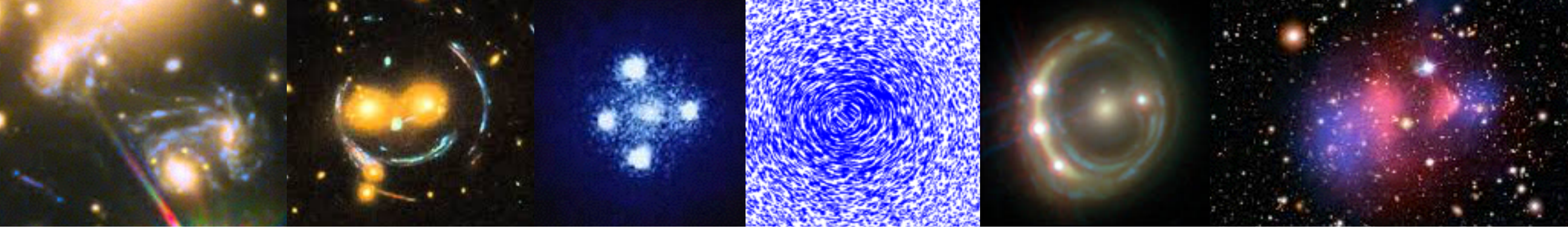


PHY-765 SS18 Gravitational Lensing Week 4

Multiple Images

Kasper B. Schmidt

Leibniz-Institut für Astrophysik Potsdam (AIP)

Last week

- We derived the lens equation:

$$\boxed{\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta})}$$

- A source with true position **β** on the sky can be seen by an observer to be located at angular position **θ** under the deflection **α**(**θ**).

- And defined (for the point mass) the

Critical Mass Surface Density

$$\Sigma_{\rm cr} \equiv \frac{c^2}{4\pi G}\frac{D_{\rm S}}{D_{\rm L}D_{\rm LS}}$$

Convergence

$$\kappa(\boldsymbol{\theta}) \equiv \frac{\Sigma(D_{\rm L}\boldsymbol{\theta})}{\Sigma_{\rm cr}}$$

Einstein Radius

$$\theta_E \equiv \sqrt{\frac{4MG}{c^2}\frac{D_{\rm LS}}{D_{\rm S}D_{\rm L}}}$$

The aim of today

- Explore the first consequence of the lens equation: **multiple images**
- Describe this for a few simplistic lens models
- Introduce the concepts of critical curves and caustics
- SN Refsdal - a spectacular example of multiple images

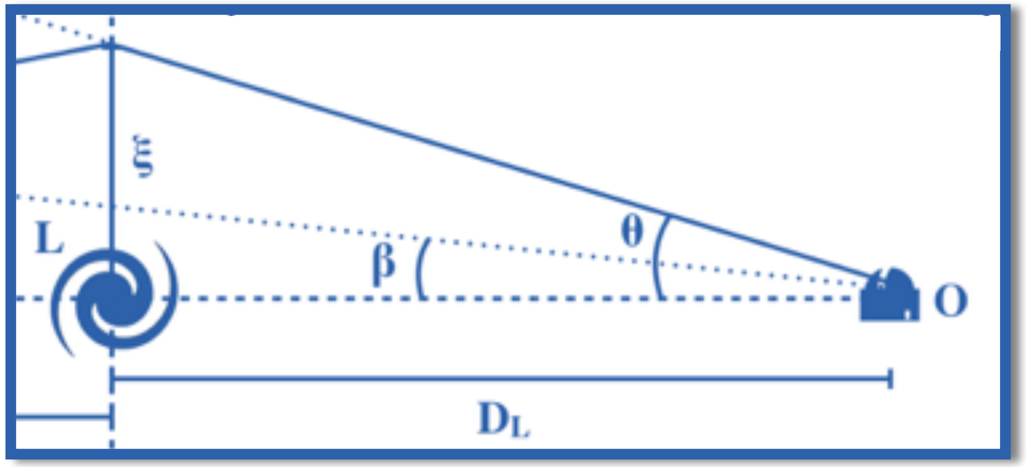
Multiple Images from the Point Mass Lens

- Last week we described the point mass lens:

$$\theta_E \equiv \sqrt{\frac{4MG}{c^2}\frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}D_{\mathrm{L}}}} \qquad \boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{4MG}{c^2}\frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}D_{\mathrm{L}}}\frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|^2} = \frac{\theta_{\mathrm{E}}^2}{|\boldsymbol{\theta}|^2}\boldsymbol{\theta}$$

- So we can write the lens equation as:

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{\theta_{\mathrm{E}}^2}{|\boldsymbol{\theta}|^2}\boldsymbol{\theta}$$



- The Einstein radius was defined at $\boldsymbol{\beta} = 0$
  - The lens equation is solved for all $\theta = \theta_{\mathrm{E}}$
- If imperfect alignment then x and y components of the lens equation are:

$$\beta = \theta_x\left[1 - \frac{\theta_{\mathrm{E}}^2}{\theta^2}\right] \qquad 0 = \theta_y\left[1 - \frac{\theta_{\mathrm{E}}^2}{\theta^2}\right]$$

  - Assuming coordinate system aligned such that $\boldsymbol{\beta} = \beta\hat{x}$ and $\beta > 0$

Multiple Images from the Point Mass Lens

- If $\theta_y \neq 0$, then we would have $\theta^2 \equiv \theta_x^2 + \theta_y^2 = \theta_{\rm E}^2$

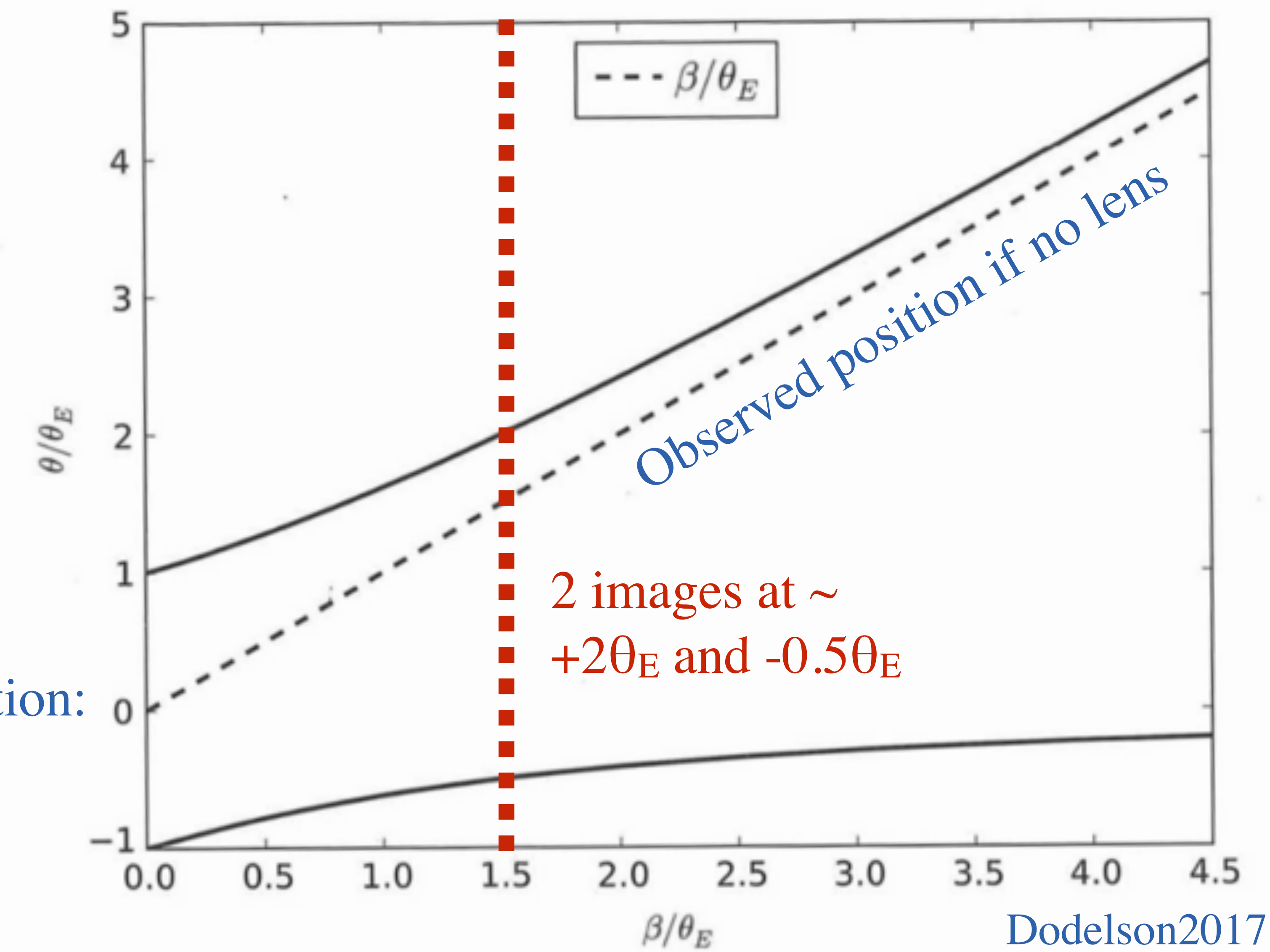
$$0 = \theta_y \left[1 - \frac{\theta_{\rm E}^2}{\theta^2}\right]$$

$$\beta = \theta_x \left[1 - \frac{\theta_{\rm E}^2}{\theta^2}\right]$$

- But then $\beta = 0$ is violating $\beta > 0$
- So we must conclude that $\theta_y = 0$ (for the simple point mass lens)
  - I.e. the lens equation mapping is determined solely by the x-component
- Hence,

$$\theta_\pm = \frac{\beta}{2}\left[1 \pm \sqrt{1 + \frac{4\theta_{\rm E}^2}{\beta^2}}\right]$$

