to show that $\delta(\mathcal{G})$ is stable under finite intersections.

Fix $D \in \delta(\mathcal{G})$. Consider $\mathcal{D}_D := \{Q \subseteq X : Q \cap D \in \delta(\mathcal{G})\}$. It suffices to show that $\delta(\mathcal{G}) \subseteq \mathcal{D}_D$. To show that it suffices to show that \mathcal{D}_D is a Dynkin system and that $\mathcal{G} \subseteq \mathcal{D}_D$.

To show that $\mathcal{G} \subseteq \mathcal{D}_D$, it suffices to show that

$$G \cap D \in \delta(\mathcal{G}) \quad \forall G \in \mathcal{G},$$

to show that it suffices to show that

$$\delta(\mathcal{G}) \subseteq \mathcal{D}_G \quad \forall G \in \mathcal{G},$$

to show that it suffices to show that (as \mathcal{D}_G is a dynkin system)

$$\mathcal{G} \subseteq \mathcal{D}_G \quad \forall G \in \mathcal{G}.$$

This follows from $\mathcal{G} \subseteq \delta(\mathcal{G})$ and \mathcal{G} is \cap -stable. □

2.15 Proposition.

$$A_j \uparrow A \implies A_j \cap B \uparrow A \cap B$$

Proof.

To show that

$$A_j \uparrow A \implies A_j \cap B \uparrow A \cap B,$$

it suffices to show that

$$A = \bigcup_j A_j \implies A \cap B = \bigcup_j A_j \cap B,$$

which is equivalent with showing that

$$\left(\bigcup_j A_j \right) \cap B = \bigcup_j A_j \cap B \quad \checkmark.$$

□

2.16 Definition.

An *exhausting sequence* $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$ is an increasing sequence of sets $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ such that $\bigcup_{j \in \mathbb{N}} A_j = X$.

2.17 Theorem.

Assume that (X, \mathcal{A}) is a measurable space and that $\mathcal{A} = \sigma(\mathcal{G})$ is generated by a family \mathcal{G} such that

- \mathcal{G} is stable under finite intersections $G, H \in \mathcal{G} \implies G \cap H \in \mathcal{G}$
- there exists an exhausting sequence $(G_j)_{j \in \mathbb{N}} \subseteq \mathcal{G}$ with $G_j \uparrow X$

Any two measure μ, ν that coincide on \mathcal{G} and are finite for all members of the exhausting sequence $\mu(G_j) = \nu(G_j) < \infty$, are equal on \mathcal{A} , i.e.

$$\mu(A) = \nu(A) \quad \forall A \in \mathcal{A}.$$

Proof.

Remember that for any increasing sequence $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$ with $A_j \uparrow A \in \mathcal{A}$ we have

$$\mu(A) = \lim_{j \in \mathbb{N}} \mu(A_j).$$

To show that

$$\mu(A) = \nu(A) \quad \forall A \in \mathcal{A}$$

it suffices to show that (as $G_j \cap A \uparrow X \cap A$)

$$\lim_{j \in \mathbb{N}} \mu(G_j \cap A) = \lim_{j \in \mathbb{N}} \nu(G_j \cap A) \quad \forall A \in \mathcal{A}$$

To show that it suffices to show that

$$\mu(G_j \cap A) = \nu(G_j \cap A) \quad \forall j \in \mathbb{N}, \quad \forall A \in \mathcal{A}.$$

Consider $\mathcal{D}_j := \{A \in \mathcal{A} : \mu(G_j \cap A) = \nu(G_j \cap A)\}$. It suffices to show that $\mathcal{A} \subseteq \mathcal{D}_j$, which is equivalent with showing $\sigma(\mathcal{G}) \subseteq \mathcal{D}_j$.

As \mathcal{G} is stable under finite intersections, it suffices to show that $\delta(\mathcal{G}) \subseteq \mathcal{D}_j$.

As \mathcal{G} is stable under finite intersections and $\mu(\mathcal{G}) = \nu(\mathcal{G})$, we have that $\mathcal{G} \subseteq \mathcal{D}_j$ and therefore it suffices to show that \mathcal{D}_j is a Dynkin system.

Which you can check. □

2.18 Theorem.

The n -dimensional Lebesgue measure λ^n is invariant under translations, i.e.

$$\lambda^n(x + B) = \lambda^n(B) \quad \forall x \in \mathbb{R}^n, \forall B \in \mathcal{B}(\mathbb{R}^n).$$

Proof.

Set $\nu(B) := \lambda^n(x + B)$ for some fixed $x \in \mathbb{R}^n$. It suffices to show that

$$\lambda^n(B) = \nu(B) \quad B \in \mathcal{B}.$$

To show that, it suffices to show that

1. \mathcal{J} is \cap -stable ✓
2. \mathcal{J} admits an exhausting sequence
 - $[-j, j) \uparrow \mathbb{R}^n$ ✓
3. $\lambda^n|_{\mathcal{J}} = \nu|_{\mathcal{J}}$

$$\begin{aligned} v([a, b)) &= \lambda^n[x + a, x + b) \\ &= \lambda^n[a, b) \end{aligned}$$

4. ν is a measure on \mathcal{B}^n

To show that ν is a measure on \mathcal{B}^n , it suffices to show that

$$\nu\left(\bigcup_{j \in \mathbb{N}} B_j\right) = \sum_{j \in \mathbb{N}} \nu(B_j),$$

which is equivalent with

$$\lambda^n\left(x + \bigcup_{j \in \mathbb{N}} B_j\right) = \sum_{j \in \mathbb{N}} \lambda^n(x + B_j).$$

It suffices to show

$$B \in \mathcal{B}^n \implies x + B \in \mathcal{B}^n.$$

Which we have already proven. □

2.19 Theorem.

Let $(X, \mathcal{A}), (X, \mathcal{A}')$ be measurable spaces and $T : X \rightarrow X'$ be an \mathcal{A}/\mathcal{A}' measurable map. For every measure μ on (X, \mathcal{A}) ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}'.$$

The measure μ' is called the image measure of μ under T and is denoted by $T \circ \mu$ or $\mu \circ T^{-1}$.

2.3 Measure Theory Chapter 7

2.20 Problem 7.7.

Use image measures to give a new proof of Problem 5.8, i.e. to show that

$$\lambda^n(t \cdot B) = t^n \lambda^n(B) \quad \forall B \in \mathcal{B}(\mathbb{R}^n), \forall t > 0$$

Proof.

Set $\nu(B) := t^n \lambda^n(B)$ for some fixed $x \in \mathbb{R}^n$. It suffices to show that

$$\lambda^n(tB) = \nu(B) \quad \forall B \in \mathcal{B}.$$

To show that, it suffices to show that

1. \mathcal{J} is \cap -stable ✓
2. \mathcal{J} admits an exhausting sequence
 - $[-j, j) \uparrow \mathbb{R}^n$ ✓
3. $\lambda^n|_{\mathcal{J}} = \nu|_{\mathcal{J}}$

$$\begin{aligned} \nu([a, b)) &= \lambda^n[ta, tb) \\ &= t^n \lambda^n[a, b) \end{aligned}$$

4. ν is a measure on \mathcal{B}^n as it is a composition of the inverse of a measurable map and a measure.

□

3 11-10-2014

3.1 Measure Theory Chapter 8

3.1 Definition.

Note that: $u^{-1}[a, \infty) = \{x \in X : u(x) \in [a, \infty)\} = \{x \in X : u(x) \geq a\}$. We define:

$$\{u(x) \geq a\} = u^{-1}[a, \infty).$$

3.2 Theorem.

Let (X, \mathcal{A}) be a measurable space. The function $u : X \rightarrow \mathbb{R}$ is \mathcal{A}/\mathcal{B} -measurable if, and only if, one, hence all, of the following conditions hold

1. $\{u \geq a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$
2. $\{u > a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$
3. $\{u \leq a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$
4. $\{u < a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$

3.3 Definition.

We define the *extended real line* $\bar{\mathbb{R}} := [-\infty, \infty]$ with the following rules for all $x \in \mathbb{R}$:

$$\begin{aligned} x + \infty &= \infty + x = \infty & x + -\infty &= -\infty + x = -\infty \\ \infty + \infty &= \infty & -\infty - \infty &= -\infty \end{aligned}$$

And for $x \in (0, \infty]$:

$$\begin{aligned} \pm x \cdot \infty &= \infty \cdot \pm x = \pm\infty \\ \pm x \cdot -\infty &= -\infty \cdot \pm x = \mp\infty \\ 0 \cdot \pm\infty &= \pm\infty \cdot 0 = 0 \\ \frac{1}{\pm\infty} &= 0 \end{aligned}$$

3.4 Definition.

Functions which take values in $\bar{\mathbb{R}}$ are called *numerical functions*.

3.5 Definition.

The Borel σ -algebra $\bar{\mathcal{B}} = \mathcal{B}(\bar{\mathbb{R}})$ is defined by:

$$\bar{\mathcal{B}} := \left\{ B \cup S : B \in \mathcal{B} \text{ and } S \in \left\{ \emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\} \right\} \right\}$$

3.6 Theorem.

We have $\mathcal{B}(\mathbb{R}) = \mathbb{R} \cap \mathcal{B}(\bar{\mathbb{R}})$. Moreover $\bar{\mathcal{B}}$ is generated by all sets of the form $[a, \infty]$ or $(a, \infty]$ or $[-\infty, a]$ or $[-\infty, a)$ where $a \in \mathbb{R}$

3.7 Definition.

Let (X, \mathcal{A}) be a measurable space. We write $\mathcal{M} := \mathcal{M}(\mathcal{A})$ and $\mathcal{M}_{\bar{\mathbb{R}}} := \mathcal{M}_{\bar{\mathbb{R}}}(\mathcal{A})$ for the families of real valued \mathcal{A}/\mathcal{B} -measurable and numerical $\mathcal{A}/\bar{\mathcal{B}}$ -measurable functions on X .

3.8 Definition.

A *simple function* $g : X \rightarrow \mathbb{R}$ on a measurable space (X, \mathcal{A}) is a function of the form

$$g(x) := \sum_{j=1}^M y_j \mathbf{1}_{A_j}(x)$$

with finitely many sets $A_1, \dots, A_M \in \mathcal{A}$ and $y_1, \dots, y_M \in \mathbb{R}$. The set of simple functions is denoted by \mathcal{E} or $\mathcal{E}(\mathcal{A})$.

If the sets A_1, \dots, A_M are mutually disjoint we call

$$\sum_{j=0}^M y_j \mathbf{1}_{A_j}(x)$$

with $y_0 := 0$ and $A_0 := (A_1 \cup \dots \cup A_M)^c$ a *standard representation* of g . Caution, this representation is not unique.

3.9 Theorem.

Let (X, \mathcal{A}) be a measurable space. Every $\mathcal{A}/\bar{\mathcal{B}}$ -measurable numerical function $u : X \rightarrow \bar{\mathbb{R}}$ is the pointwise limit of simple functions:

$$u(x) = \lim_{j \rightarrow \infty} f_j(x)$$

where $f_j \in \mathcal{E}(\mathcal{A})$ and $|f_j| \leq |u|$.

If $u \geq 0$, all f_j can be chosen to be positive and increasing towards u so that $u = \sup_{j \in \mathbb{N}} f_j$.

3.10 Theorem.

Let (X, \mathcal{A}) be a measurable space. If $u_j : X \rightarrow \bar{\mathbb{R}}, j \in \mathbb{N}$ are measurable functions, then so are

$$\sup_{j \in \mathbb{N}} u_j \quad \inf_{j \in \mathbb{N}} u_j \quad \limsup_{j \rightarrow \infty} u_j \quad \liminf_{j \rightarrow \infty} u_j$$

and whenever it exists

$$\lim_{j \rightarrow \infty} u_j.$$

3.11 Theorem.

Let u, v be $\mathcal{A}/\bar{\mathcal{B}}$ -measurable functions. Then the functions

$$u \pm v \quad uv \quad u \vee v := \max\{u, v\} \quad u \wedge v := \min\{u, v\}$$

are $\mathcal{A}/\bar{\mathcal{B}}$ -measurable (whenever they are defined).

3.12 Theorem.

A function u is $\mathcal{A}/\bar{\mathcal{B}}$ measurable if, and only if, u^\pm are $\mathcal{A}/\bar{\mathcal{B}}$ measurable.

3.13 Theorem.

Let $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$ be an \mathcal{A}/\mathcal{A}' -measurable map and let $\sigma(T) \subseteq \mathcal{A}$ be the σ -algebra generated by T . Then $u = w(T)$ for some $\mathcal{A}'/\bar{\mathcal{B}}$ measurable function $w : X' \rightarrow \bar{\mathbb{R}}$ if and only if $u : X \rightarrow \bar{\mathbb{R}}$ is $\sigma(T)/\bar{\mathcal{B}}$ -measurable.

3.14 Proposition.

Let (X, \mathcal{A}) be a measurable space. We define the indicator function:

$$1_A : X \rightarrow \mathbb{R} : x \in A \mapsto 1 \quad x \in X - A \mapsto 0$$

Show that the indicator function is measurable if, and only if, $A \in \mathcal{A}$.

Proof.

To show that 1_A is measurable, it suffices to show that

$$1_A^{-1}(a, \infty) \in \mathcal{A}.$$

Note that

$$1_A^{-1}(a, \infty) = \{x \in X : 1_A(x) \in (a, \infty)\} = \{1_A > a\}$$

If $a \geq 1$, then $1_A^{-1}(a, \infty) = \emptyset$.

If $a \in [0, 1)$, then $1_A^{-1}(a, \infty) = A$.

If $a < 0$, then $1_A^{-1}(a, \infty) = X$. □

3.15 Proposition.

Let $A_1, \dots, A_M \in \mathcal{A}$ be mutually disjoint sets and $y_1, \dots, y_M \in \mathbb{R}$. Then the function

$$g : X \rightarrow \mathbb{R} : x \mapsto \sum_{j=1}^M y_j 1_{A_j}(x)$$

is measurable.

Proof.

To show that g is measurable it suffices to show that

$$\{g > a\} \in \mathcal{A}$$

i.e.

$$\left\{x \in X : \sum_{j=1}^M y_j 1_{A_j}(x) > a\right\} = \bigcup_{j: y_j > a} A_j \in \mathcal{A}.$$

□

3.16 Problem 8.3i.

Let (X, \mathcal{A}) be a measurable space. Let $f, g : X \rightarrow \mathbb{R}$ be measurable functions. Show that for every $A \in \mathcal{A}$ the functions $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$, if $x \notin A$, is measurable.

Proof.

Note that

$$h(x) := 1_A(x)f(x) + 1_{A^c}(x)g(x).$$

And remember that sums and products of measurable functions are again measurable. □

3.17 Problem 8.3ii.

Let $(f_j)_{j \in \mathbb{N}}$ be a sequence of measurable functions and let $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$ such that $\bigcup_{j \in \mathbb{N}} A_j = X$. Suppose that $f_j|_{A_j \cap A_k} = f_k|_{A_j \cap A_k}$ for all $j, k \in \mathbb{N}$ and set $f(x) := f_j(x)$ if $x \in A_j$. Show that $f : X \rightarrow \mathbb{R}$ is measurable.

Proof.

We have that:

$$f^{-1}(B) = \bigcup_{j \in \mathbb{N}} A_j \cap f^{-1}(B) = \bigcup_{j \in \mathbb{N}} A_j \cap f_j^{-1}(B) \in \mathcal{A}$$

□

3.18 Problem 8.4.

Let (X, \mathcal{A}) be a measurable space and let $\mathcal{B} \subset \mathcal{A}$ be a sub- σ -algebra. Show that $\mathcal{M}(\mathcal{B}) \subset \mathcal{M}(\mathcal{A})$.

Proof.

To show that

$$\mathcal{M}(\mathcal{B}) \subset \mathcal{M}(\mathcal{A})$$

it suffices to show there exists a \mathcal{A} -measurable function that is not \mathcal{B} -measurable. By hypothesis, we have an element $A \in \mathcal{A}$, that is not in \mathcal{B} , i.e. $A \notin \mathcal{B}$. Since 1_A is \mathcal{B} -measurable if, and only if, $B \in \mathcal{B}$, we have find the \mathcal{A} -measurable function where we were looking for. □

3.19 Theorem.

Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable. Explain why u and $u' = du/dx$ are measurable.

Proof.

