1 Introduction

2 Preliminaries/Index

Throughout this thesis we will be using the language of category theory, for which a mild introduction is in order. To that degree we plan to define the terms used as well as prove some well known theorems which we use quite freely, so not to distract us from the argument.

We assume basic knowledge of Category theory (such as the definitions of Categories, functors, natural transformations, (filtered/directed) limits, (filtered/directed) colimits, initial object, terminal object, adjunctions, etc), and will choose to define terms as necessary as they relate to our investigation.

This thesis will be divided into two parts. In the first part we describe the general framework for arbitrary categories. We will be answering the questions of what a mathematical duality is, what makes it concrete, and also what is the minimal datum that ascertains its existence.

In the second part we should like to describe some examples. As these examples span mathematics itself, we shall assume basic knowledge in each subfield to which our respective example pertains. Before each example we will make more explicit what is assumed, as well references for background and/or further reading. One may expect general knowledge of topology, commutative algebra, group theory, lattice theory, and Galois theory to be used quite freely.

2.a Concrete Categories

In the course of this thesis we will be considering a certain type of category, called a concrete category. As we do not assume knowledge about this on behalf of the reader, we would like to introduce some definitions of the basic kinds of objects and constructions we will be working with.

First we should define a concrete category:

Definition 2.1 (Concrete Category). *Let $\mathcal{X}$ be a category. A **concrete category** over $\mathcal{X}$ is a pair $(\mathcal{A}, U)$ where $\mathcal{A}$ is a category and $U : \mathcal{A} \to \mathcal{X}$ is a faithful functor. Sometimes U is called the **underlying functor** over $\mathcal{X}$ and $\mathcal{X}$ is called the **base category** for $(\mathcal{A}, U)$. A concrete category over* Set *is called a **construct**.*

Note that from here on out, we use concrete category to mean construct, since we do not consider any other type of concrete category in this thesis.

Definition 2.2 (Concrete functor). *Let $(\mathcal{A}, U)$ and $(\mathcal{B}, V)$ be concrete categories over $\mathcal{X}$. A **concrete functor** from $(\mathcal{A}, U)$ to $(\mathcal{B}, V)$ is a functor $F : \mathcal{A} \to \mathcal{B}$ such that $U = V \circ F$.*

A concrete functor is necessarily faithful, since V is faithful. In general, for any functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, if $G \circ F$ is faithful, then F is. The proof is easy: consider elements $f, g \in \mathcal{A}(A, A')$ such that $Ff = Fg$. Then $GFf = GFg$ and since $G \circ F$ is faithful, then $f = g$.

2.b Types of arrows

Definition 2.3 (Source and sink)**.** *A **source** is a pair $(Y, (f_i))$ consisting of an object Y of a category $\mathcal{C}$, and a family of morphisms $(Y \xrightarrow{f_i} X_i)_{i\in I}$ over a class I. Equivalently it is a family of objects in the under category $\mathcal{C}_{Y/}$.*

For short we will use the notation $(Y \to X_i)_I$

*If I is a finite index set $\{1, \cdots, n\}$, we call our source an n-**source**.*

*The dual concept to a source is called a **sink**.*

Definition 2.4. *A source $\mathcal{S} = (A \xrightarrow{f_i} A_i)_I$ is called a **mono-source** if it cancels from the left, i.e., if for any two parallel morphisms $B \underset{h}{\overset{g}{\rightrightarrows}} A$ the equation $\mathcal{S} \circ g = \mathcal{S} \circ h$, that is, if $f_i \circ g = f_i \circ h$ for all $i \in I$, implies $g = h$.*

For the following definitions, let $G : \mathcal{A} \to \mathcal{B}$ be a functor, and $B \in \mathrm{Ob}(\mathcal{B})$.

Definition 2.5 (G-structured map)**.** *A G-**structured arrow with domain** B is a a pair (f, A) consisting of an $\mathcal{A}$-object A and a $\mathcal{B}$-morphism $f : B \to GA$.*

Definition 2.6 (G-structured lift)**.** *Let $(B \xrightarrow{\varphi_i} GA_i)_I$ be a G-structured source. If there exists an object A and a map of morphisms $(A \xrightarrow{f_i} A_i)_I$, for which there exists a map $GA \xrightarrow{h} B$ such that $Gf_i = \varphi_i \circ h$ for all $i \in I$, we call A a G-**structured lift** of the source.*

For notation we will interchangably refer to A or the source $(A \to A_i)_I$ as the lift.

Definition 2.7 (Morphism of G-structured lifts)**.** *We call an $\mathcal{A}$-morphism $A' \xrightarrow{\phi} A$ a **morphism of** G-**structured lifts** if there exists another lift $(A' \xrightarrow{f_i'} A_i)_I$ such that $GA' \to GA_i$ factors through $GA \to GA_i$ for all $i \in I$.*

That is, there exists a morphism $GA' \xrightarrow{h'} B$ such that $h' = h \circ G\phi$ and $f_i' = f_i \circ \phi$.

Definition 2.8. *The lift $A \in \mathrm{Ob}(\mathcal{A})$ of $(B \xrightarrow{\varphi_i} GA_i)_I$ is called a G-**initial lift** if every G-structured lift $(A' \xrightarrow{f_i'} A_i)_I$ factors uniquely through $(A \xrightarrow{f_i} A_i)_I$.*

For the previous definitions we refer to the following commutative diagram for clarity:

$$
\begin{tikzcd}
A' \arrow[r, "\phi"] \arrow[d] \arrow[rrr, bend left, "f_i'"] & A \arrow[rr, "f_i"] \arrow[d] & & A_i \arrow[d] \\
GA' \arrow[r, "G\phi"] \arrow[rr, bend right, "h'"'] & GA \arrow[r, "h"] \arrow[rr, bend right, "Gf_i"'] & B \arrow[r, "\varphi_i"] & GA_i.
\end{tikzcd}
$$

Remark. Often h is the identity map, in which case our lift is called **strict**. However in practice we assume that h is the identity unless stated otherwise.

Consider that here we mean initial in the sense of the initial or induced topology (in (Top, U) as construct), however applied to arbitrary concrete category $(\mathcal{A}, U)$ over arbitrary category $\mathcal{X}$.

That is, this lift is initial in the poset $(\mathrm{Lift}(B), \subset)$ of G-structured lifts of B where the preorder is given by $A \subset A'$ if and only if A' factors through A as a lift, i.e. the morphisms are $A' \xrightarrow{\phi} A$ such that $(GA' \xrightarrow{h'} B) = (GA' \xrightarrow{G\phi} GA \xrightarrow{h} B)$.

This is a very important concept throughout this paper, which deserves a remark about its intuition, which here should come from topology, where the initial topology is the limit topology, i.e., the coarsest topology on GA making all $(A \xrightarrow{f_i} A_i)_I$ continuous.

So for arbitrary category, we are looking for the weakest or initial $\mathcal{A}$-structure on GA such that $(A \xrightarrow{f_i} A_i)_I$ are $\mathcal{A}$-morphisms, which ensures that for any $\mathcal{A}$-structure on GA' such that $(A' \xrightarrow{f_i'} A_i)_I$ are $\mathcal{A}$-morphisms which factor through the lift $(A \xrightarrow{f_i} A_i)_I$, that this factorization $(A' \xrightarrow{\phi} A)$ is unique.

Formally this is a limit in $\mathcal{A}$ over $\mathcal{A}$-structures on GA with the property that all $(A \xrightarrow{f_i} A_i)_I$ are $\mathcal{A}$-morphisms.

Definition 2.9 (cogenerator)**.** *A* ***cogenerator*** *of a category $\mathcal{C}$ is an object $c \in \mathcal{C}$ such that the Hom-set functor $\mathcal{C}(-, c) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is injective.*

That is, given two maps $e \underset{f_2}{\overset{f_1}{\rightrightarrows}} d$, if the induced maps $\mathcal{C}(d, c) \underset{-\circ f_2}{\overset{-\circ f_1}{\rightrightarrows}} \mathcal{C}(e, c)$ are equal, then $f_1 = f_2$, i.e., for any $\phi : d \to c$ it holds that $\phi \circ f_1 = \phi \circ f_2 \Rightarrow f_1 = f_2$.

The dual concept is of a ***generator****, which applies to $\mathcal{C}(c, -)$.*

2.c Topological categories

In the following we introduce topological categories.

Let (A, U) be a concrete category over $\mathcal{X}$.

Definition 2.10 (Topological functor). *A functor $\mathcal{A} \xrightarrow{G} \mathcal{B}$ is called **topological** if every G-structured source has a unique G-initial lift.*

Definition 2.11 (Topological category). *A concrete category $(\mathcal{A}, U)$ over $\mathcal{X}$ is said to be a **topological category** if U is a topological functor.*

Replacing "source" with "mono-source" gives us the definition of a ***monotopological*** category.

Notice here again that our intuition of such a category should come from topology, which is even reflected in its name, namely, from the concrete category (Top, U), over which every source lifts initially, since the arbitrary intersection of topologies is a topology. That is, we are taking the intersection of all topologies on A such that all $(UA \xrightarrow{f_i} UA_i)_I$ are continuous.

That means there is a systematic way to choose open sets of UA. In other words, a subset $S \subset UA$ lifts to an open set in A if and only if S equals the preimage $Uf_i^{-1}(U(V))$ for some $f_i \in (UA \xrightarrow{f_i} UA_i)_I$ and $V \in \Omega(A_i)$, where $\Omega(X)$ for topological space X denotes the set of open sets of X..

Notes on free objects and the free functor

It is important to note that in some category $\mathcal{A}$ with a representing object A_0 that is a free generator on one free object, it holds that $\mathcal{A}(A_0, A) \cong A$ on the level of set, as the morphisms are uniquely determined by where they send the free generator to.

More concretely, given a forgetful functor $U : \mathcal{A} \to \mathrm{Set}$, we have $\mathcal{A}(A_0, A) = U(A)$. This due to how we construct A_0 and B_0 using the the left adjoint functor $F : \mathrm{Set} \to \mathcal{A}$ to the forgetful functor, which is called the free functor.

In general, such a functor does not have to exist, and we can still get the construction by defining A_0 as the left adjoint object of U at $\{pt\} \in \mathrm{Set}$, however, we will assume that such a functor does exist for purposes of explication: $A_0 = F(\{\mathrm{pt}\})$. Such an assumption is fair since the properties of the adjunction by definition of the left adjoint object will still hold (in particular the middle equality of the equation below), regardless of existence of the functor as a whole:

$$\mathcal{A}(A_0, A) = \mathcal{A}(F(\{\mathrm{pt}\}), A) = \mathcal{B}(\{pt\}, U(A)) = U(A).$$

The left adjoint object to the forgetful functor has the following universal property (TODO).

In the examples which are to be discussed in this thesis, we only need to check that such a left adjoint object does exist in the category, and we show what they are and that they satisfy the universal property.

3 Notes: Concrete Dualities

BIGTODO: Edit all relevant parts of this section that I wrote before I understood concrete categories and identified S and T with $\mathcal{A}(-, \tilde{A})$ and $\mathcal{B}(-, \tilde{B})$ directly, rather than by means of the forget functor.

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories with the following representable functors with representing objects $A_0 \in \mathcal{A}$, and $B_0 \in \mathcal{B}$. We assume these are free objects on one free generator, which we will later make precise.

$$\mathcal{A}(A_0, -) \cong U : \mathcal{A} \to \mathrm{Set}$$
$$\mathcal{B}(B_0, -) \cong V : \mathcal{B} \to \mathrm{Set}$$

Let $T : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{A}$ be contravariant functors with natural transformations $\eta : 1_{\mathcal{B}} \to TS$ and $\epsilon : 1_{\mathcal{A}} \to ST$. We can view this as the following *dual adjunction*

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & \mathcal{B}^{\mathrm{op}} \arrow[l, bend left, "S"] \end{tikzcd}$$

When these natural transformations are natural isomorphisms we are speaking of a *dual equivalence.*

Like any adjunction we have the following triangle equalities

$$T\epsilon_A \circ \eta_{TA} = 1_{TA} \qquad \text{and} \qquad S\eta_B \circ \epsilon_{SB} = 1_{SB}$$

and Hom set isomorphisms

$$\mathcal{A}(A, SB) \cong \mathcal{B}(B, TA) \cong \mathcal{B}^{\mathrm{op}}(TA, B)$$

Given some $\mathcal{A}$-morphism $f : A \to A'$, consider the map $Uf : UA \to UA'$. More explicitly, this is

$$\mathcal{A}(A_0, -)(f) : \mathcal{A}(A_0, A) \overset{f\circ(-)}{\to} \mathcal{A}(A_0, A')$$

We shorten this notation by using $[f] : [A] \to [A']$.

The following pair of objects will be of central importance to this these, which are defined as the following:

$$\tilde{A} := SB_0 \qquad \tilde{B} := TA_0.$$

From these characteristics we can deduce how S and T should be defined, to which a few lemmas will illuminate the bigger picture of our situation.
Lemma 3.1.

$$VT \cong \mathcal{A}(-, \tilde{A}) \qquad\qquad US \cong \mathcal{B}(-, \tilde{B})$$

Proof. As presheafs, V and VT (respectively U and US) may be computed point-wise.

$$\begin{aligned} VT(A) \cong \mathcal{B}(B_0, TA) \cong \mathcal{A}(A, SB_0) &= \mathcal{A}(A, \tilde{A}) \\ \implies VT &\cong \mathcal{A}(-, \tilde{A}) \end{aligned}$$

$$\begin{aligned} US(B) \cong \mathcal{A}(A_0, SB) \cong \mathcal{B}(B, TA_0) &= \mathcal{B}(B, \tilde{B}) \\ \implies US &\cong \mathcal{B}(-, \tilde{B}) \end{aligned}$$

■

Should we have strict identities

$$VT = \mathcal{A}(-, \tilde{A}) \qquad\qquad US = \mathcal{B}(-, \tilde{B})$$

we say that the adjunction is *strictly represented* by $\tilde{A}$ and $\tilde{B}$.

Given our assumption that A_0 and B_0 are free objects on one free generator, this result should already give us an idea of how our adjunction is to be induced, which the goal of the whole introduction to concrete dualities is to make precise.

