5) a) Argue on non-Singular values of matrix A are square roots of eigenvalues of AA^T or A^TA

Sol :- We know that Singular value Decomposition of $A = U\Sigma V^T$

where A is $m \times n$ matrix then

U is $m \times m$ orthogonal matrix

Σ is $m \times n$ diagonal matrix with Singular values

of let $\sigma_1, \sigma_2, \ldots \sigma_r$ on it's diagonal

where r is rank of A and others to 0

V^T is $n \times n$ orthogonal matrix.

$$AA^T = U\Sigma V^T (U\Sigma V^T)^T = U\Sigma V^T V \Sigma U^T \quad (\because V^T V = I)$$

$$= U\Sigma^2 U^T$$

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma U^T U \Sigma V^T \quad (\because U^T U = I)$$

$$= V\Sigma^2 V^T$$

Here, we know, Σ^2 is still a diagonal matrix

whose Singular valus are $\operatorname{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2 \ldots \sigma_r^2)$

$\rightarrow (1)$

Lets consider eigen value equations of A^TA and AA^T

A^TA : $A^TAv = \lambda v$ (v eigen vector, $\lambda \rightarrow$ eigen value)

AA^T : $AA^Tu = \lambda u$ (u eigen vector, $\lambda \rightarrow$ eigen value

$$A^TAv = V\Sigma^2 V^T v = \lambda v$$

$$= V^T(V\Sigma^2 V^T v) = V^T(\lambda v)$$

$$= \Sigma^2 V^T v = \lambda V^T v$$

Now, Considering Σ^2 as diagonal matrix, $V^T v$ is a Vector the equation $\Sigma^2(V^T v) = \lambda(V^T v)$ implies that each entry of Σ^2 scales corresponding entry of $V^T v$, implying the eigen values of $A^T A$ are $\sigma_1^2, \sigma_2^2, \ldots \sigma_r^2$ from the property of eigenvectors. which are actually the squares of Singular values of A (from ①)

Taking Square root on both sides $\sqrt{\lambda} = \sigma_1, \sigma_2, \ldots \sigma_r$. which are non-zero singular values of A.

Similarly, for AA^T, $\Rightarrow (U\Sigma^2 U^T)u = \lambda u$

$$\Rightarrow U^T(U\Sigma^2 U^T)u = U^T(\lambda u)$$

$$\Rightarrow \Sigma^2(U^T u) = \lambda(U^T u)$$

and could be proved the same

$\therefore$ The eigen values of $A^T A$ and AA^T are the squares of Singular values of A implying that square root of eigen values giving us the non-singular values of matrix A for both $A^T A$ and AA^T.

Here, we learn the relation between Singular values and eigen values through matrix transformations

5) b) Show Frobenius norm of matrix is equal to sum of squares of it's singular values.

Sol:- We know that Frobenius norm of a matrix, A is defined as sq. root of sum of square of it's elements,

Let matrix A has m rows, n columns

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2} \qquad a_{ij} \in \text{element of } A$$

SVD of matrix A :- $A = U\Sigma V^T$

Also $U \in m \times m$ orthogonal matrix

Σ: $m \times n$ diagonal matrix with ~~non~~-singular values $\sigma_1, \sigma_2, \sigma_3 \ldots \sigma_r$ (r is rank of A)

V^T: ~~m~~ $\times n$ orthogonal matrix.

Required to Proove :- $\|A\|_F^2 = \sum_{i=1}^{r}\sigma_i^2$

$$\|A\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}|(U\Sigma V^T)_{ij}|^2$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left|\sum_{k=1}^{r} U_{ik}\sigma_k V_{jk}\right|^2$$

($\because$ U and V are Orthogonal matrices, $U^TU = I$, $V^TV = I$)

$\rightarrow$ (1)

$$\|A\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \sum_{k=1}^{r} U_{ik} \sigma_k V_{jk} \right|^2$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\sum_{k=1}^{r} U_{ik} \sigma_k V_{jk} \right) \left(\sum_{l=1}^{r} U_{il} \sigma_l V_{jl} \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{r} \sum_{l=1}^{r} U_{ik} \sigma_k V_{jk} U_{il} \sigma_l V_{jl}$$

From ①, cross terms where $(k \neq l)$ will sum to '0'

thus we will be left with

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{r} \sum_{l=1}^{r} U_{ik} U_{il} V_{jk} V_{jl} \sigma_k \sigma_l$$

Now $\sum_{i=1}^{m} U_{ik} U_{il} = \delta_{kl}$ (kronecker delta) [$\because$ U, V are orthonormal]

$\sum_{j=1}^{n} V_{jk} V_{jl} = \delta_{kl}$ 11

$$\Rightarrow \|A\|_F^2 = \sum_{k=1}^{r} \sum_{l=1}^{r} \delta_{kl} \delta_{kl} \sigma_k \sigma_l .$$

$$= \sum_{k=1}^{r} \sigma_k^2$$

= sum of squares of $\sigma_1^2, \sigma_2^2, \ldots \sigma_r^2$ (singular values of A)

$\therefore$ Frobenius norm of matrix is equal to sum of squares of it's singular values; hence proved.

5c) The Problem the student is facing arises from the eigen vectors returned by 'eig' call might not always be in sorted order. So, columns of matrices U and V might not correspond to the correct singular vectors of A. So, reconstructing matrix A using U, S, V might not yield original matrix due to misalignment. Like

$[V, D1] = eig(A'*A)$ and $[U, D2] = eig(A*A')$

V, U ∈ orthogonal eigen-vector matrices of A^TA and AA^T

D1, D2 are diagonal matrices of eigen values.

We know that S's diagonal elements $S_{ii} = \sqrt{\lambda_i}$ as proved earlier and λ_i are $eig(A^TA) == eig(AA^T)$

Also; $A = USV^T \Rightarrow AV = (USV^T)V = US$

$U^T(AV) = U^T(US) = S$ $\boxed{\therefore U^TAV = S}$

when S is SVD of A

It is possible that U^TAV has negative entries in it's columns, when obtained through eig() in MATLAB, but S must have non-negative columns. So this issues might come

To solve them,

i) Sort D1 and D2 in decreasing order of λ_i's and correspondingly sorting U and V

ii) For every column with negative entry in U^TAV, switch the sign of every entry in the corresponding column of U and V.