If u is differentiable, it is continuous, hence measurable. Since u' exists, we can write it in the form

$$u'(x) = \lim_{k \rightarrow \infty} \frac{u(x + 1/k) - u(x)}{1/k}$$

i.e. as limit of measurable functions. Thus, u' is also measurable. □

3.20 Problem 8.17.

Show that the measurability of $|u|$ does not, in general, imply the measurability of u .

Proof.

Let $A \subseteq \mathbb{R}$ be such that $A \notin \mathcal{B}$. Then it is clear that

$$u(x) := 1_A(x) - 1_{A^c}(x)$$

is not measurable. Take

$$\{u = 1\} = A \notin \mathcal{A}.$$

But $|u(x)| = 1$, which is a continuous function and therefore measurable.

□

3.21 Problem 8.14.

Consider $(\mathbb{R}, \mathcal{B})$ and $u : \mathbb{R} \rightarrow \mathbb{R}$. Show that $\{x\} \in \sigma(u)$ for all $x \in \mathbb{R}$ if, and only if, u is injective.

Proof.

To show that u is injective, it suffices to assume $x, y \in \mathbb{R}$ and show that

$$u(x) = u(y) \implies x = y.$$

Showing that is equivalent with showing that

$$|\{u = u(x_0)\}| = 1.$$

We surely have that $\{x_0\} \subseteq \{u = u(x_0)\}$. And note that

$$\{x_0\} \in \sigma(u) = \sigma(u^{-1}(\mathcal{B}))$$

just means that $\{x_0\} = u^{-1}(B)$ for some $B \in \mathcal{B}$. □

Proof.

Assume that u is injective,. This means that every point in the range $u(\mathbb{R})$ comes exactly from unique defined $x \in \mathbb{R}$. This can be expressed by saying that $\{x\} = u^{-1}(\{u(x)\}) = \{u(x)\}$. But then

$$\{x\} \in \sigma(u) = \sigma(u^{-1}(\mathcal{B})).$$

□

4 12-10-2014

4.1 Measure Theory Chapter 9

4.1 Definition.

Let $f = \sum_{j=0}^M y_j 1_{A_j} \in \mathcal{E}^+$ be a simple function in standard representation. Then the number

$$I_\mu(f) := \sum_{j=0}^M y_j \mu(A_j) \in [0, \infty]$$

is called the (μ) -integral of f .

4.2 Theorem.

1. $I_\mu(1_A) = \mu(A) \quad \forall A \in \mathcal{A}$
2. $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0$
3. $I_\mu(f + g) = I_\mu(f) + I_\mu(g)$
4. $f \leq g \implies I_\mu(f) \leq I_\mu(g)$

4.3 Definition.

Let (X, \mathcal{A}, μ) be a measure space. The (μ) -integral of a positive numerical function $u \in \mathcal{M}_{\mathbb{R}}^+$ is given by

$$\int u d\mu := \sup\{I_\mu(g) : g \leq u, g \in \mathcal{E}^+\} \in [0, \infty].$$

If we need to emphasize the integration variable, we also write

$$\int u(x) \mu(dx) \quad \text{or} \quad \int u(x) d\mu(x)$$

4.4 Theorem.

For all $f \in \mathcal{E}^+$ we have $\int f du = I_\mu(f)$.

4.5 Theorem.

Let (X, \mathcal{A}, μ) be a measure space. For an increasing sequence of numerical functions $(u_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}_{\mathbb{R}}^+, 0 \leq u_j \leq u_{j+1} \leq \dots$, we have $u := \sup_{j \in \mathbb{N}} u_j \in \mathcal{M}_{\mathbb{R}}^+$ and

$$\int \sup_{j \in \mathbb{N}} u_j d\mu = \sup_{j \in \mathbb{N}} \int u_j d\mu$$

4.6 Theorem.

Let $u \in \mathcal{M}_{\mathbb{R}}^+$. Then

$$\int u d\mu = \lim_{j \rightarrow \infty} \int f_j d\mu$$

holds for every increasing sequence $(f_j)_{j \in \mathbb{N}} \subseteq \mathcal{E}^+$ with $\lim_{j \rightarrow \infty} f_j = u$.

4.7 Theorem.

Let $u, v \in \mathcal{M}_{\mathbb{R}}^+$. Then

1. $\int 1_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}$
2. $\int \alpha u d\mu = \alpha \int u d\mu$
3. $\int (u + v) d\mu = \int u d\mu + \int v d\mu$
4. $u \leq v \implies \int u d\mu \leq \int v d\mu$

4.8 Theorem.

Let $(u_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}_{\mathbb{R}}^+$. Then $\sum_{j=1}^{\infty} u_j$ is measurable and we have

$$\int \sum_{j=1}^{\infty} u_j d\mu = \sum_{j=1}^{\infty} \int u_j d\mu$$

4.9 Theorem.

Let $(u_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}_{\mathbb{R}}^+$ be a sequence of positive measurable numerical functions.

Then $u := \liminf_{j \in \mathbb{N}} \int u_j d\mu$ is measurable and

$$\int \liminf_{j \rightarrow \infty} u_j d\mu \leq \liminf_{j \rightarrow \infty} \int u_j d\mu$$

4.10 Problem 9.1.

Let $f : X \rightarrow \mathbb{R}$ be a positive simple function of the form

$$f(x) = \sum_{j=1}^m \xi_j 1_{A_j}(x) \quad \xi_j \geq 0, A_j \in \mathcal{A}.$$

Show that

$$I_{\mu}(f) = \sum_{j=1}^m \xi_j \mu(A_j)$$

Proof.

$$I_\mu(f) = I_\mu\left(\sum_{j=1}^m \xi_j 1_{A_j}\right) = \sum_{j=1}^m \xi_j I_\mu(1_{A_j}) = \sum_{j=1}^m \xi_j \mu(A_j)$$

□

4.11 Problem 9.5.

Let (X, \mathcal{A}, μ) be a measure space and $u \in \mathcal{M}^+(\mathcal{A})$. Show that the set-function

$$A \mapsto \int 1_A u d\mu \quad A \in \mathcal{A}$$

is a measure.

Proof.

Set

$$\nu : \mathcal{A} \rightarrow [0, \infty] : A \mapsto \int 1_A u d\mu.$$

1. To show that $\nu(\emptyset) = 0$. Notice that $1_\emptyset \equiv 0$.
2. Let $A = \bigcup_{j \in \mathbb{N}} A_j$ a disjoint union of sets $A_j \in \mathcal{A}$. Note that

$$\sum_{j=1}^{\infty} 1_{A_j} = 1_A$$

We have to show that

$$\begin{aligned} \nu\left(\bigcup_{j \in \mathbb{N}} A_j\right) &= \int \left(\sum_{j=1}^{\infty} 1_{A_j}\right) \cdot u d\mu \\ &= \int \left(\sum_{j=1}^{\infty} 1_{A_j} u\right) d\mu \\ &= \sum_{j=1}^{\infty} \int 1_{A_j} u d\mu \\ &= \sum_{j=1}^{\infty} \nu(A_j). \end{aligned}$$

□

4.12 Problem 9.8.

Let (X, \mathcal{A}, μ) be a measure space and $(u_j)_{j \in \mathbb{N}} \subseteq \mathcal{M}^+(\mathcal{A})$. If $u_j \leq u$ for all $j \in \mathbb{N}$ and some $u \in \mathcal{M}^+(\mathcal{A})$ with $\int u d\mu < \infty$, then

$$\limsup_{j \in \mathbb{N}} \int u_j d\mu \leq \int \limsup_{j \in \mathbb{N}} u_j d\mu.$$

Proof.

Showing that

$$\limsup_{j \in \mathbb{N}} \int u_j d\mu \leq \int \limsup_{j \in \mathbb{N}} u_j d\mu$$

is equivalent with showing that

$$-\liminf_{j \in \mathbb{N}} \int -u_j d\mu \leq -\int \liminf_{j \in \mathbb{N}} -u_j d\mu$$

which is equivalent with showing that

$$\int \liminf_{j \in \mathbb{N}} -u_j d\mu \leq \liminf_{j \in \mathbb{N}} \int -u_j d\mu$$

which is equivalent with showing that

$$\int u d\mu + \int \liminf_{j \in \mathbb{N}} -u_j d\mu \leq \int u d\mu + \liminf_{j \in \mathbb{N}} \int -u_j d\mu$$

which is equivalent with showing that

$$\int u + \liminf_{j \in \mathbb{N}} -u_j d\mu \leq \liminf_{j \in \mathbb{N}} \left(\int u d\mu + \int -u_j d\mu \right)$$

which is equivalent with showing that

$$\int \liminf_{j \in \mathbb{N}} (u - u_j) d\mu \leq \liminf_{j \in \mathbb{N}} \left(\int u - u_j d\mu \right).$$

By hypothesis, $u_j \leq u$. So we have that $u - u_j$ is a sequence of positive measurable functions and therefore our last statement follows by the theorem of Fatou. □

4.2 Measure Theory Chapter 7

4.13 Proposition.

Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \rightarrow X'$ is \mathcal{A}/\mathcal{A}' -measurable if and only if $T^{-1}(\mathcal{G}') \subseteq \mathcal{A}$.

It suffices to assume $T^{-1}(\mathcal{G}') \subseteq \mathcal{A}$ and show that

$$T^{-1}(\mathcal{A}) \subseteq \mathcal{A}.$$

Consider $\Sigma := \{A' \subseteq X' : T^{-1}(A') \in \mathcal{A}\}$. We have that $\mathcal{G}' \subseteq \Sigma$. To show that

$$\mathcal{A}' \subseteq \Sigma,$$

it suffices to show that Σ is a σ -algebra.

1. Showing that $X' \in \Sigma$, is equivalent with showing that $T^{-1}(X') \in \mathcal{A}$.
Which holds as $T^{-1}(X') = X \in \mathcal{A}$.

2. Showing that

$$A' \in \Sigma \implies A'^c \in \Sigma$$

is equivalent with showing that

$$T^{-1}(A') \in \mathcal{A} \implies T^{-1}(A'^c) \in \mathcal{A} \quad \checkmark$$

3. Showing that

$$(A'_j)_{j \in \mathbb{N}} \subseteq \Sigma \implies \bigcup_{j \in \mathbb{N}} A_j \in \Sigma$$

is equivalent with showing that

$$T^{-1}(A'_j) \in \mathcal{A} \implies T^{-1}\left(\bigcup_{j \in \mathbb{N}} A_j\right) \in \mathcal{A} \quad \checkmark$$

4.14 Proposition.

Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be measurable spaces and $T : X \rightarrow X'$ be an \mathcal{A}/\mathcal{A}' -measurable map. For every measure μ on (X, \mathcal{A}) ,

$$\mu'(A') := T(\mu)(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}'$$

defines a measure on (X', \mathcal{A}') .

Proof.

1. To show that

$$\mu(T^{-1}(\emptyset)) = 0 \quad \checkmark$$

2. Assume $(A'_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}'$ mutually disjoint sets and show that

$$\begin{aligned} \mu' \left(\bigcup_{i=1}^n A'_i \right) &= \sum_{i=1}^n \mu'(A'_i) \\ \uparrow [\text{by definition of } \mu'] \\ \mu(T^{-1} \left(\bigcup_{i=1}^n A'_i \right)) &= \sum_{i=1}^n \mu(T^{-1}(A'_i)) \\ \uparrow [T^{-1} \text{ commutes with set operations}] \\ \mu \left(\bigcup_{i=1}^n T^{-1}(A'_i) \right) &= \sum_{i=1}^n \mu(T^{-1}(A'_i)) \\ \uparrow [\mu \text{ is measure on } (X, \mathcal{A}) \text{ and } T \text{ is } \mathcal{A}/\mathcal{A}' \text{ measurable, given}] \\ T^{-1}(A'_i) \cap T^{-1}(A'_j) &= \emptyset \\ \uparrow [T^{-1} \text{ commutes with set operations}] \\ A'_i \cap A'_j &= \emptyset \text{ [by hypothesis]} \end{aligned}$$

□

4.15 Problem 7.9i.

Let μ be a measure on $(\mathbb{R}, \mathcal{B})$. Show that

$$F_\mu(x) := \begin{cases} \mu[0, x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu[x, 0) & \text{if } x < 0 \end{cases}$$

1. is monotonically increasing
2. left-continuous function

Proof.

1. Showing that F_μ is monotonically increasing is equivalent with showing that

$$x \leq y \implies F_\mu(x) \leq F_\mu(y).$$

(a) $x \leq 0 \leq y$: Then $F_\mu(x) = -\mu[x, 0] \leq 0$ and $F_\mu(y) = \mu[0, y] \geq 0$.

(b) $0 < x \leq y$: Then $[0, x] \subseteq [0, y]$. And $\mu[0, x] \leq \mu[0, y]$.

(c) $x \leq y < 0$: Then $[y, 0] \subseteq [x, 0]$. And $\mu[y, 0] \leq \mu[x, 0]$.

2. Showing that F_μ is left continuous is equivalent with assuming (x_k) a sequence such that $x_k < x$ and $x_k \uparrow x$ and showing that

$$\lim_{k \rightarrow \infty} F_\mu(x_k) = F_\mu(x).$$

If $x > 0$, it suffices to show that

$$\lim_{k \rightarrow \infty} \mu[0, x_k] = \mu[0, x].$$

If $x < 0$ it suffices to show that

$$\lim_{k \rightarrow \infty} -\mu[x_k, 0] = -\mu[x, 0].$$

If $x = 0$ it suffices to show that

$$\lim_{k \rightarrow \infty} -\mu[x_k, 0] = 0.$$

Remember that:

1. For any increasing sequence $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$ with $A_j \uparrow A \in \mathcal{A}$ we have

$$\mu(A) = \mu(\cup A_j) = \lim_{j \in \mathbb{N}} \mu(A_j)$$

2. For any decreasing sequence $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$ with $A_j \downarrow A \in \mathcal{A}$ we have

$$\mu(A) = \mu(\cap A_j) = \lim_{j \in \mathbb{N}} \mu(A_j)$$

□

4.16 Problem 7.9ii.

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a Stieltjes function. Show that

$$\nu_F[a, b) = F(b) - F(a) \quad \forall a, b \in \mathbb{R}, a < b$$

has a unique extension to a measure on \mathcal{B} .

Proof.

By theorem 6.1 it suffices to show that ν_F is a pre-measure. To show this it suffices to show that

1. $\nu_F(\emptyset) = \nu_F[a, a) = 0$
2. $\nu_F([a, b) \cup [b, c)) = \nu_F([a, b)) + \nu_F([b, c))$

•

$$\begin{aligned} \nu_F([a, b)) + \nu_F([b, c)) &= F(b) - F(a) + F(c) - F(b) \\ &= F(c) - F(a) \\ &= \nu_F[a, c) \\ &= \nu_F([a, b) \cup [b, c)) \end{aligned}$$

3. For any decreasing sequence $[a_j, b_j)_{j \in \mathbb{N}} \subseteq \mathcal{J}$ with $[a_j, b) \downarrow [a, b) \in \mathcal{J}$ we have

$$\nu_F([a, b)) = \lim_{j \in \infty} \nu_F[a_j, b).$$

This last statement is equivalent with

$$F(b) - F(a) = \lim_{j \in \infty} (F(b) - F(a_j)).$$

Note that since $[a_j, b_j) \downarrow [a, b) \in \mathcal{J}$ we have that $a_j \uparrow a, a_j \leq a$ and therefore

$$\lim_{j \in \infty} (F(b) - F(a_j)) = F(b) - F(a),$$

as F is left-continuous.

4. \mathcal{J} contains an exhausting sequence $[a_j, b_j)$ such that $[a_j, b_j) \uparrow \mathbb{R}$ and $\nu_F[a_j, b_j) < \infty$

□

5 13-10-2014

5.1 Markov Chains 1.2

5.1 Definition.

We say that i leads to j and write $i \rightarrow j$ if

$$\mathbb{P}_i(X_n = j \text{ for some } n \geq 0) > 0.$$

5.2 Definition.

We say i communicates with j and write $i \leftrightarrow j$ if both $i \rightarrow j$ and $j \rightarrow i$.

5.3 Theorem.

For distinct states i and j the following are equivalent:

1. $i \rightarrow j \iff \mathbb{P}_i(X_n = j \text{ for some } n \geq 0) > 0$
2. $p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n} > 0$ for some states i_1, \dots, i_n with $i_1 = i$ and $i_n = j$
3. $p_{ij}^n > 0$ for some $n \geq 0$

Proof.

