1 Introduction

2 Preliminaries/Index

Throughout this thesis we will be using the language of category theory, for which a mild introduction is in order. To that degree we plan to define the terms used as well as prove some well known theorems which we use quite freely, so not to distract us from the argument.

Definition 2.1. *Adjunction*

Definition 2.2 (Source and sink)**.** *A* ***source*** *is a pair* $(Y, (f_i))$ *consisting of an object* Y *of a category* $\mathcal{C}$*, and a family of morphisms* $(Y \xrightarrow{f_i} X_i)_{i \in I}$ *over a class* I*. Equivalently it is a family of objects in the under category* $\mathcal{C}_{Y/}$*.*

If I *is a finite index set* $\{1, \cdots, n\}$*, we call our source an* n***-source****.*

The dual concept to a source is called a ***sink****.*

Definition 2.3 (Concrete Category)**.** *Let* $\mathcal{X}$ *be a category. A* ***concrete category*** *over* $\mathcal{X}$ *is a pair* $(\mathcal{A}, U)$ *where* $\mathcal{A}$ *is a category and* $U : \mathcal{A} \to \mathcal{X}$ *is a faithful functor. Sometimes* U *is called the* ***underlying functor*** *over* $\mathcal{X}$ *and* $\mathcal{X}$ *is called the* ***base category*** *for* $(\mathcal{A}, U)$*. A concrete category over* Set *is called a* ***construct****.*

Notes on free objects and the free functor

It is important to note that in some category $\mathcal{A}$ with a representing object A_0 that is a free generator on one free object, it holds that $\mathcal{A}(A_0, A) \cong A$ on the level of set, as the morphisms are uniquely determined by where they send the free generator to.

More concretely, given a forgetful functor $\tilde{U} : \mathcal{A} \to \mathrm{Set}$, we have $\mathcal{A}(A_0, A) = \tilde{U}(A)$. This due to how we construct A_0 and B_0 using the the left adjoint functor $\tilde{F} : \mathrm{Set} \to \mathcal{A}$ to the forgetful functor, which is called the free functor.

In general, such a functor does not have to exist, and we can still get the construction by defining A_0 as the left adjoint object of $\tilde{U}$ at $\{pt\} \in \mathrm{Set}$, however, we will assume that such a functor does exist for purposes of explication: $A_0 = \tilde{F}(\{\mathrm{pt}\})$. Such an assumption is fair since the properties of the adjunction by definition of the left adjoint object will still hold (in particular the middle equality of the equation below), regardless of existence of the functor as a whole:

$$\mathcal{A}(A_0, A) = \mathcal{A}(\tilde{F}(\{\mathrm{pt}\}), A) = \mathcal{B}(\{pt\}, \tilde{U}(A)) = \tilde{U}(A).$$

The left adjoint object to the forgetful functor has the following universal property (TODO).

In the examples which are to be discussed in this thesis, we only need to check that such a left adjoint object does exist in the category, and we show what they are and that they satisfy the universal property.

3 Notes: Concrete Dualities

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories with the following representable functors with representing objects $A_0 \in \mathcal{A}$, and $B_0 \in \mathcal{B}$. We assume these are free objects on one free generator, which we will later make precise.

$$\mathcal{A}(A_0, -) \cong U : \mathcal{A} \to \text{Set}$$
$$\mathcal{B}(B_0, -) \cong V : \mathcal{B} \to \text{Set}$$

Let $T : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{A}$ be contravariant functors with natural transformations $\eta : 1_{\mathcal{B}} \to TS$ and $\epsilon : 1_{\mathcal{A}} \to ST$. We can view this as the following *dual adjunction*

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & \mathcal{B}^{op} \arrow[l, bend left, "S"] \end{tikzcd}$$

When these natural transformations are natural isomorphisms we are speaking of a *dual equivalence.*

Like any adjunction we have the following triangle equalities

$$T\epsilon_A \circ \eta_{TA} = 1_{TA} \qquad \text{and} \qquad S\eta_B \circ \epsilon_{SB} = 1_{SB}$$

and Hom set isomorphisms

$$\mathcal{A}(A, SB) \cong \mathcal{B}(B, TA) \cong \mathcal{B}^{op}(TA, B)$$

Given some $\mathcal{A}$-morphism $f : A \to A'$, consider the map $Uf : UA \to UA'$. More explicitly, this is

$$\mathcal{A}(A_0, -)(f) : \mathcal{A}(A_0, A) \stackrel{f\circ(-)}{\to} \mathcal{A}(A_0, A')$$

We shorten this notation by using $[f] : [A] \to [A']$.

The following pair of objects will be of central importance to this these, which are defined as the following:

$$\tilde{A} := SB_0 \qquad \tilde{B} := TA_0.$$

From these characteristics we can deduce how S and T should be defined, to which a few lemmas will illuminate the bigger picture of our situation.
Lemma 3.1.

$$VT \cong \mathcal{A}(-, \tilde{A}) \qquad US \cong \mathcal{B}(-, \tilde{B})$$

Proof. As presheafs, V and VT (respectively U and US) may be computed pointwise.

$$\begin{aligned} VT(A) = \mathcal{B}(B_0, TA) = \mathcal{A}(A, SB_0) = \mathcal{A}(A, \tilde{A}) \\ \implies VT \cong \mathcal{A}(-, \tilde{A}) \end{aligned}$$

$$\begin{aligned} US(B) = \mathcal{A}(A_0, SB) = \mathcal{B}(B, TA_0) = \mathcal{B}(B, \tilde{B}) \\ \implies US \cong \mathcal{B}(-, \tilde{B}) \end{aligned}$$

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Given our assumption that A_0 and B_0 are free objects on one free generator, this result should already give us an idea of how our adjunction is to be induced, which the goal of the whole introduction to concrete dualities is to make precise.

