On Cofinitary Groups

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Abstract

A cofinitary group is a subgroup of the symmetric group on the natural numbers in which all non-identity members have finitely many fixed points. In this note we describe some questions about these groups that interest us; questions on related cardinal invariants and isomorphism types.

1 Introduction

This paper is a writeup of some talks we have given on various occasions. We hope people reading this will become more interested in these questions and help with their resolution. We begin by defining the main notions of this paper.

Definition 1.

(i). We write $\mathrm{Sym}(\mathbb{N})$ for the symmetric group of the natural numbers; the group consisting of all bijections from the natural numbers to the natural numbers, with the operation being composition.

(ii). An element $g \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* iff it either has finitely many fixed points, or is the identity.

(iii). A group $G \leq \mathrm{Sym}(\mathbb{N})$ is *cofinitary* or a *cofinitary group* iff all of its elements are cofinitary.

(iv). A group $G \leq \mathrm{Sym}(\mathbb{N})$ is a *maximal cofinitary group (mcg)* iff it is a cofinitary group and is not properly contained in another cofinitary group.

One of the sources of interest in these groups is their connection to almost disjoint families. If we have a collection $\mathcal{A}$ of infinite objects, we call elements $x, y \in \mathcal{A}$ *almost disjoint* iff $x \cap y$ is finite. We call the family *almost disjoint* iff all distinct $x, y \in \mathcal{A}$ are almost disjoint. The family is *maximal almost disjoint* iff it is almost disjoint and not properly contained in another almost disjoint family.

If we apply the definitions in the last paragraph with $\mathcal{A} = \mathcal{P}(\mathbb{N})$ then we get the usual notion of (maximal) almost disjoint family (see, e.g. Kunen [11]).

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Next we apply these definitions with $\mathcal{A} = \mathrm{Sym}(\mathbb{N})$. Here we use the convention that $f \in \mathrm{Sym}(\mathbb{N})$ is identified with its graph, $\mathsf{graph}(f)$, which is a subset of the countable set $\mathbb{N} \times \mathbb{N}$. With this we get the notion of a *(maximal) almost disjoint family of permutations*. Requiring the group structure on top of this one obtains the notion of maximal cofinitary group as in Definition 1: We see this by considering the equivalences:

$$(f^{-1} \circ g)(n) = n \quad \Leftrightarrow \quad g(n) = f(n) \quad \Leftrightarrow (n, g(n)) \in g \cap f$$

where $f, g \in \mathrm{Sym}(\mathbb{N})$. From this equivalence you see that $f^{-1} \circ g$ has finitely many fixed points iff $g \cap f$ is finite.

Note that the existence of maximal cofinitary groups follows directly from Zorn's Lemma: the union of an increasing sequence of cofinitary groups is a cofinitary group (being cofinitary is a local property).

Some other basic results on these groups.

Theorem 2 (Adeleke [1], Truss [15]). *A countable cofinitary group is not maximal.*

This theorem can be shown using the ideas from Appendix A.2 by diagonalization.

Theorem 3 (P. Neumann). *There exists a cofinitary group of size* $|\mathbb{R}|$.

He showed this by studying cofinitary groups with all their orbits finite (see, Cameron [5] for the proof). The following result shows that these two theorems do not determine the cardinality of mcg in the context of the negation of the continuum hypothesis.

Theorem 4 (Zhang [18]). *For all κ such that $\aleph_0 < \kappa \leq 2^\omega = \lambda$ there exists a c.c.c. forcing $\mathbb{G}$ such that in $M^{\mathbb{G}}$ we have that $2^\omega = \lambda$ and there exists a maximal cofinitary group of size κ.*

This reasoning so far leads to two main motivations for work on cofinitary groups:

Motivation 1. How similar/different are (maximal) cofinitary groups from (maximal) almost disjoint families?

and

Motivation 2. What algebraic properties do (maximal) cofinitary groups have?

In the remainder of this paper we will work out some of the concrete questions this leads to. In Section 2 we look at the descriptive complexity: we explain and describe what is known about the possible complexities of maximal cofinitary groups. In Section 3 at the related cardinal invariants: we define a couple of cardinal invariants related to these families and describe some questions about them. In Section 4 we look at isomorphism types: here we explain the very algebraic question of what the possible isomorphism types of maximal cofinitary groups are. And finally in Section 5 we gather some remaining questions that did not fit in the earlier sections: questions on orbit structures and generating sets.

