CSCB63 Winter 2021

Week 2 - Balanced Trees

Anna Bretscher

January 18, 2021

Why Balanced Trees?

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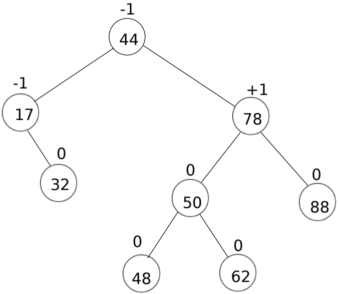
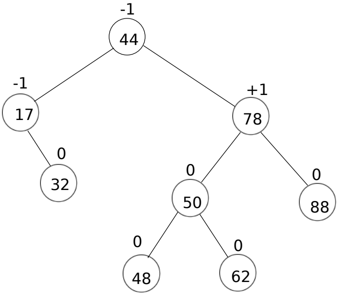
- The height of an *AVL* tree is $O(\log n)$.
- Each *internal node* has a *balance property* equal to -1, 0, 1.
- Balance value = *height* of the *left* subtree - *height* of the *right* subtree.

AVL Trees



Q. What is the purpose of the *balance* property?



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A. Ensures that the *height* is always a function of $\log n$.

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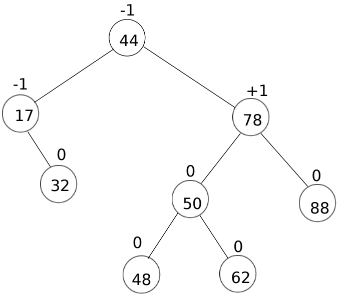

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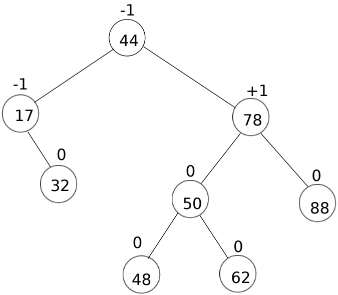

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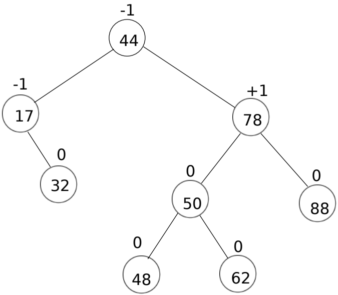
A. The *height* of the tree rooted at each node.

AVL Operations



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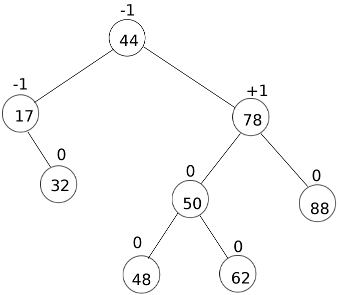

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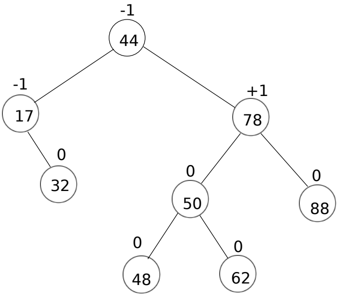

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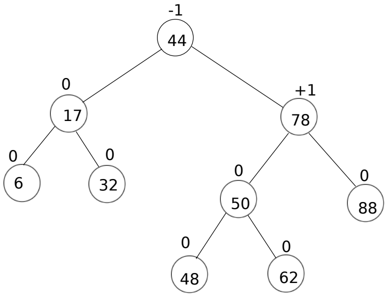

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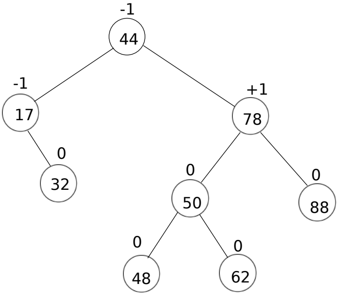
A: 17 has value 0

AVL Insertion

Q. Let's insert 35. What are the *balance factors* now?



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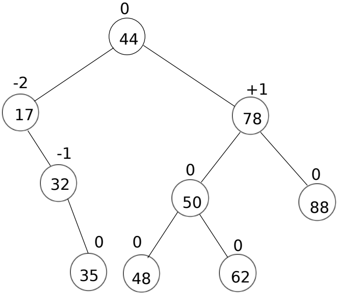

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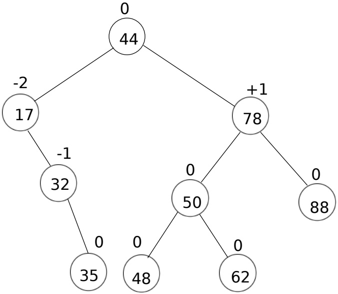

