LINEAR PROGRAMMING – SIMPLEX METHOD

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N-DIMENSIONAL LP PROBLEMS

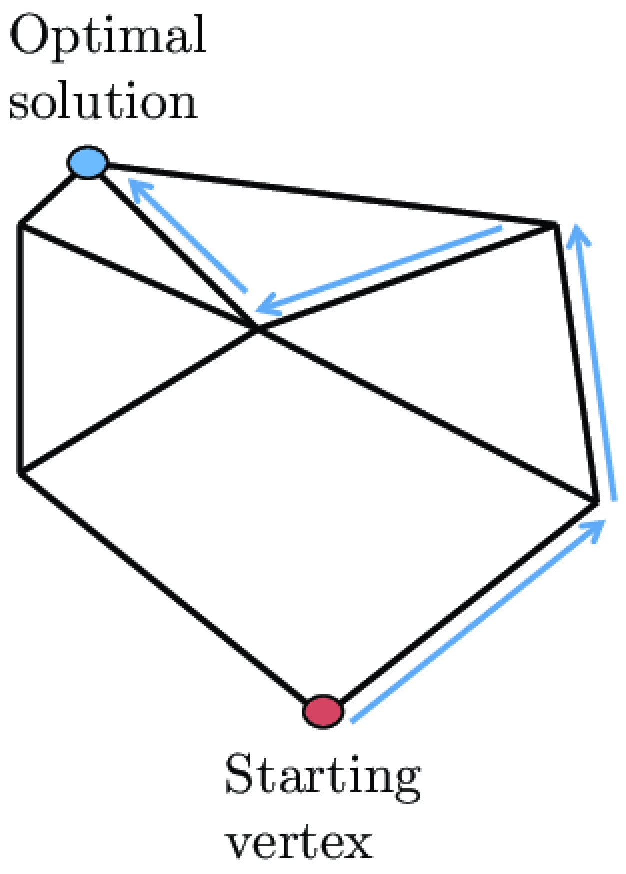
- If we have n variables, each constraint defines a half-space in n-dimensional space
- Feasible region formed by the intersection of these half-spaces is called a **simplex**
- The objective function is now a hyperplane
- An optimal solution still occurs at a vertex of the simplex

SIMPLEX METHOD

- Can use when there are more than 2 decision variables
- One of the most powerful & popular methods for linear programming
- An iterative procedure for getting the most feasible solution
- Not used to examine all the feasible solutions
- Deals only with the vertex points of the feasible space

SIMPLEX METHOD

- Steps involved:
  1. Locate an extreme point of the feasible region
  2. Examine each boundary edge intersecting at this point to see whether the movement along any edge increases the value of the objective function
  3. If the value of the objective function increase along any edge, move along this edge to the adjacent extreme point. (If several edges indicate improvement, the edge providing the greatest rate of increase is selected)
  4. Repeat steps 2 and 3 until movement along any edge no longer increases the value of the objective function.



STANDARD MAXIMIZATION PROBLEM

- In order to use the simplex method to solve a linear programming problem, we need the standard maximization problem:
  - an objective function $P = c_1x_1 + \ldots + c_nx_n$, and
  - one or more constraints of the form $a_1x_1 + a_2x_2 + \ldots a_nx_n \leq b$
    - All of the a_{number} represent real-numbered coefficients and
    - the x_{number} represent the corresponding variables.
    - x_{number} and b are non-negative (0 or larger) real numbers

SLACK VARIABLES

- Rewrite each inequality as an equation by introducing **slack variables**.

That is,

$$a_{j1}x_1 + \ldots + a_{jn}x_n \leq b_j$$

becomes

$$a_{j1}x_1 + \ldots + a_{jn}x_n + s_j = b_j$$

BASIC AND NON-BASIC VARIABLES

- Basic variable are selected arbitrarily with the restriction that there be as many as basic variables as there are equations
- The remaining variables are non-basic variables
- Example:

