Katy Stuparu
Mr. Regele
Mathematical Modeling
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First Triangle Odd Row Proof

Each cell (except the top 1) is determined by summing the cell above it and the two cells immediately to the left and right of the cell above it. Prove that every row after the second row contains an entry which is even.

Let each number in the triangle be represented by an E (for being an even number) or an O (for being an odd number). Let the blank spaces equal 0, which is an even number. Recall that each number except the first in the triangle is the sum of the three numbers above it. When even and odd numbers are added together, the sum is even or odd depending on what the numbers added are.

There are four possibilities for three numbers (even or odd) to be added together. Let each even number be represented by 2x, 2y, or 2z (since all even numbers are divisible by 2); and each odd number be represented by 2x + 1, 2y + 1, or 2z + 1 (they are not divisible by 2).

1. E + E + E
   = 2x + 2y + 2z
   = 2(x + y + z) <- This is divisible by 2 and is therefore even.
   E + E + E = E
2. O + O + O
   = 2x + 1 + 2y + 1 + 2z + 1
   = 2x + 2y + 2z + 3 <- This cannot be factored to be divisible by 2, and is therefore odd.
   O + O + O = O

3. E + O + O

   $= 2x + 2y + 1 + 2z + 1$

   $= 2x + 2y + 2z + 2$

   $= 2(x + y + z + 1)$ <- This is divisible by 2 and is therefore even.

   E + O + O = E

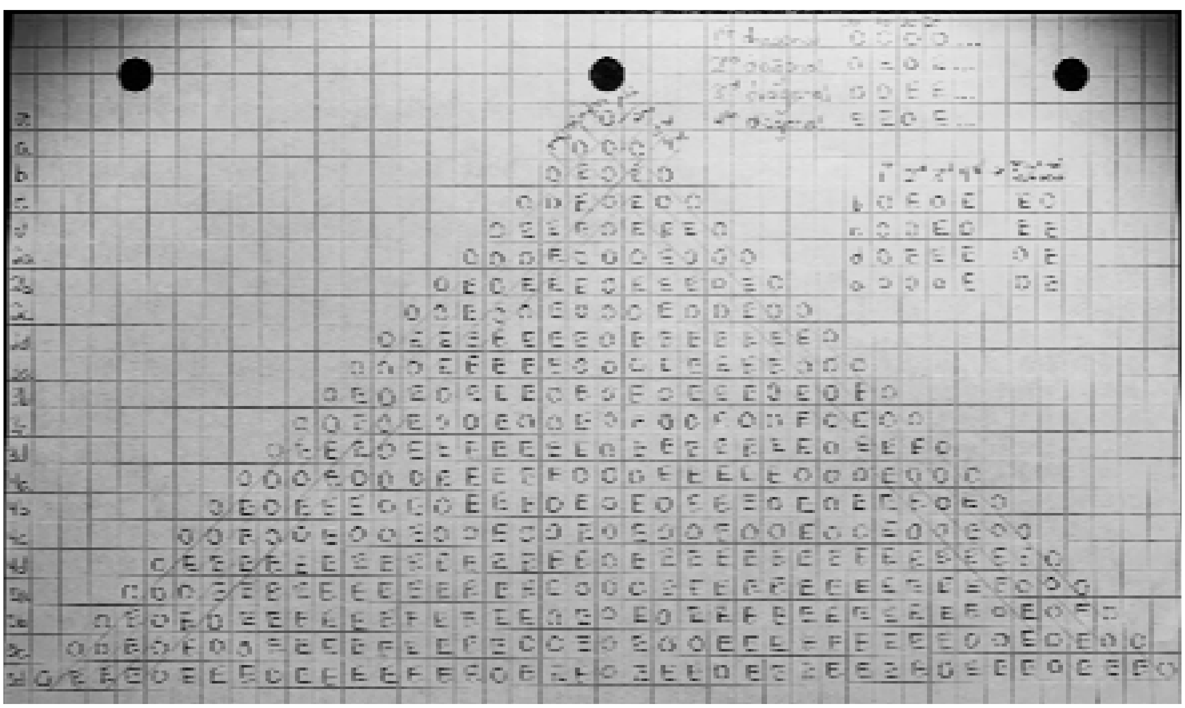
4. E + E + O

   $= 2x + 2y + 2z + 1$ <- This cannot be factored to be divisible by 2, and is therefore odd.

   E + E + O = O

Note that although these sequences of even and odd numbers are in a specific order, they may be arranged to be in any possible order due to the Commutative Property of Addition.

Using these four cases of additions of three numbers, all numbers in the infinite triangle can be represented by E's and O's. In the image below, the first 21 rows of this triangle are drawn to see any patterns that can be proved to continue to infinity.