Lens position:
$\theta = 0$



Dodelson2017

Multiple Images from the Point Mass Lens

- The limits for this setup are therefore:

$$\theta_{\pm} \simeq \pm\theta_{\mathrm{E}} + \frac{\beta}{2} \qquad (\beta \ll \theta_{\mathrm{E}})$$

$$\theta_{\pm} = \frac{\beta}{2}\left[1 \pm \sqrt{1 + \frac{4\theta_{\mathrm{E}}^2}{\beta^2}}\right]$$

$$\theta_{+} \simeq \beta + \frac{\theta_{\mathrm{E}}^2}{\beta} \quad \& \quad \theta_{-} \simeq -\frac{\theta_{\mathrm{E}}^2}{\beta} \qquad (\beta \gg \theta_{\mathrm{E}})$$

- Using the Taylor expansion $\sqrt{1+\epsilon} \simeq 1 + \frac{1}{2}\epsilon - \ldots$ for the limit $\beta \gg \theta_{\mathrm{E}}$

Spherically Symmetric Mass Distribution

- To start generalizing these ideas, we first look at the spherical distribution
- For a spherical distribution the convergence is independent on the direction

$$\kappa(\boldsymbol{\theta}) \equiv \frac{\Sigma(D_{\mathrm{L}}\boldsymbol{\theta})}{\Sigma_{\mathrm{cr}}} \quad \rightarrow \quad \kappa(\theta) \equiv \frac{\Sigma(D_{\mathrm{L}}\theta)}{\Sigma_{\mathrm{cr}}}$$

- Such that the deflection angle becomes

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi}\int d^2\theta' \kappa(\theta') \frac{\boldsymbol{\theta} - \boldsymbol{\theta}'}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^2}$$

- $\boldsymbol{\alpha}(\boldsymbol{\theta})$ is a vector. Only relevant vector is $\boldsymbol{\theta}$ ($\boldsymbol{\kappa}$ doesn't care) so we can write:

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = A(\theta)\boldsymbol{\theta}$$

- *A(θ)* can be determined considering
  - The divergence on α: $\boldsymbol{\nabla} \cdot \boldsymbol{\alpha}(\boldsymbol{\theta})$
  - Gauss' (divergence) theorem in the plane: $\int_S d^2\theta \boldsymbol{\nabla} \cdot \boldsymbol{\alpha}(\boldsymbol{\theta}) = \oint_{\partial S} \boldsymbol{\theta} \cdot \boldsymbol{\alpha}$

Spherically Symmetric Mass Distribution

- This give that $\nabla \cdot \boldsymbol{\alpha} = 2\kappa(\theta)$ $\qquad A(\theta) = \langle \kappa(\theta) \rangle$
- Where the mean normalized surface density has been defined as:

$$\langle \kappa(\theta) \rangle = \frac{1}{\pi\theta^2} \int_{\theta' - \theta} d^2\theta' \kappa(\theta')$$

- So we can express the lens equation for the spherical symmetric mass as

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \langle \kappa(\theta) \rangle \boldsymbol{\theta}$$

- As **κ** is the ratio between surface density at angular distance θ from the lens (normalized by the critical surface density) this dictates that:

The deflection (**β** - **θ**) a distance θ from the lens is governed by the mass contained within the cylinder of radius $\xi = D_{\rm L}\theta$.

- which gives that

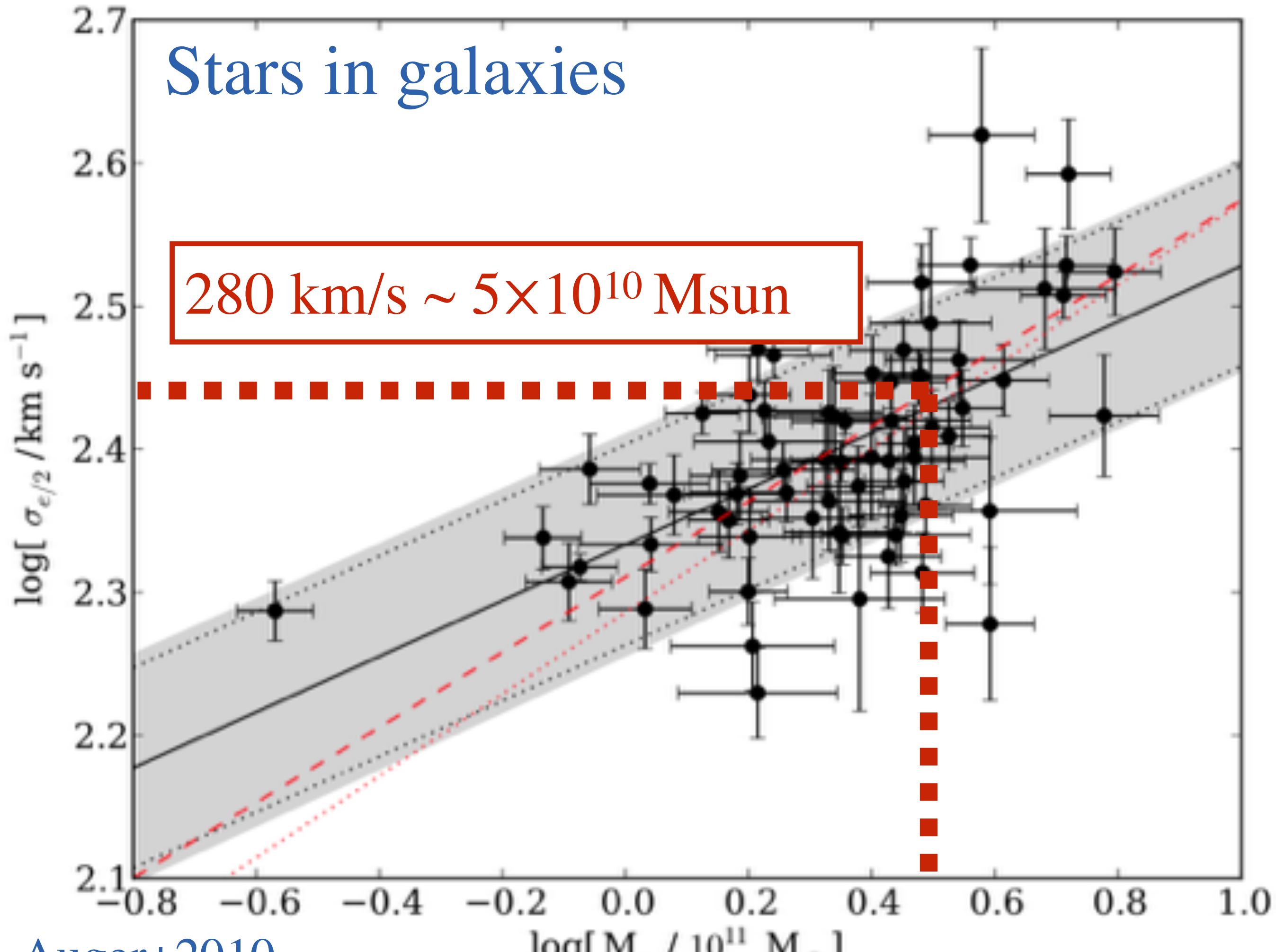
$$\langle \kappa(\theta) \rangle = \frac{M(R = D_{\rm L}\theta)}{\pi D_{\rm L}^2 \theta^2 \sum_{\rm cr}}$$

The Isothermal Sphere (IS)

