ELEMENTARY FUNCTIONS AND THEIR INVERSES*

BY

J. F. RITT

The chief item of this paper is the determination of all elementary functions whose inverses are elementary. The elementary functions are understood here to be those which are obtained in a finite number of steps by performing algebraic operations and taking exponentials and logarithms. For instance, the function

$$\tan[e^{z^2}-\log_z(1+\sqrt{z})]+[z^3+\log\operatorname{arc}\sin z]^{1/2}$$

is elementary.

We prove that *if $F(z)$ and its inverse are both elementary, there exist n functions*

$$\varphi_1(z), \quad \varphi_2(z), \quad \cdots, \quad \varphi_n(z),$$

where each $\varphi(z)$ with an odd index is algebraic, and each $\varphi(z)$ with an even index is either e^z or $\log z$, *such that*

$$F(z) = \varphi_n\varphi_{n-1}\cdots\varphi_2\varphi_1(z)$$

each $\varphi_i(z)$ $(i < n)$ being substituted for z in $\varphi_{i+1}(z)$. That every $F(z)$ of this type has an elementary inverse is obvious.

It remains to develop a method for recognizing whether a given elementary function can be reduced to the above form for $F(z)$. How to test fairly simple functions will be evident from the details of our proofs. For the immediate present, we let the general question stand.

The present paper is an addition to Liouville's work of almost a century ago on the classification of the elementary functions, on the possibility of effecting integrations in finite terms, and on the impossibility of solving certain differential equations, and certain transcendental equations, in finite terms.† Free use is made here of the ingenious methods of Liouville.

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† Journal de l'École Polytechnique, vol. 14 (1833), p. 36; Journal für die reine und angewandte Mathematik, vol. 13 (1833), p. 93; Journal de Mathématiques, vol. 2 (1837), p. 56, vol. 3 (1838), p. 523, vol. 6 (1841), p. 1. For extensions of Liouville's work on differential equations, see Lorenz, Hansen, Steen and Petersen, Tidskrift for Mathematik, 1874–1876; Koenigsberger, Mathematische Annalen, 1886; Mordukhai-Boltovski, University of Warsaw Bulletin, 1909, 1910.

Reference should also be made to the classifications by Painlevé* and by Drach† of the solutions of algebraic differential equations.

In the course of our work we prove a set of lemmas of which some are not uninteresting in themselves. That of § 14 promises to be useful in settling other questions on the elementary functions. The result on functions with elementary inverses is a corollary of a very general theorem stated in § 23.

We precede the solution of our problem by a discussion which is designed to lend rigor to our work. This discussion is more explicit, on certain points of special importance in the present paper, than that in our paper *On the integrals of elementary functions.*‡ The formal parts of our work can probably be followed without a careful reading of these preliminaries.

I. Elementary functions. Their differentiation. Liouville's principle

1. An analytic function of z will be said to be *analytic almost everywhere* if, given any element of the function $P(z-z_0)$,§ any curve

$$z = \varphi(\lambda) \qquad (0 \leqq \lambda \leqq 1),$$

where $\varphi(0) = z_0$, and any positive ε, there exists a curve

$$z = \varphi_1(\lambda) \qquad (0 \leqq \lambda \leqq 1), \tag{1}$$

where $\varphi_1(0) = z_0$, such that

$$|\varphi_1(\lambda) - \varphi(\lambda)| < \varepsilon$$

for $0 \leqq \lambda \leqq 1$, and such that the element $P(z-z_0)$ can be continued along the entire curve (1). Roughly speaking, an element of the function, if it cannot be continued along a given path, can be continued along some path in any neighborhood of the given one.

2. An algebraic function u, given by an irreducible equation

$$a_0 u^m + a_1 u^{m-1} + \cdots + a_m = 0, \tag{2}$$

* *Leçons sur les Équations Différentielles, professées à Stockholm,* Paris, 1897, p. 487.

† Annales de l'École Normale Supérieure, vol. 34 (1898), p. 243.

‡ These Transactions, vol. 25 (1923), p. 211.

§ It is to be recalled that an analytic element $P(z-z_0)$ is a convergent series of positive powers of $z-z_0$.

where each α is a polynomial in z with constant coefficients, is analytic almost everywhere, because its singularities are isolated.

In what follows the algebraic functions will frequently be called *functions of order zero,* and the variable z a *monomial of order zero.*

3. The functions e^v and $\log v$, where v is any non-constant algebraic function, are called by Liouville *monomials of the first order.* It is seen directly that e^v is analytic almost everywhere. If v is analytic, and nowhere zero, along a given curve, $\log v$ is analytic along the curve. If v should vanish for some points (necessarily isolated) of the curve, there is a curve arbitrarily close to the given one on which v is everywhere different from zero. Thus $\log v$ is analytic almost everywhere.

More generally we shall say, following Liouville, that u is a *function of the first order* if it is not algebraic and if it satisfies an equation like (2) in which each α is a rational integral combination of monomials of orders zero and one, not all α's being zero.

We mean by this that, for some point z_0, the function u and each of the monomials in the α's have analytic elements which, when combined by multiplication and addition to form the first member of (2), yield an element with coefficients all zero. We may of course assume that α_0 is not identically zero.

4. Let Γ be any area in the complex plane, and suppose that we can continue the above mentioned element of u with center at z_0 into and all over Γ, so that u has a branch which is uniform and analytic throughout Γ. Let C be some curve along which u can be continued from z_0 into Γ. Any curve which can be obtained from C by a slight deformation will serve equally well for the continuation of u into Γ. As each monomial in (2) is analytic almost everywhere, we can take a curve close to C all along which each monomial can be continued from z_0. It is easy to see that a single curve can be taken for all the monomials, because a curve which will do for one of them can be shifted slightly so as to do also for another. We conclude that in any area in which u has an analytic branch, there is an area in which all the monomials in (2) have analytic branches which satisfy (2) together with u. Evidently we can choose the smaller area in such a way that each of the algebraic functions of which the monomials in (2) are exponentials or logarithms is analytic in the smaller area.

5. Consider the domain of rationality of all of the monomials in (2). We can form this domain by taking all rational combinations of the given elements of the monomials, with centers at z_0, and continuing the functions thus obtained. If the first member of (2) is reducible in this domain, let it be replaced by that one of its irreducible factors which vanishes for the

given element of u. We may thus assume that the discriminant of (2), which is analytic in any region in which the α's (properly associated branches of them) are analytic, and in which α_0 is not zero, does not vanish for every z. We see now that u is analytic almost everywhere, since in the neighborhood of any curve there is a curve along which each α is analytic, and on which α_0 and the discriminant of (2) are everywhere different from zero.

6. Let the monomials of order one, some exponentials, some logarithms, which appear in (2) be $\theta_1(z), \cdots, \theta_r(z)$. Suppose that, in every α, we replace each θ_i by a variable z_i'. We form thus an equation

$$\alpha_0(z; z_i')v^m + \alpha_1(z; z_i')v^{m-1} + \cdots + \alpha_m(z; z_i') = 0.$$

Let a be any value of z at which the monomials are all analytic, and at which α_0 and the discriminant of (2) are not zero. Then for $z = a$, $z_i' = \theta_i(a)$ $(i = 1, \cdots, r)$, the first coefficient of the equation for v, and the discriminant, do not vanish. We obtain thus an algebraic function v of $z, z_1', \cdots, z_r'$, analytic when these variables remain in the neighborhood of $z = a$, $z_i' = \theta_i(a)$, and which, when each z_i' is replaced by $\theta_i(z)$, reduces, for a neighborhood of $z = a$, to the function u defined by (2). We observe that the equation for v is independent of the point a.

