① We say function 'f' is log-convex on real interval $D = [a,b]$ if $\forall x, y \in D$ and $\lambda \in [0,1]$, f satisfies

$$f(\lambda x + (1-\lambda)y) \leq f^{\lambda}(x) \times f^{1-\lambda}(y)$$

Require to Prove

Given, An increasing log-convex function $f: D \to \mathbb{R}$ and $0 \leq t \leq 1$

$$f\left(\frac{a+b}{2}\right) \leq \Phi(a,b) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \psi(a,b,t) \leq L(f(a), f(b)) \leq \frac{f(a)+f(b)}{2}$$

where

$$\Phi(a,b) = \sqrt{f\left(\frac{3a+b}{4}\right), f\left(\frac{a+3b}{4}\right)}$$

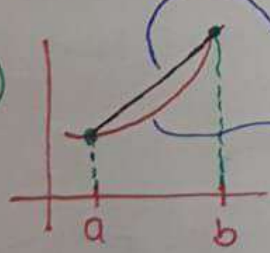
$$L(a,b) = \frac{a-b}{\ln\left(\frac{a}{b}\right)}$$

$$\psi(a,b,t) = (1-t)L\left(f(ta+(1-t)b), f(a)\right) + t \times L\left(f(b), f(ta+(1-t)b)\right)$$

Sol:-

Property Basics

(i)

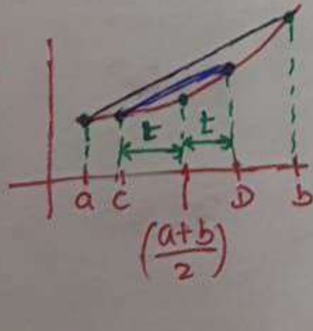


The line :- $Y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$

$f(x)$

For a Convex function $f(x)$, $Y \geq f(x)$

(ii)



$$\frac{f(c)+f(d)}{2} = \frac{f\left(\frac{a+b}{2} - t\right) + f\left(\frac{a+b}{2} + t\right)}{2}$$

$$\frac{f(c)+f(d)}{2} \geq f\left(\frac{a+b}{2}\right), \quad \left|\left(\frac{a+b}{2}\right) - a\right| \geq t$$

From (i), we know that

$$f(x) \le Y \Rightarrow f(x) \le f(a) + \frac{f(b)-f(a)}{b-a} \times (x-a)$$

Integrating on both sides:-

$$\Rightarrow \int_a^b f(x)\,dx \le f(a)\int_a^b dx + \frac{f(b)-f(a)}{b-a} \times \int_a^b (x-a)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx \le (b-a)f(a) + \frac{f(b)-f(a)}{b-a} \times \left[\frac{x^2}{2} - ax\right]_a^b$$

$$\le (b-a)f(a) + \frac{f(b)-f(a)}{(b-a)} \times \left(\frac{b^2}{2} - ba - \frac{a^2}{2} + a^2\right)$$

$$\le (b-a)f(a) + \frac{f(b)-f(a)}{(b-a)} \times \left(\frac{a^2}{2} + \frac{b^2}{2} - \frac{2ab}{2}\right)$$

$$\le (b-a)f(a) + \frac{f(b)-f(a)}{\cancel{(b-a)}} \times \frac{(a-b)^{\cancel{2}}}{2}$$

$$\le (b-a)\left[f(a) + \frac{f(b)-f(a)}{2}\right]$$

$$\boxed{\therefore \int_a^b f(x)\,dx \le (b-a)\left[\frac{f(a)+f(b)}{2}\right]} \rightarrow \text{A}$$

(iii) $$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\left[\underbrace{\int_a^{\frac{a+b}{2}} f(x)\,dx}_{1} + \underbrace{\int_{\frac{a+b}{2}}^b f(x)\,dx}_{2}\right]$$

1 Let $t = \dfrac{2x-(a+b)}{-(b-a)}$

$dx = \dfrac{-(b-a)}{2}\,dt$

2 Let $t = \dfrac{2x-(a+b)}{(b-a)}$

$dx = \dfrac{(b-a)}{2}\,dt$

From (iii)

