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Q3)
1)

$P = A^T A, \quad Q = AA^T.$ $\quad P: \mathbb{C}^n \to \mathbb{C}^n$
$\quad Q: \mathbb{C}^m \to \mathbb{C}^m.$

$y \in \mathbb{R}^n, z \in \mathbb{R}^m \Rightarrow$ Should be..

$y^T P y = y^T A^T A y$ —— ①

$z^T Q z = z^T A A^T z$ —— ②

from ① $= (Ay)^T Ay = \|Ay\|^2$
hence $y^T P y \geq 0$ {because of square}.

Similary $= (A^T z)^T A^T z = \|A^T z\|^2$

from ② hence $A^T z \geq 0 \cong \underbrace{z^T Q z \geq 0}$
{square property}

$\Rightarrow$ Since $y^T P y \geq 0$
P is ~~semi~~ positive Semi-definite.
and also $z^T Q z \geq 0$.
Q is also positive Semi-definite.
& hence eigen vector of positive semi-definite matrix is always non-negative
or since $\|Ay\|^2 \geq 0$ {eigen vector non-negative}.

Or :-

we have scaler eigen value λ for P and μ for Q.

$$P y = \lambda y \quad \& \quad Q z = \mu z$$

$$y^t P y = \lambda y^t y \quad \cancel{z^t Q z = \mu z^t z}$$

$$z^t Q z = \mu z^t z$$

hence from the ~~result~~ above result.

for any vector x, $z^t z \geq 0$

hence $\boxed{\lambda \geq 0, \quad \mu \geq 0}$

↳ hence eigen vector P & Q are ~~Essent~~ Essentially non-negative.

2) If v is an eigen vector of P with eigen value λ, $PV = \lambda V$.

$A^T A V = \lambda V$, [Multiply by A both side]

we get. :-

$$A(A^T A V) = A(\lambda V)$$

$$= (A A^T) A V = \lambda (A V)$$

$$= Q(A V) = \lambda (A V).$$

$\therefore$ AV is an eigen vector of Q with eigenvalue λ.

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Similarly :-

V is an eigen vector of Q with eigen value μ, $QV = \mu V$.

$$AA^TV = \mu V$$

$$\Rightarrow A^T(AA^TV) = A^T(\mu V)$$

$$\Rightarrow (A^TA)A^TV = \mu(A^TV)$$

$$\Rightarrow P(A^TV) = \mu(A^TV).$$

hence A^TV eigen vector of P with eigen value (μ) μ.

~~Hence~~ Proved

no. of element U = n (since $P = n \times n$)

no. of element V = m ($Q = m \times m$)

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(c)
Ans

$\Rightarrow$ v_i is a eigen vector of Q,

$\Rightarrow$ $u_i = \dfrac{A^T v_i}{\|A^T v_i\|_2}$ (given)

$$A u_i = A \dfrac{A^T v_i}{\|A^T v_i\|_2}$$

$\Rightarrow$ $Q v_i = \cancel{\quad} \lambda v_i$.

$$A u_i = \dfrac{Q v_i}{\|A^T v_i\|_2} \qquad (Q = A^T A)$$

$$A u_i = \cancel{\quad} \dfrac{\lambda v_i}{\|A^T v_i\|^2} \qquad \{Q v_i = \lambda v_i\}$$

$$A u_i = \gamma v_i \Rightarrow \boxed{\gamma_i = \dfrac{\lambda_i}{\|A^T v_i\|^2}}$$

now we have to show γ_i is non-negative.
$\lambda_i \geq 0$, $\|A^T v_i\|^2 \geq 0$ hence
γ_i is real and non-negative.

(d)
Ans

$U = [v_1 | v_2 | v_3 | \ldots | v_m]$, $V = [u_1 | u_2 | u_3 \ldots | u_m]$
$u_i^T u_j = 0 \; (i \neq j)$ $v_i^T v_j = 0 \; (i \neq j)$.

~~to show~~:-

$$A = U \Gamma V^T$$

$\Rightarrow$ Γ is a diagonal matrix containing the non-negative values $\gamma_1, \gamma_2, \ldots \gamma_m$. we also have that the vector v_i are orthogonal to each other.

$$\Rightarrow U_i^t U_j = \frac{v_i^t A A^T v_j}{\|A^T v_i\|_2 \|A^T v_j\|_2}$$

$$\Rightarrow U_i^t U_j = \frac{v_i^t Q v_j}{\|A^T v_i\|_2 \|A^T v_j\|_2} \quad \{Q = AA^T\}$$

$$\Rightarrow U_i^t U_j = \frac{\mu \, v_i^t v_j}{\|A^T v_i\|_2 \|A^T v_j\|_2}$$

$$= \boxed{U_i^t U_j = 0}$$

$\Rightarrow$ from part (c) proof $\Rightarrow \underline{A U_i = \gamma_i v_i}$

for diagonal matrix, Γ such that $\Rightarrow$

$\Gamma_{ij} = \gamma_i$

$\Gamma_{ij} = 0$ when $i \neq 0$.

$$AV = U\Gamma$$

$$AVV^T = U\Gamma V^T$$

$$\boxed{A = U\Gamma V^T}$$

hence, singular value decomposition of the matrix A in the desired form.