7. Comparing § 4 and § 6, we see that if u is a function of order 1, then for any area in which (some branch of) u is analytic, there exist

(0) a point a interior to the area, a $\varrho > 0$ and a $\varrho_1 > \varrho$;

(I) r algebraic functions of z, each analytic for $|z-a| < \varrho_1$;

(I′) r monomials, $\theta_1, \cdots, \theta_r$, each either an exponential or a logarithm of one of the r functions in (I), each analytic for $|z-a| < \varrho$, and such that $|\theta_i(z) - \theta_i(a)| < \varrho_1$ for $|z-a| < \varrho$ $(i = 1, \cdots, r)$;

(II) an algebraic function of the variables $z, z_1', \cdots, z_r'$ which is analytic for $|z-a| < \varrho_1$, $|z_i' - \theta_i(a)| < \varrho_1$ $(i = 1, \cdots, r)$, and which reduces to (the given branch of) u for $|z-a| < \varrho$, if each z is replaced by θ_i.*

Furthermore, the integer r, the algebraic equations satisfied by the functions in (I) and that in (II), and the exponential or logarithmic characters of the θ's, are independent of the area in which u is considered and of the branch of u.

8. We now define, by induction, functions of any order n. The exponential or a logarithm of a function of order $n-1$ will be called a *monomial of order* n, provided that it is not among the functions of orders $0, 1, \cdots, n-1$.

* The fact that this algebraic function may actually depend on z explains our insistence that ρ_1 exceed ρ.

With the same reservation, any function defined by an equation like (2), in which each α is a rational integral combination of monomials of order $0, 1, \cdots, n$, is a *function of order* n*. As above, we may assume that the discriminant of (2) does not vanish identically. One sees by a quick induction that a function of any order n is analytic almost everywhere.

9. As in § 7, we find by induction that if u is a function of any order n, then, for any area in which some branch of u is analytic, there exist

(0) a point a interior to the area, a $\varrho > 0$ and a $\varrho_1 > \varrho$;

(I) r_1 algebraic functions of z, each analytic for $|z-a| < \varrho_1$;

(I′) r_1 monomials, $\theta_1', \cdots, \theta_{r_1}'$, each either an exponential or a logarithm of one of the functions in (I), each analytic for $|z-a| < \varrho$, and such that $|\theta_i'(z) - \theta_i'(a)| < \varrho_1$ for $|z-a| < \varrho$ and for every i;

(II) r_2 algebraic functions of z and of r_1 other variables $z_1', \cdots, z_{r_1}'$, each analytic for $|z-a| < \varrho_1$, $|z_i' - \theta_i'(a)| < \varrho_1$;

(II′) r_2 monomials, $\theta_1'', \cdots, \theta_{r_2}''$, each either an exponential or a logarithm of one of the functions of order 1 to which the algebraic functions in (II) reduce when each z_i' is replaced by θ_i'; each θ_i'' is analytic for $|z-a| < \varrho$, and also $|\theta_i''(z) - \theta_i''(a)| < \varrho_1$ for $|z-a| < \varrho$;

(III) r_3 algebraic functions of $z, z_1', \cdots, z_{r_1}'$ and of r_2 variables $z_1'', \cdots, z_{r_2}''$, each analytic for $|z-a| < \varrho_1$, $|z_i' - \theta_i'(a)| < \varrho_1$, $|z_i'' - \theta_i''(a)| < \varrho_1$;

. . . .

(N+I) an algebraic function of $z; \cdots; z_1^{(n)}, \cdots, z_{r_n}^{(n)}$, analytic for $|z-a| < \varrho_1, \cdots, |z_i^{(n)} - \theta_i^{(n)}(a)| < \varrho_1$, which reduces to the given branch of u for $|z-a| < \varrho$, when each variable z is replaced by the monomial which corresponds to it.

Furthermore the integers r_i, the algebraic equations satisfied by the functions in (I), $\cdots$, (N+I), and the character of the θ's as exponentials or logarithms are independent of the areas in which u is considered, and of the branch of u.

We see that an accented z may be used in forming a monomial of higher order than that to which it corresponds, and be used again by itself.†
We have chosen a symbolism which allows this, for the purposes of § 11.

10. For any n, the functions of orders $0, 1, \cdots, n$ form a set which is closed with respect to all algebraic operations. That is, a function defined by an equation like (2), in which each α is a rational integral combination of *functions* of orders $0, 1, \cdots, n$, is itself a function of one of those orders. This follows immediately from (N+I) of § 9, if one considers that an algebraic function of algebraic functions is also algebraic.

* The existence of functions of all orders is proved by Liouville.

† Consider $\log(e^z + 1) + e^z$.

The functions to which orders are assigned by the preceding definitions will be called *elementary functions* of z.

11. We consider now the differentiation of the elementary functions. Of the algebraic functions introduced in (I), $\cdots$, (N) of § 9, there are possibly some which are used for forming logarithmic monomials. As each monomial is analytic at a, such an algebraic function cannot vanish when z is a, and each accented z is its $\theta(a)$; the function is therefore distinct from zero if the z's are close to these values. If now ϱ_1 is taken sufficiently small, and if ϱ is made correspondingly small, so as to limit the variation of the monomials, we may assume that none of the algebraic functions which give logarithmic monomials vanish when z differs from a, and each accented z from its $\theta(a)$, by an amount smaller than ϱ_1 in modulus.

This understood, the formulas for the differentiation of composite functions show that *if u is an elementary function, described as in* (N+I) *of* § 9, *there exists an algebraic function of the z's, analytic for $|z-a|<\varrho_1$, $\cdots$, $|z_i^{(n)}-\theta_i^{(n)}(a)|<\varrho_1$, which reduces to the derivative of u for $|z-a|<\varrho$, when each variable is replaced by the monomial which corresponds to it.* A similar result holds for the higher derivatives of u.

12. The equation (2) which defines a function u of order n is never unique, except for $n=0$. But of all the equations (2) which determine u, there are some which involve a minimum number of monomials of order n; that is, the r_n in (N + I) of § 9 is a minimum. In that case, no algebraic relation can exist between these r_n monomials of order n and monomials of order less than n. We mean by this that if $\xi_1, \cdots, \xi_p$ are monomials of order less than n, analytic at $z=a$, and if a function

$$f(z_1^{(n)}, \cdots, z_{r_n}^{(n)};\ x_1, \cdots, x_p),$$

algebraic in all its variables, and analytic for $z_i^{(n)}=\theta_i^{(n)}(a)$, $x_i=\xi_i(a)$, should vanish for the neighborhood of a when each $z_i^{(n)}$ is replaced by $\theta_i^{(n)}$ and each x_i by ξ_i, then the function vanishes for any $z^{(n)}$'s close to the values $\theta^{(n)}(a)$, if only each x_i is replaced by ξ_i.

For suppose that this is not so. Then there is a point b, close to a, such that for $x_i=\xi_i(b)$ $(i=1, 2, \cdots, p)$, and for certain values of the $z^{(n)}$'s close to the $\theta^{(n)}(a)$'s, f does not vanish. Consider the partial derivatives of f, of all orders, with respect to the $z^{(n)}$'s.* Not all of them can vanish for $x_i=\xi_i(b)$, $z_i^{(n)}=\theta_i^{(n)}(b)$, else we could not make f different from zero by varying the $z^{(n)}$'s slightly from the $\theta^{(n)}(b)$'s. (Each $\theta^{(n)}(b)$ is close to $\theta^{(n)}(a)$.)