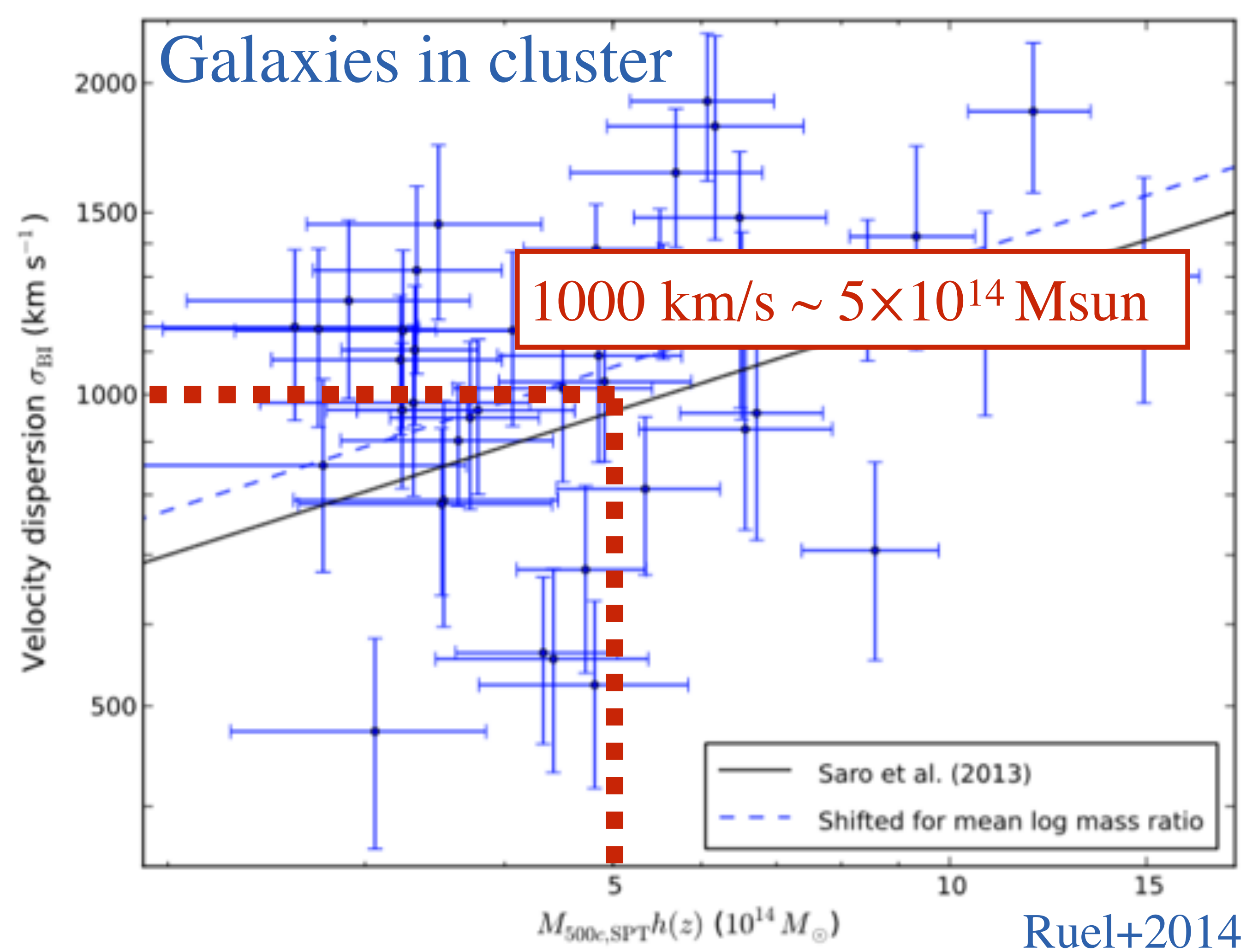
- The result of the spherical symmetric mass distribution can be applied to

$$\rho(r) = \frac{\sigma^2}{2\pi G(r^2 + r_{\rm core}^2)}$$

  - $r_{\rm core}$ is the core radius (the density profile turns flat in the core)
  - σ is the velocity dispersion of the lens



Auger+2010



Ruel+2014

The Isothermal Sphere (IS)

- Using this density profile we have the surface density:

$$\Sigma(R) = \frac{\sigma^2}{2\pi G}\int_{-\infty}^{+\infty}\frac{dz}{R^2+z^2+r_{\rm core}^2} \quad \rightarrow \quad \Sigma(R) = \frac{\sigma^2}{2G\sqrt{R^2+r_{\rm core}^2}}$$

- Which can be used to express the average surface density within a radius R

$$M(R) = 2\pi\int_0^R dR' R'\Sigma(R') \quad \rightarrow \quad M(R) = \frac{\pi\sigma^2}{G}\left[\sqrt{R^2+r_{\rm core}^2} - r_{\rm core}\right]$$

- Using $\theta = R/D_{\rm L}$ and $\theta_{\rm core} \equiv r_{\rm core}/D_{\rm L}$ we get the lens equation

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{\theta_0}{\theta^2}\left[\sqrt{\theta^2+\theta_{\rm core}^2} - \theta_{\rm core}\right]\boldsymbol{\theta}$$

  - by defining:

$$\theta_0 \equiv \frac{4\pi\sigma^2 D_{\rm LS}}{D_{\rm S}c^2}$$

The Singular Isothermal Sphere (SIS)

- For an isothermal sphere with no core ($\theta_{\text{core}} = 0$) the lens equation becomes

$$\boldsymbol{\beta} = \boldsymbol{\theta}\left[1 - \frac{\theta_0}{|\boldsymbol{\theta}|}\right]$$

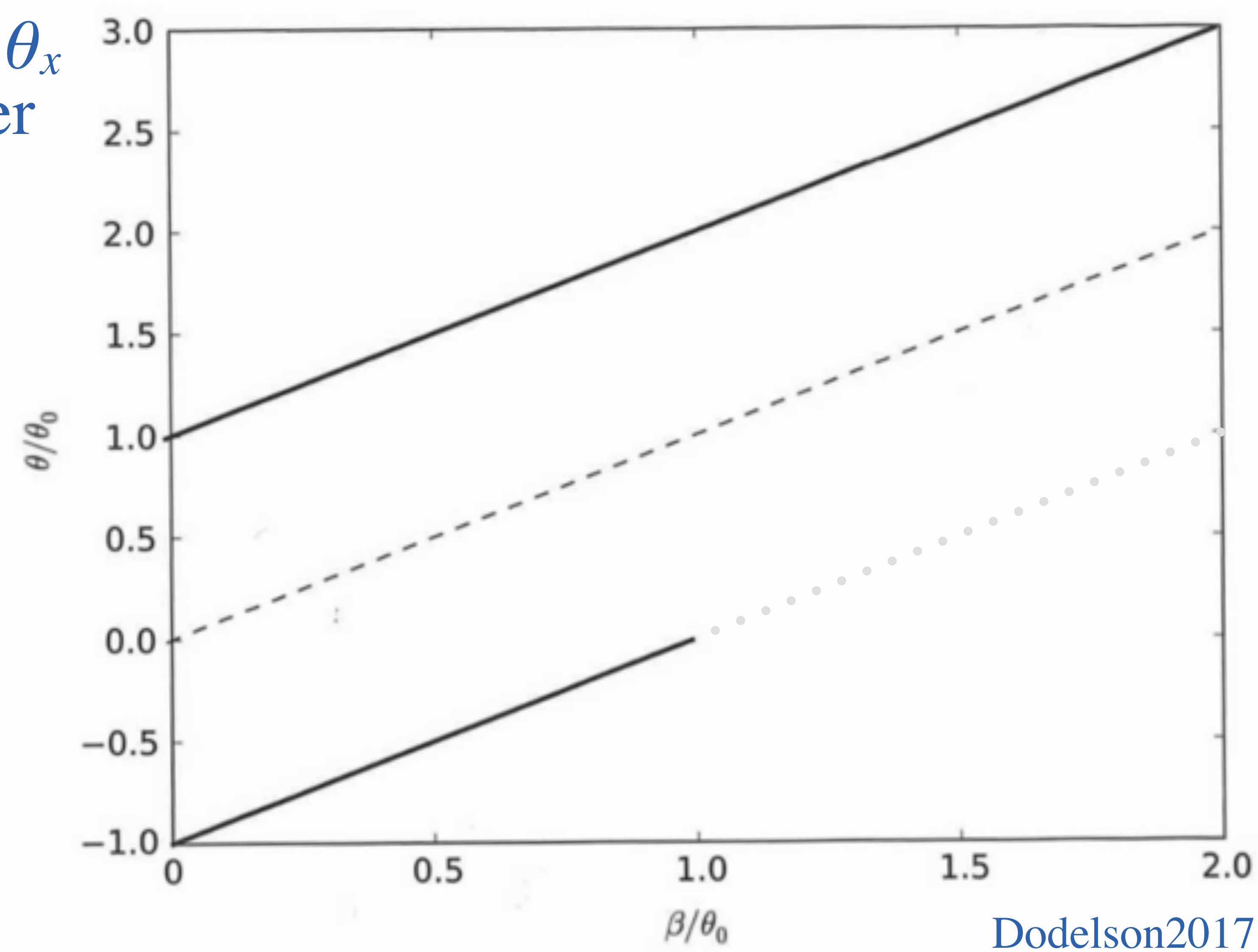
  - where $\beta = 0$ generates an image (Einstein) ring motivating the θ_0 definition

- Like for the point mass only the θ_x component is relevant to consider

$$\theta_+ = \beta + \theta_0$$

$$\theta_- = \beta - \theta_0$$

(but only for $\beta < \theta_0$)



Dodelson2017

Cored Isothermal Sphere (CIS)

- The mass density of real galaxies does not rise all the way into the center
- So even though SIS is simple, the assumption that $r_{\rm core} = 0$ is poor.
- Using the definition of θ_0 and solving the lens equation for $\beta = 0$ we have

$$\theta^2 = \theta_0 \left[\sqrt{\theta^2 + \theta_{\rm core}^2} - \theta_{\rm core}\right] \qquad \theta_{\rm E} = \theta_0 \sqrt{1 - 2\frac{\theta_{\rm core}}{\theta_0}}$$

- Hence, the size of the core determines when an Einstein ring can exist
  - I.e., if $2\theta_{\rm core} > \theta_0$ then an Einstein ring cannot be formed
- Happens at ~1" for lens at 1Gpc
  - Actual physical size differes as θ_0 depends on $z_{\rm L}, z_{\rm S}$ and σ
- If the lens-source alignment is not perfect the lens equation becomes