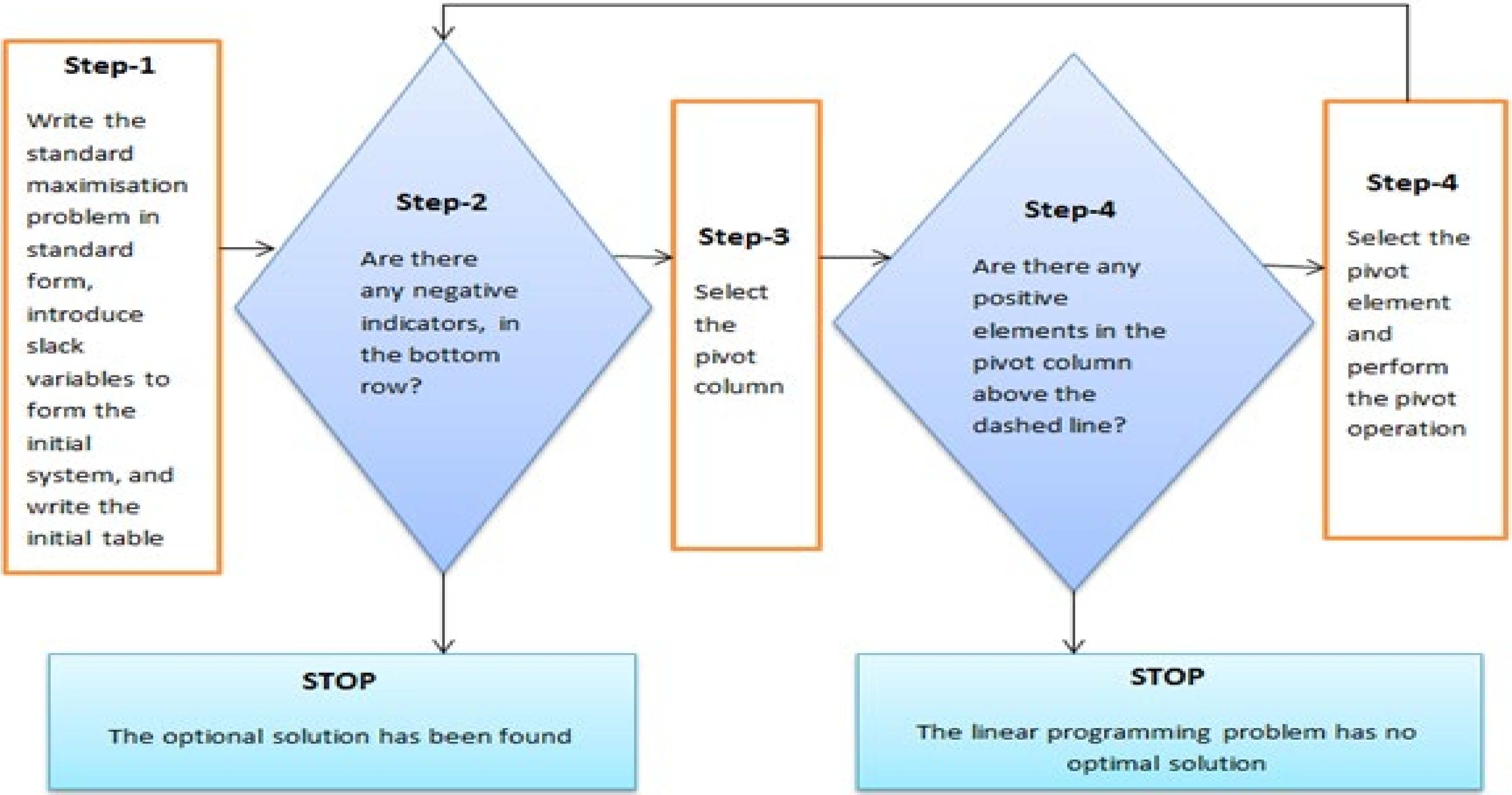
$$x_1 + 2x_2 + s_1 \quad = 32$$

$$3x_1 + 4x_2 + s_2 \quad = 84$$

Two equations:
Two basic variables
Two non-basic variables

- A solution found by setting the two non-basic variables to 0 and solving the basic variables is a basic solution
- If a basic solution has no negative values it is a basic feasible solution

SIMPLEX METHOD



SIMPLEX METHOD

- To solve a LP problem in standard form,
  1. Convert each inequality in the set of constraints to an equation by adding slack variables
  2. Create the initial simplex tableau
  3. Select the pivot column (The column with the 'most negative value' in the last row)
  4. Select the pivot row (The row with the smallest non-negative result when the last element in the row is divided by the corresponding in the pivot column)
  5. Use elementary row operations to calculate the new values for the pivot row so that the pivot is one (Divide every number in the row by the pivot number)
  6. Use elementary row operations to make all numbers in the pivot column equal to 0 except for the pivot number. If all entries in the bottom row are 0 or positive, this is the final tableau. If not go back to step 3
  7. If you obtain a final tableau, then the linear programming problem maximum solution which is given by the entry in the lower-right corner of the tableau.