2 Concrete Example

We observed above that settling the existence of maximal cofinitary groups is easy, Zorn's Lemma provides a maximal cofinitary group (in fact any cofinitary group can be extended to a maximal cofinitary group with the same reasoning). An object so constructed is one that is usually extremely non-constructive and the result therefore usually hard to describe. The following are some well-known examples of this phenomenon:

- (Suslin [10]) No well ordering of an uncountable set of reals is analytic.
- (Sierpinski) No free ultrafilter is measurable or has the property of Baire.
- (Talagrand [14]) The intersection of countably many nonmeasurable filters is nonmeasurable.
- (Mathias [12]) There is no analytic maximal almost disjoint family.

The last of these items determines the least possible complexity of maximal almost disjoint families when combined with the following theorem.

Theorem 5 (Miller [13])**.** *The axiom of constructibility implies the existence of a coanalytic maximal almost disjoint family.*

These ideas and results together with Motivation 1 immediately give rise to the following question.

Question 6. What is the least possible complexity of a maximal cofinitary group?

The result analogous to Miller's result has been obtained for maximal cofinitary groups. This was done in two steps.

Theorem 7 (Gao and Zhang [6])**.** *The axiom of constructibility implies the existence of a maximal cofinitary group with a coanalytic generating set.*

They used the method developed by Miller and an interesting and ingenious coding: the key to applying this method is in proving a lemma of the following general form.

Lemma Pattern 8 (Form of Key Lemma in Miller's method)**.** *Given*

- *a countable family of the right type A, and*
- *a countable object f.*

We can construct a new element g such that

- *$A \cup g$ is a family of the right type,*
- *$f \leq_T g$ uniformly, and*
- *if we iterate this construction (with some bookkeeping) $\aleph_1$ many times we get a maximal family of the right type (for this item note that we are in the context of the axiom of constructibility).*

Part of this requires being able to construct a maximal family of the right type under the continuum hypothesis. This is done here by the method of *good extensions* as described in [6] and here summarized in Appendix A.

The family is then constructed by iterating the key lemma for ω_1 many steps at every step encoding the construction performed so far into the next element. By decoding this information we can then from an element decide if it belongs to the family.

Doing this for maximal cofinitary groups you do the encoding into the generators you construct. In [6], we performed this construction with a very nice encoding, obtaining the above result.

The difficulty with cofinitary groups is that adding a generator (which you construct) also forces lots of other elements to be added. These you have less control over; however, these would also need to encode the construction up to that point.

It can be shown that the Key Lemma fails for cofinitary groups (see Kastermans [8]), that is, there does not exist a way to find g such that not only it encodes the construction up to this point, but also other new elements do. This means that Miller's method as it stands does not work. However the coding requirement can be relaxed, to have non-uniform encoding. Then by using a simple coding we can perform the construction and obtain the following theorem.

Theorem 9 (Kastermans [8])**.** *The axiom of constructibility implies the existence of a coanalytic maximal cofinitary group.*

From the ideas that made us use Motivation 1 we believe that this is the best possible result in this direction. That is, we believe that the result analogous to Mathias result mentioned above should hold for mcg. There are some weak results in the this direction, but really the following question (also on Veličković problem list) is very open

Question 10. Can there exist Borel maximal cofinitary groups?

This is the right question since Blass, see Gao and Zhang [6], has observed that any analytic maximal cofinitary group is already Borel.

3 Cardinal Invariants

In this section we describe a question on cardinal invariants related to maximal almost disjoint families. For a good general overview of results and ideas around cardinal invariants see Blass [2]. Here we focus on cardinal invariants related to different types of maximal almost disjoint families. These invariants are usually written $\mathfrak{a}$ with some subscript. We give the definitions next.

Definition 11.

(i). $\mathfrak{a}$ is the least cardinality of a maximal almost disjoint family.

(ii). $\mathfrak{a}_p$ is the least cardinality of a maximal almost disjoint family of permutations.

(iii). $\mathfrak{a}_g$ is the least cardinality of a maximal cofinitary group.

We think the most interesting question about these cardinal invariants is the following.