Remember that

$$p_{ij}^n = \mathbb{P}_i(X_n = j) = \sum_{i_2, \dots, i_{n-1}} p_{ii_2} p_{i_2 i_3} \cdots p_{i_{n-1} j}.$$

From this everything follows. □

5.4 Proposition.

Show that $i \rightarrow i$.

Proof.

$$i \rightarrow i \iff \mathbb{P}_i(X_n = i \text{ for some } n \geq 0) > 0$$

This follows, as $\mathbb{P}_i(X_0 = i) = 1$. □

5.5 Definition.

The equivalence classes of the equivalence relations \leftrightarrow are called *communicating classes*. We say that a class C is closed if

$$i \in C, i \rightarrow j \implies j \in C$$

5.6 Definition.

A state i is *absorbing* if $\{i\}$ is a closed class.

5.7 Definition.

A chain or transition matrix P where I is a single class is called *irreducible*.

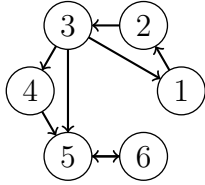
5.8 Example 1.2.2.

Find the communicating classes associated to the stochastic matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Proof.

The solution is obvious from the diagram. The classes being $\{1, 2, 3\}$, $\{4\}$ and $\{5, 6\}$. With only $\{5, 6\}$ closed.



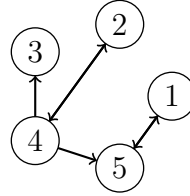
□

5.9 Exercise 1.2.1.

Identify the communicating classes of the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

The solution is obvious from the diagram. The classes being $\{1, 5\}$, $\{2, 4\}$



and $\{3\}$. With $\{1, 5\}$ closed and $\{3\}$ absorbing.

5.2 Markov Chains 1.3

5.10 Definition.

Let X_n be a Markov chain with transition matrix P . The *hitting time* of a subset A of I is the random variable

$$H^A : \Omega \rightarrow \{0, 1, 2, \dots, \infty\}$$

given by

$$H^A(\omega) = \inf\{n \geq 0 : X_n(\omega) \in A\}$$

where we agree that the infimum of the empty set \emptyset is ∞ . The probability starting from i that $(X_n)_{n \geq 0}$ ever hits A is then

$$h_i^A = \mathbb{P}_i(H^A < \infty).$$

When A is a closed class, h_i^A is called the *absorption probability*. The mean time taken for X_n to reach A is given by

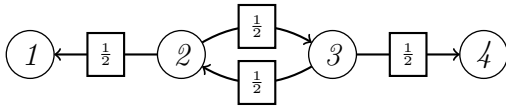
$$k_i^A = E_i(H^A) = \sum_{n < \infty} n \mathbb{P}(H^A = n) + \infty \mathbb{P}(H^A = \infty).$$

We shall often write less formally

$$h_i^A = \mathbb{P}_i(\text{hit } A) \quad k_i^A = E_i(\text{time to hit } A).$$

5.11 Example 1.3.1.

Consider the chain with following diagram:



1. Starting from 2, what is the probability of absorption in 4?
2. How long does it take until the chain is absorbed in 1 or 4 ?

Proof.

1. Note that $A = \{4\}$ is a closed class. The absorption probability is defined as

$$h_i := h_i^{\{4\}} = \mathbb{P}_i(\text{hit } 4).$$

We have

$$\begin{aligned} h_1 &= 0 \\ h_4 &= 1 \\ h_2 &= \frac{1}{2}h_1 + \frac{1}{2}h_3 = \frac{1}{2}h_3 \\ h_3 &= \frac{1}{2}h_2 + \frac{1}{2}h_4 = \frac{1}{2}h_2 + \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} h_2 &= \frac{1}{4}h_2 + \frac{1}{4} \\ \implies \frac{3}{4}h_2 &= \frac{1}{4} \\ \implies h_2 &= \frac{1}{3} \end{aligned}$$

2. We need to compute

$$k_2 = k_2^{\{1,4\}} = E_2(\text{time to hit } \{1,4\}).$$

We have

$$\begin{aligned} k_1 &= 0 \\ k_4 &= 0 \\ k_2 &= 1 + \frac{1}{2}k_3 \\ k_3 &= 1 + \frac{1}{2}k_2 \end{aligned}$$

Hence

$$\begin{aligned} k_2 &= \frac{3}{2} + \frac{1}{4}k_2 \\ \implies \frac{3}{4}k_2 &= \frac{3}{2} \\ \implies k_2 &= 2 \end{aligned}$$

□

5.12 Theorem 1.3.2.

The vector of hitting probabilities $h^A = (h_i^A : i \in I)$ is the minimal non-negative solution to the system

$$\begin{cases} h_i^A = 1 & \text{for } i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & \text{for } i \notin A. \end{cases}$$

Minimality means that if $x = (x_i : i \in I)$ is another solution with $x_i \geq 0$ for all i , then $x_i \geq h_i$ for all i .

5.13 Example 1.3.1(continued).

Use theorem 1.3.2 to compute h_2 again.

Proof.

The vector of hitting probabilities $h^{\{4\}} = (h_1^{\{4\}}, h_2^{\{4\}}, h_3^{\{4\}}, h_4^{\{4\}})$ is the minimal non-negative solution to the system

$$\begin{aligned}h_1^{\{4\}} &= \sum_{j \in I} p_{ij} h_j^{\{4\}} = h_1^{\{4\}} \\h_2^{\{4\}} &= \sum_{j \in I} p_{ij} h_j^{\{4\}} = \frac{1}{2} h_1^{\{4\}} + \frac{1}{2} h_3^{\{4\}} \\h_3^{\{4\}} &= \sum_{j \in I} p_{ij} h_j^{\{4\}} = \frac{1}{2} h_2^{\{4\}} + \frac{1}{2} h_4^{\{4\}} \\h_4^{\{4\}} &= 1\end{aligned}$$

The minimality condition gives, that $h_1^{\{4\}} = 0$. So that

$$\begin{aligned}h_2^{\{4\}} &= \frac{1}{2} h_3^{\{4\}} \\h_3^{\{4\}} &= \frac{1}{2} h_2^{\{4\}} + \frac{1}{2}\end{aligned}$$

which gives:

$$\begin{aligned}h_2^{\{4\}} &= \frac{1}{4} h_2^{\{4\}} + \frac{1}{4} \implies h_2^{\{4\}} = \frac{1}{3} \\h_3^{\{4\}} &= \frac{1}{4} h_3^{\{4\}} + \frac{1}{2} \implies h_3^{\{4\}} = \frac{2}{3}\end{aligned}$$

□

5.14 Theorem.

Consider a recurrence relation of the form

$$ax_{n+1} + bx_n + cx_{n-1} = 0 \quad a, c \neq 0.$$

Let α, β be the roots of the quadratic equation

$$ax^2 + bx + c.$$

Then the general solution is given by

$$x_n = \begin{cases} A\alpha^n + B\beta^n & \text{if } \alpha \neq \beta \\ (A + nB)\alpha^n & \text{if } \alpha = \beta \end{cases}$$

5.15 Proposition.

Give a general solution for the recurrence relation

$$\begin{aligned} h_0 &= 1 \\ h_i &= ph_{i+1} + qh_{i-1} \end{aligned}$$

Proof.

Note that we have $-ph_{i+1} + h_i - qh_{i-1} = 0$. Consider

$$-px^2 + x - 1 + p = 0$$

We have the roots $\alpha = 1, \beta = \frac{q}{p}$. If $q \neq p$, this gives

$$h_i = A\alpha^i + B\beta^i = A + B\left(\frac{q}{p}\right)^i.$$

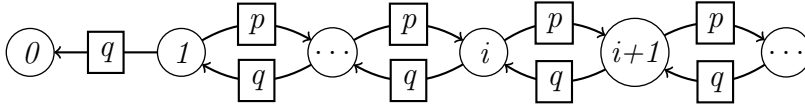
And if $p = q$, then $\alpha = \beta = 1$, and we have

$$h_i = A\alpha^i + B\beta^i = A + iB$$

□

5.16 Example 1.3.3.

Consider the Markov chain with diagram



where $0 < p = 1 - q < 1$. What is $h_i = \mathbb{P}_i(\text{hit } 0)$?

Proof.

We know that h is the minimal non-negative solution to

$$\begin{aligned} h_0 &= 1 \\ h_i &= ph_{i+1} + qh_{i-1} \end{aligned}$$

We consider some cases:

- Suppose $p = q$, then we have

$$h_i = A\alpha^i + B\beta^i = A + iB$$

and as $0 \leq h_i \leq 1$ is a probability, we must have $B = 0$. We then have

$$h_i = A,$$

and as $h_0 = 1$, we must have $h_i = 1$.

- Suppose $p \neq q$, we then have

$$h_i = A\alpha^i + B\beta^i = A + B\left(\frac{q}{p}\right)^i.$$

If $\frac{q}{p} > 1$, then we must set $B = 0$ again.

- Suppose $\frac{q}{p} < 1$. We have that $h_0 = 1$, and therefore $A + B = 1$. Hence:

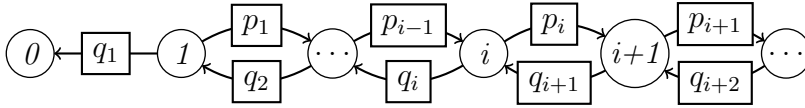
$$h_i = \left(\frac{q}{p}\right)^i + A\left(1 - \left(\frac{q}{p}\right)^i\right)$$

So the minimal non-negative solutions is $h_i = (q/p)^i$.

□

5.17 Example 1.3.4.

Consider the Markov chain with diagram



where for $i = 1, 2, \dots$, we have $0 < p_i = 1 - q_i < 1$. As in the preceding example, 0 is the absorbing state, and we wish to calculate the absorption probability starting from i .

Proof.

Consider the system of equations

$$\begin{aligned} h_0 &= 1 \\ h_i &= p_i h_{i+1} + q_i h_{i-1} \quad \text{for } i = 1, 2, \dots \end{aligned}$$

Consider

$$u_i := h_{i-1} - h_i,$$

then

$$\begin{aligned} p_i u_{i+1} &= p_i h_i - p_i h_{i+1} \\ q_i u_i &= q_i h_{i-1} - q_i h_i \end{aligned}$$

$$\implies q_i u_i - p_i u_{i+1} = h_i - q_i h_i - p_i h_i = 0$$

Therefore $p_i u_{i+1} = q_i u_i$ and we have

$$u_{i+1} = \left(\frac{q_i}{p_i}\right) u_i = \prod_{j=1}^i \frac{q_j}{p_j} u_1 = \gamma_1 u_1$$

We also have

$$u_1 + \dots + u_i = h_0 - h_i$$

so

$$h_i = h_0 - (u_1 + \dots + u_i) = 1 - u_1(\gamma_0 + \dots + \gamma_{n-1}).$$

At this point A remains to be determined. In the case that $\sum_{i=0}^{\infty} \gamma_i = \infty$, we must have $A = 0$. In the other case, we can't take $A < 0$, but we can take $A > 0$ so long as

$$h_i = 1 - A \sum_{i=0}^{i-1} \gamma_i \geq 0.$$

Which means that

$$A \leq \left(\sum_{i=0}^{i-1} \gamma_i\right)^{-1}$$

but still as big as possible. So we get

$$A = \left(\sum_{i=0}^{\infty} \gamma_i\right)^{-1}.$$

And therefore

$$h_i = \frac{\sum_{j=i}^{\infty} \gamma_j}{\sum_{j=0}^{\infty} \gamma_j}$$

□

5.18 Theorem 1.3.5.

The vector of mean hitting times $k^A = (k^A : i \in I)$ is the minimal non-negative solution to the system of linear equations

$$\begin{cases} k_i^A = 0 & \text{for } i \in A \\ k_i^A = 1 + \sum_{j \notin A} p_{ij} k_j^A & \text{for } i \notin A \end{cases}$$

6 14-10-2014

6.1 Markov Chains 1.4

6.1 Definition.

A random variable $T : \Omega \rightarrow \{0, 1, 2, \dots, \infty\}$ is called a *stopping time* if the event $\{T = n\}$ depends only on X_0, X_1, \dots, X_n for $n = 0, 1, 2, \dots$. Intuitively, by watching the process, you know at the time when T occurs. If asked to stop at T , you know when to stop.

6.2 Proposition.

The first passage time

$$T_j = \inf\{n \geq 1 : X_n = j\}$$

is a stopping time.

Proof.

To show that T_j is a stopping time, we have to show that

$$\{T_j = n\}$$

depends only on X_0, \dots, X_n .

This follows from

$$\{T_j = n\} = \{X_1 \neq j, \dots, X_{n-1} \neq j, X_n = j\}$$

□

6.3 Proposition.

The first hitting time

$$H^A = \inf\{n \geq 0 : X_n \in A\}$$

is a stopping time.

Proof.

To show that H^A is a stopping time, we have to show that

$$\{H^A = n\}$$

depends only on X_0, \dots, X_n . This follows from

$$\{H^A = n\} = \{X_0 \notin A, \dots, X_{n-1} \notin A, X_n \in A\}$$

□

6.4 Proposition.

The last exit time

$$L^A = \sup\{n \geq 0 : X_n \in A\}$$

is not in general a stopping time.

Proof.

The event $\{L^A = n\}$ depends on whether $(X_{n+m})_{m \geq 0}$ visits A or not. So we don't have a stopping time. □

6.5 Theorem 1.4.2 (Strong Markov property).

Let $(X_n)_{n \geq 0}$ be Markov(λ, P) and let T be a stopping time of $(X_n)_{n \geq 0}$. Then, conditional on $T < \infty$ and $X_T = i$, $(X_{T+n})_{n \geq 0}$ is Markov(δ_i, P) and independent of X_0, \dots, X_T .

We now consider an application of the strong Markov property to a Markov chain $(X_n)_{n \geq 0}$ observed only at certain times. In the first instance suppose that J is some subset of the state space I and that we observe the chain only when it takes values in J .

6.6 Proposition.

Let $(X_n)_{n \geq 0}$ be a Markov chain. Consider

$$T_0 = \inf\{n \geq 0 : X_n \in J\}$$

and, for $m = 0, 1, 2, \dots$

$$T_{m+1} = \inf\{n > T_m : X_n \in J\}.$$

Assume $P(T_m < \infty) = 1$ for all m . Show that $Y_m = X_{T_m}$ is a Markov chain and compute its transition matrix in terms of the transition matrix P of X_n .

Proof.

Showing that (Y_m) is a markov chain is equivalent with showing that

$$\begin{aligned}\mathbb{P}(Y_{m+1} = i_{m+1} | Y_0 = i_1, \dots, Y_m = i_m) \\ = \mathbb{P}(Y_{m+1} = i_{m+1} | Y_m = i_m)\end{aligned}$$

which in turn is equivalent with showing that

$$\begin{aligned}\mathbb{P}(X_{T_{m+1}} = i_{m+1} | X_{T_0} = i_1, \dots, X_{T_m} = i_m) \\ = \mathbb{P}(X_{T_{m+1}} = i_{m+1} | X_{T_m} = i_m).\end{aligned}$$

The Markov property gives that $(X_{T_m+n})_{n \geq 0}$ is a markov chain and independent of X_0, \dots, X_{T_m} , and so surely independent of $X_{T_0} = i_1, \dots, X_{T_{m-1}}$. Now $X_{T_{m+1}} = X_{T_m+n}$ for some n . So the equality follows.

Now the question is, starting from $i \in J$ what is the chance that we hit $j \in J$ the first time we hit J ? Call this chance h_i^j . Well this chance is surely greater than p_{ij} as there is also a chance that we first get outside of J and then next time hit J , and so on. With a similar reasoning as in Theorem 1.3.2 we can show that for $j \in J$ the vector $(h_i^j : i \in I)$ is the minimal non-negative solution to

$$h_i^j = p_{ij} + \sum_{k \notin J} p_{ik} h_k^j.$$

□

6.7 Proposition.

Let $(X_n)_{n \geq 0}$ be a markov chain. Consider

$$T_0 = \inf\{n \geq 0 : X_n \neq X_0\}$$

and, for $m = 0, 1, 2, \dots$

$$T_{m+1} = \inf\{n \geq T_m : X_n \neq X_{T_m}\}.$$

Assume $\mathbb{P}(T_m < \infty) = 1$ for all m . Show that $Y_m = X_{T_m}$ is a markov chain and compute its transition matrix in terms of the transition matrix P of X_n .