SIMPLEX TABLEAU

- A simplex tableau is a way to systematically evaluate variable mixes in order to find the best one

	All variables	Solution
Basic variables	coefficients	
		0

PIVOT

- **Pivot column** – The column of the tableau representing the variable to be entered in to the solution mix.
- **Pivot row** – The row of the tableau representing the variable to be replaced in the solution mix
- **Pivot number** – The element in both the pivot column and the pivot row

EXAMPLE

Problem Definition: The Cannon Hill furniture company produces tables and chairs. Each table takes 4 hours of labour from the carpentry department and 2 hours of labour from the finishing department. Each chair requires 3 hours of carpentry and 1 hour of finishing. During the current week, 240 hours of carpentry time are available and 100 hours of finishing time. Each table produced gives a profit of $70 and each chair a profit of $50. How many chairs and tables should be made to maximize the profit?

SIMPLEX METHOD - STEP 1

LP program formulation

All information about example			
Resource	Table s (x_1)	Chairs (x_2)	Constraints
Carpentry (hr)	4	3	240
Finishing (hr)	2	1	100
Unit Profit	\$70	\$50	

Objective function $\quad$ Max $P = 70x_1 + 50x_2$

Carpentry constraint $\quad 4x_1 + 3x_2 \leq 240$

Finishing constraint $\quad 2x_1 + 1x_2 \leq 100$

$x_1, x_2 \geq 0$

SIMPLEX METHOD - STEP 1

- Convert inequality to an equality

Suppose s_1 carpentry hours and s_2 finishing hours remain unused in a week.

So the constraints become:

$$4x_1 + 3x_2 + s_1 = 240$$
$$2x_1 + 1x_2 + s_2 = 100$$

or

$$4x_1 + 3x_2 + s_1 + 0s_2 = 240$$
$$2x_1 + 1x_2 + 0s_1 + s_2 = 100$$

Slack variables also could be included in the objective function with 0 coefficients:

$$P = 70x_1 + 50x_2 + 0s_1 + 0s_2$$

$$P - 70x_1 - 50x_2 - 0s_1 + 0s_2 = 0$$

SIMPLEX METHOD - STEP 1

- The problem can now be think as solving a system of 3 linear equations involving the 5 variables (x_1, x_2, s_1, s_2, P) in such a way that P has the maximum value.

$$4x_1 + 3x_2 + s_1 + 0s_2 = 240$$

$$2x_1 + 1x_2 + 0s_1 + s_2 = 100$$

$$P - 70x_1 - 50x_2 - 0s_1 + 0s_2 = 0$$

- This system of linear equations can be written in matrix form or as a 3x6 augmented matrix.

SIMPLEX METHOD - STEP 2

- The initial tableau is:

Basic Variables	x_1	x_2	S_1	S_2	P	Right Hand Side
S_1	4	3	1	0	0	240
S_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0

$4x_1 + 3x_2 + s_1 + 0s_2 = 240$
$2x_1 + 1x_2 + 0s_1 + s_2 = 100$
$P - 70x_1 - 50x_2 - 0s_1 + 0s_2 = 0$

- This tableau represents the initial solution

$x_1=0 \quad x_2=0 \quad s_1=240 \quad s_2=100 \quad P=0$

SIMPLEX METHOD - STEP 3

- Select the pivot column (determine which variable to enter into the solution mix)
  - The column with the 'most negative value' in the last row

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	4	3	1	0	0	240
S_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0

Pivot Column

- x_1 should enter into the solution mix

SIMPLEX METHOD - STEP 4

- Select pivot row (determine which variable to replace in the solution mix)
  - The row with the smallest non-negative result when the last element in the row is divided by the corresponding in the pivot column

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side	
S_1	4	3	1	0	0	240	240 / 4 = 60
S_2	2	1	0	1	0	100	100 / 2 = 50
P	-70	-50	0	0	1	0	

Pivot Column

SIMPLEX METHOD – STEP 4

- Select pivot row (determine which variable to replace in the solution mix)

Enter

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	4	3	1	0	0	240
S_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0

Exit

Pivot number

Pivot Column

Pivot row

- S_2 should be replaced by x_1

SIMPLEX METHOD – STEP 5

- Calculate the new values for the pivot row so that the pivot is one.
  - Divide every number in the row by the pivot number

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	4	3	1	0	0	240
x_1	1	1/2	0	1/2	0	50
P	-70	-50	0	0	1	0

$\frac{R_2}{2}$

SIMPLEX METHOD – STEP 6

- Make all numbers in the pivot column equal to 0 except for the pivot number.