However the broad strokes of it uses the free-forget adjunction ($\bar{F} \dashv \bar{U}$) to show via

$$\mathcal{A}(A, \tilde{A}) = VT(A) = \mathcal{B}(B_0, TA) = \bar{U}(TA)$$

that the underlying sets of T and S respectively, are $\mathrm{Hom}_A(-, \tilde{A})$ and $\mathcal{B}(-, \tilde{B})$. [1]

Lemma 3.2. $V\tilde{B} \cong U\tilde{A}$

Proof.

$$\begin{aligned} V\tilde{B} \overset{Def.V}{=} \mathcal{B}(B_0, \tilde{B}) \overset{Def.\tilde{B}}{=} \mathcal{B}(B_0, TA_0) \\ \overset{Adjunction}{\cong} \mathcal{A}(A_0, SB_0) \overset{Def.\tilde{A}}{=} \mathcal{A}(A_0, \tilde{A}) \overset{Def.U}{=} U\tilde{A} \end{aligned}$$

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The aim of the following will be to show how the objects $\tilde{A}$ and $\tilde{B}$ actually induce the adjunction $T \dashv S$, while showing that the units and counits of the adjunction are given *by evaluation*, that is

$$\begin{aligned} \epsilon_A(x) : \mathcal{A}(A, \tilde{A}) &\to \tilde{B} & \eta_B(y) : \mathcal{B}(B, \tilde{B}) &\to \tilde{A} \\ f &\mapsto f(x) & g &\mapsto g(y). \end{aligned}$$

In the following we define the canonical "evaluation" maps, $\varphi_{A,x}$ and $\psi_{B,y}$ and the canonical bijections τ and σ:

$$\begin{aligned} \varphi_{A,x} : \mathcal{A}(A, \tilde{A}) &\to [\tilde{A}] & \psi_{B,y} : \mathcal{B}(B, \tilde{B}) &\to [\tilde{B}] \\ s &\mapsto [s](x) & t &\mapsto [t](y) \end{aligned}$$

$$\begin{aligned} \tau : [\tilde{A}] &\to [\tilde{B}] & \sigma : [\tilde{B}] &\to [\tilde{A}] \\ \tilde{x} &\mapsto [[\epsilon_{\tilde{A}}](\tilde{x})](1_{\tilde{A}}) & \tilde{y} &\mapsto [[\eta_{\tilde{B}}](\tilde{y})](1_{\tilde{B}}) \end{aligned}$$

[1] In the course of this thesis we will often prove results about $\mathcal{A}$, respectively $T : \mathcal{A} \to \mathcal{B}$ with unit $\epsilon : 1_{\mathcal{A}} \to ST$, from which the results about $\mathcal{B}$, respectively $S : \mathcal{B} \to \mathcal{A}$ with counit $\eta : 1_{\mathcal{B}} \to TS$, follow completely analogously. Unless we state that results do not follow analogously, we assume this to be the case.

which evaluate the maps $[s]$ and $[t]$ at x and y respectively:

$$\begin{aligned}[s] : [A] &\to [\tilde{A}] \\ x &\mapsto [s](x)\end{aligned} \qquad\qquad \begin{aligned}[t] : [B] &\to [\tilde{B}] \\ y &\mapsto [t](y)\end{aligned}$$

as for any $s \in \mathcal{A}(A, \tilde{A})$, we have the induced map $[s] : [A] \to [\tilde{A}]$.

So for every $x \in \mathcal{A}(A_0, A)$ we have the following diagram.

$$\begin{tikzcd}
s \arrow[r, mapsto] & {[s](x)} \arrow[r, mapsto] & \tau([s](x)) \\
\mathcal{A}(A, \tilde{A}) \arrow[r, "\varphi_{A,x}"] & \mathcal{A}(A_0, \tilde{A}) \arrow[r, "\tau"] \arrow[d, "\cong"] & \mathcal{B}(B_0, \tilde{B}) \arrow[d, "\cong"] \\
& \mathcal{A}(A_0, SB_0) \arrow[r] & \mathcal{B}(B_0, TA_0)
\end{tikzcd}$$

By the dual adjunction we have $\mathcal{A}(A, SB_0) \cong \mathcal{B}(B_0, TA)$.

Lemma 3.3. *τ and σ are inverses, and the following identities hold:*

$$[[\epsilon_A](x)] = \tau\varphi_{A,x} \qquad\qquad [\eta_B](y)] = \sigma\psi_{B,y}$$

Proof. First we check the identities, as understanding them will help us prove that τ and σ are inverses. We only check the left identity, and as the right identity will follow analogously.

First we have $\tau\varphi_{A,x}(s) = \tau([s](x))$ by definition of $\varphi_{A,x}$. But then by definition of τ we have $\tau([s](x)) = [[\epsilon_{\tilde{A}}][s](x)](1_{\tilde{A}})$.

Since ϵ is a natural transformation, we have the following commutative diagram for all $A, A' \in \mathcal{A}$ such that there exists a map $A \to A'$. In particular, given $s : A \to \tilde{A}$ we have:

$$\begin{tikzcd}
1_{\mathcal{A}}(A) \arrow[r, "\epsilon_A"] \arrow[d, "s"'] & ST(A) \arrow[d, "STs"] \\
1_{\mathcal{A}}(\tilde{A}) \arrow[r, "\epsilon_{\tilde{A}}"'] & ST(\tilde{A})
\end{tikzcd}$$

so that $[[\epsilon_{\tilde{A}}][s](x)](1_{\tilde{A}}) = [[STs][\epsilon_A](x)](1_{\tilde{A}})$.