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\left[① + ② \right]$$

① $\displaystyle\int_a^{\frac{a+b}{2}} f(x)\,dx = \int_{\cancel{0}}^{1} f\left(\frac{a+b}{2} - t\times\left(\frac{b-a}{2}\right)\right)dx$

$$= \int_0^1 f\left(\frac{a+b}{2} - t\left(\frac{b-a}{2}\right)\right)\times -\frac{(b-a)}{2}\,dt$$

$x=a \Rightarrow t=1$

$x=\frac{a+b}{2} \Rightarrow t=0$

$t = \dfrac{2x-(a+b)}{-(b-a)}$

$\Rightarrow x = \left(\frac{a+b}{2} - \frac{(b-a)t}{2}\right)$

② $\displaystyle\int_{\frac{a+b}{2}}^{b} f(x)\,dx = \int_0^1 f\left(\frac{a+b}{2} + t\left(\frac{b-a}{2}\right)\right)\times\frac{(b-a)}{2}\,dt$

$t = \dfrac{2x-(a+b)}{+(b-a)}$

$$\Rightarrow \frac{1}{(b-a)}\int_a^b f(x)\,dx = \frac{1}{b-a}\times\frac{(b-a)}{2}\times\int_0^1\left[f\left(\frac{a+b}{2} - t\left(\frac{b-a}{2}\right)\right)dt + f\left(\frac{a+b}{2} + t\left(\frac{b-a}{2}\right)\right)dt\right]$$

$k = \dfrac{(b-a)t}{2}$

$$= \int_0^1 \frac{1}{2}\left[f\left(\frac{a+b}{2} - \frac{t(b-a)}{2}\right) + f\left(\frac{a+b}{2} + \frac{t(b-a)}{2}\right)\right]dt$$

From (ii)

$$\frac{f(c)+f(d)}{2} = \frac{f\left(\frac{a+b}{2} - k\right) + f\left(\frac{a+b}{2} - k\right)}{2} \geq f\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \frac{1}{(b-a)}\int_a^b f(x)\,dx) = \int_0^1 \frac{1}{2}\left[f\left(\frac{a+b}{2} - k\right) + f\left(\frac{a+b}{2} + k\right)\right]dt$$

$$\geq \int_0^1 f\left(\frac{a+b}{2}\right)dt = f\left(\frac{a+b}{2}\right)$$

$$\therefore \frac{1}{(b-a)}\int_a^b f(x)\,dx \geq f\left(\frac{a+b}{2}\right) \rightarrow Ⓑ$$

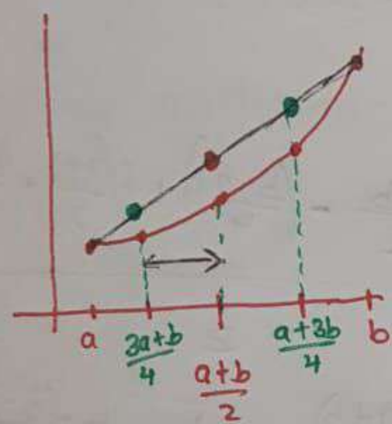
From A and B, we proved Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \rightarrow \text{C}$$

which is one of the part of solution that needs to be shown.

and other parts are improvements to the core concept of the above equation.

3 Showing, $f\left(\frac{a+b}{2}\right) \le \sqrt{f\left(\frac{3a+b}{4}\right) \times f\left(\frac{a+3b}{4}\right)} \le \frac{1}{(b-a)}\int_a^b f(x)\,dx$



From C, if $a \rightarrow a$, $b \rightarrow \frac{a+b}{2}$

$$f\left(\frac{a+\frac{a+b}{2}}{2}\right) \le \frac{1}{\left(\frac{a+b}{2}-a\right)}\int_a^{\frac{a+b}{2}} f(x)\,dx \le \frac{f(a)+f\left(\frac{a+b}{2}\right)}{2}$$

$$\Rightarrow f\left(\frac{3a+b}{4}\right) \le \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx \le \frac{f(a)+f\left(\frac{a+b}{2}\right)}{2} \quad \text{(i)}$$

similarly (when $a \rightarrow \frac{a+b}{2}$, $b \rightarrow b$)

$$f\left(\frac{a+3b}{4}\right) \le \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)\,dx \le \frac{f\left(\frac{a+b}{2}\right)+f(b)}{2} \quad \text{(ii)}$$

