Outline: Acting Under Uncertainty

Decisions and Inference

- Basic Probability Notation
- Inference Using Full Joint Distributions
- Bayes' Rule
- Bayes’ Networks

Acting Under Uncertainty

- Real world problems contain **uncertainties due to:**
  - partial observability,
  - nondeterminism, or
  - adversaries.

- Example of dental diagnosis using propositional logic

$$Toothache \Rightarrow Cavity.$$

- However inaccurate, not all patients with toothaches have cavities

$$Toothache \Rightarrow Cavity \lor GumProblem \lor Abscess\ldots$$

- In order to make the rule **true**, we would have to add a very large list of possible problems.

- The only way to fix the rule is to make it logically exhaustive

Acting Under Uncertainty

- An agent strives to choose the right thing to do—the rational decision—depends on both the relative importance of various goals and the likelihood that, and degree to which, they will be achieved.

- Models for large domains such as medical diagnosis fail to three main reasons:
  - **Modeling limitations:** It is too much work to list the complete set of antecedents or consequents needed to ensure an exceptionless rule (also computational feasibility)
  - **Theoretical ignorance:** Medical science has no complete theory for the domain
  - **Practical ignorance:** Even if we know all the rules, we might be uncertain about a particular patient because not all the necessary tests have been or can be run.

$\Rightarrow$ An agent has a degree of belief in the relevant sentences.

Basic Probability Notation: Review

For our agent to represent and use probabilistic information, we need a formal language.

- Sample space: the set of all possible worlds (outcomes) of interest
- The possible worlds are mutually exclusive and exhaustive
- A fully specified probability model associates a numerical probability $P(\omega)$ with each possible world.
- The basic axioms of probability theory say that every possible world has a probability between 0 and 1 and that the total probability of the set of possible worlds is 1:
- $0 \leq P(\omega) \leq 1$ for every ω and $\omega \in \Omega$
- Unconditional or prior probability: degrees of belief in propositions in the absence of any other information

Uncertainty in the real (and modeled world)

- General situation:
  - **Observed variables (evidence)**: Agent knows certain things about the state of the world (e.g., sensor readings or symptoms)
  - **Unobserved variables**: Agent needs to reason about other aspects (e.g. where an object is or what disease is present)
  - **Model**: Agent knows something about how the known variables relate to the unknown variables

- Probabilistic reasoning gives us a framework for managing our beliefs and knowledge

Random Variables

- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
- We denote random variables with capital letters

- Like variables in a CSP, random variables have domains
  - R in {true, false}   (often write as {+r, -r})
  - T in {hot, cold}
  - D in $[0, \infty)$

Formally: Basic Probability Notation

- **Factored representation:** possible world is represented by a set of variable/value pairs.
  - Variables in probability theory are called random variables, and their names begin with an uppercase letter. ($Total$ and Die_1)

- Sometimes we will want to talk about the probabilities of all the possible values of a random variable. We could write:

$$P(Weather = sun) = 0.6$$
$$P(Weather = rain) = 0.1$$
$$P(Weather = cloud) = 0.29$$
$$P(Weather = snow) = 0.01\ ,$$

- Abbreviation of this will be:

$$\mathbf{P}(Weather) = \langle 0.6, 0.1, 0.29, 0.01 \rangle,$$

This defines a **probability distribution** for the random variable *Weather*

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Shorthand notation:

$$P(hot) = P(T = hot),$$
$$P(cold) = P(T = cold),$$
$$P(rain) = P(W = rain),$$
$$\ldots$$

OK *if* all domain entries are unique

- A distribution is a TABLE of probabilities of values

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n) \quad \text{(shorthand)}$$

  – Must obey: $P(x_1, x_2, \ldots x_n) \geq 0$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- Size of distribution if n variables with domain sizes d?
  – For all but the smallest distributions, impractical to write out!

A probabilistic model is a joint distribution over a set of random variables

- Probabilistic models:
  - (Random) variables with domains
  - Assignments are called outcomes
  - Joint distributions: say whether assignments (outcomes) are likely
  - Normalized: sum to 1.0
  - Ideally: only certain variables directly interact

Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Events in a probabilistic model

- An event is a set E of outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?