However the broad strokes of it uses the free-forget adjunction ($\bar{F} \dashv \bar{U}$) to show via

$$\mathcal{A}(A, \tilde{A}) = VT(A) = \mathcal{B}(B_0, TA) = \bar{U}(TA)$$

that the underlying sets of T and S respectively, are $\mathrm{Hom}_A(-, \tilde{A})$ and $\mathcal{B}(-, \tilde{B})$. [1]
Lemma 3.2. $V\tilde{B} \cong U\tilde{A}$

Proof.

$$\begin{aligned} V\tilde{B} &\overset{Def.V}{=} \mathcal{B}(B_0, \tilde{B}) \overset{Def.\tilde{B}}{=} \mathcal{B}(B_0, TA_0) \\ &\overset{Adjunction}{\cong} \mathcal{A}(A_0, SB_0) \overset{Def.\tilde{A}}{=} \mathcal{A}(A_0, \tilde{A}) \overset{Def.U}{=} U\tilde{A} \end{aligned}$$

■

[1] In the course of this thesis we will often prove results about $\mathcal{A}$, respectively $T : \mathcal{A} \to \mathcal{B}$ with unit $\epsilon : 1_{\mathcal{A}} \to ST$, from which the results about $\mathcal{B}$, respectively $S : \mathcal{B} \to \mathcal{A}$ with counit $\eta : 1_{\mathcal{B}} \to TS$, follow completely analogously. Unless we state that results do not follow analogously, we assume this to be the case.

The aim of the following will be to show how the objects $\tilde{A}$ and $\tilde{B}$ actually induce the adjunction $T \dashv S$. In doing so, we first show that the units and counits of the adjunction are given *by evaluation*, that is

$$\begin{aligned} \epsilon_A(x) : \mathcal{A}(A, \tilde{A}) &\to \tilde{B} \\ f &\mapsto f(x) \end{aligned} \qquad \begin{aligned} \eta_B(y) : \mathcal{B}(B, \tilde{B}) &\to \tilde{A} \\ g &\mapsto g(y). \end{aligned}$$

In the following we define the canonical "evaluation" maps, $\varphi_{A,x}$ and $\psi_{B,y}$ and the canonical bijections τ and σ:

$$\begin{aligned} \varphi_{A,x} : \mathcal{A}(A, \tilde{A}) &\to [\tilde{A}] \\ s &\mapsto [s](x) \end{aligned} \qquad \begin{aligned} \psi_{B,y} : \mathcal{B}(B, \tilde{B}) &\to [\tilde{B}] \\ t &\mapsto [t](y) \end{aligned}$$

$$\begin{aligned} \tau : [\tilde{A}] &\to [\tilde{B}] \\ \tilde{x} &\mapsto [[\epsilon_{\tilde{A}}](\tilde{x})](1_{\tilde{A}}) \end{aligned} \qquad \begin{aligned} \sigma : [\tilde{B}] &\to [\tilde{A}] \\ \tilde{y} &\mapsto [[\eta_{\tilde{B}}](\tilde{y})](1_{\tilde{B}}) \end{aligned}$$

which evaluate the maps $[s]$ and $[t]$ at x and y respectively:

$$\begin{aligned} [s] : [A] &\to [\tilde{A}] \\ x &\mapsto [s](x) \end{aligned} \qquad \begin{aligned} [t] : [B] &\to [\tilde{B}] \\ y &\mapsto [t](y) \end{aligned}$$

as for any $s \in \mathcal{A}(A, \tilde{A})$, we have the induced map $[s] : [A] \to [\tilde{A}]$.

So for every $x \in \mathcal{A}(A_0, A)$ we have the following diagram.

$$\begin{tikzcd} s \arrow[r, mapsto] & {[s](x)} \arrow[r, mapsto] & \tau([s](x)) \\ \mathcal{A}(A, \tilde{A}) \arrow[r, "\varphi_{A,x}"] & \mathcal{A}(A_0, \tilde{A}) \arrow[r, "\tau"] \arrow[d, "\cong"] & \mathcal{B}(B_0, \tilde{B}) \arrow[d, "\cong"] \\ & \mathcal{A}(A_0, SB_0) \arrow[r] & \mathcal{B}(B_0, TA_0) \end{tikzcd}$$

By the dual adjunction we have $\mathcal{A}(A, SB_0) \cong \mathcal{B}(B_0, TA)$.

Lemma 3.3. *τ and σ are inverses, and the following identities hold:*

$$[[\epsilon_A](x)] = \tau\varphi_{A,x} \qquad [[\eta_B](y)] = \sigma\psi_{B,y}$$

Proof. First we check the identities, as understanding them will help us prove that τ and σ are inverses. We only check the left identity, and as the right identity will follow analogously.

First we have $\tau\varphi_{A,x}(s) = \tau([s](x))$ by definition of $\varphi_{A,x}$. But then by definition of τ we have $\tau([s](x)) = [[\epsilon_{\tilde{A}}][s](x)](1_{\tilde{A}})$.

Since ϵ is a natural transformation, we have the following commutative diagram for all $A, A' \in \mathcal{A}$ such that there exists a map $A \to A'$. In particular, given $s : A \to \tilde{A}$

we have:

$$\begin{tikzcd} 1_{\mathcal{A}}(A) \arrow[r, "\epsilon_A"] \arrow[d, "s"'] & ST(A) \arrow[d, "STs"] \\ 1_{\mathcal{A}}(\tilde{A}) \arrow[r, "\epsilon_{\tilde{A}}"'] & ST(\tilde{A}) \end{tikzcd}$$

so that $[[\epsilon_{\tilde{A}}][s](x)](1_{\tilde{A}}) = [[STs][\epsilon_A](x)](1_{\tilde{A}})$.

By Lemma 3.1, we know that $[STs] = US(Ts) = \mathcal{B}(Ts, \tilde{B})$, which is just a map $\mathcal{B}(TA, \tilde{B}) \to \mathcal{B}(T\tilde{A}, \tilde{B})$, induced by $Ts : T\tilde{A} \to TA$. *Notice that $US(-)$ and $T(-)$ are both contravariant, so that $UST(-)$ is covariant.*

Now as $[\epsilon_A](x) \in \mathcal{B}(TA, \tilde{B})$, we have the induced precomposition

$$\begin{tikzcd} T\tilde{A} \arrow[r, dashed, "{[\epsilon_A](x) \circ Ts}"] \arrow[d, "Ts"'] & \tilde{B} \\ TA \arrow[ur, "{[\epsilon_A](x)}"'] & \end{tikzcd}$$

which can be otherwise phrased as a right action of Ts on $[\epsilon_A](x)$ so that

$$[[STs][\epsilon_A](x)](1_{\tilde{A}}) = [[\epsilon_A](x) \circ Ts](1_{\tilde{A}}). \tag{1}$$

From Lemma 3.1 we know that $VT = \mathcal{A}(-, \tilde{A})$ so that we get the induced diagram

$$\begin{tikzcd} \mathcal{A}(\tilde{A}, \tilde{A}) \arrow[r, dashed, "{[[\epsilon_A](x) \circ Ts]}"] \arrow[d, "{[Ts]}"'] & {[\tilde{B}]} \\ \mathcal{A}(A, \tilde{A}) \arrow[ur, "{[[\epsilon_A](x)]}"'] & \end{tikzcd}$$

Notice that then $[Ts]$ becomes an evaluation of the precomposition functor $\mathcal{A}(-, \tilde{A})$ at s, i.e. $[Ts] = - \circ s$, which sends $1_{\tilde{A}} \mapsto s$. Therefore it holds that

$$[[\epsilon_A](x) \circ Ts](1_{\tilde{A}}) = [[\epsilon_A](x)](s).$$

All together we have

$$\begin{aligned}\tau \circ \varphi_{A,x}(s) =&\tau([s](x))\\ =&[[\epsilon_{\tilde{A}}][s](x)](1_{\tilde{A}})\\ =&[[STs][\epsilon_A](x)](1_{\tilde{A}})\\ =&[[\epsilon_A](x)\circ Ts](1_{\tilde{A}})\\ =&[[\epsilon_A](x)](s)\end{aligned}$$

which gives us the desired identity.

Now we check that τ and σ are inverses.

The above identity gives us the particular instance

$$\tau\varphi_{S\tilde{B},1_{\tilde{B}}}(s) = [[\epsilon_{S\tilde{B},\tilde{y}}](1_{\tilde{B}})](s)$$

for all $s \in \mathcal{A}(S\tilde{B}, \tilde{A})$.

For $s = [\eta_{\tilde{B}}](\tilde{y})$ with $\tilde{y} \in [\tilde{B}]$, noticing the maps

$$\begin{aligned}[\eta_{\tilde{B}}](\tilde{y}) : S\tilde{B} &\to \tilde{A} \\ 1_{\tilde{B}} &\mapsto 1_{\tilde{B}}(\tilde{y})\end{aligned} \qquad\qquad \begin{aligned}\varphi_{S\tilde{B},1_{\tilde{B}}} : \mathcal{A}(S\tilde{B},\tilde{A}) \to&[\tilde{A}]\\ s \mapsto&[s](1_{\tilde{B}})\end{aligned}$$

we see that $[[\eta_{\tilde{B}}](\tilde{y})](1_{\tilde{B}}) = \varphi_{S\tilde{B},1_{\tilde{B}}}(s)$ and so we have

$$\begin{aligned}\tau\sigma(\tilde{y}) =&\tau([[\eta_{\tilde{B}}](\tilde{y})](1_{\tilde{B}}))\\ =&\tau(\varphi_{S\tilde{B},1_{\tilde{B}}}(s))\\ =&[[\epsilon_{S\tilde{B}}](1_{\tilde{B}})][\eta_{\tilde{B}}](\tilde{y}) = \tilde{y}.\end{aligned}$$

We only need to show the last equality.

Consider the triangle equality $S\eta_{\tilde{B}} \circ \epsilon_{S\tilde{B}} = 1_{S\tilde{B}}$ which induces $[S\eta_{\tilde{B}}][\epsilon_{S\tilde{B}}] = 1_{[S\tilde{B}]}$.

We can extrapolate from (1) using the result $US = \mathcal{B}(-, \tilde{B})$ from Lemma 3.1 that for some $A \in \mathcal{A}$, $x \in A$, $B \in \mathcal{B}$ and $f \in \mathcal{B}(B, TA)$ the left action of $[Sf]$ on $[\epsilon_A](x) \in \mathcal{B}(TA, \tilde{B})$ becomes a right action of f on $[\epsilon_A](x)$, i.e.

$$[Sf][\epsilon_A](x) = [\epsilon_A](x) \circ f \in \mathcal{B}(B, \tilde{B}).$$

Therefore

$$1_{\tilde{B}} = 1_{\mathcal{B}(\tilde{B},\tilde{B})}(1_{\tilde{B}}) = 1_{[S\tilde{B}]}(1_{\tilde{B}}) = [S\eta_{\tilde{B}}][\epsilon_{S\tilde{B}}](1_{\tilde{B}}) = [\epsilon_{S\tilde{B}}](1_{\tilde{B}}) \circ \eta_{\tilde{B}}.$$

In other words, the induced map is the identity on $[\tilde{B}]$:

$$[[\epsilon_{S\tilde{B}}](1_{\tilde{B}})][\eta_{\tilde{B}}] = 1_{[\tilde{B}]}.$$

Thus we have $\tau\sigma = 1_{[\tilde{B}]}$ as desired. ∎

Lemma 3.3 makes precise the notion that our unit and counit are given "by evaluation", given

$$[[\epsilon_A](x)] : \mathcal{A}(A, \tilde{A}) \to [\tilde{B}] \qquad [[\eta_B](y)] : \mathcal{B}(B, \tilde{B}) \to [\tilde{A}]$$
$$f \mapsto f(x) \qquad g \mapsto g(y)$$

via canonical bijections τ and σ. From here we can (TODO: deduce or allude to?) deduce the fact that $T = \mathcal{A}(-, \tilde{A})$ and $S = \mathcal{B}(-, \tilde{B})$, as τ and σ send the precise evaluation maps to the other pairing, and as such we can view the unit and counit as such:

$$\epsilon_A : A \to \mathcal{B}(\mathcal{A}(A, \tilde{A}), \tilde{B}) \qquad \eta_B : B \to \mathcal{A}(\mathcal{B}(B, \tilde{B}), \tilde{A})$$
$$x \mapsto (f \to f(x)) \qquad y \mapsto (g \to g(y))$$

4 Schizophrenic Objects

4.a Natural dual adjunction

What we want for our set up is that the composition of these maps induces a U-structured lift, i.e. for every $A \in \mathcal{A}$ and $x \in A$ there exists an $e_{A,x} \in \mathcal{B}(TA, \tilde{B})$, that induces $[e_{A,x}] : [TA] \to [\tilde{B}]$, such that $[e_{A,x}] = \tau\varphi_{A,x}$, i.e., the following diagram commutes:

$$\begin{tikzcd}
 & TA \arrow[r, dashed, "e_{A,x}"] \arrow[d, "U"] & \tilde{B} \arrow[d, "U"] \\
\mathcal{B}(B_0, TA) = & \mathcal{A}(A, \tilde{A}) \arrow[r, "\tau\varphi_{A,x}"] & \mathcal{B}(B_0, \tilde{B})
\end{tikzcd}$$

We will shortly see that to obtain the desired dual adjunction, we may additionally require such a lift to be intial, which lends itself to a precise definition of the *schizophrenic object*, which is the central notion of this thesis:

Definition 4.1. *A triple $(\tilde{A}, \tau, \tilde{B})$ with a pair of objects $(\tilde{A}, \tilde{B}) \in \mathcal{A} \times \mathcal{B}$ and a bijective map $\tau : [\tilde{A}] \to [\tilde{B}]$ is called a* schizophrenic object *(for concrete categories $\mathcal{A}$ and $\mathcal{B}$) if the following conditions are satisfied:*

SO1. For every $A \in \mathcal{A}$ the family $(\tau\varphi_{A,x} : \mathcal{A}(A, \tilde{A}) \to [\tilde{B}])_{x\in[A]}$ admits a V-initial lifting $(e_{A,x} : TA \to \tilde{B})_{x\in[A]}$

SO2. For every $B \in \mathcal{B}$ the family $(\sigma\psi_{B,y} : \mathcal{B}(B, \tilde{B}) \to [\tilde{A}])_{y\in[B]}$ admits a U-initial lifting $(d_{B,y} : SB \to \tilde{A})_{y\in[B]}$

What these conditions actually mean is two-fold: Firstly, the V-structured lifting property yields the existence of a $\mathcal{B}$-morphism $e_{A,x} \in \mathcal{B}(TA, \tilde{B})$ for every $A \in \mathcal{A}$ and $x \in A$, such that $[TA] = \mathcal{A}(A, \tilde{A})$ and $[e_{A,x}] = \tau\varphi_{A,x}$.

