Discrete Math midterm answers

CS161

July 22, 2016

1 Simple counting

6% of grade

1. $\frac{2^{16}}{2^{12}} = 2^4$
2. $\binom{13}{13} = 1$
3. ${}^4P_3 = \frac{4!}{1!}$
4. ${}^4P_2 = \frac{4!}{2!}$
5. ${}^{13}P_4 = \frac{13!}{9!}$
6. $\binom{13}{4} = \frac{{}^xP_y}{{}^wP_z}$: what are x, y, w, and z? $x = 13, y = 4, w = 4, z = 4$

$$\binom{13}{4} = \frac{{}^{13}P_4}{{}^4P_4}$$

2 Counting

20% of grade

1. I have 8 shirts, one of which is gray. I have 3 pairs of pants, one of which is gray. How many shirt-pants combinations can I make that involve a gray article of clothing?

$$3 + 8 - 1$$

2. License plates have three uppercase letters followed by three digits. How many different license plates are there that include a nine? Hint: first calculate how many three digit strings include a nine.

$$26^3 * (10^3 - 9^3)$$

3. If all students have one of five majors and at least one of three minors, how many students are needed in a room to guarantee that at least three of them in the room share a major-minor combination? (ie. three people majoring in astronomy with a minor in physical education)

$$(5 * 3) * 2 + 1$$

4. A group of 14 friends put in a pizza order for pick-up. Somebody has to go pick up the pizza. But nobody wants to go alone and no pair of people want to go as just the two of them. At least three people need to go on the pizza run. How many different subsets of people could go on the pizza run?

$$2^{14} - \binom{14}{2} - \binom{14}{1} - \binom{14}{0}$$

5. How many permutations of the letters ABCDEFG have C and D right next to each other? (ie. CD or DC) Equivalently, if you are taking photos of a wedding party of three bridesmaids, two groomsmen, and the happy couple, how many ways can you arrange the people from left to right in the photo given that the bride and groom must stand next to each other?

$${}^{6}P_{6} * 2$$

3 Advanced Counting

24% of grade

1. There are four books, called A, B, C, D. How many ways can you put these books on a shelf such that A is to the left of C?

$$\binom{4}{2} * {}^{2}P_{2}$$

2. How many distinct five card hands could you draw from a normal 52 card deck that have a four-of-a-kind? (ie. four twos and some other card, or four kings and some other card)

$$\binom{13}{1} * \binom{48}{1}$$

3. Find n if

   (a) ${}^{n}P_{2} = 110$, then $n = 11$

   (b) ${}^{n}P_{n} = 5040$, then $n = 7$

   (c) ${}^{n}P_{4} = 12 * {}^{n}P_{2}$, then $n = 6$

4 Proofs

45% of grade

4.1 Series sum proof

Show that $1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for every positive integer n.

4.1.1 Definitions

Let:

$$F(n) = 1^3 + 2^3 + ... + n^3$$

$$P(n) : F(n) = \left(\frac{n(n+1)}{2}\right)^2$$

4.1.2 Goal

We will prove

$$\forall n \in \mathbb{N} : P(n)$$

4.1.3 Proof by induction

Base case

$$P(1) : F(1) = \left(\frac{1 * (1+1)}{2}\right)^2$$

$$1^3 = 1^2$$

Inductive step We will prove:

$$P(k) \implies P(k+1), \forall k \in \mathbb{N}$$

Inductive hypothesis, assume:

$$P(k) : F(k) = \left(\frac{k(k+1)}{2}\right)^2$$

Now prove:

$$P(k+1) : F(k+1) = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2$$

Starting with the left side:

$$F(k+1) = 1^3 + 2^3 + ... + k^3 + (k+1)^3$$

$$F(k+1) = F(k) + (k+1)^3$$

By our inductive hypothesis:

$$F(k+1) = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$F(k+1) = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$F(k+1) = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$F(k+1) = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$F(k+1) = \frac{(k+1)^2(k^2 + 4(k+1))}{4}$$

$$F(k+1) = \frac{(k+1)^2(k^2 + 4k + 4)}{4}$$

$$F(k+1) = \frac{(k+1)^2(k+2)^2}{4}$$

$$F(k+1) = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

$$F(k+1) = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2$$

4.1.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(1)$ and that for all $k \in \mathbb{N}$, $P(k)$ implies $P(k+1)$. Therefore $P(n)$ holds for all $n > 0$.