- **Typically, the events we care about are partial assignments, like P(T=hot)**

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Quiz: Events: 2 Boolean random variable

- P(+x, +y) ?
- P(+x) ?
- P(-y OR +x) ?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Quiz: Events

- P(+x, +y) ? **.2**
- P(+x) ? **.2+.3=.5**
- P(-y OR +x) ? **.1+.3+.2=.6**

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_s P(t, s)$$

$$P(s) = \sum_t P(t, s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Quiz: Marginal Distributions

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x,y)$$

$$P(y) = \sum_x P(x,y)$$

$P(X)$

X	P
+x	
-x	

$P(Y)$

Y	P
+y	
-y	

Quiz: Marginal Distributions

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

$P(X)$

X	P
+x	**.5**
-x	**.5**

$P(Y)$

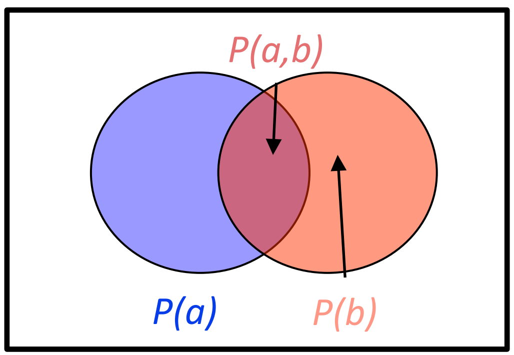
Y	P
+y	**.6**
-y	**.4**

Conditional Probabilities

Conditional Probability: The probability of ***a*** given ***b***

- The definition of a conditional probability

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

P(W=s,T=c)

P(T=c)

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

Quiz: Conditional Probabilities

$$P(X,Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?
- P(-x | +y) ?
- P(-y | +x) ?

Quiz: Conditional Probabilities

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?

.2/.6=1/3

- P(-x | +y) ?

.4/.6=2/3

- P(-y | +x) ?

.3/.5=.6

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$P(W|T)$

$P(W|T = hot)$

W	P
sun	0.8
rain	0.2

$P(W|T = cold)$

W	P
sun	0.4
rain	0.6

Joint Distribution

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Using Definition of Conditional Probability

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$\begin{aligned} P(W=s|T=c) &= \frac{P(W=s,T=c)}{P(T=c)} \\ &= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.2}{0.2+0.3} = 0.4 \end{aligned}$$

$$\longrightarrow$$

$$\begin{aligned} P(W=r|T=c) &= \frac{P(W=r,T=c)}{P(T=c)} \\ &= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.3}{0.2+0.3} = 0.6 \end{aligned}$$

$$P(W|T=c)$$

W	P
sun	0.4
rain	0.6

Normalization Trick

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$
$$= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.2}{0.2+0.3} = 0.4$$

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence →

$P(c,W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one) →

$P(W|T=c)$

W	P
sun	0.4
rain	0.6

$$P(W=r|T=c) = \frac{P(W=r,T=c)}{P(T=c)}$$
$$= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.3}{0.2+0.3} = 0.6$$

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)

$P(W|T = c)$

W	P
sun	0.4
rain	0.6

- Why does this work? Sum of selection is P(evidence)! (P(T=c), here)

$$P(x_1|x_2) = \frac{P(x_1, x_2)}{P(x_2)} = \frac{P(x_1, x_2)}{\sum_{x_1} P(x_1, x_2)}$$

Quiz: Normalization Trick

- P(X | Y=-y) ?

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence

NORMALIZE the selection (make it sum to one)

Quiz: Normalization Trick

- P(X | Y=-y) ?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence →

X	Y	P
+x	-y	0.3
-x	-y	0.1

NORMALIZE the selection (make it sum to one) →

X	P
+x	0.75
-x	0.25

To Normalize

- (Dictionary) To bring or restore $\boxed{\text{to a normal condition}}$

All entries sum to ONE

- Procedure:
  - Step 1: Compute Z = sum over all entries
  - Step 2: Divide every entry by Z

- Example 1

W	P
sun	0.2
rain	0.3

Normalize

Z = 0.5

W	P
sun	0.4
rain	0.6

- Example 2

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

Normalize

Z = 50

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)

- We generally compute conditional probabilities
  – P(on time | no reported accidents) = 0.90
  – These represent the agent's beliefs given the evidence

- Probabilities change with new evidence:
  – P(on time | no accidents, 5 a.m.) = 0.95
  – P(on time | no accidents, 5 a.m., raining) = 0.80
  – Observing new evidence causes beliefs to be updated

Q: Inference by Enumeration

- P(W)?

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Q: Inference by Enumeration

- P(W)?

P(sun)=.3+.1+.1+.15=.65

P(rain)=1-.65=.35

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Quiz: Inference by Enumeration

- P(W | winter, hot)?

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W | winter, hot)?

