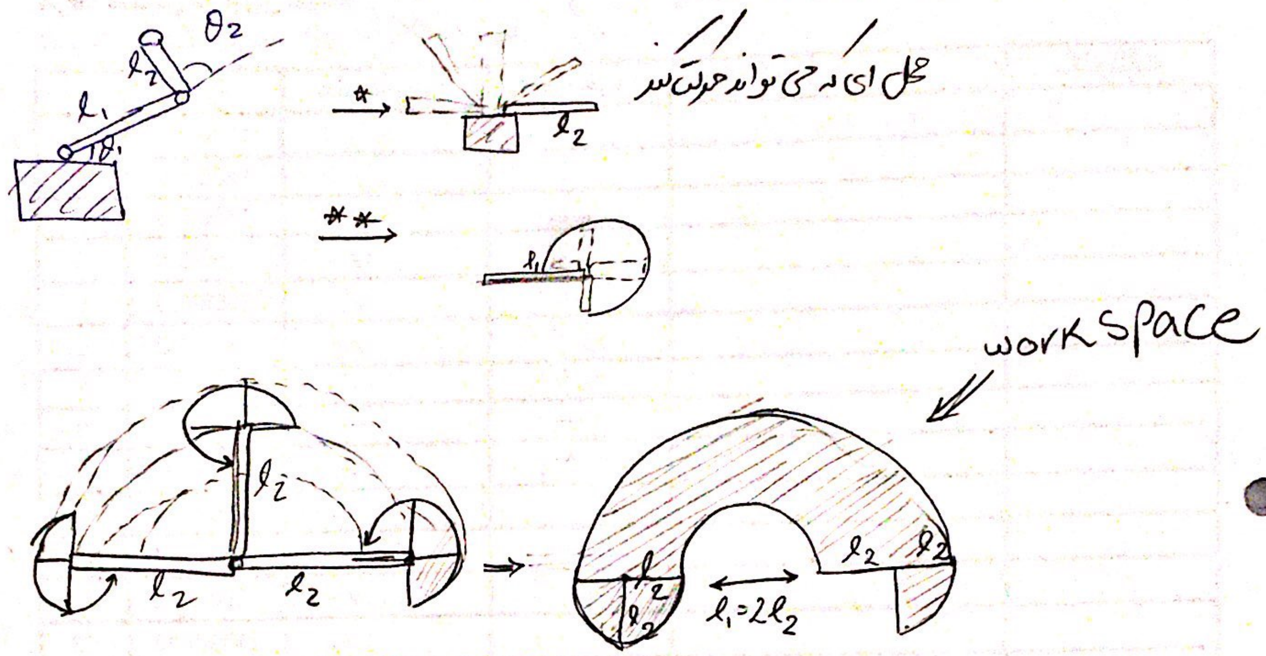
کسری خلفی   تمرین سری اول رباتیک پیشرفته دکتر شریفی

(1) $l_1 = 2l_2$, $0 \leq \theta_1 \leq 180$ (*)

$-90 \leq \theta_2 \leq 180$ (**)



(2) (a) $\|RP_1 - RP_2\| = \|P_1 - P_2\|$

for proving this: $\|RP_1 - RP_2\| = \|R\| \, \|P_1 - P_2\|$ $\Rightarrow$ so we should prove that Norm of R is $\underline{1}$

for proving: if $M \in R^{3\times3}$ $m = [\hat{a} \;\; \hat{b} \;\; \hat{c}]$

fact: $\det(m) = \hat{a}^T(\hat{b} \times \hat{c})$

$\Rightarrow R = [\hat{x} \;\; \hat{y} \;\; \hat{z}] \Rightarrow \det(R) = \|R\| = \hat{x}^T \underbrace{(\hat{y} \times \hat{z})}_{\hat{x}} = \hat{x}^T\hat{x} = \underline{1}$ Proved!

so $\Rightarrow \|RP_1 - RP_2\| = \|P_1 - P_2\|$

(b) $R^T R = I$: ← Prove

$$R = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \Rightarrow$$

if we calculate R^T & R^{-1} $\Rightarrow$ both will be equal!