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side	
S_1	4	3	1	0	0	240	$-4R_2 + R_1$
x_1	1	1/2	0	1/2	0	50	
P	-70	-50	0	0	1	0	$70R_2 + R_3$

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	0	1	1	-2	0	40
x_1	1	1/2	0	1/2	0	50
P	0	-15	0	35	1	3500

SIMPLEX METHOD – STEP 6

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	0	1	1	-2	0	40
x_1	1	1/2	0	1/2	0	50
P	0	-15	0	35	1	3500

$x_1=50$ $x_2=0$ $s_1=40$ $s_2=0$ $P=3500$

- Repeat the steps until the final tableau is achieved (until there are no negative numbers in the last row)

SIMPLEX METHOD – REPEAT STEPS

- Select the new pivot column - x_2 should enter into the solution mix
- Select the new pivot row - s_1 should be replaced by x_2 in the solution mix

Enter

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	0	1	1	-2	0	40
x_1	1	1/2	0	1/2	0	50
P	0	-15	0	35	1	3500

Exit

40/1 = 40

50/0.5 = 100

New Pivot number

New Pivot Column

New Pivot row

SIMPLEX METHOD – REPEAT STEPS

- Calculate new values for pivot row
- Make all numbers in the pivot column equal to 0 except for the pivot number

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	0	1	1	-2	0	40
x_1	1	1/2	0	1/2	0	50
P	0	-15	0	35	1	3500

$-\frac{1}{2}R_1 + R_2$

$15R_1 + R_3$

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
x_2	0	1	1	-2	0	40
x_1	1	0	-1/2	3/2	0	30
P	0	0	15	5	1	4100

SIMPLEX METHOD – RESULT

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
x_2	0	1	1	-2	0	40
x_1	1	0	-1/2	3/2	0	30
P	0	0	15	5	1	4100

- This simplex tableau represent the optimal solution to the LP problem and is interpreted as:

x_1=30 x_2=40 s_1=0 s_2=0

and profit or P=$4100

MINIMIZATION WITH PROBLEM CONSTRAINTS OF THE FORM ≥

- The Dual Problem
  - Duality relationship of LP problems
    - Every maximization problem is associated with a minimization problem
    - Every minimization problem is associated with a maximization problem
  - Problem with its objective function as maximization can be written in its minimization version
    - Original LP problem is known as primal problem
    - Derived LP problem is known as dual problem

The Duality Principle: The objective function of the minimization problem reaches its minimum if and only if the objective function of its dual reaches its maximum. And when they do, they are equal.

STANDARD MINIMIZATION PROBLEM

- A **standard minimization problem** is a linear programming problem in which we seek to minimize an objective function

$$C = c_1x_1 + \dots + c_nx_n$$

subject to the constraints

$$a_{11}x_1 + \dots + a_{1n}x_n \geq b_1$$

$$\vdots$$

$$a_{m1}x_1 + \dots + a_{mn}x_n \geq b_m$$

where all $x_i \geq 0$ and $b_i \geq 0$

STANDARD MINIMIZATION PROBLEM

- Primal and Dual problems

Consider the following minimization problem

Minimize	$C = 4x + 2y$
Subject to:	$5x + y \geq 5$
	$5x + 3y \geq 10$
	$x \geq 0,$
	$y \geq 0$

STANDARD MINIMIZATION PROBLEM

- Primal and Dual problems

Minimize	$C = 4x + 2y$
Subject to:	$5x + y \geq 5$
	$5x + 3y \geq 10$
	$x \geq 0,$
	$y \geq 0$

Step 1: We first write down the following tableau for the given primal problem:

x	y	
5	1	5
5	3	10
4	2	

- Not a typical Simplex tableau.
- There are no slack variables,
- Don't rewrite the objective function for the last row

STANDARD MINIMIZATION PROBLEM

- Primal and Dual problems

Minimize $C = 4x + 2y$

Subject to:
$5x + y \geq 5$
$5x + 3y \geq 10$
$x \geq 0,$
$y \geq 0$

Step 2: Get the transpose of the tableau:

x	y	
5	1	5
5	3	10
4	2	

u	v	
5	5	4
1	3	2
5	10	

Note: The **transpose** of matrix A, denoted A^T, is the matrix that switches the rows and columns of matrix A

STANDARD MINIMIZATION PROBLEM

- Primal and Dual problems

Minimize	$C = 4x + 2y$
Subject to:	$5x + y \geq 5$
	$5x + 3y \geq 10$
	$x \geq 0,$
	$y \geq 0$