By Lemma 3.1, we know that $[STs] = US(Ts) = \mathcal{B}(Ts, \tilde{B})$, which is just a map $\mathcal{B}(TA, \tilde{B}) \to \mathcal{B}(T\tilde{A}, \tilde{B})$, induced by $Ts : T\tilde{A} \to TA$. *Notice that $US(-)$ and $T(-)$ are both contravariant, so that $UST(-)$ is covariant.*

Now as $[\epsilon_A](x) \in \mathcal{B}(TA, \tilde{B})$, we have the induced post composition

$$\begin{tikzcd} T\tilde{A} \arrow[r, dashed, "{[\epsilon_A](x)\circ Ts}"] \arrow[d, "Ts"'] & \tilde{B} \\ TA \arrow[ur, "{[\epsilon_A](x)}"'] & \end{tikzcd}$$

which can be otherwise phrased as a right action of Ts on $[\epsilon_A](x)$ so that

$$[[STs][\epsilon_A](x)](1_{\tilde{A}}) = [[\epsilon_A](x) \circ Ts](1_{\tilde{A}}). \tag{1}$$

From Lemma 3.1 we know that $VT = \mathcal{A}(-, \tilde{A})$ so that we get the induced diagram

$$\begin{tikzcd} \mathcal{A}(\tilde{A}, \tilde{A}) \arrow[r, dashed, "{[[\epsilon_A](x)\circ Ts]}"] \arrow[d, "{[Ts]}"'] & \tilde{B} \\ \mathcal{A}(A, \tilde{A}) \arrow[ur, "{[[\epsilon_A](x)]}"'] & \end{tikzcd}$$

Notice that then $[Ts]$ becomes an evaluation of the precomposition functor $\mathcal{A}(-, \tilde{A})$ at s, i.e. $[Ts] = - \circ s$, which sends $1_{\tilde{A}} \mapsto s$. Therefore it holds that

$$[[\epsilon_A](x) \circ Ts](1_{\tilde{A}}) = [[\epsilon_A](x)](s).$$

All together we have

$$\begin{aligned} \tau \circ \varphi_{A,x}(s) =& \tau([s](x)) \\ =& [[\epsilon_{\tilde{A}}][s](x)](1_{\tilde{A}}) \\ =& [[STs][\epsilon_A](x)](1_{\tilde{A}}) \\ =& [[\epsilon_A](x) \circ Ts](1_{\tilde{A}}) \\ =& [[\epsilon_A](x)](s) \end{aligned}$$

which gives us the desired identity.

Now we check that τ and σ are inverses.

The above identity gives us the particular instance

$$\tau\varphi_{S\tilde{B},1_{\tilde{B}}}(s) = [[\epsilon_{S\tilde{B},\tilde{y}}](1_{\tilde{B}})](s)$$

for all $s \in \mathcal{A}(S\tilde{B}, \tilde{A})$.

For $s = [\eta_{\tilde{B}}](\tilde{y})$ with $\tilde{y} \in [\tilde{B}]$, noticing the maps

$$[\eta_{\tilde{B}}](\tilde{y}) : S\tilde{B} \to \tilde{A} \qquad \text{and} \qquad \varphi_{S\tilde{B},1_{\tilde{B}}} : \mathcal{A}(S\tilde{B}, \tilde{A}) \to [\tilde{A}]$$
$$1_{\tilde{B}} \mapsto 1_{\tilde{B}}(\tilde{y}) \qquad\qquad s \mapsto [s](1_{\tilde{B}})$$

we see that $[[\eta_{\tilde{B}}](\tilde{y})](1_{\tilde{B}}) = \varphi_{S\tilde{B},1_{\tilde{B}}}(s)$ and so we have

$$\begin{aligned}\tau\sigma(\tilde{y}) =& \tau([[\eta_{\tilde{B}}](\tilde{y})](1_{\tilde{B}})) \\ =& \tau(\varphi_{S\tilde{B},1_{\tilde{B}}}(s)) \\ =& [[\epsilon_{S\tilde{B}}](1_{\tilde{B}})][\eta_{\tilde{B}}](\tilde{y}) = \tilde{y}.\end{aligned}$$

We only need to show the last equality.

Consider the triangle equality $S\eta_{\tilde{B}} \circ \epsilon_{S\tilde{B}} = 1_{S\tilde{B}}$ which induces $[S\eta_{\tilde{B}}][\epsilon_{S\tilde{B}}] = 1_{[S\tilde{B}]}$.

We can extrapolate from (1) using the result $US = \mathcal{B}(-, \tilde{B})$ from Lemma 3.1 that for some $A \in \mathcal{A}$, $x \in A$, $B \in \mathcal{B}$ and $f \in \mathcal{B}(B, TA)$ the left action of $[Sf]$ on $[\epsilon_A](x) \in \mathcal{B}(TA, \tilde{B})$ becomes a right action of f on $[\epsilon_A](x)$, i.e.

$$[Sf][\epsilon_A](x) = [\epsilon_A](x) \circ f \in \mathcal{B}(B, \tilde{B}).$$

Therefore

$$1_{\tilde{B}} = 1_{\mathcal{B}(\tilde{B},\tilde{B})}(1_{\tilde{B}}) = 1_{[S\tilde{B}]}(1_{\tilde{B}}) = [S\eta_{\tilde{B}}][\epsilon_{S\tilde{B}}](1_{\tilde{B}}) = [\epsilon_{S\tilde{B}}](1_{\tilde{B}}) \circ \eta_{\tilde{B}}.$$

In other words, the induced map is the identity on $[\tilde{B}]$:

$$[\epsilon_{S\tilde{B}}](1_{\tilde{B}})][\eta_{\tilde{B}}] = 1_{[\tilde{B}]}.$$

Thus we have $\tau\sigma = 1_{[\tilde{B}]}$ as desired.

∎

Lemma 3.3 makes precise the notion that our unit and counit are given "by evaluation", given

$$[[\epsilon_A](x)] : \mathcal{A}(A, \tilde{A}) \to [\tilde{B}] \qquad [[\eta_B](y)] : \mathcal{B}(B, \tilde{B}) \to [\tilde{A}]$$
$$f \mapsto f(x) \qquad\qquad g \mapsto g(y)$$

via canonical bijections τ and σ. From here we can (TODO: deduce or allude to?) deduce the fact that $T = \mathcal{A}(-, \tilde{A})$ and $S = \mathcal{B}(-, \tilde{B})$, as τ and σ send the precise evaluation maps to the other pairing, and as such we can view the unit and counit as such:

$$\epsilon_A : A \to \mathcal{B}(\mathcal{A}(A, \tilde{A}), \tilde{B}) \qquad \eta_B : B \to \mathcal{A}(\mathcal{B}(B, \tilde{B}), \tilde{A})$$
$$x \mapsto (f \to f(x)) \qquad\qquad y \mapsto (g \to g(y))$$