But secondly, the V-initiality means that for any $Z \in \mathcal{B}$ and a map $h : [Z] \to [TA]$, if all composite maps $\tau\varphi_{A,x} \circ h$ are the underlying-set maps for $\mathcal{B}$-morphisms in

$\mathcal{B}(Z, \tilde{B})$, then there exists a unique $\mathcal{B}$-morphism $h' \in \mathcal{B}(Z, TA)$ whose underlying set map is h.

In other words, the V-structured lift is initial among all such lifts: if Z is any other $\mathcal{B}$-object whose underlying set maps into $\mathcal{A}(A, \tilde{A})$ in a way that is compatible with all $\tau\varphi_{A,x}$ composites, then that map factors uniquely through TA in $\mathcal{B}$.

$$\begin{tikzcd}
Z \arrow[r, "h'"] \arrow[d] \arrow[rr, bend left, "(\tau\varphi_{A,x} \circ h)'"] & TA \arrow[r, "e_{A,x}"] \arrow[d, "V"] & \tilde{B} \arrow[d, "V"] \\
{[Z]} \arrow[r, "h"] & \mathcal{A}(A, \tilde{A}) \arrow[r, "\tau\varphi_{A,x}"] & {[\tilde{B}]}
\end{tikzcd}$$

We now show a central theorem to this thesis.

Theorem 4.1. *Every schizophrenic object $(\tilde{A}, \tau, \tilde{B})$ induces a natural dual adjunction strictly represented by $(\tilde{A}, \tilde{B})$, such that τ and $\sigma = \tau^{-1}$ are the canonical bijections defined in the previous section.*

Proof. First we show that T and S are well defined functors. The conditions (SO1.) and (SO2.) show us how T and S act on objects up to underlying-set isomorphism.

Now we show how T acts on morphisms. To that effect, given some $f : A \to A'$, we seek to show the existence of $Tf : TA' \to TA$ whose underlying set map is $[Tf] = \mathcal{A}(f, \tilde{A}) : \mathcal{A}(A', \tilde{A}) \to \mathcal{A}(A, \tilde{A})$, which sends $s \mapsto s \circ f$. As we have just seen, by $(SO1)$ it suffices to show that $\tau\varphi_{A,x} \circ \mathcal{A}(f, \tilde{A})$ are the underlying set maps of $\mathcal{B}$-morphisms in $\mathcal{B}(TA', \tilde{B})$.

Considering that $[Tf]$ is simply the precomposition map $- \circ f$, we see that given some $s \in \mathcal{A}(A', \tilde{A})$, it holds that $\tau\varphi_{A,x} \circ \mathcal{A}(f, \tilde{A})(s) = \tau\varphi_{A,x}(sf)$. But since $[sf](x) = [s][f](x)$, where $[f](x) \in [A']$, we have $\tau\varphi_{A,x}(sf) = \tau\varphi_{A',[f](x)}(s) = [e_{A',[f](x)}](s)$, which is the underlying set map of a $\mathcal{B}$-morphism in $\mathcal{B}(TA', \tilde{B})$ by definition, and which exists by the lifting property given by $(SO1.)$.

Therefore T and S are well-defined functors, where preservation of the identity and the composition law follow the same logic as above, using V-initiality and the fact that the underlying-set map is defined by precomposition.

Now we show that T and S are adjoint, and to do that we shall construct unit and counit maps ϵ and η. In order to establish the existence of η_B by playing the same game we first define $[\eta_B] : [B] \to [TSB]$ and show that each $\tau\varphi_{SB,t} \circ [\eta_B]$ with $t \in [SB]$, can be lifted along V, i.e. that it is the underlying set map of a $\mathcal{B}$-morphism in $\mathcal{B}(B, \tilde{B})$. So we define in light of Lemma 3.3 under (SO2.)

$$\begin{aligned} [\eta_B] : [B] &\to \mathcal{A}(SB, \tilde{A}) \\ y &\mapsto d_{B,y}. \end{aligned}$$

Then by definitions and (SO2.) we have

$$\begin{aligned}\tau\varphi_{SB,t}\circ[\eta_B](y) &= \tau\varphi_{SB,t}(d_{B,y})\\ &= \tau[d_{B,y}](t)\\ &= \tau\sigma\psi_{B,y}(t)\\ &= [t](y)\end{aligned}$$

which shows that $\tau\varphi_{SB,t}\circ[\eta_B] : [B] \to [\tilde{B}]$ is the underlying set map of a $\mathcal{B}$-morphism in $\mathcal{B}(B, \tilde{B})$, proving the existence of η_B.

Furthermore we see that

$$e_{SB,t}\circ\eta_B = t \quad \text{for all } t \in [SB] = \mathcal{B}(B, \tilde{B}) \tag{2}$$

Since $[\eta_B]$ lifts for all B, we may verify naturality on underlying-set maps; we verify that given a $\mathcal{B}$-morphism $f : B \to B'$, that we have $[\eta_{B'}]\circ(f\circ -) = [TSf]\circ[\eta_B]$, or in other words:

$$d_{B',f\circ y} = [TSf]\circ d_{B,y} \tag{3}$$

Remember that the left action of $[TSf]$ on $d_{B,y}$ is a right action of Sf on $d_{B,y}$. But the underlying map of $d_{B,y}\circ Sf$ is $\sigma\psi_{B,y}\circ\mathcal{B}(f,\tilde{B})$ which is, up to the bijection σ, just the evaluation map of a morphism in $\mathcal{B}(B,\tilde{B})$ at some $y \in [B]$ precomposed with $f \in \mathcal{B}(B, B')$, which yields the evaluation map of a morphism in $\mathcal{B}(B', \tilde{B})$ at $f \circ y \in [B']$. In other words $[d_{B,y}][Sf] = [\sigma\psi_{B',f\circ y}] = [d_{B',f\circ y}]$, which is clearly the underlying set map of $d_{B',f\circ y}$, giving us (3) by uniqueness of the lift.

The definition of S gives us that $[S\eta_B][\epsilon_{SB}](t) = \mathcal{B}(\eta_B, \tilde{B})(e_{SB,t})$ and by (2) we have $\mathcal{B}(\eta_B, \tilde{B})(e_{SB,t}) = e_{SB,t}\circ\eta_B = t$. Since U is faithful, any map $[Sf] \in \mathrm{Set}([SB'], [SB])$ is the underlying-set map to a unique map $Sf \in \mathcal{A}(SB', SB)$, and we deduce that $[S\eta_B][\epsilon_{SB}] = 1_{[SB]}$ is the underlying set map of the triangle identity $S\eta_B\circ\epsilon_{SB} = 1_{SB}$.

Finally, to show that τ is induced by this adjunction it suffices to see that it maps $\tilde{x} \mapsto [[\epsilon_{\tilde{A}}](\tilde{x})](1_{\tilde{A}})$ as desired. For every $\tilde{x} \in [\tilde{A}]$ we have

$$[[\epsilon_{\tilde{A}}](\tilde{x})](1_{\tilde{A}}) = \tau\varphi_{\tilde{A},\tilde{x}}(1_{\tilde{A}}) = \tau([1_{\tilde{A}}](\tilde{x})) = \tau(\tilde{x}).$$

■

A dual adjunction induced by a schizophrenic object in this way is called *natural*. However there exist also dual adjunctions which are not natural in this sense, although some modifications are in order to make it natural.

4.b Non-natural dual adjunction

Dual adjunctions which are not induced by a schizophrenic object are called *non-natural*. The key difference here is initiality of the lifts. That is, an arbitrary dual adjunction (S', T') in the situation described in §3 already determines a triple $(\tilde{A}, \tau, \tilde{B})$ such that the following weakened conditions of $SO1$ and $SO2$ are fulfilled:

WSO1. For every $A \in \mathcal{A}$ the family $(\tau\varphi_{A,x} : \mathcal{A}(A, \tilde{A}) \to [\tilde{B}])_{x\in[A]}$ admits a V-structured lift $(e_{A,x} : T'A \to \tilde{B})_{x\in[A]}$ which extends functorially, i.e., for every $A \xrightarrow{f} A'$ in $\mathcal{A}$ there exists a $\mathcal{B}$-morphism $T'A' \xrightarrow{T'f} T'A$ with $[T'f] = \mathcal{A}(f, \tilde{A})$.

WSO2. For every $B \in \mathcal{B}$ the family $(\sigma\psi_{B,y} : \mathcal{B}(B, \tilde{B}) \to [\tilde{A}])_{y\in[B]}$ admits a U-structured lift $(d_{B,y} : S'B \to \tilde{A})_{y\in[B]}$ which extends functorially.

Though (S', T') determins a triple, such a triple does not necessarily induce (S', T') like the schizophrenic object induces (S, T).

However, if we are in this situation, there are potential modifications which may give us a natural dual adjunction. There are in particular two methods which we will remark.

Firstly, one may use the triple $(\tilde{A}, \tau, \tilde{B})$ to induce a natural dual adjunction (S, T) on the concrete categories $(\mathcal{A}, U)$ and $(\mathcal{B}, V)$. Such a method requires additional assumptions on $(\mathcal{A}, U)$ and $(\mathcal{B}, V)$ which we will not discuss. For more details, see [1 1-D].

The second method is the one which we will later see in action in our examples, which is to restrict our adjunction to full subcategories of our categories under which we have an equivalence.

In the next part we will discuss a situation which induces a non-natural dual adjunction between concrete categories.

4.c Internal Hom-Functors

Let $(\mathcal{A}, U)$ be a concrete category. An **internal hom-functor** is a functor $H : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathcal{A}$ such that $UH = \mathcal{A}(-,-)$. Moreover

$$\text{all evaluation maps } \phi_{A,A',x} : \mathcal{A}(A, A') \to [A']$$
$$h \mapsto [h](x)$$
$$\text{lift to } \mathcal{A}\text{-morphisms } p_{A,A',x} : H(A, A') \to A' \text{ for all } A, A' \in \mathcal{A}, x \in [A]$$

Any cartesian closed concrete category which admits function spaces is a good example of a category with internal hom-functors. Recall the following definitions (from [2]):

Definition 4.2 (Cartesian closed category)**.** *A category $\mathcal{A}$ is **cartesian closed** if it has finite products and for each $\mathcal{A}$-object A the functor $(A \times -)$ has a right adjoint $(-)^A$, called the **Heyting implication**. For $B \in \mathcal{A}$ we call B^A an **exponentiable object**.*

Definition 4.3. *A construct $(\mathcal{A}, U)$ is said to admit **function spaces** if $\mathcal{A}$ is cartesian closed, $(\mathcal{A}, U)$ admits finite concrete products, and the evaluation morphisms $A \times B^A \xrightarrow{\text{ev}} B$ can be chosen in such a way that $U(B^A) = \mathcal{A}(A, B)$ where* ev *is the restriction of the canonical evaluation map in* Set.

A cartesian closed concrete category which admits function spaces admits an internal hom-functor by definition, since we can choose B^A to be our lift.

Now we want to see how and under what conditions can a category which admits an internal hom-functor induce a dual adjunction. Let $(\mathcal{A}, U)$ admit an internal hom-functor H, and let there be a concrete category $(\mathcal{B}, V)$ and a concrete functor $|-|$: $\mathcal{B} \to \mathcal{A}$ such that $\mathcal{B}(B, C) \hookrightarrow \mathcal{A}(|B|, |C|)$ lift to $\mathcal{A}$-morphisms[2] $\gamma_{B,C} : \mathcal{B}_{\mathcal{A}}(B, C) \to H(|B|, |C|)$. In a monotopological category, this can be done by lifting initially.

Given $\tilde{B} \in \mathcal{B}$ with $\tilde{A} := |\tilde{B}|$ and $\tau = 1_{[\tilde{A}]}$, we can check that $WSO2$ is fulfilled. In other words we are seeking a lift of the map $\sigma\psi_{B,y} : \mathcal{B}(B, \tilde{B}) \to [\tilde{A}]$. But such a lift is given by the internal hom-functor, given that the bottom part of the following diagram commutes:

$$\begin{tikzcd}
\mathcal{B}_{\mathcal{A}}(B, \tilde{B}) \arrow[r, "\gamma_{B,\tilde{B}}"] \arrow[d] & H(|B|, \tilde{A}) \arrow[r, "p_{|B|,\tilde{A},y}"] \arrow[d] & \tilde{A} \arrow[d] \\
\mathcal{B}(B, \tilde{B}) \arrow[r, hook, "\iota"] \arrow[rr, bend right, "\sigma\varphi_{B,y}"'] & \mathcal{A}(|B|, \tilde{A}) \arrow[r, "\phi_{|B|,\tilde{A},x}"] & {[\tilde{A}]}
\end{tikzcd}$$

That is, if $\sigma\varphi_{B,y} = \phi_{|B|,\tilde{A},x} \circ \iota$ then we have $d_{B,y} = p_{|B|,\tilde{A},y} \circ \gamma_{B,\tilde{B}}$, which gives us $WSO2$.

The question is if the concrete functor commutes with the evaluation map, since $\tilde{A} = |\tilde{B}|$ and τ (and therefore also σ) is the identity.

$$\begin{tikzcd}
\mathcal{B}(B, \tilde{B}) \arrow[r, "\varphi_{B,y}"] \arrow[d, hook, "i"'] & {[\tilde{B}]} \arrow[d, "\sigma"] \\
\mathcal{A}(|B|, \tilde{A}) \arrow[r, "\phi_{|B|,\tilde{A},x}"'] & {[\tilde{A}]}
\end{tikzcd}$$

In other words, we check $\sigma([f](y)) = [|f|](\iota(y))$, but this follows from faithfulness of the concrete functor $|-|$. (TODO/Question: is this true/really so immediate?)

Therefore we have a contravariant functor $S(B) = \mathcal{B}_{\mathcal{A}}(B, \tilde{B})$.

Now if for every $A \in \mathcal{A}$ we can lift the $\mathcal{A}$-source $(p_{A,\tilde{A},x} : H(A, \tilde{A}) \to \tilde{A})$ along $|-|$ functorially, then we also have WSO1, inducing the contravariant functor $T : \mathcal{A} \to \mathcal{B}$ such that $|T(A)| = H(A, \tilde{A})$, and as such we have the desired dual adjunction (S, T). We will however only show this example-wise.

