③ a) Show that following definitions are equivalent.

A Function is L-Smooth with Lipschitz constant $L>0$, if

(i) $\forall x,y \in dom(f)$, $\|\nabla f(x) - \nabla f(y)\| \leq L\|x-y\|$

(ii) $|f(y) - f(x) - \langle \nabla f(x), y-x \rangle| \leq \frac{L}{2}\|x-y\|_2^2$

R.T.P :- ∇f is L-Lipschitz continous $\cong$ A quadratic function Upper bounds f

(i) → (ii) Proof

Let ∇f is Lipschitz continous with parameter $L>0$ and so

$\|\nabla f(x) - \nabla f(y)\| \leq L\|x-y\|$ for $\forall x,y \in dom f$

Since $\|\cdot\|$ are dual norms, based on generalized Cauchy-Schwarz inequality, we have.

$$(\nabla f(x) - \nabla f(y))^T(x-y) \leq L\|x-y\|^2 \longrightarrow ①$$

$$\|u\| = \sup_{v\neq 0} \frac{u^T v}{\|v\|} \Rightarrow |u^T v| \leq \|u\|.\|v\|$$

(Cauchy - Schwarz Inequality)

$\because \|\nabla f(x) - \nabla f(y)\| \leq L\|x-y\| \Rightarrow (\nabla f(x) - \nabla f(y))^T(x-y) \leq L\|x-y\|^2$

Now, let us consider an arbitrary $x,y \in dom f$ and

Let $g(t) = f(x+t(y-x))$, which is defined for $t\in[0,1]$

$\because$ dom f is convex, if ① holds, then

$$g'(t) - g'(0) = (\nabla f(x + t(y-x)) - \nabla f(x))^T (y-x) \leq t L \|x-y\|^2$$

Integrating from $t=0$ to $t=1$ gives

$$f(y) = g(1) = g(0) + \int_0^1 g'(t)\,dt \leq g(0) + g'(0) + \frac{L}{2}\|x-y\|^2$$

$$= f(x) + \nabla f(x)^T (y-x) + \frac{L}{2}\|x-y\|^2$$

$$\Rightarrow f(y) \leq f(x) + \nabla f(x)^T (y-x) + \frac{L}{2}\|x-y\|^2$$

$$\Rightarrow f(y) - f(x) - \nabla f(x)^T \cdot (y-x) \leq \frac{L}{2}\|x-y\|^2 \rightarrow ②$$

Hence (i) $\rightarrow$ (ii), ~~p~~roved.

(ii) $\rightarrow$ (i) Proof

Let $|f(y) - f(x) - \nabla f(x)^T \cdot (y-x)| \leq \frac{L}{2}\|x-y\|_2^2$ $\rightarrow$ ①

hold true,

consider the same inequality by switching x, y

$$\Rightarrow f(x) - f(y) - \nabla f(y)^T \cdot (x-y)| \leq \frac{L}{2}\|y-x\|_2^2$$

$$\leq \frac{L}{2}\|x-y\|_2^2 \rightarrow ②$$

Combining ① & ② gives

$$\left[\cancel{f(y)}-\cancel{f(x)}-\nabla f(x)^T(y-x)\right]+\left[\cancel{f(x)}-\cancel{f(y)}-\nabla f(y)^T(x-y)\right]$$
$$\leq \frac{L}{2}\|x-y\|^2+\frac{L}{2}\|x-y\|^2$$

$$\Rightarrow \quad \nabla f(x)^T(x-y)-\nabla f(y)^T(x-y) \leq L\|x-y\|^2$$

$$\Rightarrow \quad (\nabla f(x)-\nabla f(y))^T(x-y) \leq L\|x-y\|^2 \longrightarrow (1)$$

Hence (ii) $\rightarrow$ (i) is proved

Since (i) $\rightarrow$ (ii) and (ii) $\rightarrow$ (i), both (i) & (ii) are equivalent, hence proved.

