Mathematics for Computing I - PMAT 11212
Department of Mathematics
University of Kelaniya

Dr M.H.L Weerasinghe

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Outline

Logic

- Any formal system can be considered a **logic** if it has a well-defined syntax or a well-defined proof-theory.
- The simplest, and most abstract logic we can study is called **propositional logic**.

Propositions

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Eg: Following are propositions.

- Washington, D.C., is the capital of the United States of America.
- $1 + 1 = 2$.
- $2 + 2 = 3$.

Following are not propositions.

- What time is it?
- Read this carefully.
- $x + 1 = 2$.
- $x + y = z$.

Notation

It is standard practice to use the lower-case roman letters $p, q, r, \ldots$ to stand for propositions.

$p : 1 + 1 = 3$, to define p to be the proposition $1 + 1 = 3$.

Truth Values

- The truth value of a proposition is true, denoted by **T**, if it is a true proposition.
- The truth value of a proposition is false, denoted by **F**, if it is a false proposition.

Logical Connectives

Now, the study of propositions is pretty boring. We therefore now introduce a number of connectives which will allow us to build up complex propositions.

The connectives we introduce are:

- $\neg$ Negation [read as not] ($\sim$)
- $\wedge$ Conjunction [read as and] ($\cdot$, $\&$)
- $\vee$ Disjunction [read as or] ($|$, $+$)
- $\rightarrow$ Implication[read as implies] ($\Rightarrow$, $\supset$)
- $\leftrightarrow$ Equivalence [read as if and only if, iff]

Negation

Let p be a proposition. The negation of p denoted by $\neg p$, is the statement "It is not the case that p".

i.e. $\neg p$: "It is not the case that p."

The proposition $\neg p$ is read "not p". The truth value of the negation of p, is the opposite of the truth value of p.

Find the negation of the proposition.
"Michael's PC runs Linux" and express this in simple English.

The negation is "It is not the case that Michael's PC runs Linux."

"Michael's PC does not run Linux."

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"Michael's PC does not run Linux."

Find the negation of the proposition “Ann’s smartphone has at least 32GB of memory”.

What is the negation of each of these propositions?

- $3 + 2 = 5$
- $3 > 5$
- $3 \leq 5$

Truth Tables

The truth table for the negation of a proposition *p*.

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

Conjunction of two propositions

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Find the conjunction of the propositions p and q where p is the proposition “Rebecca’s PC has more than 16 GB free hard disk space” and q is the proposition “The processor in Rebecca’s PC runs faster than 1 GHz.”

Solution: “Rebecca’s PC has more than 16 GB free hard disk space, and the processor in Rebecca’s PC runs faster than 1 GHz.”

“Rebecca’s PC has more than 16 GB free hard disk space, and its processor runs faster than 1GHz.”

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Disconjunction of two propositions

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Find the disconjunction of the propositions p and q where p is the proposition “Rebecca’s PC has more than 16 GB free hard disk space” and q is the proposition “The processor in Rebecca’s PC runs faster than 1 GHz.”

Solution: “Rebecca’s PC has more than 16 GB free hard disk space, or the processor in Rebecca’s PC runs faster than 1 GHz.”

Find the disconjunction of the propositions p and q where p is the proposition “Rebecca’s PC has more than 16 GB free hard disk space” and q is the proposition “The processor in Rebecca’s PC runs faster than 1 GHz.”

Solution: “Rebecca’s PC has more than 16 GB free hard disk space, or the processor in Rebecca’s PC runs faster than 1 GHz.”

Connective or - Inclusive or

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"Students who have taken calculus or computer science can take this class."

we mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects

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Eg: “Students who have taken calculus or computer science, but not both, can enroll in this class.”

we mean that students who have taken both calculus and a computer science course cannot take the class. Only those who have taken exactly one of the two courses can take the class.

Eg: “Coca-Cola or Sprite comes with a large pizza.”

This means that customers can have either Coca-Cola or Sprite, but not both.
Hence, this is an exclusive or.

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Exclusive or

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statements

Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q."

The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise.

In the conditional statement $p \to q$, p is called the hypothesis (or condition) and q is called the conclusion (or consequence).

TABLE 5 The Truth Table for the Conditional Statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Variety of terminology is used to express $p \to q$

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"

"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

Eg: “If I am elected, then I will lower taxes.”