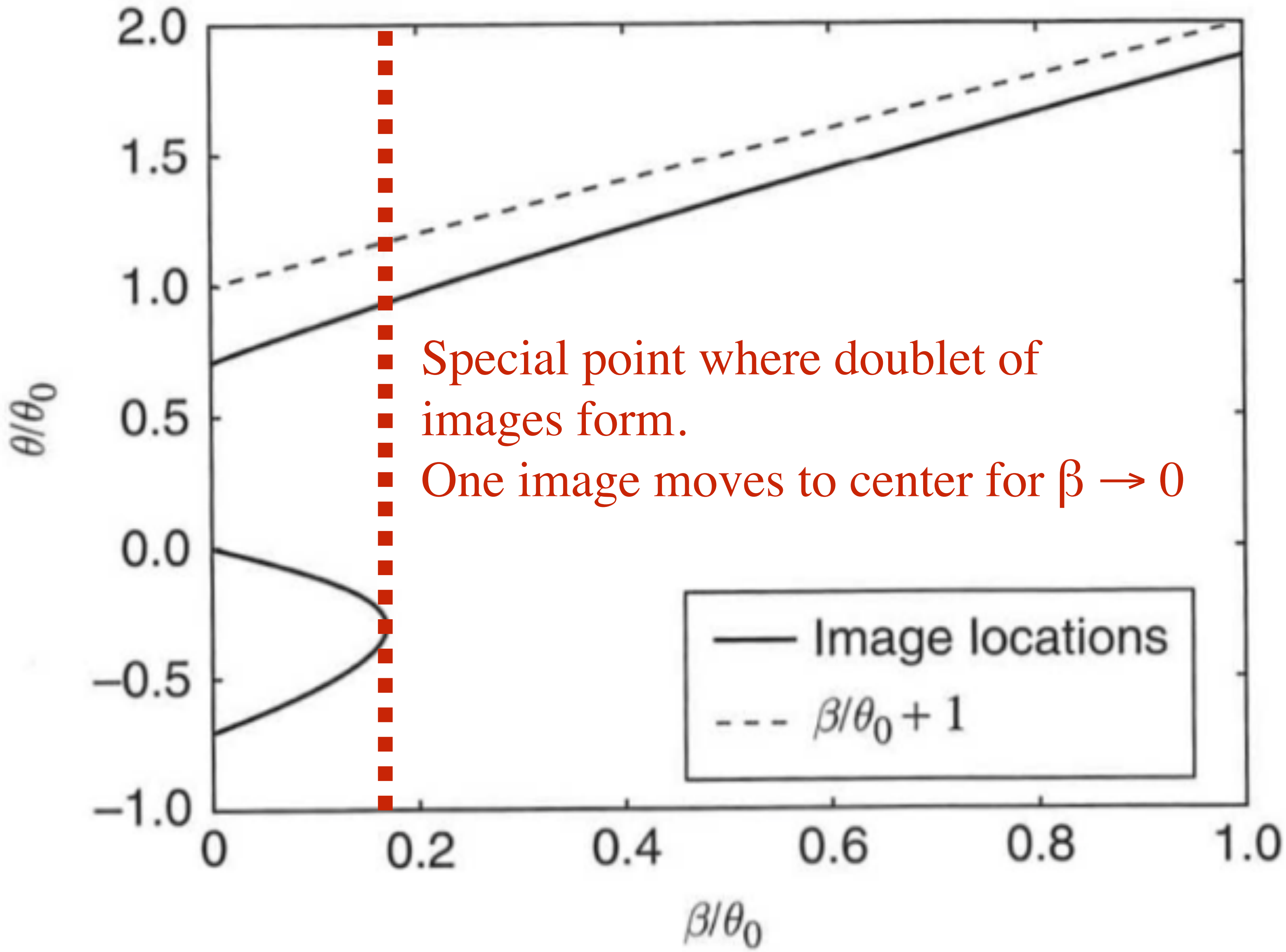
$$\theta(\beta - \theta) = -\theta_0 \left[\sqrt{\theta^2 + \theta_{\rm core}^2} - \theta_{\rm core}\right]$$

  - Rather complex to solve for individual images.

Cored Isothermal Sphere (CIS)

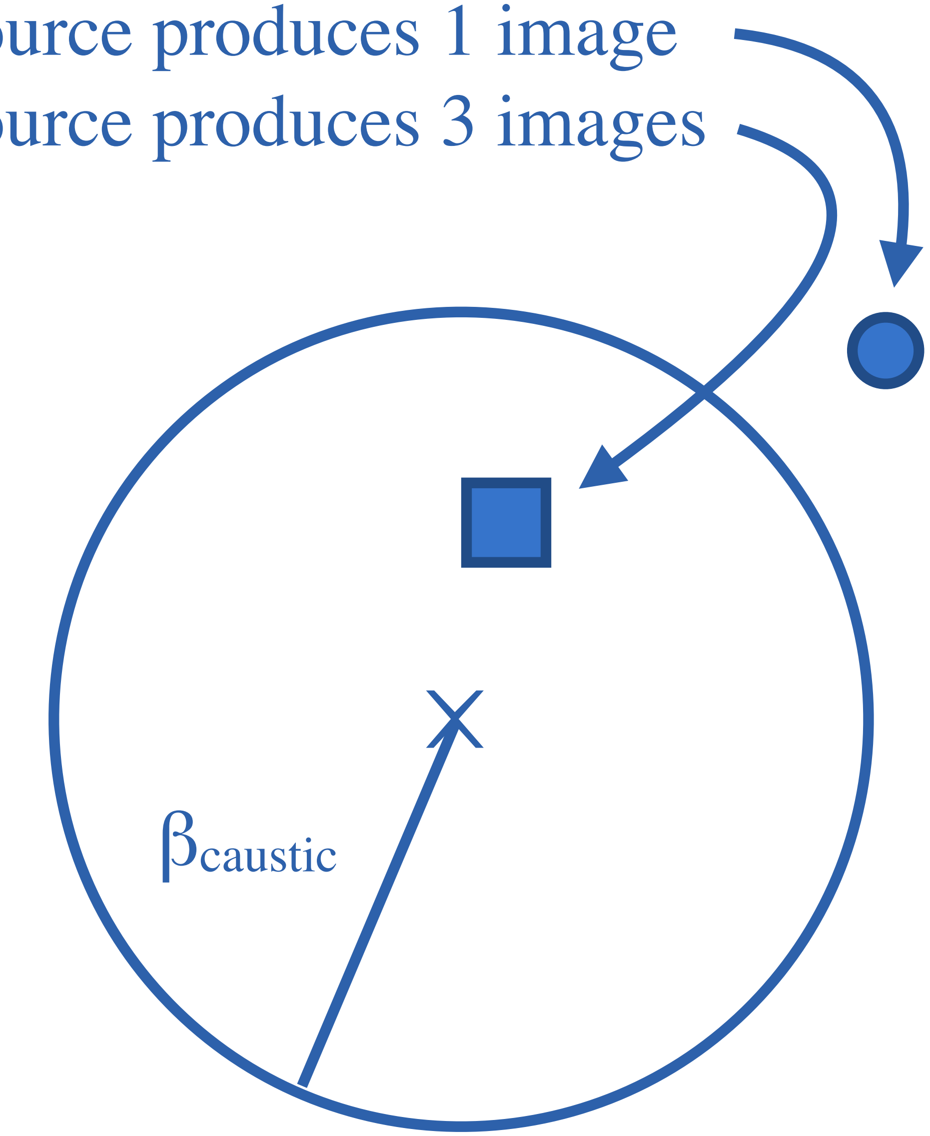
- But plotting β as a function of θ and flipping the axes is easy:

