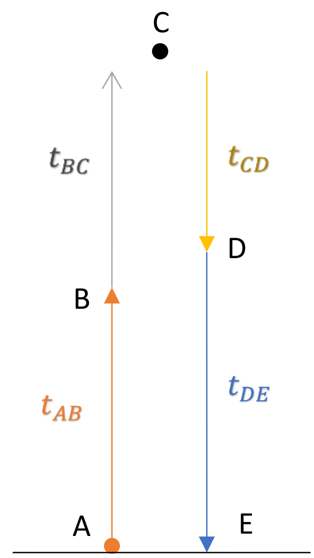
Uber Rocket | Calculus

One calm afternoon Calculus Cam decides to launch Hamster Huey into the air using a model rocket. The rocket is launched straight up off the ground, from rest. The rocket engine is designed to burn for $6.5\ s$ while producing non-constant net acceleration given by the equation $a_{AB}[t_{AB}] = -0.5{t_{AB}}^2 + 15$. After the engine stops the rocket continues upward in free-fall. A parachute opens after the rocket falls $121\ m$ from its maximum height. When the parachute opens, assume the rocket instantly stops, and then increases speed to a terminal velocity given by the equation $v_{DE}[t_{DE}] = -12\left[1 - e^{-\frac{t_{DE}}{9}}\right]$. Assume the air resistance affects the rocket only during the parachute stage.



Key:

Bold – subheading

<u>Underline</u> – a sub answer that will be used later, and where it is used

Green highlight – time sub answer

Cyan highlight – used to generate equation for graphs

$\boxed{\text{Yellow, boxed}}$ – final answer

Givens:

$a_{AB}[t_{AB}] = -0.5{t_{AB}}^2 + 15$

$t_{AB} = 6.5\ s$

$a_{BD} = -9.8\ {}^{m}/_{s^2}$

$y_{CD} = -121\ m$

$v_{DE}[t_{DE}] = -12\left[1 - e^{-\frac{t_{DE}}{9}}\right]$

Strategy:

This problem can be split up into four stages, as labeled on the diagram.

Stage AB:

The acceleration of this stage can be represented by the equation given in the problem statement: $a_{AB}[t_{AB}] = -0.5{t_{AB}}^2 + 15$. By taking the integral once to find the equation for velocity of this stage, and once again for change in position, the equations can be evaluated for $t = 6.5$ as given in the problem to find the velocity and position at point B which will carry into the next stage.

Stage BC:

The acceleration of stage can be represented by the constant $-9.8\ {}^{m}/_{s}$. By setting the velocity and position from point B obtained from the last stage as initial values, the kinematic equations can be used to find the maximum height and the time at the maximum height.

Stage CD:

This stage is a continuation of the last stage, meaning the acceleration is still constant at $-9.8\ {}^{m}/_{s}$. Any final values from last stage can be carried on to the initial values of this stage. It is also important to note from the problem that the distance is $121\ m$. The time of this stage can then be solved for using a kinematic equation.

Stage DE:

In the final stage, the velocity can be represented by the equation $v_{DE}[t_{DE}] = -12\left[1 - e^{-\frac{t_{DE}}{9}}\right]$, as stated in the problem. By taking the integral of this equation and evaluating it at the final time of the stage and 0 and setting the displacement to 121 from the previous stage minus the maximum height, the time can be solved for.

Finally, all of the times from the stages can be added together to find the total time of the object in the air

Stage AB:

First, integrate the given acceleration equation to obtain the equation for velocity. Substitute the given time for this stage to find the velocity at point B.

$$a_{AB}[t_{AB}] = -0.5{t_{AB}}^2 + 15$$
$$\int a_{AB}[t_{AB}]\,\mathrm{d}t_{AB} = \int -0.5{t_{AB}}^2 + 15\,\mathrm{d}t_{AB}$$
$$v_{AB}[t_{AB}] = -\frac{{t_{AB}}^3}{6} + 15t_{AB}$$
$$v_B[6.5] = -\frac{6.5^3}{6} + 15[6.5]$$
$$\underline{v_B[6.5] = 51.729\ {}^{m}/_{s}}$$