Proof.

Showing that (Y_m) is a markov chain is equivalent with showing that

$$\begin{aligned}\mathbb{P}(Y_{m+1} = i_{m+1} | Y_0 = i_1, \dots, Y_m = i_m) \\ = \mathbb{P}(Y_{m+1} = i_{m+1} | Y_m = i_m)\end{aligned}$$

which in turn is equivalent with showing that

$$\begin{aligned}\mathbb{P}(X_{T_{m+1}} = i_{m+1} | X_{T_0} = i_1, \dots, X_{T_m} = i_m) \\ = \mathbb{P}(X_{T_{m+1}} = i_{m+1} | X_{T_m} = i_m).\end{aligned}$$

The Markov property gives that $(X_{T_m+n})_{n \geq 0}$ is a markov chain and independent of X_0, \dots, X_{T_m} , and so surely independent of $X_{T_0} = i_1, \dots, X_{T_{m-1}}$. Now $X_{T_{m+1}} = X_{T_m+n}$ for some n . So the equality follows.

Now, the question is, starting from i what is the chance to go to j now, if we set the chance $p_{ii} = 0$. Call this chance \tilde{p}_{ij} . We have

$$\tilde{p}_{ij} = \frac{p_{ij}}{\sum_{k \neq i} p_{ik}}$$

□

6.2 Markov Chains 1.5

6.8 Definition.

Let $(X_n)_{n \geq 0}$ be a Markov chain with transition matrix P . We say that a state i is *recurrent* if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 1.$$

A recurrent state is a state i where you keep coming back.

6.9 Definition.

We say that a state i is *transient* if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 0.$$

A transient state is a state i which you eventually leave for ever.

6.10 Theorem.

A state i is either recurrent or transient.

6.11 Definition.

Recall that the first passage time to a state i is the random variable T_i defined by

$$T_i(\omega) = \inf\{n \geq 1 : X_n(\omega) = i\}$$

where $\inf \emptyset = \infty$. We now define inductively the r th passage time $T_i^{(r)}$ to state i by

$$T_i^{(0)}(\omega) = 0 \quad T_i^{(1)}(\omega) = T_i(\omega)$$

and for $r = 0, 1, 2, \dots$,

$$T_i^{(r+1)}(\omega) = \inf\{n \geq T_i^{(r)}(\omega) + 1 : X_n(\omega) = i\}.$$

The length of the r th excursion to i is then

$$S_i^{(r)} = \begin{cases} T_i^{(r)} - T_i^{(r-1)} & \text{if } T_i^{(r-1)} < \infty \\ 0 & \text{otherwise} \end{cases}.$$

6.12 Lemma 1.5.1.

For $r = 2, 3, \dots$, conditional on $T_i^{(r-1)} < \infty$, $S_i^{(r)}$ is independent of $\{X_m : m \leq T_i^{(r-1)}\}$ and

$$\mathbb{P}(S_i^{(r)} = n | T_i^{(r-1)} < \infty) = \mathbb{P}_i(T_i = n)$$

6.13 Definition.

The number of visits to i is denoted by

$$V_i = \sum_{n=0}^{\infty} 1_{\{X_n=i\}}.$$

6.14 Theorem.

$$E_i(V_i) = \sum_{n=0}^{\infty} p_{ii}^{(n)}$$

Proof.

We have

$$\begin{aligned}
 E_i(V_i) &= \sum_{n=0}^{\infty} E_i(1_{\{X_n=i\}}) \\
 &= \sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) \\
 &= \sum_{n=0}^{\infty} p_{ii}^{(n)}
 \end{aligned}$$

□

6.15 Definition.

The *return probability* of i is denoted by

$$f_i = \mathbb{P}_i(T_i < \infty).$$

6.16 Lemma 1.5.2.

For $r = 0, 1, 2, \dots$, we have $\mathbb{P}_i(V_i > r) = f_i^r$.

Proof.

Showing that

$$\mathbb{P}_i(V_i > r) = f_i^r$$

is equivalent with showing that

$$\mathbb{P}_i(V_i > r) = \mathbb{P}_i(T_i < \infty)^r$$

which in turn is equivalent with

$$\mathbb{P}_i(T_i^{(r)} < \infty) = \mathbb{P}_i(T_i < \infty)^r.$$

This last statement can be proven by induction.

□

6.17 Theorem 1.5.3.

The following dichotomy holds:

1. if $\mathbb{P}_i(T_i < \infty) = 1$, then i is recurrent and $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$
2. if $\mathbb{P}_i(T_i < \infty) < 1$, then i is transient and $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$

6.18 Theorem 1.5.4.

Let C be a communicating class. Then either all states in C are transient or all are recurrent.

6.19 Theorem 1.5.5.

*Every recurrent class is closed. And the contrapositive:
Every class that is not closed, is transient.*

6.20 Theorem 1.5.6.

*Every finite closed class is recurrent. And the contrapositive:
Every transient class is either infinite or not closed..*

6.21 Theorem 1.5.7.

Suppose P is irreducible and recurrent. Then for all $j \in I$ we have

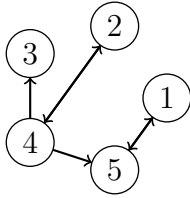
$$\mathbb{P}(T_j < \infty) = 1.$$

6.22 Exercise 1.5.1.

Identify the recurrent and transient states of the Markov chain with the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

The solution is obvious from the diagram:



The classes being $\{1, 5\}$, $\{2, 4\}$ and $\{3\}$. With $\{1, 5\}$ closed and finite, and therefore recurrent. The class $\{3\}$ is absorbing, so closed and finite, and therefore recurrent. The other class $\{2, 4\}$ is not closed, and therefore, not recurrent. So we have that $\{2, 4\}$ is transient.

7 15-10-2014

7.1 Markov Chains 1.6

7.1 Theorem.

A state i is recurrent if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$.

7.2 Example 1.6.1 The simple random walk on \mathbb{Z} .

Compute $\sum_{n=0}^{\infty} p_{00}^{(n)}$.

Proof.

First note that $p_{00}^{(2n+1)} = 0$.

Any given sequence of steps of length $2n$ from 0 to 0 occurs with probability $p^n q^n$, there being n steps up and n steps down. And the number of such sequences is the number of ways of choosing the n steps up from $2n$. Thus

$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{n!^2} (pq)^n.$$

Remember that

$$n! \simeq \sqrt{2\pi n} (n/e)^n \quad n \rightarrow \infty.$$

So

$$\begin{aligned} (2n)! &\simeq \sqrt{4\pi n} (2n/e)^{2n} & n \rightarrow \infty \\ n!^2 &\simeq 2\pi n (n/e)^{2n} & n \rightarrow \infty. \end{aligned}$$

And therefore

$$\frac{(2n)!}{n!^2} (pq)^n \simeq \frac{(4pq)^n}{\sqrt{\pi n}} \quad n \rightarrow \infty$$

- $p = q$: Then $p = q = 1/2$, so $4pq = 1$. And we have

$$p_{00}^{(2n)} \simeq \frac{1}{\sqrt{\pi n}} \quad n \rightarrow \infty$$

which is equivalent with

$$\forall \epsilon > 0 \exists N : n \geq N \implies \frac{1}{\sqrt{\pi n}} - \epsilon < p_{00}^{(2n)} < \frac{1}{\sqrt{\pi n}} + \epsilon.$$

So there exists a N such that for all $n \geq N$ we have

$$p_{00}^{(2n)} > \frac{1}{2\sqrt{n}}.$$

Therefore

$$\sum_{n=0}^{\infty} p_{00}^{(n)} > \sum_{n=N}^{\infty} p_{00}^{(2n)} > \sum_{n=N}^{\infty} \frac{1}{2\sqrt{n}} = \infty$$

- $p \neq q$: Then $4pq = r < 1$. And we have

$$p_{00}^{(2n)} \simeq \frac{r^n}{\sqrt{\pi n}} \quad n \rightarrow \infty$$

which is equivalent with

$$\forall \epsilon > 0 \exists N : n \geq N \implies \frac{r^n}{\sqrt{\pi n}} - \epsilon < p_{00}^{(2n)} < \frac{r^n}{\sqrt{\pi n}} + \epsilon.$$

So there exists a N such that for all $n \geq N$ we have

$$p_{00}^{(2n)} < r^n$$

Therefore

$$\sum_{n=N}^{\infty} p_{00}^{(n)} = \sum_{n=N}^{\infty} p_{00}^{(2n)} < \sum_{n=N}^{\infty} r^n < \infty.$$

And therefore

$$\sum_{n=0}^{\infty} p_{00}^{(n)} < \infty.$$

□

7.3 Example 1.6.2 The simple random walk on \mathbb{Z}^2 .

Compute $\sum_{n=0}^{\infty} p_{00}^{(n)}$.

Proof.

By rotating \mathbb{Z}^2 we get that each step is like moving in one step in each of the independent, one dimensional, simple random walks. Hence

$$p_{00}^{(2n)} \simeq \left(\frac{1}{\sqrt{\pi n}} \right)^2 = \frac{1}{\pi n}$$

which is equivalent with

$$\forall \epsilon > 0 \exists N : n \geq N \implies \frac{1}{\pi n} - \epsilon < p_{00}^{(2n)} < \frac{1}{\pi n} + \epsilon.$$

So there exists a N such that for all $n \geq N$ we have

$$p_{00}^{(2n)} > \frac{1}{4n}.$$

Therefore

$$\sum_{n=0}^{\infty} p_{00}^{(n)} > \sum_{n=N}^{\infty} p_{00}^{(2n)} > \frac{1}{4} \sum_{n=N}^{\infty} \frac{1}{n} = \infty$$

□

7.2 Markov Chains 1.7

7.4 Definition.

We say that a measure $\lambda = (\lambda_i : i \in I)$ where $\lambda_i \geq 0$ is *invariant* if

$$\lambda P = \lambda.$$

7.5 Theorem 1.7.1.

Let $(X_n)_{n \geq 0}$ be a Markov (λ, P) and suppose that λ is invariant for P . Then $(X_{n+m})_{n \geq 0}$ is also Markov (λ, P) .

7.6 Theorem 1.7.2.

Let I be finite. Suppose for some $i \in I$ that for all $j \in J$

$$p_{ij}^{(n)} \rightarrow \pi_j \quad n \rightarrow \infty.$$

Then $\pi = (\pi_j : j \in I)$ is an invariant distribution.

7.7 Proposition.

Give a example of infinite state space I where Theorem 1.7.2 doesn't hold.

Proof.

For the random walk in \mathbb{Z} we have for all $i, j \in I$

$$p_{ij}^n \rightarrow 0 \quad n \rightarrow \infty.$$

Note however that $(0, 0, \dots)$ is not a distribution. As the total mass

$$\sum_{i \in I} \lambda_i = 0 \neq 1.$$

□

7.8 Theorem.

Consider a recurrence relation of the form

$$x_{n+1} = ax_n + b.$$

The general solution is

$$x_n = \begin{cases} Aa^n + b/(1-a) & a \neq 1 \\ x_n = x_0 + nb & a = 1 \end{cases}$$

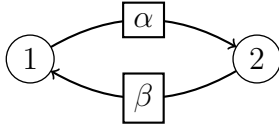
7.9 Example 1.1.4.

The most general two-state chain has transition matrix of the form

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

Draw the diagram, and find $p_{11}^{(n)}$.

Proof.



We have that

$$P^{n+1} = P^n \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

And therefore

$$p_{11}^{(n+1)} = p_{11}^{(n)}(1-\alpha) + p_{12}^{(n)}\beta.$$

Note that we have

$$p_{12}^{(n)} = 1 - p_{11}^{(n)}.$$

So we have

$$p_{11}^{(n+1)} = p_{11}^{(n)}(1-\alpha-\beta) + \beta.$$

The general solution of this recurrence relation is

$$p_{11}^{(n)} = \begin{cases} A(1 - \alpha + \beta)^n + \alpha/(\alpha + \beta) & (1 - \alpha + \beta) \neq 1 \\ 1 + n\beta & 1 - \alpha + \beta = 1 \end{cases}$$

This reduces to:

$$p_{11}^{(n)} = \begin{cases} \frac{\beta}{\alpha + \beta}(1 - \alpha + \beta)^n + \alpha/(\alpha + \beta) & \alpha + \beta > 0 \\ 1 & \alpha = \beta = 0 \end{cases}$$

□

7.10 Proposition.

Consider the two-state Markov chain with transition matrix

$$P = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}$$

Ignore the trivial cases $\alpha = \beta = 0$ and $\alpha = \beta = 1$.

Proof.

From example 1.1.4 we have

$$p_{11}^{(n)} = \begin{cases} \frac{\alpha}{\alpha + \beta}(1 - \alpha - \beta)^n + \beta/(\alpha + \beta) & \alpha + \beta > 0 \\ 1 & \alpha = \beta = 0 \end{cases}$$

In a similar we could have shown that

$$p_{12}^{(n)} = \begin{cases} \frac{\alpha}{\alpha + \beta}(1 - \alpha - \beta)^n + \alpha/(\alpha + \beta) & \alpha + \beta > 0 \\ 1 & \alpha = \beta = 0 \end{cases}$$

Therefore $p_{11}^{(n)} \rightarrow \frac{\beta}{\alpha + \beta}$ and $p_{12}^{(n)} \rightarrow \frac{\alpha}{\alpha + \beta}$. And by theorem 1.7.2 we get that

$$\left(\frac{\beta}{\alpha + \beta}, \frac{\alpha}{\alpha + \beta} \right)$$

is an invariant distribution.

□

7.11 Definition.

For a fixed state k , consider for each i the expected time spent in i between visits to k :

$$\gamma_i^k = E_k \sum_{n=0}^{T_k-1} 1_{\{X_n=i\}}$$

7.12 Theorem 1.7.5.

Let P be irreducible and recurrent. Then

1. $\gamma_k^k = 1$
2. $\gamma^k = (\gamma_i^k : i \in I)$ satisfies $\gamma^k P = \gamma^k$
3. $0 < \gamma_i^k < \infty$ for all $i \in I$

7.13 Theorem 1.7.6.

Let P be irreducible and let λ be an invariant measure for P with $\lambda_k = 1$. Then $\lambda \geq \gamma^k$. If in addition P is recurrent, then $\lambda = \gamma^k$.

Recall that a state i is recurrent if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 1$$

and we showed in Theorem 1.5.3 that this is equivalent to

$$\mathbb{P}_i(T_i < \infty) = 1.$$

7.14 Definition.

We call a recurrent state i positive recurrent if

$$m_i = E_i(T_i) < \infty.$$

If a recurrent state fails to have this stronger property we call it *null recurrent*.

7.15 Theorem 1.7.7.

Let P be irreducible. Then the following are equivalent:

1. every state is positive recurrent
2. some state i is positive recurrent
3. P has an invariant distribution π

7.16 Theorem.

If P has an invariant distribution π , we have that $m_i = E_i(T_i) = \frac{1}{\pi_i}$ for all i .

7.17 Example 1.7.8.

Show that the simple symmetric random walk on \mathbb{Z} is null recurrent.

Proof.

The simple symmetric random walk on \mathbb{Z} is irreducible and, by example 1.6.1, it also recurrent. Remember that

$$(\pi P)_j = \sum_{i \in I} \pi_i p_{ij}$$

and this equals to

$$\sum_{i \in I} \pi_i p_{ij} = 1/2\pi_{j-1} + 1/2\pi_{j+1}.$$

So

$$\pi_i = \frac{1}{2}\pi_{i-1} + \frac{1}{2}\pi_{i+1}.$$

We get the equation

$$x^2 - 2x + 1 = 0.$$

And so we have

$$\alpha = \beta = 1.$$

And so the general solutions is

$$\pi_i = A + iB.$$

and the total mass is then

$$\sum_i \pi_i = A \sum_i 1 + B \sum_i i = B\infty.$$

So whatever A, B we choose, we never get the total mass to be zero. So there doesn't exist an invariant distribution, and by Theorem 1.7.7 we have that all states must be null recurrent. \square

7.18 Example 1.7.9.

Show that the existence of an invariant measure does not guarantee recurrence.

Proof.