Adding (i) and (ii)

$$\Rightarrow f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \le \frac{2}{(b-a)}\left[\int_a^{\frac{a+b}{2}} f(x)\,dx + \int_{\frac{a+b}{2}}^{b} f(x)\,dx\right.$$

$$\Rightarrow f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \leq \frac{2}{(b-a)} \int_a^b f(x)\,dx$$

$$\Rightarrow \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \leq \frac{1}{(b-a)} \int_a^b f(x)\,dx$$

We Know through Arithmetic - Geometric mean inequality,

$$\frac{c+d}{2} \geq \sqrt{c \times d} \Rightarrow \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \geq \sqrt{f\left(\frac{3a+b}{4}\right) \times f\left(\frac{a+3b}{4}\right)}$$

$$\Rightarrow \sqrt{f\left(\frac{3a+b}{4}\right) \times f\left(\frac{a+3b}{4}\right)} \leq \frac{1}{(b-a)} \int_a^b f(x)\,dx$$

Also, $$\frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} = \frac{f\left(\frac{a+b}{2} - \left(\frac{b-a}{4}\right)\right) + f\left(\frac{a+b}{2} + \left(\frac{b-a}{4}\right)\right)}{2}$$

From Basics (ii) Property. $\geq f\left(\frac{a+b}{2}\right)$ when $k = \left(\frac{b-a}{4}\right)$

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1st Page

$$\therefore f\left(\frac{a+b}{2}\right) \leq \sqrt{f\left(\frac{3a+b}{4}\right) \times f\left(\frac{a+3b}{4}\right)} \leq \frac{1}{(b-a)} \int_a^b f(x)\,dx$$

F

So, till now, $$f\left(\frac{a+b}{2}\right) \leq \phi(a,b) \leq \frac{1}{(b-a)} \int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

has been Prooved

where $$\phi(a,b) = \sqrt{f\left(\frac{3a+b}{4}\right) \times f\left(\frac{a+3b}{4}\right)}$$

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$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \psi(a,b,t) \le L(f(a), f(b)) \le \frac{f(a)+f(b)}{2}$$

Convex Properties

(i) We know that function $f: D \to \mathbb{R}$ is said to be convex on D if $\forall x, y \in D$ and $t \in [0,1]$,

$$f(tx+(1-t)y) \le t f(x) + (1-t) f(y)$$

(ii) Also, $f: D \to \mathbb{R}$ is said to be log-convex here if

$$f(tx+(1-t)y) \le [f(x)]^t \times [f(y)]^{(1-t)}$$

From, arithmetic-geometric mean inequality, we also have

$$[f(x)]^t \times [f(y)]^{(1-t)} \le t f(x) + (1-t) f(y)$$

Sol:-

Since given that f is log-convex on D, $D \in [a,b]$

we have,

$$f(ta+(1-t)b) \le [f(a)]^t \times [f(b)]^{1-t}$$

$$f((1-t)a+tb) \le [f(a)]^{1-t} \times [f(b)]^t$$

Let's proove,

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le L(f(a), f(b)) \le \frac{f(a)+f(b)}{2}$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_0^1 f(ta+(1-t)b)\times(b-a)\,dt$$

$\because x = ta+(1-t)b \qquad x=a \Rightarrow t=1$

$dx = (a-b)\,dt \qquad x=b \Rightarrow t=0$

$$\Rightarrow \frac{1}{b-a}\int_a^b f(x)\,dx = \int_0^1 f(ta+(1-t)b)\,dt$$

$$\because f(ta+(1-t)b) \leq [f(a)]^t \times [f(b)]^{1-t} \leq t\times f(a) + (1-t)f(b)$$

$$\Rightarrow \frac{1}{b-a}\int_a^b f(x)\,dx = \int_0^1 f(ta+(1-t)b)\,dt \leq \underbrace{\int_0^1 [f(a)]^t\times[f(b)]^{(1-t)}\,dt}_{(i)} \leq \underbrace{\int_0^1 tf(a)+(1-t)\ldots}_{(ii)}$$