[2] $\mathcal{B}_{\mathcal{A}}(-, -)$ is notation for the hom set in $\mathcal{B}$ as $\mathcal{A}$-object

5 Motivation

We are now in the position to understand what this schizophrenic object really affords us, however first we will explicate what that is by saying a few words about motivation that will hopefully make the above seemingly abstract definition of the schizophrenic object more understandable.

First remember that there is a bijection between $[\tilde{A}]$ and $[\tilde{B}]$, since

$$[\tilde{A}] = \mathcal{A}(A_0, SB_0) = \mathrm{Hom}_B(B_0, TA_0) = [\tilde{B}].$$

In particular, we have shown that τ and σ are necessarily such bijections and inverses of one another.

What is interesting about the schizophrenic object is that the object itself defines the adjunction via Hom sets in our respective categories. In other words, our question, in general, is that given a schizophrenic object $(\tilde{A}, \tau, \tilde{B})$ do the sets $\mathrm{Hom}_A(A, SB_0)$ have $\mathcal{B}$ structure for every $A \in \mathcal{A}$ and $\mathcal{B}(B, TA_0)$ have $\mathcal{A}$ structure for every $B \in \mathcal{B}$, and moreover, are they the initial objects with $\mathcal{B}$-structure.

More explicitly, do we have initial lifts via the forgetful functor, such that the following diagrams commute for every $A \in \mathcal{A}$ and for every $B \in \mathcal{B}$?

$$\begin{tikzcd}
 & \mathcal{B}^{\mathrm{op}} \arrow[d, "\tilde{U}(-)"] \\
\mathcal{A} \arrow[ur, dashed, "\exists"] \arrow[r, "\mathcal{A}(-,\tilde{A})"'] & \mathrm{Set} \\
A \arrow[r, mapsto] & \mathcal{A}(A, \tilde{A})
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & \mathcal{A}^{\mathrm{op}} \arrow[d, "\tilde{U}(-)"] \\
\mathcal{B} \arrow[ur, dashed, "\exists"] \arrow[r, "\mathcal{B}(-,\tilde{B})"'] & \mathrm{Set} \\
B \arrow[r, mapsto] & \mathcal{B}(B, \tilde{B})
\end{tikzcd}$$

If such lifts exist, then by construction, we will have the following adjunction

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left, "T:=\mathcal{A}(-,\tilde{A})"] \arrow[rr, phantom, "\perp"] & & \mathcal{B}^{\mathrm{op}} \arrow[ll, bend left, "S:=\mathcal{B}(-,\tilde{B})"]
\end{tikzcd}$$

To understand this adjunction and see that it does indeed correspond to the morphisms explained in the first section, we will go through several examples.

6 Examples

Our goal in the following will be to build intuition for the schizophrenic object through a careful analysis of examples. In particular, we are going to start with a subexample, if you will, of a more profound duality called the Stone duality, simply by elucidating its schizophrenic nature as it ties directly to the generality we have learned, so that we can get a sense for in what direct way we can lift hom sets.

Consequently, in discussing the Stone duality we plan to give concrete descriptions of these Hom set functors, for which we had just given a more general description using universal properties and hom sets. Though we do lead with the general into the concrete, we may otherwise view our elucidation of the concrete as the generalization, since we must show the actual mechanics behind why we can think of these concrete objects as lifts of the hom sets.

This will then lend itself to the intuition of the following examples, which we can in some sense think of extensions or special cases of the given dualities.

For the examples note that we may sometimes by abuse of notation and for lack of a better functorial description use $\mathcal{A}(-, \tilde{A})$ to denote T, unless we denote otherwise.

7 The $\mathrm{Top} \rightleftarrows \mathrm{Frm}^{\mathrm{op}}$ duality

The aim of discussing this example is two-fold: first we want to concretize the maps using our general description in the first part of this paper. To this end we aim to be very explicit in this example, so that we just once get to see what these maps look like explicitly, then later we will drop the explicicity for readability.

Secondly, we want to provide argumentative insight for the following examples.

For this reason we will come to refer to this example as the leading example, as it will serve as a leading example for the following.

We begin with the adjunction

$$\begin{tikzcd} \mathrm{Top} \arrow[rr, bend left, "{[\mathrm{T}] = \mathrm{Top}(-,\, \mathbb{S})}"] \arrow[rr, phantom, "\perp"] & & \mathrm{Frm}^{\mathrm{op}} \arrow[ll, bend left, "{[\mathrm{S}] = \mathrm{Frm}(-,\, \mathbb{2})}"] \end{tikzcd}$$

with the following bijections induced by the unit and counit

$$\begin{aligned} \tau : [\mathbb{S}] &\to [\mathbb{2}] \\ \tilde{x} &\mapsto [[\epsilon_{\mathbb{S}}](\tilde{x})](1_{\mathbb{S}}) \end{aligned} \qquad\qquad \begin{aligned} \sigma : [\mathbb{2}] &\to [\mathbb{S}] \\ \tilde{y} &\to [[\eta_{\mathbb{2}}](\tilde{y})](1_{\mathbb{2}}) \end{aligned}$$

In this setting the respective representable objects are $A_0 = \{pt\}$ and $B_0 = b$, where b is here notation for the free frame generated by one object, i.e., the free 3-chain $\perp - b - \top$.

Now we know that $[TA_0] = \mathrm{Top}(\{pt\}, \mathbb{S}) = \mathbb{S}$ and $[SB_0] = \mathrm{Frm}(b, \mathbb{2}) = \mathbb{2}$, however it less immediate why $\mathrm{Top}(A, \mathbb{S})$ lifts to the category Frm and why $\mathrm{Frm}(B, \mathbb{2})$ lifts to the category Top.

For a Frm-structure on $\mathrm{Top}(A, \mathbb{S})$ we desire a lift which preserves the evaluation map $\sigma\varphi_{A,x}$, and which is initial among such lifts. This means we want the weakest Frm-structure such that for all $x \in A$ and $A \in \mathrm{Top}$ evaluation maps $\sigma\varphi_{A,x}$ lift to frame homomorphisms $e_{A,x}$.

We know that the evaluation is a frame homomorphism if and only if it preserves pointwise order in $\mathbb{2}$, since $e_{A,x}(u \leq v) = e_{A,x}(u) \leq e_{A,x}(v) = u(x) \leq v(x)$, and conversely a pointwise order in $\mathbb{2}$ determines limits and colimits by definition; $u(x) \wedge v(x) \leq (u \wedge v)(x)$ holds since the pointwise intersection is less than or equal to $u(x)$ and $v(x)$ respectively, and $(u \wedge v)(x) \leq u(x) \wedge v(x)$ holds since the intersection in the frame is less than or equal to u and v respectively and evaluating preserves pointwise order.

(TODO does a pointwise order preserving evaluation morphism into $\mathbb{2}$ preserve finite limits and colimits by some pointwise argument in the presheaf category? in other words, is there a way to abstract this argument so i dont have to prove it on elements?)

So for $A \in \mathrm{Top}$ we construct the Frm-structure on $\mathrm{Top}(A, \mathbb{S})$ by a preorder that preserves the pointwise order in $\mathbb{2}$ for all $x \in A$. As we want the weakest such structure, we define it such that for arbitrary $u, v \in \mathrm{Top}(A, \mathbb{S})$ we have $u \leq v$ if and only if $u(x) \leq v(x)$ for all $x \in A$.

This is the initial Frm-structure making all evaluation maps frame homomorphisms.

For a topology on $\mathrm{Frm}(A, \mathbb{2})$, we consider the family

$$\{\ \{p \in \mathrm{Frm}(A, \mathbb{2}) \mid p(x) = 1\} \mid x \in A\ \}$$

Our adjunction has the counit

$$\begin{aligned} \eta : & 1_{\mathrm{Frm}} \to TS \\ & B \mapsto TSB \\ & B \mapsto \mathrm{Top}(\mathrm{Frm}(B, \mathbb{2}), \mathbb{S}) \end{aligned}$$

so that

$$\eta_B : B \to \mathrm{Top}(\mathrm{Frm}(B, \mathbb{2}), \mathbb{S})$$

is such that its underlying map

$$[\eta_B] : [B] \to [\mathrm{Top}(\mathrm{Frm}(B, \mathbb{2}), \mathbb{S})] = \mathrm{Top}(\mathrm{Frm}(B, \mathbb{2}), \mathbb{S})$$

where

$$\begin{aligned} [\eta_B](y) : \mathrm{Frm}(B, \mathbb{2}) & \to \mathbb{S} \\ p & \mapsto p(y) \end{aligned}$$

is given by evaluation.

(TODO work out how to call these maps underlying set maps or not)

Notice that the topology on $\mathrm{Frm}(B, \mathbf{2})$ is the initial topology making all $(\eta_B(y))_{y \in B}$ continuous, since continuous maps $\eta_B(y)$ are uniquely determined by the preimage of $\{1\}$.

For each side of this adjunction respectively, we will see this kind of reasoning more often to argue initiality of lifts.

We would like, only on this example, to explicate exactly what the bijection of the schizophrenic object is, since this example is quite straightforward and leads to a very familiar map, namely the identity.

When passing to U and V, we have

$$\begin{aligned}
[\eta_{\tilde{B}}] : [\tilde{B}] &\to [\mathrm{Top}(\mathrm{Frm}(\tilde{B}, \mathbb{2}), \mathbb{S})] = \mathrm{Top}(\mathrm{Frm}(\tilde{A}, \mathbb{2}), \mathbb{S}) \\
(\tilde{y} : b \to y) &\mapsto (p \to p(y))
\end{aligned}$$

which is equal to

$$\begin{aligned}
[\eta_{\mathbb{2}}] : [\mathbb{2}] &\to [\mathrm{Top}(\mathrm{Frm}(\mathbb{2}, \mathbb{2}), \mathbb{S})] = \mathrm{Top}(\mathrm{Frm}(\mathbb{2}, \mathbb{2}), \mathbb{S}) && (4) \\
(\tilde{y} : b \to y) &\mapsto (1_{\mathbb{2}} \to 1_{\mathbb{2}}(y)) && (5)
\end{aligned}$$

So we have

$$\begin{aligned}
[\eta_{\mathbb{2}}](\tilde{y}) : \mathrm{Frm}(\mathbb{2}, \mathbb{2}) &\to \mathbb{S} \\
1_{\mathbb{2}} &\mapsto 1_{\mathbb{2}}(y)
\end{aligned}$$

and now

$$\begin{aligned}
[[\eta_{\mathbb{2}}](\tilde{y})] : [\mathrm{Frm}(\mathbb{2}, \mathbb{2})] &\to [\mathbb{S}] \\
(b \to 1_{\mathbb{2}}) &\mapsto (\{pt\} \to 1_{\mathbb{2}}(y))
\end{aligned}$$

so that

$$[[\eta_{\mathbb{2}}](\tilde{y})](1_{\mathbb{2}}) = (\{pt\} \to 1_{\mathbb{2}}(y)).$$

Now we can see that the map

$$\begin{aligned}
\sigma : [\mathbb{2}] &\to [\mathbb{S}] \\
(\tilde{y} : b \to y) &\mapsto (\{pt\} \to 1_{\mathbb{2}}(y))
\end{aligned}$$

is the identity in Set, as the underlying set on both sides is $[\mathbb{2}] = [\mathbb{S}] = \{0, 1\}$ so that we are looking at the set map

$$\begin{aligned}
\{0, 1\} &\to \{0, 1\} \\
y &\mapsto 1_{\{0,1\}}(y) = y
\end{aligned}$$

(TODO: Change all maps in $[-]$ from $A_0 \to x$ to simply x and make a note about it when talking about why we can make this identification in the beginning.)

This map is clearly the identity. This boils down to the fact that our choice of morphism from $\mathrm{Frm}(\mathbb{2}, \mathbb{2})$ was easy to determine since the set $\mathrm{Frm}(\mathbb{2}, B) = \{pt\}$, as $\mathbb{2}$ is an initial object in Frm, and in particular it is clear in (1) that $\mathrm{Frm}(\mathbb{2}, \mathbb{2}) = \{1_{\mathbb{2}}\}$. From this we can deduce that σ is the identity on $\bar{U}(\mathbb{2})$.

In general, our schizophrenic object will not necessarily be initial in arbitrary concrete duality, and as such, our choice $p \in \mathcal{A}(\tilde{A}, \bar{A})$ may not be unique nor easy to determine, so that σ, though always a bijection, is not necessarily always the identity.

8 Stone duality

(TODO: introduce boolean algebras, as well as ideals, filters, ultrafilters, principle ultrafilters, maybe in the beginning? question for Georg)

Before we begin we would like to call attention to the fact that the Stone duality is wrought with interesting subdualities, some of which are themselves examples. It is not explicit yet what we mean by subduality, since here we are using it in a more broad sense, however in this section we will make that explicit. Though describing these subdualities may feel like a diversion, we find they actually elucidate the structure of the Stone duality as well as provide us with intuition for the following examples, so we plan to describe them.

Our strategy is as follows: first we describe a the natural dual adjunction between FinSet and FinBool, which we will later see, can be obtained by restricting the Stone duality to full subcategories. However that is not how we will be constructing the Stone duality.

We start with this example because it is easier to understand and provides some groundwork for the following dualities.

On the other hand, we will actually obtain the Stone duality via restriction and composition of some different dualities, namely the dualities $\mathrm{Top} \rightleftarrows \mathrm{Loc}$, which we just saw, and $\mathrm{CohLoc}^{\mathrm{op}} \rightleftarrows \mathrm{DLat}$, between the category of coherent locales and coherent maps between them, which we will later define, and the category of distributive lattices with lattice morphisms.[3]

And finally we will show why the finite duality is a restriction of the Stone duality.

8.a Duality between finite boolean algebras and finite sets

Before we speak about the Stone duality we plan to discuss one of its subdualities. However we should be clear what we mean by subduality.

[3]Some authors choose to differentiate between DLat and $\mathrm{DLat}_{\mathrm{bdd}}$, but we will go by the convention of [Joh] who chooses to define semi-lattices to include the bottom and top elements, respectively.