so: $\boxed{R^T = R^{-1}}$

$\Rightarrow R^T R = R^{-1} R = I$

③ $R_{x,\theta}\, R_{y,\varphi}\, R_{z,\pi}\, R_{y,-\varphi}\, R_{x,-\theta}$

we Know $\boxed{R_{z,-\theta} = R_{z,\theta}^{-1}}$ $\longrightarrow$ but let's compute it directly!

$$R_{z,\pi} = \begin{bmatrix} C\pi & -S\pi & 0 \\ S\pi & C\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cos(-\varphi) = \cos\varphi$, $\sin(-\varphi) = -\sin\varphi$, $\cos(-\theta) = \cos(\theta)$, $\sin(-\theta) = -\sin(\theta)$

$\downarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}\begin{bmatrix} C\varphi & 0 & S\varphi \\ 0 & 1 & 0 \\ -S\varphi & 0 & C\varphi \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C\varphi & 0 & -S\varphi \\ 0 & 1 & 0 \\ S\varphi & 0 & C\varphi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & S\theta \\ 0 & -S\theta & C\theta \end{bmatrix}$$

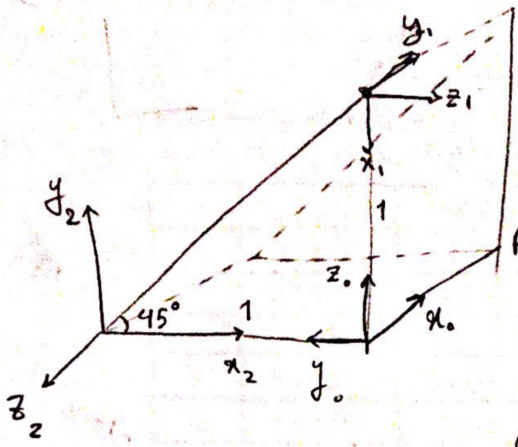
$$= \begin{bmatrix} C\varphi & 0 & S\varphi \\ (S\theta)(S\varphi) & C\theta & -(S\theta)(C\varphi) \\ -(C\theta)(S\varphi) & +S\theta & (C\theta)(C\varphi) \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C\varphi & (S\varphi)(S\theta) & -(S\varphi)(C\theta) \\ 0 & C\theta & S\theta \\ S\varphi & -(S\theta)(C\varphi) & (C\varphi)(C\theta) \end{bmatrix}$$

$$= \begin{bmatrix} -C\varphi & 0 & S\varphi \\ -(S\theta)(S\varphi) & -C\theta & (S\theta)(C\varphi) \\ (C\theta)(S\varphi) & -S\theta & (C\theta)(C\varphi) \end{bmatrix}\begin{bmatrix} C\varphi & (S\varphi)(S\theta) & -(S\varphi)(C\theta) \\ 0 & C\theta & S\theta \\ S\varphi & -(S\theta)(C\varphi) & (C\varphi)(C\theta) \end{bmatrix}$$

$$= \begin{bmatrix} S^2\varphi - C^2\varphi & -(C\varphi)(S\varphi)(S\theta) - (S\varphi)(C\varphi)(S\theta) & (C\varphi)(S\varphi)(C\theta) + (S\varphi)(C\varphi)(C\theta) \\ -(S\theta)\cancel{(S\varphi)}(C\varphi) + (S\theta)\cancel{(S\varphi)}(S\varphi) & -\underline{(S^2\theta)(S^2\varphi)} - C^2\theta - \underline{S^2\theta C^2\varphi} & (S\theta)(C\theta)(S^2\varphi) - (C\theta)(S\theta) + (S\theta)(C\theta)(C^2\varphi) \\ (C\theta)(S\varphi)(C\varphi) + (C\theta)(C\varphi)(S\varphi) & (S^2\varphi)(C\theta)(S\theta) - (S\theta)(C\theta) - (C^2\varphi)(C\theta)(S\theta) & -(C^2\theta)(S^2\varphi) - S^2\theta + (C^2\theta)(C^2\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\cos(2\varphi)}{2} & -(S\theta)(C(2\varphi)) & \cos\theta \times \sin(2\varphi) \\ 0 & -1 & 0 \\ (C\theta)(S(2\varphi)) & -\frac{\sin 2\theta}{2}(1+\cos 2\varphi) & (\cos^2\theta)(\cos 2\varphi) - S^2\theta \end{bmatrix} \longleftarrow \text{Rotation Product}$$