The simple symmetric random walk on \mathbb{Z}^3 has an invariant measure. Consider:

$$(\pi P)_j = \sum_{i \in I} \pi_i p_{ij} = 1/4(\pi_a + \pi_b + \pi_c + \pi_d)$$

So if we set $\pi = (1, 1, 1, \dots)$. Then π is invariant. But \mathbb{Z}^3 is also transient. \square

7.19 Example 1.7.10.

Consider the asymmetric random walk on \mathbb{Z} with transition probabilities $p_{i,i-1} = q < p = p_{i,i+1}$. Show that the walk is null recurrent.

Proof.

We have

$$\begin{aligned} \pi_j = (\pi P)_j &= \sum_{i \in I} \pi_i p_{ij} = \pi_{j-1} p_{j-1,j} + \pi_{j+1} p_{j+1,j} \\ &= \pi_{j-1} p + \pi_{j+1} q \end{aligned}$$

This is a recurrence relation and we have the equation

$$\begin{aligned} qx^2 - x + p &= 0 \\ D &= \sqrt{1 - 4(1-p)p} = \sqrt{1 - 4p + 4p^2} = 1 - 2p \\ \alpha &= \frac{-1 + 1 - 2p}{-2p} = 1 \quad \beta = \frac{-2 + 2p}{-2p} = \frac{1-p}{p} = \frac{q}{p} \end{aligned}$$

So we have the general solution:

$$\pi_j = A\alpha^j + B\beta^j = A + B\left(\frac{p}{q}\right)^j.$$

And the total mass is therefore

$$\sum_{j=0}^{\infty} \pi_j = \sum_{j=0}^{\infty} A + B\left(\frac{p}{q}\right)^j = B\infty.$$

So whatever A, B we choose, we never get the total mass to be zero. So there doesn't exist an invariant distribution, and by Theorem 1.7.7 we have that all states must be null recurrent.

□

7.20 Example 1.7.11.

Consider a success-run chain on \mathbb{Z}^+ , whose transition probabilities given by

$$p_{i,i+1} = p_i \quad p_{i0} = q_i = 1 - p_i.$$

And where

$$p = \prod_{i=0}^{\infty} p_i > 0.$$

Show that every state is transient.

Proof.

We have

$$\begin{aligned} \pi_j &= (\pi P)_j = \sum_{i \in I} \pi_i p_{ij} \\ &= \pi_{j-1} p_{j-1} \\ &= \pi_0 \prod_{i=0}^{j-1} p_i \end{aligned}$$

and

$$\begin{aligned} \pi_0 &= \sum_{i=0}^{\infty} \pi_i p_{i0} \\ &= \sum_{i=0}^{\infty} (1 - p_i) \pi_i \\ &= \pi_0 \sum_{i=0}^{\infty} (1 - p_i) \prod_{j=0}^{i-1} p_j \\ &= \pi_0 \sum_{i=0}^{\infty} \prod_{j=0}^{i-1} p_j - \prod_{j=0}^i p_j \\ &= \pi_0 (1 - \prod_{j=0}^{\infty} p_j). \end{aligned}$$

The last equality need some thinking, but notice that it's a telescoping sum. Define

$$a_i := \prod_{j=0}^{i-1} p_j,$$

the sum is

$$\sum_{i=0}^{\infty} (a_i - a_{i+1}) = a_0 - \lim_{k \rightarrow \infty} a_k = 1 - \prod_{j=0}^{\infty} p_j$$

The equation

$$\pi_0 = \pi_0 \left(1 - \prod_{j=0}^{\infty} p_j\right)$$

forces π_0 to be 0. And therefore all π_i are zero. So there is no invariant distribution and we have that P is transient by theorem 1.7.7. \square

8 16-10-2014

8.1 Markov Chains 1.8

8.1 Proposition.

Give an example where the limit $p_{ij}^{(n)}$ fails to converge.

Proof.

Consider the two-state chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$P^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = I.$$

So $P^2 = I$, so $P^{2n} = I$. And $P^{2n+1} = P$. Thus $p_{ij}^{(n)}$ fails to converge for all i, j . \square

8.2 Definition.

We call a state i *aperiodic* if $p_{ii}^{(n)} > 0$ for all sufficiently large n .

8.3 Theorem.

A state i is aperiodic if and only if the set $\{n \geq 0 : p_{ii}^{(n)} > 0\}$ has no common divisor other than 1.

8.4 Lemma 1.8.2.

Suppose P is irreducible and has an aperiodic state i . Then, for all states j and k , $p_{jk}^{(n)} > 0$ for all sufficiently large n . In particular, all states are aperiodic.

8.5 Theorem 1.8.3.

Let P be irreducible and aperiodic, and suppose that P has an invariant distribution π . Let λ be any distribution. Suppose that $(X_n)_{n \geq 0}$ is Markov(λ, P). Then for all j

$$\mathbb{P}(X_n = j) \rightarrow \pi_j \quad n \rightarrow \infty.$$

In particular, for all i, j

$$p_{ij}^{(n)} \rightarrow \pi_j \quad n \rightarrow \infty$$

8.2 Representation Theory Reader Week 39

Let k be a field. And let A be a k -algebra.

8.6 Definition.

An element $a \in A$ is called an *idempotent element* or an *idempotent* if $a^2 = a$.

8.7 Proposition.

We have that 0 and 1 are idempotents of A .

8.8 Definition.

Two idempotents $a, b \in A$ are called *orthogonal* if $ab = 0$ and $ba = 0$.

8.9 Proposition.

If a, b are orthogonal, then $a + b$ is an idempotent.

Proof.

Showing that $a + b$ is idempotent, is equivalent with showing that

$$(a + b)^2 = a + b$$

which is equivalent with showing that

$$a + ab + ba + b = a + b$$

which in turn is equivalent with showing that

$$ab + ba = 0.$$

It suffices to show that

$$ab = 0 = ba.$$

By hypothesis we have that a, b are orthogonal, and therefore our last statement follows by definition. \square

8.10 Definition.

A set of mutually orthogonal idempotents $a_1, \dots, a_n \in A$ is called *complete* if

$$1 = a_1 + \dots + a_n.$$

8.11 Definition.

An idempotent $a \in Z(A)$ is called a *central idempotent* of A .

8.12 Definition.

A nonzero idempotent $a \in A$ is called *minimal* if any decomposition of a as a sum of two orthogonal idempotents $a = b + c$ implies that $b = 0$ or $c = 0$.

8.13 Proposition.

If V is a k -vector space, then $\text{End}_k(V)$ is a k -algebra.

Proof.

Showing that $\text{End}_k(V)$ is a k -algebra is equivalent with showing that there exists a ring homomorphism

$$f : k \rightarrow Z(\text{End}_k(V))$$

which is equivalent with showing that there exists a map f such that

1. $f(\alpha)L = Lf(\alpha) \quad \forall \alpha \in k, \forall L \in \text{End}_k(V)$
2. $f(\alpha)f(\beta) = f(\alpha\beta) \quad \forall \alpha, \beta \in k$
3. $f(\alpha) + f(\beta) = f(\alpha + \beta) \quad \forall \alpha, \beta \in k$

Remember that the condition in 1) implies that $f(\alpha)$ must be a scalar. Let's try the mapping $f(\alpha) = \alpha I$ and rewrite our conditions:

1. $\alpha I \circ L = L \circ \alpha I \quad \forall \alpha \in k, \forall L \in \text{End}_k(V)$
2. $\alpha I \beta I = (\alpha\beta)I \quad \forall \alpha, \beta \in k$
3. $(\alpha + \beta)I = \alpha I + \beta I \quad \forall \alpha, \beta \in k$

All those conditions hold directly. □

8.14 Proposition.

If V is a representation of a k -algebra A , then $\text{End}_A(V)$ is a k -algebra.

Proof.

Showing that $\text{End}_A(V)$ is a k -algebra is equivalent with showing that there exists a ring homomorphism

$$f : k \rightarrow Z(\text{End}_A(V))$$

which is equivalent with showing that there exists a map f such that

1. $f(\alpha)L = Lf(\alpha) \quad \forall \alpha \in k, \forall L \in \text{End}_A(V)$

2. $f(\alpha)f(\beta) = f(\alpha\beta) \quad \forall \alpha, \beta \in k$
3. $f(\alpha) + f(\beta) = f(\alpha + \beta) \quad \forall \alpha, \beta \in k$

Remember that the condition in 1) implies that $f(\alpha)$ must be a scalar. Let's try the mapping $f(\alpha) = \alpha.I$ and rewrite our conditions:

1. $\alpha.I \circ L = L \circ \alpha.I \quad \forall \alpha \in k, \forall L \in \text{End}_k(V)$
2. $\alpha.I\beta.I = (\alpha\beta).I \quad \forall \alpha, \beta \in k$
3. $(\alpha + \beta).I = \alpha.I + \beta.I \quad \forall \alpha, \beta \in k$

All those conditions hold directly. □

8.15 Proposition.

$$\text{End}_A(V) \subseteq \text{End}_k(V)$$

Proof.

Showing that $\text{End}_A(V) \subseteq \text{End}_k(V)$ is equivalent with showing that

$$\phi \in \text{End}_A(V) \implies \phi \in \text{End}_k(V).$$

In other words, does A -linearity imply k -linearity? This holds as $k1 \subseteq A$. □

8.16 Proposition.

Given a representation V of A . Then V is also a representation of $\text{End}_A(V)$

Proof.

To show that V is a representation of $\text{End}_A(V)$ it suffices to show that there exists a homomorphism of algebras:

$$\rho : \text{End}_A(V) \rightarrow \text{End}_k(V).$$

Remember that we have $\text{End}_A(V) \subseteq \text{End}_k(V)$. So we can define ρ just as the inclusion map, which is clearly a homomorphism of algebras. □

8.17 Definition.

Given a representation V of A . Then V as representation of $A' := \text{End}_A(V)$ is called the *centralizer module*. If we need to distinguish the two module structures on V , we will write ${}_A V$ to denote V as a module over A , and ${}_{A'} V$ to denote the centralizer module.

8.18 Proposition.

The image of an idempotent element under an algebra homomorphism is again an idempotent element.

Proof.

Let $a \in A : a^2 = a$. It suffices to show that

$$f(a)^2 = f(a).$$

This holds as:

$$f(a)^2 = f(a)f(a) = f(a^2) = f(a).$$

□

8.19 Proposition.

For any idempotent $e \in A$ we have that

1. $\rho(e) \in \text{End}_k(V)$ is idempotent
2. if e is central, then $\rho(e) \in \text{End}_A(V)$

Proof.

1. The statement holds as ρ is an algebra homomorphism. And remember that the image of an idempotent element under an algebra homomorphism is again an idempotent element.
2. Showing that

$$\rho(e) \in \text{End}_A(V)$$

is equivalent with showing that

$$a.\rho(e)(v) = \rho(e)(a.v)$$

which in turn is equivalent with

$$\rho(a)(\rho(e)(v)) = \rho(e)(\rho(a)(v)).$$

In other notation

$$\rho(a) \circ \rho(e)(v) = \rho(e) \circ \rho(a)(v).$$

As ρ is an algebra homomorphism this is equivalent with

$$\rho(ae)(v) = \rho(ea)(v).$$

By hypothesis, we have that e is central, so $ea = ae$ and therefore our last statement holds.

□

8.20 Proposition.

For any $x \in \text{End}_A(V)$ the subspace $\text{Ker}(x) \subseteq V$ is a A -submodule.

Proof.

Showing that $\text{Ker}(x)$ is a subrepresentation of V is equivalent with showing that

$$a \cdot \text{Ker}(x) \subseteq \text{Ker}(x) \quad \forall a \in A.$$

which is equivalent with showing that

$$x(a.v) = 0 \quad \forall a \in A, \forall v \in \text{Ker}(x)$$

As $x \in \text{End}_A(V)$, this is equivalent with showing that

$$a.x(v) = 0 \quad \forall a \in A, \forall v \in \text{Ker}(x) \quad \checkmark.$$

□

8.21 Proposition.

For any $x \in \text{End}_A(V)$ the subspace $\text{Im}(x) \subseteq V$ is a A -submodule.

Proof.

Showing that $\text{Im}(x)$ is a subrepresentation is equivalent with showing that

$$a \cdot \text{Im}(x) \subseteq \text{Im}(x)$$

which is equivalent with showing that

$$\forall a \in A \quad \forall v \in V \quad \exists w \in W \quad a.x(v) = x(w)$$

As $x \in \text{End}_A(V)$, this is equivalent with showing that

$$\forall a \in A \quad \forall v \in V \quad \exists w \in W \quad x(a.v) = x(w) \quad \checkmark.$$

□

8.22 Proposition.

If $V = U \oplus W$ is a decomposition of ${}_A V$ as direct sum of A -submodules. Then the projection e_U of V onto U along W and the projection e_W of V onto W along U satisfy:

1. $e_U, e_W \in \text{End}_A(V)$
2. e_U and e_W are orthogonal idempotent elements
3. $1 = e_U + e_W$

Remember that if $V = U \oplus W$, then we can regard U, W as subspaces of V such that $U \cap W = 0$ and for any $v \in V$ we have that $v = u + w$ where $u \in U, w \in W$. And $a.v = a.u + a.w$ where $a.u \in U$

Proof of 1.

To show that $e_U \in \text{End}_A(V)$ it suffices to show that

$$a.e_U(v) = e_U(a.v) \quad \forall a \in A, v \in V$$

which is equivalent with showing that

$$a.e_U(u + w) = e_U(a.(u + w)) \quad \forall a \in A, u \in U, w \in W$$

which is equivalent with showing that

$$a.u = e_U(a.u + a.w) \quad \forall a \in A, u \in U, w \in W.$$

With holds as U, W are subrepresentations. □

Proof of 2.

Showing that e_U is idempotent, is equivalent with showing that

$$e_U \circ e_U = e_U$$

which is equivalent with showing that

$$e_U \circ e_U(v) = e_U(v) \quad \forall v \in V$$

which is equivalent with showing that

$$e_U \circ e_U(u + w) = e_U(u + w) \quad \forall u \in U, w \in W \quad \checkmark.$$

Showing that e_U and e_W are orthogonal is equivalent with showing that

$$e_U \circ e_W = 0 \quad e_W \circ e_U = 0$$

which is equivalent with showing that

$$e_U \circ e_W(u+w) = 0(u+w) \quad e_W \circ e_U(u+w) = 0(u+w) \quad \forall u \in U, w \in W \quad \checkmark.$$

□

Proof of 3.

Showing that

$$e_U + e_W = 1$$

is equivalent with showing that

$$e_U(u+w) + e_W(u+w) = 1(u+w) \quad \forall u \in U, w \in W \quad \checkmark$$

□

8.23 Proposition.

If $e \in A'$ is an idempotent element then

1. $1 - e \in A'$ idempotent
2. $1 = e + (1 - e)$ is a decomposition as sum of orthogonal elements
3. $V = eV \oplus (1 - e)V$ is a decomposition of ${}_A V$ as a direct sum of A -submodules

Proof of 1.

Showing that $1 - e$ is idempotent is equivalent with showing that

$$(1 - e)^2 = 1 - e$$

which is equivalent with showing that

$$1 - 2e + e^2 = 1 - e \quad \checkmark.$$

□

Proof of 2.

To show that $1 = e + (1 - e)$ is a decomposition of sum of orthogonal elements, it suffices to show that

$$e(1 - e) = 0 \quad (1 - e)e = 0.$$

which is equivalent with showing that

$$e - e^2 = 0. \quad \checkmark$$

□

Proof of 3.

Since $1 = e + (1 - e)$ is a decomposition as sum of orthogonal elements, we have that

$$v = e(v) + (1 - e)(v).$$

Hence if $v \in \text{Im}(e) \cap \text{Im}(1 - e)$ then

$$v = v + v = 2v.$$

And then $v = 0$.

□

8.24 Theorem.

${}_A V$ is an indecomposable module iff $1 \in A'$ is a minimal idempotent

9 17-10-2014

9.1 Representation Theory Reader Week 39

9.1 Exercise 1.

Let V be an A -module. Suppose that e_1, \dots, e_n is a complete set of orthogonal idempotents of A . Show that $V = \bigoplus_{i=1}^n e_i V$ is a decomposition of V as direct sum of A -submodules.

Proof.

To show that $V = \bigoplus_{i=1}^n e_i V$ is a decomposition of V as direct sum of A -submodules, it suffices to show that

1. $e_i V \cap e_j V = 0 \quad \forall i, j.$
2. $V \subseteq \bigoplus_{i=1}^n e_i V$

Note that 1) is equivalent with

$$v \in \text{Im}(e_i) \cap \text{Im}(e_j) \implies v = 0.$$

Assume that $v \in \text{Im}(e_i) \cap \text{Im}(e_j)$. Then $v = e_i(v) = e_j(v)$. And so $e_i(v) = v$ but also $e_i(v) = 0$. So $v = 0$.