* Cross-derivatives included.

Suppose then that all of the derivatives up to and including those of order j vanish over the neighborhood of a when the variables are replaced by their monomials, but that some derivative of order $j+1$ does not vanish for a b close to a. To fix our ideas, suppose that

$$g(z_1^{(n)}, \cdots, z_{r_n}^{(n)}; x_1, \cdots, x_p)$$

is a partial derivative which vanishes over the neighborhood of a, but that the derivative of g with respect to $z_1^{(n)}$ does not vanish at b. Then the equation $g = 0$ determines $z_1^{(n)}$ as an algebraic function of $z_2^{(n)}, \cdots, x_p$, analytic in the neighborhood of $\theta_2^{(n)}(b), \cdots, \xi_p(b)$, which reduces to $\theta_1^{(n)}$ for the familiar replacements. If we substitute this algebraic function for $z_1^{(n)}$ in (N+I) of § 9, we find a contradiction of the assumption that r_n is a minimum.

The foregoing principle is due to Liouville, and underlies all of his work on the elementary functions.

II. Some lemmas

13. By a *logarithmic sum of order* n, we shall mean a function of order n of the form

$$c_1 \log \varphi_1(z) + \cdots + c_m \log \varphi_m(z) \qquad (m \geqq 1),$$

where each c is a constant, and each $\varphi(z)$ a function of order not exceeding $n-1$. Of course, at least one $\varphi(z)$ is of order $n-1$.*

If we assume that m is a minimum, it follows that no relation $\sum p_i c_i = 0$ can exist with the p's integral and not all zero. For if, for instance, $c_m = \sum_{i=1}^{m-1} q_i c_i$, with the q's rational, the sum could be written $\sum_{i=1}^{m-1} c_i \log \varphi_i \varphi_m^{q}$.

A function defined by an equation (2) in which each α is a rational integral combination of exponential monomials of order n, of logarithmic sums of order n and of monomials of order less than n, will be of order n or less. We may reword (N+I) of § 9, (also (N) and (N′)), so as to permit the substitution of logarithmic sums of order n, with any number of terms, for some of the variables $z^{(n)}$.† The results of §§ 11, 12 evidently hold for this new type of substitution.

14. The proof of the following lemma will sharpen its statement.

Lemma. *If, in the expression for a function u of order n, the number of exponentials of order n plus the number of logarithmic sums of order n*

* The present investigation seems to be the first in which sums of logarithms play the rôle of monomials.

† For the $z^{(p)}$ with $p < n$, we shall continue to substitute only monomials.

is a minimum, each exponential of order n and each logarithmic sum of order n is an algebraic function of u, a certain number of the derivatives of u and the monomials of order less than n which appear in the expression for u.

We represent the derivatives of u by u', u'', etc. According to § 11, there exists an infinite sequence of algebraic functions

$$
(3)\qquad
\begin{aligned}
v &= f_0(z_1^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z),\\
v' &= f_1(z_1^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z),\\
v'' &= f_2(z_1^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

which reduce respectively to u, u', u'', etc., for the neighborhood of $z = a$ ((I) of § 9), when each z is replaced by its corresponding monomial or logarithmic sum. The functions of (3) are analytic when the variables are close to the values which their corresponding monomials or sums assume at $z = a$.

Consider any r_n functions of (3). The functional determinant of these functions with respect to $z_1^{(n)}, \cdots, z_{r_n}^{(n)}$ is algebraic in all the z's. If, for some b close to a, this jacobian does not vanish when the z's are replaced by the values which their monomials or sums assume at b, we can solve for the $z^{(n)}$'s; each $z^{(n)}$ will be an algebraic function of $z_1^{(n-1)}, \cdots, z$ and a certain number of v's, which reduces to the exponential or logarithmic sum corresponding to that $z^{(n)}$ when $z_1^{(n-1)}, \cdots, z$ are properly replaced and when each $v^{(i)}$ is replaced by $u^{(i)}$.* This is the state of affairs sought in the lemma.

We are going to show that, because r_n is a minimum, there must be r_n functions in (3) whose jacobian does not vanish throughout the neighborhood of a.

Let the contrary be assumed. We observe first that the derivative of v with respect to $z_1^{(n)}$ cannot vanish for every z close to a. If it did, then, according to Liouville's principle, it would vanish for any $z_1^{(n)}$ close to $\theta_1^{(n)}(a)$, if only the other z's are replaced by their monomials or logarithmic sums. This would mean that u is obtained from

$$f(\theta_1^{(n)}(a),\ z_2^{(n)}, \cdots, z_{r_n}^{(n)};\ z_1^{(n-1)}, \cdots, z)$$

by the familiar replacements, and that r_n is not a minimum.

* $v^{(0)} = v$, $u^{(0)} = u$.

Suppose then that for certain $m < r_n$ of the functions (3), the jacobian with respect to m of the $z^{(n)}$'s, say $z_1^{(n)}, \cdots, z_m^{(n)}$, does not vanish for some b close to a, but that the jacobian of any $m+1$ functions of (3) with respect to any $m+1$ of the $z^{(n)}$'s vanishes for every z close to a. Let the m functions be

$$
\begin{aligned}
v^{(i_1)} &= f_{i_1}(z_1^{(n)}, \cdots, z_m^{(n)}; z_{m+1}^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
v^{(i_m)} &= f_{i_m}(z_1^{(n)}, \cdots, z_m^{(n)}; z_{m+1}^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z).
\end{aligned} \tag{4}
$$

In (4), let $z_1^{(n-1)}, \cdots, z$ be replaced by the values of their monomials at $z = b$, and each $v^{(p)}$ by the value of $u^{(p)}$ at $z = b$. Then $z_1^{(n)}, \cdots, z_m^{(n)}$ become functions of $z_{m+1}^{(n)}, \cdots, z_{r_n}^{(n)}$, analytic when the latter variables stay in the neighborhood of $\theta_{m+1}^{(n)}(b), \cdots, \theta_{r_n}^{(n)}(b)$. We are going to allow $z_{m+1}^{(n)}, \cdots, z_{r_n}^{(n)}$ to vary in the following way as functions of a parameter μ. Suppose that $\theta_{m+1}^{(n)}, \cdots, \theta_p^{(n)}$ are exponentials, while $\theta_{p+1}^{(n)}, \cdots, \theta_{r_n}^{(n)}$ are logarithmic sums. We put

$$
\begin{aligned}
z_{m+1}^{(n)} &= (1+\mu)\,\theta_{m+1}^{(n)}(b), \cdots, z_p^{(n)} = (1+\mu)\,\theta_p^{(n)}(b), \\
z_{p+1}^{(n)} &= \theta_{p+1}^{(n)}(b) + \mu, \qquad \cdots, z_{r_n}^{(n)} = \theta_{r_n}^{(n)}(b) + \mu.
\end{aligned} \tag{5}
$$

Then $z_1^{(n)}, \cdots, z_m^{(n)}$ become functions of μ, analytic for $\mu = 0$. Suppose that $\theta_1^{(n)}, \cdots, \theta_l^{(n)}$ are exponentials, while $\theta_{l+1}^{(n)}, \cdots, \theta_m^{(n)}$ are logarithmic sums. We define functions $\beta(\mu)$, analytic at $\mu = 0$, by

$$
\begin{aligned}
z_1^{(n)} &= \beta_1(\mu)\,\theta_1^{(n)}(b), \qquad \cdots, z_l^{(n)} = \beta_l(\mu)\,\theta_l^{(n)}(\mu), \\
z_{l+1}^{(n)} &= \theta_{l+1}^{(n)}(b) + \beta_{l+1}(\mu), \cdots, z_m^{(n)} = \theta_m^{(n)}(b) + \beta_m(\mu).^*
\end{aligned} \tag{6}
$$

Thus if all the $z^{(n)}$'s in (4) are replaced by the functions of μ associated with them, the other z's by the values of their monomials at b, the second members in (4) stay constant as μ ranges over the neighborhood of zero.