A. 32 has -1, 17 has -2. 44 has 0.

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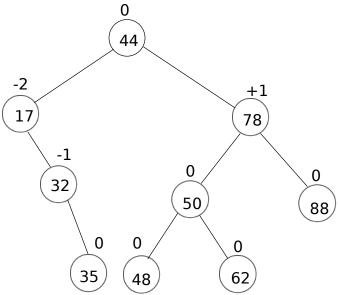

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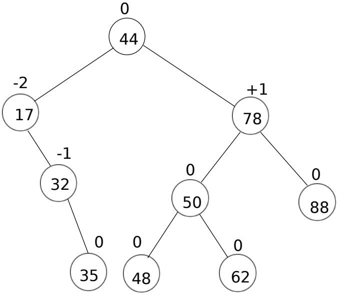

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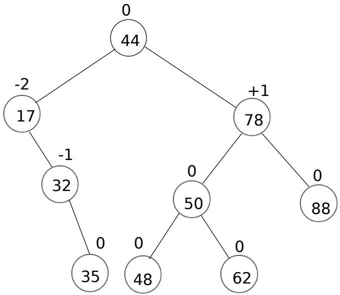

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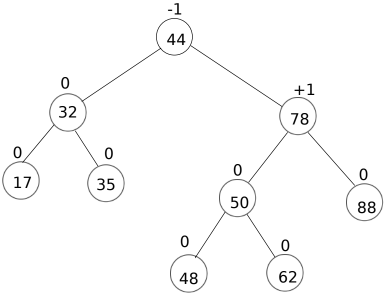

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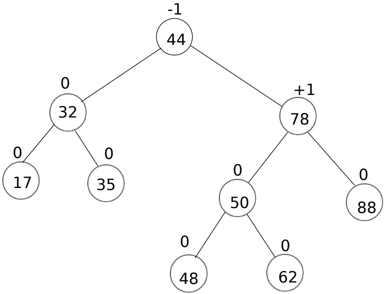

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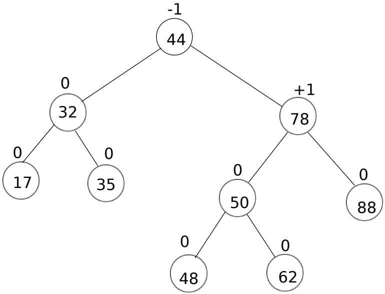
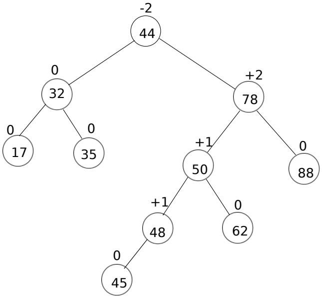


Q. How do we update the *balance factors*?

A. Update the heights first and then update balance factors.

AVL Insertion

Now let's insert *45*.



- Notice the *balance factors* now. How should we resolve the problem?

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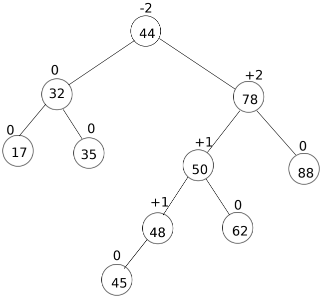

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A. Do a *single rotation* clock-wise about the *78*. 50 goes *up*, 78 *down*.

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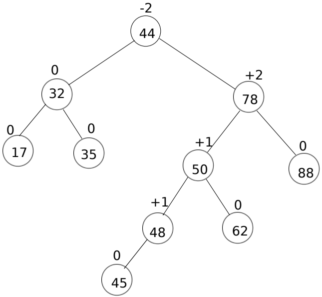

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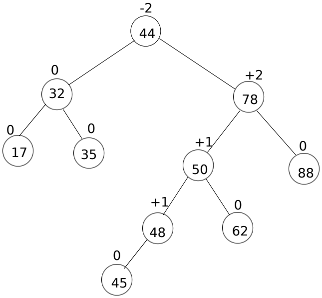

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Q. What happens to the subtree rooted at 62?

