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Assignment 2

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Question 2: In bicubic interpolation, the image intensity value is expressed in the form $v(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x^i y^j$ where a_{ij} are the coefficients of interpolation and (x,y) are spatial coordinates. This uses sixteen nearest neighbors of a point (x,y). Given the intensity values of these 16 neighbors, explain with the help of matrix based equations, how one can determine the coefficients a_{ij} that determine the function $v(x,y)$? Why do you require 16 neighbors for determining the coefficients?

Solution:

In bicubic interpolation, the image intensity value is expressed in the form

$$v(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x^i y^j$$

where a_{ij} are the coefficients of interpolation and (x,y) are spatial coordinates. Expanding above equation gives:

$$\begin{aligned}
v(x,y) = {} & a_{00}x^0y^0 + a_{01}x^0y^1 + a_{02}x^0y^2 + a_{03}x^0y^3 + \\
& a_{10}x^1y^0 + a_{11}x^1y^1 + a_{12}x^1y^2 + a_{13}x^1y^3 + \\
& a_{20}x^2y^0 + a_{21}x^2y^1 + a_{22}x^2y^2 + a_{23}x^2y^3 + \\
& a_{30}x^3y^0 + a_{31}x^0y^1 + a_{32}x^3y^2 + a_{33}x^3y^3 \\
v(x,y) = {} & a_{00} + a_{01}y + a_{02}y_2 + a_{03}y^3 + \\
& a_{10}x + a_{11}xy + a_{12}xy^2 + a_{13}xy^3 + \\
& a_{20}x^2 + a_{21}x^2y + a_{22}x^2y^2 + a_{23}x^2y^3 + \\
& a_{30}x^3 + a_{31}x^3y + a_{32}x^3y^2 + a_{33}x^3y^3
\end{aligned}$$

Now we have 16 coeficients represented by a_{ij}. So finding out these we need 16 equations. Therefore, we have 16 neighbouring pixel intensity values:

$$\begin{aligned}
& v(x_1,y_1), v(x_1,y_2), v(x_1,y_3), v(x_1,y_4), \\
& v(x_2,y_1), v(x_2,y_2), v(x_2,y_3), v(x_2,y_4), \\
& v(x_3,y_1), v(x_3,y_2), v(x_3,y_3), v(x_3,y_4), \\
& v(x_4,y_1), v(x_4,y_2), v(x_4,y_3), v(x_4,y_4)
\end{aligned}$$

The above intensity value $v(x_1,y_1)$ is given below:

$$
\begin{aligned}
v(x_1, y_1) = \; & a_{00} + a_{01}y_1 + a_{02}y_1^2 + a_{03}y_1^3 + \\
& a_{10}x_1 + a_{11}x_1y_1 + a_{12}x_1y_1^2 + a_{13}x_1y_1^3 + \\
& a_{20}x_1^2 + a_{21}x_1^2y_1 + a_{22}x_1^2y_1^2 + a_{23}x_1^2y_1^3 + \\
& a_{30}x_1^3 + a_{31}x_1^3y_1 + a_{32}x_1^3y_1^2 + a_{33}x_1^3y_1^3
\end{aligned}
$$

$$
v(x_1, y_1) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_1^i y_1^j
$$

Similarly the other intensity values will be given.

$$
\begin{aligned}
v(x_1, y_1) &= \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_1^i y_1^j \\
v(x_2, y_2) &= \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_2^i y_2^j \\
v(x_3, y_3) &= \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_3^i y_3^j \\
v(x_4, y_4) &= \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_4^i y_4^j
\end{aligned}
$$

This can be represented as form of matrix.

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & y_1 & x_1y_1 & x_1^2y_1 & \dots & x_1^2y_1^3 & x_1^3y_1^3 \\ 1 & x_1 & x_1^2 & x_1^3 & y_2 & x_1y_2 & x_1^2y_2 & \dots & x_1^2y_2^3 & x_1^3y_2^3 \\ 1 & x_1 & x_1^2 & x_1^3 & y_3 & x_1y_3 & x_1^2y_3 & \dots & x_1^2y_3^3 & x_1^3y_3^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_4 & x_4^2 & x_4^3 & y_3 & x_4y_3 & x_4^2y_3 & \dots & x_4^2y_3^3 & x_1^3y_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 & y_4 & x_4y_4 & x_4^2y_4 & \dots & x_4^2y_4^3 & x_4^3y_4^3 \end{bmatrix}$$

$$x = \begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ \vdots \\ a_{23} \\ a_{33} \end{bmatrix}, \quad b = \begin{bmatrix} v\left(x_1, y_1\right) \\ v\left(x_1, y_2\right) \\ v\left(x_1, y_3\right) \\ \vdots \\ v\left(x_4, y_3\right) \\ v\left(x_4, y_4\right) \end{bmatrix},$$

where A is the coefficient matrix for linear equations of $v(x_i, y_j)$, x is coefficients of interpolation which needs to be solved and b is given intensity values. For solving this we need to solve:

$$Ax = b$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & y_1 & x_1y_1 & x_1^2y_1 & \dots & x_1^2y_1^3 & x_1^3y_1^3 \\ 1 & x_1 & x_1^2 & x_1^3 & y_2 & x_1y_2 & x_1^2y_2 & \dots & x_1^2y_2^3 & x_1^3y_2^3 \\ 1 & x_1 & x_1^2 & x_1^3 & y_3 & x_1y_3 & x_1^2y_3 & \dots & x_1^2y_3^3 & x_1^3y_3^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_4 & x_4^2 & x_4^3 & y_3 & x_4y_3 & x_4^2y_3 & \dots & x_4^2y_3^3 & x_1^3y_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 & y_4 & x_4y_4 & x_4^2y_4 & \dots & x_4^2y_4^3 & x_4^3y_4^3 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ \vdots \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} v\left(x_1, y_1\right) \\ v\left(x_1, y_2\right) \\ v\left(x_1, y_3\right) \\ \vdots \\ v\left(x_4, y_3\right) \\ v\left(x_4, y_4\right) \end{bmatrix},$$

Hence the solution for x is given by:

$$x = A^{-1}b$$

We need 16 neighbors for determining the coefficients for bicubic interpolation. The reason for this is given below:

i. In Bicubic interpolation the polynomial is used that is having cubic degree in both the x and y direction. A cubic polynomial has four coefficients, and since we are using it in both x and y directions, we have a total of 4x4 = 16 coefficients (aij) to determine. The same is seen in the above equation.

ii. The aim of Bicubic interpolation is to produce a smooth and continuous interpolated surface. The more neighboring points help achieve smoothness by ensuring that the polynomial function can adapt to local variations in the data.

iii. Local method, bicubic interpolation, calculates the pixel value at a position (x, y) depending on the values of the pixels nearby. There must be sufficient data points to accurately represent a cubic polynomial function. In the case of bicubic interpolation, 16 neighboring pixels offer sufficient details to allow a bicubic polynomial function to be fitted smoothly through these points.