IV.10.38

Gelte $\mu(S) < \infty$ und $1 \leq r \leq p$. Dann folgt $\mathscr{L}^p(\mu) \subset \mathscr{L}^r(\mu)$ und $\|f\|_r \leq \mu(S)^{1/r-1/p}\|f\|_p$ für $f \in \mathscr{L}^p(\mu)$.

$$\text{Seien } f \in \mathscr{L}^p,\ \beta := \inf_{\substack{N \in \mathscr{A} \\ \mu(N)=0}} \sup_{s \in S \setminus N} |f(s)|$$

$$\xRightarrow{\mu(S)<\infty} \beta < \infty \Rightarrow \beta^x < \infty\ \forall x \in [1,p]$$

$$\xRightarrow[klar]{geometrisch} \int_S |f|^p d\mu \leq \beta^p \mu(S) < \infty$$

$$\overset{\beta^r<\infty}{\Longleftrightarrow} \int_S |f|^r d\mu \leq \beta^r \mu(S) < \infty$$

$$\Rightarrow f \in \mathscr{L}^r \Rightarrow \mathscr{L}^p(\mu) \subset \mathscr{L}^r(\mu)$$

Nun ist zu zeigen: $\|f\|_r \leq \mu(S)^{1/r-1/p}\|f\|_p$

$$\text{Wähle } f_0 := |f|^r, g_0 = 1, p_0 = \frac{p}{r}, \frac{1}{p_0} + \frac{1}{q} = 1$$

$$\xRightarrow{Hölder} \|f_0 g_0\|_{\mathscr{L}^1} \leq \|f_0\|_{\mathscr{L}^{p_0}} \|g_0\|_{\mathscr{L}^q}$$

$$\Rightarrow \int_S |f|^r\, d\mu \leq \Big(\int_S \left(|f|^r\right)^{p/r} d\mu \Big)^{1/p_0} \Big(\int_S (1)^q d\mu \Big)^{1/q}$$

$$\Longleftrightarrow \int_S |f|^r\, d\mu \leq \Big(\int_S |f|^p\, d\mu \Big)^{r/p} \Big(\int_S d\mu \Big)^{1-r/p}$$

$$\overset{\cdot 1/r}{\Longleftrightarrow} \Big(\int_S |f|^r\, d\mu \Big)^{1/r} \leq \Big(\int_S |f|^p\, d\mu \Big)^{1/p} \Big(\int_S d\mu \Big)^{1/r-1/p}$$

$$\Longleftrightarrow \|f\|_r \leq \|f\|_p\ \mu(S)^{1/r-1/p}$$

IV.10.40

Es gilt weder $\mathscr{L}^r(\mathbb{R}) \subset \mathscr{L}^p(\mathbb{R})$ noch $\mathscr{L}^p(\mathbb{R}) \subset \mathscr{L}^r(\mathbb{R})$ für $1 \leq r < p \leq \infty$.

$$\begin{aligned}
\text{Seien } & f \in \mathscr{L}^r(\mathbb{R}), g \in \mathscr{L}^p(\mathbb{R}) \\
\text{Wähle } & f := \frac{1}{x^{1/p}}, g := \frac{1}{x^{1/r}} \\
\text{Dann ist } & \int_{\mathbb{R}} |f|^r d\lambda = \int_{\mathbb{R}} \left| \frac{1}{x^{1/p}} \right|^r d\lambda \\
&= \int_{\mathbb{R}} \left| \frac{1}{x} \right|^{r/p} d\lambda < \infty \left(\text{da } \frac{r}{p} < 1 \right) \\
\text{Beziehungsweise } & \int_{\mathbb{R}} |g|^p d\lambda = \int_{\mathbb{R}} \left| \frac{1}{x^{1/r}} \right|^p d\lambda \\
&= \int_{\mathbb{R}} \left| \frac{1}{x} \right|^{p/r} d\lambda < \infty \left(\text{da } \frac{p}{r} > 1 \right) \\
\text{Aber } & \int_{\mathbb{R}} |f|^p d\lambda = \int_{\mathbb{R}} \left| \frac{1}{x^{1/p}} \right|^p d\lambda = \int_{\mathbb{R}} \left| \frac{1}{x} \right|^{p/p} d\lambda \\
&= \int_{\mathbb{R}} \left| \frac{1}{x} \right| d\lambda \\
&= \int_{\mathbb{R}} \left| \frac{1}{x} \right|^{r/r} d\lambda = \int_{\mathbb{R}} \left| \frac{1}{x^{1/r}} \right|^r d\lambda = \int_{\mathbb{R}} |g|^r d\lambda \\
&= \infty \\
&\Rightarrow \mathscr{L}^r(\mathbb{R}) \not\subset \mathscr{L}^p(\mathbb{R}) \wedge \mathscr{L}^p(\mathbb{R}) \not\subset \mathscr{L}^r(\mathbb{R})
\end{aligned}$$

