All of Statistic
2 - Random Variable

Levi Kassel

May 15, 2023

Summary

Random Variable: A Random variable is a mapping

$$X : \Omega \to \mathbb{R}$$

that assigns a real number $X(\omega)$ to each number ω.

Cumulative Distribution Funtion (CDF): The cumulative distribution function, or cdf, is the function $F_X(x) : \mathbb{R} \to [0, 1]$ defined by:

$$F_X(x) = \mathbb{P}(X \leq x)$$

Theorem: Let X have cdf F and let Y have cdf G. If $F(x) = G(x)$ for all x, then $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for all A.

Theorem: A function F mapping the real line to $[0, 1]$ is a cdf for some probability $\mathbb{P}$ if and only if F satisfies the following three conditions:

- F is non-decreasing: $x_1 < x_2$ inplies that $F(x_1) \leq F(x_2)$
- F is normelized:
$$\lim_{x \longrightarrow -\infty} F(x) = 0$$
and:
$$\lim_{x \longrightarrow \infty} F(x) = 1$$
- F is right-continuous: $F(x) = F(x^+)$ for all x, where:
$$F(x^+) = \lim_{x \longrightarrow y, y > x} F(y)$$

Probability Funtion: When X is **discrete**, We define the probability function or probability mass function for X by

$$f_X(x) = \mathbb{P}(X = x)$$

. Thus, $f_X(x) \geq 0$ for all $x \in R$ and $\sum_i f_X(xi) = 1$.

Probability Density Function (PDF): when X is **continuous** we define the Probability Density Function $f_X(x)$ as follow:

- $f_X(x) \geq 0, \forall x$.
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- for every $a \leq b$:
$$\mathbb{P}(a < X < b) = \int_a^b f_X(x)dx.$$
- $f_X(x) = F_X^{'}(x)$

Inverse CDF: Let X be a random variable with CDF F. The inverse CDF or **quantile** function is defined by:

$$F^{-1}(q) = inf \left\{ \ x : F(x) > q \ \right\}$$

for $q \in [0, 1]$.

Some Important Discrete Random Variables

The Point Mass Distribution: $X \sim \delta_a$

$$F(x) = \begin{cases} 0, & x < a \\ 1, & x \geq a \end{cases}$$

The Discrete Uniform Distribution: Let$k > 1$ be a given integer. Suppose that X has probability mass function given by

$$f(x) = \begin{cases} \frac{1}{k}, & \text{for } x = 1, ..., k \\ 0, & \text{otherwise} \end{cases}$$

The Bernoulli Distribution: Let X represent a binary coin flip. Then $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$ for some $p \in [0, 1]$. We say that X has a Bernoulli distribution written $X \sim Bernoulli(p)$. The probability function is

$$f(x) = p^x \cdot (1-p)^{(1-x)} \text{ for } x \in \{0, 1\}.$$

The Binomial Distribution: Flip the coin n times and let X be the number of heads.Assume that the tosses are independent. $X \sim Binomial(n, p)$

$$f(x) = \begin{cases} \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}, & \text{for } x = 0, ..., n \\ 0, & \text{otherwise} \end{cases}$$

If $X_1 \sim Binomial(n_1, p)$ and $X_2 \sim Binomial(n_2, p)$ then $X_1 + X_2 \sim Binomial(n_1+$

$n_2, p)$.
The Geometric Distribution: X has a geometric distribution with parameter $p \in (0,1)$, written $X \sim Geom(p)$, if

$$\mathbb{P}(X = k) = p \cdot (1-p)^{k-1}, \quad k \geq 1$$

Think of X as the number of flips needed until the first head when flipping a coin.
The Poisson Distribution: X has a Poisson distribution with parameter λ, written $X \sim Poisson(\lambda)$ if:

$$f(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}, \quad x \geq 0$$

If $X_1 \sim Poisson(\lambda_1)$ and $X_2 \sim Poisson(\lambda_2)$ then $X_1 + X_2 \sim Poisson(\lambda_1 + \lambda_2)$.

