Why is $\frac{\partial J(W)}{\partial W_{i,j}^{(out)}} = \left(A^{(h)}\right)^T \delta^{(out)}$?

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1 Introduction

In this LaTeX document, I want to show my step-by-step process to derive the partial derivative of all the weights in $W^{(out)}$.

I'll start simple with a 2-2-2 MLP, which is a Multi-Layer Perceptron with 2 inputs, 2 hidden units, and 2 outputs, and then generalize what we learn to a general $m - d - t$ MLP.

In both cases, I will use the following loss/error function:

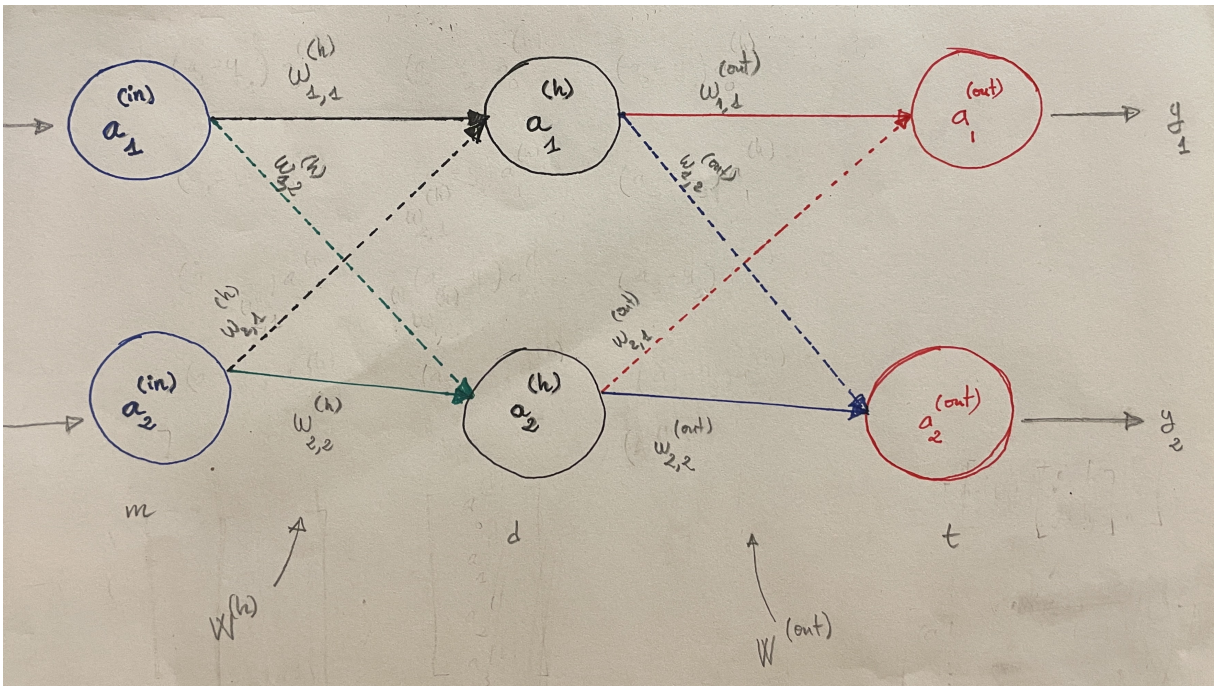
$$J(w) = -\sum_{i=1}^{n} \sum_{j=1}^{t} y_j^{[i]} ln(a_j^{[i]}) + (1 - y_j^{[i]}) ln(1 - a_j^{[i]})$$

Where the superscript $[i]$ is an index for training examples, and j is the number of output units.

Ready? Let's go!