We also have that

$$1 = e_1 + \dots + e_n$$

and therefore

$$v = e_1 v + \dots + e_n v \quad \forall v \in V.$$

Which shows that

$$V \subseteq \bigoplus_{i=1}^n e_i V.$$

□

9.2 Theorem.

Let U, V be finite dimensional k -vector spaces with direct sum decompositions $U = \oplus_{j=1}^n U_j$ and $V = \oplus_{i=1}^m V_i$, and with corresponding complete sets of orthogonal idempotents $e_j \in \text{End}_k(U)$ and $f_i \in \text{End}_k(V)$. We have a k -linear isomorphism M between $\text{Hom}_k(U, V)$ and the k -vector space consisting of $m \times n$ matrices

$$\begin{pmatrix} \phi_{1,1} & \cdots & \phi_{1,n} \\ \vdots & & \vdots \\ \phi_{m,1} & \cdots & \phi_{m,n} \end{pmatrix}.$$

The isomorphism M is defined by $\phi \rightarrow (\phi_{i,j})$ where $\phi \in \text{Hom}_k(U, V)$ and $\phi_{i,j} \in \text{Hom}_k(U_j, V_i)$ such that $\phi_{i,j} = f_i \phi|_{U_j}$.

Show that the matrix $M(\phi) = (\phi_{i,j})$ is characterized by its property that if we decompose $u = \sum_{j=1}^n u_j \in U$ and $\phi(u) = \sum_{i=1}^m \phi(u)_i$ with $\phi(u)_i \in V_i$ we have:

$$\begin{pmatrix} \phi(u)_1 \\ \vdots \\ \phi(u)_m \end{pmatrix} = \begin{pmatrix} \phi_{1,1} & \cdots & \phi_{1,n} \\ \vdots & & \vdots \\ \phi_{m,1} & \cdots & \phi_{m,n} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

In particular, if also $W = \oplus_{l=1}^p$ is a direct sum of finite dimensional k -modules, and if $\psi \in \text{Hom}_k(V, W)$, then $M(\psi \circ \phi) = M(\psi) \circ M(\phi)$. Also notice that $M(\text{id}_V)$ is the $n \times n$ identity matrix with id_{V_i} in the i th diagonal position and 0 of the diagonal.

9.3 Proposition.

Let U, V be finite dimensional A -modules with direct sum decompositions $U = \oplus_{j=1}^n U_j$ and $V = \oplus_{i=1}^m V_i$ as A -modules, with corresponding complete sets of orthogonal idempotents $e_j \in \text{End}_A(U)$ and $f_i \in \text{End}_A(V)$. Given $\phi \in \text{Hom}_k(U, V)$ with $M(\phi) = (\phi_{i,j})$ we have that

$\phi \in \text{Hom}_A(U, V)$ if and only if $\phi_{i,j} \in \text{Hom}_A(U_j, V_i)$ for all i, j

In other words, we can restrict the isomorphism M to A -linear maps.

9.4 Proposition.

Given $U = \bigoplus_{j=1}^n U_j$, a direct sum decomposition of A -modules, M defines an isomorphism between $A'_U := \text{End}_A(U)$ and the algebra of $n \times n$ matrices

$$\begin{pmatrix} \phi_{1,1} & \cdots & \phi_{1,n} \\ \vdots & & \vdots \\ \phi_{n,1} & \cdots & \phi_{n,n} \end{pmatrix}$$

where $\phi_{i,j} \in \text{Hom}_A(U_j, U_i)$. In particular, if V is an A -module, and $U = V \oplus \cdots \oplus V = V^n$, then M defines an isomorphism of $A'_U = \text{End}_A(V^n)$ with the algebra $\text{Mat}_n(A')$ of $n \times n$ matrices with coefficients in $A'_V := \text{End}_A(V)$

9.5 Definition.

An A -module V is called *semisimple* if V can be decomposed as direct sum of irreducible submodules.

9.6 Definition.

Let $\text{Irr}(A)$ denote a complete set of representatives for the equivalence classes of irreducible representations of A .

9.7 Proposition.

Let $V = V_1 \oplus \cdots \oplus V_n$ be a finite dimensional semisimple A -module. For each $U \in \text{Irr}(A)$ let

$$n_U = \dim_k(\text{Hom}_A(U, V)) \in \mathbb{Z}_{\geq 0}.$$

Show that n_U is equal to the number of V_i that are equivalent to U .

9.8 Proposition.

Let $V = V_1 \oplus \cdots \oplus V_n$ be a finite dimensional semisimple A -module. For each $U \in \text{Irr}(A)$ let

$$n_U := \dim_k(\text{Hom}_A(U, V)).$$

Then $\{U \in \text{Irr}(A)\} = \{U_1, \dots, U_l\}$ is a finite set and we have

$$\begin{aligned} V &= n_1 U_1 \oplus \cdots \oplus n_l U_l \\ \text{End}_A(V) &\cong \text{Mat}_{n_1}(k) \oplus \cdots \oplus \text{Mat}_{n_l}(k) \end{aligned}$$

9.9 Proposition.

Let $V = V_1 \oplus \cdots \oplus V_n$ be a finite dimensional semisimple A -module. For each $U \in \text{Irr}(A)$ let

$$n_U = \dim_k(\text{Hom}_A(U, V)) \in \mathbb{Z}_{\geq 0}.$$

Show that n_U is equal to the number of V_i that are equivalent to U .

9.10 Proposition.

Let $V = V_1 \oplus \cdots \oplus V_n$ be a finite dimensional semisimple A -module. For each $U \in \text{Irr}(A)$ let

$$n_U := \dim_k(\text{Hom}_A(U, V)).$$

Then $\{U \in \text{Irr}(A) : n_U \neq 0\} = \{U_1, \dots, U_l\}$ is a finite set and we have

$$V = n_{U_1}U_1 \oplus \cdots \oplus n_{U_l}U_l$$

9.11 Proposition.

Let $V = V_1 \oplus \cdots \oplus V_n$ be a finite dimensional semisimple A -module. For each $U \in \text{Irr}(A)$ let

$$n_U := \dim_k(\text{Hom}_A(U, V)).$$

Show that

$$\text{End}_A(V) \cong \text{Mat}_{n_1}(k) \oplus \cdots \oplus \text{Mat}_{n_l}(k)$$

Proof.

Remember that

$$\text{End}_A(n_{U_1}U_1 \oplus \cdots \oplus n_{U_l}U_l) \cong \text{End}_A(n_{U_1}U_1) \oplus \cdots \oplus \text{End}_A(n_{U_l}U_l)$$

Remember also that

$$\text{End}_A(n_{U_1}U_1) = \text{Mat}_{n_{U_1}}(\text{End}_A(U_1))$$

And lastly remember that if U_1 is irreducible then $\text{End}_A(U_1) \cong k$. □

9.12 Definition.

Let U be a subrepresentation of V , a complement W of U is a subrepresentation such that $V = U \oplus W$.

9.13 Proposition.

Let A be a k -algebra. And let V be a finite dimensional A -module. The following are equivalent.

1. V is a sum of irreducible submodules
2. V is semisimple, i.e. V is direct sum of irreducible submodules
3. Every submodule $U \subseteq V$ admits a complement

9.14 Proposition.

Let V be a finite dimensional semisimple A module, and let $U \subseteq V$ be a submodule. Then U is also semisimple.

Proof.

Showing that U is semisimple, is equivalent with showing that any submodule $U' \subseteq U$ admits a complement $W' \subseteq U$.

By hypothesis, we have that V is semisimple, and so U' as submodule of V has an complement $W \subseteq V$. Define now $W' = U \cap W$. We need to show that

1. $U' \cap W' = 0$
2. W' is a submodule of U
3. $U' + W' = U$

Here is the proof:

1. $U' \cap W' = U' \cap (U \cap W) = U' \cap W = 0$
2. Showing that W' is a submodule of U is equivalent with showing that

$$a.W' \subseteq W' \quad \forall a \in A$$

It suffices to assume $a \in A$ and $w' \in W'$ and show that

$$a.w' \in W'$$

which is equivalent with showing that

$$a.w' \in U \cap W$$

which is equivalent with showing that

$$a.w' \in U \wedge a.w' \in W.$$

This holds as U and W are both A -submodules.

3. To show that $U' + W' = U$ it suffices to assume $u \in U$ and show that

$$\exists u' \in U, w' \in W' : u = u' + w'.$$

By hypothesis, we have that $U' + W = V$ and as $U \subseteq V$ we have that $u = u' + w$ for some $u' \in U', w \in W$. But then w must also be in U , and therefore $W' = U \cap W$.

□

9.15 Proposition.

Let V be a finite dimensional semisimple A module, and let $U \subseteq V$ be a submodule. Then V/U is also semisimple. And V/U is the complement of U .

Proof.

Showing that V/U is semisimple is equivalent with showing that there exists a complement V/U .

As U is semisimple, it has a complement W so that

$$V/U = (U \oplus W)/U \cong W$$

Therefore V/U and U are complement of each other. □

9.16 Exercise 5.

Let U be a submodule of a finite dimensional A -module. If U and V/U are semisimple, show that V is semisimple.

Proof.

Showing that V is semisimple is equivalent with showing that V is the sum of irreducible submodules. We already shown that if U and V/U are semisimple then

$$V = U \oplus V/U.$$

But besides that, we also have that U and V/U are sums of irreducible submodules. And therefore V is also a sum of irreducible submodules. And therefore V is semisimple. □

9.17 Definition.

Let k be a field. A finite dimensional k -algebra A is called semisimple if all its finite dimensional representations are semisimple.

9.18 Proposition.

A finite dimensional k -algebra A is semisimple if and only if the left regular representation (A, ρ) of A is semisimple.

Proof.

If A is semisimple, then all its finite dimensional representations are semisimple. And as A is finite-dimensional, the regular representation is finite dimensional, and so (A, ρ) is indeed semisimple.

The other side seems difficult. □

9.19 Theorem.

Let A be a k -algebra for a field k . And let V be a representation of A . Define $A'' = \text{End}_{A'}(V) = \text{End}_{\text{End}_A(V)}(V)$. Show that A'' is a k -algebra.

Proof.

To show that A'' is a k -algebra, it suffices to show that there exists a ring homomorphism

$$f : k \rightarrow Z(A'').$$

To show that f is a ringhomomorphism it suffices to show that

1. $f(\alpha) \circ L = L \circ f(\alpha) \quad \forall \alpha \in k, \forall L \in \text{End}_{A'}(V)$
2. $f(\alpha) + f(\beta) = f(\alpha + \beta) \quad \forall \alpha, \beta \in k$
3. $f(\alpha)f(\beta) = f(\alpha\beta) \quad \forall \alpha, \beta \in k$

So I think first condition forces $f(\alpha)$ to be a scalar of $I_{A''}$. But what are the scalars in $A' = \text{End}_A(V)$? So if understand correctly the scalars in A are $\alpha 1_A$. So are the scalars in A' then $\alpha 1_A I_{A'}$? If that is true it would define

$$f : k \rightarrow Z(A'') : \alpha \mapsto \alpha 1_A I_{A'} I_{A''}$$

□

9.20 Proposition.

Show that V is an A'' module.

Proof.

To show that V is an A'' module, it suffices to show that there exists a homomorphism of algebras

$$\rho : A'' \rightarrow \text{End}_k(V).$$

Note that any $\phi \in A''$ is a linear endomorphism of V such that

$$\phi(a' \cdot v) = a' \cdot \phi(v) \quad \forall a' \in A', v \in V.$$

But is ϕ also in $\text{End}_k(V)$? This holds as $\alpha 1_A 1_{A'}$ acts the same way as $\alpha \in k$ does. Therefore we can just choose the inclusion map for ρ which is an algebra homomorphism. \square

9.21 Proposition.

Show that $\rho(A) \subseteq A''$.

Proof.

Showing that $\rho(A) \subseteq A''$ is equivalent with showing that

$$\phi_a = \rho(a) \in A'' \quad \forall a \in A.$$

Assume $a \in A$. It suffices to show that ϕ_a is A' linear, i.e.

$$\phi_a(a'.v) = a'.\phi_a(v) \quad \forall a' \in \text{End}_A(V), \forall v \in V.$$

Note that the action of A' on V is defined as $a'.v = a'(v)$. So it suffices to show that

$$\phi_a(a'(v)) = a'(\phi_a(v)) \quad \forall a' \in \text{End}_A(V), \forall v \in V.$$

This is equivalent with showing that

$$a.a'(v) = a'(a.v) \quad \forall a' \in \text{End}_A(V), \forall v \in V.$$

This clearly holds. \square

9.22 Definition.

We say that ${}_A V$ has the double centralizer property if $\rho(A) = A''$.

9.23 Theorem.

Let V be a finite dimensional semisimple A -module. Then V has the DCP.

9.24 Theorem.

Let (V, ρ) be a finite dimensional simple A -module. Then $\rho(A) = \text{End}_k(V)$.

9.25 Theorem.

Let (V, ρ) be a finite dimensional semisimple A -module such that $V = V_1 \oplus \cdots \oplus V_n$ is a direct sum of mutually inequivalent irreducible submodules. Then

$$\rho(A) = \text{End}_k(V_1) \oplus \cdots \oplus \text{End}_k(V_n)$$

is a direct sum of matrix algebras.

9.26 Theorem.

A finite dimensional k -algebra A has finitely many equivalence classes of irreducible A -modules, each of which is finite dimensional over k . If we write $\text{Irr}(A) = \{(V_1, \rho_1), \dots, (V_n, \rho_n)\}$ then

$$\dim_k(A) \geq \sum_{i=1}^n \dim_k(V_i)^2.$$

9.2 Representation Theory Reader Week 40**9.27 Theorem.**

Let k be a field, and let D be a finite dimensional k -division algebra. Then $A := \text{Mat}_n(D)$ is a semisimple k -algebra with a single isomorphism class of simple A -modules, namely $V = D^n$. We have $A' = D^{\text{op}}$.

9.28 Definition.

A finite dimensional k -algebra A is called simple if it is semisimple and if it has an single equivalence class of irreducible modules.

9.29 Theorem.

If a finite dimensional k -algebra A is simple, then the only two-sided ideals of A are 0 and A .

9.30 Theorem.

A finite dimensional k -algebra of the form

$$A := \text{Mat}_{d_1}(D_1) \oplus \dots \oplus \text{Mat}_{d_n}(D_n)$$

with D_i a finite dimensional division algebra over k , is a semisimple k -algebra with $\text{Irr}(A) = \{V_1, \dots, V_n\}$ where $V_i = D_i^{d_i}$ is equipped with the matrix multiplication action of the summand $\text{Mat}_{d_i}(D_i)$.

9.31 Definition.

Let A be a k -algebra (with k a field). Let W be an A -module. A subquotient of W is an A module of the form V/U where $U \subseteq V \subseteq W$ are submodules of W . If $V = V_0 \supseteq V_1 \supseteq \dots \supseteq V_n := 0$ is a descending filtration of V by submodules, then we call the subquotients $U_{i-1} := V_{i-1}/V_i$ the consecutive subquotients of the filtration. Similarly, if $0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n := V$ is an ascending filtration of V by submodules then the consecutive subquotients of the filtration are the subquotients $U_i := V_i/V_{i-1}$ for $i = 1, \dots, n$.

9.32 Proposition.

If V is finite dimensional, then there exists a finite descending filtration with irreducible consecutive subquotients. Similarly such finite ascending filtrations exists (in this case the sequence of consecutive irreducible subquotients is called a Jordan-Holder sequence for V).

9.33 Definition.

Let A be a finite dimensional k algebra. The radical $\text{Rad}(A) \subseteq A$ of A is the intersection of the kernels of the irreducible representations of A .

9.34 Definition.

A two sided ideal $I \subseteq A$ is called nilpotent if there exists an $n \in \mathbb{N}$ such that $I^n = 0$.

9.35 Proposition.

$\text{Rad}(A)$ is the largest two-sided nilpotent ideal of A .