Definition 8.1 (subduality)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two categories for which there exists an duality, i.e., an adjunction $\mathcal{A}^{\mathrm{op}} \rightleftarrows \mathcal{B}$, given by left and right adjoint functors $T : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{A}^{\mathrm{op}}$, respectively.*

If $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{B}' \subset \mathcal{B}$ are subcategories on which S and T restrict to a duality, we call $\mathcal{A}'^{\mathrm{op}} \rightleftarrows \mathcal{B}'$ given by $T|_{\mathcal{A}'}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ and $S|_{\mathcal{B}'}\colon \mathcal{B}' \to \mathcal{A}'^{\mathrm{op}}$ a **subduality** *of $\mathcal{A}^{\mathrm{op}} \rightleftarrows \mathcal{B}$.*

Note that we may change perspective from talking about subdualities of the leading example, or of the Stone duality.

The first subduality we will discuss is a subduality of the Stone duality.

Consider the category Bool of Boolean algebras and boolean algebra homomorphisms between them. We claim there is an adjunction between FinBool and FinSet given by $\mathrm{Ult}(B)$, which sends a boolean algebra B to the set $\mathrm{Ult}(B)$ of ultrafilters on B, and by $\mathcal{P}(X)$, which sends a set X to its power-set.

$$\begin{tikzcd}
\mathrm{FinSet} \arrow[rr, bend left, "\mathcal{P}(-)"] \arrow[rr, phantom, "\perp"] && \mathrm{FinBool}^{\mathrm{op}} \arrow[ll, bend left, "\mathrm{Ult}(-)"]
\end{tikzcd}$$

Notice that $\mathcal{P}(X)$ is a boolean algebra, since it has distributive lattice structure and every subset S has a well-defined complement $X - S$.

It is also the case that $\mathrm{Ult}(-)$ is fully faithful as long as A is a finite boolean algebra, since ultrafilters on those are principle, meaning that they are generated by a single element, an element which we can then identify with the points of X in a faithful way. And so the unit $\mathrm{Ult}(\mathcal{P}(X)) \to X$ of the adjunction is an isomorphism.

Similarly, $\mathcal{P}(-)$ is fully faithful , as sets have unique power sets up to isomorphisms, and their maps induce unique boolean algebra homomorphisms (TODO show this), so that the counit $B \to \mathcal{P}(\mathrm{Ult}(B))$ is an isomorphism, and so the adjunction above is a dual equivalence.

To determine the schizophrenic object first consider the free boolean algebra on one generator b is the set $\diamondsuit := \{\bot, b, \neg b, \top\}$, since our generator needs to induce complements (TODO and has finite limits and arbitrary colimits), and the free set on one generator is $\{pt\}$.

Now an ultrafilter on $\diamondsuit$ is determined by which element the bottom (and analogously top) associates to, which is an element of the set $\{b, \neg b\} = 2$, which we will prove in the following. But from that it holds that $\mathrm{Ult}(\diamondsuit) = 2$.

Meanwhile $\mathcal{P}(\{pt\}) = \{\emptyset, \{pt\}\} = 2$ as boolean algebra, since the point and the empty set are complements, and thus form the truth-value boolean algebra which is an initial object of Bool.

Now in the following we want to show that such lifts exist, and we check that by showing that the adjunction we gave serves as a lift of the respective hom sets.

For the power-set functor, it is easy to see that $\mathcal{P}(X) = \mathrm{Set}(X, 2)$ since any set map $X \to 2$ is given uniquely by a subset $S \subset X$, via characteristic functions

$$\mathcal{X}_S(x) = \begin{cases} 1 & x \in S \\ 0 & \text{else.} \end{cases}$$

On the other hand, for a boolean algebra A we have the following bijection

$$\begin{aligned} \phi : \mathrm{Ult}(A) &\to \mathrm{Bool}(A, 2) \\ F &\mapsto f_F \qquad\qquad f_F(x) = \begin{cases} 1 & x \in F \\ 0 & \text{else} \end{cases} \end{aligned}$$

The proof is straightforward and boils down to an equivalence between properties of boolean algebra homomorphisms and prime filter axioms.

Lattice homomorphisms are order preserving, finite limit preserving, and finite colimit preserving.

Through the above map, we see that this corresponds to axioms of being a prime filter F, which are being a sub meet-semilattice (closed under finite meets), upwards closed, and having the property of being prime: $\bot \notin F$ and $a \vee b \in F \implies a \in F$ or $b \in F$.

Now if our setting is Boolean, then we have that prime filters correspond to ultrafilters, which, given the axioms of being a filter, will additionally require that $A \in F \iff \forall B \in F : B \cap A \neq \emptyset$.

The property that F is prime is equivalent to F being given above by $f^{-1}(1)$ for $f \in \mathrm{DLat}(A, 2)$ which is further equivalent to the fact that its complement $I := f^{-1}(0)$ is an ideal [Joh].

So if F is prime, then $\emptyset \notin F$ gives the forwards direction of the ultrafilter condition and for the other direction supposing $A \in I$, then $\neg A \in F$, but $A \cap \neg A = \emptyset$ contradicting our assumption implying $A \in F$.

Conversely if F is an ultrafilter, then it cannot contain the complement of any of its elements, which is just $\neg f(A) = f(\neg A)$.

Note that frame homomorphisms additionally require preservation of arbitrary colimits, which simply upgrades the prime condition to completely prime, i.e. $\bigvee a_i \in F \implies \exists i \in I : a_i \in F$. *However in our above setting of* A *in* FinBool*, this is no condition at all.*

Ultimately, what a point map $F \overset{f}{\rightleftarrows} 2$ in DLat represents is the question of which subsets of our lattice are compatible with our desired structure restrictions.

As we have just seen, that means if our lattice is a frame, the points correspond to completely prime filters on F, and if it is a boolean algebra, they correspond to ultrafilters.

Note that the unit and counit maps then follow the exact same logic as the leading example, and in particular, $\mathbb{2}$ is also initial in Bool so that by the same logic, our σ is equal to the set identity $1_{\mathbb{2}}$.

8.b The subdualities Top $\rightleftarrows$ Frm$^{\mathrm{op}}$ and CohLoc $\rightleftarrows$ DLat

In the following we will want to give a general overview of how we obtain the Stone duality in the first place.

Before we define it we first focus on the following adjunction

$$\begin{tikzcd} \mathrm{Top} \arrow[rr, bend left, "\Omega(-)"] \arrow[rr, phantom, "\perp"] & & \mathrm{Loc} \arrow[ll, bend left, "\mathrm{pt}(-)"] \end{tikzcd}$$

which is the adjunction in our leading example. This is indeed given by $\mathrm{pt}(A) = \mathrm{Frm}(A, \mathbb{2})$, and $\mathrm{Top}(X, \mathbb{S}) = \Omega(X)$, which is the frame of opens on X, since a continuous map into the Sierpinski space is uniquely determined by the preimage of $\{1\} \subset \mathbb{S}$, which corresponds to a unique open set of X.

Indeed, we may define the category of ***spatial locales*** and of ***sober spaces*** respectively to be the largest subcategories of $\mathrm{Loc} = \mathrm{Frm}^{\mathrm{op}}$ and Top, such that the adjunction is an equivalence. So now we have

$$\begin{tikzcd} \mathrm{SobTop} \arrow[rr, bend left, "\Omega(-)"] \arrow[rr, phantom, "\overset{\cong}{\perp}"] & & \mathrm{SpatLoc} \arrow[ll, bend left, "\mathrm{pt}(-)"] \end{tikzcd}$$

Note that this further restricts to the duality CohTop $\rightleftarrows$ CohLoc, whose categories we define in the following:

Let B be a locale. Then
Definition 8.2. *We call an element $b \in B$ **finite**, if for all $S \subset A$ such that $\bigvee S \geq b$ there exists a finite $F \subset S$ such that $\bigvee F \geq b$.*

This definition is equivalent to the following two statements [Joh]:

1. For all directed $S \subset B$ such that $\bigvee S \geq b$ there exists $s \in S$ such that $s \geq b$.
2. For all ideals $I \subset B$ such that $\bigvee I \geq b$ it holds that $b \in I$.

Notice that if B is spatial, which implies the existence of a topological space $X = \mathrm{pt}(B)$ such that $B = \Omega(X)$, then the finite elements are precisely the compact open subsets of X.
Definition 8.3. *We call B **coherent** if the following conditions hold:*

1. *Every element $b \in B$ can be given as a join of finite elements*
2. *The finite elements $K(B)$ make up a sublattice of B.*

Since $K(B)$ is a sub join-semilattice of B (closed under finite joins) [Joh], the second condition is equivalent to $1 \in K(B)$ and $K(B)$ is closed under finite meets.

If B is spatial, this means that $K(B)$ forms a basis for the topology on X. In fact this is an equivalence: coherent locales are spatial [Joh].

On the other hand, any sober topological space X can be given as $X = \mathrm{pt}(\Omega(X))$, but are not necessarily generated by compact elements. This motivates the definition of a coherent topological space to focus on topologies on X where the compact open subsets, or $K(\Omega(A))$, are closed under finite intersection and generate the topology of X.

So a ***coherent space*** is a sober space which satisfies the above condition.

Given $A, B \in \mathrm{CohLoc}$, we know that any lattice homomorphism $K(A) \to K(B)$ extends to a frame homomorphism $A \to B$, since $K(A)$ freely generates A (TODO: make sense of this for yourself), however the converse is not true, frame homomorphisms do not necessarily preserve finiteness.

In other words, the preimage of compact subsets under continuous maps is not necessarily compact.

So we may define a locale map $B \xrightarrow{f} A$ between coherent locales to be ***coherent*** if f^* maps $K(A)$ to $K(B)$. Similarly a continuous map $B \xrightarrow{f} A$ is coherent if $f^{-1}(K\Omega(A)) \subset K\Omega(B)$.

Unlike the other examples, that means the restriction of SpatLoc to CohLoc is not full.

In particular, we should be careful regarding how this affects how we interpret the schizophrenic object of this adjunction.

The preimage of coherent topological maps must preserve compact opens, and $\{1\}$ is a compact open set in $\mathbb{S}$, so that $\mathrm{CohTop}(X, \mathbb{S}) = K\Omega(X)$.

Strictly speaking, this adjunction does not have a schizophrenic object, since we cannot obtain the opens by lifting hom-sets anymore, and we choose not to restrict the image of the adjunction. However we may loosely interpret the adjunction as having a schizophrenic object in that it is a subdual equivalence of a duality with a schizophrenic object, namely, of $\mathrm{Top} \rightleftarrows \mathrm{Loc}$.

That is to say, we have a dual equivalence

$$\begin{tikzcd} \mathrm{CohTop} \arrow[rr, bend left, "\Omega(-)"] \arrow[rr, phantom, "\overset{\cong}{\perp}"] & & \mathrm{CohLoc} \arrow[ll, bend left, "\mathrm{pt}(-)"] \end{tikzcd}$$

Now we will want to compose. Consider the fact that a locale is coherent if and only if it is isomorphic to the locale of ideals of a distributive lattice [Joh]. This gives us a functor $\mathrm{Idl}(-) : \mathrm{DLat} \to \mathrm{CohLoc}$ (TODO: check that this is a functor).

We can also view $K(-)$ as a functor by sending frame homomorphisms to their restrictions on finite elements, which we can do since, as we have seen above, we have defined CohLoc to be exactly the category where such a restriction is well defined.

This shows us that we have a duality (TODO: show that this is an adjunction)

$$\begin{tikzcd} \mathrm{CohLoc}^{\mathrm{op}} \arrow[rr, bend left, "K(-)"] \arrow[rr, phantom, "\perp"] & & \mathrm{DLat} \arrow[ll, bend left, "\mathrm{Idl}(-)"] \end{tikzcd}$$

Now we want to compose these dualities $\mathrm{CohTop}^{\mathrm{op}} \rightleftarrows \mathrm{CohLoc}^{\mathrm{op}} \rightleftarrows \mathrm{DLat}$.

This duality sends a distributive lattice A to what we call its ***spectrum***, in other words, $\mathrm{Spec}(A) := \mathrm{pt}(\mathrm{Idl}(A))$. On the other side we shall send a coherent space X to its lattice of compact open subsets $K(\Omega(X))$.

But we aren't done yet, as we will then want to restrict the composition.

Seeing that $\mathrm{Spec}(A)$ is Hausdorff if and only if A is a boolean algebra [Joh], we shall see that we can restrict this equivalence to a duality between Bool and the category of coherent, Hausdorff spaces which are called ***Stone spaces***.

Note that Stone spaces are compact and totally disconnected [Joh] making them a subcategory of kHaus. Moreover, if X is Hausdorff, then $K(\Omega(X)) = \mathrm{Clop}(X)$, since compact sets are exactly the closed sets of a compact Hausdorff space.

8.c Schizophrenic object of the Stone duality

As we have seen, we have the following adjunction

$$\begin{tikzcd} \mathrm{Stone}^{\mathrm{op}} \arrow[rr, bend left, "\mathrm{Clop}(-)"] \arrow[rr, phantom, "\perp"] & & \mathrm{Bool} \arrow[ll, bend left, "\mathrm{Spec}(-)"] \end{tikzcd}$$

Now remember our schizophrenic object $(2, 1_2, 2)$. The question is then twofold:

1. Do the hom-sets lift?
   (a) Can we give the spectrum as a lift of the set of points of a boolean algebra, i.e. $[\mathrm{Spec}(B)] = \mathrm{Bool}(B, 2)$?

  (b) Can we give the boolean algebra of clopens of a stone space as a lift of the set of continuous maps over compact Hausdorff spaces, i.e., does $[\mathrm{Clop}(X)] = \mathrm{Stone}(X, \mathbb{2})$?

2. Are these lifts initial?

In order to answer these questions, we will first show that the composite subduality CohTop $\rightleftarrows$ DLat has a schizophrenic object, and then see what happens when we restrict to the Stone duality.

First we look to the leading example and remember our points functor. Remember at the outset when we identified ultrafilters of a finite set X with points of a finite boolean algebra, we showed that points of any lattice can be identified with its prime filters.

But in the background of that example we actually sent a boolean algebra B to its frame of ideals $\mathrm{Idl}(B)$, and then considered the points of that frame, which correspond to completely prime filters of $\mathrm{Idl}(B)$. These correspond to prime filters P of B by the bijective assignments $P \mapsto \{I \in \mathrm{Idl}(B) | I \cap P \neq \emptyset\}$ and $F \mapsto \{b \in B | \downarrow(b) \in F\}$ [Joh] so that the prime filters of B correspond to the spectrum. There was nothing about this argumentation that required B to be boolean, as it works for any distributive lattice.

