IV.10.38

Gelte $\mu(S) < \infty$ und $1 \leq r \leq p$. Dann folgt $\mathscr{L}^p(\mu) \subset \mathscr{L}^r(\mu)$ und $\|f\|_r \leq \mu(S)^{1/r-1/p}\|f\|_p$ für $f \in \mathscr{L}^p(\mu)$.

$$\text{Seien } f \in \mathscr{L}^p,\ \beta := \inf_{\substack{N \in \mathscr{A} \\ \mu(N)=0}} \sup_{s \in S \setminus N} |f(s)|$$

$$\xRightarrow{\mu(S)<\infty} \beta < \infty \Rightarrow \beta^x < \infty\ \forall x \in [1, p]$$

$$\Rightarrow \int_S |f|^p d\mu \leq \beta^p \mu(S) < \infty$$

$$\iff \int_S |f|^r d\mu \leq \beta^r \mu(S) < \infty$$

$$\Rightarrow f \in \mathscr{L}^r \Rightarrow \mathscr{L}^p(\mu) \subset \mathscr{L}^r(\mu)$$

Nun ist zu zeigen: $\|f\|_r \leq \mu(S)^{1/r-1/p}\|f\|_p$

$$\iff \Big(\int_S |f|^r d\mu\Big)^{1/r} \leq \frac{\mu(S)^{1/r}}{\mu(S)^{1/p}}\Big(\int_S |f|^p d\mu\Big)^{1/p}$$

$$\iff \Big(\frac{\int_S |f|^r d\mu}{\mu(S)}\Big)^{1/r} \leq \Big(\frac{\int_S |f|^p d\mu}{\mu(S)}\Big)^{1/p}$$

$$\iff \Big(\int_S |f|^r d\mu\Big)^{1/r} \leq \Big(\int_S |f|^p d\mu\Big)^{1/p}$$

IV.10.40

Es gilt weder $\mathscr{L}^r(\mathbb{R}) \subset \mathscr{L}^p(\mathbb{R})$ noch $\mathscr{L}^p(\mathbb{R}) \subset \mathscr{L}^r(\mathbb{R})$ für $1 \leq r < p \leq \infty$.

$$
\begin{aligned}
\text{Seien } & f \in \mathscr{L}^r(\mathbb{R}), g \in \mathscr{L}^p(\mathbb{R}) \\
\text{Wähle } & f := \frac{1}{x^{1/p}}, g := \frac{1}{x^{1/r}} \\
\text{Dann ist } & \int_{\mathbb{R}} |f|^r d\lambda = \int_{\mathbb{R}} \left|\frac{1}{x^{1/p}}\right|^r d\lambda \\
& = \int_{\mathbb{R}} \left|\frac{1}{x}\right|^{r/p} d\lambda < \infty \left(\text{da } \frac{r}{p} < 1\right) \\
\text{Beziehungsweise } & \int_{\mathbb{R}} |g|^p d\lambda = \int_{\mathbb{R}} \left|\frac{1}{x^{1/r}}\right|^p d\lambda \\
& = \int_{\mathbb{R}} \left|\frac{1}{x}\right|^{p/r} d\lambda < \infty \left(\text{da } \frac{p}{r} > 1\right) \\
\text{Aber } & \int_{\mathbb{R}} |f|^p d\lambda = \int_{\mathbb{R}} \left|\frac{1}{x^{1/p}}\right|^p d\lambda = \int_{\mathbb{R}} \left|\frac{1}{x}\right|^{p/p} d\lambda \\
& = \int_{\mathbb{R}} \left|\frac{1}{x}\right| d\lambda \\
& = \int_{\mathbb{R}} \left|\frac{1}{x}\right|^{r/r} d\lambda = \int_{\mathbb{R}} \left|\frac{1}{x^{1/r}}\right|^r d\lambda = \int_{\mathbb{R}} |g|^r d\lambda \\
& = \infty \\
& \Rightarrow \mathscr{L}^r(\mathbb{R}) \not\subset \mathscr{L}^p(\mathbb{R}) \wedge \mathscr{L}^p(\mathbb{R}) \not\subset \mathscr{L}^r(\mathbb{R})
\end{aligned}
$$