Step 3: Construct the dual problem:

u	v	
5	5	4
1	3	2
5	10	

Maximize	$P = 5u + 10v$
Subject to:	$5u + 5v \leq 4$
	$1u + 3v \leq 2$
	$u \geq 0,$
	$v \geq 0$

STANDARD MINIMIZATION PROBLEM

- Solve dual problem using simplex method

Maximize $P = 5u + 10v$

Subject to:
$5u + 5v \leq 4$
$1u + 3v \leq 2$
$u \geq 0,$
$v \geq 0$

Convert inequalities into equalities

$$5u + 5v + 1x + 0y = 4$$

$$1u + 3v + 0x + 1y = 2$$

$$P - 5u - 10v + 0x + 0y = 0$$

STANDARD MINIMIZATION PROBLEM

- Solve dual problem using simplex method

$5u + 5v + 1x + 0y = 4$
$1u + 3v + 0x + 1y = 2$
$P - 5u - 10v + 0x + 0y = 0$

Initial simplex tableau

Basic Variables	u	v	x	y	P	Right Hand Side
x	5	5	1	0	0	4
y	1	3	0	1	0	2
P	-5	-10	0	0	1	0

Initial solution: $u = 0$ $v = 0$ $x = 4$ $y = 2$ $P = 0$

STANDARD MINIMIZATION PROBLEM

- Solve dual problem using simplex method

Select Pivot column

Basic Variables	u	v	x	y	P	Right Hand Side
x	5	5	1	0	0	4
y	1	3	0	1	0	2
P	-5	-10	0	0	1	0

Pivot column

STANDARD MINIMIZATION PROBLEM

- Solve dual problem using simplex method

Select Pivot row

Basic Variables	u	v	x	y	P	Right Hand Side	
x	5	5	1	0	0	4	4/5 = 0.8
y	1	3	0	1	0	2	2/3 = 0.66
P	-5	-10	0	0	1	0	

Pivot column

Pivot row

STANDARD MINIMIZATION PROBLEM

- Solve dual problem using simplex method

Calculate new values for pivot row so pivot number is one

Basic Variables	u	v	x	y	P	Right Hand Side
x	5	5	1	0	0	4
v	1/3	1	0	1/3	0	2/3
P	-5	-10	0	0	1	0

STANDARD MINIMIZATION PROBLEM

- Solve dual problem using simplex method

Make all numbers in the pivot column equal to 0 except for the pivot number

Basic Variables	u	v	x	y	P	Right Hand Side
x	10/3	0	1	-5/3	0	2/3
v	1/3	1	0	1/3	0	2/3
P	-5/3	0	0	10/3	1	20/3

STANDARD MINIMIZATION PROBLEM

- Solve dual problem using simplex method

Select new pivot column

Basic Variables	u	v	x	y	P	Right Hand Side
x	10/3	0	1	-5/3	0	2/3
v	1/3	1	0	1/3	0	2/3
P	-5/3	0	0	10/3	1	20/3

STANDARD MINIMIZATION PROBLEM

- Solve dual problem using simplex method

Select new pivot row

Basic Variables	u	v	x	y	P	Right Hand Side	
x	10/3	0	1	-5/3	0	2/3	2/3 ÷ 10/3 = 1/5
v	1/3	1	0	1/3	0	2/3	2/3 ÷ 1/3 = 2
P	-5/3	0	0	10/3	1	20/3	

STANDARD MINIMIZATION PROBLEM

- Solve dual problem using simplex method

Calculate new values for pivot row so pivot number is one

Basic Variables	u	v	x	y	P	Right Hand Side
u	1	0	3/10	-1/2	0	1/5
v	1/3	1	0	1/3	0	2/3
P	-5/3	0	0	10/3	1	20/3

STANDARD MINIMIZATION PROBLEM

- Solve dual problem using simplex method

Make all numbers in the pivot column equal to 0 except for the pivot number

Basic Variables	u	v	x	y	P	Right Hand Side
u	1	0	3/10	-1/2	0	1/5
v	0	1	-1/10	1/2	0	3/5
P	0	0	1/2	5/2	1	7

Final tableau

Solution for the dual problem: u = 1/5 v = 3/5 P = 7

STANDARD MINIMIZATION PROBLEM

- Solution for the primal problem

Basic Variables	u	v	x	y	P	Right Hand Side
u	1	0	3/10	-1/2	0	1/5
v	0	1	-1/10	1/2	0	3/5
P	0	0	1/2	5/2	1	7

Final tableau

Solution for the primal problem: $x = 1/2$ $y = 5/2$ $C = 7$

SUMMARY

- Simplex method
- Solving standard maximization problem
- Solving standard minimization problem