Consider now any $v^{(q)}$ of (3) where q is distinct from every i of (4). The jacobian of $v^{(q)}$ and of the functions of (4) with respect to any $m+1$ of the $z^{(n)}$'s vanishes for the neighborhood of a. By Liouville's principle, such a jacobian must vanish for arbitrary $z^{(n)}$'s close to their respective $\theta^{(n)}(b)$'s if only the other z's are replaced by their monomials and sums.

* $\beta_i(0)$ is zero or one according as i does or does not exceed l.

It follows from well known theorems on functional dependence that if, in $v^{(q)}$, $z^{(n-1)}, \cdots, z$ are replaced by the values of their monomials at b and if $z_1^{(n)}, \cdots, z_{r_n}^{(n)}$ vary, according to (5) and (6) for instance, so as to keep the functions in (4) constant, $v^{(q)}$ will also stay constant.

The function v' of (3) is derived from v by a formula

$$v' = \sum \frac{\partial v}{\partial z_i^{(n)}} z_i^{(n)} \varphi_i + \sum \frac{\partial v}{\partial z_j^{(n)}} \varphi_j + \text{other terms},$$

where the $z_i^{(n)}$'s correspond to exponentials and the $z_j^{(n)}$'s to logarithmic sums. Each φ_i is a function of $z_1^{(n-1)}, \cdots, z$ which reduces to the derivative of the exponent in $\theta_i^{(n)}$ for the proper replacements; each φ_j is a function of $z^{(n-1)}, \cdots, z$ which reduces to the derivative of $\theta_j^{(n)}$. The "other terms" are derivatives of v with respect to $z_1^{(n-1)}, \cdots, z$ times algebraic functions which reduce to the derivatives of $\theta_1^{(n-1)}, \cdots, z$. It follows that if each $z_i^{(n)}$ is replaced in v' by $k_i \theta_i^{(n)}$ (k_i a constant close to unity) and each $z_j^{(n)}$ by $\theta_j^{(n)} + k_j$ (k_j a constant close to zero), and $z^{(n-1)}, \cdots, z$ by their monomials, the function obtained is the derivative of the function obtained from v by these same replacements. Similarly, v'' etc. will give the higher derivatives of the new function obtained from v.

If, in (5) and (6), we write z in place of b, the $z^{(n)}$'s are associated with functions of z and μ, analytic for $z = b$, $\mu = 0$. If, in v, we replace the $z^{(n)}$'s by these functions, and the other z's by their monomials, we obtain, for any μ, a function u_μ of z. By what we have just seen, the derivatives of u_μ with respect to z for $z = b$ are obtained by making the substitutions (5) and (6), and replacing $z_1^{(n-1)}, \cdots, z$ by $\theta_1^{(n-1)}(b), \cdots, b$ in the functions of (3). Thus the discussion of $v^{(q)}$ above shows that

$$u_\mu(b) = u(b), \qquad u'_\mu(b) = u'(b), \qquad u''_\mu(b) = u''(b), \cdots.$$

Hence, as u_μ and u are analytic in z, *they are identical.*

Thus the partial derivative of u_μ with respect to μ is zero for every admissible z and μ. We equate to zero this partial derivative for $\mu = 0$, and find, using (5) and (6) with b replaced by z,

$$(7) \quad \sum_{i=1}^{l} \beta_i'(0) \frac{\partial v}{\partial z_i^{(n)}} z_i^{(n)} + \sum_{i=l+1}^{m} \beta_i'(0) \frac{\partial v}{\partial z_i^{(n)}} + \sum_{i=m+1}^{p} \frac{\partial v}{\partial z_i^{(n)}} z_i^{(n)} + \sum_{i=p+1}^{r_n} \frac{\partial v}{\partial z_i^{(n)}} = 0.$$

In (7), each z is to be replaced by its monomial or logarithmic sum. But, according to Liouville's principle, (7) will also hold for arbitrary $z^{(n)}$'s if the other z's are replaced by their monomials.

The fact that some of the coefficients $\beta_i'(0)$ may be zero makes a change of notation desirable. Every $z^{(n)}$ of (7), for which $\beta_i'(0)=0$, we replace by a symbol w_q. Every other $z^{(n)}$ we replace by an x_q or a y_q, according as it corresponds to an exponential or to a logarithmic sum. If there are j of the w's, h of the x's, k of the y's, we have $j+h+k=r_n$. The first function of (3) becomes

$$(8)\qquad v=f(w_1,\cdots,w_j;\ x_1,\cdots,x_h;\ y_1,\cdots,y_k;\ z_1^{(n-1)},\cdots,z),$$

the order of its arguments probably being disturbed, while (7) assumes the form

$$(9)\qquad \gamma_1 x_1\frac{\partial v}{\partial x_1}+\cdots+\gamma_h x_h\frac{\partial v}{\partial x_h}+\delta_1\frac{\partial v}{\partial y_1}+\cdots+\delta_k\frac{\partial v}{\partial y_k}=0.$$

Here each γ or δ is either unity or a $\beta'(0)\neq 0$. Also (9) holds for arbitrary, but admissible, w's, x's and y's if $z_1^{(n-1)},\cdots,z$ are replaced by their monomials.

Suppose first that some x's are actually present in (9). We may, after a division, assume that $\gamma_1=1$. Consider then the following $h+k-1$ solutions of (9):

$$(10)\qquad \begin{aligned} s_2&=x_2x_1^{-\gamma_2}, &&\cdots,\ s_h=x_hx_1^{-\gamma_h},\\ t_1&=y_1-\delta_1\log x_1, &&\cdots,\ t_k=y_k-\delta_k\log x_1.\end{aligned}$$

These solutions are analytic for the values in which we are interested of the x's and the y's, because x_1, which is associated with the exponential of an analytic function, does not become zero. The jacobian of these solutions with respect to $x_2,\cdots,y_k$ is $x^{-(\gamma_2+\cdots+\gamma_h)}$, which is not zero. Consequently if the w's are given arbitrary fixed values, and if $z_1^{(n-1)},\cdots,z$ are held fast at the values of their monomials for any fixed z, v in (8) becomes an analytic function of the functions (10). If we replace $x_2,\cdots,y_k$ by their values obtained from (10), we find

$$(11)\quad v=f(w_1,\cdots;\ x_1,\ x_1^{\gamma_2}s_2,\cdots;\ t_1+\delta_1\log x_1,\cdots;\ z^{(n-1)},\cdots,z).$$

By what precedes, the second member of (11) is independent of x_1, so that

$$(12)\qquad v=f(w_1,\cdots;\ c_1,\ c_2s_2,\cdots;\ t_1+d_1,\cdots;\ z_1^{(n-1)},\cdots,z),$$

where the c's and d's are constants.

