CSCB63 Winter 2021

Week 5 Lecture 2 - Minimum Cost Spanning Trees

Anna Bretscher

February 8, 2021

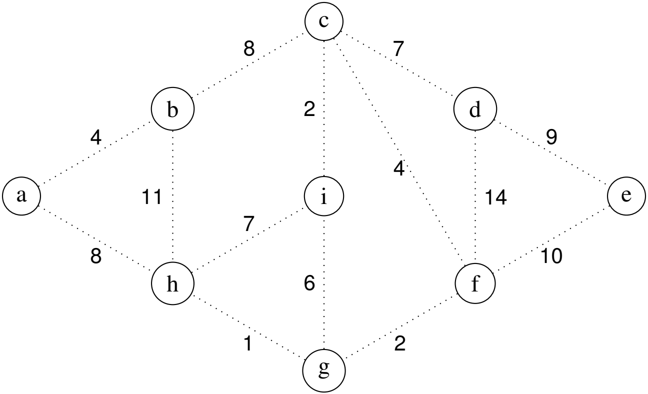
TODAY

Kruskals Algorithm

Prims Algorithm

Dijkstra's Algorithm

INTRODUCTION: (EDGE-)WEIGHTED GRAPHS



These are computers and costs of direct connections. What is a cheapest way to network them?

(EDGE-)WEIGHTED GRAPH

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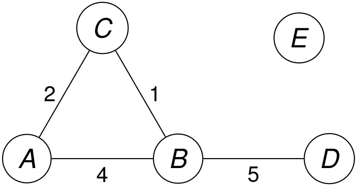
- **Notation**: $w(u, v)$ or $w(e)$ or $weight(u, v)$ etc.

Storing a Weighted Graph



Adjacency matrix:

	A	B	C	D	E
A	0	4	2	∞	∞
B	4	0	1	5	∞
C	2	1	0	∞	∞
D	∞	5	∞	0	∞
E	∞	∞	∞	∞	0

Adjacency lists:

	adjacency list
A	$(B,4)$, $(C,2)$
B	$(A,4)$, $(C,1)$, $(D,5)$
C	$(A,2)$, $(B,1)$
D	$(B,5)$
E	

Common Task #1 on Weighted Graphs - Minimum Cost Spanning Trees

Let $G = (V, E)$ be a *connected, undirected* graph with *edge weights* $w(e)$ for each edge $e \in E$.

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A *spanning tree* is a tree A such that *every vertex* $v \in V$ is an *endpoint* of at least *one edge* in A.

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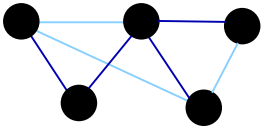

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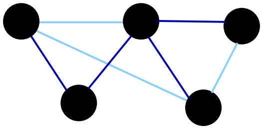
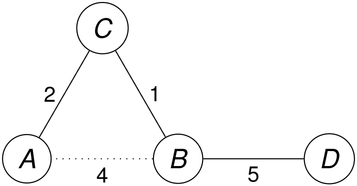


A.

A *minimum cost spanning tree* (**MST**) is a spanning tree A such that the *sum of the weights is minimum* for all possible spanning trees B.

$$w(A) = \sum_{e \in A} w(e) \leq w(B)$$

EXAMPLE

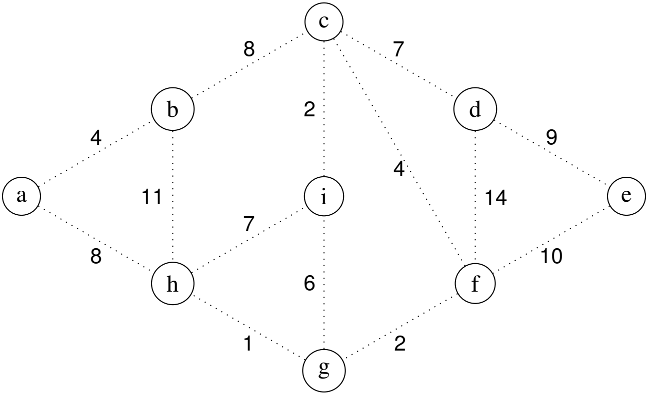


Usually just for *undirected, connected graphs*.

