1 Background Materials

This section covers relevant background material to understanding the rest of the course and this script.

1.1 Set Theory

Material for this section is drawn from [1, Chapter 1] and [2, Chapters 1.2 and 6].

Intuitively, sets are collections of objects, which are called *elements* of the set. For example, one could consider a class of students to be a set – where each student is an element of that set.

1.1 Definition (Common sets and their notation)

- $\mathbb{N}$: the natural numbers
- $\mathbb{Z}$: the integers
- $\mathbb{Q}$: the rational numbers
- $\mathbb{R}$: the real numbers
- $\mathbb{C}$: the complex numbers

1.2 Remark (Notation)
Sets are denoted using braces: $\{$ and $\}$. There are two common ways of describing sets, as follows.

1.3 Definition (Set-Roster Notation)
The elements of a set may be listed explicitly: starting and ending with braces, and with the elements comma separated. Ellipses may be used for infinite sets.

1.4 Example (Finite)

$$\begin{aligned} A &= \{a, b, c, d\} \\ B &= \{1, 2, 3, 4\} \\ S &= \{a, 1, \%\} \end{aligned}$$

1.5 Remark (Use of "...")
The elements of a set be defined using clear patterns and ellipses ("..."). This method is generally discouraged in all but the most obvious of use cases.

1.6 Example (Infinite)

$$\begin{aligned} C &= \{\dots, -1, 0, 1, \dots\} \\ D &= \{2, 4, 6, 8, \dots\} \end{aligned}$$

1.7 Example (Ambiguous Pattern)

$$E = \{2, 4, \ldots\}$$

Exercise 1 Identify at least two patterns that can fit the set-roster notation in the above example.

1.8 Definition (The Empty Set)
The set with no elements in it is denoted as $\emptyset$.

1.9 Remark (Alternate Notation)
The empty set may sometimes be denoted as a pair of braces with nothing in between.

$$\emptyset = \{\}$$

1.10 Definition (Set-Builder Notation)
The most powerful – and common – way of desribing a set is through set-builder notation. Essentially, it is a way to describe a set via some number of rules that all elements in the set must adhere.

1.11 Example (Representing Even Integers)
Recall that $\mathbb{Z}$ is the set of all integers.

$$\mathbb{Z}^{\text{even}} = \{2x \mid x \in \mathbb{Z}, x \geq 0\}$$

1.12 Example (The Rational Numbers)
We can desribe the set of all rational numbers as follows.

$$\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}.$$

1.13 Remark (Reading Notation)

- The vertical line or "pipe", $|$, is read as "such that", "when" or "where,"; sometimes a colon is used instead.
- Commas in mathematics are read as "and."

1.14 Definition (Cardinality)
Cardinality of a set X, denoted as $|X|$, is the measure of a set representing the number of unique, distinct elements in that set.

1.15 Example (Cardinality)
Let $X = \{1, 2, a, \{3, 4\}, 9\}$. Then, the cardinality $|X| = 5$.

Exercise 2 Compute $|\emptyset|$.

Exercise 3 Compute $|\{\mathcal{R}\}|$.

1.16 Definition (Universal Quantifier)
The universal quantifier $\forall$ is read as "for all", indicating that the corresponding condition holds for all cases.

1.17 Example (Using $\forall$)
For example, we may use this qualifer as an indexing operator, similar to a `for` loop in programming where we may want to compute a value for each natural number less than ten. That is, $\forall n \in \mathbb{N},$ s.t. $n < 10, f(n)$.

1.18 Definition (Existential Quantifier)
The exisential quanitifer $\exists$ is read as “exists”, indicating that an arbitrary but particular value exists according to the corresponding condition.

1.19 Example (Using $\exists$)
For example, we may say that there exists a global minima in $\mathbb{N}$, as follows.

$$\exists n \text{ s.t. } \forall n' \in \mathbb{N}, n \leq n'$$

Exercise 4 Prove Example 1.19.

1.20 Definition (Improper Subset)
$A \subseteq B \leftrightarrow \{a \mid a \in A, a \in B\}$

1.21 Example (Improper Subset)
Text

1.22 Definition (Proper Subset)
A is a proper subset of B, when $A \subset B$ iff $A \subseteq B$ and $\exists b \in B$ s.t. $b \notin A$.

1.23 Remark (Superset)
Terminologically, the inverse of a subset is the superset.

1.24 Definition (Set Union)
The Set Union C is denoted as

$$C = A \cup B$$

, where A, B, C are sets. It is defined as

$$C = \{c \mid c \in A \text{ or } c \in B\}$$

1.25 Example (Set Union)
Let $A = \{..., -2, -1, 0\}$ and $B = \{0, 1, 2, ...\}$. Then, $A \cup B = \mathbb{Z}$

1.26 Definition (Set Intersection)
The Set Union C is denoted as

$$C = A \cap B$$

, where A, B, C are sets. It is defined as

$$C = \{c \mid c \in A \text{ and } c \in B\}$$

1.27 Example (Set Intersection)

$$\mathbb{Z} \cap \mathbb{N} = \{0\}$$

Exercise 5 Compute

$$|\mathbb{Q} \cap \mathbb{Z} \cap \mathbb{N}|$$

1.28 Definition (Ordered Tuple)
An ordered tuple is a finite set of objects that has a static order. This is denoted with parentheses.

1.29 Example (Ordered Pair)
$(1, 2)$ is an example of an ordered pair (or an ordered 2-tuple).

1.30 Definition (Cartesian Product)
The Cartesian product $A \times B$ is defined as the set

$$A \times B = \{(a, b) \mid \forall a \forall b \ a \in A, b \in B\}$$

. Note that this generalizes to any number of sets in the Cartesian product.

1.31 Example (Cartesian Product)
Let $A = \{a, b\}$ and $B = \{1, 2\}$. Then, $A \times B = \{(a, 1), (b, 2)\}$.

Exercise 6 (Cartesian Product) Let $A = \{r, s\}, B = \{6, 7\}$, and $C = \emptyset$. Compute

$$A \times B \times C$$

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