Question 12 (Zhang [19])**.** What is the relationship between $\mathfrak{a}_p$ and $\mathfrak{a}_g$?

Other than the obvious (they can be equal) nothing is known. We mention some results related to this that are known.

Related to this Zhang [17] and [20] has shown that it is consistent that there exists a maximal cofinitary group G contained in a maximal almost disjoint family of permutations P where $|G| < |P|$, but that in the model for Theorem 7 $\mathfrak{a}_p$ and $\mathfrak{a}_g$ do not differ.

The consistency of $\mathfrak{a} < \mathfrak{a}_g$ was established in Zhang [16] and Hrušák, Steprans, and Zhang [7], the consistency of $\mathfrak{a} < \mathfrak{a}_p$ in Brendle, Spinas, and Zhang [4]. There is an obvious question should be answered.

Question 13. Can we prove the consistency of $\mathfrak{a}_p$, $\mathfrak{a}_g < \mathfrak{a}$?

J. Brendle once made a conjecture that it should be provable from ZFC that $\mathfrak{a} \leq \mathfrak{a}_p$, $\mathfrak{a}_g$.

A different question on these invariants is whether they can be singular. Brendle [3] proves using the method of template forcing that $\mathfrak{a}$ can be singular. We believe, but have not yet worked through the details, that the same result can be obtained for $\mathfrak{a}_g$ by adjusting the method to work with groups.

4 Isomorphism Types

This is the most immediate question following from Motivation 2. Say two groups have the same isomorphism type iff they are isomorphic. Write $T(G)$ for the isomorphism type of the group G. Then the question is the following

Question 14. What is the collection $\{T(G) \mid G \text{ is a maximal cofinitary group}\}$?

One restriction we know follows from the fact that a cofinitary group with all orbits finite is not maximal (this follows from the fact that a maximal cofinitary group cannot have infinitely many orbits). From this we see that any maximal cofinitary group has an infinite orbit, which (as Andreas Blass observed) quickly implies that a maximal cofinitary group cannot be Abelian: Supposed G is an Abelian cofinitary group with an infinite orbit O, $k \in O$, and $g_n \in G$, $n \in \mathbb{N}$ such that $O = \{g_n(k) \mid n \in \mathbb{N}\}$. Then $g \in G$ has $g \restriction O$ completely determined by where it maps k, since if $l \in O$, then $l = g_n(k)$ for some n, therefore $g(k) = g(g_n(k)) = g_n(g(k))$. From this you see that if $g, h \in G$ have $g(k) = h(k)$, then $g \restriction O = h \restriction O$ which (since G is cofinitary) means $g = h$. I.e. we have shown $G = \{g_n \mid n \in \mathbb{N}\}$. Then G is not maximal since no maximal cofinitary group is countable.

This, together with the obvious restrictions on cardinality, are the only restrictions known.

Using the method of good extensions the resulting groups have a lot of freeness in them. At every step in the construction, all the newly constructed elements are free over the earlier part of the group. In the notation of the appendix (starting on page 7) we have that $G_{\alpha+1} \cong G_\alpha * H$ for some group H.

On the positive side we know that Martin's axiom implies there exists a locally finite maximal cofinitary group (locally finite means that any finite subset generates a finite subgroup). In this proof we do not extend finite partial functions, but finite group actions. The locally finite isomorphism type is not determined a priori, but is determined, mostly outside of our control, during the construction. See Kastermans [9].

5 Miscellany

Following from Motivation 2 we are also interested in the orbit structure of maximal cofinitary groups. Since a cofinitary group is a subset of $\mathrm{Sym}(\mathbb{N})$ any such group has a natural action on the natural numbers: $(f, n) \mapsto f(n)$. The question then becomes the following.

Question 15. What are the possible orbit structures of maximal cofinitary groups?

Above we already mentioned part of the answer, a maximal cofinitary group cannot have infinitely many orbits. We have shown though that from Martin's Axiom a maximal cofinitary group can be constructed with any finite number of finite orbits and any non-zero finite number of infinite orbits. The orbit structure of the diagonal actions has not yet been determined (here by a diagonal action I mean an action on $\mathbb{N}^k$ for some k defined by $(f, (n_1, \ldots, n_k)) \mapsto (fn_1, \ldots, fn_k)$.