A. It becomes the left subtree of 78.

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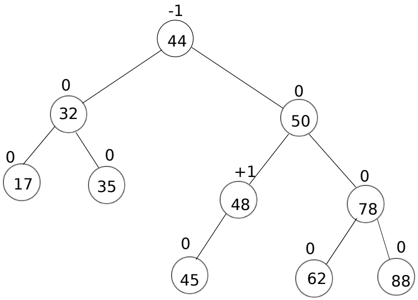

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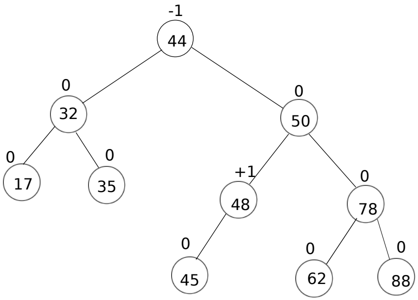
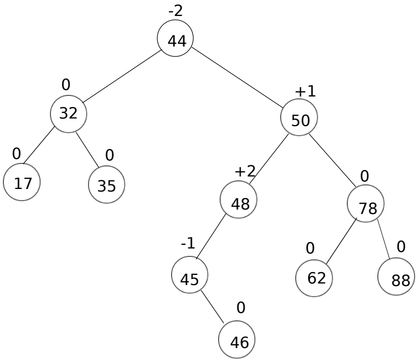


- Notice the updated *balance factors*.
- Let's insert *46* this time.

AVL Insertion



Q. Can we do a rotation about 48?

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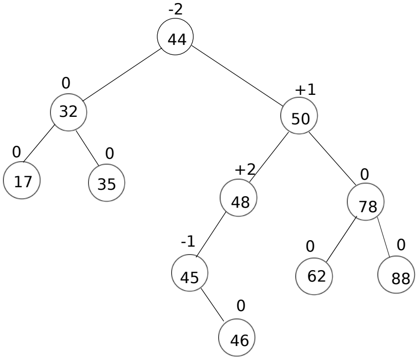
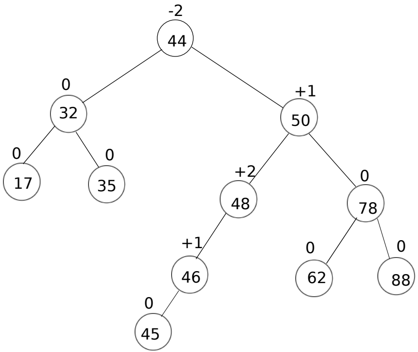


Q. Can we do a rotation about 48?

A. NO. Need a **double rotation**.

Double Rotation



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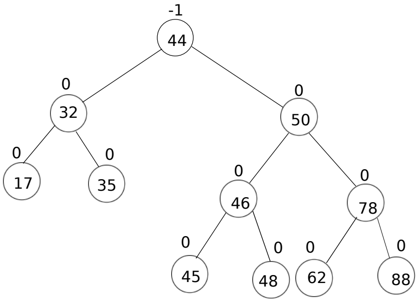

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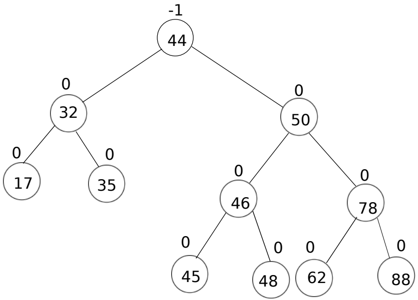

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A. Have change in sign of the balance factors (+2, -1 or -2, +1).

Delete

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- If the key is an internal node, replace with predecessor/successor and rebalance.

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- This means *insert*, *delete* and *search* are all $O(\log n)$.
- Searching for the location to *insert* or *delete*, takes $O(\log n)$.
- *Rebalancing* takes at most $O(\log n)$.

If there are *n* nodes, what is the *maximum* possible *height h*?

Q. What is *another* way to say this in terms of the number of nodes *n*?

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AVL Tree Height

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Then, we can say:

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Q. Does this *remind* you of anything?

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$$minsize(h) = \frac{\phi^{h+2} - (1-\phi)^{h+2}}{\sqrt{5}} - 1$$

AVL Tree Height

$$\begin{aligned} minsize(h) &= \frac{\phi^{h+2}}{\sqrt{5}} - \frac{(1-\phi)^{h+2}}{\sqrt{5}} - 1 \\ &> \frac{\phi^{h+2}}{\sqrt{5}} - 1 - 1 \\ n \geq minsize(h) &> \frac{\phi^{h+2}}{\sqrt{5}} - 1 - 1 \end{aligned}$$

Solving for h:

$$\frac{\phi^{h+2}}{\sqrt{5}} - 2 < n$$

$$\begin{aligned} h &< \frac{\lg(n+2)}{\lg\phi} + \frac{\sqrt{5}}{\lg\phi} + 2 \\ &= 1.44\lg(n+2) + \text{constant} \\ &\in O(\lg n) \end{aligned}$$