4 Schizophrenic Objects

What we want for our set up is that the composition of these maps induces a lift, i.e. for every $A \in \mathcal{A}$ and $x \in A$ there exists an $f \in \mathcal{B}(TA, \tilde{B})$, that induces $[f] : [TA] \to [\tilde{B}]$, such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{B}(B_0, TA) \arrow[d, dashed] \\
\mathcal{A}(A, \tilde{A}) \arrow[ur, "\cong"] \arrow[r, "\tau\circ(\varphi_{A,x})"] & \mathcal{B}(B_0, \tilde{B})
\end{tikzcd}$$

This lends itself to a precise definition of the *schizophrenic object*, which is the central notion of this thesis:

Definition 4.1. *A triple $(\tilde{A}, \tau, \tilde{B})$ with a pair of objects $(\tilde{A}, \tilde{B}) \in \mathcal{A} \times \mathcal{B}$ and a bijective map $\tau : [\tilde{A}] \to [\tilde{B}]$ is called a* schizophrenic object *(for concrete categories $\mathcal{A}$ and $\mathcal{B}$) if the following conditions are satisfied:*

SO1. For every $A \in \mathcal{A}$ the family $(\tau\varphi_{A,x} : \mathcal{A}(A, \tilde{A}) \to [\tilde{B}])_{x\in[A]}$ admits a V-initial lifting $(e_{A,x} : TA \to \tilde{B})_{x\in[A]}$

SO2. For every $B \in \mathcal{B}$ the family $(\sigma\psi_{B,y} : \mathcal{B}(B, \tilde{B}) \to [\tilde{A}])_{y\in[B]}$ admits a U-initial lifting $(d_{B,y} : SB \to \tilde{A})_{y\in[B]}$

What these conditions actually mean is two-fold: Firstly, the V-structured lifting property yields the existence of a $\mathcal{B}$-morphism $e_{A,x} \in \mathcal{B}(TA, \tilde{B})$ for every $A \in \mathcal{A}$ and $x \in A$, such that $[TA] = \mathcal{A}(A, \tilde{A})$ and $[e_{A,x}] = \tau\varphi_{A,x}$.

But secondly, the V-initiality means that for any $Z \in \mathcal{B}$ and a map $h : [Z] \to [TA]$, if all composite maps $\tau\varphi_{A,x} \circ h$ are the underlying-set maps for $\mathcal{B}$-morphisms in $\mathcal{B}(Z, \tilde{B})$, then there exists a unique $\mathcal{B}$-morphism in $\mathcal{B}(Z, TA)$ whose underlying set map is h.

In other words, the V-structured lift is initial among all such lifts: if Z is any other $\mathcal{B}$-object whose underlying set maps into $\mathcal{A}(A, \tilde{A})$ in a way that is compatible with all $\tau\varphi_{A,x}$ composites, then that map factors uniquely through TA in $\mathcal{B}$.

$$\begin{tikzcd}
 & & {[TA]} \arrow[d, "{[e_{A,x}]}"] \\
{[Z]} \arrow[r, "h"'] & \mathcal{A}(A, \tilde{A}) \arrow[ur] \arrow[r, "\tau\varphi_{A,x}"'] & {[\tilde{B}]}
\end{tikzcd}$$

We now show a central theorem to this thesis.

Theorem 4.1. *Every schizophrenic object $(\tilde{A}, \tau, \tilde{B})$ induces a natural dual adjunction strictly represented by $(\tilde{A}, \tilde{B})$ (TODO: define strictly represented), such that τ and $\sigma = \tau^{-1}$ are the canonical bijections defined in the previous section.*

Proof. First we show that T and S are well defined functors. The conditions (SO1.) and (SO2.) show us how T and S act on objects up to underlying-set isomorphism.

Now we show how T acts on morphisms. To that effect, given some $f : A \to A'$, we seek to show the existence of $Tf : TA' \to TA$ whose underlying set map is $[Tf] = \mathcal{A}(f, \tilde{A}) : \mathcal{A}(A', \tilde{A}) \to \mathcal{A}(A, \tilde{A})$. As we have just seen, by $(SO1)$ it suffices to show that $\tau\varphi_{A,x} \circ \mathcal{A}(f, \tilde{A})$ are the underlying set maps of $\mathcal{B}$-morphisms in $\mathcal{B}(TA', \tilde{B})$.

Considering that $[Tf]$ is simply the precomposition map $- \circ f$, we see that given some $s \in \mathcal{A}(A', \tilde{A})$, it holds that $\tau\varphi_{A,x} \circ \mathcal{A}(f, \tilde{A})(s) = \tau\varphi_{A,x}(sf)$. But since $[sf](x) = [s][f](x)$, where $[f](x) \in [A']$, we have $\tau\varphi_{A,x}(sf) = \tau\varphi_{A',[f](x)}(s) = [e_{A',[f](x)}](s)$, which is the underlying set map of a $\mathcal{B}$-morphism in $\mathcal{B}(TA', \tilde{B})$ by definition, and which exists by the lifting property given by $(SO1.)$.

Therefore T and S are well-defined functors, where preservation of the identity and composition are derived from the underlying set-map given by precomposition (TODO: is identity and composition actually clear because of this? ties to main question of why we can derive functor properties from underlying set map properties)

Now we show that T and S are adjoint, and to do that we shall construct unit and counit maps ϵ and η. In order to establish the existence of η_B by playing the same game we first define $[\eta_B] : [B] \to [TSB]$ and show that each $\tau\varphi_{SB,t} \circ [\eta_B]$ with $t \in [SB]$, can be lifted along V, i.e. that it is the underlying set map of a $\mathcal{B}$-morphism in $\mathcal{B}(B, \tilde{B})$. So we define in light of Lemma 3.3 under (SO2.)

$$\begin{aligned} [\eta_B] : [B] &\to \mathcal{A}(SB, \tilde{A}) \\ y &\mapsto d_{B,y}. \end{aligned}$$

Then by definitions and (SO2.) we have

$$\begin{aligned} \tau\varphi_{SB,t} \circ [\eta_B](y) &= \tau\varphi_{SB,t}(d_{B,y}) \\ &= \tau[d_{B,y}](t) \\ &= \tau\sigma\psi_{B,y}(t) \\ &= [t](y) \end{aligned}$$

which shows that $\tau\varphi_{SB,t} \circ [\eta_B] : [B] \to [\tilde{B}]$ is the underlying set map of a $\mathcal{B}$-morphism in $\mathcal{B}(B, \tilde{B})$, proving the existence of η_B.