Some Important Continuous Random Variables

The Uniform Distribution: X has a $Uniform(a,b)$ distribution, written $X \sim Uniform(a,b)$, if

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{for } x \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & \text{for } x \in [a,b] \\ 1, & x > b \end{cases}$$

Normal (Gaussian): X has a Normal (or Gaussian) distribution with parameters μ and σ, denoted by $X \sim N(\mu, \sigma^2)$, if

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{s\sigma^2}(x-\mu)^2 \right\}.$$

We say that X has a standard Normal distribution if $\mu = 0$ and $\sigma = 1$ (denoted by Z).
The PDF and CDF of a standard Normal are denoted by $\phi(z)$ and $\Phi(z)$.
There is no closed-form expression for . Here are some useful facts:

1. if $X \sim N(\mu, \sigma^2), \quad then \quad Z = (X - \mu)/\sigma \sim N(0,1)$.
2. if $Z \sim N(0,1)' \quad then = \mu + \sigma \cdot Z \sim N(\mu, \sigma^2)$.
3. if $X \sim N(\mu_i, \sigma_i^2), \quad i = 1, ..n$ are independent, then

$$\sum_{i=1}^{n} X_i \sim N(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2)$$

4. if $X \sim N(\mu, \sigma^2)$, then:

$$\mathbb{P}(a < X < b) = \mathbb{P}(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}) = \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})$$

Exponential Distribution: X has an Exponential distribution with parameter β, denoted by $X \sim Exp(\beta)$, if

$$f(x) = \frac{1}{\beta}e^{-x/\beta}, \quad x > 0$$

where β ¿ 0.
Gamma Distribution: For $\alpha > 0$, the Gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha 1} e^y dy$$

. X has a Gamma distribution with parameters α and β, denoted by $X \sim Gamma(\alpha, \beta)$, if

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \quad x > 0$$

Bivariate Distributions

joint mass function Given a pair of discrete random variables X and Y , define the joint mass function by $f(x, y) = \mathbb{P}(X = x \text{ and } Y = y)$.

$f(x, y)$ is a **PDF** for the **continuous** random variables (X, Y) if:

1. $f(x, y) \geq 0$ for all (x, y)
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$
3. for any set $A \subset \mathbb{R} \times \mathbb{R}, \mathbb{P}((X, Y) \in A) = \int \int_A f(x, y)dxdy$

Marginal Distributions: If (X, Y) have joint distribution with mass function $f_{X,Y}$, then the marginal mass function for X is defined by:
For **discrete** variables:

$$f_X(x) = P(X = x) = \sum_y \mathbb{P}(X = x, Y = y) = \sum_y f(x, y)$$

For **continuous** variables:

$$f_X(x) = \int_y f(x, y)dy$$

Independent Random Variables Two random variables X and Y are independent if, for every A and B,

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A) \cdot \mathbb{P}(Y \in B)$$

Theorem: Let X and Y have joint PDF $f_{X,Y}$. Then $X \amalg Y$ if and only if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all values x and y. **Theorem:** Suppose that the range of X and Y is a (possibly infinite) rectangle. If $f(x,y) = g(x)h(y)$ for some functions g and h (not necessarily probability density functions) then X and Y are independent.

Conditional Distributions: The conditional probability mass function is:
For **discrete** variables:

$$f_{X|Y} = \mathbb{P}(X - x|Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad \text{if } f_Y(y) > 0$$