In a word, the points of the locale of ideals of A, or the *spectrum* of A, are precisely its set of prime ideals, given by the lift of the functor $\mathrm{DLat}(-, \mathbb{2})$.

Such a lift is initial by the same argumentation as in the leading example: we simply take the initial topology such that all evaluation maps $\mathrm{DLat}(B, \mathbb{2}) \to \mathbb{2}$ are continuous, which have as a subbasis $\{\ \{p \in \mathrm{DLat}(B, \mathbb{2}) | p(b) = 1\} \mid b \in B\ \}$.

(TODO Compute/confirm that this lifts initially to CohTop)

For the second map, consider that we also have the initial lift of the adjunction $\mathrm{Top} \overset{\mathrm{Top}(-,\mathbb{S})}{\rightleftarrows} \mathrm{Frm}^{\mathrm{op}}$.

Remember that each restriction is simply the free generator or left adjoint of the forgetful functor, whose underlying functors into subcategories of Top compose with the underlying functor into Top.

Since coherentifying a topology on X doesn't do anything to the Sierpinski space (all elements in $\mathbb{S}$ are compact open, and taking unions of finite intersections of $\mathbb{S}$ gives us back $\mathbb{S}$), we first notice that after composing, we have a functor CohTop $\rightleftarrows$ DLat that lifts $\mathrm{CohTop}(X, \mathbb{S})$ to DLat, since coherent maps are precisely those which preserve compact opens, i.e., as we have already seen $\mathrm{CohTop}(X, \mathbb{S}) = K\Omega(X)$, so that $\mathrm{CohTop}(X, \mathbb{S})$ lifts to DLat: it is exactly the distributive lattice of compact open subsets of X.

Now we see what happens when applying the restrictions.

On the topological side, there is the restriction $\mathrm{CohTop} \overset{\mathrm{Stone}(-)}{\rightleftarrows} \mathrm{Stone}$, which is called the stone space reflection, or stoneification. On the other side, DLat restricts to Bool by the corresponding boolean algebra reflection functor, or Booleanification.

To understand the Stoneification, we may first want to booleanify the lattice of compact open subsets of a space X, which we will call A. Let A^* be the set of complements in $\mathcal{P}(X)$ of memebers of A. Then the patch topology, which has as a base $C = \{U \cap V | U \in A, V \in A^*\}$, will be a Stone space (notice that we are taking the boolean completion B of A and then taking its spectrum) [Joh].

We must be careful here, because $\mathbb{S} \notin$ Stone, since $\mathbb{S}$ is clearly not Hausdorff. So we will need to take the boolean completion of $\mathbb{S}$ in $\mathcal{P}(\{0, 1\})$ which is simply 2_s, the two point discrete space, since by booleanifying we must generate the complement of $\{1\}$.

The argumentation however works the same way as it would if $\mathbb{S}$ was in Stone: the lift of $\mathrm{CohTop}(X, \mathbb{S})$ equals $[\mathrm{Clop}(X)]$ if X is a Stone space, since coherent maps into $\mathbb{S}$ are uniquely determined by the preimage of $\{1\}$, which are simply the compact open subsets of a compact Hausdorff space, which are its clopens.

By the same argument we have $\mathrm{Stone}(X, 2_s) = [\mathrm{Clop}(X)]$, and in fact, since we have Stonefied all objects, we are back to being in a full subcategory of Top, since continuous maps between compact Hausdorff spaces must be coherent (continuity means the preimage of every clopen set is clopen, and in this setting the clopens are the compact opens).

This is in particular true for any compact Hausdorff space, so that one has $\mathrm{kHaus}(X, 2_s) = \mathrm{Stone}(X, 2_s)$ for all $X \in$ Stone.

Moreover, considering all n-ary operations $2_s^n \xrightarrow{\gamma} 2_s$, where 2_s^n is given the product topology on 2_s, we get an induced map

$$\mathrm{Stone}(X, 2_s)^n \cong \mathrm{Stone}(X, 2_s^n) \xrightarrow{\gamma \circ -} \mathrm{Stone}(X, 2_s)$$

where the first isomorphism is the fact that Hom-sets preserve limits in the second argument, which is given by a natural map that combines components and its inverse which separates them.

This shows us that the left adjoint of the Stone duality is given as a lift of $\mathrm{Stone}(-, 2_s)$, and in particular, this lift is initial, by a similar argument as in the leading example.

$$
\begin{tikzcd}
\mathrm{Top} \arrow[rr, bend left, "\Omega(-)"] \arrow[rr, phantom, "\perp"] \arrow[d, bend left, "\mathrm{Sob}(-)"] \arrow[d, phantom, "\vdash"] & & \mathrm{Loc} \arrow[ll, bend left, "\mathrm{pt}(-)"] \arrow[d, bend left, "\mathrm{Spat}(-)"] \arrow[d, phantom, "\vdash"] & \\
\mathrm{SobTop} \arrow[u, hook, bend left] \arrow[rr, bend left, "\Omega(-)"] \arrow[rr, phantom, "\overset{\cong}{\perp}"] \arrow[d, bend left, "\mathrm{Coh}(-)"] \arrow[d, phantom, "\vdash"] & & \mathrm{SpatLoc} \arrow[u, hook, bend left] \arrow[ll, bend left, "\mathrm{pt}(-)"] \arrow[d, bend left, "\mathrm{Coh}(-)"] \arrow[d, phantom, "\vdash"] & \\
\mathrm{CohTop} \arrow[u, hook, bend left] \arrow[rr, bend left, "\Omega(-)"] \arrow[rr, phantom, "\perp"] \arrow[d, bend left, "\mathrm{Stone}(-)"] \arrow[d, phantom, "\vdash"] & & \mathrm{CohLoc} \arrow[u, hook, bend left] \arrow[ll, bend left, "\mathrm{pt}(-)"] \arrow[rr, bend left, "K(-)"] \arrow[rr, phantom, "\perp"] & & \mathrm{DLat}^{\mathrm{op}} \arrow[ll, bend left, "\mathrm{Idl}(-)"] \arrow[d, bend left, "\mathrm{Bool}(-)"] \arrow[d, phantom, "\vdash"] \\
\mathrm{Stone} \arrow[u, hook, bend left] \arrow[rrrr, bend left, "\mathrm{Clop}(-)"] \arrow[rrrr, phantom, "\overset{\cong}{\perp}"] & & & & \mathrm{Bool}^{\mathrm{op}} \arrow[u, hook, bend left] \arrow[llll, bend left, "\mathrm{Spec}(-)"]
\end{tikzcd}
$$

As a side remark, another reason our original equivalence of categories $\mathrm{FinSet}^{\mathrm{op}} \cong \mathrm{FinBool}$ might be interesting to us now is that by considering the Ind and Pro categories, we get the following equivalence:

$$\mathrm{Bool} \cong \mathrm{Ind}(\mathrm{FinBool}) \cong \mathrm{Ind}(\mathrm{FinSet}^{\mathrm{op}}) \cong \mathrm{Pro}(\mathrm{FinSet})^{\mathrm{op}}$$

which means we can identify Stone spaces with profinite sets.

9 Rings and Affine schemes

We now turn to an example that any graduate Algebra student has encountered, the duality between the category of rings and the category of affine schemes. We will use the more general category of commutative R-algebras, which we denote $\mathrm{CAlg}_{\mathrm{R}}$. Notice that if $R = \mathbb{Z}$, then $\mathrm{CAlg}_{\mathbb{Z}} = \mathrm{Ring}$.

This example is actually a non-example which we find nevertheless pedagogic, as it has a schizophrenic object on one side of the duality, but not the other. Moreover, given the nature of the functors at play it shall take the form of being a schizophrenic object without explicitly being one, and we shall elucidate why that is, and why we have nevertheless included it.

We already know from commutative algebra the adjunction that gives a dual equivalence, and we can easily show that one of these adjoint functors is isomorphic to the Hom functor.

We will nevertheless try to give some intuition about this isomorphism.

Firstly, we discuss the adjunction that one might learn in Algebra:

$$\begin{tikzcd} {\mathrm{CAlg_R}}^{\mathrm{op}} \arrow[rr, bend left, "X(-)"] \arrow[rr, phantom, "\perp"] & & \mathrm{Aff} \arrow[ll, bend left, "\mathcal{O}(-)"] \end{tikzcd}$$

Recall that an affine scheme $X \in \mathrm{Aff}$ is defined as a representable functor in the functor category $\mathrm{Fun}(\mathrm{CAlg_R}, \mathrm{Set})$.

Now we see that there is a natural choice for our adjunction given by sending a representable object to its functor, and sending that representable functor to its object. In other words we have $X : {\mathrm{CAlg_R}}^{\mathrm{op}} \to \mathrm{Aff}$ that sends $A \mapsto X_A = \mathrm{CAlg_R}(A, -)$ and $\mathcal{O} : \mathrm{CAlg_R} \to \mathrm{Set}$, which sends $X(-) = \mathrm{CAlg_R}(\mathcal{O}(X), -)$ to $\mathcal{O}(X)$, its representable object.

The equivalence is clear, since by construction our unit and counit are isomorphisms, i.e. $X_{\mathcal{O}(X)} = X$ and $\mathcal{O}(X_A) = A$.

Now on the one hand, the Yoneda lemma shows us that $\mathrm{Aff}(X, \mathbb{A}^1) \cong \mathbb{A}^1(\mathcal{O}(X)) = \mathrm{CAlg_R}(R[x], \mathcal{O}(X)) \cong [\mathcal{O}(X)]$. The final set isomorphism is due to the fact that $R[x]$ is a free commutative R-algebra on one free generator. Now we see that for any $X_A \in \mathrm{Aff}$ it holds that $[\mathcal{O}(X_A)] = \mathrm{Aff}(X_A, \mathbb{A}^1)$, and as such our natural candidate for a schizophrenic object is $(R[x], \tau, \mathbb{A}^1)$.

$$\begin{tikzcd} A \arrow[r, dashed, "1_A"] \arrow[d, "U"'] & SX_A \arrow[r, "d_{X_A,y}"] \arrow[d, "U"'] & R[t] \arrow[d, "U"] \\ {[A]} \arrow[r, "\cong"'] & \mathrm{Aff}(X_A, \mathbb{A}^1) \arrow[r, "\sigma\psi_{X_A,y}"'] & {[R[t]]} \end{tikzcd}$$

Since we have a canonical choice $[A] \cong \mathrm{Aff}(X_A, \mathbb{A}^1)$ of set isomorphisms, fully faithfulness of the Yoneda embedding U ensures that any morphism of lifts into X_A and A respectively are unique, and as such the lifts and $(A \overset{d_{X_A,y}}{\to} R[t])_{y \in [X_A]}$ is initial.

The problem on the other side however is that we cannot find an appropriate concrete functor V over which $\mathrm{CAlg_R}(A, R[t]) \overset{\tau\varphi_{A,x}}{\to} [R[t]]$ lifts to the category of affine schemes initially.

By Yoneda we see that

$$\mathrm{CAlg}_{\mathrm{R}}(A, R[x]) \cong X_A(R[x]) = \mathrm{Aff}(\mathbb{A}^1, X_A)$$

so if we could find such a functor we would have $\mathrm{Aff}(\mathbb{A}^1, X_A) \cong [X_A]$ suggesting that $\mathbb{A}^1$ is a free object on one free generator, however there does not exist a functor V so that this isomorphism holds. Although Aff is concrete (just take the functor $[\mathcal{O}(-)]$), such a functor does not lift $\tau\varphi_{A,x}$ in a way that can identify the set $\mathrm{CAlg}_{\mathrm{R}}(A, R[t])$ with the functor X_A.

10 Gelfand Duality

In order to describe the following duality, some context is in order. In setting up a more general dual adjunction, whose restriction to an equivalence later defines the *Gelfand duality*, we must set the scene, and in doing so we first define the following category, *Kelley spaces.*

10.a Kelley Spaces

Of primary importance to a Kelley space is the notion of a compactly generated topological space.

Definition 10.1 (k-continuous)**.** *A function $f : X \to Y$ of underlying sets of a topological space is said to be k-**continuous** if for all compact K and continuous functions $t : K \to X$ the composition $f \circ t$ is continuous.*

Definition 10.2 (k-space)**.** *A topological space X is said to be a k-**space**, or a **compactly generated topological space**, if for all spaces Y and underlying-set functions $f : X \to Y$, it holds that f is continuous if and only if f is k-continuous.*

That is to say a space is compactly generated if all its continuous functions are continuous on compact subspaces. Note that the domain in the definition of arbitrary compact space Y could be restricted to compact subsets $K \subset X$ since images of compact spaces by continuous maps $f(Y) \subset X$ are homeomorphic to compact subspaces $K \subset X$, so in particular, f factors through the inclusion $\iota : K \hookrightarrow X$.

For the following we will assume that k-spaces are additionally Hausdorff, and refer to kSp as the category of compactly generated Hausdorff spaces, whose morphisms are the continuous functions between them, making it a full subcategory of Top, and in particular, of Haus $\subset$ Top.

If given a topological space which may or may not be a k-space, we may force the compactly generated condition on it through a process which we call the *Kelleyfication* of a topological space.

That is, given the set of inclusions $(K \overset{t_i}{\hookrightarrow} X)_{i\in I}$ of compact subspaces $K \subset X$, we give X the finest topology making all t_i continuous.

That is, given an underlying-set function $X \xrightarrow{f} Y$, we give X the topology such that f is continuous if and only if $f \circ t_i$ is continuous.

But this is just the universal property of the colimit applied to topological spaces (and their full subcategories): for any set of continuous functions $K_i \xrightarrow{\varphi_i} Y$ into some topological space Y that satisfy commutativity (i.e., if there is a continuous map $K_i \xrightarrow{h} K_j$ for some $i, j \in I$, then $\varphi_i = \varphi_j \circ h$), then there exists a unique continuous map $X \xrightarrow{f} Y$ such that $\varphi = f \circ t_i$.

This is reflected by commutativity of the following diagram:

$$\begin{tikzcd}
& Y & \\
& X \arrow[u, dashed, "f"] & \\
K_i \arrow[ur, "t_i"] \arrow[uur, bend left, "f\circ t_i"] \arrow[rr, "h"'] & & K_j \arrow[ul, "t_j"'] \arrow[uul, bend right, "f\circ t_j"']
\end{tikzcd}$$

That is if $f \circ t_i$ is continuous, preimages of open sets $V \subset Y$ must be open under composition, in other words $t_i^{-1}(f^{-1}(V)) \subset K_i$ is open.