9.36 Theorem.

Let A be a finite dimensional k -algebra. Then all irreducible A -modules factor through $A/\text{Rad}(A)$ and $A/\text{Rad}(A)$ is a finite dimensional semisimple k -algebra. In particular, if $\text{Irr}(A) = \{(V_1, \rho_1), \dots, (V_n, \rho_n)\}$, then we have an isomorphism of k -algebras.

$$\bigoplus_{i=1}^n \overline{\rho_i} : A/\text{Rad}(A) \rightarrow \text{End}_k(V_1) \oplus \dots \oplus \text{End}_k(V_n)$$

10 18-10-2014

10.1 Representation Theory Reader Week 40

10.1 Proposition.

Let A be simple. Show that $A = \sum_J J$ where J runs over the set of minimal nonzero left ideals of A .

Proof.

As A is simple, we know that

1. A is finite
2. The left regular representation (A, ρ) of A is semisimple.
3. There is a single equivalence class of irreducible modules

So 1) and 2) gives that A as representation is a finite sum of irreducible submodules $A = \sum_{k=1}^n V_k$, and 3) gives that $A = V_1^n$, where V_1 is an irreducible submodule of the regular representation. And that is the same as a minimal nonzero left ideal J . \square

10.2 Proposition.

Let $I \subseteq A$ be minimal nonzero left ideal. Show that there exists an idempotent $e_I \in \text{End}_A(A) = A^{\text{op}}$ such that $I = e_I(A) = Ae_I$

Proof.

We have that $A = I^n$ is a decomposition of A as direct sum of A -submodules. Then the projection e_I is an idempotent element in $\text{End}_A(A)$. So clearly $e_I(A) = I$ But e_I can also be seen as an element of A^{op} with the isomorphism

$$f : A^{\text{op}} \rightarrow \text{End}_A(A) : a \mapsto (\phi_a : A \rightarrow A : b \mapsto b \cdot a)$$

Clearly $e_I = \phi_i$ for some $i \in I$. So we have that e_I acts as $i \in A^{\text{op}}$. And therefore $Ae_i = I$ as well. \square

10.3 Theorem.

Let A be a finite dimensional k -algebra, and $\text{Irr}(A) = \{(V_1, \rho_1), \dots, (V_n, \rho_n)\}$. The following are equivalent:

1. A is semisimple
2. $\text{Rad}(A) = 0$
3. $\dim_k(A) = \sum_{i=1}^n \dim_k(V_i)^2$
4. $A \cong \text{Mat}_{d_1}(D_1) \oplus \dots \oplus \text{Mat}_{d_m}(D_m)$ for some division k -algebras D_i

10.4 Theorem.

Let A be a finite dimensional semisimple k -algebra, and suppose that $\text{Irr}(A) = \{(V_1, \rho_1), \dots, (V_n, \rho_n)\}$. Then we have an isomorphism of k algebras

$$\bigoplus_{i=1}^n \rho_i : A \rightarrow \text{End}_k(V_1) \oplus \dots \oplus \text{End}_k(V_n)$$

Assume that k is algebraically closed, A is finite dimensional k -algebra, and $\text{Irr}(A) = \{(V_1, \rho_1), \dots, (V_n, \rho_n)\}$.

10.5 Definition.

Let (V, ρ) be a finite dimensional representation of A . The character of V is the functional $\chi_V \in A^* := \text{Hom}_k(A, k)$ defined by $\chi_V(a) := \text{Tr}(\rho(a))$. A character χ_V is always a central functional on A , i.e. $\chi_V([A, A]) = 0$. Hence we often consider χ_V as an element of the subspace $(A, [A, A])^* = \text{Hom}_k(A/[A, A], k) \subseteq \text{Hom}_k(A, k)$.

10.6 Proposition.

Let U, V be equivalent finite dimensional A -modules. Then $\chi_U = \chi_V$.

10.7 Proposition.

If U is an irreducible A -module, then its character $\chi_U = \chi_{V_i}$ for some $V_i \in \text{Irr}(A)$.

10.8 Definition.

We call $\chi_i = \chi_{V_i}$ an irreducible character of A , and $\{\chi_1, \dots, \chi_n\}$ the set of irreducible characters of A .

10.9 Proposition.

Let $U \subseteq V$ be a subrepresentation. Then $\chi_V = \chi_U + \chi_{V/U}$.

10.10 Proposition.

Let W be a finite dimensional A -module, and suppose that $0 = W_0 \subseteq W_1 \subseteq \dots \subseteq W_N := W$ is an ascending filtration of W with consecutive subquotients $U_i = W_i/W_{i-1}$. Let $U = \bigoplus_{i=1}^N U_i$. Then $\chi_V = \chi_U$.

10.11 Theorem.

Let $0 = W_0 \subseteq W_1 \subseteq \dots \subseteq W_N := W$ be a Jordan-Holder filtration of W . The character of a finite dimensional representation W of A can be written as a sum of irreducible characters of A . If $m_i = |\{k \in \{1, \dots, N\} | U_k \cong V_i\}| \in \mathbb{Z}_{\geq 0}$ (the so-called Jordan Holder multiplicity of V_i in W with respect to the given Jordan-Holder filtration) then $\chi_W = \sum_{i=1}^n m_i \chi_i$.

10.12 Theorem.

The characters of a collection of mutually inequivalent irreducible finite dimensional representations of A are linearly independent. If A is a finite dimensional semisimple algebra then the characters of the elements of $\text{Irr}(A)$ form a basis of $(A/[A, A])^*$.

10.13 Theorem.

Suppose that k is algebraically closed and of characteristic zero. If V is a finite dimensional semisimple A -module then up to equivalence, V is determined by its character χ_V .

10.14 Jordan-Holder Theorem.

Suppose that k is algebraically closed and of characteristic ≥ 0 , and W a finite dimensional A -module. Let $0 = W_0 \subseteq W_1 \subseteq \dots \subseteq W_N := W$ be an ascending Jordan-Holder filtration of W , with consecutive subquotients $U_i = W_i/W_{i-1}$ (with $i = 1, \dots, N$). Let U be a semisimple A -module $U = \bigoplus_{i=1}^n U_i$. Then U is, up to equivalence, determined by V (in particular the equivalence class of U is independent of the chosen ascending filtration). The Jordan-Holder multiplicities of W are independent of the chosen filtrations.

10.15 Definition.

The semisimple A -module $U = \bigoplus_{i=1}^n U_i$ of the Jordan Holder Theorem is called the semisimplification of V . It is determined up to equivalence by V (hence independent of the chosen filtration).

10.2 Measure Theory Chapter 8

10.16 Proposition.

Show that $\bar{\mathcal{B}}$ is generated by $\{[a, \infty], a \in \mathbb{R}\}$

Proof.

Showing that $\bar{\mathcal{B}}$ is generated by $\{[a, \infty], a \in \mathbb{R}\}$ is equivalent with showing that

$$\bar{\mathcal{B}} = \sigma\{[a, \infty], a \in \mathbb{R}\}$$

which is equivalent with showing that

$$\bar{\mathcal{B}} \subseteq \sigma\{[a, \infty], a \in \mathbb{R}\} \quad \wedge \quad \sigma\{[a, \infty], a \in \mathbb{R}\} \subseteq \bar{\mathcal{B}}$$

1. To show that

$$\sigma\{[a, \infty], a \in \mathbb{R}\} \subseteq \bar{\mathcal{B}} \quad \forall a \in \mathbb{R}$$

As $\bar{\mathcal{B}}$ is a σ -algebra it suffices to show

$$[a, \infty] \subseteq \bar{\mathcal{B}} \quad \forall a \in \mathbb{R} \quad \checkmark.$$

2. To show that

$$\bar{\mathcal{B}} \subseteq \sigma\{[a, \infty], a \in \mathbb{R}\}$$

it suffices to show that

$$\forall B \in \bar{\mathcal{B}} : B, B \cup \{\infty\}, B \cup \{-\infty\}, B \cup \{-\infty, \infty\} \in \sigma\{[a, \infty], a \in \mathbb{R}\}.$$

As $\sigma\{[a, \infty], a \in \mathbb{R}\}$ is a σ -algebra it suffices to show that

$$\bar{\mathcal{B}} \subseteq \sigma\{[a, \infty], a \in \mathbb{R}\} \quad \{\infty\}, \{-\infty\} \in \sigma\{[a, \infty], a \in \mathbb{R}\}.$$

- (a) To show that $\bar{\mathcal{B}} \subseteq \sigma\{[a, \infty], a \in \mathbb{R}\}$ it suffices to show that

$$[a, b] \in \sigma\{[a, \infty], a \in \mathbb{R}\} \quad \forall a, b \in \mathbb{R}.$$

This holds as $[a, b] = [a, \infty] - [b, \infty]$.

- (b) To show that $\{\infty\}, \{-\infty\} \in \sigma\{[a, \infty], a \in \mathbb{R}\}$. Note that

$$\{\infty\} = \bigcap_{j \in \mathbb{N}} [j, \infty] \quad \{-\infty\} = \bigcap_{j \in \mathbb{N}} [-j, \infty]^c$$

□

10.17 Proposition.

If a measurable function $h : X \rightarrow \mathbb{R}$ attains only finitely many values $y_1, \dots, y_M \in \mathbb{R}$, then it is a simple function and it has a standard representation.

Proof.

Showing that h is a simple function is b.d.e. with showing that

$$h(x) = \sum_{j=1}^M y_j 1_{A_j}(x) \quad \exists y_1, \dots, y_M \in \mathbb{R} \exists A_1, \dots, A_m \in \mathcal{A}.$$

Set $A_i := \{h = y_i\} = \{h \leq y_i\} - \{h < y_i\} \in \mathcal{A}$. Note that those sets are mutually disjoint, otherwise h would map some point in the domain to two points y_i, y_j . Therefore we have that

$$h(A_i) = y_i \quad \sum_{j=1}^M y_j 1_{A_j}(A_i) = y_i$$

and the statement follows. □

10.18 Proposition.

Every simple function has at least one standard representation. And this standard representation is measurable, i.e. $\mathcal{E}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$.

10.19 Definition.

For a function $u : X \rightarrow \mathbb{R}$ we write

$$u^+(x) := \max\{u(x), 0\} \quad u^-(x) := \min\{u(x), 0\}$$

for the positive u^+ and negative u^- parts of u .

10.20 Proposition.

We have

$$u = u^+ - u^- \quad |u| = u^+ + u^-$$

10.21 Proposition.

If $f, g \in \mathcal{E}(\mathcal{A})$, then

$$f \pm g, fg, f^+, f^-, |f| \in \mathcal{E}(\mathcal{A})$$

10.22 Definition.

The expression $\sup_{j \in \mathbb{N}} u_j$ is short for

$$\sup_{j \in \mathbb{N}} u_j(x) := \sup\{u_j(x) : j \in \mathbb{N}\} \quad \forall x$$

10.23 Definition.

The expression $u_j \xrightarrow{j \rightarrow \infty} u$ is short for

$$\lim_{j \rightarrow \infty} u_j(x) = u(x) \quad \forall x$$

10.24 Proposition.

$$\inf_{j \in \mathbb{N}} u_j(x) = -\sup_{j \in \mathbb{N}} (-u_j(x))$$

10.25 Definition.

Definition of the lower resp. upper limits

$$\liminf_{j \rightarrow \infty} u_j(x) := \sup_{k \in \mathbb{N}} \left(\inf_{j \geq k} u_j(x) \right) = \lim_{k \rightarrow \infty} \left(\inf_{j \geq k} u_j(x) \right)$$

$$\limsup_{j \rightarrow \infty} u_j(x) := \inf_{k \in \mathbb{N}} \left(\sup_{j \geq k} u_j(x) \right) = \lim_{k \rightarrow \infty} \left(\sup_{j \geq k} u_j(x) \right)$$

10.26 Proposition.

If u, v are $\mathcal{A}/\bar{\mathcal{B}}$ -measurable numerical functions, then

$$\{u < v\}, \quad \{u \leq v\}, \quad \{u = v\}, \quad \{u \neq v\} \in \mathcal{A}$$

11 19-10-2014

11.1 Representation Theory Reader Week 40

11.1 Proposition.

$$\dim_D(V) = \dim_k(V) / \dim_k(D)$$

11.2 Proposition.

Let k be a field, and let D be a finite dimensional k -division algebra. Then

1. $A := \text{Mat}_n(D)$ is a semisimple k -algebra
2. with a single isomorphism class of simple A -modules, namely $V = D^n$
3. $A' = D^{\text{op}}$

Proof of 1.

To show that $\text{Mat}_n(D)$ is a semisimple k -algebra, it suffices to show that the regular representation is semisimple.

Showing that the regular representation is semisimple, is equivalent with showing that the regular representation is a direct sum of irreducible submodules. It suffices to show that

1. $\text{Mat}_n(D) \cong V^n$ $V := D^n$.
2. D^n is an irreducible submodule of the regular representation

To show that D^n is irreducible submodule, it suffices to show that every nonzero vector $v \in D^n$ is cyclic. To show that it suffices to show that

$$\text{Mat}_n(D).(d_1, \dots, d_n) = D^n \quad \forall d_1, \dots, d_n \in D.$$

To show this it suffices to show that

$$D^n \subseteq \text{Mat}_n(D).(d_1, \dots, d_n) \quad \forall d_1, \dots, d_n \in D.$$

In other words, for any $v, w \in D^n$ there exists a matrix such that

$$v = Mw \quad \checkmark.$$

□

Proof of 2.

Remember that

$$\dim_k(A) \geq \sum_{i=1}^n \dim_{D_i}(V_i)^2 \dim_k(D_i).$$

To show that the only irreducible representation of $\text{Mat}_n(D)$ is $V = D^n$ it suffices to show that

$$\dim_D(V)^2 \dim_k(D) = \dim_k(\text{Mat}_n(D))$$

which is equivalent with showing that

$$\dim_D(D^n)^2 = \dim_D(\text{Mat}_n(D)) \quad \checkmark.$$

□

Proof of 3.

$$A' = D^{op}$$

$$\uparrow [\text{by the DCP we have } \text{End}_{A'}(V) = \rho(A) \stackrel{?}{\cong} A]$$

$$\text{End}_{D^{op}}(V) = A$$

$$\uparrow [\text{given}]$$

$$\text{End}_{D^{op}}(V) = \text{Mat}_n(D)$$

$$\uparrow [\text{End}_D(D) = D^{op} \text{ and } \text{End}_{D^{op}}(D) = D]$$

$$\text{End}_{D^{op}}(V) = \text{Mat}_n(\text{End}_{D^{op}}(D))$$

$$\uparrow [\text{by prop 4.9}]$$

V is a finite dimensional semisimple D^{op} module and any irreducible module is equivalent to D

$$\uparrow$$

$V = D^n e_1 \oplus \cdots \oplus D^n e_n$ is a decomposition of V as direct sum of irreducible D^{op} modules where e_1, \dots, e_n is the standard D -basis of $V = D^n$ □

11.3 Proposition.

A finite dimensional k -algebra of the form

$$A := \text{Mat}_{d_1}(D_1) \oplus \cdots \oplus \text{Mat}_{d_n}(D_n)$$

with D_i a finite dimensional algebra over k , is a semisimple k -algebra with $\text{Irr}(A) = \{V_1, \dots, V_n\}$, where $V_i = D_i^{d_i}$ is equipped with the matrix multiplication action of the summand $\text{Mat}_{d_i}(D_i)$.

Proof.

Showing that A is a semisimple k -algebra

\Uparrow by P4.16

The left regular representation of A is semisimple

\Uparrow direct sum of semisimple modules are semisimple

Every $\text{Mat}_{d_i}(D_i)$ is semisimple. (by Lemma 5.1)

Showing that V_i, V_j are inequivalent representations of A

\Uparrow [by definition]

$V_i = D_i^{d_i}$ and $V_j = D_j^{d_j}$ are inequivalent representations of

$$\text{Mat}_{d_1}(D_1) \oplus \cdots \oplus \text{Mat}_{d_n}(D_n)$$

\Uparrow [$\text{Mat}_{d_1}(D_1) = V_1^{d_1} = D_1^{d_1^2}$]

V_i and V_j are inequivalent representations of

$$V_1^{d_1} \oplus \cdots \oplus V_n^{d_n}$$

\Uparrow

Let $v_i \in V_i - 0$ and $v_j \in V_j - 0$. There exists an $M \in \text{Mat}_{d_i}(D_i)$ such that $M(v_i) \neq 0$, but $M(v_j) = 0$ for any $M \in \text{Mat}_{d_i}(D_i)$. \square

11.4 Proposition.

If V is a finite dimensional, then there exists a finite descending filtration with irreducible consecutive subquotients.