(i) $$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \int_0^1 [f(a)]^t \times [f(b)]^{(1-t)}\,dt$$

Let $m = f(a)$

$n = f(b)$

$$\leq \int_0^1 m^t \times n^{(1-t)}\,dt$$

$$\leq \int_0^1 \left(\frac{m}{n}\right)^t \times n\,dt$$

$$\leq n\times\int_0^1 \left(\frac{m}{n}\right)^t dt$$

$\left[\because \int a^x\,dx = \frac{a^x}{\log x}\right]$

$$\leq n\left[\frac{\left(\frac{m}{n}\right)^t}{\ln\left(\frac{m}{n}\right)}\right]_0^1$$

$$\leq n\left[\frac{\frac{m}{n}}{\ln\left(\frac{m}{n}\right)} - \frac{1}{\ln\left(\frac{m}{n}\right)}\right]$$

$\left[\because \left(\frac{m}{n}\right)^0 = 1\right]$

$$\Rightarrow \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{m}{\ln\left(\frac{m}{n}\right)} - \frac{n}{\ln\left(\frac{m}{n}\right)} = \frac{m-n}{\ln\left(\frac{m}{n}\right)}$$

$$\le \frac{f(a)-f(b)}{\ln\left(\frac{f(a)}{f(b)}\right)} \quad \left[\because \begin{array}{l} m=f(a) \\ n=f(b) \end{array}\right]$$

$$\le L\left(f(a), f(b)\right) = (i)$$

(ii) $(i) \le (ii) \Rightarrow \int_0^1 [f(a)]^t \times [f(b)]^{(1-t)}\,dt \le \int_0^1 t\,f(a) + (1-t)f(b)\,dt$

$$\Rightarrow L\left(f(a), f(b)\right) \le \int_0^1 t\,f(a) + (1-t)\,f(b)\,dt$$

$$\Rightarrow L\left(f(a), f(b)\right) \le f(a)\int_0^1 t\,dt + f(b)\int_0^1 dt - f(b)\int_0^1 t\,dt$$

$$\le f(a)\left[\frac{t^2}{2}\right]_0^1 + f(b)\left[t\right]_0^1 - f(b)\left[\frac{t^2}{2}\right]_0^1$$

$$\le \frac{f(a)}{2} + f(b) - \frac{f(b)}{2}$$

$$\le \frac{f(a)+f(b)}{2}$$

$$\therefore \frac{1}{b-a}\int_a^b f(x)\,dx \le L\left(f(a), f(b)\right) \le \frac{f(a)+f(b)}{2}$$

$\rightarrow$ (D)

Now, we have to show,

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \psi(a,b,t) \le \frac{f(a)-f(b)}{\ln\left(\frac{f(a)}{f(b)}\right)}$$

$$\psi(a,b,t) = t\,L\big(f(b), f(ta+(1-t)b)\big) + (1-t)\,L\big(f(ta+(1-t)b), f(a)\big)$$

$\psi(a,b,t)$ is a convex combination of two logarithmic means, i.e., $L(f(b), f(c))$ and $L(f(c), f(a))$

where $c = ta+(1-t)b$

$\because t \in [0,1]$, we can get that $a \le c \le b$ and c lies between a and b.

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\left[\int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right]$$

$\because \frac{1}{(b-a)}\int_a^b f(x)\,dx \le L(f(a), f(b))$, From (I).

$$\Rightarrow \quad \le \frac{1}{(b-a)} \times \left[(c-a)\,L(f(a), f(c)) + (b-c)\,L(f(c), f(b))\right]$$

$$\le \frac{1}{(b-a)}\left[(ta+(1-t)b-a)\,L(f(a), f(c)) + (b-ta-(1-t)b) \times L(f(c), f(b))\right]$$

$$\le \frac{1}{(b-a)}\left[(1-t)(b-a)\,L(f(a), f(c)) + (\cancel{b}-ta-\cancel{b}+bt)\,L(f(c), f(b))\right]$$

$$\le \frac{1}{\cancel{(b-a)}}\left[(1-t)\cancel{(b-a)}\,L(f(a), f(c)) + t\cancel{(b-a)}\,L(f(c), f(b))\right]$$

$$\Rightarrow \frac{1}{(b-a)}\int_a^b f(x)\,dx \le t\,L(f(c), f(b)) + (1-t)\,L(f(a), f(c))$$