This means; If the politician is elected, voters would expect this politician to lower taxes. Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes. It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge. This last scenario corresponds to the case when p is true but q is false in $p \to q$.

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"If you get 100 on the final, then you will get an A."

This means; If you manage to get a 100 on the final, then you would expect to receive an A. If you do not get 100 you may or may not receive an A depending on other factors. However, if you do get 100, but the professor does not give you an A, you will feel cheated.

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Converse, Contrapositive, Inverse

- The proposition $q \to p$ is called the **converse** of $p \to q$.
- The proposition $\neg q \to \neg p$ is called the **contrapositive** of $p \to q$.
- The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$.

What are the contrapositive, the converse, and the inverse of the conditional statement
"The home team wins whenever it is raining."

Solution:

- "q whenever p" is one of the ways to express the conditional statement $p \to q$. The original statement can be rewritten as "If it is raining, then the home team wins."
- The contrapositive is "If the home team does not win, then it is not raining."
- The converse is "If the home team wins, then it is raining."
- The inverse is "If it is not raining, then the home team does not win."

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Biconditional statements

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

There are some other common ways to express $p \leftrightarrow q$:

- "p is necessary and sufficient for q"
- "if p then q, and conversely"
- "p iff q".

Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \land (q \rightarrow p)$.

There are some other common ways to express $p \leftrightarrow q$:

- "*p* is necessary and sufficient for *q*"
- "if *p* then *q*, and conversely"
- "p iff q".

Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$.

"You can take the flight if and only if you buy a ticket."

This means; This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight.
It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket but you cannot take the flight (such as when the airline bumps you).

"You can take the flight if and only if you buy a ticket."

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Truth Tables of Compound Propositions

We have now introduced important logical connectives; negation, conjunctions, disjunctions, conditional statements, and biconditional statements. We can use these connectives to build up complicated compound propositions involving any number of propositional variables. We can construct truth tables to determine the truth values of these compound propositions.

Construct the truth table of the compound proposition $(p \to q) \to (p \wedge q)$.

TABLE 7 The Truth Table of $(p \vee \neg q) \to (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operators

TABLE 8
Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Example

- $(p \vee q) \wedge (\neg r)$:
- $\neg p \wedge q$:
- $p \wedge q \vee r$:
- $p \vee q \rightarrow r$:

Construct the truth table of the compound proposition $(p \to q) \lor (\neg p \to r)$.

Tautology, Contradiction, Contingency

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.
- A compound proposition that is always false is called a contradiction.
- A compound proposition that is neither tautology nor contradiction is called a contingency.

Construct the truth table of the compound proposition $(p \vee \neg p)$ and $(p \wedge \neg p)$.

Applications in propositional logic

How can this English sentence be translated into a logical expression?
Eg: “You can access the Internet from campus only if you are a computer science major or you are not a freshman.”
Solution :

Eg:

“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.”

Solution :

Eg:

- “The diagnostic message is stored in the buffer or it is retransmitted.”
  Solution :
- “The diagnostic message is not stored in the buffer.”
  Solution :
- “If the diagnostic message is stored in the buffer, then it is retransmitted.”
  Solution :

Express these system specifications using the propositions p “The message is scanned for viruses” and q “The message was sent from an unknown system” together with logical connectives (including negations).

a) “The message is scanned for viruses whenever the message was sent from an unknown system.”
b) “The message was sent from an unknown system but it was not scanned for viruses.”
c) “It is necessary to scan the message for viruses whenever it was sent from an unknown system.”
d) “When a message is not sent from an unknown system it is not scanned for viruses.”

Propositional Equivalences

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Remark: The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology.

Show that $\neg p \vee q$ and $p \to q$ are logically equivalent.

TABLE 4 Truth Tables for $\neg p \vee q$ and $p \to q$.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Show that $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent.

TABLE 4 Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Some Important equivalences

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Use De Morgon's laws to express the negation of " Migara has a cell phone and he has a laptop computer".
Solution:

Constructing New Logical Equivalences

Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

$$
\begin{aligned}
\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{by Example 3} \\
&\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan law} \\
&\equiv p \wedge \neg q && \text{by the double negation law}
\end{aligned}
$$

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent. ◀

Show that $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent.

End of Propositional Logic ....

Next week we will discuss Predicates and Quantifiers...