IV.10.41

$\mathscr{L}^\infty[0,1] \subsetneq \bigcup_{p<\infty} \mathscr{L}^p[0,1]$

Seien $S := [0,1], f \in \mathscr{L}^\infty(S) \Rightarrow \exists \alpha \geq 0 : \lambda(\{|f| > \alpha\}) = 0$

$$\begin{aligned}
\text{Wähle } & p := \frac{1}{\alpha} \\
&\Rightarrow \int_S |f|^p d\lambda = \int_S |f|^{\frac{1}{\alpha}} d\lambda < \infty \big(\text{da } \lambda(S') \leq 1 \ \forall S' \subset S, \ |f|^{\frac{1}{\alpha}} \leq 1 \big) \\
&\Rightarrow \exists p < \infty : f \in \mathscr{L}^p(S) \Rightarrow f \in \bigcup_{p<\infty} \mathscr{L}^p(S) \\
&\Rightarrow \mathscr{L}^\infty(S) \subset \bigcup_{p<\infty} \mathscr{L}^p(S)
\end{aligned}$$

Betrachte $f := \frac{1}{x}$

$$\begin{aligned} \int_S |f|^2 \, d\lambda &= \int_S \left|\frac{1}{x}\right|^2 d\lambda < \infty \\ \Rightarrow & f \in \mathscr{L}^2(S) \Rightarrow f \in \bigcup_{p<\infty} \mathscr{L}^p(S) \\ \text{Aber } \nexists \alpha \geq 0 : \lambda(\{|f| > \alpha\}) &= 0 \ \big(\lambda(\{|f| = \infty\}) = 0\big) \\ \Rightarrow & f \notin \mathscr{L}^\infty \\ \Rightarrow & \bigcup_{p<\infty} \mathscr{L}^p(S) \not\subset \mathscr{L}^\infty(S) \end{aligned}$$

$\Rightarrow \mathscr{L}^\infty(S) \subsetneq \bigcup_{p<\infty} \mathscr{L}^p(S)$

Für $f \in \mathscr{L}^\infty[0,1]$ gilt $\lim_{p\to\infty} \|f\|_p = \|f\|_{\mathscr{L}^\infty}$

Seien $S := [0,1],\ f \in \mathscr{L}^\infty(S) \Rightarrow \exists \beta \geq 0 : \lambda(\{|f| > \beta\}) = 0$

$$zz: \lim_{p\to\infty} \Big(\int_S |f|^p d\lambda \Big)^{1/p} = \inf_{\substack{N\in\mathscr{A} \\ \lambda(N)=0}} \sup_{s\in S\setminus N} |f(s)|$$

$$\lim_{p\to\infty} \Big(\int_S |f|^p d\lambda \Big)^{1/p}$$

$$\overset{IV.4.6}{=} \lim_{p\to\infty} \big(\lim_{n\to\infty} \sum_{j=1}^{n} |\alpha_j|^p \lambda(A_j \cap S) \big)^{1/p}$$

$$= \lim_{p\to\infty} \big(\lim_{n\to\infty} \sum_{j=1}^{n} |\beta|^p \left|\frac{\alpha_j}{\beta}\right|^p \lambda(A_j \cap S) \big)^{1/p}$$

Seien $\beta_n := \inf_{\substack{N\in\mathscr{A} \\ \lambda(N)=0}} \sup_{\substack{j\in 4^n - 1 \\ f^{-1}(\alpha_j^p)\in A_j\setminus N}} |\alpha_j|,$

$$\beta := \lim_{n\to\infty} \beta_n = \inf_{\substack{N\in\mathscr{A} \\ \lambda(N)=0}} \sup_{s\in S\setminus N} |f(s)|,$$

$$f_{p,n} := \big(\sum_{j=1}^{n} \beta_n^p \left|\frac{\alpha_j}{\beta_n}\right|^p \lambda(A_j \cap S) \big)^{1/p}$$

$$\xRightarrow{IV.4.6} f_{p,1} \leq f_{p,2} \leq \ldots \leq \lim_{n\to\infty} f_{p,n} =: f_p$$

$$\xRightarrow[\beta_n \leq \beta]{\left|\frac{\alpha_j}{\beta}\right|^p \leq \left|\frac{\alpha_j}{\beta_n}\right|^p \leq 1} f_{p,n} \leq g_p := \big(\sum_{j=1}^{\infty} \beta^p \lambda(A_j \cap S) \big)^{1/p} = \sup_{n\in\mathbb{N}} f_{p,n} < \infty$$

$$\xRightarrow[Konvergenz]{Monoton} f_p = g_p\ \forall p \in [1,\infty)$$

$$\text{Also} \lim_{p\to\infty} f_p = \lim_{p\to\infty} g_p = \lim_{p\to\infty} \big(\sum_{j=1}^{\infty} \beta^p \lambda(A_j \cap S) \big)^{1/p}$$

$$= \lim_{p\to\infty} \big(\beta^p \lambda(A_1 \cap S) + \beta^p \lambda(A_2 \cap S) + \ldots + \beta^p \lambda(A_i \cap S) + \ldots \big)^{1/p}$$

wobei $A_i = \{|f| > \beta\}$

$$\overset{\sigma-Additivität}{=} \lim_{p\to\infty} \big(\beta^p \lambda(S) \big)^{1/p} = \lim_{p\to\infty} \big(\beta^p \big)^{1/p} \lim_{p\to\infty} \big(\lambda(S) \big)^{1/p}$$

$$\overset{\lambda(S)=1<\infty}{=} \lim_{p\to\infty} \big(\beta^p \big)^{1/p}$$

$$= \beta \qquad \blacksquare$$