Solution : 6

Given :- Image :: $I(x) = cx + d$, c, d are scalar coefficient

a) Filtering Image I with zero-mean Gaussian

Filtered output $[J(x)] = I(x) * F_G(x)$

we know that $F_G(x) = \frac{1}{\sqrt{2\pi}\sigma} \times e^{\left(\frac{-x^2}{2\sigma^2}\right)}$ [$F_G(x)$ = Gaussian Filter]

$\Rightarrow J(x) = (cx + d)$ convolusion with $F_G(x)$

$$\Rightarrow J(x) = \int I(x) \times F_G(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\left(\frac{t^2}{2\sigma^2}\right)} \times (cx - ct + d)\, dt$$ (Using t as variable for integration)

$$= \frac{1}{\sqrt{2\pi}\sigma} \left[(cx+d) \int_{-\infty}^{\infty} e^{-\left(\frac{t^2}{2\sigma^2}\right)} dt - c \int_{-\infty}^{\infty} e^{\frac{-t^2}{2\sigma^2}} \times t\, dt \right.$$

$$= \frac{1}{\sqrt{2\pi}\sigma} (cx+d) \times \int_{-\infty}^{\infty} e^{\frac{-t^2}{2\sigma^2}} dt$$ [$\because$ Odd function, integration = 0]

$$= \frac{1}{\sqrt{2\pi}\sigma} ((cx+d)) \times \left(\sqrt{2\pi}\,\sigma\right)$$ [$\because$ Integral of Gaussian with $\mu = 0$ & SD $= \sigma$ $= \sqrt{2\pi}\sigma$]

$$= (cx+d)$$

$\therefore J(x) = cx + d = I(x)$

$\therefore$ For Gaussian Filtering, Output image is same as input image

b) Filtering image I with Bilateral Filter :-

$$J(x) = \int_{-\infty}^{\infty} I(t) \times \left(\frac{1}{\sqrt{2\pi}\sigma_s} e^{-\left(\frac{x^2}{2\sigma_s^2}\right)} \times \frac{1}{\sqrt{2\pi}\sigma_r} \times e^{-\left(\frac{I^2(x)}{2\sigma_r^2}\right)}\right) dt$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_s} \times \frac{1}{\sqrt{2\pi}\,\sigma_r} \times \int_{-\infty}^{\infty} (ct+d) \times e^{-\frac{(x-t)^2}{2\sigma_s^2}} \times e^{-\frac{c^2(x-t)^2}{2\sigma_r^2}} dt$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_s} \times \frac{1}{\sqrt{2\pi}\,\sigma_r} \times \int_{-\infty}^{\infty} (ct+d) \times e^{-\left[\frac{(x-t)^2}{2\sigma_s^2} + \frac{c^2(x-t)^2}{2\sigma_r^2}\right]} \times dt$$

$$= \frac{1}{2\pi\sigma_s\sigma_r} \int_{-\infty}^{\infty} (ct+d) \times e^{-\frac{(x-t)^2}{2}\left(\frac{1}{\sigma_s^2} + \frac{c^2}{\sigma_r^2}\right)} \times dt$$

The equation is equivalent to convolution of image with Gaussian filter with $\sigma_{std} = \frac{2\sigma_s\sigma_r}{\sqrt{2\sigma_r^2 + 2c^2\sigma_s^2}}$

$$\Rightarrow \quad J(x) = \frac{1}{2\pi\sigma_s\sigma_r} \times \left(\sqrt{2\pi} \times \sigma_{std}\right) \times I(x)$$

[$\because$ Integral of Guassian with $= \sqrt{2\pi}\,\sigma_{std}$ from the ① case]

$$= \frac{1}{2\pi\sigma_s\sigma_r} \times \sqrt{2\pi} \times \left(\frac{2\sigma_s\sigma_r}{\sqrt{2\sigma_r^2 + 2c^2\sigma_s^2}}\right) \times I(x)$$

$$= \frac{1}{\sqrt{\pi(\sigma_r^2 + c^2\sigma_s^2)}} \times I(x)$$

$\therefore$ Here we can see that output image is same as Input image with a scaling factor.