We notice that when the x's are replaced by their exponentials, each s becomes an exponential of a function which is at most of order $n-1$, and each t a logarithmic sum of order n plus a function of order $n-1$. If then we replace the variables in (12) by the functions of z to which they correspond, we have u expressed in terms of fewer than r_n exponentials and sums of order n. This contradiction of the assumption that r_n is a minimum implies the truth of the lemma.

If no x's are present in (9) ($h=0$), we use the independent solutions of (9),

$$t_2 = \delta_2 y_2 - \delta_1 y_1, \quad \cdots, \quad t_k = \delta_k y_k - \delta_1 y_1 .$$

As above, we find that r_n is no minimum. This completes the proof of the lemma.

15. We shall call any set of numbers, $c_1, \cdots, c_m$ *dependent* or *independent* according as there do or do not exist integers $p_1, \cdots, p_m$, not all zero, such that $\sum p_i c_i = 0$.

LEMMA. *A function* $\sum_{i=1}^{m} c_i \log \varphi_i(z)$, *with no* $\varphi_i(z)$ *of order greater than* $n-1$, *with at least one* $\log \varphi_i(z)$ *of order* n, *and with independent* c*'s, is a function of order* n.

We begin by proving the theorem for the case of $n=1$. Suppose then that each $\varphi_i(z)$ is an algebraic function, that some $\varphi_i(z)$ is not constant, but that the sum of logarithms is an algebraic function $\psi(z)$. Differentiating, we have

$$c_1 \frac{\varphi_1'(z)}{\varphi_1(z)} + \cdots + c_m \frac{\varphi_m'(z)}{\varphi_m(z)} = \psi'(z). \tag{13}$$

Suppose that a function $\varphi_i(z)$ has a zero or a pole at some point a, which may or may not be a branch point of the function. Then $\varphi_i'(z)/\varphi_i(z)$ will have a pole at a in which the coefficient of $1/(z-a)$ is a rational number; the coefficient may be a fraction if a is a branch point, but otherwise it is an integer. Thus the first member of (13) has a development at a in which the coefficient of $1/(z-a)$ is a linear combination of the c's with rational coefficients, some coefficients distinct from zero. But we cannot get a term in $1/(z-a)$ by differentiating an algebraic function $\psi(z)$, so that (13) is impossible.

Suppose now that the lemma is untrue for some $n>1$, so that there exists a class of functions $\psi = \sum_{i=1}^{m} c_i \log \varphi_i$ of order less than n, with each φ_i of order less than n, with some $\log \varphi_i$ of order n, and with independent c's. Here m may depend on ψ, but this is not of importance.

For any ψ of this class, let r represent the minimum number of monomials of order $n-1$ in terms of which, with monomials of lower order, ψ and all of the functions φ_i can be expressed. Consider the subclass formed by those functions ψ whose r is not greater than the r of any other ψ. We may assume that the functions of this subclass are so expressed that of the r monomials of order $n-1$ appearing in $\psi, \varphi_1, \cdots, \varphi_m$, the number s of those which appear in $\varphi_1, \cdots, \varphi_m$ is a minimum. In the subclass there are certain functions ψ whose s is not greater than the s of any other function of the subclass. We assume that we have in hand a ψ of this type, and proceed to force a contradiction.

Writing w for $z^{(n-1)}$, we have, for the familiar replacements,

$$(14)\quad \sum_{i=1}^{m} c_i \log f_i(w_1, \cdots, w_s; z^{(n-2)}, \cdots, z) = g(w_1, \cdots, w_r; z^{(n-2)}, \cdots, z),$$

each φ_i resulting from f_i, and ψ from g. After differentiation, (14) gives, for the replacements,

$$(15)\qquad \sum_{i=1}^{m} c_i \frac{f_i'}{f_i} = g',$$

where the significance of f_i' and g' is obvious. As usual, (15) holds for arbitrary w's.

We shall prove first that none of $w_1, \cdots, w_s$ can be associated with a logarithm. Suppose, for instance, that w_1 corresponds to a logarithm, θ. As seen in § 15, if, in (14) and (15), w_1 is replaced by $\theta + \mu$ (μ constant and small), the other w's and z's by their monomials, the members of (15) will still be the derivatives of those of (14). Also, (15) will remain an equation. Consequently, for these replacements, we have

$$(16)\qquad \sum c_i \log f_i = g + \beta(\mu),$$

where $\beta(\mu)$, being the difference of two analytic functions of μ, is analytic for μ small. We differentiate with respect to μ in (16), and put $\mu = 0$, obtaining

$$(17)\qquad \sum c_i \frac{1}{f_i} \frac{\partial f_i}{\partial w_1} = \frac{\partial g}{\partial w_1} + \beta'(0).$$

Again, (17) holds for arbitrary w's, but we consider only w_1 arbitrary, and replace the other w's. Integrating (17), we have, for w_1 arbitrary,

$$(18)\qquad \sum c_i \log f_i = g + \beta'(0) w_1 + \gamma(z),$$

where it will be unnecessary to determine $\gamma(z)$.

By what we know for the case of $r=1$, (18) shows that when $w_2, \cdots, z$ are replaced by their monomials, each $\log f_i$ (and also $g+\beta'(0)w_1$) becomes independent of w_1. But this contradicts the assumption that s is a minimum, so that w_1 cannot stand for a logarithm.

Suppose then that w_1 corresponds to an exponential, θ. We find that (16) holds when w_1 is replaced by $\mu\theta$, with μ close to 1. Differentiating with respect to μ, and putting $\mu=1$, we have

$$\sum c_i \frac{1}{f_i} \frac{\partial f_i}{\partial w_1} w_1 = \frac{\partial g}{\partial w_1} w_1 + \beta'(1).$$

Letting w_1 be arbitrary, and integrating, we find

$$\sum c_i \log f_i - \beta'(1) \log w_1 = g + \gamma(z). \tag{19}$$

If $\beta'(1)$ were not a linear combination of the c's with rational coefficients we would have, on fixing z, a contradiction. Thus, let

$$\beta'(1) = q_1 c_1 + \cdots + q_m c_m$$

with rational q's. Then (19) gives, for w_1 arbitrary,

$$\sum c_i \log \frac{f_i}{w_1^{q_i}} = g + \gamma(z).$$

Consequently, for every i, $f_i/w_1^{q_i}$ is independent of w_1, and if we write (14)

$$\sum c_i \log \frac{f_i}{w_1^{q_i}} = g - \beta'(1) \log w_1, \tag{20}$$

we may replace w_1 in the first member by a constant instead of by θ.

Now some term in the first member of (20) is of order n, because we have subtracted from each $\log \varphi_i$ a function of order $n-2$. Also the order of the second member is less than n. This contradiction of the assumption that s is a minimum proves the lemma.

16. If $\varphi(z)$ is of order n, $\log \varphi(z)$ may be of any of the orders $n-1$, n, $n+1$. We prove the

LEMMA. *If $\varphi(z)$ and $\log \varphi(z)$ are both of order $n>0$, $\varphi(z)$ is of the form $\xi_1(z)e^{\xi_2(z)}$, where $\xi_1(z)$ and $\xi_2(z)$ are each of order $n-1$.*

Let $\psi = \log \varphi$. We choose expressions for φ and ψ such that the total number r of monomials of order n appearing in both of them is a minimum, and this condition being first satisfied, we suppose further that we have expressions such that s, the number of monomials of order n appearing in φ, is a minimum.

