A1	A2	A3	Σ
	5.75	4.5	10.

Aufgabe 24 SMD Schröer, Appel, Kasper

a) $f_1(\varphi) = \cos\varphi \quad f_2(\varphi) = \sin\varphi$

$$A = \begin{pmatrix} \cos(\varphi_1) & \sin(\varphi_1) \\ \vdots & \vdots \\ \cos(\varphi_{12}) & \sin(\varphi_{12}) \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & \sqrt{3}/2 & 1/2 & 0 & -1/2 & -\sqrt{3}/2 & -1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 1/2 & \sqrt{3}/2 & 1 & \sqrt{3}/2 & 1/2 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$ ✓ 1.5P.

b) $A^T \cdot A = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \qquad (A^T \cdot A)^{-1} = \begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix}$

b) $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (\underline{\underline{A}}^T \cdot \underline{\underline{A}})^{-1} A \vec{Y} = \begin{pmatrix} 0{,}0375 \\ 0{,}0774 \end{pmatrix}$

$$(A^T A)^{-1} \cdot A^T = \begin{pmatrix} 1/6 & \sqrt{3}/12 & 1/12 & 0 & -\frac{1}{12} & -\frac{\sqrt{3}}{2} & -\frac{1}{6} & -\frac{\sqrt{3}}{2} & -\frac{1}{12} & 0 & \frac{1}{12} & \frac{\sqrt{3}}{12} \\ 0 & 1/12 & \sqrt{3}/12 & 1/6 & \frac{\sqrt{3}}{12} & \frac{1}{12} & 0 & -\frac{1}{12} & -\frac{\sqrt{3}}{12} & -\frac{1}{6} & -\frac{\sqrt{3}}{12} & -\frac{1}{12} \end{pmatrix}$$

y_i
-0,32
0,088
0,037
-0,050
0,070
0,076
0,005
-0,052
0,057
0,050
-0,011
-0,094

$(A^T A^{-1}) \cdot A^T \vec{y} = \vec{a}$ $\qquad y_i \hat{=}$ Asymmetrie zu MW:

$\Rightarrow \vec{a} \approx \begin{pmatrix} 0{,}0375 \\ 0{,}0774 \end{pmatrix}$ ✓ 1.5P.

c) $V[\vec{a}] = (A^T A)^{-1} A^T \; V[\vec{y}] A \underbrace{(A^T A)^{-1}}_{= \begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix}}$

NR:

$V[\vec{y}] = \sigma^2 \mathbb{1}_{12\times 12}$ da $\sigma = 0{,}011$ und MW unkorreliert

$= (A^T A^{-1}) A^T \; \sigma^2 \mathbb{1}_{12\times 12} \; A \; (A^T A)^{-1}$

$= (A^T A)^{-1} A^T \sigma^2 A (A^T A)^{-1}$

$= \sigma^2 (A^T A)^{-1} A^T A (A^T A)^{-1}$

$= \sigma^2 \begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix} = \sigma^2 \begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix}$

$\approx 2{,}016 \times 10^{-5} \; \mathbb{1}_{2\times 2}$ ✓

$$\sigma_{a_1 a_2} = \rho \underbrace{\sqrt{\sigma_{a_1}^2 \sigma_{a_2}^2}}_{\neq 0}$$

$$\sigma_{a_1 a_2} = 0 \quad \Rightarrow \rho = 0$$

Nebendiagonalelemente der Kovarianzmatrix

✓ 1.5P.

d)

$$f(x) = A_0 \cos(\Psi)\cos(\delta) - A_0 \sin(\Psi)\sin(\delta)$$

4.5 / 8P.