Proof.

There exists a finite descending filtration with irreducible consecutive subquotients.

↑ by definition

There exists submodules V_0, \dots, V_n such that

$$V = V_0 \supseteq V_1 \supseteq \dots \supseteq V_n := 0 \quad V_{i-1}/V_i \text{ irreducible}$$

- Base case: If $\dim_k(V) = 0$ then V itself is already irreducible.
- Induction hypothesis: Suppose that the statement is true for modules of dimension smaller than n .
- Induction step: Let V be a submodule of dimension $< n + 1$. It suffices to show the case that V is of dimension n . Let V_1 be a proper submodule of V of maximal dimension. Then $V_0/V_1 = V/V_1$ is irreducible. And by induction hypothesis we can construct a finite descending filtration now.

□

11.5 Proposition.

$\text{Rad}(A)$ is a two sided ideal of A

Proof.

\Uparrow [by definition]

The set of all elements of A which act by 0 in all irreducible representations of A is a two sided ideal of A

\Uparrow [by definition]

$\text{Rad}(A)a \subseteq \text{Rad}(A)$ en $a \text{Rad}(A) \subseteq \text{Rad}(A)$ for all $a \in A$

- $\text{Rad}(A)a \subseteq \text{Rad}(A) \quad \forall a \in A$

\Uparrow

$$ra(v) = 0 \quad \forall a \in A, \forall v \in V \subseteq \text{Irr}(A), \forall r \in \text{Rad}(A)$$

\Uparrow [$a(v) \in V$ for any $v \in V \subseteq \text{Irr}(A)$]

$$r(w) = 0 \quad \forall w \in V \subseteq \text{Irr}(A), \forall r \in \text{Rad}(A)$$

\Uparrow by definition of $\text{Rad}(A)$

- $a \text{Rad}(A) \subseteq \text{Rad}(A)$ **for all** $a \in A$

\Uparrow

$$ar(v) = 0 \quad \forall a \in A, \forall v \in V \subseteq \text{Irr}(A), \forall r \in \text{Rad}(A)$$

\Uparrow

$$a(r(v)) = 0 \quad \forall a \in A, \forall v \in V \subseteq \text{Irr}(A), \forall r \in \text{Rad}(A)$$

\Uparrow

$$a(0) = 0 \quad \forall a \in A$$

□

12 20-10-2014

12.1 Representation Theory Reader Week 40

12.1 Proposition.

Let A be a finite dimensional k -algebra such that $\text{Irr}(A) = \{(V_1, \rho_1), \dots, (V_n, \rho_n)\}$.
Then

$$\text{Irr}(A/\text{Rad}(A)) = \{(V_1, \overline{\rho}_1), \dots, (V_n, \overline{\rho}_n)\}.$$

Proof.

First, how is $\overline{\rho}_i : A/\text{Rad}(A) \rightarrow \text{End}_k(V_i)$ defined? Let's define it like this: $\overline{\rho}_i(\overline{a}).v = av$. At least we end up in the codomain this way. We also have to check that: $a, b \in x + \text{Rad}(A) \implies av = bv$. In other words, we have to check that:

$$x + r_1(v) = x + r_2(v) \quad \forall v \in V_i, \forall r_1, r_2 \in \text{Rad}(A)$$

We can rewrite this equation as:

$$x(v) + r_1(v) = x(v) + r_2(v) \quad \forall v \in V_i, \forall r_1, r_2 \in \text{Rad}(A)$$

This equation holds, as $r_1(v) = r_2(v) = 0$ for any element of an irreducible module.

- $\{(V_1, \overline{\rho}_1), \dots, (V_n, \overline{\rho}_n)\} \subseteq \text{Irr}(A/\text{Rad}(A))$
- \Uparrow
- $(V_i, \overline{\rho}_i)$ is an irreducible representation of $A/\text{Rad}(A)$
- $(V_i, \overline{\rho}_i)$ is a representation of $A/\text{Rad}(A)$
- \Uparrow [by definition]
- $(\overline{a}\overline{b}).v_i = \overline{a}.(\overline{b}.v_i) \quad \forall \overline{a}, \overline{b} \in A/\text{Rad}(A) \quad \forall v_i \in V_i$
- \Uparrow
- $\overline{a}.v_i = a.v_i$
- \Uparrow [by definition of $\overline{\rho}_i$]
- $(V_i, \overline{\rho}_i)$ is a irreducible $A/\text{Rad}(A)$ -module
- \Uparrow [by definition]

$A/\text{Rad}(A).v_i \supseteq V_i$ for all nonzero vectors $v_i \in V_i$

\uparrow [Let $v \in V_i$ then there exists a nonzero vector $v \in V_i$ and $a \in A$ such that $v_i = a.v$. But $a.v = \bar{a}v$. So $v \in A/\text{Rad}(A).v_i$.]

$A.v_i \supseteq V_i$ for all nonzero vectors $v_i \in V_i$

- $\text{Irr}(A/\text{Rad}(A)) \subseteq \{(V_1, \bar{\rho}_1), \dots, (V_n, \bar{\rho}_n)\}$

\uparrow

every irreducible representation of $A/\text{Rad}(A)$ lifts to an irreducible representation of A

\uparrow [Let V be an irreducible representation of $A/\text{Rad}(A)$. Then $A/\text{Rad}(A).v \supseteq V$ for any nonzero vector $v \in V$. But then surely $A.v \supseteq V$. So V is an irreducible representation of A as well.]

□

12.2 Proposition.

We have an isomorphism of k -algebras:

$$\bigoplus_{i=1}^n \bar{\rho}_i : A/\text{Rad}(A) \rightarrow \text{End}_{D_1}(V_1) \oplus \dots \oplus \text{End}_{D_n}(V_n)$$

where $D_i := \text{End}_A(V_i)$ a finite dimensional division algebra over k . And $A/\text{Rad}(A)$ is semisimple.

Proof.

$\bigoplus_{i=1}^n \bar{\rho}_i : A/\text{Rad}(A) \rightarrow \text{End}_{D_1}(V_1) \oplus \dots \oplus \text{End}_{D_n}(V_n)$ is an isomorphism of k -algebras

\uparrow [isomorphism theorem]

- $\bigoplus_{i=1}^n \rho_i : A \rightarrow \text{End}_{D_1}(V_1) \oplus \dots \oplus \text{End}_{D_n}(V_n)$ is surjective

\uparrow

$$\bigoplus_{i=1}^n \rho_i(A) = \text{End}_{D_1}(V_1) \oplus \dots \oplus \text{End}_{D_n}(V_n)$$

\uparrow [C4.20: If V_1, \dots, V_n are mutually inequivalent irreducible A -modules, then $\rho(A) = \text{End}_{D_1}(V_1) \oplus \dots \oplus \text{End}_{D_n}(V_n)$.]

- the kernel of $\bigoplus_{i=1}^n \rho_i$ is $\text{Rad}(A)$

\Uparrow [by definition of $\text{Rad}(A)$]

□

Proof.

$A/\text{Rad}(A)$ is semisimple

\Uparrow [P5.2: If $A := \text{Mat}_{d_1}(D_1) \oplus \cdots \oplus \text{Mat}_{d_n}(D_n)$, with D_i a finite dimensional division algebra over k , then A is semisimple.]

There exists some division algebras D_i such that

$$A/\text{Rad}(A) \cong \text{Mat}_{d_1}(D_1) \oplus \cdots \oplus \text{Mat}_{d_n}(D_n)$$

\Uparrow [As $A/\text{Rad}(A) \cong \text{End}_{D_1}(V_1) \oplus \cdots \oplus \text{End}_{D_n}(V_n)$]

There exists some division algebras D_i such that

$$\text{End}_{D_1}(V_1) \oplus \cdots \oplus \text{End}_{D_n}(V_n) \cong \text{Mat}_{d_1}(D_1) \oplus \cdots \oplus \text{Mat}_{d_n}(D_n)$$

\Uparrow

Choose $D_i = \text{End}_A(V_i)$. Then we have for some $d_i \in \mathbb{N}$ that

$$\text{End}_{D_i}(V_i) \cong \text{Mat}_{d_i}(D_i)$$

□

12.3 Theorem.

Let A be a finite dimensional k -algebra, and $\text{Irr}(A) = \{(V_1, \rho_1), \dots, (V_n, \rho_n)\}$. The following are equivalent:

1. A is semisimple
2. $\text{Rad}(A) = 0$
3. $\dim_k(A) = \sum_{i=1}^n \dim_{D_i}(V_i)^2 \dim_k(D_i)$ with $D_i = \text{End}_A(V_i)$
4. $A \cong \text{Mat}_{d_1}(D_1) \oplus \cdots \oplus \text{Mat}_{d_m}(D_m)$ for some k division algebras

Proof.

A is semisimple $\implies \text{Rad}(A) = 0$

↓

$\text{Rad}(A)$ has an complement B such that $A = \text{Rad}(A) \oplus B$ is a direct sum of of A -modules

↓

There exists an corresponding decomposition $1 = e_R + e_B$ in idempotents of $A' = \text{End}_A(A) = A^{\text{op}}$

$$\Downarrow [e_R = e_R^2 = e_R^3 = e_R^n = 0]$$

$$A = B$$

$$\Downarrow [A = \text{Rad}(A) \oplus B]$$

$$\text{Rad}(A) = 0$$

□

Proof.

$$\dim_k(A) = \sum_{i=1}^n \dim_{D_i}(V_i)^2 \dim_k(D_i)$$

$$\Uparrow [\text{Rad}(A) = 0 \text{ by hypothesis}]$$

$$\dim_k(\text{End}_{D_1}(V_1) \oplus \cdots \oplus \text{End}_{D_n}(V_n)) = \sum_{i=1}^n \dim_{D_i}(V_i)^2 \dim_k(D_i)$$

$$\Uparrow$$

$$\dim_k(\text{End}_{D_i}(V_i)) = \dim_{D_i}(V_i)^2 \dim_k(D_i)$$

$$\Uparrow$$

$$\dim_{D_i}(\text{End}_{D_i}(V_i)) \dim_k(D_i) = \dim_{D_i}(V_i)^2 \dim_k(D_i)$$

$$\Uparrow$$

$$\dim_{D_i}(\text{End}_{D_i}(V_i)) = \dim_{D_i}(V_i)^2$$

□

12.4 Definition.

Let (V, ρ) be a finite dimensional representation of A . The character of V is the functional $\chi_V \in \text{Hom}_k(A, k)$ defined by $\chi_V(a) := \text{Tr}(\rho(a))$.

12.5 Proposition.

The character χ_V is always a central functional on A , i.e. $\chi_V([A, A]) = 0$. Hence we often consider χ_V as an element of the subspace $\text{Hom}_k(A/[A, A], k)$.

12.6 Proposition.

Let U, V be equivalent finite dimensional A -modules. Then $\chi_U = \chi_V$.

Proof.

Let $\phi : U \rightarrow V$ be an interwining isomorphism. Then

$$\phi(\rho_U(a)(u)) = \rho_V(a)(\phi(u))$$

$$\phi \circ \rho_U(a) = \rho_V(a) \circ \phi$$

$$\chi_U = \chi_V$$

\Uparrow

$$\chi_U(a) = \chi_V(a) \quad \forall a \in A$$

\Uparrow

$$\text{Tr}(\rho_V(a)) = \text{Tr}(\rho_U(a)) \quad \forall a \in A$$

\Uparrow

$$\bullet \text{Tr}(\rho_V(a)) = \text{Tr}(\phi \circ \rho_U(a) \circ \phi^{-1})$$

\Uparrow

$$\phi \circ \rho_U(a) = \rho_V(a) \circ \phi$$

$$\bullet \text{Tr}(\rho_U(a)) = \text{Tr}(\phi \circ \rho_U(a) \circ \phi^{-1})$$

\Uparrow

Remember the general property: $\text{Tr}(L \circ K) = \text{Tr}(K \circ L)$

□

12.7 Proposition.

Let W be a finite dimensional A -module, and suppose that

$$0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n := W$$

is an ascending filtration of W with consecutive subquotients $U_i = W_i/W_{i-1}$.

Let $U = \bigoplus_{i=1}^n U_i$. Then $\chi_W = \chi_U$.

Proof.

- Base case: $0 = W_1 \subseteq W_1 := W$. We have $U_1 = W/0 = W$. So $U = W$
- Induction hypothesis: $\chi(W_{n-1}) = \chi(\bigoplus_{i=1}^{n-1} U_i)$

- Induction step: $\chi(W) = \chi(W_{n-1}) + \chi(W/W_{n-1})$ by prop. 5.14 and so $\chi(W) = \chi(W_{n-1}) + \chi(U_n) = \chi(\oplus_{i=1}^{n-1} U_i) + \chi(U_n)$. It suffices to show that $\chi(\oplus_{i=1}^n U_i) = \chi(\oplus_{i=1}^{n-1} U_i) + \chi(U_n)$. To show that it suffices to show that $\oplus_{i=1}^n U_i / U_n = \oplus_{i=1}^{n-1} U_i$. Which follows from exercise 5d week 36.

□

12.8 Theorem.

The characters χ_i of a collection V_1, \dots, V_n of mutually inequivalent irreducible finite dimensional representations of A are linearly independent. If A is a finite dimensional semisimple algebra then the characters of the elements of $\text{Irr}(A)$ form a basis of $(A/[A, A])^$*

Proof.

$$\sum_{j=1}^n c_j \chi_j = 0 \implies c_i = 0 \quad \forall i = 1, \dots, n$$

↑↑

Let $i = 1, \dots, n$. We are going to show that

$$c_i = 0$$

↑↑

$$\exists a_i \in A : \chi_j(a_i) = \delta_{ij} \quad \forall j = 1, \dots, n$$

↑↑

$$\exists a_i \in A : \text{Tr}(\rho_j(a_i)) = \delta_{ij} \quad \forall j = 1, \dots, n$$

↑↑

$\rho_i : A \rightarrow \text{End}_k(V_i)$ is surjective

□

Proof.

If A is a finite dimensional semisimple algebra then the characters of the elements of $\text{Irr}(A)$ form a basis of $(A/[A, A])^*$

↑↑

$(A/[A, A]^*)$ has dimension n

↑↑

$(A/[A, A])$ has dimension n

↑

$$(A/[A, A]) \cong \bigoplus_{i=1}^n M_i/[M_i, M_i]$$

↑

$$A \cong \bigoplus_{i=1}^n M_i \text{ and } [M_i, M_i] = \{m \in M_i : \text{Tr}(m) = 0\} \quad \square$$

12.9 Definition.

A representation of a group G over k is a pair (V, ρ_G) where V is a k -vector space, and $\rho_G : G \rightarrow GL_k(V)$ a homomorphism of groups.

12.10 Definition.

The group algebra $k[G]$ over k is the finite dimensional k -algebra with k -basis $\{a_g : g \in G\}$ and multiplication of basis elements is given by $a_g a_h = a_{gh}$.

12.11 Theorem.

If (V, ρ_G) is a representation of G over k then there exists a unique representation of (V, ρ) of $k[G]$ such that $\rho(a_g) := \rho_G(g)$ for all $g \in G$. Conversely, if (V, ρ) is a representation of $k[G]$ then there exists a unique representation (V, ρ_G) of G on V defined by $\rho_G(g) = \rho(a_g)$ for all $g \in G$. This sets up a bijection between the set of representations of G over k and the set of left $k[G]$ modules. Moreover, the notions homomorphism of representations of G over k and homomorphism of left $k[G]$ modules coincide.

12.12 Definition.

Given a $k[G]$ module (V, ρ) , we will often not distinguish between ρ and ρ_G and just write $\rho(g)$ instead of $\rho(a_g) = \rho_G(g)$ if no confusion is possible.

12.13 Theorem.

Assume that $\text{Char}(k)$ does not divide $|G|$. Then $k[G]$ is semisimple. If $\text{Irr}(G) = \{(V_1, \rho_1), \dots, (V_n, \rho_n)\}$ then there exists a k -algebra isomorphism

$$\rho : k[G] \rightarrow \bigoplus_{i=1}^n \text{End}_k(V_i)$$

given by $\rho(a_g) = \bigoplus_{i=1}^n \rho_i(g)$. In particular, the left regular representation of $k[G]$ is equivalent with $\bigoplus_{i=1}^n \dim_k(V_i) V_i$ and we have $|G| = \sum_{i=1}^n \dim_k(V_i)^2$.