For **continuous** variables:

$$\mathbb{P}(X \in A|Y = y) = \int_A f_{X|Y}(x|y)dx$$

Multivariate Distributions and iid Samples If $X_1, ..., X_n$ are independent and each has the same marginal distribution with CDF F, we say that $X_1, ..., X_n$ are iid (independent and identically distributed) and we write:

$$X_1, ...X_n \sim F$$

Multinomial The multivariate version of a Binomial is called a Multinomial.
We say that X has a$Multinomial(n,p)$ distribution written $X \sim Multinomial(n,p)$.
The probability function is:

$$f(x) = \binom{n}{x_1, ..x_n} \cdot p^{x_1} \cdots p^{x_n}$$

Lemma: Suppose that $X \sim Multinomial(n,p)$ where $X = (X_1, ..., X_k)$ and $p = (p_1, ..., p_k)$. The marginal distribution of X_j is $Binomial(n, p_j)$.
Multivariate Normal let:

$$\mathbf{Z} = \begin{pmatrix} z_1 \\ . \\ . \\ . \\ z_k \end{pmatrix}$$

where$Z_1, ..., Z_k \sim N(0,1)$ are independent. The density of Z is :

$$f(z) = \prod_{i=1}^{k} f(z_i) = \frac{1}{2\pi^{k/2}} exp\left\{ -\tfrac{1}{2}\textstyle\sum_{j=1}^{k} z_j^2 \right\} = \frac{1}{2\pi^{k/2}} exp\left\{ -\tfrac{1}{2} z^T z \right\}$$

More generally, a vector X has a multivariate Normal distribution, denoted by $X \sim N(\mu, \Sigma)$, if it has density

$$f(x;\mu,\Sigma) = \frac{1}{2\pi^{k/2}|\Sigma|^{1/2}} exp\left\{ -\tfrac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu) \right\}$$

Theorem: If $Z \sim N(0, I)$ and $X = \mu + \Sigma^{-1/2} \cdot Z$ then $X \sim N(\mu, \Sigma)$. Conversely, if $X \sim N(\mu, \Sigma)$, then $\Sigma^{1/2}(X - \mu) \sim N(0, I)$

Theorem: Let $X \sim N(\mu, \Sigma)$. Then:

1. The marginal distribution of X_a is $X_a \sim N(\mu_a, \Sigma_{aa})$.
2. The conditional distribution of X_b given $X_a = x_a$ is:

$$X_b | X_a = x_a \sim N\left(\mu_b + \Sigma_{ba}\Sigma_{aa}^{1} \cdot (x_a - \mu_a), \Sigma_{bb} - \Sigma_{ba}\Sigma_{aa}^{1}\Sigma_{ab}\right).$$

3. If a is a vector then $a^T X \sim N(a^T \mu, a^T \Sigma a)$.
4. $V = (X\mu)^T \Sigma^1 (X\mu) \sim x_k^2$.

Transformations of Random Variables

Let $Y = r(X)$ be a function of X, for example, $Y = X^2$ or $Y = e^X$. We call $Y = r(X)$ a transformation of X.
How do we compute the pdf and cdf of Y ?
Descrete case: The mass function of Y is given by:

$$f_Y(y) = \mathbb{P}(Y = y) = \mathbb{P}(r(X) = y) = \mathbb{P}(\{x; r(x) = y\}) = \mathbb{P}(X \in r^1(y)).$$

Continuous case:

1. For each y, find the set $A_y = x : r(x) \leq y$.
2. Find the CDF:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(r(X) \leq y) = \mathbb{P}(\{x; r(x) \leq y\}) = \int_{Ay} f_X(x) dx$$

3. The PDF is $f_Y(y) = F_Y(y)$.

Transformations of Several Random Variables

X and Y are given random variables, we might want to know the distribution of X/Y , X + Y , maxX, Y or minX, Y . Let Z = r(X, Y) be the function of interest. The steps for finding f_Z are the same as before:

1. For each z, find the set $A_z = \{(x, y) : r(x, y) \leq z\}$.
2. Find the CDF:

$$F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(r(X, Y) \leq z) = \mathbb{P}(\{(x, y); r(x, y) \leq z\}) = \int\int_{A_z} f_{X,Y}(x, y) dx dy.$$

3. Then $f_Z(z) = F_Z(z)$.