IV.10.41

$\mathscr{L}^\infty[0,1] \subsetneq \bigcup_{p<\infty} \mathscr{L}^p[0,1]$

Seien $S := [0,1], f \in \mathscr{L}^\infty(S) \Rightarrow \exists \alpha \geq 0 : \lambda(\{|f| > \alpha\}) = 0$

$$
\begin{aligned}
\text{Wähle } & p := \frac{1}{\alpha} \\
& \Rightarrow \int_S |f|^p d\lambda = \int_S |f|^{\frac{1}{\alpha}} d\lambda < \infty \big(\text{da } \lambda(S' \subset S), |f|^{\frac{1}{\alpha}} \leq 1\big) \\
& \Rightarrow \exists p < \infty : f \in \mathscr{L}^p(S) \Rightarrow f \in \bigcup_{p<\infty} \mathscr{L}^p(S) \\
& \Rightarrow \mathscr{L}^\infty(S) \subset \bigcup_{p<\infty} \mathscr{L}^p(S)
\end{aligned}
$$

Betrachte $f := \frac{1}{x}$

$$\begin{aligned}
&\int_S |f|^2 \, d\lambda = \int_S \left|\frac{1}{x}\right|^2 d\lambda < \infty \\
&\Rightarrow f \in \mathscr{L}^2(S) \Rightarrow f \in \bigcup_{p<\infty} \mathscr{L}^p(S) \\
\text{Aber } &\nexists \alpha \geq 0 : \lambda(\{|f| > \alpha\}) = 0 \; \big(\lambda(\{|f| = \infty\}) = 0\big) \\
&\Rightarrow f \notin \mathscr{L}^\infty \\
&\Rightarrow \bigcup_{p<\infty} \mathscr{L}^p(S) \not\subset \mathscr{L}^\infty(S)
\end{aligned}$$

$\Rightarrow \mathscr{L}^\infty(S) \subsetneq \bigcup_{p<\infty} \mathscr{L}^p(S)$

Für $f \in \mathscr{L}^\infty[0,1]$ gilt $\lim_{p\to\infty}\|f\|_p = \|f\|_{\mathscr{L}^\infty}$

Seien $S := [0,1],\ f \in \mathscr{L}^\infty(S) \Rightarrow \exists\beta \geq 0 : \lambda(\{|f| > \beta\}) = 0$

$$zz : \lim_{p\to\infty}\Big(\int_S |f|^p d\lambda\Big)^{1/p} = \inf_{\substack{N\in\mathscr{A}\\ \lambda(N)=0}} \sup_{s\in S\setminus N} |f(s)|$$

$$\lim_{p\to\infty}\Big(\int_S |f|^p d\lambda\Big)^{1/p}$$

$$\overset{IV.4.6}{=} \lim_{p\to\infty}\big(\lim_{n\to\infty}\sum_{j=1}^{n} |\alpha_j|^p \lambda(A_j \cap S)\big)^{1/p}$$

$$= \lim_{p\to\infty}\big(\lim_{n\to\infty}\sum_{j=1}^{n} |\beta|^p \left|\frac{\alpha_j}{\beta}\right|^p \lambda(A_j \cap S)\big)^{1/p}$$

