CS 663, Fall 2023

Assignment 2

Jahanvi Rajput 23D0378
Badri Vishal Kasuba 22M2119
Abhishek Kumar Singh 22M210

Question 7: Prove that the Laplacian operator is rotationally invariant. For this consider a rotation of the coordinate system from $(x, y) tou = x \cos\theta y \sin\theta, v = x \sin\theta + y \cos\theta$**, and show that** $f_{xx} + f_{yy} = f_{uu} + f_{vv}$ **for any image f.**

Need to show Laplacian operator is rotationally Invariant.

(x, y) rotated into system $u = x\cos\theta - y\sin\theta$, — ①
$v = x\sin\theta + y\cos\theta$.

to show : $f_{xx} + f_{yy} = f_{uu} + f_{vv}$

---

we have, $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that ;

$$f(x,y) = f(u(x,y), v(x,y))$$ [where u & v are fun^n of x & y given by ①)

Now first let us find f_x & f_y.

$$f_x = \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(u(x,y), v(x,y))$$

$$= f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x}$$ (where $f_u = \frac{\partial f}{\partial u}$, $f_v = \frac{\partial f}{\partial v}$)

$$f_y = \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial y}(u(x,y), v(x,y))$$

$$= f_u \frac{\partial u}{\partial y} + f_v \frac{\partial u}{\partial y}$$

Now finding f_{xx} & f_{yy}

$$f_{xx} = \frac{\partial (f_x)}{\partial x} = \frac{\partial}{\partial x}\left(f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(f_u \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(f_v \frac{\partial v}{\partial x}\right)$$

$$f_{xx} = f_{uu}\left(\frac{\partial u}{\partial x}\right)^2 + f_u \frac{\partial^2 u}{\partial x^2} + f_{vv}\left(\frac{\partial v}{\partial x}\right)^2 + f_{uv}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + f_{vu}\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + f_v \frac{\partial^2 v}{\partial x^2} \quad \text{— (ii)}$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(f_u \frac{\partial u}{\partial y} + f_v \frac{\partial u}{\partial y}\right)$$

$$= f_{uu}\left(\frac{\partial u}{\partial y}\right)^2 + f_u \frac{\partial^2 u}{\partial y^2} + f_{vv}\left(\frac{\partial u}{\partial y}\right)^2 + f_v \frac{\partial^2 u}{\partial y^2} + f_{uv}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + f_{vu}\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \quad \text{— (iii)}$$

(Here product rule of derivative is used, i.e. $d(u \cdot v) = v\,du + u\,dv$).

we have, $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 v}{\partial x^2}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ in (ii) & (iii).

finding these values for (i).

$$u = x\cos\theta - y\sin\theta \Rightarrow \frac{\partial u}{\partial x} = \cos\theta, \quad \frac{\partial u}{\partial y} = -\sin\theta$$

$$\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (A)}$$

$$v = x\sin\theta + y\cos\theta$$

$$\Rightarrow \frac{\partial v}{\partial x} = \sin\theta, \quad \frac{\partial v}{\partial y} = \cos\theta, \quad \frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{—— (B)}$$

Substitute (A) & (B) into (ii).

$$f_{xx} = f_{uu}(\cos^2\theta) + f_u \cdot 0 + f_{vv}(\sin^2\theta) + 2 f_{uv}\cos\theta \cdot \sin\theta + f_v \cdot 0$$

$$f_{xx} = f_{uu}\cos^2\theta + f_{vv}\sin^2\theta + 2 f_{uv}\sin\theta\cos\theta \quad \text{—— (C)}$$

Substitute (A) & (B) into (iii).

$$f_{yy} = f_{uu}(\sin^2\theta) + f_u \cdot 0 + f_{vv}\cos^2\theta + 2 f_{uv}(-\sin\theta)\cos\theta + f_v \cdot 0$$

$$= f_{uu}\sin^2\theta + f_{vv}\cos^2\theta - 2 f_{uv}\sin\theta\cos\theta \quad \text{—— (D)}$$

Adding (C) & (D),

$$f_{xx} + f_{yy} = f_{uu}\cos^2\theta + f_{vv}\sin^2\theta + \cancel{2f_{uv}\sin\theta\cos\theta} + f_{vv}\cos^2\theta - \cancel{2f_{uv}\sin\theta\cos\theta} + f_{uu}\sin^2\theta$$

$$\Rightarrow f_{xx} + f_{yy} = f_{uu}(\cos^2\theta + \sin^2\theta) + f_{vv}(\sin^2\theta + \cos^2\theta)$$

$$= (f_{uu} + f_{vv})(\cos^2\theta + \sin^2\theta) = (f_{uu} + f_{vv}) \cdot 1$$

$$(\because \cos^2\theta + \sin^2\theta = 1)$$

$$\Rightarrow \boxed{f_{xx} + f_{yy} = f_{uu} + f_{vv}}$$

Hence laplacian operator is rotationally Invariant.