Furthermore we see that

$$e_{SB,t} \circ \eta_B = t \quad \text{for all } t \in [SB] = \mathcal{B}(B, \tilde{B}) \tag{2}$$

To verify naturality, we verify that given a $\mathcal{B}$-morphism $f : B \to B'$, that we have $f \circ [\eta_B] = [TSf] \circ [\eta_B]$ (TODO: can we simply use the left action becomes right action becomes left action? Or is this what I only have by proving the way i did below?), or in other words:

$$d_{B',f \circ y} = [TSf] \circ d_{B,y} \tag{3}$$

Remember that the left action of $[TSf]$ on $d_{B,y}$ is a right action of Sf on $d_{B,y}$. But the underlying map of $d_{B,y} \circ Sf$ is $\sigma\psi_{B,y} \circ \mathcal{B}(f, \tilde{B})$ which is, up to the bijection σ, just the evaluation map of a morphism in $\mathcal{B}(B, \tilde{B})$ at some $y \in [B]$ precomposed with $f \in \mathcal{B}(B, B')$, which yields the evaluation map of a morphism in $\mathcal{B}(B', B)$ at $f \circ y \in [B']$. In other words $[d_{B,y}][Sf] = [\sigma\psi_{B',f\circ y}] = [d_{B',f\circ y}]$, which is clearly the underlying set map of $d_{B',f\circ y}$, giving us (3).

The definition of S gives us that $[S\eta_B][\epsilon_{SB}](t) = \mathcal{B}(\eta_B, \tilde{B})(e_{SB,t})$ and by (2) we have $\mathcal{B}(\eta_B, \tilde{B})(e_{SB,t}) = e_{SB,t} \circ \eta_B = t$. Since U is faithful, any map $[Sf] \in \mathrm{Set}([SB'], [SB])$ is the underlying-set map to a unique map $Sf \in \mathcal{A}(SB', SB)$, and we deduce that $[S\eta_B][\epsilon_{SB}] = 1_{[SB]}$ is the underlying set map of the triangle identity $S\eta_B \circ \epsilon_{SB} = 1_{SB}$.

Finally, to show that τ is induced by this adjunction it suffices to see that it maps $\tilde{x} \mapsto [[\epsilon_{\tilde{A}}](\tilde{x})](1_{\tilde{A}})$ as desired. For every $\tilde{x} \in [\tilde{A}]$ we have

$$[[\epsilon_{\tilde{A}}](\tilde{x})](1_{\tilde{A}}) = \tau\varphi_{\tilde{A},\tilde{x}}(1_{\tilde{A}}) = \tau([1_{\tilde{A}}](\tilde{x})) = \tau(\tilde{x}).$$

■

5 Motivation

We are now in the position to understand what this schizophrenic object really affords us, however first we will explicate what that is by saying a few words about motivation that will hopefully make the above seemingly abstract definition of the schizophrenic object more understandable.

First remember that there is a bijection between $[\tilde{A}]$ and $[\tilde{B}]$, since

$$[\tilde{A}] = \mathcal{A}(A_0, SB_0) = \mathrm{Hom}_B(B_0, TA_0) = [\tilde{B}].$$

In particular, we have shown that τ and σ are necessarily such bijections and inverses of one another.

What is interesting about the schizophrenic object is that the object itself defines the adjunction via Hom sets in our respective categories. In other words, our question, in general, is that given a schizophrenic object $(\tilde{A}, \tau, \tilde{B})$ do the sets $\mathrm{Hom}_A(A, SB_0)$ have $\mathcal{B}$ structure for every $A \in \mathcal{A}$ and $\mathcal{B}(B, TA_0)$ have $\mathcal{A}$ structure for every $B \in \mathcal{B}$.

More explicitly, do we have lifts via the forgetful functor, such that the following diagrams commute for every $A \in \mathcal{A}$ and for every $B \in \mathcal{B}$?

$$\begin{tikzcd} & \mathcal{B}^{op} \arrow[d, "\tilde{U}(-)"] \\ \mathcal{A} \arrow[ur, dashed, "\exists"] \arrow[r, "\mathcal{A}(-{,}\tilde{A})"'] & \text{Set} \\ A \arrow[r, mapsto] & \mathcal{A}(A,\tilde{A}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathcal{A}^{op} \arrow[d, "\tilde{U}(-)"] \\ \mathcal{B} \arrow[ur, dashed, "\exists"] \arrow[r, "\mathcal{B}(-{,}\tilde{B})"'] & \text{Set} \\ B \arrow[r, mapsto] & \mathcal{B}(B,\tilde{B}) \end{tikzcd}$$

If such lifts exist, then by construction, we will have the following adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "T:=\mathcal{A}(-{,}\tilde{A})"] \arrow[r, phantom, "\perp"] & \mathcal{B}^{op} \arrow[l, bend left, "S:=\mathcal{B}(-{,}\tilde{B})"] \end{tikzcd}$$

To understand this adjunction and see that it does indeed correspond to the morphisms explained in the first section, we will go through several examples.

6 Leading example

We begin with the adjunction

$$\begin{tikzcd} \text{Frm} \arrow[r, bend left, "\mathrm{T} = \mathrm{Frm}(-{,}\ \mathbb{S})"] \arrow[r, phantom, "\perp"] & \text{Top}^{op} \arrow[l, bend left, "\mathrm{S} = \mathrm{Top}(-{,}\ 2)"] \end{tikzcd}$$

with the following unit and counit

$$\begin{aligned} \tau :&[2] \to [\mathbb{S}] \\ &\tilde{x} \to [[\epsilon_2](\tilde{x})](1_2) \end{aligned} \qquad \begin{aligned} \sigma :& [\mathbb{S}] \to [2] \\ &\tilde{y} \mapsto [[\eta_{\mathbb{S}}](\tilde{y})](1_{\mathbb{S}}) \end{aligned}$$

In this setting the respective representable objects are $A_0 = a$ and $B_0 = \{pt\}$, where a is here notation for the free frame generated by one object, i.e., the free 3-chain $\perp - a - \top$.

