Painted Tetrahedron

Problem Statement

A tetrahedron with all side lengths being n is comprised of several smaller tetrahedra (and possibly other shapes). If the large tetrahedron is dipped in paint, then how many of the shapes that it is made up of have paint on 0, 1, 2, 3, or 4 faces?

Process

Before starting to solve the problem, I tried to visualize the tetrahedron. Originally, I thought that by making n - 1 cuts parallel to each face would create several smaller tetrahedra. I made several paper tetrahedra but became quite confused as to why they did not fit together perfectly like the model in class. I thought that it was because they were not exactly perfect. After cutting some play dough, I realized that the interior of a tetrahedron with n = 2 was an octahedron.

After examining the tetrahedron model in class, I began to draw out several layers of the tetrahedron (bottom view). I realized that these drawings displayed all of the octahedra but not all of the interior tetrahedra (they can only be viewed on one side), so I drew the top views of each row as well. From there, it was easy to count the number of tetrahedra and octahedra in each row with 0, 1, 2, 3, or 4 sides exposed to the paint. It was also easy to understand how many of each shape were in a layer of the large tetrahedron.

I considered a solution with Sierpinski's Triangle but I thought that my layers solution as previously described was much simpler. However, it was interesting to visualize the tetrahedron in this way. I also considered removing the outside layer of the tetrahedron (a strategy I used with the cube problem) to count the shapes with zero faces exposed to paint; however, this was very tricky for me to imagine and I thought that the interior shape was probably not a tetrahedron (the interior shape of the cube was a cube), so it would not be as simple to calculate. Finally, I considered using volume to find the number of shapes with 0 sides painted. Once I came across the octahedra and understood that there were two different shapes, I realized that this strategy would be overcomplicating the problem.

Solution

When the large tetrahedron of side length n is split up, there are several smaller tetrahedra of side length 1 and several octahedra of side length 1. This solution counts the number of faces exposed of tetrahedra and octahedra separately.

The upcoming solutions are only for a large tetrahedron with $n \geq 2$. For $n = 1$, there is one small tetrahedron with all 4 sides painted and 0 with all other numbers of sides painted. There are no octahedra. For $n = 2$, there is one octahedron with 4 sides painted and 4 sides hidden. For all other cases, the number of sides painted of all small shapes are between 0 and 3.

The explanations below utilize "layers," or horizontal slices of the large tetrahedron. For each layer, it is necessary to look at the top and bottom view. The octahedra in each layer are represented by one triangle on the top and bottom view; however, tetrahedra are only visible from either the top or bottom view as a triangle. Each layer is solid (there are no gaps or room for additional shapes) and they neatly stack on top of each other to form the large tetrahedron. The tetrahedra explanations refer to the bottom layer view (unless specified) and the octahedra explanations utilize the top layer view.

Tetrahedra with 3 Sides Painted

The tetrahedra with 3 sides painted are the shapes from the corners of the large tetrahedron. Since the large tetrahedron has 4 vertices, the number of small tetrahedra with 3 sides painted is always 4 for $n \geq 2$.

Tetrahedra with 2 Sides Painted

The tetrahedra with 2 sides painted are all of the tetrahedra on the edges of the large tetrahedron (not including the tetrahedra with 3 sides painted at the vertices). Using layers, the edges are considered to be the three triangles at the vertices of each bottom view layer triangle (except for the first layer which is just one tetrahedron exposed on 3 sides and the final layer where the edges of the layer triangle are counted rather than the corners because it is the bottom layer). Thus, the number of tetrahedra is $3(n - 2) + 3(n - 2) = 6(n - 2)$, where the 3 in the first term represents the number of vertices of a triangle, the 2 in the first term represents the number of layers that it applies to, the 3 in the second term represents the number of edges of the layer triangle, and the 2 in the second term represents the number of triangles on the edge of the layer triangle (not including vertices triangles).

Tetrahedra with 1 Side Painted

The tetrahedra with 1 side painted are all of the tetrahedra on the faces of the large tetrahedron (not including the tetrahedra on edges or vertices). With layers, the faces are represented by the edges of each bottom view of the layer triangle (not including vertices of the layer triangles). For the final layer, the faces are represented by all the interior triangles of the layer triangle (not touching the edge of the layer triangle). The tetrahedra with 1 side painted sum up to be $3 \cdot \frac{(n-2)(n-3)}{2} + \frac{(n-2)(n-3)}{2} = 2(n - 2)(n - 3)$. The 3 in the first term represents the number of edges of the layer triangle and $\frac{(n-2)(n-3)}{2}$ represents the sum of whole numbers starting at n - 2 (the number of triangles on the edge in the nth layer triangle).

$\frac{(n-2)(n-3)}{2}$ in the second term represents the sum of the interior triangles of the final layer.