$$\Rightarrow a_1 = A_0 \cos(\delta) \qquad a_2 = -A_0 \sin(\delta)$$

$$\Rightarrow A_0^2 = a_1^2 + a_2^2 \qquad \delta = -\arctan\left(\frac{a_2}{a_1}\right)$$

$$\Rightarrow A_0 = -\sqrt{a_1^2 + a_2^2}$$

Aus vergleich der Plots zu den Daten

$$\sigma_{A_0} = \sqrt{\frac{a_1^2}{a_1^2 + a_2^2}\sigma_{a_1}^2 + \frac{a_2^2}{a_1^2 + a_2^2}\sigma_{a_2}^2}$$

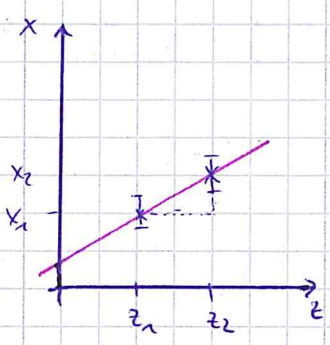
$$\sigma_{\delta} = \sqrt{\frac{a_2^2}{(a_1^2 + a_2^2)^2}\sigma_{a_1}^2 + \frac{a_1^2}{(a_1^2 + a_2^2)^2}\sigma_{a_2}}$$

$$\Rightarrow A_0 = -0{,}086 \pm 0{,}004$$

$$\delta = 1{,}12 \pm 0{,}05$$

Aufgabe 23

a) $x = az + b$ Messung: $p_1 = \begin{pmatrix} z_1 \\ x_1 \pm \sigma_{x_1} \end{pmatrix}$ $p_2 = \begin{pmatrix} z_2 \\ x_2 \pm \sigma_{x_2} \end{pmatrix}$



$$x_1 = az_1 + b \quad \Rightarrow b = x_1 - az_1$$

$$x_2 = az_2 + x_1 - az_1 \quad \Rightarrow \boxed{\frac{x_2 - x_1}{z_2 - z_1} = a}$$ ✓

$$b = x_1 - \frac{x_2 - x_1}{z_2 - z_1} \cdot z_1$$

$$= \frac{x_1(z_2 - z_1)}{z_2 - z_1} - \frac{x_2 z_1 - x_1 z_1}{z_2 - z_1}$$

$$= \frac{x_1 z_2 - x_1 z_1 - x_2 z_1 + x_1 z_1}{z_2 - z_1}$$

$$\boxed{b = \frac{x_1 z_2 - x_2 z_1}{(z_2 - z_1)}}$$ ✓

Geradengleichung:

$$\boxed{x = \underbrace{\left(\frac{x_2 - x_1}{z_2 - z_1}\right)}_{a} z + \underbrace{\frac{x_1 z_2 - x_2 z_1}{(z_2 - z_1)}}_{b}}$$ ✓

Fehler mit Gaußscher Fehlerfortpflanzung (weil ohne Korrelation von x_1 & x_2)

$$a = \frac{x_2 - x_1}{z_2 - z_1} \qquad \sigma_a = \sqrt{\left(\frac{\partial a}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial a}{\partial x_2}\sigma_{x_2}\right)^2}$$

$$= \sqrt{\left(-\frac{1}{z_2 - z_1} \cdot \sigma_{x_1}\right)^2 + \left(\frac{1}{z_2 - z_1}\sigma_{x_2}\right)^2}$$

$$= \sqrt{\frac{1}{(z_2 - z_1)^2}\left(\sigma_{x_1}^2 + \sigma_{x_2}^2\right)} = \frac{1}{z_2 - z_1}\sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2}$$

$$b = \frac{x_1 z_2 - x_2 z_1}{(z_2 - z_1)} \qquad \sigma_b = \sqrt{\left(\frac{\partial b}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial b}{\partial x_2}\sigma_{x_2}\right)^2}$$

$$= \sqrt{\left(\frac{z_2}{(z_2 - z_1)}\sigma_{x_1}\right)^2 + \left(\frac{-z_1}{(z_2 - z_1)}\sigma_{x_2}\right)^2}$$

$$= \frac{1}{z_2 - z_1}\sqrt{\left(z_2^2 \sigma_{x_1}^2 + z_1^2 \sigma_{x_2}^2\right)}$$

Kovarianzmatrix: nächste Seite :)

10.