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b) Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be such that: f is convex function

∇f is Lipschitz-continous with Lipschize constant 2μ

R.T.P:- $\frac{1}{4\mu}\|\nabla f(x)-\nabla f(y)\|_2^2 \leq |f(y)-f(x)-\nabla f(x)^T(y-x)|$
$\leq \mu\|y-x\|^2$

i) $\frac{1}{4\mu}\|\nabla f(x)-\nabla f(y)\|_2^2 \leq |f(y)-f(x)-\nabla f(x)^T(y-x)|$
$\leq \mu\|y-x\|^2$

sol:-
We know that, by lipschitz continuity and Lipschitz constant 2μ

$$\|\nabla f(x)-\nabla f(y)\| \leq L\|x-y\|$$

$$\Rightarrow \|\nabla f(x)-\nabla f(y)\| \leq 2\mu\|x-y\|$$

Squaring on both sides

$$\Rightarrow \|\nabla f(x) - \nabla f(y)\|_2^2 \leq 4\mu^2 \|x-y\|_2^2$$

$$\Rightarrow \frac{1}{4\mu^2} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq \|x-y\|_2^2 \rightarrow ①$$

We also know that,

$$f(y) - f(x) - \nabla f(x)^T (y-x) \leq \frac{L}{2} \|x-y\|_2^2$$

$$\Rightarrow f(y) - f(x) - \nabla f(x)^T (y-x) \leq \frac{2\mu}{2} \|x-y\|_2^2$$

$$\leq \mu \|x-y\|_2^2 \rightarrow ②$$

From ① & ②, we can say that

$$\frac{1}{4\mu} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq \mu \|y-x\|_2^2$$

and $|f(y) - f(x) - \nabla f(x)^T (y-x) \leq \mu \|y-x\|_2^2$

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ii) $\frac{1}{4\mu} \|\nabla f(x) - \nabla f(y)\|_2^2 \leq |f(y) - f(x) - \nabla f(x)^T (y-x)|$

~~we know that, from earlier derivation,~~

$$\cancel{\|\nabla f(x) - \nabla f(y)\|_2^2 \leq 4\mu^2 \|y-x\|_2^2} \rightarrow ①$$

The above inequality is the co-coercivity property of ∇f

Proof :-

Let f_x, f_y be two convex functions with domain $\mathbb{R}^n$

$$f_x(z) = f(z) - \nabla f(x)^T z$$

$$f_y(z) = f(z) - \nabla f(y)^T z$$

The two functions have L-Lipschitz continous gradients

When $z = x$

$$f_x(y) - f_x(x) = [f(y) - \nabla f(x)^T \cdot y] - [f(x) - \nabla f(x)^T \cdot x]$$

$$= f(y) - f(x) - \nabla f(x)^T \cdot (y - x)$$

$\because$ When $z = x$, minimizes $f_x(z)$, so f has a minimizer x^*

$$\Rightarrow \frac{1}{2L} \|\nabla f(z)\|^2 \leq f(z) - f(x^*)$$

[Property by minimizing quadratic upper bound for $x = z$]

$$\therefore f_x(y) - f_x(x) = f(y) - f(x) - \nabla f(x)^T \cdot (y - x)$$

$$\geq \frac{1}{2L} \|\nabla f_x(y)\|^2$$

$$\geq \frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2$$

[$\because \nabla f_x(y) = \nabla f(y) - \nabla f(x)$]

$$\therefore f(y) - f(x) - \nabla f(x)^T \cdot (y - x) \geq \frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2$$

~~Similarly when Lips~~

$\because$ Lipschitz constant $= 2\mu$

$$\geq \frac{1}{4\mu} \|\nabla f(y) - \nabla f(x)\|^2$$

therefore, ~~Prove~~ - $\frac{1}{4\mu} \|\nabla f(x) - \nabla f(y)\|^2 \leq |f(y) - f(x) - \nabla f(x)^T (y-x)|$

Also, we proved $|f(y) - f(x) - \nabla f(x)^T (y-x)| \leq \mu \|y-x\|^2$

$$\therefore \frac{1}{4\mu} \|\nabla f(x) - \nabla f(y)\|^2 \leq |f(y) - f(x) - \nabla f(x)^T (y-x)| \leq \mu \|y-x\|^2$$

Comments about 'f' in this case

we know that f is Lipschitz continous with Param $L > 0 = 2\mu$.

Since 'f' holds this property, f is L-smooth function

Also, since being L-Lipscitz contionous, is equivalent to the quadratic function upper bound f, for this consequence, when f has a minimize x^*, then

$$\frac{1}{2L} \|\nabla f(z)\|^2 \leq f(z) - f(x^*) \leq \frac{L}{2} \|z - x^*\|^2$$

for $\forall z$ Property holds

through which we show Co-coercivity Property of gradient for funtion 'f' (Part-b of Solution). So we can say that

when Lipschitz continuity of ∇f holds, then upper bound property, Co-coercivity of ∇f and Lipschity continity of ∇f holds true and all are equivalent properties