Seien $\beta_n := \inf_{\substack{N\in\mathscr{A}\\ \lambda(N)=0}} \sup_{\substack{j\in 4^n-1\\ f^{-1}(\alpha_j^p)\in A_j\setminus N}} |\alpha_j|,$

$$\beta := \lim_{n\to\infty}\beta_n = \inf_{\substack{N\in\mathscr{A}\\ \lambda(N)=0}} \sup_{s\in S\setminus N} |f(s)|,$$

$$f_n := \lim_{p\to\infty}\big(f_n{}' := \sum_{j=1}^{n} \beta_n^p \left|\frac{\alpha_j}{\beta_n}\right|^p \lambda(A_j \cap S)\big)^{1/p}$$

$$\xRightarrow{|\frac{\alpha_j}{\beta}|^p \leq 1} f_n \leq g_+ := \lim_{p\to\infty}\big(\sum_{j=1}^{\infty} \beta^p \lambda(A_j \cap S)\big)^{1/p}$$

$$= \lim_{p\to\infty}\big(\beta^p\lambda(A_1\cap S) + \beta^p\lambda(A_2\cap S) + \ldots + \beta^p\lambda(A_{j_i}\cap S) + \ldots\big)^{1/p}$$

Wobei $i \in I$ Indexmenge, indem $(A_{j_i} \cap S) \subset N$ $(Nullmenge)$

$$= \lim_{p\to\infty}\big(\beta^p\lambda(S)\big)^{1/p} = \lim_{p\to\infty}\big(\beta^p\big)^{1/p} \lim_{p\to\infty}\big(\lambda(S)\big)^{1/p}$$

$$=\beta$$

$$\Rightarrow g_+ \ \lambda - integrierbar$$

Sei $f = \lim_{n\to\infty} f_n = \lim_{n\to\infty}\Big(\lim_{p\to\infty}\big(\sum_{j=1}^{n} \beta_n^p \left|\frac{\alpha_j}{\beta_n}\right|^p \lambda(A_j\cap S)\big)^{1/p}\Big)$

$$\overset{\substack{I_0\ Indexmenge\\ wobei\ \alpha_j<\beta_n}}{\underset{\substack{I_1\ Indexmenge\\ wobei\ \alpha_j=\beta_n}}{=}} \lim_{n\to\infty}\Bigg(\Big(\lim_{p\to\infty}\big(\sum_{j\in I_0} \beta_n^p \left|\frac{\alpha_j}{\beta_n}\right|^p \lambda(A_j\cap S)\big)^{1/p}\Big) + \Big(\lim_{p\to\infty}\big(\sum_{j\in I_1} \beta_n^p \left|\frac{\alpha_j}{\beta_n}\right|^p \lambda(A_j\cap S)\big)^{1/p}\Big)\Bigg)$$

$$\overset{|\frac{\alpha_j}{\beta_n}|^p \xrightarrow[j\in I_0]{p\to\infty} 0}{\underset{|\frac{\alpha_j}{\beta_n}|^p \xrightarrow[j\in I_1]{p\to\infty} 1}{=}} \lim_{n\to\infty}\Bigg(\lim_{p\to\infty}\Big(|I_1|^{1/p}\,(\beta_n^p)^{1/p}(1)^{1/p}\lambda(S)^{1/p}\Big) + \Big(\lim_{p\to\infty}\big(\sum_{j\in I_1} 0\big)^{1/p}\Big)\Bigg)$$

$$\overset{p\to\infty}{=} \lim_{n\to\infty}\beta_n \xrightarrow{n\to\infty} \beta$$

$$\Rightarrow \exists f(s) = \lim_{n\to\infty} f_n(s)\big(\text{per Konstruktion } \forall s \in S\big)$$

$$\xRightarrow{f.ü.Dom.Konv.} \lim_{n\to\infty}\int_S f_n d\lambda = \int_S f d\lambda \iff \lim_{n\to\infty}\Big(\int_S \left|f_n{}'\right|^p d\lambda\Big)^{1/p} = \Big(\int_S |f|^p\, d\lambda\Big)^{1/p} \forall p \in [1,\infty]$$

$$\xRightarrow[in\ n,\ p]{Mon.Konv.} \text{Limites vertauschbar}$$

$$\Rightarrow \lim_{p\to\infty}\Big(\int_S |f|^p d\lambda\Big)^{1/p} \lim_{n\to\infty}\lim_{p\to\infty} f_n{}' = \lim_{n\to\infty} f_n = \beta \quad \blacksquare$$