Dodelson2017



Source produces 1 image

Source produces 3 images

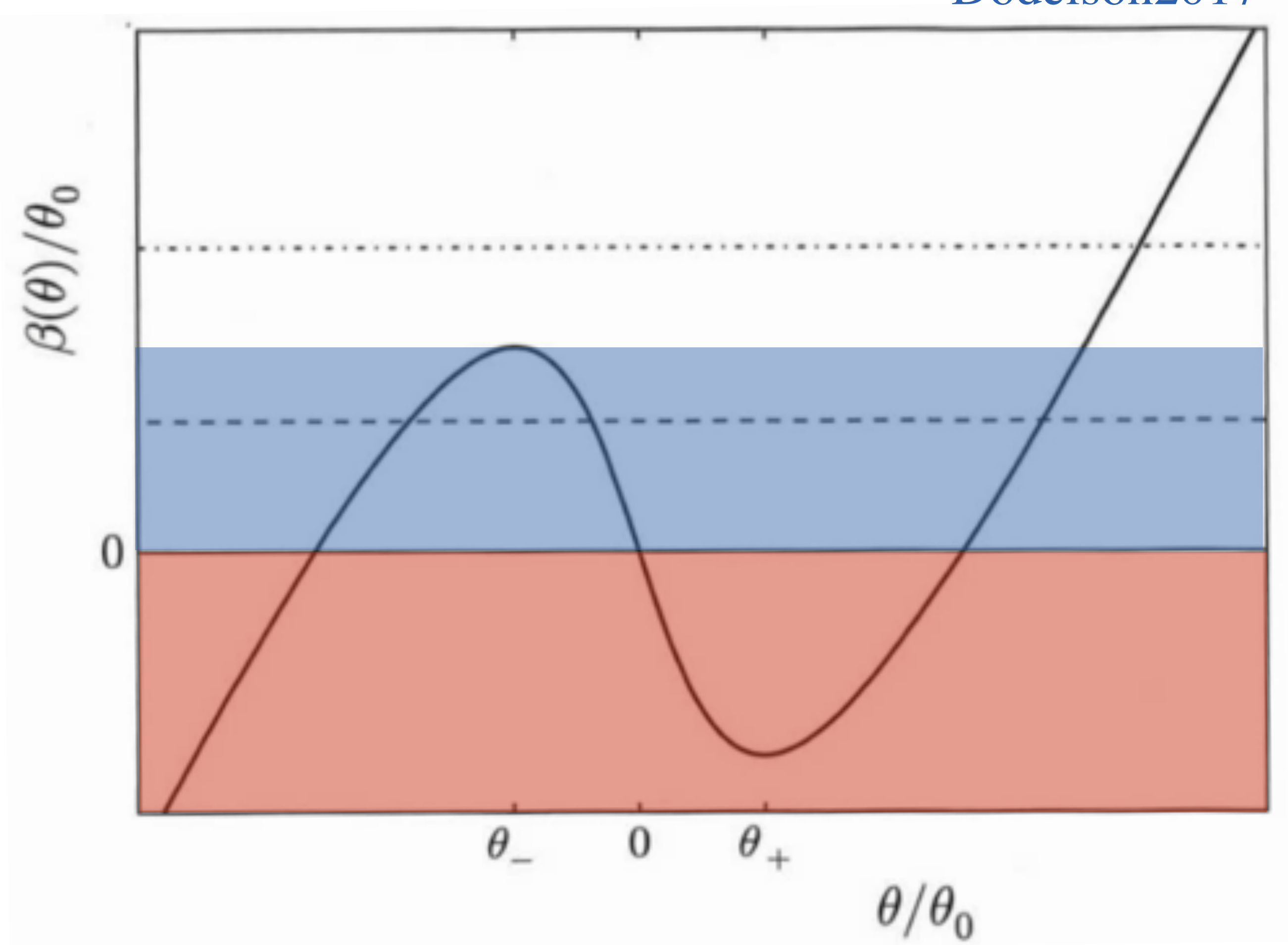


- In 1D this special value $\beta_{caustic}$ is on a curve, but more generally it defines a circle of radius $\beta_{caustic}$ in the source plane

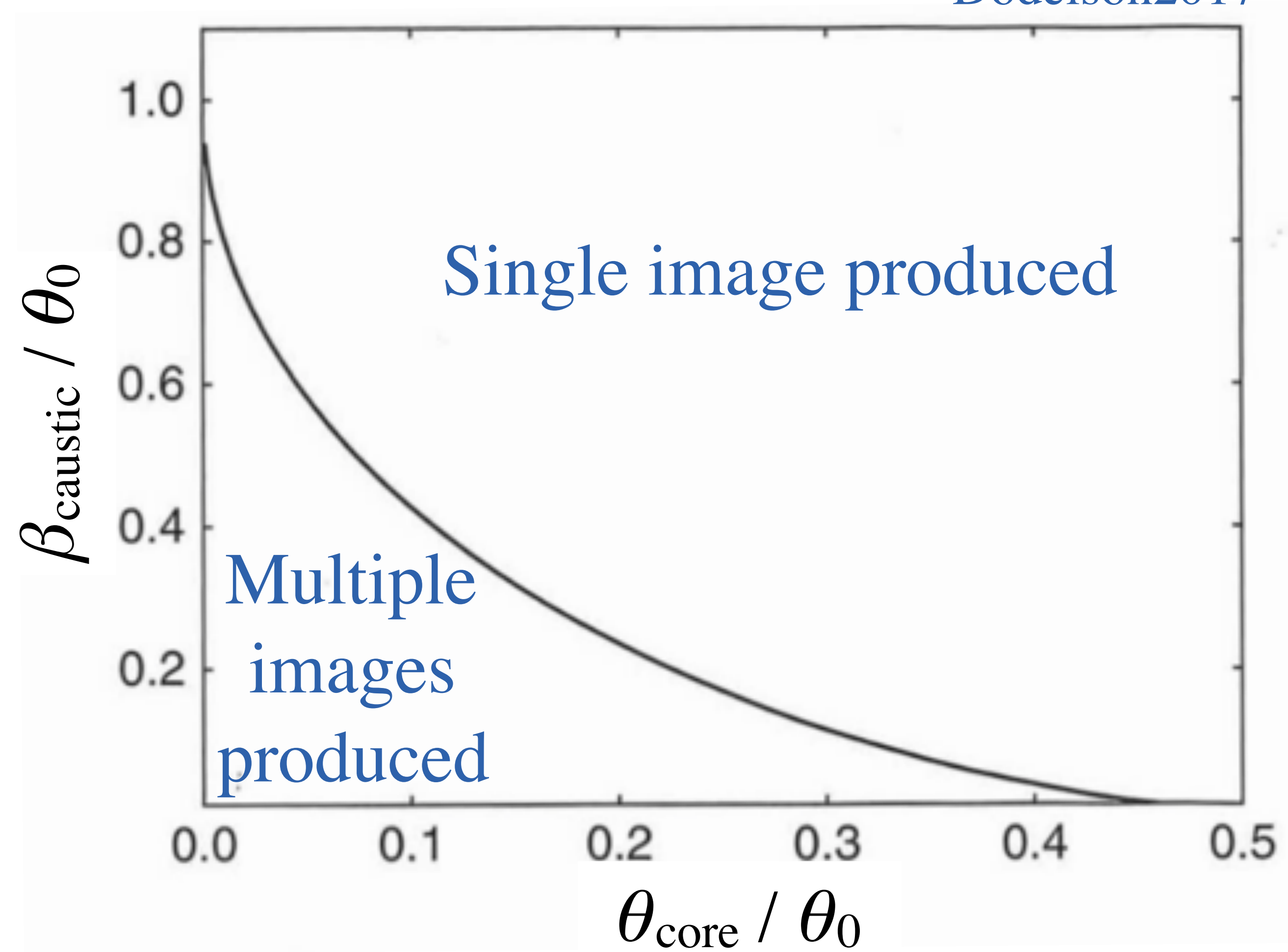
Caustics

- We want to determine the source position $\beta_{caustic}$
- Returning to the lens eq. on the form $\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{\theta_0}{\theta^2}\left[\sqrt{\theta^2 + \theta_{core}^2} - \theta_{core}\right]\boldsymbol{\theta}$
- $\beta \rightarrow \pm\infty$ for $\theta \rightarrow \pm\infty$ so for three solutions there must be 2 extrema
- All values of β below $\beta(\theta_-) \equiv \beta_{caustic}$ produces 3 images (where $\beta > 0$)
- Finding the extrema is done by obtaining the solutions to: $\frac{d\beta}{d\theta} = 0$