Now it is easy to see that $TA_0 = \mathrm{Frm}(a, 2) = 2$ and $SB_0 = \mathrm{Top}(\{pt\}, \mathbb{S}) = \mathbb{S}$, however it is not easy to see in general why $\mathrm{Frm}(A, 2) \in \mathrm{Top}$ and $\mathrm{Top}(B, \mathbb{S}) \in \mathrm{Frm}$.

Given that $\mathbb{S}$ is a topology on the set $\{0, 1\}$, we endow $\mathrm{Top}(B, \mathbb{S})$ with an order $u \leq v \iff u(x) \leq v(x)$ for all $x \in B$ given a pointwise order in 2.

For a topology on $\mathrm{Frm}(A, \mathbb{2})$, we consider the family

$$\{\ \{p \in \mathrm{Frm}(A, \mathbb{2}) \mid p(x) = 1\} \mid x \in A\ \}$$

Our adjunction has the unit

$$\begin{aligned}\epsilon :& 1_{\mathrm{Frm}} \to ST\\ & A \mapsto STA\\ & A \mapsto \mathrm{Top}(\mathrm{Frm}(A, \mathbb{2}), \mathbb{S})\end{aligned}$$

so that

$$\epsilon_A : A \to \mathrm{Top}(\mathrm{Frm}(A, \mathbb{2}), \mathbb{S})$$

where

$$\begin{aligned}\epsilon_A(x) : & \mathrm{Frm}(A, \mathbb{2}) \to \mathbb{S}\\ & p \mapsto p(x)\end{aligned}$$

Notice that the topology on $\mathrm{Frm}(A, \mathbf{2})$ is the initial topology making all $(\epsilon_A(x))_{x\in A}$ continuous.

When passing to U and V, we have

$$\begin{aligned}[\epsilon_{\tilde{A}}] :& [\tilde{A}] \to [\mathrm{Top}(\mathrm{Frm}(\tilde{A}, \mathbb{2}), \mathbb{S})] = \mathrm{Top}(\mathrm{Frm}(\tilde{A}, \mathbb{2}), \mathbb{S})\\ & (\tilde{x} : a \to x) \mapsto (p \to p(x))\end{aligned}$$

which is equal to

$$[\epsilon_{\mathbb{2}}] : [\mathbb{2}] \to [\mathrm{Top}(\mathrm{Frm}(\mathbb{2}, \mathbb{2}), \mathbb{S})] = \mathrm{Top}(\mathrm{Frm}(\mathbb{2}, \mathbb{2}), \mathbb{S}) \tag{4}$$
$$(\tilde{x} : a \to x) \mapsto (1_{\mathbb{2}} \to 1_{\mathbb{2}}(x)) \tag{5}$$

So we have

$$\begin{aligned}[\epsilon_{\mathbb{2}}](\tilde{x}) : & \mathrm{Frm}(\mathbb{2}, \mathbb{2}) \to \mathbb{S}\\ & 1_{\mathbb{2}} \mapsto 1_{\mathbb{2}}(x)\end{aligned}$$

and now

$$\begin{aligned}[[\epsilon_{\mathbb{2}}](\tilde{x})] :& [\mathrm{Frm}(\mathbb{2}, \mathbb{2})] \to [\mathbb{S}]\\ & (a \to 1_{\mathbb{2}}) \mapsto (\{pt\} \to 1_{\mathbb{2}}(x))\end{aligned}$$

so that

$$[[\epsilon_{\mathbb{2}}](\tilde{x})](1_{\mathbb{2}}) = (\{pt\} \to 1_{\mathbb{2}}(x)).$$

Now we can see that the map

$$\begin{aligned}\tau : & [\mathbb{2}] \to [\mathbb{S}]\\ (\tilde{x} : a \to x) & \mapsto (\{pt\} \to 1_{\mathbb{2}}(x))\end{aligned}$$

is the identity in Set, as the underlying set on both sides is $[\mathbb{2}] = [\mathbb{S}] = \{0, 1\}$ so that we are looking at the set map

$$\begin{aligned} \{0,1\} &\to \{0,1\} \\ x &\mapsto 1_{\{0,1\}}(x) = x \end{aligned}$$

This map is clearly the identity. This boils down to the fact that our choice of morphism from $\mathrm{Frm}(\mathbb{2}, \mathbb{2})$ was easy to determine since the set $\mathrm{Frm}(\mathbb{2}, A) = \{pt\}$, as $\mathbb{2}$ is an initial object in Frm, and in particular it is clear in (1) that $\mathrm{Frm}(\mathbb{2}, \mathbb{2}) = \{1_{\mathbb{2}}\}$. From this we can deduce that τ is the identity on $\bar{U}(\mathbb{2})$. In general, our schizophrenic object will not necessarily be initial in arbitrary concrete duality, and as such, our choice $p \in \mathcal{A}(\tilde{A}, \tilde{A})$ may not be unique nor easy to determine, so that τ, though always a bijection, is not necessarily always the identity.

7 A more general example

The leading example is modeled from a more general example from which many examples to be discussed in this thesis are structured.

This is the duality between the category of Boolean Algebras and the category of sets. This duality is actually given by the forgetful functor and it's left adjoint the free functor. In this case, this free functor is necessarily the power-set functor $\mathcal{P}(-) : \mathrm{Set} \to \mathrm{Bool}$, given by $S \mapsto \mathcal{P}(S)$, which sends the set S to its powerset.

(TODO: introduce boolean algebras, and show that the power set forms a boolean algebra)

(TODO: clarify about finite boolean algebras/finite sets, why this works/doesn't work (dont know yet) in the infinite case, and how to mend such a construction).

This duality is easy to see, as all we need to see is that the adjunction isomorphic to the expected ones on the Hom sets given by the schizophrenic object $(\mathbb{2}, 1_{\mathbb{2}}, \mathbb{2})$.