~~$V[\vec{a}] = B V[?] B^T$ oder $V[\vec{z}] = B V[x] B^T$?~~

~~$\vec{a} = \begin{pmatrix} a \\ b \end{pmatrix}$ $\vec{p} = \begin{pmatrix} z \\ x \end{pmatrix}$~~

$$V[\vec{a}] = \left(A^T W A\right)^{-1}$$

$$A = \begin{pmatrix} \underbrace{1}_{\text{für } b} & \underbrace{z_1}_{\text{für } a} \\ 1 & z_2 \end{pmatrix}, \quad W = \begin{pmatrix} \sigma_{x_1}^{-2} & 0 \\ 0 & \sigma_{x_2}^{-2} \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 1 \\ z_1 & z_2 \end{pmatrix}$$

$$V[\vec{a}] = \left(A^T W A\right)^{-1}$$

$$= \left(\begin{pmatrix} 1 & 1 \\ z_1 & z_2 \end{pmatrix} \begin{pmatrix} \sigma_{x_1}^{-2} & 0 \\ 0 & \sigma_{x_2}^{-2} \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ 1 & z_2 \end{pmatrix} \right)^{-1}$$

$$= \left(\begin{pmatrix} 1 & 1 \\ z_1 & z_2 \end{pmatrix} \begin{pmatrix} \sigma_{x_1}^{-2} & \frac{z_1}{\sigma_{x_1}^2} \\ \sigma_{x_2}^{-2} & \frac{z_2}{\sigma_{x_2}^2} \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} \sigma_{x_1}^{-2} + \sigma_{x_2}^{-2} & \frac{z_1}{\sigma_{x_1}^2} + \frac{z_2}{\sigma_{x_2}^2} \\ \frac{z_1}{\sigma_{x_1}^2} + \frac{z_2}{\sigma_{x_2}^{2}} & \frac{z_1^2}{\sigma_{x_1}^2} + \frac{z_2^2}{\sigma_{x_2}^2} \end{pmatrix}^{-1}$$

$$= \frac{1}{\left(\left(\sigma_{x_1}^{-2} + \sigma_{x_2}^{-2}\right)\left(\frac{z_1^2}{\sigma_{x_1}^2} + \frac{z_2^2}{\sigma_{x_2}^2}\right)\right) - \left(\frac{z_1}{\sigma_{x_1}^2} + \frac{z_2}{\sigma_{x_2}^2}\right)^2} \overbrace{\begin{pmatrix} \frac{z_1^2}{\sigma_{x_1}^2} + \frac{z_2^2}{\sigma_{x_2}^2} & -\frac{z_1}{\sigma_{x_1}^2} - \frac{z_2}{\sigma_{x_2}^2} \\ -\frac{z_1}{\sigma_{x_1}^2} - \frac{z_2}{\sigma_{x_2}^2} & \sigma_{x_1}^{-2} + \sigma_{x_2}^{-2} \end{pmatrix}}^{*}$$

$$= \frac{1}{\left(\frac{z_1^2}{\sigma_{x_1}^4} + \frac{z_2^2}{\sigma_{x_1}^2\sigma_{x_2}^2} + \frac{z_1^2}{\sigma_{x_2}^2\sigma_{x_1}^2} + \frac{z_2^2}{\sigma_{x_2}^4}\right) - \frac{z_1^2}{\sigma_{x_1}^4} - \frac{2 z_1 z_2}{\sigma_{x_1}^2\sigma_{x_2}^2} - \frac{z_2^2}{\sigma_{x_2}^4}} \cdot \begin{pmatrix} * \end{pmatrix}$$

$$= \frac{1}{\frac{z_2^2}{\sigma_{x_1}^2\sigma_{x_2}^2} + \frac{z_1^2}{\sigma_{x_2}^2\sigma_{x_1}^2} - \frac{2 z_1 z_2}{\sigma_{x_1}^2\sigma_{x_2}^2}} \begin{pmatrix} * \end{pmatrix}$$