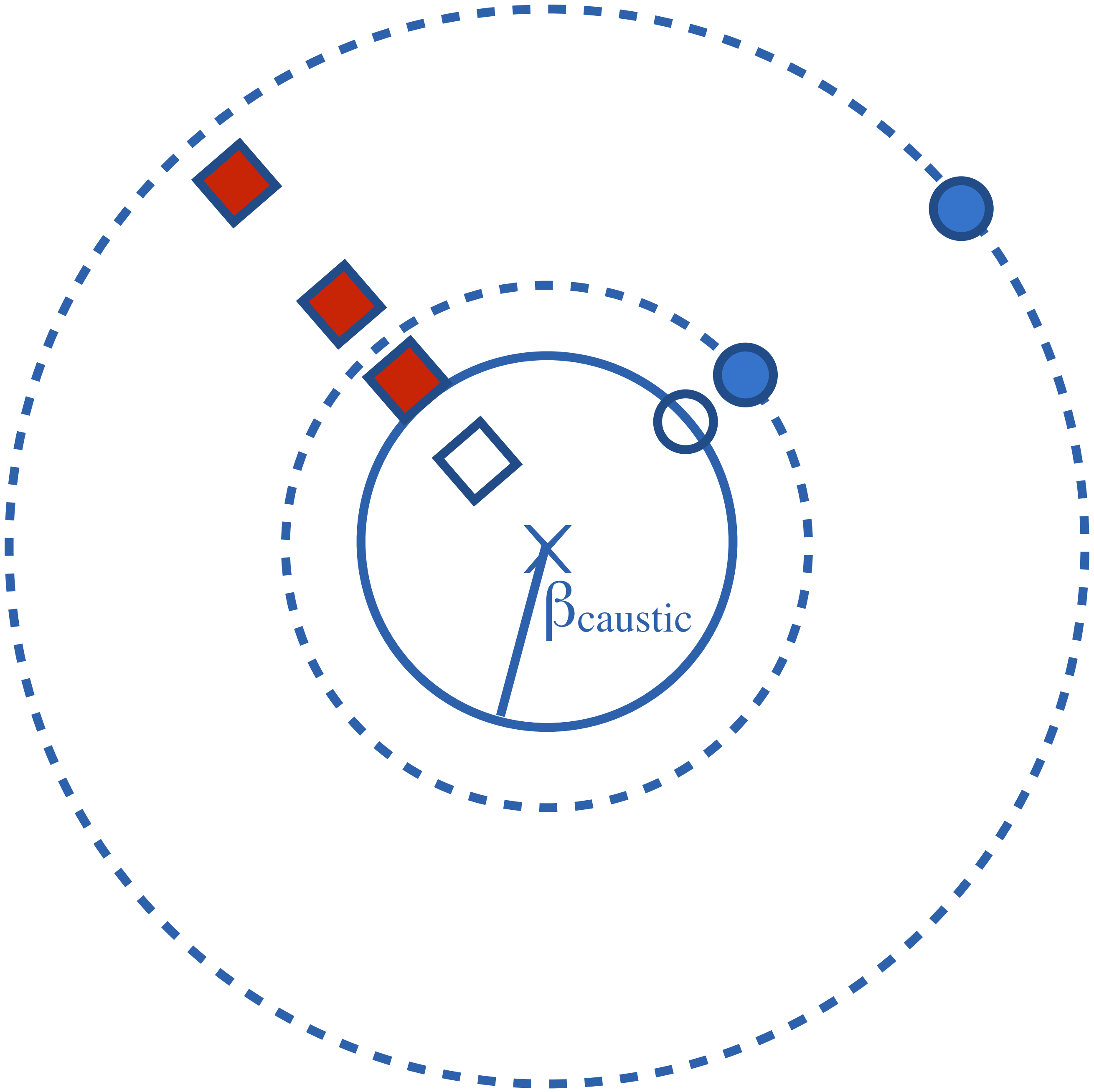
Dodelson2017

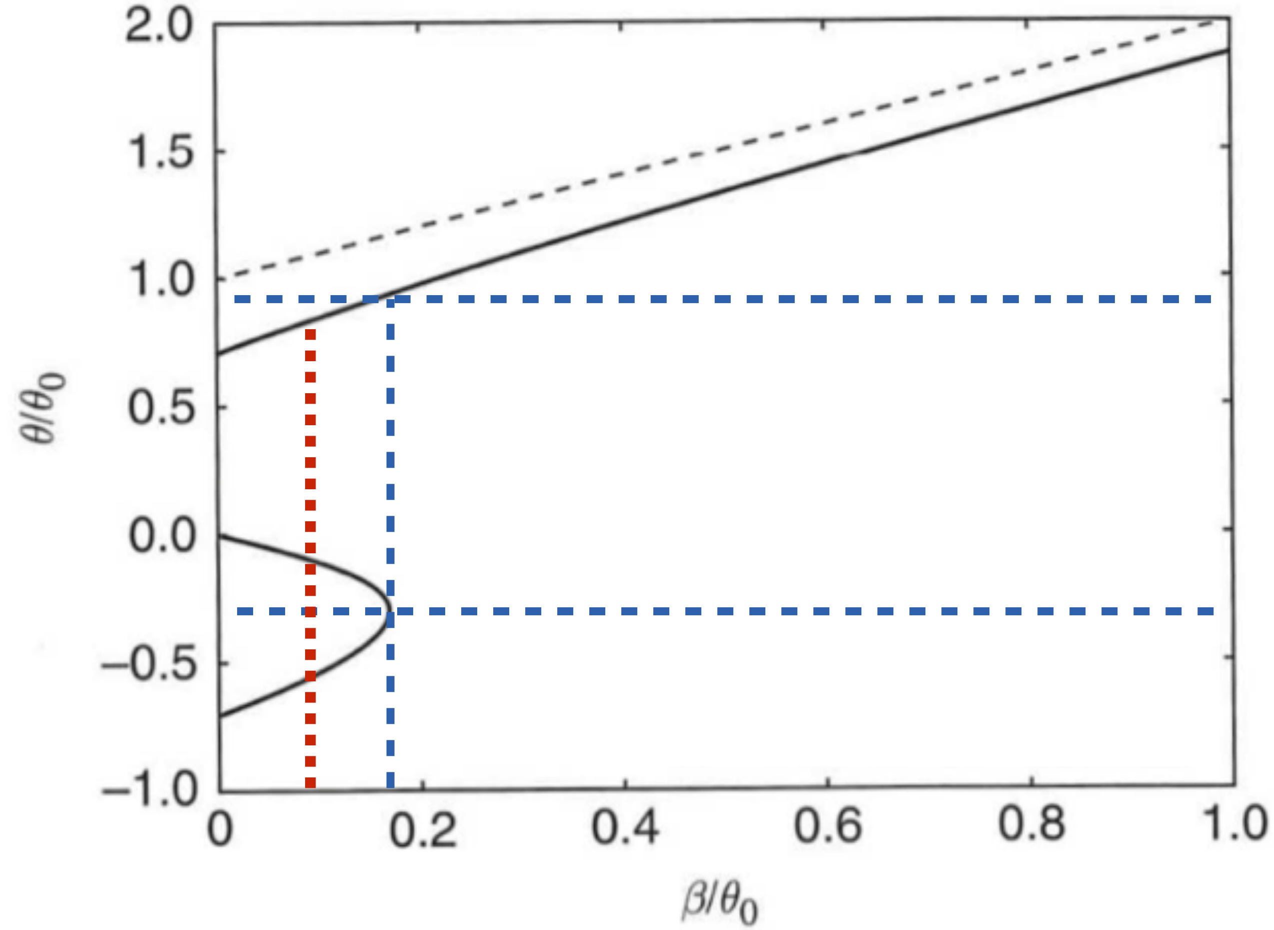


Dodelson2017



Critical Curves



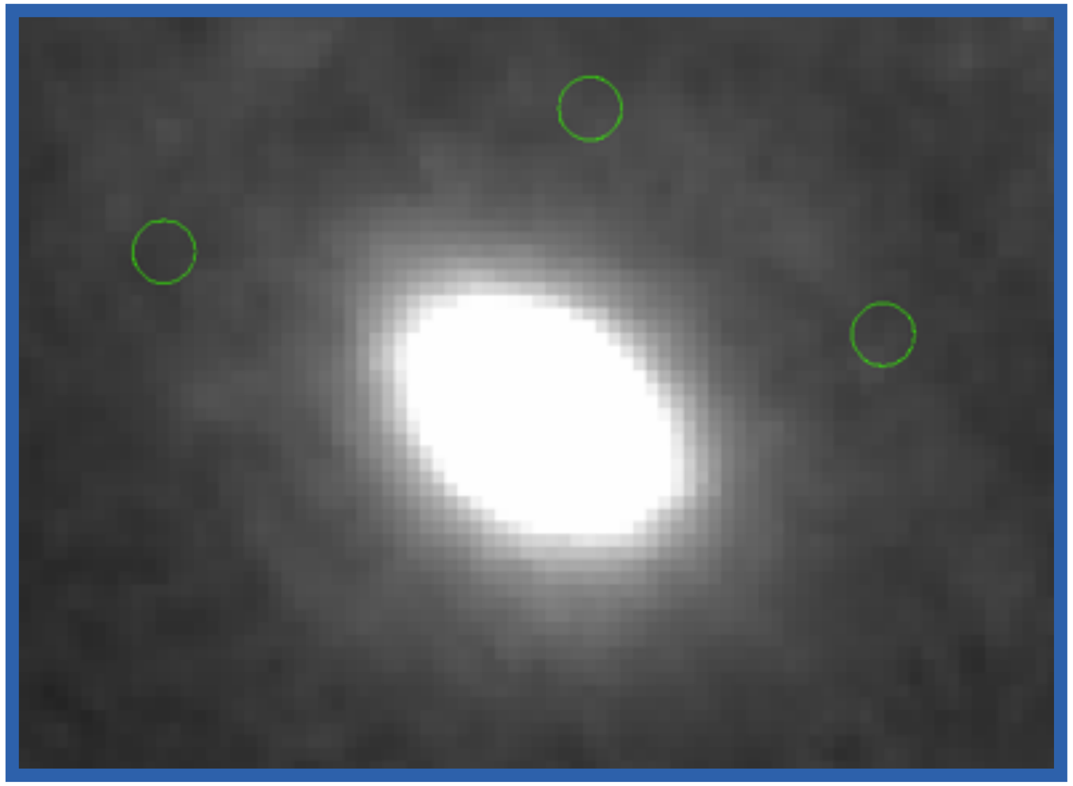


- The critical curves are define as the curves in the *lens* (image) plane where the images fall if the source is on the caustic in the source plane.