We have, for the replacements,

$$\text{(21)}\quad \log f(w_1, \cdots, w_s;\; z^{(n-1)}, \cdots, z) = g(w_1, \cdots, w_r;\; z^{(n-1)}, \cdots, z),$$

φ resulting from f and ψ from g.

Precisely as in § 15, we prove that $w_1, \cdots, w_s$ cannot correspond to logarithms. Suppose that w_1 corresponds to an exponential, θ_1. We find the equation

$$\log f = g + \beta(\mu)$$

to hold when w_1 is replaced by $\mu\theta_1$. Then

$$\frac{1}{f}\frac{\partial f}{\partial w_1} w_1 = \frac{\partial g}{\partial w_1} w_1 + \beta'(1),$$

so that, for w_1 arbitrary,

$$\log f - \beta'(1)\log w_1 = g + \gamma(z).$$

This means that $\beta'(1)$ is a rational number q_1, and that $f/w_1^{q_1}$ is independent of w_1 when $w_2, \cdots, z$ are replaced. Writing (21)

$$\text{(22)}\qquad \log\frac{f}{w_1^{q_1}} = g - q_1 \log w_1,$$

replacing w_1 by a constant in the first member and by θ_1 in the second, we have again, if $s > 1$, a function of order n whose logarithm is also of order n. Continuing thus, we find that $\varphi(z)$ divided by $\theta_1^{q_1}\theta_2^{q_2}\cdots\theta_s^{q_s}$ is a function of order $n-1$ at most, and this proves the lemma.

As an immediate consequence of the above result, we have the

LEMMA. *If $\varphi(z)$ and $e^{\varphi(z)}$ are both of order $n > 0$, $\varphi(z) = \xi_1(z) + \log\xi_2(z)$, where $\xi_1(z)$ and $\xi_2(z)$ are each of order $n-1$.*

17. We record here two results, easily proved, of which we shall later use the second.

If $\varphi(z)$ is of order n, and if $\varphi'(z)$ is of order less than n, $\varphi(z) = \varphi_1(z) + \varphi_2(z)$, where $\varphi_1(z)$ is of order less than n, and where $\varphi_2(z)$ is a sum of logarithms of order n multiplied by constants.

If $\varphi(z)$ is of order n, and if the logarithmic derivative of $\varphi(z)$ is of order less than n, $\varphi(z) = \varphi_1(z)e^{\varphi_2(z)}$ where $\varphi_1(z)$ is of order less than n, and where $\varphi_2(z)$ is of order $n-1$.

III. Composite elementary functions

18. In what follows, we shall discontinue the "replacement" language, and speak of the arguments w, z etc. in our algebraic functions as "being" monomials. What precedes indicates sufficiently how everything we say is to be taken.

Lemma. *Given a function $\varphi(z)$ of order m, if a function $\psi(z)$ of order $n > 1$ exists such that the order of $\psi[\varphi(z)]$ does not exceed $m+n-2$, there exists a monomial of order n, $\theta(z)$, such that $\theta[\varphi(z)]$ is at most of order $m+n-2$.*

According to § 14, if w is one of a minimum number of monomials and sums of order n in the expression for $\psi(z)$, we have, with f algebraic,

$$w = f(z_1^{(n-1)}, \cdots, z; \psi, \psi', \cdots, \psi^{(p)}). \tag{23}$$

From § 11, we see that the order of the derivative of a function does not exceed the order of the function. Thus, since

$$\psi'[\varphi(z)] = \frac{1}{\varphi'(z)} \frac{d}{dz} \psi[\varphi(z)],$$

the order of $\psi'[\varphi]$ does not exceed the greater of $m+n-2$ and m. By induction, the order of every $\psi^{(i)}[\varphi]$ is seen not to exceed the greater of these integers. As n is now at least 2, the order of no $\psi^{(i)}[\varphi]$ exceeds $m+n-2$.

Thus, by (23), $w[\varphi]$ is at most of order $m+n-1$. Its order will be even less if no $z^{(n-1)}[\varphi]$ is of order $m+n-1$.

Suppose first that $w = e^u$, where u depends on $z_1^{(n-1)}, \cdots, z$. If the order of $w[\varphi]$ does not exceed $m+n-2$, w is the monomial sought in the lemma. In what follows, we assume the order of $w[\varphi]$ to be $m+n-1$.

If a $z^{(n-1)}[\varphi]$ is of order $m+n-1$, it is a *monomial.** Hence $u[\varphi]$ has an expression in which all monomials of order $m+n-1$, if indeed there be any, are of the form $z^{(n-1)}[\varphi]$. By (23), the same is true of $w[\varphi]$.

We choose expressions for $u[\varphi]$ and $w[\varphi]$ such that the total number r of monomials of order $m+n-1$, all of the form $z^{(n-1)}[\varphi]$, appearing in

* When $n > 1$, as the hypothesis stipulates.

both of them is a minimum, and this condition being first satisfied, we suppose further that we have expressions such that s, the number of monomials of order $m+n-1$ in $w[\varphi]$, is a minimum.

Let $W = w[\varphi]$, $U = u[\varphi]$. We write x for the monomials of order $m+n-1$, and omit symbols for monomials of lower order. We have

$$\log W(x_1, \cdots, x_s; z) = U(x_1, \cdots, x_r; z).$$

Precisely as in § 16, we prove that $x_1, \cdots, x_s$ are exponentials, and that

$$W = x_1^{q_1} \cdots x_s^{q_s} V,$$

where the q's are rational, and where V is of order $m+n-2$ at most. Now $x_1^{q_1} \cdots x_s^{q_s}$ is of the form $\zeta[\varphi]$, where ζ is an exponential of order $n-1$. Let $\zeta = e^{u_1}$, where u_1 is of order $n-2$. Then $v = u - u_1$ is of order $n-1$ while its exponential is of order n, and we have

$$e^{v[\varphi]} = V,$$

as the lemma requires.

Suppose now that, in (23), w is a logarithmic sum of order n, and that the order of $w[\varphi]$ is $m+n-1$. We shall later cover the case in which the order is less.

Let $w = \sum c_i \log u_i$, with independent c's, where no u_i is of order greater than $n-1$. We put

$$W = w[\varphi], \qquad U_i = u_i[\varphi],$$

observing that W and each U_i have expressions in which every monomial of order $m+n-1$ is of the form $z^{(n-1)}[\varphi]$. Introducing x's, with r a minimum for W and the U_i's and then s a minimum for W alone, we write

$$W(x_1, \cdots, x_s; z) = \sum c_i \log U_i(x_1, \cdots, x_r; z).$$

We prove quickly that $x_1, \cdots, x_s$ are not exponentials. Let x_1 be a logarithm. We find, for x_1 arbitrary,

$$W = \sum c_i \log U_i + \beta'(0) x_1 + \gamma(z),$$

so that, by § 15, $W - \beta'(0)x_1$, and each U_i, are independent of x. Continuing, we find that W less a linear combination of the x's is independent of the x's, and is hence of order $m+n-2$ at most. But the linear combination of the x's is of the form $\xi[\varphi]$, where ξ is a logarithmic sum of order $n-1$. Let

$$\xi(z) = \sum d_i \log v_i(z),$$

where no v_i is of order greater than $n-2$. Let $\zeta = w - \xi$, so that

$$\zeta(z) = \sum c_i \log u_i(z) - \sum d_i \log v_i(z). \tag{24}$$

Then ζ is a logarithmic sum of order n, and $\zeta[\varphi]$ is at most of order $m+n-2$.