So $f \circ t_i$ induces a continuous $f : X \to Y$ in the following way: given a topological space with underlying set X, the colimit out of compact Hausdorff subspaces will be its *Kelleyfication*, which is necessarily a refinement of the topology of X, since continuous maps $X \xrightarrow{f} Y$ necessarily satisfy commutativity of the above diagram, so we want to add opens to X which satisfy commutativity for arbitrary function $X \xrightarrow{f} Y$.

That is, if X is not already Kelleyfied, we add opens $U = f^{-1}(V)$, for functions f such that $t_i^{-1}(U) \subset K$ is open but $U \subset X$ is not. This is the universality condition, since commutativity must be satisfied for arbitrary function f. But this is our original statement: we want a topology on X such that f is continuous if and only if $f \circ t_i$ is continuous.

Thus we can understand a k-space as a colimit of compact Hausdorff spaces, or specifically, $X \in \mathrm{kSp}$ if and only if $X = \mathrm{colim}_{K\subset X \text{ compact}} K$.

We may otherwise view the *Kelleyfication* as the left adjoint $k(-)$ to the forgetful functor. As kSp is a full subcategory of Haus, this puts us in the setting of a coreflective subcategory of Haus $\subset$ Top.

Another way to think about kSp is as the coreflective hull of kHaus in Haus. That is to say, the forgetful functor kHaus $\hookrightarrow$ Haus does not have a left adjoint (TODO: Why? Does it have something to do with that kHaus is not closed under colimits), however we may take the intersection of all coreflective subcategories of Haus which contain kHaus, and this will precisely be the category generated under colimits of kHaus (TODO: make this precise.)

Note that products are given as the Kelleyfications of topological products, and furthermore, that for all locally compact spaces X, we have $X \times_k Y = X \times Y$ for all $Y \in \mathrm{kSp}$. (TODO show this).

It is important that our category admits function spaces, or in other words, that we are in a closed category (i.e. all exponentiables exist). In the following we will show why kSp is a closed category. This will indeed give us intuition of why we have even defined kSp the way we have, as Haus is not a closed category.

What we want is the following adjunction:

$$\mathrm{Haus}(A \times_{\mathrm{Haus}} C, B) \cong \mathrm{Haus}(C, B^A)$$

For all $A, B, C \in \mathrm{Ob}(\mathrm{Haus})$. However this does not hold in general.

However in kSp function spaces are given as Kelleyfications of the set $\mathrm{Top}(X, Y)$ with the compact-open topology. (TODO why?). ToDO Show that this means that kSp is cartesian closed concrete category which admits function spaces. And as per [Dub], we see that kSp admits function spaces.

First we note that kSp is a coreflective subcategory of Top. Since kSp is a coreflective subcategory of a topological category, it is itself a topological category (Herrlichs theorem TODO Find and cite this), and in particular it is monotopological.

Note that Top is a topological category since for any source in Set we can always take the initial topology making every map in the source continuous.

(ToDO Show that kSp admits function spaces)

Now let us define the category kAlg of complex k-algebras, whose objects are $\mathbb{C}$-algebras endowed with a Kelley topology, whose morphisms are the continuous algebra homomorphisms, and whose algebra operations are continuous with respect to k-products.

Now since kSp is a monotopological category which admits function spaces, we are in the setting of a category which admits an internal hom-functor.

To put ourselves completely in the situation of **4.c** we notice that $\mathrm{kAlg} \subset \mathrm{kSp}$ is a full subcategory, so that we have an underlying functor $U : \mathrm{kAlg} \to \mathrm{kSp}$, which satisfies the conditions of $4.c$ in the following. As $\mathbb{C}$ is a colimit of its compact Hausdorff subspaces which is moreover a $\mathbb{C}$-algebra, we see that $\mathbb{C} \in \mathrm{kAlg}$, and we call it $\mathbb{C}_a$, so that $U(\mathbb{C}_a) = \mathbb{C}_s$, by analogous notation, and clearly $\tau = 1_{\mathbb{C}}$.

Since we are in a monotopological category, then we know that $\mathrm{kAlg}(A, \mathbb{C}_a) \to [\mathbb{C}_s]$ lifts initially to a kSp-morphism, which we will denote $\mathrm{Hom}_k(A, \mathbb{C}_a) \to \mathbb{C}_s$.

Now all we need to see is that $\mathrm{kSp}(X, \mathbb{C}_s) \to [\mathbb{C}_a]$ lifts along U functorially to a kAlg-morphism $C_k(X, \mathbb{C}_s) \to \mathbb{C}_a$, which we must show directly.

To show this we first want to see that $C_k(X, \mathbb{C}_s)$ has the necessary internal operations. For that we first notice that the functor $C_k(X, -)$ is product preserving (as a coreflection we can compute limits in Top and then Kelleyfy).

So for instance we obtain addition by applying $C_k(X, -)$ to the corresponding addition operation $+ : \mathbb{C}_a \times \mathbb{C}_a \to \mathbb{C}_a$ in $\mathbb{C}_a$ but viewed as $\mathbb{C}_s$, so that $C_k(X, +) : C_k(X, \mathbb{C}_s \times \mathbb{C}_s) \to C_k(X, \mathbb{C}_s)$ is isomorphic to

$$C_k(X, +) : C_k(X, \mathbb{C}_s) \times C_k(X, \mathbb{C}_s) \to C_k(X, \mathbb{C}_s).$$

For scalar multiplication we notice that the composition

$$\mathbb{C}_s \times C_k(X, \mathbb{C}_s) \times X \overset{1_{\mathbb{C}_s} \times \mathrm{ev}}{\longrightarrow} \mathbb{C}_s \times \mathbb{C}_s \overset{m}{\to} \mathbb{C}_s$$

induces a continuous scalar multiplication on $C_k(X, \mathbb{C}_s)$, by applying the Heyting implication to $\Big(\mathbb{C}_s \times C_k(X, \mathbb{C}_s)\Big) \times X \to \mathbb{C}_s$, to get

$$s : \mathbb{C}_s \times C_k(X, \mathbb{C}_s) \to C_k(X, \mathbb{C}_s).$$

As such we have an adjunction

$$\begin{aligned} C : \mathrm{kSp} &\to \mathrm{kAlg} \\ X &\mapsto C_k(X, \mathbb{C}_s) \end{aligned} \qquad\qquad \begin{aligned} S : \mathrm{kAlg} &\to \mathrm{kSp} \\ A &\mapsto \mathrm{Hom}_k(A, \mathbb{C}_a) \end{aligned}$$

which is not natural.

This is called the *generalized Gelfand-Naimark Duality.*

In the following we will want to restrict kSp and kAlg to their full subcategories under which the above adjunction is a dual equivalence. For that we will introduce the category of C^*-algebras.

10.b C^*-algebras

Consider the functor $C^* : \mathrm{Top} \to \mathrm{Set}$, which sends $X \mapsto C^*(X) = \mathrm{Top}_{bd.}(X, \mathbb{C})$ for all $X \in \mathrm{Top}$. From pointwise operations $C^*(X)$ can be seen to be an associative, commutative, unital $\mathbb{C}$-algebra. Pointwise conjugation gives us the operation $f \mapsto f^*$, so that $C^*(X)$ is an involutive algebra, and the supremum norm $\|f\| = \sup_{x \in X} |f(x)|$ turns $C^*(X)$ into a normed algebra satisfying $\|f\|^2 = \|f \cdot f^*\|$. If we consider the involution preserving unital $\mathbb{C}$-algebra homomorphisms, we obtain a category C^*.

Note that $C^* \subset \mathrm{kAlg}$ is a full subcategory.

We could view C^* as a concrete category via the usual underlying-set functor, however we might find it more interesting in this case to consider a different faithful functor, namely the functor $\bigcirc : C^* \to \mathrm{Set}$ which sends any C^*-algebra A to it's unit ball $\{a \in A \mid \|a\| \leq 1\}$ and each morphism f to its restriction $f_\bigcirc$ to the unit ball of its domain.

10.b.1 Gelfand-Naimark Duality

For any compact Hausdorff space X, we see that the C^*-algebra $C^*(X)$ and the function k-algebra $C(X) = C_k(X, \mathbb{C}_s)$ coincide algebraically, since continuous functions over compact spaces are bounded in $\mathbb{C}$ and since $X \times_k X = X \times X$ (TODO: why is $C^*(X) \subset C(X)$. Also make a comment about why $X \in \mathrm{kSp}$ for X locally compact). Moreover the topology of $C^*(X)$ is the compact open topology, since X is compact, and therefore they also coincide topologically. That means we have can restrict C to a functor $C : \mathrm{kHaus} \to C^*$.

Can we restrict S accordingly? For every C^*-algebra A, the space $S(A) = \mathrm{Hom}_k(A, \mathbb{C}_a)$ is compact, and its topology is that of pointwise convergence, due to basic results about the topology of function spaces. (TODO cite basic results). We call $S : C^* \to \mathrm{kHaus}$ the *spectrum*- functor, and it follows that the generalized Gelfand-Naimark adjunction restricts to a dual adjunction

$$
\begin{aligned}
C : \mathrm{kHaus} &\to C^* \\
X &\mapsto C_k(X, \mathbb{C}_s)
\end{aligned}
\qquad\qquad
\begin{aligned}
S : C^* &\to \mathrm{kHaus} \\
A &\mapsto \mathrm{Hom}_k(A, \mathbb{C}_a).
\end{aligned}
$$

However this still doesn't give us the equivalence, as $\mathbb{C}_s \notin \mathrm{kHaus}$. Remember that we want to consider C^* as a concrete category via the functor $\bigcirc$, so that for any compact Hausdorff space X, one has $\bigcirc C(X) = \mathrm{kHaus}(X, D)$ where $D = \{c \in \mathbb{C}_s \mid \|c\| \leq 1\}$.

Thus we can conclude that there is a natural dual adjunction between concrete categories (kHaus, U) and $(C^*, \bigcirc)$ with schizophrenic object $(D, 1_D, \mathbb{C}_a)$.

11 Galois theory

Let us dive into another familiar example called the *Galois correspondence*. The example we want to discuss is actually a generalization of the usual Galois correspondence from the fundamental theorem of Galois theory, given by the following contravariant adjunction between the directed posets (TODO define this) of subfield extensions of the finite field extension $k \hookrightarrow M \hookrightarrow L$ and of the corresponding subgroups H of the Galois group $\mathrm{Gal}(L/k)$:

$$
\begin{tikzcd}
\{L|M|k\}^{\mathrm{op}} \arrow[rr, bend left, "\mathrm{Gal}(L/-)"] \arrow[rr, phantom, "\perp"] & & \{H \leq \mathrm{Gal}(L/k)\} \arrow[ll, bend left, "L^{(-)}"]
\end{tikzcd}
$$

That is to say $\{L|M|k\}^{\mathrm{op}}$ is codirected since $L|k$ is an initial object of the poset, and similarly $\{H \leq \mathrm{Gal}(L/k)\}$ is directed since $\mathrm{Gal}(L/k)$ is terminal, corresponding to the full subgroup $\mathrm{Gal}(L/k)$.

It is easily checked that this defines an adjunction, however the schizophrenic object doesn't live here, so we would like to focus on a more general adjunction, which we can derive from the above adjunction in the following way.

The first important consideration is that we want our left (and by consequence, our right) category to include the separable closure of k, which we denote k_s, and naturally, all its intermediate finite field extensions. Intuitively, we want a field extension k_s whose intermediate field extensions $k^s|L$ are all normal *and* separable.

That's why we take k_s and not $\bar{k}$, the algebraic closure of k. For perfect fields, where every algebraic extension is separable, we have $k_s = \bar{k}$, by definition. However in general, we only have $k_s \subset \bar{k}$ so we cannot guarantee that an arbitrary algebraic extension of k is separable.

Now from the above left adjoint functor one may obtain an adjunction with respect to the set of cosets of each subgroup H in $\mathrm{Gal}(k_s/k) =: G$, which has a canonical G-action on it.

Explicitly this is a map $L|k \mapsto \mathrm{CAlg_k}(L, k_s)$, where $\mathrm{CAlg_k}(L, k_s)$ can be given by $\mathrm{Gal}(k_s/L) \setminus \mathrm{Gal}(k_s/k)$ (this shall be clear by the end of this section).

For any finite field extension $L|k$, homomorphisms into the closure k_s are given by maps which permute the roots of the minimal polynomials over k of the finite generators of $L|k$. For background reading one may consider any introductory textbook on Galois theory, such as *Bosch - Algebra*, where this claim follows directly from Lemma 3.4/8.

Since every element of k_s is a separable root over k, the canonical G-action on k_s which defines G in the first place is completely determined by how its elements permute the roots of separable minimal polynomials with coefficients in k.

Schizophrenic object of $\mathrm{CAlg_k} \rightleftarrows \text{G-Set}$ duality

In the following we shall want to show explicitly that the generalized Galois adjunction fits the schizophrenic framework to be considered a concrete duality.

To be sufficiently general we consider the category $\mathrm{CAlg_k}$ of commutative unital k-algebras with k-algebra homomorphisms, and the category G-Set of sets with a $\mathrm{Gal}(k_s/k)$-action and G-equivariant maps between them, and we plan to show that $(k_s, 1_{k_s}, k_s)$ is a schizophrenic object which induces the following adjunction

$$\begin{tikzcd} \mathrm{CAlg_k} \arrow[rr, bend left, "\mathrm{CAlg_k}(-{,}k_s)"] \arrow[rr, phantom, "\perp"] & & \text{G-Set} \arrow[ll, bend left, "\text{G-Set}(-{,}k_s)"] \end{tikzcd}.$$

First we should seek G-equivariant lifts of the evaluation maps $\mathrm{CAlg_k}(L, k_s) \to [k_s]$, i.e. lifts that preserves the G-action, which just means we can define the G-action of our lift of $\mathrm{CAlg_k}(L, k_s)$ pointwise on k_s, since

$$(g\varphi)(x) := \mathrm{ev}_{x,L}(g \cdot \varphi) = g \cdot \mathrm{ev}_{x,L}(\varphi) = g \cdot \varphi(x).$$

Since k_s comes with a canonical action, we know $L|k \mapsto \mathrm{CAlg_k}(L, k_s)$ lifts to the category G-Set.