In other words, we check that for all $A \in \mathrm{Bool}$, it holds that $U(A) = \mathrm{Bool}(A, \mathbb{2})$, and that for all $S \in \mathrm{Set}$, it holds that $\mathcal{P}(S) = \mathrm{Set}(S, \mathbb{2})$.

Morphisms in $\mathrm{Bool}(A, \mathbb{2})$ are left exact, colimit preserving, and complement preserving.

Lemma 7.1. *We claim that there is a bijection*

$$\begin{aligned} \phi : U(A) &\to \mathrm{Bool}(A, \mathbb{2}) \\ a &\mapsto f_a^* \end{aligned} \qquad\qquad f_a^*(x) = \begin{cases} 1 & x \geq a \\ 0 & else \end{cases}$$

Proof. First we show that f_a^* is finite limit, colimit, and complement preserving.

It holds that

$$\begin{aligned} f_a^*(x \wedge y) = 1 &\iff x \wedge y \geq a \\ &\iff x \geq a \text{ and } y \geq a \\ &\iff f_a^*(x) \wedge f_a^*(y) = 1 \end{aligned}$$

and similarly that

$$\begin{aligned} f_a^*(x \wedge y) = 0 &\iff x \wedge y \ngeq a \\ &\iff x \ngeq a \text{ or } y \ngeq a \\ &\iff f_a^*(x) \wedge f_a^*(y) = 0 \end{aligned}$$

Now we check

$$\begin{aligned} f_a^*(x \vee y) = 1 &\iff x \vee y \geq a \\ &\iff x \geq a \text{ or } y \geq a \\ &\iff f_a^*(x) \vee f_a^*(y) = 1 \end{aligned}$$

and similarly

$$\begin{aligned} f_a^*(x \vee y) = 0 &\iff x \vee y \ngeq a \\ &\iff x \ngeq a \text{ and } y \ngeq a \\ &\iff f_a^*(x) \vee f_a^*(y) = 0 \end{aligned}$$

Moreover it is clear from the definition that $f_a^*(\neg x) = \neg f_a^*(x)$.

This shows that ϕ is well defined, and injectivity follows from definition of ϕ.

To show surjectivity we consider a boolean algebra homomorphism $f \in \mathrm{Bool}(A, 2)$, and define

$$a = \bigwedge\{x \in A | f(x) = 1\}$$

Now we show that $f = f_a^*$. (TODO: Finish this proof) ■

This implies that morphisms in $\mathrm{Bool}(A, 2)$ are uniquely determined by which $a \in A$ is severed by the boolean algebra homomorphism.

The second equation is given by the fact that any set map $S \to 2$ is given uniquely by a subset $\tilde{S} \subset S$, via characteristic functions

$$\mathcal{X}_{\tilde{S}}(x) = \begin{cases} 1 & x \in \tilde{S} \\ 0 & \text{else.} \end{cases}$$

The unit and counit maps then follow the exact same logic as the leading example, and in particular, 2 is also initial in Bool so that by the same logic, our τ is equal to the set identity 1_2.

(TODO: talk about free object and show why $(2, 1_2, 2)$ is the schizophrenic object).

8 Rings and Affine varieties

We now turn to an example that any graduate Algebra student has encountered, the duality between the category of rings and the category of affine schemes. We will use the more general category of commutative R-algebras, which we denote $\mathrm{CAlg}_{\mathrm{R}}$. Notice that if $R = \mathbb{Z}$, then $\mathrm{CAlg}_{\mathbb{Z}} = \mathrm{Ring}$.

In this example we already know from commutative algebra the adjunction that gives a dual equivalence, and we can easily show that one of these adjoint functors is isomorphic to the Hom functor. For the other adjoint, it is not so easy to see that the two functors are isomorphic directly, though we can easily conclude from uniqueness of the adjoint. We will nevertheless try to give some intuition about this isomorphism.

Firstly, we discuss the adjunction that one might learn in Algebra:

$$\begin{tikzcd} {\mathrm{CAlg}_{\mathrm{R}}}^{op} \arrow[r, bend left, "X(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Aff} \arrow[l, bend left, "\mathcal{O}(-)"] \end{tikzcd}$$

Recall that an affine scheme $X \in \mathrm{Aff}$ is defined as a representable functor in the functor category $\mathrm{Fun}(\mathrm{CAlg}_{\mathrm{R}}, \mathrm{Set})$ (*to avoid confusion, we use the notation* $\mathrm{Nat}^{\mathcal{B}}_{\mathcal{A}}(-,-)$, *or more often when the context is clear simply* $\mathrm{Nat}(-,-)$, *to refer to the set of morphisms, or natural transformations, in an arbitrary functor category* $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$.)

Now we see that there is a natural choice for our adjunction given by sending a representable object to its functor, and sending that representable functor to its object. In other words we have $X : {\mathrm{CAlg}_{\mathrm{R}}}^{op} \to \mathrm{Aff}$ that sends $A \to X_A = \mathrm{CAlg}_{\mathrm{R}}(A, -)$ and $\mathcal{O} : \mathrm{CAlg}_{\mathrm{R}} \to \mathrm{Set}$, which sends $X(-) = \mathrm{CAlg}_{\mathrm{R}}(\mathcal{O}(X), -)$ to $\mathcal{O}(X)$, its representable object.

The equivalence is clear, since by construction our unit and counit are isomorphisms, i.e. $X_{\mathcal{O}(X)} = X$ and $\mathcal{O}(X_A) = A$.

Now on the one hand, the Yoneda lemma shows us that $\mathrm{Aff}(X, \mathbb{A}^1) = \mathbb{A}^1(\mathcal{O}(X)) = \mathrm{CAlg}_{\mathrm{R}}(R[X], \mathcal{O}(X)) \cong \mathcal{O}(X)$. The final set isomorphism is due to the fact that $R[X]$ is a free commutative R-algebra on one free generator. Now we see that for any $X_A \in \mathrm{Aff}$ it holds that $\mathcal{O}(X_A) = \mathrm{Aff}(X_A, \mathbb{A}^1)$, and as such we have a natural choice for a schizophrenic object $(R[X], \tau, \mathbb{A}^1)$.