Note that this relates to the question of isomorphism types, and the descriptive complexity of maximal cofinitary groups. Since if the answer is that all diagonal actions have only finitely many orbits, then these groups are *oligomorphic*. This in turn means that if they are closed they are the automorphism group of a $\aleph_0$-categorical structure.

Above we mentioned the result of Gao and Zhang that under the axiom of constructibility there exists a maximal cofinitary group with a coanalytic generating set, and our result that then there exists a coanalytic maximal cofinitary group. We do not know that these results are in fact different results; it is conceivable that every maximal cofinitary group with a coanalytic generating set is already coanalytic. The only approximation to showing that they are different is our positive answer to the following question by Verhik: does there exist a computable set of generators that generate a cofinitary group whose isomorphism type is not computable. See Kastermans [9] for this result. This is still far removed from the question about coanalytic generating sets and groups.

A Construction from CH and MA

In this section we describe some of the combinatorics involved in establishing the results mentioned above; this is just to give some of the flavor. We first establish some notation.

If $G \leq H$ and $g \in H$ we write $\langle G, g\rangle$ for the subgroup of H generated by the set $\{G, g\}$. $F(x)$ denotes the free group on the generator x. If G and H are groups, we write $G * H$ for their free product. We write W_G for $G * F(x)$, which can be identified with the set of reduced words in x and elements of G, that is expressions of the form

$$g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_l} g_{l+1},$$

where $g_i \in G$ for $i \leq l+1$, $g_i \neq \mathrm{id}$ for $1 \leq i \leq l$, and $k_i \in \mathbb{Z} \setminus \{0\}$.

$p : A \rightharpoonup B$ is the notation for a partial function from A to B (as usual, $p : A \to B$ is the notation for a total function).

A.1 Really easy from CH

First observe that with CH we do not need to do a complicated construction. Enumerate $\mathrm{Sym}(\mathbb{N})$ by $\langle f_{\alpha+1} \mid \alpha \in \omega_1\rangle$, and do the following inductive construction of a sequence of cofinitary groups $\langle G_\alpha \mid \alpha \in \omega_1 \cup \{\omega_1\}\rangle$:

- $G_0 = \{\mathrm{id}\}$,
- $G_{\alpha+1} = \begin{cases} \langle G_\alpha, f_{\alpha+1}\rangle & \text{if } \langle G_\alpha, f_{\alpha+1}\rangle \text{ is cofinitary;} \\ G_\alpha & \text{otherwise.} \end{cases}$
- $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$, if λ is a limit ordinal.

Then the group $G = G_{\omega_1}$ is a maximal cofinitary group: (1) it is cofinitary since it is the union of an increasing sequence of cofinitary groups. (2) is it maximal: suppose towards a contradiction that there is a cofinitary group H of which G is a proper subgroup. Choose $g \in H \setminus G$, note that $g = f_{\alpha+1}$ for some $\alpha \in \omega_1$. Note that $\langle G, g\rangle$ is cofinitary. From this we see that $\langle G_\alpha, g\rangle = \langle G_\alpha, f_{\alpha+1}\rangle$ is also cofinitary. But then by the inductive construction $f_{\alpha+1} \in G_{\alpha+1} \leq G$ which is a contradiction with $g \in H \setminus G$.

This construction is of very little use in answering questions like the ones in this paper since the group is really out of our control. CH gives us an enumeration, and this arbitrary enumeration determines which elements are in and out. The same method clearly works using AC, then however it is easier to use Zorn's Lemma.

This is why in the next paragraph we describe a more complicated construction. This construction and the ideas therein can be tweaked to be useful to many of the above (that is, many of the obtained results above use these ideas).

A.2 Good Extensions

The above easy construction from CH clearly does not help us to prove a lemma of the form of Lemma 8. The enumeration that is axiomatically obtained from CH determines which elements are in the group. In terms of definability of the resulting group this is as bad as constructing it using AC.

Gao and Zhang [6] describe a more concrete construction fitting with Lemma 8. Given a countable cofinitary group G and an element $f \in \mathrm{Sym}(\mathbb{N})$, we want to construct an element $g \in \mathrm{Sym}(\mathbb{N})$ such that $\langle G, g\rangle$ is cofinitary, and $\langle G, g, f\rangle$ is either equal to $\langle G, g\rangle$ or is not cofinitary. The first case applies if $f \in G$, so suppose that is not the case. Then it suffices to construct the element g such that $\langle G, g\rangle$ is cofinitary and $f \cap g$ is infinite but not equal to g.