Note that there is nothing specific about the canonical action for the above equalities to hold, so that for any G-action $\star$ on k_s there automatically exists a lift of $\mathrm{CAlg_k}(L, k_s) \to [k_s]$ whose G-action is defined pointwise by $\star$.

But we have good reason to consider the canonical G-action on k_s. Remember as a schizophrenic object this G-action must be fixed.

The question then becomes, given this G-action on k_s, can we derive a obtain a different G-action on $\mathrm{CAlg_k}(L, k_s)$ than the one we defined above?

Explicitly the inhereted G-action on $\mathrm{CAlg_k}(L, k_s)$ is given by postcomposition, i.e., $g \cdot_{\mathrm{CAlg_k}(L,k_s)} \varphi = g \circ \varphi$ for each $L \in \mathrm{CAlg_k}$. As G is a group of autormorphisms on k_s, a G-equivariant map $\mathrm{CAlg_k}(f, k_s) : \mathrm{CAlg_k}(L', k_s) \overset{-\circ f}{\to} \mathrm{CAlg_k}(L, k_s)$ means

$$g \cdot (\varphi \circ f) = (g \cdot \varphi) \circ f$$

holds. For postcomposition the equality is satisfied for all $g \in G$.

Notice that such a G-equivariant map which is compatible with lifts of the evaluation maps is given by a natural transformation $\mathrm{CAlg_k}(-, k_s) \overset{\eta}{\Longrightarrow} \mathrm{CAlg_k}(-, k_s)$, who by Yoneda is given by unique k_s-automorphisms $u_g : k_s \to k_s$, so that such an action is given by a group homomorphism $\theta : G \to G$, which sends $g \mapsto u_g$. As such, a G-equivariant map f satisfies the equality

$$g \cdot (\varphi \circ f) = (\theta(g) \circ \varphi)(f).$$

We want to show that $\theta = 1_G$.

In our set up we want a G-action on $\mathrm{CAlg_k}(L, k_s)$ such that for all $g \in G$ and $\varphi \in \mathrm{CAlg_k}(L, k_s)$ we have

$$g \cdot \varphi(x) = g \cdot \mathrm{ev}_{x,L}(\varphi) = \mathrm{ev}_{x,L}(\theta(g) \circ \varphi) = (\theta(g) \circ \varphi)(x).$$

Since this must hold for all $L \in \mathrm{CAlg_k}$, $x \in L$ and all $\varphi \in \mathrm{CAlg_k}(L, k_s)$, it must hold in particular for $L = k_s$ and $\varphi = 1_{k_s}$, then the above equality becomes $g(x) = \theta(g)(x)$ for all $x \in k_s$ and $g \in G$. Indeed this lift only exists if it is the canonical one, i.e. if $\theta = 1_G$.

That means there is only one way to lift $\mathrm{CAlg_k}(L, k_s) \to [k_s]$ to G-Set if k_s is considered with the canonical G-action, and therefore such a lift is initial.

On the other side, we want to show that $\text{G-Set}(H, k_s)$ lifts initially to $\mathrm{CAlg_k}$, where we take k_s to be equipped with the canonical G-action.

It is easily checked that the ring axioms can be defined pointwise in k_s.

Since G fixes k, then we have $\text{const}_a \in \text{G-Set}(H, k_s)$ for each $a \in k$, since $\text{const}_a(\theta(g) \cdot y) = g \circ \text{const}_a(y)$ holds true for all $g \in G$ and independent of θ. That means for any G-equivariant map $H \to H'$ the induced map $\text{G-Set}(H', k_s) \to \text{G-Set}(H, k_s)$ is stable on these constant functions. In other words it commutes with the map $k \xrightarrow{\gamma} \text{G-Set}(H, k_s)$, which sends $a \mapsto \text{const}_a$, which defines a ring homomorphism, so that $\text{G-Set}(H, k_s) \in \text{CAlg}_\text{k}$. Moreover this lift is clearly functorial in H, since composition preserves all pointwise operations.

Therefore we can determine our CAlg_k object by the datum $(\text{G-Set}(H, k_s), +, \cdot, \gamma)$. And let $e_{y,H}$ be the lift of $\text{ev}_{y,H}$ to CAlg_k.

Now let $(\oplus, \otimes, \gamma')$ be any other CAlg_k structure on $\text{G-Set}(H, k_s)$ such that every $\text{ev}_{y,H}$ lifts to a CAlg_k-morphism $e'_{y,H}$. Then

$$\begin{aligned}(f \oplus g)(y) = e'_{y,H}(f \oplus g) =& e'_{y,H}(f) + e'_{y,H}(g) \\ =& f(y) + g(y) = e_{y,H}(f) + e_{y,H}(g) \\ =& e_{y,H}(f + g) = (f + g)(y)\end{aligned}$$

for all $y \in H$ and $H \in \text{G-Set}$, so that $f \oplus g = f + g$. Similarly we get $f \otimes g = f \cdot g$ and $\gamma = \gamma'$. Therefore our pointwise algebra structure is the unique CAlg_k structure making all evaluation maps CAlg_k-morphisms, and in particular, it is the *initial* such lift.

This is analogous to many familiar examples where we derived our structure pointwise from the structure of our schizophrenic object, and as a result such a structure is the weakest such compatible structure, i.e. an initial lift.

Therefore our triple $(k_s, 1_{k_s}, k_s)$ is indeed a schizophrenic object, which means the diagram we gave at the beginning of this section is in fact a well defined adjunction, as it is the induced adjunction of our schizophrenic object.

Restricting to an equivalence of categories

Even though at this point the goal of this example, showing that we have a concrete duality, is complete, we would like to continue our discussion to bring us back to concrete Galois theory.

That is we would like to work only with the objects that we see in Galois theory, in particular, with finite extensions of k, as opposed to general algebras over k.

Given a finite separable field extension $L|k$, the set $\text{CAlg}_\text{k}(L, k_s)$ is finite and the G-action it inherits is transitive [Szam]. In particular, our above adjunction restricts to the following dual equivalence:

$$\begin{tikzcd}
\{\text{finite separable extensions of } k\}^{\mathrm{op}} \arrow[rr, bend left, "\mathrm{CAlg}_{\mathrm{k}}(-{,}k_s)"] \arrow[rr, phantom, "\overset{\cong}{\perp}"] & & \text{G-FinSet}_{\text{transitive}} \arrow[ll, bend left, "\text{G-Set}(-{,}k_s)"]
\end{tikzcd}$$

One may refer to Theorem 1.5.2 in *Szamuely* to see that the left adjoint functor induces a dual equivalence. We will indeed inadvertantly end up reproving this, however, our goal in this part is rather to underline what some constituents of this adjunction look like as we're now in a position to see our lifts a little more concretely.

Firstly, for a finite field extension $L|k$ we have an explicit description of $\mathrm{CAlg}_{\mathrm{k}}(L, k_s)$ as a set of homomorphisms which permute the roots of minimal polynomials of the generators of L. This is clearly a transitive finite G-set, since G permutes these finite roots transitively.

From the proof of Theorem 1.5.2 we also know that any transitive G-set H is given as $\mathrm{CAlg}_{\mathrm{k}}(L, k_s)$ for some finite field extension L by the assignment $g \circ \iota = g \cdot x$ for some $x \in H$, where $\iota : L \to k_s$ is the inclusion homomorphism. It is a group theoretic fact that then this is isomorphic to the left coset space $U_x \setminus G$, where U_x is the stabilizer of x.

That means a transitive G-set is encoded by the information of the stabilizer U_x of $x \in H$. This will be important to understand what finite field extension $\text{G-Set}(H, k_s)$ is.

Consider that $\text{G-Set}(H, k_s) = \text{G-Set}(\mathrm{CAlg}_{\mathrm{k}}(L, k_s), k_s)$ for some finite separable field extension $L|k$. We want both an explicit description of L that only depends on the G-action of H, and to see why $\text{G-Set}(\mathrm{CAlg}_{\mathrm{k}}(L, k_s), k_s) = L$.

Notice that a G-equivariant map $H \xrightarrow{f} k_s$ is completely determined by the image of a single element $x \in H$, since transitivity of H and the G action on k_s give us a full description of f via

$$f(g \cdot x) = g \cdot f(x).$$

Remember that U_x is a subgroup of G, so that by the fundamental theorem it fixes a Galois extension $L' = (k_s)^{U_x}$. We want to see that indeed $L' = L$.

We show first that the map $\text{G-Set}(H, k_s) \xrightarrow{\psi} (k_s)^{U_x}$ given by $f \mapsto f(x)$ is a bijection.

For any $h \in U_x$ we have $f(x) = f(h \cdot x) = h \cdot f(x)$ by G-equivariance, so that U_x fixes $f(x)$.

For an inverse consider $a \mapsto f_a$ where $f_a(x) = a$, given by $f_a(g \cdot x) = g \cdot a$. We just want to see that f_a is well defined and G-equivariant.

If $g_1 \cdot x = g_2 \cdot x$ then $g_2^{-1} \cdot g_1 \in U_x$, which means $g_1 \cdot a = g_2(g_2^{-1}g_1) \cdot a = g_2 \cdot a$.

Given $g' \in G$ and that $y = g \cdot x$, we have

$$f(g' \cdot y) = f(g' \cdot (g \cdot x)) = f((g'g) \cdot x) = (g'g) \cdot a = g' \cdot (g \cdot a) = g' \cdot (f(g \cdot x)) = g' \cdot f(y).$$

It is easily checked that these maps are inverse to one another.

Now replacing H with $\mathrm{CAlg_k}(L, k_s)$, we can use an analog to the above well-definedness argument to show why the stabilizer of ι is exactly U_x. And as ι is just the inclusion we have then $h(a) = a$ for any $a \in L$ and all $h \in U_x$, so that $L \subset (k_s)^{U_x}$.

For the other inclusion we simply see use the fact that any finite field extension is given as a fixed field of some subgroup of G, so that our stabilizer $U_\iota = \{g \in G | g \circ \iota = \iota\}$ is exactly $\mathrm{Gal}(k_s/L)$. That means if $g(a) = a$ for all $a \in (k_s)^{U_x}$, then $a \in L$.

Therefore $L|k = \text{G-Set}(\mathrm{CAlg_k}(L, k_s), k_s)$. Using this we can also derive directly that $H = \mathrm{CAlg_k}(\text{G-Set}(H, k_s), k_s))$, since restating gives us

$$\mathrm{CAlg_k}(\text{G-Set}(H, k_s), k_s)) = \mathrm{CAlg_k}(k_s^{U_x}, k_s) = \mathrm{Gal}(k_s/k_s^{U_x}) \setminus \mathrm{Gal}(k_s/k) = U_x \setminus G = H$$

This adjunction will then extend to another subdual equivalence of the Galois correspondence, namely between the full subcategory $\mathrm{k_{ét}} \subset \mathrm{CAlg_k}$ of finite dimensional algebras over k, defined to be isomorphic to a finite product of finite separable field extensions, and the full subcategory G-FinSet $\subset$ G-Set of finite G-sets:

$$\begin{tikzcd} {\mathrm{k_{ét}}}^{\mathrm{op}} \arrow[rr, bend left, "\mathrm{CAlg_k}(-{,}k_s)"] \arrow[rr, phantom, "\overset{\cong}{\perp}"] & & \text{G-FinSet} \arrow[ll, bend left, "\text{G-Set}(-{,}k_s)"] \end{tikzcd}$$

For this we only remark that any G-set can be seen as a disjoint union of transitive G-sets over each G-orbit, and that

$$\mathrm{CAlg_k}(\prod_I L_i, k) = \coprod_I \mathrm{CAlg_k}(L_i, k)$$

where I is the finite index set of the set of orbits $G \setminus \mathrm{CAlg_k}(\prod_I L_i, k)$. For details see *Szamuely*.

As a side remark, if our subfield extension $L|k$ is *Galois*, that is, it is a *normal* (separable, algebraic) field extension, i.e. L contains the roots of every polynomial over itself, then the fundamental theorem of Galois theory states that $\mathrm{Gal}(k_s/L) \trianglelefteq \mathrm{Gal}(k_s/k)$. This gives us a Galois group over a different functor, namely the functor $\mathrm{Gal}(-/k)$ which yields a profinite system $(k_s|L|k, \mathrm{Gal}(L/k))$.

Consider that $k_s|k = \operatorname{colim}_{\{k_s|L|k\}} L|k$, then we have

$$\begin{aligned}\mathrm{Gal}(k_s/k) &= \mathrm{CAlg_k}(\operatorname{colim}_{\{k_s|L|k\}} L, k_s) \\ &= \lim_{\{k_s|L|k\}} \mathrm{CAlg_k}(L, k_s) \\ &= \lim_{\{k_s|L|k\}} \mathrm{Gal}(k_s/k)/\,\mathrm{Gal}(k_s/L) \\ &= \lim_{\{k_s|L|k\}} \mathrm{Gal}(L/k)\end{aligned}$$

so that $\mathrm{Gal}(k_s/k)$ can be realized as a profinite group.

12 Questions

1. How do I approach the preliminaries part? Just a list of definitions? Or should I write some kind of narrating text? Should I move it to the appendix?
2. Ask Georg how to differentiate between k-Alg in terms of kelleyfied complex algebras and k-Alg in terms of algebras over a field k. We know that $\mathrm{kAlg} \subset \mathrm{CAlg_k}$ as subcategory (not full), should this be reflected in the notation?
3. How do I introduce subjects and where do i draw the line for assumed knowledge and not.
4. $\mathbb{A}^1$ is a free object on one free generator in Aff
5. Why do we know that Kelleyfied products and kelleyfied function spaces with the compact open topology are the categorical products and exponentials in kSp
6. G-Set(H, k_s) lifts to $\mathrm{CAlg_k}$ (possible for later)
7. $X \times_k Y = X \times Y$ for X locally compact, $Y \in \mathrm{kSp}$
8. ask Georg if pointwise order preserving evaluation morphism into $\mathbb{2}$ preserves finite limits and colimits by some pointwise argument in the presheaf category
9. why is $C^*(X) \subset C(X)$
10. $B \cap A \neq \emptyset$ for all $B \in F$ (filter) implies $B \cap A \in F$
11. Why is kSp a monotopological category? Why does it have function spaces?
12. can same schizophrenic object induce different dualities when looked at different categories? In what sense am I supposed to think of sub-dualities? (I guess so yeah)
13. Show that $C_k(X, \mathbb{C}) = C(X, \mathbb{C})$ if X is locally compact.