$\because$ logarithmic mean is symmetric, $L(a,b) = L(b,a)$
for log-convex functions,

$$\Rightarrow \frac{1}{(b-a)}\int_a^b f(x)\,dx \le t \times L(f(b), f(c)) + (1-t)\,L(f(c), f(a))$$

$$\Rightarrow \boxed{\frac{1}{(b-a)}\int_a^b f(x)\,dx \le \psi(a,b,t)} \longrightarrow (i)$$

Let's prove $\psi(a,b,t) \le L(f(a), f(b))$

Let us assume, for simplicity $f(a) = l$, $f(ta+(1-t)b) = f(c) = m$, $f(b) = n$

$$\psi(a,b,t) = t\,[L(n,m)] + (1-t)\,[L(m,l)]$$

For convex function, $f(x)$, we know that

$$(1-t)\,f(x) + t\,f(y) \ge f(ty + (1-t)x)$$

$$\Rightarrow \psi(a,b,t) = (1-t)\,L(m,l) + t\,L(n,m)$$

$$\ge$$

Now Let's prove $\psi(a,b,t) \leq L(f(a), f(b))$

We know, for any value, $c = ta + (1-t)b \quad t \in [0,1]$

$$a \leq c \leq b \rightarrow (i)$$

Since f is an increasing log-convex function,

$$f(a) \leq f(c) \leq f(b) \rightarrow (ii)$$

$\psi(a,b,t) =$
$t \times L(f(b), f(c)) + (1-t) L(f(c), f(a))$ is a convex combination $L(f(b), f(c))$ and $L(f(c), f(a))$. So it lies between these to logarithmic-mean values $\rightarrow (iii)$

Also, when (ii) holds, with the logarithmic mean property
we have $L(f(a), f(c)) + L(f(c), f(b)) \leq L(f(a), f(b))$
$\rightarrow (iv)$

From (iii), we know that

$$\psi(a,b,t) \leq \left[L(f(b), f(c)), L(f(c), f(a))\right]$$ max value of one of them

$$\leq \left(L(f(b), f(c)) + L(f(c), f(a))\right)$$

$$\leq L(f(a), f(c)) + L(f(c), f(b)) \quad \left[\because L(f(x), f(y)) = L(f(y), f(x)) \text{ for log-convex functions}\right]$$

$$\leq L(f(a), f(b)) \quad [\because \text{from (iv)}]$$

$$\Rightarrow \psi(a,b,t) \leq L(f(a), f(b)) \rightarrow (ii)$$

From (i) and (ii) of 5^{th} part, we have

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \psi(a,b,t) \leq L(f(a),f(b)) \quad \rightarrow (E)$$

From (C), (D), (E), (F) below, derivations

$$(C) \rightarrow f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

$$(D) \rightarrow \frac{1}{(b-a)}\int_a^b f(x)dx \leq L(f(a),f(b)) \leq \frac{f(a)+f(b)}{2}$$

$$(E) \rightarrow \frac{1}{(b-a)}\int_a^b f(x)dx \leq \psi(a,b,t) \leq L(f(a),f(b))$$

$$(F) \rightarrow f\left(\frac{a+b}{2}\right) \leq \sqrt{f\left(\frac{3a+b}{4}\right) \times f\left(\frac{a+3b}{4}\right)} \leq \frac{1}{b-a}\int_a^b f(x)dx$$

We can conclude that,

$$f\left(\frac{a+b}{2}\right) \leq \phi(a,b) \leq \frac{1}{(b-a)}\int_a^b f(x)dx \leq \psi(a,b,t) \leq L(f(a),f(b)) \leq \frac{f(a)+f(b)}{2}$$

Hence Proved