⑤ show that : $H_2^0 = H_1^0 H_2^1$ → we have to calculate each H and then calculate product of them



$$H = \left[\begin{array}{ccc|c} & R & & T \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

(*) H_2^0 → transfer of 1 meter in $\hat{y}$

$R_{z,-90}$ ← current Rotation z for -90 degrees

$R_{x,90}$ ← current Rotation for x axis and 90 degrees

(**) H_1^0 → transfer of 1 meter in $\hat{z}$

$R_{z,-90}$ ← current Rotation in z axis and -90 degrees

$R_{y,90}$ ← " " y axis and 90 degrees

(***) H_2^1 → transfer of 1 meter in $+\hat{x}$ and $-\hat{z}$

$R_{x,90}$ ← current Rotation x axis and 90 degrees

$R_{z,90}$ ← " " z axis and 90 degrees

$$(*) \rightarrow H_2^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-90) & -\sin(-90) & 0 & 0 \\ \sin(-90) & \cos(-90) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(90) & -\sin(90) & 0 \\ 0 & \sin(90) & \cos(90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with matlab $\begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = H_2^0$

$$(**) \rightarrow H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-90) & -\sin(-90) & 0 & 0 \\ \sin(-90) & \cos(-90) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 90 & 0 & \sin 90 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 90 & 0 & \cos 90 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

MATLAB $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = H_1^0$

$$\circledast \longrightarrow H_2^1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90 & -\sin 90 & 0 \\ 0 & \sin 90 & \cos 90 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 90 & -\sin 90 & 0 & 0 \\ \sin 90 & \cos 90 & 0 & 0 \\ 0 & 0 & 1 & - \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>MATLAB</u>

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = H_2^1$$

$$H_1^0 H_2^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = H_2^0$$

proved!

4

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

a) → show it's a rotation Matrix!

① $\det(R) \stackrel{?}{=} 1 \Rightarrow \|R\| = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{vmatrix} = \frac{1}{\sqrt{2}}\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right)$

$$= \frac{1}{\sqrt{2}}\left(\frac{2}{2\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{2}{2\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

② $R \in SO\ ? \Rightarrow$

$$\begin{array}{ccc} V_1 & V_2 & V_3 \\ \downarrow & \downarrow & \downarrow \end{array}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow V_1^T V_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = 0$$

$$\Rightarrow V_2^T V_3 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{-1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = 0$$

$$V_1^T V_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} - \frac{1}{4} - \frac{1}{4} = \frac{1}{2} - \frac{1}{2} = 0$$

$$R^{-1} = \begin{bmatrix} 0.7071 & -0.5 & -0.5 \\ 0 & 0.7071 & -0.7071 \\ 0.7071 & 0.5 & 0.5 \end{bmatrix} = R^{*}$$

with Matlab

③ $R^{-1} = R^{T}$ ✓ ✓ as shown above

④ $R^{-1} \in SO \rightarrow R^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -1/2 & -1/2 \\ 0 & \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & 1/2 \end{bmatrix}$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 \end{array}$$

$$v_1^T v_2 = \left[\tfrac{1}{\sqrt{2}} \quad 0 \quad \tfrac{1}{\sqrt{2}}\right] \begin{bmatrix} -1/2 \\ \frac{1}{\sqrt{2}} \\ 1/2 \end{bmatrix} = \frac{-1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = 0$$

$$v_1^T v_3 = \left[\tfrac{1}{\sqrt{2}} \quad 0 \quad \tfrac{1}{\sqrt{2}}\right] \begin{bmatrix} -1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix} = \frac{-1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = 0$$

$$v_2^T v_3 = \left[\tfrac{-1}{2} \quad \tfrac{1}{\sqrt{2}} \quad \tfrac{1}{2}\right] \begin{bmatrix} -1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix} = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = \frac{1}{2} - \frac{1}{2} = 0$$