Definition 16.

(i). Let $p, q : \mathbb{N} \rightharpoonup \mathbb{N}$ be finite partial injective functions, and $w \in W_G$. Then q is a *good extension* of p with respect to w iff $p \subseteq q$ and for every $n \in \mathbb{N}$ such that $w(q)(n) = n$ there exist $l \in \mathbb{N}$, and $u, z \in W_G$ such that

- $w = u^{-1}zu$ without cancellation,
- $z(p)(l) = l$, and
- $u(q)(n) = l$.

Note that if $w(p)(n) = n$ we can choose $z = w$ and $u = \mathrm{id}$.

With these definitions the following lemmas can be proved, see Gao and Zhang [6].

Lemma 17. *Let $G \leq \mathrm{Sym}(\mathbb{N})$, $p : \mathbb{N} \rightharpoonup \mathbb{N}$ finite and injective, $f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ with $\langle G, f\rangle$ cofinitary, and $w \in W_G$. Then*

- *(Domain Extension Lemma) For each $n \in \mathbb{N} \setminus \mathrm{dom}(p)$, for all but finitely many $k \in \mathbb{N}$, the extension $p \cup \{(n, k)\}$ is a good extension of p with respect to w.*
- *(Range Extension Lemma) For each $k \in \mathbb{N} \setminus \mathrm{ran}(p)$, for all but finitely many $n \in \mathbb{N}$, the extension $p \cup \{(n, k)\}$ is a good extension of p with respect to w.*

- *(Hitting f Lemma) For all but finitely many $n \in \mathbb{N}$, the extension $p \cup \{(n, f(n))\}$ is a good extension of p with respect to w.*

With these lemmas we can do the construction by iterating the following lemma.

Lemma 18. *Let $G \leq \mathrm{Sym}(\mathbb{N})$ be countable, $f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f\rangle$ cofinitary. Then we can construct a $g \in \mathrm{Sym}(\mathbb{N})$ such that $\langle G, g\rangle$ is cofinitary, $\langle G, g\rangle \cong G * F(x)$ and $f \cap g$ is infinite.*

We enumerate W_G as $E = \langle w_n \mid n \in \mathbb{N}\rangle$. This lemma is then proved by iterating the Domain/Range and Hitting f lemmas with enough bookkeeping to ensure the result is a permutation and Hitting f is used infinitely often, and taking good extensions w.r.t. larger initial segments of the enumeration E. Since Hitting f is used infinitely often the resulting g satisfies $f \cap g$ is infinite. Because of this either $g = f$ or $\langle G, g, f\rangle$ is not cofinitary (since $f^{-1}g$ has infinitely many fixed points, but is not the identity). We see that $\langle G, g\rangle \cong G * F(x)$ since for every $w \in W_G$ from some point we are only taking good extensions w.r.t. it and all of its subwords. Then all fixed points that will appear in $w(g)$ are already present in root.

We can use these same ideas on constructions from MA. Given a group G we define the partial order $\mathbb{P}_G$:

- $\langle p, F\rangle \in \mathbb{P}_G$ if $p : \mathbb{N} \rightharpoonup \mathbb{N}$ finite, and $G \subseteq W_G$ finite.
- $\langle p_1, F_1\rangle \leq \langle p_0, F_0\rangle$ iff $p_0 \subseteq p_1$, $F_0 \subseteq F_1$, and p_1 is a good extension of p_0 w.r.t. all $w \in F_0$.

This partial order is c.c.c. since all elements with the same first element are compatible. The Domain, Range, and Hitting f Lemmas give the denseness of the sets $\{\langle p, F\rangle \mid n \in \mathrm{dom}(p)\}$, $\{\langle p, F\rangle \mid n \in \mathrm{ran}(p)\}$, and $\{\langle p, F \mid |p \cap f| \geq n\}$. Finally the obvious denseness of $\{\langle p, F\rangle \mid w \in F\}$ for $w \in W_G$ replaces the initial segments of the enumeration above.

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