2 With a 2-2-2 MLP

For this example, we will use the following Neural network:

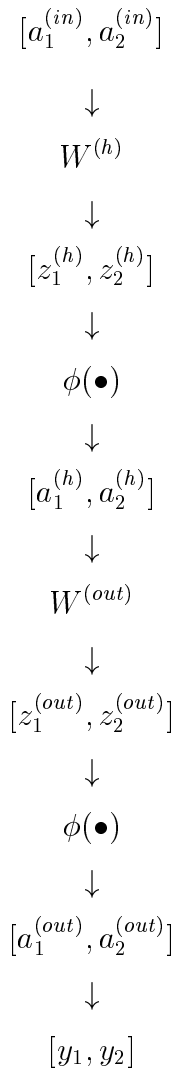


We'll ignore biase units in the input and hidden layers, and consider only ONE training example for simplicity purposes.

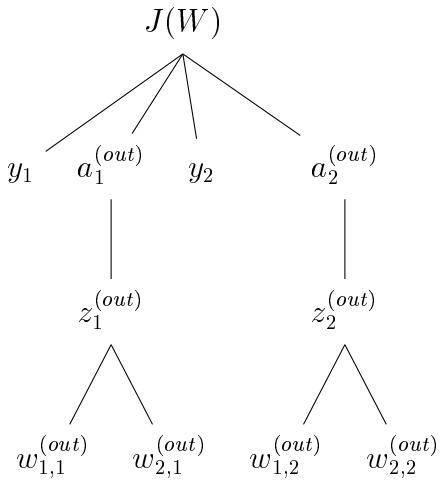
Since one training example is considered and the network has two output units, then $n = 1$ and $t = 2$. So, the loss function becomes like this:

$$J(w) = -\sum_{j=1}^{t} y_j^{[1]} ln(a_j^{[1]}) + (1 - y_j^{[1]}) ln(1 - a_j^{[1]})$$

The journey of a SINGLE training example from the input layer to the output layer of our network goes like this:


$[a_1^{(in)}, a_2^{(in)}]$
$W^{(h)}$
$[z_1^{(h)}, z_2^{(h)}]$
$\phi(\bullet)$
$[a_1^{(h)}, a_2^{(h)}]$
$W^{(out)}$
$[z_1^{(out)}, z_2^{(out)}]$
$\phi(\bullet)$
$[a_1^{(out)}, a_2^{(out)}]$
$[y_1, y_2]$

Now, let's compute the derivative of each weights in the $W^{(out)}$ matrix. To help myself, I drew the following tree diagram to see how the variables in $J(W)$ relate to the weights in $W^{(out)}$:



Our MLP has two outputs. If you look at $J(W)$ expression closely, we compute the cost for each output, then add the results together. And that's how we get the cost for the current training example. With that in mind, we can now compute the partial derivatives:

$$\frac{\partial J(W)}{\partial w_{1,1}^{(out)}} = \frac{\partial J(W)}{\partial a_1^{(out)}} \times \frac{\partial a_1^{(out)}}{\partial z_1^{(out)}} \times \frac{\partial z_1^{(out)}}{\partial w_{1,1}^{(out)}}$$

$$\frac{\partial J(W)}{\partial w_{2,1}^{(out)}} = \frac{\partial J(W)}{\partial a_1^{(out)}} \times \frac{\partial a_1^{(out)}}{\partial z_1^{(out)}} \times \frac{\partial z_1^{(out)}}{\partial w_{2,1}^{(out)}}$$

$$\frac{\partial J(W)}{\partial w_{1,2}^{(out)}} = \frac{\partial J(W)}{\partial a_2^{(out)}} \times \frac{\partial a_2^{(out)}}{\partial z_2^{(out)}} \times \frac{\partial z_2^{(out)}}{\partial w_{1,2}^{(out)}}$$

$$\frac{\partial J(W)}{\partial w_{2,2}^{(out)}} = \frac{\partial J(W)}{\partial a_2^{(out)}} \times \frac{\partial a_2^{(out)}}{\partial z_2^{(out)}} \times \frac{\partial z_2^{(out)}}{\partial w_{2,2}^{(out)}}$$

3 With a general $m - d - t$ MLP