Of course, it might have been that $w[\varphi]$ above was itself of order not exceeding $m+n-2$. If such be the case, ζ is to stand for w in what follows.

Let ζ be reduced to the form $\sum e_i \log t_i$ with no t_i of order greater than $n-1$, and with independent e's. We put $T_i = t_i[\varphi]$. Then each T_i has an expression in which all monomials of order $m+n-1$, if there are any, are of the form $z^{(n-1)}[\varphi]$. We assume that the T's are so expressed that the total number r of such monomials appearing in all of them is a minimum, and putting $Z = \zeta[\varphi]$, we write

$$\sum e_i \log T_i(x_1, \cdots, x_r;\, z) = Z. \tag{25}$$

We prove quickly that no x is a logarithm. Let x_1 be an exponential. We find, for x_1 arbitrary,

$$\sum e_i \log T_i = \beta'(1)\log x_1 + \gamma(z).$$

By § 15, we must have $\beta'(1) = \sum q_i e_i$ with rational q's, and each $T_i/x_1^{q_i}$ must be independent of x_1. Now x_1 is of the form $\tau[\varphi]$, where τ is an exponential of order $n-1$. We put

$$T_i' = t_i'[\varphi] = \frac{T_i}{x_1^{q_i}}, \qquad Z' = \zeta'[\varphi] = Z - \beta'(1)\log x_1.$$

Then, since $\log\tau$ is of order $n-2$, $\zeta' = \sum e_i \log t_i'$ is a logarithmic sum of order n. Also $\zeta'[\varphi]$ is at most of order $m+n-2$. Finally each $t_i'[\varphi]$ involves only $x_2, \cdots, x_r$ and not x_1.

It is evident that if this process is gone through r times, we will arrive at a logarithmic sum of order n, $\zeta^{(r)} = \sum e_i \log t_i^{(r)}$, such that $\zeta^{(r)}[\varphi]$ and also each $t^{(r)}[\varphi]$ are at most of order $m+n-2$. It follows by § 15 that no $\log t_i^{(r)}[\varphi]$ has an order greater than $m+n-2$. Since some $\log t_i^{(r)}$ is of order n, the lemma is proved.

The fact that the order of $t_i^{(r)}[\varphi]$ does not exceed $m+n-2$ will be used in the next section.

19. LEMMA. *Given a function $\varphi(z)$ of order m, if a $\psi(z)$ of order $n>1$ exists such that the order of $\psi[\varphi(z)]$ does not exceed $m+n-2$, there exists a function $\psi_1(z)$ of order $n-1$, where either $\log\psi_1(z)$ or $e^{\psi_1(z)}$ is of order n, such that the order of $\psi_1[\varphi(z)]$ does not exceed $m+n-2$.*

According to the preceding section, we may assume that ψ is a monomial, and indeed, the final remark of that section disposes of the case in which ψ is a logarithm.

Suppose that ψ is an exponential e^w. We have to discuss the case in which $w[\varphi]$ is of order $m+n-1$. Of course, $w[\varphi]$ has an expression in which every monomial of order $m+n-1$ is of the form $z^{(n-1)}[\varphi]$. This is because $n>1$. Let $W=w[\varphi]$, and suppose that W is expressed in terms of a minimum number r of monomials of order $m+n-1$, all of the form $z^{(n-1)}[\varphi]$. Writing

$$W(x_1,\cdots,x_r;\ z) = \log\psi[\varphi],$$

we prove that the x's are logarithms, and that $W=\sum c_i x_i+\xi$, where ξ is at most of order $m+n-2$. Here the c's are independent, since r is a minimum. Let $x_i=\log v_i$, where v_i is of order $n-2$. We have

$$c_1\log v_1[\varphi]+\cdots+c_r\log v_r[\varphi]-\log\psi[\varphi] = -\xi.$$

Hence, by § 15, we must have $1=\sum q_i c_i$, with rational q's, so that

$$c_1\log\frac{v_1[\varphi]}{(\psi[\varphi])^{q_1}}+\cdots+c_r\log\frac{v_r[\varphi]}{(\psi[\varphi])^{q_r}} = -\xi.$$

Furthermore, by § 15, no logarithms in the equation just written can be of order greater than $m+n-2$. Thus, considering the first term, we see that

$$q_1\log\psi[\varphi]-\log v_1[\varphi] = q_1 w[\varphi]-\log v_1[\varphi]$$

is of order $m+n-2$ at most.

Hence $q_1 w-\log v_1$ is the function we seek, unless its order is less than $n-1$. But then

$$e^{q_1 w} = v_1 e^{q_1 w-\log v_1}$$

would be of order less than n. As e^w is of order n, and as q_1 is rational, and clearly not zero, this is impossible. The lemma is proved.

20. LEMMA. *Given a function* $\varphi(z)$ *of order* m, *if a* $\psi(z)$ *of order* $n>2$ *exists such that the order of* $\psi[\varphi(z)]$ *does not exceed* $m+n-2$, *there exists a function* $\psi_1(z)$ *of order* $n-1$ *such that the order of* $\psi_1[\varphi(z)]$ *does not exceed* $m+n-3$.

According to §§ 18, 19, we may assume that ψ is a monomial, and that if w is the function whose exponential or logarithm is taken, $w[\varphi]$ is at most of order $m+n-2$.

First let $\psi = e^w$, and suppose that $w[\varphi]$ is of order $m+n-2$. According to § 16, if $\psi[\varphi]$ is of order $m+n-2$, we have

$$w[\varphi] = \xi_1 + \log \xi_2 \tag{26}$$

where ξ_1 and ξ_2 are of order $m+n-3$. If $\psi[\varphi]$ is of order $m+n-3$, $\xi_1 = 0$ in (26).

Let x be one of a minimum number of exponentials and *logarithmic sums* of order $n-1$ in w. Then x is algebraic in w, w', w'', etc., and in monomials of order less than $n-1$. Thus, as $n>2>1$, $x[\varphi]$ is at most of order $m+n-2$; suppose it is actually of order $m+n-2$.

First let x be an exponential, e^v. If $v[\varphi]$ is of order $m+n-3$, $x[\varphi]$ is an exponential of order $m+n-2$. If $v[\varphi]$ is of order $m+n-2$, then, by § 16, $x[\varphi] = \zeta_1 e^{\zeta_2}$ with ζ_1 and ζ_2 of order $m+n-3$.

Again, let x be a logarithmic sum, $\sum c_i \log v_i$, with independent c's. According to § 15, no $\log v_i[\varphi]$ can be of order $m+n-1$. Let $\log v_i[\varphi]$ be of order $m+n-2$. If $v_i[\varphi]$ is of order $m+n-3$, $\log v_i[\varphi]$ is a logarithmic monomial of order $m+n-2$. Otherwise $\log v_i[\varphi] = \zeta_1 + \log \zeta_2$, with ζ_1 and ζ_2 of order $m+n-3$.

If $z^{(n-2)}$ is a monomial of order $n-2$ in the expression for w, and if $z^{(n-2)}[\varphi]$ is of order $m+n-2$, then, since $n>2$, $z^{(n-2)}[\varphi]$ is a *monomial* of order $m+n-2$.