We may use the same argument to show why $\mathrm{CAlg}_{\mathrm{R}}(A, R[X])$ lifts to the category of affine schemes, and that it is isomorphic to $X(A) = \mathrm{CAlg}_{\mathrm{R}}(A, -)$, however it is not obvious how to think about $\mathbb{A}^1$ as a free functor on one free generator, as elements of affine schemes are not given in the same way as they are for $R[X]$, where the element X is clearly our free generator.

We may however still use Yoneda to see that $\mathrm{CAlg}_{\mathrm{R}}(A, R[X]) = X_A(R[X]) = \mathrm{Aff}(\mathbb{A}^1, X_A) \cong X_A$, and as such our intuition thus must come from the fact that

$\mathbb{A}^1$ is our free affine scheme on one free generator and that the above equality is given, so that elements of our affine scheme can be thought of as global functions from A to $R[X]$.

Now we shall try to understand the unit and counits.

9 Gelfand Duality

In order to describe the following duality, some context is in order. In setting up a more general dual adjunction, whose restriction to an equivalence later defines the *Gelfand duality*, we must set the scene, and in doing so we first define the following category, *Kelley spaces*.

9.a Kelley Spaces

Of primary importance to a Kelley space is the notion of a compactly generated topological space.

Definition 9.1 (k-continuous)**.** *A function $f : X \to Y$ of underlying sets of a topological space is said to be k-**continuous** if for all compact Hausdorff $K \subset X$ and continuous functions $t : K \to X$ the composition $f \circ t$ is continuous.*

Definition 9.2 (k-space)**.** *A topological space X is said to be a k-**space**, or a **compactly generated topological space**, if for all spaces Y and underlying-set functions $f : X \to Y$, it holds that f is continuous if and only if f is k-continuous.*

That is to say a space is compactly generated if all its continuous functions are continuous on compact subspaces. Note that in the definition the requirement that $K \subset X$ could be lifted to arbitrary compact Hausdorff space Y, since images of compact spaces by continuous maps $f(Y) \subset X$ are homeomorphic to compact subspaces $K \subset X$, so in particular, f factors through the inclusion $\iota : K \hookrightarrow X$.

For the following we will assume that k-spaces are additionally Hausdorff, and refer to kSp as the category of compactly generated Hausdorff spaces, whose morphisms are the continuous functions between them, making it a full subcategory of Top, and in particular, of Haus $\subset$ Top.

If given a topological space which may or may not be a k-space, we may force the compactly generated condition on it through a process which we call the *Kelleyfication* of a topological space.

That is, given a set(TODO: proper class?) $(K \xrightarrow{t_i} X)_{i \in I}$ of functions on compact subspaces $K \subset X$, we give X the finest topology making all t_i continuous.

That is, given an underlying-set function $X \xrightarrow{f} Y$, we give X the topology such that f is continuous if and only if $f \circ t_i$ is continuous.

But this is just the universal property of the colimit applied to topological spaces (and their full subcategories): for any set of continuous functions $K_i \xrightarrow{\varphi_i} Y$ into some topological space Y that satisfy commutativity (i.e., if there is a continuous map $K_i \xrightarrow{h} K_j$ for some $i, j \in I$, then $\varphi_i = \varphi_j \circ h$), then there exists a unique continuous map $X \xrightarrow{f} Y$ such that $\varphi = f \circ t_i$.

This is reflected by commutativity of the following diagram:

$$\begin{tikzcd}
& Y & \\
& X \arrow[u, dashed, "f"] & \\
K_i \arrow[ur, "t_i"] \arrow[rr, "h"'] \arrow[uur, bend left, "f\circ t_i"] & & K_j \arrow[ul, "t_j"'] \arrow[uul, bend right, "f\circ t_j"']
\end{tikzcd}$$

That is if $f \circ t_i$ is continuous, preimages of open sets $V \subset Y$ must be open under composition, in other words $t_i^{-1}(f^{-1}(V)) \subset K_i$ is open.

So $f \circ t_i$ induces a continuous $f : X \to Y$ in the following way: given a topological space with underlying set X, the colimit out of compact Hausdorff subspaces will be its *Kelleyfication*, which is necessarily a refinement of the topology of X, since continuous maps $X \xrightarrow{f} Y$ necessarily satisfy commutativity of the above diagram, so we want to add opens to X which satisfy commutativity for arbitrary function $X \xrightarrow{f} Y$.

That is, if X is not already Kelleyfied, we add opens $U = f^{-1}(V)$, for functions f such that $t_i^{-1}(U) \subset K$ is open but $U \subset X$ is not. This is the universality condition, since commutativity must be satisfied for arbitrary function f. But this is our original statement: we want a topology on X such that f is continuous if and only if $f \circ t_i$ is continuous.

Thus we can understand a k-space as a colimit of compact Hausdorff spaces, or specifically, $X \in \text{kSp}$ if and only if $X = \text{colim}_{K \subset X \text{ compact}} K$.

Question: are compact subspaces in X compact if and only if they are compact in $\text{Kelley}(X)$?

We may otherwise view the *Kelleyfication* as the left adjoint $k(-)$ to the forgetful functor. As kSp is a full subcategory of Haus, this puts us in the setting of a coreflective subcategory of Haus $\subset$ Top.

Another way to think about kSp is as the coreflective hull of kHaus in Haus. That is to say, the forgetful functor kHaus $\hookrightarrow$ Haus does not have a left adjoint (TODO: Why? Does it have something to do with that kHaus is not closed under colimits), however we may take the intersection of all coreflective subcategories of Haus which contain kHaus, and this will precisely be the category generated under colimits of kHaus (TODO: make this precise.)

As it is important that our category admits function spaces (which we will later discuss), in the following we will show why kSp admits such spaces. In doing so we will introduce monotopological categories.

Let (A, U) be a concrete category.

Definition 9.3 (Topological functor)**.** *A functor $A \overset{G}{\to} B$ is called **topological** if every G-structured source has a unique G-initial lift.*