Let the first row be called "0", the second "a", the third "b", the fourth "c", the fifth "d", the sixth "2a", and so on. Every row up to 5d is pictured here. There are also some faint lines on the triangle indicating the diagonals. Let the outermost diagonal (on the left side) be called 1st, the second from the outside be called 2nd, the third from the outside be called 3rd, and the fourth from the outside be called 4th.

When this many rows of the infinite triangle are written out as even or odd, there are many patterns that can be seen. The patterns relevant for this proof shall be pointed out.

In the 1st diagonal, the pattern is OOOOOOOO… for all cases seen here.
In the 2nd diagonal, the pattern is OEOEOEOE… for all cases seen here.
In the 3rd diagonal, the pattern is OOEEOOEE… for all cases seen here.
In the 4th diagonal, the pattern is OEEEOEEE… for all cases seen here.

Assume that these patterns continue to infinity (they will be proved later). Take a look at the first four letters in each row (the first letter is in the 1st diagonal, the second letter is in the 2nd diagonal, and so on). The sequence along diagonals is the pattern described above, and they make up the sequence of all the 1st letters, 2nd, 3rd, and 4th. Starting from row b (the first row with four letters), take a look at the pattern repeating every four rows.

Row	1st letter	2nd letter	3rd letter	4th letter	->	3rd letter of next row	4th letter of next row
b	O	E	O	E		E	O
c	O	O	E	O		E	E
d	O	E	E	E		O	E
2a	O	O	O	E		O	E

This pattern continues for all of the rows pictured in this diagram. Notice that among the first four letters of each row, there is at least one even number. If this pattern can be proven to continue to infinity, then it can be proven that there is an even number in every row (more specifically, there is an even number in the first four numbers of every row).

For the remaining part of the proof, the first pattern (which makes up the second pattern (will be proven to be true). If each pattern in each diagonal is proven true, then there is an even number in every row.

1. In the 1st diagonal, the pattern is OOOOOOOO… (every number is odd)

Let the number in the 1st diagonal row 0 = 1 (as per rules). Let the number in the 1st diagonal row a = v_a. Let the number in the 1st diagonal row b = v_b, and so on.

- $v_a = 0 + 0 + 1 = 1$ (it is equal to the three numbers above it; the zeros come from blanks… recall that much earlier each blank was said to be represented by 0 = E)
- $v_b = 0 + 0 + v_a = 0 + 0 + 1 = 1$, and so on

Thus, every number in the 1st diagonal is 1 and therefore odd.

2. In the 2nd diagonal, the pattern is OEOEOEOE…

Let the number in the 2nd diagonal row a = 1 (since the three numbers above it are 0 + 1 + 0 = 1). Let the number in the 2nd diagonal row b = x_b, the one in row c = x_c, and so on.

- $x_b = 0 + v_a + 1 = E + O + O = E$ (it is equal to the sum of the three numbers above it)
- $x_c = 0 + v_b + x_b = E + O + E = O$
- $x_d = 0 + v_c + x_c = E + O + O = E$, and so on

Thus, the pattern in the 2nd diagonal is OEOEOEOE…

3. In the 3rd diagonal, the pattern is OOEEOOEE…

Let the number in the 3rd diagonal row a = 1 (since the three numbers above it are 1 + 0 + 0 = 1). Let the number in the 3rd diagonal row b = y_b, the one in row c = y_c, and so on.

- $y_b = n_a + x_a + 1 = O + O + O = O$ (it is equal to the sum of the three numbers above it)
- $y_c = n_b + x_b + y_b = O + E + O = E$
- $y_d = n_c + x_c + y_c = O + O + E = E$

- $y_{2a} = n_d + x_d + y_d = O + E + E = O$
- $y_{2b} = n_{2a} + x_{2a} + y_{2a} = O + O + O = O$, and so on

Thus, the pattern in the third diagonal is OOEEOOEE…

4. In the 4th diagonal, the pattern is OEEEOEEE…

Let the number in the 4th diagonal row b = 2 (since the three numbers above it are 1 + 1 + 0 = 2). Let the number in the 4th diagonal row c = z_c, the one in row d = z_d, and so on.

- $z_c = x_b + y_b + 2 = E + O + E = O$
- $z_d = x_c + y_c + z_c = O + E + O = E$
- $z_{2a} = x_d + y_d + z_d = E + E + E = E$
- $z_{2b} = x_{2a} + y_{2a} + z_{2a} = O + O + E = E$
- $z_{2c} = x_{2b} + y_{2b} + z_{2b} = E + O + E = O$, and so on

Thus, the pattern in the fourth diagonal is OEEEOEEE…

The patterns in the four diagonals have been proven to continue to infinity. Since these patterns directly make up the four possibilities of the first letter sequences in each row, those four sequences also repeat to infinity. Due to the fact that there is an even number in each sequence, it has been proven that there is an even number in every row of this triangle (except rows 0 and a).