Q. How might we find a *minimum spanning tree*?

A.

Sample Graph



Kruskal's Algorithm: Proof of Correctness

Kruskal's algorithm finds an *MST* by

Proof by Contradiction.

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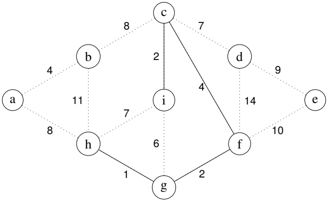

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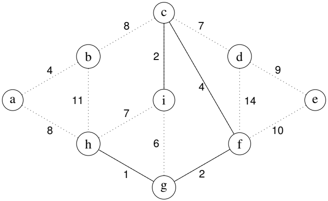

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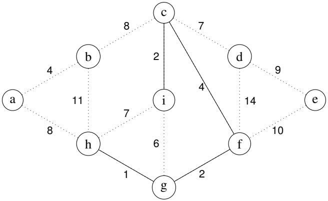

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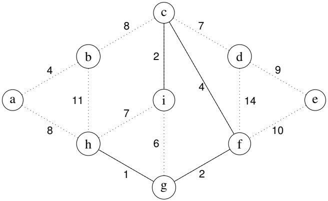

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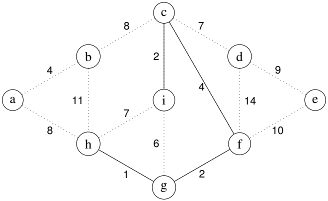

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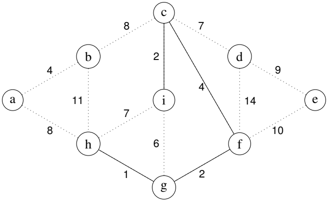

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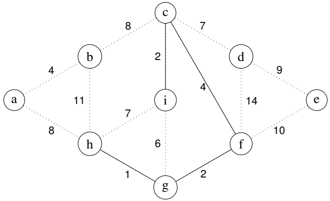

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Kruskal's algorithm finds an *MST* by repeatedly adding the *least weight edge* that does *not induce* a *cycle*.

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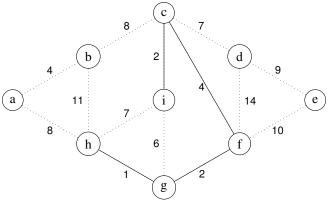

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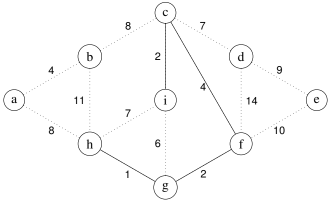

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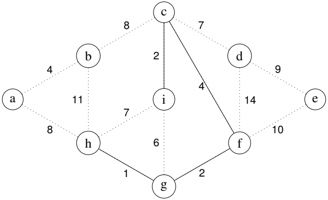

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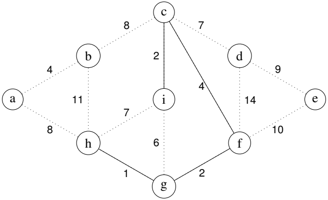
Case 1. $w(e') = w_i$.

Case 2. $w(e') > w_i$.



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```
Kruskal(E, V)
    S := new container() for chosen edges
    PQ := min priority queue of edges and weights
    for each vertex v:
        v.cluster := {v}
    while not PQ.is_empty():
        {u,v} = PQ.extract_min():
        if u.cluster ≠ v.cluster:
            S.add({u,v})
            union(u.cluster, v.cluster)
    return S
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Storing Clusters: Easy Way - Linked Lists

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We will see a *faster* way *later* in this course.

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Total $\mathcal{O}(n \lg n + m \lg m)$ time worst case.