P(sun|winter,hot)~.1

P(rain|winter,hot)~.05

P(sun|winter,hot)=2/3

P(rain|winter,hot)=1/3

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W | winter)?

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W | winter)?

  P(sun, winter) = .1+.15=.25

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W | winter)?

  P(rain, winter)=.05+.2=.25

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W | winter)?

  P(sun|winter)~.25
  P(rain|winter)~.25

  P(sun|winter)=.5
  P(rain|winter)=.5

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- General case:
  - Evidence variables: $E_1 \ldots E_k = e_1 \ldots e_k$
  - Query* variable: Q
  - Hidden variables: $H_1 \ldots H_r$

$$\left.\begin{array}{l} E_1 \ldots E_k = e_1 \ldots e_k \\ Q \\ H_1 \ldots H_r \end{array}\right\} X_1, X_2, \ldots X_n$$

All variables

- We want: $P(Q|e_1 \ldots e_k)$

- Step 1: Select the entries consistent with the evidence
- Step 2: Sum out H to get joint of Query and evidence
- Step 3: Normalize

$$P(Q, e_1 \ldots e_k) = \sum_{h_1 \ldots h_r} P(\underbrace{Q, h_1 \ldots h_r, e_1 \ldots e_k}_{X_1, X_2, \ldots X_n})$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

Inference by Enumeration

- Obvious problems:
  - Worst-case time complexity $O(d^n)$
  - Space complexity $O(d^n)$ to store the joint distribution

The Product Rule: Computing Joint Distributions

$$P(y)P(x|y) = P(x, y)$$

- Example:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$\Leftrightarrow$

$P(D, W)$

D	W	P
wet	sun	
dry	sun	
wet	rain	
dry	rain	

The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x,y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Enables us build one conditional from its reverse
  - Often one conditional is difficult but the other one is easy

Conditional Probabilities and Bayes' Rule

- Conditional Probability: **The probability of an event given that another event has occurred**. E.g. Rolling a pair of dice (red, blue)
  - Two random variables: sum of dice, value of red die
  - P(sum = 4) = 3/36 = 1/12 unconditional probability
  - P(sum=4 | red=1) = 1/6 P(red=1 | sum=4) = 1/3
- Conditional Probability Definition:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ if } P(B) \neq 0, \qquad P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ if } P(A) \neq 0.$$

- Bayes' Rule/Theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}, \text{ if } P(B) \neq 0.$

Quiz: Bayes' Rule

- Given:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is P(W | dry) ?

Quiz: Bayes' Rule

- Given:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is P(W | dry) ?

P(sun|dry) ~ P(dry|sun)P(sun) = .9*.8 = .72
P(rain|dry) ~ P(dry|rain)P(rain) = .3*.2 = .06
normalize
P(sun|dry)=12/13
P(rain|dry)=1/13

Bayes' Rule and Its Use

- Bayes’ rule is derived from the product rule
  $P(a \wedge b) = P(a|b)P(b)$ and $P(a \wedge b) = P(b|a)P(a)$.

- Equating the two right-hand sides and dividing by P(a), we get
  $P(a|b)P(b) = P(b|a)P(a) \Rightarrow P(a|b) = P(b|a)P(a) \,/\, P(b)$

- Often, we perceive as evidence the effect of some unknown cause and we would like to determine that cause. In that case, Bayes’ rule becomes:
  P(cause|effect) =P(effect |cause)P(cause) **/** P(effect)

- The conditional probability P(effect|cause) quantifies the relationship in the causal direction (usually easier to compute) ,
  whereas P(cause|effect) describes the diagnostic direction.

Bayes' Rule and Its Use: Meningitis

The disease meningitis causes a patient to have a stiff neck, say, 70% of the time. The are also unconditional facts: the prior probability that any patient has meningitis is 1/50,000, and the prior probability that any patient has a stiff neck is 1%. **Set up:**

- S be the proposition that the patient has a stiff neck
- M be the proposition that the patient has meningitis, we have

Known:

- $P(S|M) = 0.7$ $P(M) = 1/50000 = .00002$ $P(S) = 0.01$

Compute the “posterior probability that the patient has meningitis

- $P(M|S) = P(S|M)\ P(M) / P(S) = (0.7 \times 1/50000) / 0.01 = 0.0014$
- Note: Estimate only 0.14% of patients with a stiff neck to have meningitis.
- Even though a stiff neck is quite strongly indicated by meningitis (with probability 0.7), the probability of meningitis in patients with stiff necks remains small.
- This is because the prior probability of stiff necks (from any cause) is much higher than the prior for meningitis.