In all, we see that $w[\varphi]$ has an expression in which every monomial of order $m+n-2$ is either the product or the sum of a function of order $m+n-3$ at most, and a function $\tau[\varphi]$, where τ is a monomial of order $n-1$ or $n-2$. It is the product if it is an exponential, the sum if a logarithm. Also, the monomial of order $m+n-2$ and τ are either both exponentials or both logarithms.

Of all expressions for $w[\varphi]$ in which the monomials of order $m+n-2$, $y_1, \cdots, y_r$, are of the rather complicated type just described, consider one for which r is a minimum. We prove quickly, using (26), that each y is a logarithm, and that

$$(27) \qquad w[\varphi] = \sigma + c_1 y_1 + \cdots + c_r y_r,$$

where the order of σ does not exceed $m+n-3$.* An easy discussion would show that because r is a minimum, the c's are independent, but we get along more simply as follows. We note that each y is a logarithmic monomial of order $m+n-2$, and differs by a function of order less than $m+n-2$ from a function $\tau[\varphi]$, where τ is a logarithm of a function of order not exceeding $n-2$. The fact that τ is a monomial, we do not stress. Of all the representations of $w[\varphi]$ of the form (27) with y's of this type, we take one for which r is a minimum. In that case the c's are evidently independent.

Using (26) and § 15, and the representation just obtained for $w[\varphi]$, we prove that a rational q_1 exists such that y_1 and $q_1 \log \xi_2$ differ by a function of order $m+n-3$ at most. Hence, remembering that y is of the form $\log v[\varphi] - \zeta$ where v is of order $n-2$ or $n-3$, and ζ is of order less than $m+n-2$, we find by (26) that $q_1 w[\varphi] - \log v[\varphi]$ is at most of order $m+n-3$. Thus $q_1 w - \log v$ is the function sought in our lemma, unless its order is less than $n-1$. This is seen, as in § 19, to be impossible.

We take the case in which $\psi = \log w$, with $w[\varphi]$ at most of order $m+n-2$. We could use a discussion similar to that for the exponential case, but the following method is shorter.

If $w[\varphi]$ is of order $m+n-2$, it is of the form e^{ξ_2} or $\xi_1 e^{\xi_2}$, (ξ_1 and ξ_2 of order $m+n-3$), according as log $w[\varphi]$ is of order $m+n-3$ or $m+n-2$. In any case, the logarithmic derivative of $w[\varphi]$ is of order $m+n-3$ at most. Hence, as $n>2$, the logarithmic derivative of w is the function sought, unless it is of order less than $n-1$. In the latter case, according to § 17, we would have $w = \zeta_1 e^{\zeta_2}$, where ζ_2 is of order $n-2$, and ζ_1 of order less than $n-1$, so that $\log w$ could not be of order n.

This completes the proof of the lemma.

21. Lemma. *Given a $\varphi(z)$ of order $m>0$, if a $\psi(z)$ of order two exists such that $\psi[\varphi(z)]$ is at most of order m, then a $\psi_1(z)$ of order one exists such that $\psi_1[\varphi(z)]$ is at most of order $m-1$.*

According to §§ 18, 19, we may assume that $\psi(z)$ is a monomial e^w or $\log w$, with $w[\varphi]$ at most of order m.

Let θ be one of a minimum number of exponentials and sums in w. Then θ is algebraic in z, w, etc., so that $\theta[\varphi]$ is at most of order m. If θ is a logarithmic sum with independent c's, none of the terms in it can become of order greater than m when z is replaced by $\varphi(z)$.

* Liouville's principle applies as usual.

If, when we substitute $\varphi(z)$ into one of the monomials in w, we obtain a function of order $m-1$, the monomial is the function sought in the lemma.

Suppose that this is not so. Then if one of the monomials θ is an exponential, φ is an algebraic function of $\xi_1+\log\xi_2$ with ξ_1 and ξ_2 of order $m-1$, whereas if θ is a logarithm, φ is an algebraic function of $\xi_1 e^{\xi_2}$.* But φ cannot have both forms, so that w cannot contain both logarithms and exponentials.

Suppose first that all monomials are exponentials. If an algebraic function of $\xi_1+\log\xi_2$ (as above) is also of the form $\xi_1+\log\xi_2$, the algebraic function is of the form $az+b$, with a rational. Hence w can contain only one exponential, essentially.

Similarly, w cannot contain more than one logarithm.

The results just obtained are a consequence of the mere fact that the order of $w[\varphi]$ does not exceed m. This will be made use of in the following section.

Consider the case of $\psi = e^w$.

Let, then, $w=f(\theta, z)$, and suppose first that θ is an exponential. If $w[\varphi]$ is of order m, it is of the form $\log\xi$ or $\xi_1+\log\xi_2$, with ξ's of order $m-1$, because $e^{w[\varphi]}$ is at most of order m. Then $\theta[\varphi]$, which has to be of the form $\xi_1 e^{\xi_2}$, cannot be so, for it is algebraic in $w[\varphi]$ and φ. Thus $w[\varphi]$ is at most of order $m-1$.

Thus, if $w[\varphi]$ is of order m, θ cannot be an exponential. If θ is a logarithm, we find quickly that $f(\theta, z)=a\theta+b$, with a rational, so that e^w is not of order 2.

Hence, when ψ is an exponential, $w[\varphi]$ is of order $m-1$ at most. The logarithmic case goes through with only slight changes.

22. Lemma. *If $\varphi(z)$ is of order $m>0$, and if a $\psi(z)$ of order one exists such that $\psi[\varphi(z)]$ is of order not exceeding $m-1$, then $\varphi(z)$ is an algebraic function of a monomial of order m.*

As noted in § 21, the fact that $\psi[\varphi]$ is of order not exceeding m implies either that ψ has a monomial θ such that $\theta[\varphi]$ is of order $m-1$, or else that ψ is of the form $f(\theta, z)$. In the former case, we have what the lemma requires. In the second case, $\theta[\varphi]$ is algebraic in $\psi[\varphi]$ and φ, so that, by arguments like those of § 21, $\theta[\varphi]$ cannot be of order m. This settles the lemma.

23. Comparing the lemmas of §§ 20—22, we find the

Theorem. *Given a function $\varphi(z)$ of order m, if a function $\psi(z)$ of order $n>0$ exists such that $\psi[\varphi(z)]$ is at most of order $m+n-2$, then $\varphi(z)$ is an algebraic function of a monomial of order m.*

* The hypothesis prevents φ from being algebraic.

This theorem permits us to determine all elementary functions with elementary inverses. For if $\varphi(z)$ is such a function, of order $m>0$, since $\varphi^{-1}\varphi(z)$ is of order zero, $\varphi(z)$ is an algebraic function of a monomial of order m. But the function of order $m-1$ of which the monomial is an exponential or a logarithm also has an elementary inverse, and is thus algebraic in a monomial of order $m-1$. Continuing thus, we find the result stated in the introduction.

With a set of lemmas only slightly different from those above (the changes are all simplifications), we obtain the

THEOREM. *Given a function* $\varphi(z)$ *of order* $m>0$, *if a function* $\psi(z)$ *of order* $n>0$ *exists such that* $\psi[\varphi(z)]$ *is precisely of order* $m+n-1$, *then* $\varphi(z)$ *is an algebraic function of a function of one of the forms* $\xi_1(z)+\log\xi_2(z)$ *or* $\xi_1(z)e^{\xi_2(z)}$, *where* $\xi_1(z)$ *and* $\xi_2(z)$ *are of order* $m-1$.

COLUMBIA UNIVERSITY,
NEW YORK, N. Y.