⑤ each columns and rows of R is a unit vector

Norm of each columns and rows are $\underline{1}$

columns: $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{-1}{\sqrt{2}} \\ 1/2 \end{bmatrix} \Rightarrow \text{Norm} = 1 \rightsquigarrow$ for instance: $\begin{bmatrix} -1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \Rightarrow \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \boxed{1}$

rows $\begin{matrix} \left[\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}\right] \\ \left[-\frac{1}{2} \quad \frac{1}{\sqrt{2}} \quad \frac{1}{2}\right] \\ \left[-\frac{1}{2} \quad \frac{-1}{\sqrt{2}} \quad \frac{1}{2}\right] \end{matrix} \Rightarrow \text{Norm} = 1 \rightsquigarrow$ for instance: $\left[\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}\right] \Rightarrow \frac{1}{2} + \frac{1}{2} = \boxed{1}$

(4)

b) what is the Euler Parameters representing R?

for calculating Euler Angles, as written in craig we have:

$$R_{zyz} = R_z(\alpha) R_y(\beta) R_z(\gamma)$$

$$\beta = Atan2\left(\pm\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right) \Rightarrow \beta = Atan2\left(\pm\sqrt{\tfrac{1}{2}+\tfrac{1}{4}}, \tfrac{1}{2}\right) = \pm 60^\circ$$

$$\alpha = Atan2\left(\frac{r_{23}}{s\beta}, \frac{r_{13}}{s\beta}\right) \rightarrow \alpha = Atan2\left(\frac{\frac{-1}{\sqrt{2}}}{\pm\frac{\sqrt{3}}{2}}, \frac{-\frac{1}{2}}{\pm\frac{\sqrt{3}}{2}}\right) = 54^\circ \text{ or } -125^\circ$$

$$\gamma = Atan2\left(\frac{r_{32}}{s\beta}, \frac{-r_{31}}{s\beta}\right) \Rightarrow \gamma = Atan2\left(\frac{\frac{1}{2}}{\pm\frac{\sqrt{3}}{2}}, \frac{\frac{1}{\sqrt{2}}}{\pm\frac{\sqrt{3}}{2}}\right) = 144^\circ \text{ or } 35^\circ$$

Also as written in craig

In terms of equivalent axis, $\hat{K} = [K_x \ K_y \ K_z]^T$ and equivalent angle θ

$$(*)\begin{cases} \epsilon_1 = K_x \sin\frac{\theta}{2} \\ \epsilon_2 = K_y \sin\theta/2 \\ \epsilon_3 = K_z \sin\frac{\theta}{2} \\ \epsilon_4 = \cos\frac{\theta}{2} \end{cases} \xrightarrow{\text{and}} \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$$

$$\theta = A\cos\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) = A\cos\left(\frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{1}{2}-1}{2}\right)$$

$$= A\cos\left(\frac{4-\sqrt{2}}{4\sqrt{2}}\right) = \boxed{62^\circ} = \theta$$

$$\text{and } \hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{\sqrt{2}} \\ -\frac{1}{2}-\frac{1}{\sqrt{2}} \\ 0+\frac{1}{2} \end{bmatrix} \frac{1}{2\sin 60}$$

$\frac{1}{\sqrt{3}}$

$$= \left[\frac{1.2}{\sqrt{3}} \ \frac{-1.2}{\sqrt{3}} \ \frac{0.5}{\sqrt{3}}\right]^T \simeq [0.7 \ -0.7 \ 0.3]^T$$

Euler Parameters $\xrightarrow{(*)}$ $\epsilon_1 = 0.7 \times \frac{1}{2} \simeq 0.35$ & $\epsilon_2 = -0.7 \times \frac{1}{2} \simeq -0.35$

& $\epsilon_3 = 0.3 \times \frac{1}{2} \simeq 0.15$ & $\epsilon_4 \simeq \frac{\sqrt{3}}{2} = 0.866$