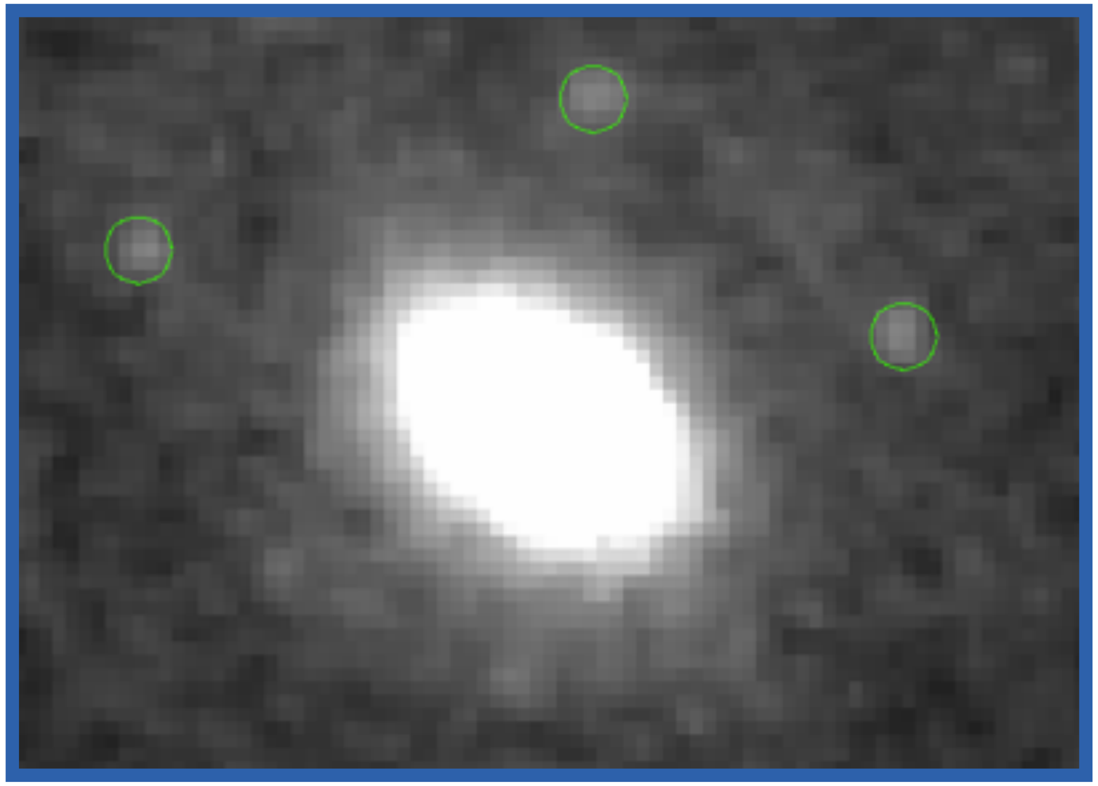
- Will return to caustic and critical curves when talking about magnification…

SN Refsdal

- Nov 2014: Discovered in MACS1149 data from the GLASS program



Existing Imaging



GLASS F104W

- Dec 2016: Imaging and spectroscopic follow-up of MACS1149 FoV
  - HFF, MOSFIRE, X-SHOOTER, DEIMOS, WFC3-G141

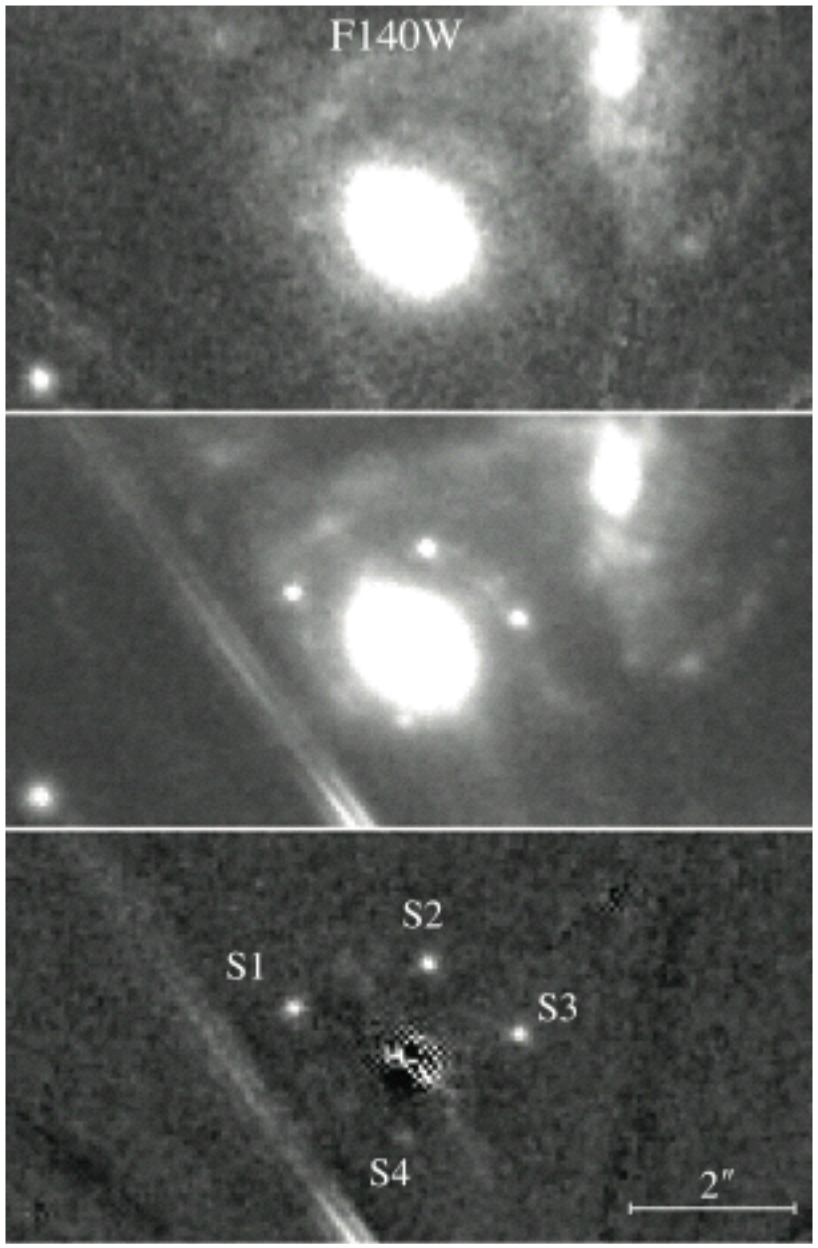
SN Refsdal

GLASS+ CLASH

GLASS+ HFF Dec ‘14

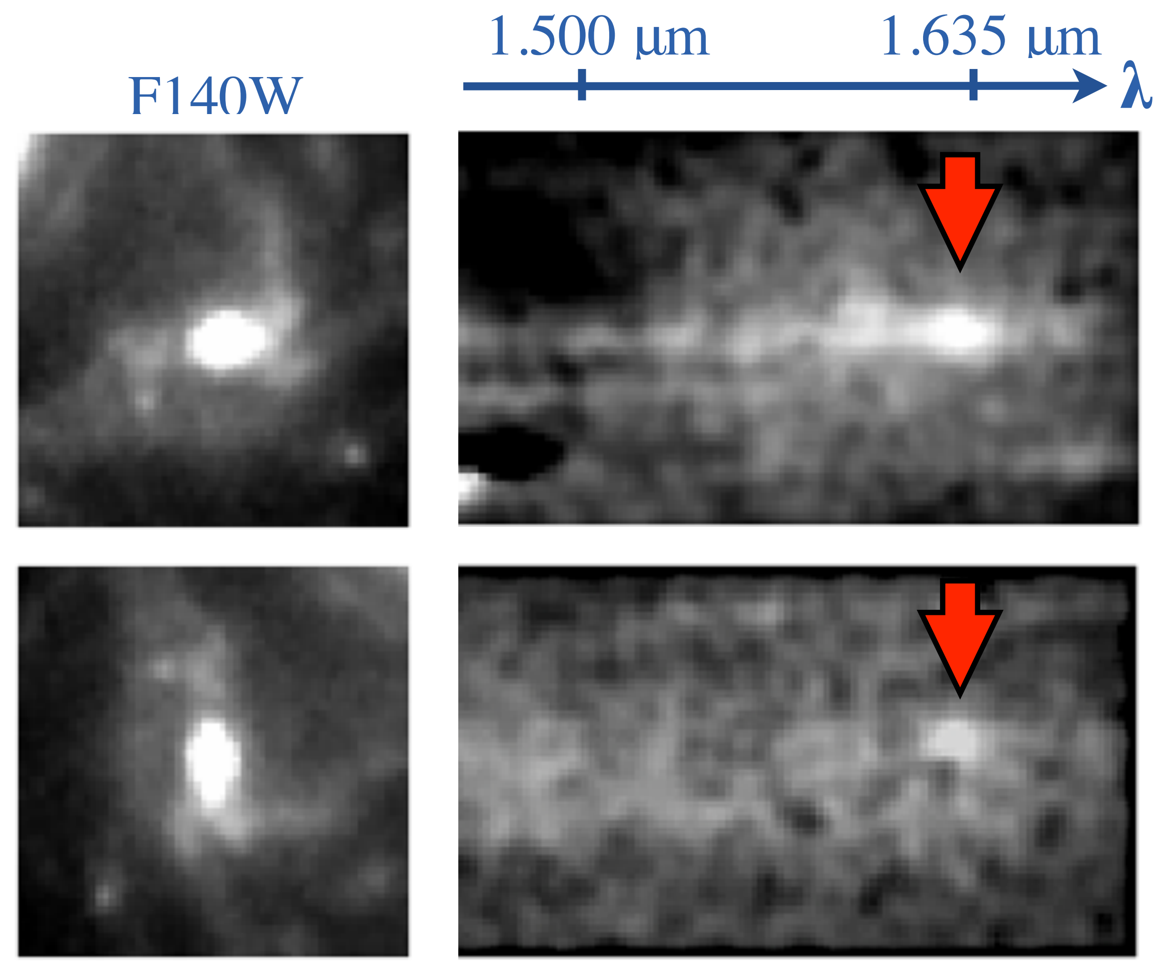
Diff.