Similarly, integrate the velocity equation to obtain the displacement equation and find the displacement at point B.

$$\int v_{AB}[t_{AB}]\,\mathrm{d}t_{AB} = \int -\frac{{t_{AB}}^3}{6} + 15t_{AB}\,\mathrm{d}t_{AB}$$
$$y_{AB}[t_{AB}] = -\frac{{t_{AB}}^4}{24} + \frac{15{t_{AB}}^2}{2}$$
$$y_B[6.5] = -\frac{6.5^4}{24} + \frac{15[6.5]^2}{2}$$
$$\underline{y_{AB}[6.5] = 242.50\ m}$$

Stage BC:

Using the following kinematic formulas, find the time between the two points and the maximum height.

$$v_C = a_{BD}t_{BC} + v_B$$
$$v_C = -9.8t_{BC} + 51.729$$
$$0 = -9.8t_{BC} + 51.729$$
$$\underline{t_{BC} = 5.2785\ s}$$

$$y_C = \frac{1}{2}a_{BD}{t_{BC}}^2 + v_B t_{BC} + y_B$$
$$y_C = \frac{1}{2} \times -9.8{t_{BC}}^2 + 51.729t_{BC} + 242.50$$
$$y_C = -4.9{t_{BC}}^2 + 51.729t_{BC} + 242.50$$
$$y_C = \frac{1}{2} \times -9.8 \times 5.2785^2 + 51.729 \times 5.2785 + 242.50$$
$$\underline{y_C = 379.023\ m}$$

Stage CD:

Use the following kinematic formula to find the velocity at point D and the time between points C and D. Find the displacement.

$${v_D}^2 = {v_C}^2 + 2a_{BD}y_{CD}$$
$${v_D}^2 = 0^2 + 2 \times -9.8 \times -121$$
$${v_D}^2 = 2371.6$$
$$\cancel{v_D = 48.6991} \qquad \underline{v_D = -48.6991\ {}^{m}/_{s}}$$

$$y_D = y_C - 121$$
$$y_D = 379.023 - 121$$
$$\underline{y_D = -258.023\ m}$$

$$v_D = a_{BD}t_{CD} + v_C$$
$$-48.6991 = -9.8t_{CD} + 0$$
$$t_{CD} = 4.9693\ s$$

Stage DE:

Differentiate and integrate the given equation for velocity to find the equations for acceleration and displacement, respectively. Evaluate this equation at t_{DE} and 0, with the displacement set as the height at point D.

$$v_{DE}[t_{DE}] = -12\left[1 - e^{-\frac{t_{DE}}{9}}\right]$$
$$\frac{d}{dt_{DE}}[v_{DE}[t_{DE}]] = \frac{d}{dt_{DE}}\left[-12\left[1 - e^{-\frac{[t_{DE}-16.748]}{9}}\right]\right]$$
$$a_{DE}[t_{DE}] = -\frac{4}{3} \times [0.99956]^{250t_{DE}-4187}$$

$$\int_0^{t_{DE}} v_{DE}[t_{DE}]\,dt_{DE} = \int_0^{t_{DE}} -12\left[1 - e^{-\frac{t_{DE}}{9}}\right]dt_{DE}$$
$$y_{DE} = -12\left[9e^{-\frac{t_{DE}}{9}} + t_{DE}\right]\Big|_0^{t_{DE}}$$
$$-258.023 = -12\left[9e^{-\frac{t_{DE}}{9}} + t_{DE}\right] - \left[-12\left[9e^{-\frac{0}{9}} + 0\right]\right]$$
$$-258.023 = -108e^{\frac{-t_{DE}}{9}} - 12t^{DE} + 108$$
$$0 = -108e^{\frac{-t_{DE}}{9}} - 12t^{DE} + 366.023$$
$$\cancel{t_{DE} = -14.4817}$$
$$t_{DE} = 30.1875\ s$$