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Faster if faster cluster implementation.

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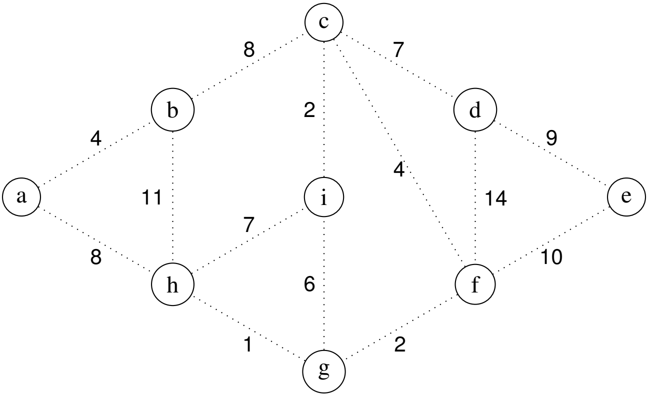
Priority of vertex v =

Let's step through the example again...

Prim's Algorithm: A Few Example Steps



vertex	a	b	c	d	e	f	g	h	i
priority	0	∞	∞	∞	∞	∞	∞	∞	∞
pred									

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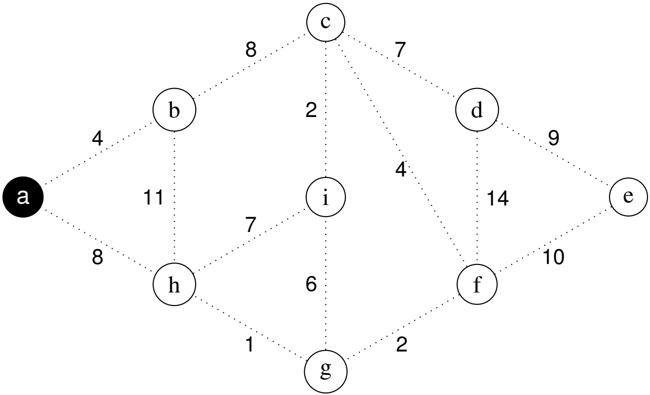

vertex	b	h	c	d	e	f	g	i
priority	4	8	∞	∞	∞	∞	∞	∞
pred	a	a						

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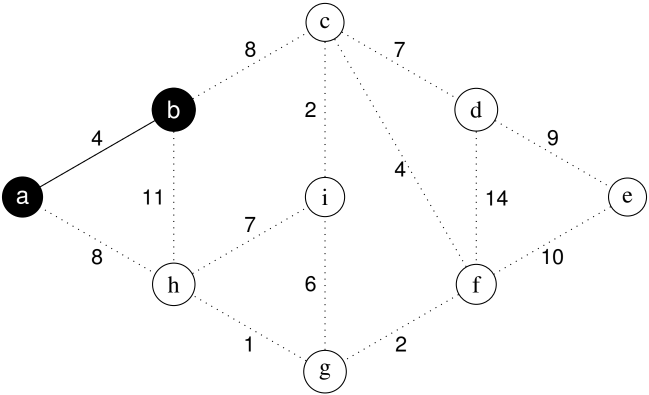
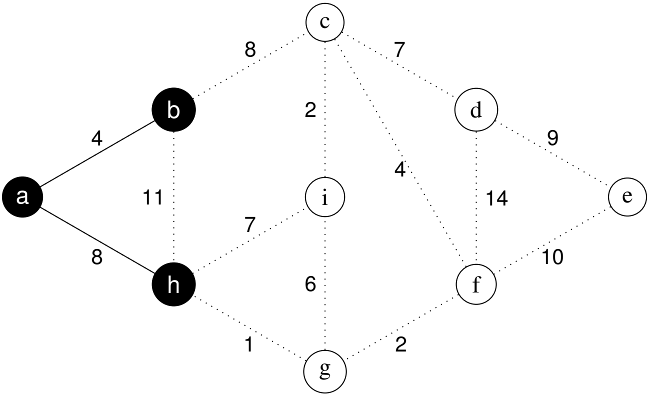