$$= \frac{\sigma_{x_1}^2\sigma_{x_2}^2}{\left(z_2 - z_1\right)^2} \begin{pmatrix} \sigma_{x_1}^{-2} + \sigma_{x_2}^{-2} \checkmark & -\frac{z_1}{\sigma_{x_1}^2} - \frac{z_2}{\sigma_{x_2}^2} \checkmark \\ -\frac{z_1}{\sigma_{x_1}^2} - \frac{z_2}{\sigma_{x_2}^2} & \frac{z_1^2}{\sigma_{x_1}^2} + \frac{z_2^2}{\sigma_{x_2}^2} \checkmark \end{pmatrix}$$

NP.

$$\mathrm{cov}(a,b) = V_{12} = V_{21} = -\frac{\sigma_{x1}^2 \sigma_{x2}^2}{(z_2-z_1)^2}\left(\frac{z_1}{\sigma_{x1}^2} + \frac{z_2}{\sigma_{x2}^2}\right)$$ ✓

$$\rho = \frac{\mathrm{cov}(a,b)}{\sqrt{V_{11}}\sqrt{V_{22}}} = \frac{\sigma_{x1}^2\sigma_{x2}^2}{(z_2-z_1)^2} \frac{(z_2-z_1)^2}{\sigma_{x1}^2\sigma_{x2}^2} \, ? \quad \frac{\left(\frac{z_1}{\sigma_{x1}^2}+\frac{z_2}{\sigma_{x2}^2}\right)}{\sqrt{\sigma_{x1}^{-2}+\sigma_{x2}^{-2}} \cdot \sqrt{\frac{z_1^2}{\sigma_{x1}^2}+\frac{z_2^2}{\sigma_{x2}^2}}}$$

Ah. verstehe

$$= \frac{\frac{z_1}{\sigma_{x1}^2}+\frac{z_2}{\sigma_{x2}^2}}{\sigma_{x1}^{-1}\,\sigma_{x2}^{-1}\cdot\sqrt{\frac{z_1^2}{\sigma_{x1}^2}+\frac{z_2^2}{\sigma_{x2}^2}}}$$

$$= \frac{\left(\frac{z_1}{\sigma_{x1}^2}+\frac{z_2}{\sigma_{x2}^2}\right)\sigma_{x1}\,\sigma_{x2}}{\sqrt{\frac{z_1^2}{\sigma_{x1}^2}+\frac{z_2^2}{\sigma_{x2}^2}}}$$

ƒ

0.75P.

b) Position des Teilchens:

$$x_3 = \underbrace{\left(\frac{x_2-x_1}{z_2-z_1}\right)}_{a} z_3 + \underbrace{\frac{x_1 z_2 - x_2 z_1}{(z_2-z_1)}}_{b}$$ ✓

1P.

Fehler auf x_3:

$$\sigma_{x_3} = \sqrt{\left(\frac{\partial x}{\partial a}\sigma_a\right)^2 + \left(\frac{\partial x}{\partial b}\sigma_b\right)^2 + 2\frac{\partial x}{\partial a}\cdot\frac{\partial x}{\partial b}\cdot \mathrm{cov}(a,b)}$$

$$= \sqrt{z_3^2\cdot\left(\frac{1}{z_2-z_1}\sqrt{\sigma_{x1}^2+\sigma_{x2}^2}\right)^2 + 1\cdot\left(\frac{1}{z_2-z_1}\sqrt{z_2^2\sigma_{x1}^2+z_1^2\sigma_{x2}^2}\right)^2 + 2\cdot z_3\cdot 1\cdot\left(-\frac{\sigma_{x1}^2\sigma_{x2}^2}{(z_2-z_1)^2}\left(\frac{z_1}{\sigma_{x1}^2}+\frac{z_2}{\sigma_{x2}^2}\right)\right)}$$ ✓

1P.

c) Wenn man die Korrelation nicht berücksichtigt, fällt der letzte Summand in der obrigen Formel weg. Damit wird der Fehler größer, da

$$2z_3\cdot\left(-\frac{\sigma_{x1}^2\sigma_{x2}^2}{(z_2-z_1)^2}\left(\frac{z_1}{\sigma_{x1}^2}+\frac{z_2}{\sigma_{x2}^2}\right)\right)$$ negativ ist.