Add up all of the times to find the total time.

$$t_{AE} = t_{AB} + t_{BC} + t_{CD} + t_{DE}$$
$$t_{AE} = 6.5 + 5.27849 + 4.9693 + 30.1875$$
$$\boxed{t_{AE} = 46.}$$

t (s)	a (m/s²)	v (m/s)	y (m)
0.00	15.000	0.00	0.000
1.00	14.500	14.83	7.458
2.00	13.000	28.67	29.333
3.00	10.500	40.50	64.125
4.00	7.000	49.33	109.333
5.00	2.500	54.17	161.458
6.00	-3.000	54.00	216.000
6.50	-6.125	51.73	242.497
6.50	-9.800	51.73	242.497
7.00	-9.800	46.83	267.137
8.00	-9.800	37.03	309.066
9.00	-9.800	27.23	341.195
10.00	-9.800	17.43	363.524
11.00	-9.800	7.63	376.054
12.00	-9.800	-2.17	378.783
12.00	-9.800	-2.17	378.783
13.00	-9.800	-11.97	371.712
14.00	-9.800	-21.77	354.841
15.00	-9.800	-31.57	328.170
16.00	-9.800	-41.37	291.699
16.75	-9.800	-48.72	257.915
16.75	-1.333	-48.72	257.915
17.00	-1.297	-49.05	257.874
18.00	-1.160	-50.28	256.920
19.00	-1.038	-51.38	254.805
20.00	-0.929	-52.36	251.650
21.00	-0.832	-53.24	247.565
22.00	-0.744	-54.02	242.647
23.00	-0.666	-54.73	236.985
24.00	-0.596	-55.36	230.656
25.00	-0.533	-55.92	223.731
26.00	-0.477	-56.43	216.273
27.00	-0.427	-56.88	208.336
28.00	-0.382	-57.28	199.973
29.00	-0.342	-57.64	191.227
30.00	-0.306	-57.97	182.138
31.00	-0.274	-58.26	172.744
32.00	-0.245	-58.52	163.075
33.00	-0.219	-58.75	153.162
34.00	-0.196	-58.96	143.029
35.00	-0.176	-59.14	132.699
36.00	-0.157	-59.31	122.194
37.00	-0.141	-59.46	111.532
38.00	-0.126	-59.59	100.729
39.00	-0.113	-59.71	89.800
40.00	-0.101	-59.81	78.759
41.00	-0.090	-59.91	67.616
42.00	-0.081	-60.00	56.384
43.00	-0.072	-60.07	45.071
44.00	-0.065	-60.14	33.685
45.00	-0.058	-60.20	22.235
46.00	-0.052	-60.26	10.727
46.94	-0.047	-60.30	-0.137

Values were obtained with the highlights in cyan:

Acceleration:

AB: $a_{AB}[t_{AB}] = -0.5{t_{AB}}^2 + 15$

BC/CD: $a_{BD} = -9.8\ {}^{m}/{}_{s^2}$

DE: $a_{DE}[t_{DE}] = -\frac{4}{3} \times [0.99956]^{250t_{DE}-4187}$

Velocity:

AB: $v_{AB}[t_{AB}] = -\frac{{t_{AB}}^3}{6} + 15t_{AB}$

BC/CD: $v_C = -9.8[t_{BC} - 6.5] + 51.729$

DE: $v_{DE}[t_{DE}] = -12\left[1 - e^{-\frac{[t_{DE}-16.748]}{9}}\right] - 48.699$

Displacement:

AB: $y_{AB}[t_{AB}] = -\frac{{t_{AB}}^4}{24} + \frac{15{t_{AB}}^2}{2}$

BC/CD: $y_C = -4.9(t_{BC} - 6.5)^2 + 51.729(t_{BC} - 6.5) + 242.50$

DE: $y_{DE} = -12\left[9\left[e^{-\frac{[t_{DE}-16.478]}{9}} - 1\right] + [t_{DE} - 16.478]\right] + 258.02$