② Let $D = [a,b]$ and $f: D \to \mathbb{R}$ be a convex or concave C^2 class function.

[Note:- Considering 'ζ' notation as 'μ' $(\zeta = \mu)$]

Given :- $|f'(x)| \geq \mu \quad \forall x \in D$

$\mu > 0$

Proove :- $\left| \int_a^b e^{if(x)} dx \right| \leq \frac{2}{\mu}$

Sol :- $\left| \int_a^b e^{if(x)} dx \right| \leq \frac{2}{\mu} \Rightarrow \left| \int_a^b e^{if(x)} dx \right| \times \mu \leq 2$

$\Rightarrow \left| \int_a^b e^{if(x)} \times \mu \, dx \right| \leq 2$ is the inequality to be Proved.

L.H.S :- $\left| \int_a^b e^{if(x)} \mu \, dx \right| \leq \left| \int_a^b e^{if(x)} |f'(x)| \, dx \right|$

$(\because$ given that $\mu \leq |f'(x)|)$

Let $t = f(x)$

$dt = f'(x) dx$

[$\because$ we need absolute value, $f(x)$ being either convex or concave and respective $f'(x) = +ve/-ve$ we can show for modulus]

$\int_a^b e^{if(x)} f'(x) dx = \int_{f(a)}^{f(b)} e^{it} dt \quad [\because t = f(x)]$

$= \int_c^d (\cos t + i \sin t) \, dt \quad \begin{bmatrix} d = f(b) \\ c = f(a) \end{bmatrix}$

$= [\sin t]_c^d - i [\cos t]_c^d$

$= \sin d - \sin c - i\cos d + i\cos c$

$$= 2\cos\left(\frac{D+C}{2}\right)\sin\left(\frac{D-C}{2}\right) - 2i\sin\left(\frac{D+C}{2}\right)\sin\left(\frac{B-C}{2}\right)$$

$$= 2\sin\left(\frac{D-C}{2}\right)\left[\cos\left(\frac{D+C}{2}\right) - i\sin\left(\frac{D+C}{2}\right)\right]$$

$$\therefore \int_C^D e^{it}\,dt = 2\sin\left(\frac{D-C}{2}\right) \times e^{-i\left(\frac{D+C}{2}\right)} \qquad \left[\because e^{-i\theta} = \cos\theta - i\sin\theta\right]$$

$$\left|\int_a^b e^{if(x)} f'(x)\,dx\right| = \left|\int_C^D e^{it}\,dt\right| = \left|2\sin\left(\frac{D-C}{2}\right) \times e^{-i\left(\frac{D+C}{2}\right)}\right| \rightarrow (1)$$

Absolute value of $e^{-i\theta}$ is $\sqrt{\cos^2\theta + \sin^2\theta} = 1$,

So, (1), yields an absolute value $\left|2\sin\left(\frac{D-C}{2}\right)\right| \times \left|e^{-i\left(\frac{D+C}{2}\right)}\right|$

$$= 2\left|\sin\left(\frac{D-C}{2}\right)\right|$$

we know $-1 \leq \sin\theta \leq 1 \Rightarrow 2\left|\sin\left(\frac{D-C}{2}\right)\right| \leq 2$

$$\therefore \left|\int_a^b e^{if(x)} f'(x)\,dx\right| = 2\left|\sin\left(\frac{D-C}{2}\right)\right| \leq 2$$

$$\Rightarrow \left|\int_a^b e^{if(x)} f'(x)\,dx\right| \leq 2 \qquad \left[\because \mu \leq |f'(x)|\right]$$

$$\Rightarrow \left|\int_a^b e^{if(x)} \mu\,dx\right| \leq 2$$

$$\Rightarrow \left|\int_a^b e^{if(x)}\,dx\right| \leq \frac{2}{\mu}$$, hence proved.