CS 663, Fall 2023

Assignment 2

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Question 1: Consider a $1D$ convolution mask given as $(w_0, w_1, ..., w_6)$. Express the convolution of the mask with a $1D$ image f as the multiplication of a suitable matrix with the image vector f. What are the properties of this matrix? What could be a potential application of such a matrix-based construction?

Solution:

To express the convolution of a 1D convolution mask with a 1D image as the multiplication of a suitable matrix with the image vector. Let's assume you have a 1D convolution mask given as $(w_0, w_1, ..., w_6)$ and a 1D image f of length N, i.e. $(f_0, f_1, f_2, \ldots, f_{N-1})$. Let's assume the resultant convulation represented by g and given as $(g_0, g_1, g_2, \ldots, g_{N-1})$. Now for finding values of g_0 we will use matrix multiplication with f which is given below.

$$g_0 = \mathbf{f} \begin{bmatrix} w_3 \\ w_2 \\ w_1 \\ w_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$g_0 = [f_0, f_1, f_2, \ldots, f_{N-1}] \begin{bmatrix} w_3 \\ w_2 \\ w_1 \\ w_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0w_6 + 0w_5 + 0w_4 + f_0w_3 + f_1w_2 + f_2w_1 + f_3w_0$$

Similarly for g_1 we will adding before and after zero's to $\mathbf{f}$. Hence we have:

$$g_1 = \mathbf{f} \begin{bmatrix} w_4 \\ w_3 \\ w_2 \\ w_1 \\ w_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$g_1 = [f_0, f_1, f_2, \ldots, f_{N-1}] \begin{bmatrix} w_4 \\ w_3 \\ w_2 \\ w_1 \\ w_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0w_6 + 0w_5 + f_0 w_4 + f_1 w_3 + f_2 w_2 + f_3 w_1 + f_4 w_0$$

These expressions are obatined by flipping of mask 180° and multiply elements of mask with image, followed by summation of these as obtained above. Hence for all g_i's we can obtain generalize expressions and give it in the form of a matrix by adding require number of zeroes to $\mathbf{f}$.

multiplication

$$
\mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g_{N-2} \\ g_{N-1} \end{bmatrix} = \mathbf{f} \begin{bmatrix} w_3 & w_4 & w_5 & w_6 & 0 & 0 & 0 & \dots & 0 & 0 \\ w_2 & w_3 & w_4 & w_5 & w_6 & 0 & 0 & \dots & 0 & 0 \\ w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & 0 & \dots & 0 & 0 \\ w_0 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & \dots & 0 & 0 \\ 0 & w_0 & w_1 & w_2 & w_3 & w_4 & w_5 & \dots & 0 & 0 \\ 0 & 0 & w_0 & w_1 & w_2 & w_3 & w_4 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & w_5 & w_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & w_4 & w_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & w_3 & w_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & w_2 & w_3 \end{bmatrix}
$$

Hence the above matrix gives as the desired result which is obatained by Band matrix,i.e. a **Toeplitz matrix**. A Toeplitz matrix is a matrix in which each descending diagonal from left to right is constant(In our case iot is zero). In the context of convolution, this type of matrix is used to perform convolution as a matrix-vector multiplication.

Properties of the Toeplitz matrix:

Toeplitz Structure: The matrix A may have a Toeplitz structure, meaning that each diagonal of the matrix contains the same value or set of values. This property can be useful for efficient matrix-vector multiplications.

Potential Applications of the Toeplitz matrix:

- **Image Processing:** Matrix-based convolution is commonly used in image processing for tasks like filtering and feature extraction. Convolutional operations are fundamental in techniques like edge detection, blurring, and sharpening.

- **Machine Learning:** In deep learning, convolutional neural networks (CNNs) leverage convolution operations with learnable kernels. The convolution operation in CNNs can be represented as a matrix-vector multiplication, making it computationally efficient for training and inference on images and other structured data.

- **Audio Processing:** Convolution is applied in audio processing for tasks like audio effects, speech recognition, and music analysis.
- **Time Series Analysis:** It can be used for various time series analysis tasks, including smoothing, trend analysis, and anomaly detection.
- **Signal Processing:** It is also used in signal processing for tasks such as noise reduction, smoothing, and pattern recognition.