Tetrahedra with 0 Sides Painted

The tetrahedra with no sides painted are all the interior tetrahedra not seen on the outside. They are the interior triangles for each (front and bottom view) layer triangle (except there are none for the first and last layer). The number of tetrahedra with 0 sides painted is the sum of natural numbers $\frac{(n-2)(n-3)}{2}$ (visible from the bottom view of each layer) added to the sum of natural numbers $\frac{(n-1)(n-2)}{2}$ (visible from the top view of each layer). The final formula is $\frac{(n-2)(n-3)(n-4)}{6} + \frac{n(n-1)(n-2)}{6} = \frac{(n-3)(n^2-4n+6)}{3}$.

Tetrahedra Final Results

All formulas found above were confirmed with a table of tetrahedra data from n = 1 to n = 7:

Out[19]=

n	1	2	3	4	5	6	7	formulas
0	0	0	1	4	11	24	45	$\frac{1}{3}(n-3)(n^2-4n+6)$
1	0	0	0	4	12	24	40	$2(n-2)(n-3)$
2	0	0	6	12	18	24	30	$6(n-2)$
3	0	4	4	4	4	4	4	4
4	1	0	0	0	0	0	0	0

Octahedra with 3 Sides Painted

The octahedra with 3 sides painted are the shapes nearest to the corners of the large tetrahedron. Since the large tetrahedron has 4 vertices, the number of octahedra with 3 sides painted is always 4 for $n \geq 2$.

Octahedra with 2 Sides Painted

The octahedra with 2 sides painted are all of the octahedra closest to the edges of the large tetrahedron (not including the octahedra with 3 sides painted near the vertices). Using layers, the edges are considered to be the three triangles at the vertices of each top view layer triangle (except for the first and final layer; the number of triangles on the edges excluding vertices should be counted instead). Thus, the number of octahedra is 3(n - 3) + 3(n - 3) = 6(n - 3), where the 3 in the first term represents the number of vertices of a triangle, the 2 in the first term represents the number of layers that it applies to, the 3 in the second term represents the number of edges of the layer triangle, and the 2 in the second term represents the number of triangles on the edge of the layer triangle (not including vertices triangles).

Octahedra with 1 Side Painted

The octahedra with 1 side painted are all of the octahedra on the faces of the large tetrahedron (not including the octahedra closest to edges or vertices). With layers, the faces are represented by the edges of each top view of the layer triangle (not including vertices of the layer triangles). For the final layer, the faces are represented by all the interior triangles of the layer triangle (not touching the edge of the layer triangle). The tetrahedra with 1 side painted sum up to be $3\cdot\frac{(n-3)(n-4)}{2}+\frac{(n-3)(n-4)}{2}$ = 2(n - 3)(n - 4). The 3 in the first term represents the number of edges of the layer triangle and $\frac{(n-3)(n-4)}{2}$ represents the sum of whole numbers starting at n - 3 (the number of triangles on the edge in the nth layer triangle). $\frac{(n-3)(n-4)}{2}$ in the second term represents the sum of the interior triangles of the final layer.

Octahedra with 0 Sides Painted

The octahedra with no sides painted are all the interior octahedra not seen on the outside. They are the interior triangles for each top view layer triangle (except there are none for the first and last layer). The number of tetrahedra with 0 sides painted is the sum of natural numbers $\frac{(n-3)(n-4)}{2}$.The final formula is $\frac{(n-3)(n-4)(n-5)}{6}$.

Octahedra Final Results

All formulas found above were confirmed with a table of octahedra data from n = 1 to n = 7:

Out[20]=

n	1	2	3	4	5	6	7	formulas
0	0	0	0	0	0	1	4	$\frac{1}{6}(n-3)(n-4)(n-5)$
1	0	0	0	0	4	12	24	2 (n - 3) (n - 4)
2	0	0	0	6	12	18	24	6 (n - 3)
3	0	0	4	4	4	4	4	4
4	0	1	0	0	0	0	0	0

Extensions

1. We have dipped a cube and a tetrahedron composed of smaller shapes in paint. What do the smaller shapes of other 3D regular shapes look like? Is there a pattern or a formula for the number of sides of each type of shape? How many of each small shape have x amount of sides covered in paint?

2. For the cube and tetrahedron, the degree of formulas has been 3 for 0 sides, 2 for 1 side, 1 for 2 sides, and 0 for 3 sides. Why does this occur? Is this true for all other 3D regular shapes?

Evaluation

I enjoyed this final tetrahedron problem very much. The visualization aspect was the most challenging for me, but the model in class and discussions with my peers helped me to understand the behavior of the smaller tetrahedra and octahedra. Besides improving visualization skills, my knowledge of Mahler polynomials and definite sums helped me develop and understand my formulas (instead of simply calculating the formulas from a difference table). Once I understood how the smaller shapes came together, the problem was very simple with my strategy. Perhaps a different way of splitting up the tetrahedron (something similar to the Sierpinski Triangle?) would have been easier on the visualization but more difficult with calculations. If not the unique geometry of this problem, my favorite part of this problem is that it came full circle to the tetrahedron problem at the very beginning of the year, or should I say full triangle.