1.2 Relations, Functions, and their Properties

Material for this section is drawn from [1, Chapters 1.3, 6, and 7] and [2, Chapters 7 and 8].

1.32 Definition (Relation)
A *binary* relation R is defined between two sets A, B such that $R = \{(a, b) \mid a \in A, b \in B\}$. Notice the lack of quantifiers.

1.33 Remark (Alternate specification)
We say that a relation R is defined over $A \times B$.

1.34 Remark (Relating elements)
We say that two elements a, b are *related* if $(a, b) \in R$ and may be written as aRb.

1.35 Remark (Omitting the sets)
If we define a relation R but omit which sets it's defined over, we may assume that R is defined over $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.

1.36 Remark (Single Set Relation)
If we define a relation over a single set, we assume that R is defined over that set's Cartesian product with itself. That is a relation R defined over A means that $R \subseteq A \times A$.

Exercise 7 (Relation) Let $A = \{3, 4, 5\}$ and S be a relation defined over $A \times B$ such that $S = \{(a, b) \mid a \in A, b \in B, b = a^2\}$. Provide a possible set B.

1.37 Remark
Arrow Diagrams Arrow Diagrams may be used to depict a relation. See notes for an example.

1.38 Remark
Directed Graphs When a relation is defined over a single set (i.e., $A \times A$), we may use a directed graph to depict the relation. See notes for an example.

1.39 Definition (Reflexivity)
A relation R defined over an arbitrary set A is *reflexive* iff $\forall a \in A, aRa$.

1.40 Definition (Symmetry)
A relation R over A is *symmetric* iff $\forall a, b \in A$, if aRb, then bRa.

1.41 Definition (Transitivity)
A relation R over A is *transitive* iff $\forall a, b, c \in A$, if aRb and bRc, then aRc.

1.42 Definition (Function)
A function f from a set A to a set B, which we denote as $f : A \to B$, is a relation from A, which we call the domain, to B, which we call the co-domain. The following must hold:

1. $\forall a \in A, \exists f(a) \in B$.

2. $\forall a, b \in A$ if $f(a) \in B$ and $f(b) \in B$, then $a = b$.

1.43 Remark (Notation)
We will commonly say that f maps from the domain to the co-domain, and similarly for elements. The latter is denoted using lower-case letters, e.g., $f : a \to b$.

1.44 Definition (Injective)
A function $f : A \to B$ is injective if $f(x) = f(y)$, then $x = y$.

1.45 Definition (Surjective)
A function $f : A \to B$ is surjective if $\forall b \in B, \exists a \in A$, s.t. $f(b) = a$.

1.46 Definition (Bijective)
A function is bijective if it is surjective and injective.