✓ 1P.

~~1P.~~

5.75 / 6P.

BRUNNEN

SMD Übung 18.12.18

Aufgabe 22

a) $y = a_0 + a_1 x$

ohne Korrelation: $\sigma_y = \sqrt{\left(\frac{\partial y}{\partial a_0}\cdot \sigma_{a_0}\right)^2 + \left(\frac{\partial y}{\partial a_1}\,\sigma_{a_1}\right)^2}$

$$\sigma_y = \sqrt{0{,}2^2 + x^2\, 0{,}2^2} = 0{,}2\sqrt{1+x^2}$$

mit Korrelation: $\sigma_y = \sqrt{\sum_{i=0}^{m}\left(\frac{\partial y}{\partial a_i}\sigma_i\right)^2 + 2\sum_{i=0}^{m-1}\sum_{k=i+1}^{m}\frac{\partial y}{\partial a_i}\frac{\partial y}{\partial a_k}\cdot \mathrm{cov}(a_i,a_k)}$

$$\mathrm{cov}(a_i,a_k) = \rho(a_i,a_k)\,\sigma_{a_i}\sigma_{a_k}$$

$$\sigma_y = \sqrt{0{,}2^2(1+x) + 2x\rho\,\sigma_{a_0}\sigma_{a_1}}$$

$$= 0{,}2\sqrt{x^2 + 2x\rho + 1}$$

Aufgabe 23 mit BuB-Formel

$\vec{y} = \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad y = B\vec{x}$

$$B = \begin{pmatrix} \frac{1}{z_1 - z_2} & \frac{-1}{z_1 - z_2} \\ \frac{-z_2}{z_1 - z_2} & \frac{z_1}{z_1 - z_2} \end{pmatrix}, \quad V(\vec{x}) = \begin{pmatrix} \sigma_{x_1}^2 & \overbrace{\mathrm{cov}(x_1,x_2)}^{0} \\ \underbrace{\mathrm{cov}(x_1,x_2)}_{0} & \sigma_{x_2}^2 \end{pmatrix}$$

$$V(\vec{y}) = \begin{pmatrix} \sigma_a^2 & \mathrm{cov}(a,b) \\ \mathrm{cov}(a,b) & \sigma_b^2 \end{pmatrix}$$

$$V(\vec{y}) = B V(\vec{x}) B^T = \begin{pmatrix} \frac{\sigma_{x_1}^2}{(z_1-z_2)^2} + \frac{\sigma_{x_2}^2}{(z_1-z_2)^2} & -\frac{\sigma_{x_1}^2 z_1}{(z_1-z_2)^2} - \frac{z_1\sigma_{x_2}^2}{(z_1-z_2)^2} \\ \underbrace{\frac{-\sigma_{x_1}^2 z_2}{(z_1-z_2)^2} - \frac{z_1\sigma_{x_2}^2}{(z_1-z_2)^2}}_{\mathrm{cov}(a,b)} & \frac{\sigma_{x_1}^2 z_2^2}{(z_1-z_2)^2} + \frac{\sigma_{x_2}^2 z_1^2}{(z_1-z_2)^2} \end{pmatrix}$$

$$\rho(a,b) = \frac{\mathrm{cov}(a,b)}{\sigma_a\sigma_b} = \frac{-z_2\sigma_{x_1}^2 - z_1\sigma_x^2}{\sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2}\sqrt{(-z_2\sigma_{x_1})^2 + (z_1\sigma_{x_2})^2}}$$