4.2 Inequality proof

Prove that for all integers $n > 1$ that $n! < n^n$

4.2.1 Definitions

Let:

$$P(n) : n! < n^n$$

4.2.2 Goal

We will prove

$$\forall n \in \mathbb{Z}_{>1} : P(n)$$

4.2.3 Proof by induction

Base case

$$P(2) : 2! < 2^2$$

$$2 < 4$$

Inductive step We will prove:

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{>1}$$

Inductive hypothesis, assume:

$$P(k) : k! < k^k$$

Now prove:

$$P(k+1) : (k+1)! < (k+1)^{k+1}$$

Starting with the left side:

$$(k+1)! = (k+1)k!$$

By our inductive hypothesis:

$$(k+1)! = (k+1)k! < (k+1)k^k$$

So:

$$(k+1)! < (k+1)k^k$$

Since $k > 1$, $(k+1)^a \geq k^a$ if $a \geq 0$. So:

$$(k+1)! < (k+1)k^k \leq (k+1)(k+1)^k$$

$$(k+1)! < (k+1)(k+1)^k$$

$$(k+1)! < (k+1)^{k+1}$$

4.2.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(2)$ and that for all $k \in \mathbb{Z}_{>1}$, $P(k)$ implies $P(k+1)$. Therefore $P(n)$ holds for all $n > 1$.

4.3 Divisibility proof

Prove that 6 evenly divides $n^3 - n$ for every non-negative integer n.

4.3.1 Definitions

Let:

$$P(n) : n^3 - n \equiv 0 \mod 6 \tag{1}$$

4.3.2 Goal

We will prove that $P(n)$ holds for all values of n greater than or equal to 0.

$$\forall n \in \mathbb{Z}_{\geq 0} : P(n) \tag{2}$$

4.3.3 Proof by induction

Base case

$$P(0) : 0^3 - 0 \equiv 0 \mod 6 \tag{3}$$
$$0^3 - 0 = 0$$

Inductive step We will prove:

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{\geq 0} \tag{4}$$

Inductive hypothesis, assume:

$$P(k) : k^3 - k \equiv 0 \mod 6 \tag{5}$$

Now prove:

$$P(k+1) : (k+1)^3 - (k+1) \equiv 0 \mod 6 \tag{6}$$

Starting with the left side:

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$$
$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 2k$$
$$(k+1)^3 - (k+1) = k^3 - k + 3k^2 + 3k$$
$$(k+1)^3 - (k+1) = (k^3 - k) + 3(k^2 + k)$$

By our *inductive hypothesis* the expression $k^3 - k$ is divisible by 6. If we can prove that $k^2 + k$ is a multiple of 2 (even), then the expression $3(k^2 + k)$ is also divisible by 6, proving $P(k+1)$.

If k is even, then $k^2 + k$ is even, and if k is odd then $k^2 + k$ is also even. Therefore $k^2 + k$ must be a multiple of 2.

4.3.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(0)$ and that for all $n \in \mathbb{Z}_{\geq 0}$, $P(n)$ being true implies that $P(n+1)$ is also true. Therefore $P(n)$ holds for all $n \geq 0$.

5 Recursion

5% of grade

1. Write a method in Java that calculates factorial of a number n recursively.

```
public static int factorial(int n) {
    if(n == 0) return 1;
    return n * factorial(n - 1);
}
```

6 Extra credit

5% extra credit for being awesome

1. If you want to get a seven scoop bowl of ice cream from an ice cream shop with 14 flavors of ice cream, how many different bowls could you get? All scoops of a flavor are indistinguishable and the order of scoops is not important (Remember that you would create your bowl by working down the line saying just two different commands to the ice cream scooper. How many commands would you say in total?)

$$\binom{14+7-1}{7} = \binom{14+7-1}{14-1}$$