Kelly+15



G141 PA1

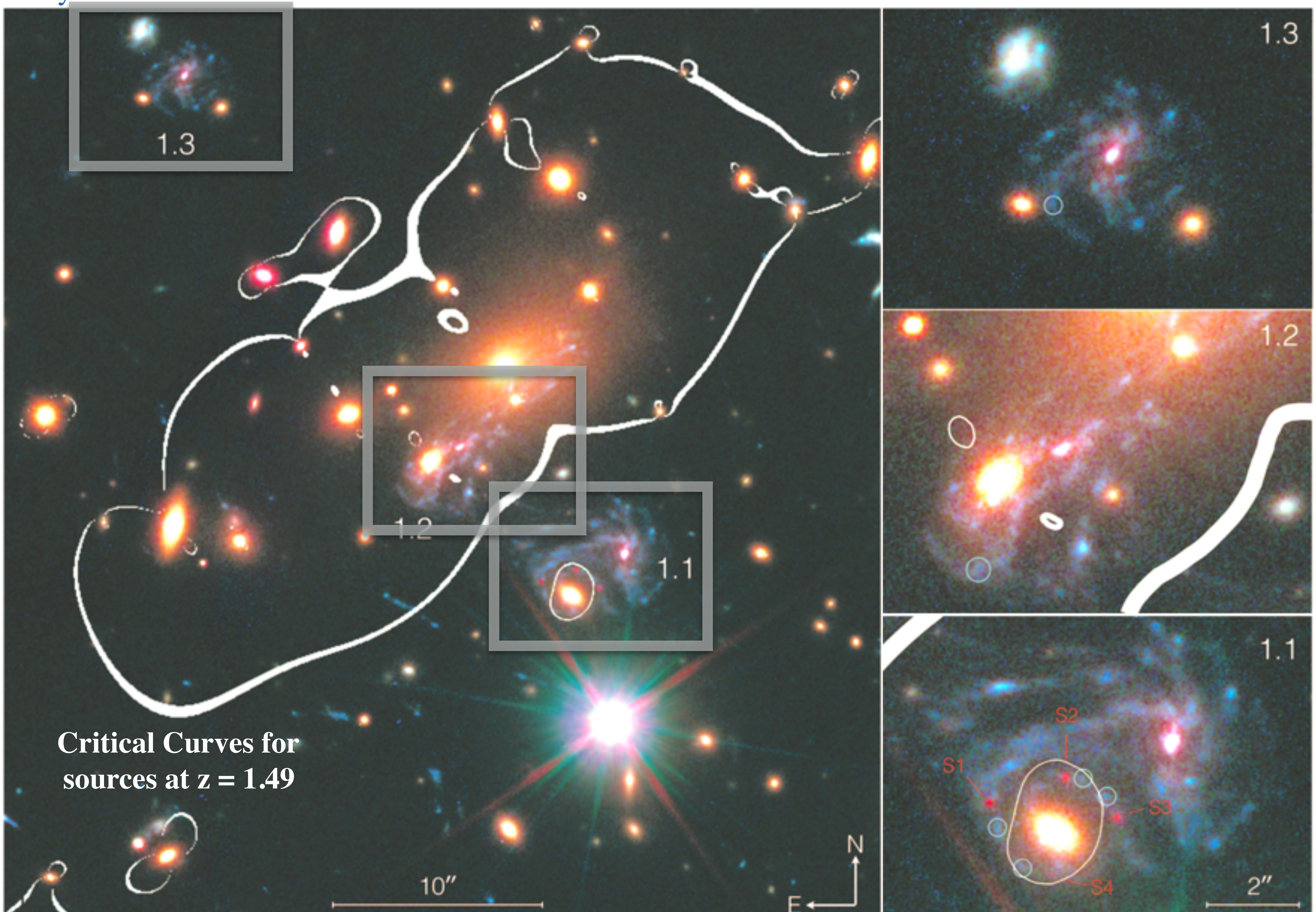
G141 PA2



$z_{\mathrm{grism}}(\mathrm{H}\alpha) = 1.491$

SN Refsdal - Multiple images

Kelly+2015



Critical Curves for sources at z = 1.49

So in summary…

- The multiple images occurring from a point mass lens are given by

$$\theta_{\pm} = \frac{\beta}{2}\left[1 \pm \sqrt{1 + \frac{4\theta_{\mathrm{E}}^2}{\beta^2}}\right]$$

- The lens equation for the more general 'spherical lens' is

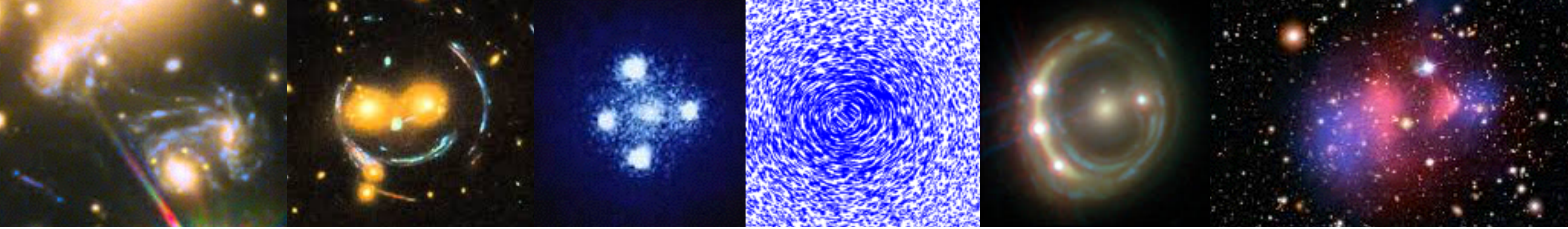
$$\boldsymbol{\beta} = \boldsymbol{\theta} - \langle\kappa(\theta)\rangle\boldsymbol{\theta} \quad \text{where} \quad \langle\kappa(\theta)\rangle = \frac{1}{\pi\theta^2}\int_{\theta'-\theta} d^2\theta'\kappa(\theta') = \frac{M(R = D_{\mathrm{L}}\theta)}{\pi D_{\mathrm{L}}^2\theta^2\sum_{\mathrm{cr}}}$$

- This leads to the lens equations for the CIS and SIS:

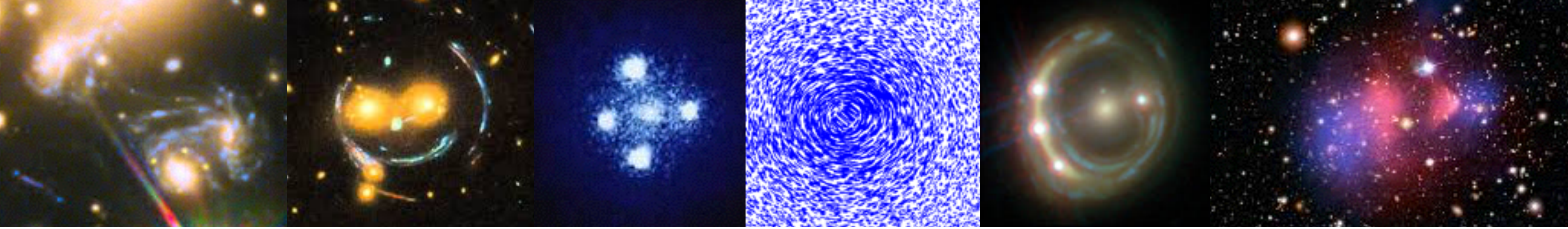
$$\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{\theta_0}{\theta^2}\left[\sqrt{\theta^2 + \theta_{\mathrm{core}}^2} - \theta_{\mathrm{core}}\right]\boldsymbol{\theta} \qquad\qquad \boldsymbol{\beta} = \boldsymbol{\theta}\left[1 - \frac{\theta_0}{|\theta|}\right]$$

- Solving for $\beta = 0$ reveals the multiple images for these lens models
- *Caustics* and *Critical curves* describe the source and image positions of multiple-image geometries in the source and lens (image) planes, respectively.

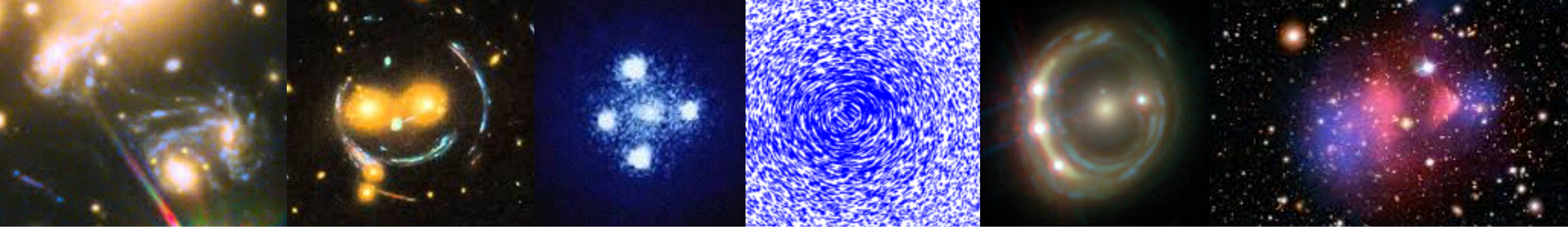
- SN Refsdal: a spectacular case of multiple images on galaxy and lens scales.



Questions?



Last Week’s Worksheet



This Week's Worksheet