vertex	h	c	d	e	f	g	i
priority	8	8	∞	∞	∞	∞	∞
pred	a	b					

PRIM'S ALGORITHM: A FEW EXAMPLE STEPS



vertex	g	i	c	d	e	f
priority	1	7	8	∞	∞	∞
pred	h	h	b			

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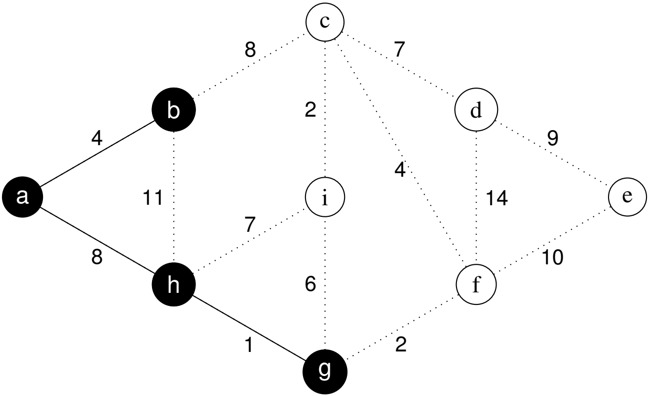

vertex	f	i	c	d	e
priority	2	6	8	∞	∞
pred	g	g	b		

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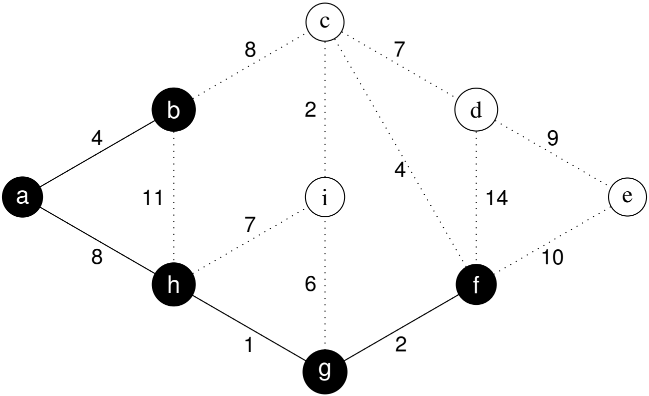
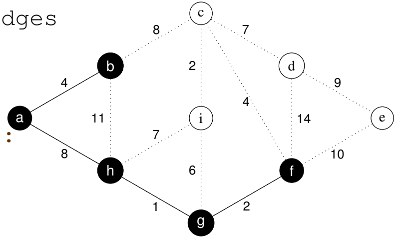


vertex	c	i	e	d
priority	4	6	10	14
pred	f	g	f	f

Prim's Algorithm

```
Prim(V, E)
    S := new container() for edges
    PQ := new min-heap()
    start := pick a vertex
    PQ.insert(start, 0)
     for each vertex v ≠ start:
        # initialize pq
        PQ.insert(v, ∞)
     while not PQ.is_empty():
        # add least edge to grow the tree
         u := PQ.extract_min()
         S.add({u.pred, u})
     for each z in u's adjacency list:
        # update priorities based on u now in S
        if z in PQ && weight(u,z) < priority of z:
            PQ.decrease_priority(z, weight(u,z))
            z.pred := u
    return S
```



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- Everything else, can be done in $\Theta(1)$ per *vertex* or per *edge*
- Total $O((n+m)\lg n)$ time worst case.

PRIM'S CORRECTNESS PROOF

To begin with we will first prove a useful property:

Cut Property: Let S be a nontrivial subset of V in G (i.e. $S \neq \emptyset$ and $S \neq V$). If (u, v) is the *lowest-cost edge* crossing $(S, V - S)$, then (u, v) is in *every MST* of G.

Proof.

- Suppose there exists an *MST* T that does not contain (u, v).

Prim's Correctness Proof

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★ L02's notes have a different but similar template - another perspective.